Notes on breeze velocity, temperature, and density profiles

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1 Basics of the outflows from January 6, 2020

1.1 General setup

Let us establish some basic properties of subsonic outflows from a protoneutron star surface. We will consider an idealized model, with entropy per baryon set at the beginning and constant throughout the flow. In this case, we have a single governing ordinary differential equation,

$$\frac{dv}{dr} = \frac{v}{r}\frac{2v_s^2 - GM/r}{v^2 - v_s^2}, \tag{1}$$

supplemented by an algebraic constraint

$$I \equiv v^2/2 - GM/r + 3v_s^2 = v^2/2 - GM/r + TS/m_N = const. \tag{2}$$

Observe that the terms making up I are the kinetic energy, gravitational potential energy, and enthalpy of a unit test mass. The constraint says that the sum of these terms is conserved along flow lines.

The quantity S is the entropy of radiation per baryon. Explicitly,

$$S = \frac{4aT^3}{3}\frac{1}{n_N} = \frac{4aT^3}{3}\frac{m_N}{\rho}. \tag{3}$$

In turn,

$$a = \frac{g_\star \pi^2}{30}, \tag{4}$$

which follows from the energy density calculation

$$U = \frac{g_\star}{(2\pi)^3}\int_0^\infty \frac{4\pi E^2 E}{\exp(E/T)-1}dE = \frac{g_\star T^4}{2\pi^2}\int_0^\infty \frac{x^3}{\exp(x)-1}dx = \frac{g_\star T^4}{2\pi^2}\frac{\pi^4}{15}. \tag{5}$$

Eq. (2) can be used to eliminate the sound speed v_s from the differential equation, $v_s^2 = (I - v^2 + GM/r)/3$, resulting in

$$\frac{dv}{dr} = \frac{2v}{r}\frac{(2I - v^2 - GM/r)}{(-2I + 7v^2 - 2GM/r)}. \tag{6}$$

Given initial I and v at $r = r_0$, as well as the protoneutron star M, this equation gives us a velocity profile, $v(r)$.

1.2 Application to our conditions

As a practical matter, to obtain semi-realistic solutions, we start at $r_0 = 11$ km and assume the temperature there is 3 MeV. For the PNS mass, we take $M = 1.4M_\odot$ and for the entropy per baryon $S = 60$.

Of course, as already mentioned, this is an idealization. In reality, the outflow starts at the gain radius, where $S \sim 1$. The value of S then grows to 60 over tens of km.

While the choice of $S \sim 60$ is motivated by numerical simulations, it has an important physical meaning. The gravitational term at the surface r_0, in SI units, is

$$\frac{GM}{r} = \frac{6.674\times 10^{-11}(m^3kg^{-1}s^{-2})1.4\times 2\times 10^{30}(kg)}{11\times 10^3(m)} = 1.7\times 10^{16}(m^2/s^2). \tag{7}$$

In natural units, this is $1.7\times 10^{16}(m^2/s^2)/c^2 \simeq 0.19$.

Now, $S = 60$ is precisely the value which gives, for $T = 3$ MeV, the enthalpy term

$$\frac{TS}{m_N} = \frac{3(MeV)\times 60}{938(MeV)} \simeq 0.19. \tag{8}$$

In other words, when enthalpy is enough to overcome gravitational binding, the plasma starts to flow out.

As the flow merges into the medium of finite density and temperature, the final enthalpy is nonzero. Accordingly, the value of I is actually greater than zero and initial enthalpy has to be a bit greater than the absolute values

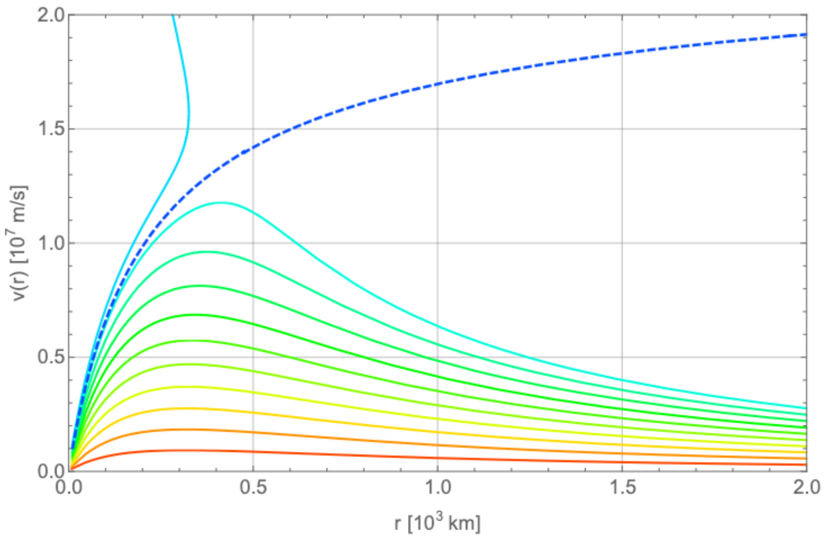

Figure 1: A family of subsonic curves for the chosen parameters. The corresponding supersonic outflow is shown by by a dashed curve.

of gravitational binding. However, the relative difference is small—most of the initial enthalpy goes to overcoming gravity.

Incidentally, this shows the futility of the old efforts to imagine that somehow $S \sim 300$: this implies that either $M \sim 7M_\odot$ or initial $T \sim 0.6$ MeV: clearly untenable. Once again, when enthalpy is enough to overcome gravitational binding, the plasma starts to flow out.

The last quantity to specify is the value of the initial velocity, $v(r_0)$. For the next, illustration, let us choose values $v_n(r_0) = n \times 10^5\ m/s$. The first ten values of n produce a family of subsonic curves. The next value, $n = 11$, exceeds the maximum (supersonic) value, giving an unphysical curve.

The solutions of Eq. (6) for these initial conditions are shown in Fig. 1. The outflow accelerates to 250-400 km, then decelerates to zero.

The corresponding speed of sound for each solution is easy to plot using the definition of I:

$$v_s = \left(\frac{I - v^2/2 + GM/r}{3} \right)^{1/2} \tag{9}$$

For illustration, Fig. 2 shows the v and v_s profiles for the $n = 10$ curve, as

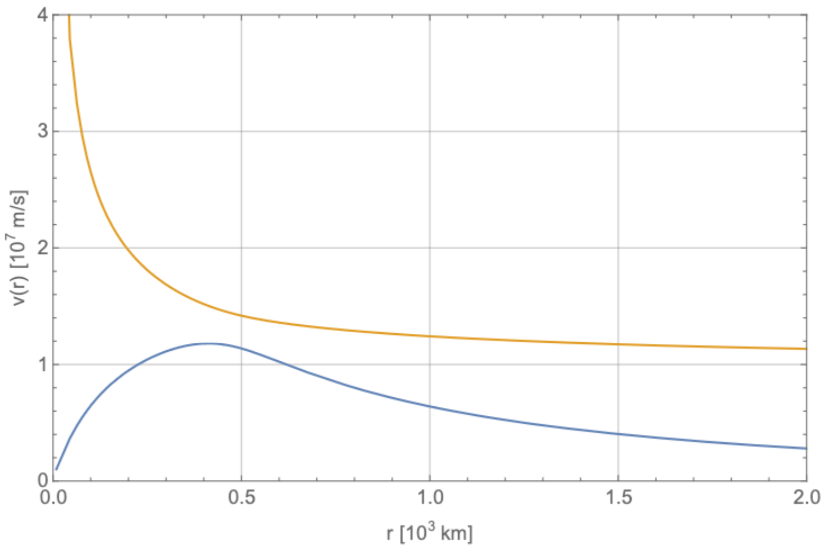

Figure 2: Velocity profile for $n = 10$ and the corresponding sound speed curve.

defined above.

The sound speed is, in general, $\sim 10^7$ m/s, or a few percent of the speed of light. The exception is close to the starting point, where it exceeds 10% of c. One may worry whether the treatment we follow here is consistent in this regime. Indeed, it is known from the CMB physics that the maximum speed of sound in plasma is $c/\sqrt{3}$. We will return to this below.

For now, let us suppose that $3v_s^2 = TS/m_N$ is valid throughout the range of r. Then, for the temperature, we immediately get

$$T = \frac{3v_s^2 m_N}{S} = \frac{(I - v^2/2 + GM/r)m_N}{S}. \tag{10}$$

The resulting temperature profiles are shown in Fig. 3. We see that actually the curves are remarkably close.

This can also be seen in the behavior of the density profile, which can be obtained starting from Eq. (3). We have,

$$\rho = \frac{4aT^3}{3}\frac{m_N}{S} = \frac{2g_\star \pi^2}{45}\frac{(I - v^2/2 + GM/r)^3 m_N^4}{S^4}. \tag{11}$$

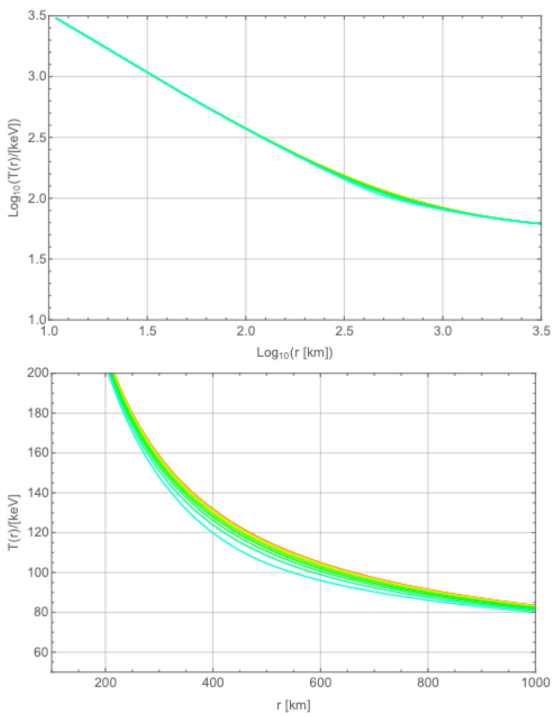

Figure 3: Temperature profiles corresponding to the velocity profiles shown in Fig. 1. The top panel shows the log-log scale, while the bottom one shows a linear zoom-in into the transition region from acceleration to deceleration, where the differences are largest.

The density curves are shown in Fig. 4.

The final density corresponding to the solutions we are considering is very low, $\mathcal{O}(10)$ g/cm^3.

This behavior can be understood, once again, by examining the terms that make up the invariant I: in units of 10^{14} m^2/s^2, the values are 2.92733, 2.92748, 2.92773, 2.92808, 2.92853, 2.92908, 2.92973, 2.93048, 2.93133, 2.93228, 2.93333. The change in v_i^2 makes a very small contribution and since I determines the final temperature (and density), all the solutions result in approximately the same final state. It should be noted here that even a small relative change in S will have a larger impact on the value of I, and hence

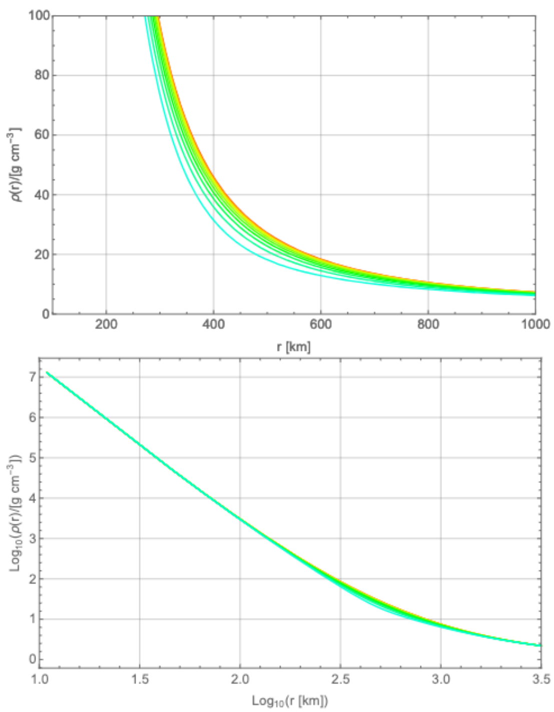

Figure 4: Density profiles corresponding to the velocity profiles shown in Fig. 1. The top panel shows the log-log scale, while the bottom one shows a linear zoom-in into the transition region from acceleration to deceleration, where the differences are largest.

on the final temperature. The details require further investigation.

1.3 Comment on the sound speed in plasma

Finally, let us briefly return to the validity of the sound speed expression, $3v_s^2 = TS/m_N$. It grows without limit with T and, clearly, cannot apply at very high temperatures. Let us recall its derivation:

$$v_s^2 = \left(\frac{dP}{dV}\right)\left(\frac{dV}{d\rho}\right) = \left(\frac{4aT^3}{3}\frac{dT}{dV}\right)\left(-\frac{V}{\rho}\right) = \frac{4aT^3}{3}\frac{T}{3\rho}. \tag{12}$$