Notes on breeze velocity, temperature, and density profiles

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1 Basics of the outflows from January 6, 2020

1.1 General setup

Let us establish some basic properties of subsonic outflows from a protoneutron star surface. We will consider an idealized model, with entropy per baryon set at the beginning and constant throughout the flow. In this case, we have a single governing ordinary differential equation,

$$\frac{dv}{dr} = \frac{v}{r}\frac{2v_s^2 - GM/r}{v^2 - v_s^2}, \tag{1}$$

supplemented by an algebraic constraint

$$I \equiv v^2/2 - GM/r + 3v_s^2 = v^2/2 - GM/r + TS/m_N = const. \tag{2}$$

Observe that the terms making up I are the kinetic energy, gravitational potential energy, and enthalpy of a unit test mass. The constraint says that the sum of these terms is conserved along flow lines.

The quantity S is the entropy of radiation per baryon. Explicitly,

$$S = \frac{4aT^3}{3}\frac{1}{n_N} = \frac{4aT^3}{3}\frac{m_N}{\rho}. \tag{3}$$

In turn,

$$a = \frac{g_\star \pi^2}{30}, \tag{4}$$

which follows from the energy density calculation

$$U = \frac{g_\star}{(2\pi)^3}\int_0^\infty \frac{4\pi E^2 E}{\exp(E/T)-1}dE = \frac{g_\star T^4}{2\pi^2}\int_0^\infty \frac{x^3}{\exp(x)-1}dx = \frac{g_\star T^4}{2\pi^2}\frac{\pi^4}{15}. \tag{5}$$

Eq. (2) can be used to eliminate the sound speed v_s from the differential equation, $v_s^2 = (I - v^2 + GM/r)/3$, resulting in

$$\frac{dv}{dr} = \frac{2v}{r}\frac{(2I - v^2 - GM/r)}{(-2I + 7v^2 - 2GM/r)}. \tag{6}$$

Given initial I and v at $r = r_0$, as well as the protoneutron star M, this equation gives us a velocity profile, $v(r)$.

1.2 Application to our conditions

As a practical matter, to obtain semi-realistic solutions, we start at $r_0 = 11$ km and assume the temperature there is 3 MeV. For the PNS mass, we take $M = 1.4M_\odot$ and for the entropy per baryon $S = 60$.

Of course, as already mentioned, this is an idealization. In reality, the outflow starts at the gain radius, where $S \sim 1$. The value of S then grows to 60 over tens of km.

While the choice of $S \sim 60$ is motivated by numerical simulations, it has an important physical meaning. The gravitational term at the surface r_0, in SI units, is

$$\frac{GM}{r} = \frac{6.674\times10^{-11}(m^3kg^{-1}s^{-2})1.4\times2\times10^{30}(kg)}{11\times10^3(m)} = 1.7\times10^{16}(m^2/s^2). \tag{7}$$

In natural units, this is $1.7\times10^{16}(m^2/s^2)/c^2 \simeq 0.19$.

Now, $S = 60$ is precisely the value which gives, for $T = 3$ MeV, the enthalpy term

$$\frac{TS}{m_N} = \frac{3(MeV)\times60}{938(MeV)} \simeq 0.19. \tag{8}$$

In other words, when enthalpy is enough to overcome gravitational binding, the plasma starts to flow out.

As the flow merges into the medium of finite density and temperature, the final enthalpy is nonzero. Accordingly, the value of I is actually greater than zero and initial enthalpy has to be a bit greater than the absolute values

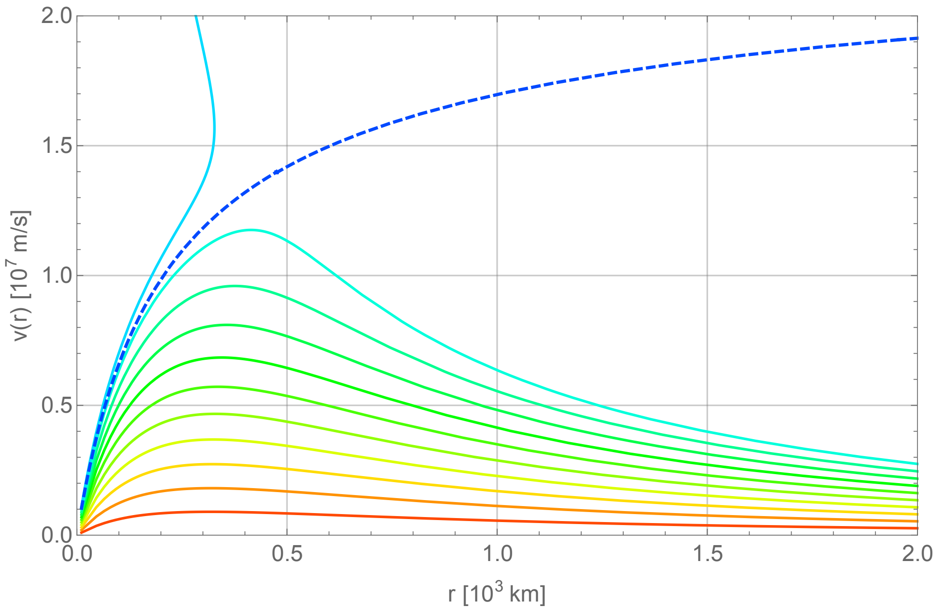

Figure 1: A family of subsonic curves for the chosen parameters. The corresponding supersonic outflow is shown by by a dashed curve.

of gravitational binding. However, the relative difference is small—most of the initial enthalpy goes to overcoming gravity.

Incidentally, this shows the futility of the old efforts to imagine that somehow $S \sim 300$: this implies that either $M \sim 7M_\odot$ or initial $T \sim 0.6$ MeV: clearly untenable. Once again, when enthalpy is enough to overcome gravitational binding, the plasma starts to flow out.

The last quantity to specify is the value of the initial velocity, $v(r_0)$. For the next, illustration, let us choose values $v_n(r_0) = n \times 10^5\ m/s$. The first ten values of n produce a family of subsonic curves. The next value, $n = 11$, exceeds the maximum (supersonic) value, giving an unphysical curve.

The solutions of Eq. (6) for these initial conditions are shown in Fig. 1. The outflow accelerates to 250-400 km, then decelerates to zero.

The corresponding speed of sound for each solution is easy to plot using the definition of I:

$$v_s = \left(\frac{I - v^2/2 + GM/r}{3} \right)^{1/2} \tag{9}$$

For illustration, Fig. 2 shows the v and v_s profiles for the $n = 10$ curve, as

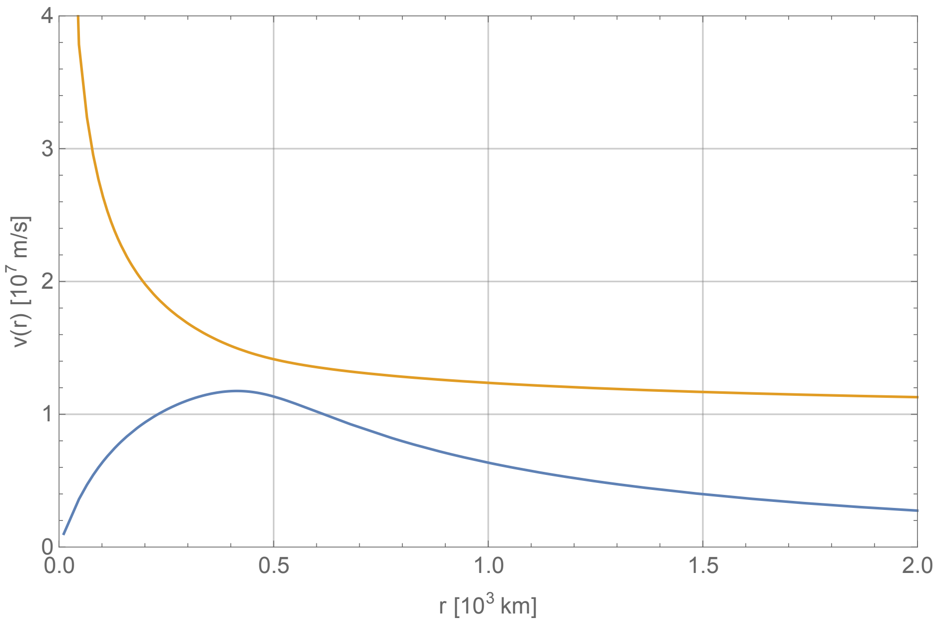

Figure 2: Velocity profile for $n = 10$ and the corresponding sound speed curve.

defined above.

The sound speed is, in general, $\sim 10^7$ m/s, or a few percent of the speed of light. The exception is close to the starting point, where it exceeds 10% of c. One may worry whether the treatment we follow here is consistent in this regime. Indeed, it is known from the CMB physics that the maximum speed of sound in plasma is $c/\sqrt{3}$. We will return to this below.

For now, let us suppose that $3v_s^2 = TS/m_N$ is valid throughout the range of r. Then, for the temperature, we immediately get

$$T = \frac{3v_s^2 m_N}{S} = \frac{(I - v^2/2 + GM/r)m_N}{S}. \tag{10}$$

The resulting temperature profiles are shown in Fig. 3. We see that actually the curves are remarkably close.

This can also be seen in the behavior of the density profile, which can be obtained starting from Eq. (3). We have,

$$\rho = \frac{4aT^3}{3}\frac{m_N}{S} = \frac{2g_\star \pi^2}{45}\frac{(I - v^2/2 + GM/r)^3 m_N^4}{S^4}. \tag{11}$$

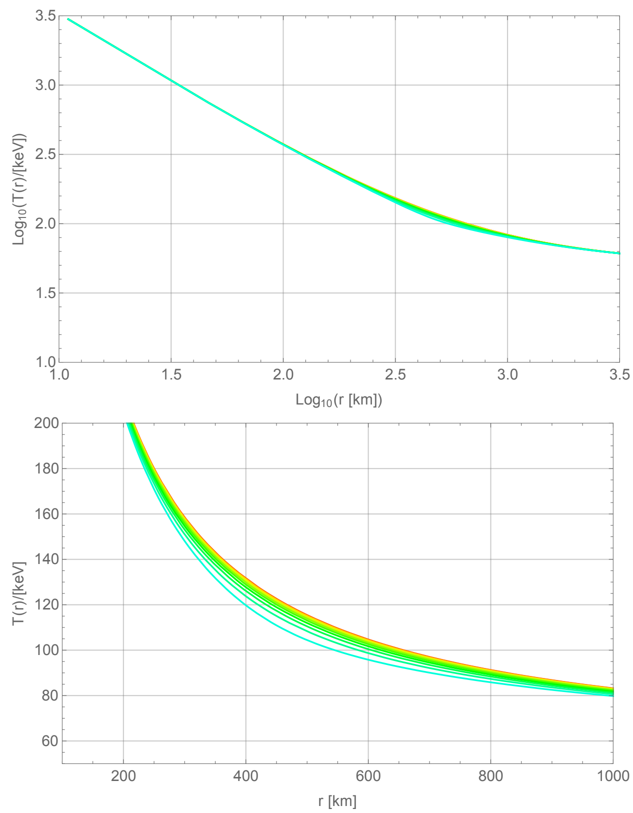

Figure 3: Temperature profiles corresponding to the velocity profiles shown in Fig. 1. The top panel shows the log-log scale, while the bottom one shows a linear zoom-in into the transition region from acceleration to deceleration, where the differences are largest.

The density curves are shown in Fig. 4.

The final density corresponding to the solutions we are considering is very low, $\mathcal{O}(10)\ g/cm^3$.

This behavior can be understood, once again, by examining the terms that make up the invariant I: in units of $10^{14}\ m^2/s^2$, the values are 2.92733, 2.92748, 2.92773, 2.92808, 2.92853, 2.92908, 2.92973, 2.93048, 2.93133, 2.93228, 2.93333. The change in v_i^2 makes a very small contribution and since I determines the final temperature (and density), all the solutions result in approximately the same final state. It should be noted here that even a small relative change in S will have a larger impact on the value of I, and hence

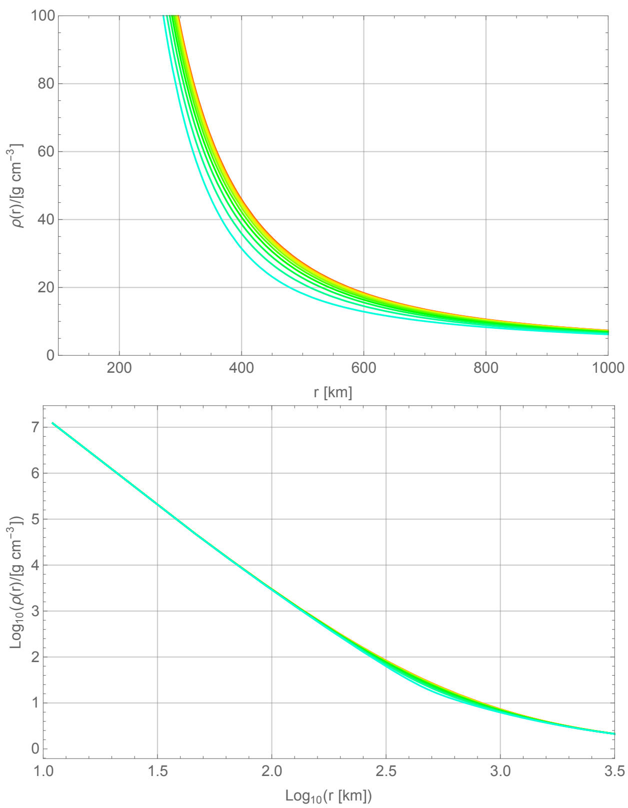

Figure 4: Density profiles corresponding to the velocity profiles shown in Fig. 1. The top panel shows the log-log scale, while the bottom one shows a linear zoom-in into the transition region from acceleration to deceleration, where the differences are largest.

on the final temperature. The details require further investigation.

1.3 Comment on the sound speed in plasma

Finally, let us briefly return to the validity of the sound speed expression, $3v_s^2 = TS/m_N$. It grows without limit with T and, clearly, cannot apply at very high temperatures. Let us recall its derivation:

$$v_s^2 = \left(\frac{dP}{dV}\right)\left(\frac{dV}{d\rho}\right) = \left(\frac{4aT^3}{3}\frac{dT}{dV}\right)\left(-\frac{V}{\rho}\right) = \frac{4aT^3}{3}\frac{T}{3\rho}. \tag{12}$$

In evaluating the derivative $d\rho/dV$, we have assumed that the energy density is dominated by baryons, so that $\rho V = const$. In the high-temperature limit, however, one also has to take into account the contribution to the density of radiation. One has,

$$\left(\frac{d\rho}{dV}\right) = \left(\frac{d\rho_{rad}}{dV} + \frac{d\rho_{bar}}{dV}\right) = \left(4aT^3\frac{dT}{dV} - \frac{\rho_{bar}}{V}\right), \tag{13}$$

and therefore

$$3v_s^2 = \left(1 - \frac{\rho_{bar}}{V}\frac{dV/dT}{4aT^3}\right)^{-1}. \tag{14}$$

By entropy conservation, $T^3V = const$, so that $3dT/T = -dV/V$. This means,

$$3v_s^2 = \left(1 + \frac{\rho_{bar}}{V}\frac{3V/T}{4aT^3}\right)^{-1} = \left(1 + \frac{3\rho_{bar}}{4aT^4}\right)^{-1}. \tag{15}$$

This is the famous sound speed formula used in CMB physics. For CMB, the first term in parentheses makes a leading contribution, with the second one describing "baryon loading". In our case, it is the second term that dominates. The expression we have been using, $3v_s^2 = TS/m_N$, corresponds to neglecting 1 compared to $3\rho_{bar}/4aT^4$. By including 1, we limit the growth of v_s^2 with T, so that it never exceeds 1 (c^2 in the SI units).

2 Physics of the termination shock

2.1 Motivation

So far, we have been studying subsonic outflows by imposing the boundary condition at infinity. We found that there exists a minimal value of the far density (or, more accurately, far pressure) beyond which no solutions with the shooting method could be obtained. We argued that beyond this critical value ρ_{crit} the solution goes transonic and we have a qualitatively different regime, with a termination shock located some distance away from the PNS. Two questions now arise:

- How can one understand and model the full flow in the transonic regime, including the shock?
- Is there any notion of continuity between the two regimes?

Let us start with the second question. At first sight, it is not obvious that such continuity should exist. After all, the density profiles—one with a finite density jump and the other smooth everywhere—are obviously qualitatively different. And such differences are expected on general physical grounds: in the subsonic case, the entire flow is in causal contact, while in the transonic case the information about the conditions at infinity travel only up to the point of the shock. Nevertheless, let us pursue this issue a bit deeper.

One may ask the question this way: what happens when the far density is only slightly below the critical value, i.e., as $\rho_\infty \to \rho_{critical}$ from below? Does one still get a finite density jump? Does the solution abruptly jump from "a jump" to "no jump" as $\rho_\infty = \rho_{crit}$?

A crucial clue comes from examining the critical solution. That solution is obtained as a limit of the family of subsonic curves. We have seen that its velocity hits the sound speed at a single point. This means that information travels from infinity only to that point. Thus, it may be reasonable to think that it is in the same general class as the transonic solutions and view the kink at the sonic point as a version of a shock. A "gentle" shock, since in this case we do not have a discontinuity in the function $v(r)$, but only in its derivative, $dv(r)/dr$.

Calling this curve *critical* is actually very helpful, as this brings to mind an analogy with a textbook problem of a phase transition between a gas and a liquid phases of water. Far below the critical point, one has a clear gas-liquid separation, with a density jump between the phases, while far above, there is no notion of different phases. At the critical point, which is the end of the line on a phase digram separating the gas and liquid phases, the phase transition becomes second order. This means quantities such as density become continuous, but the discontinuity in their derivatives persists.

This analogy, in fact, turns out to be very fruitful. The relevant observation is what happens as one approaches the critical point from the low temperature direction: the density contrast between gas and liquid decreases. Accordingly, we conjecture that in our system as the far density approaches the critical value from below, the discontinuity across the shock decreases: $\rho_2/\rho_1 \to 1$ as $\rho_\infty \to \rho_{crit}$. Moreover, the location of the disconinuity should smoothly approach the sonic point, such that in the critical case it merges with the sonic point.

2.2 Shock conditions

Like any shock phanomenon, the termination shock is characterized by the three Rankine-Hugoniot conditions:

$$\rho_1 v_1 = \rho_2 v_2, \tag{16}$$

$$\rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2, \tag{17}$$

$$\frac{v_1^2}{2} + h_1 = \frac{v_2^2}{2} + h_2. \tag{18}$$

Here, the first equation expresses conservation of mass, the second one conservation of momentum, and the last one conservation of energy, in the form of the sum of specific kinetic energy and specific enthalpy, $h = (U + P)/\rho$.

An important observation for our problem can be immediately made by examining this last condition. Since in our case specific entropy is given by $TS/m_N = 3v^2$, we can rewrite the equation as $v^2/2+3v_s^2 = const$. Adding the gravitation term, GM/r, on both sides, reveals that this equation is nothing but the conservation of our invariant I across the shock. Hence, the invariant I is conserved always, with or without the presence of the termination shock. Since the shock involves non adiabatic compression that does not conserve entropy, we see that conservation of I does not rely on conservation of S. The only condition is the absence of neutrino heating.

This observation immediately allows us to write down expressions for the temperature and density profiles in the shocked flow using Eqs. (10) and (11):

$$T = \frac{(I - v^2/2 + GM/r)m_N}{S}, \tag{19}$$

$$\rho = \frac{2g_\star\pi^2(I - v^2/2 + GM/r)^3}{45}\frac{m_N^4}{S^4}. \tag{20}$$

The discontinuities in both T and ρ at the shock front appear as a result of the jumps in S and v.

We can use the above expressions to give us to match to the boundary conditions on density or pressure at large radii,

$$T_\infty = \frac{Im_N}{S}, \tag{21}$$

$$\rho_\infty = \frac{2g_\star\pi^2 I^3}{45}\frac{m_N^4}{S^4}. \tag{22}$$

A crucial point here is this. While for subsonic flows, the value of I varies depending on the boundary conditions at large radii, all cases when the termination shock is present, I has the same value, which is independent of T_∞ or ρ_∞. It is the value one finds for the critical curve, or for the transonic solution expanding in vacuum.

This means that the only way to satisfy the far boundary conditions when T_∞ or ρ_∞ are below their critical values is to generate additional entropy, since I is at its saturation value. This is why we need a termination shock.

To reiterate: *the role of the termination shock is simply to generate the necessary amount of additional entropy to match the conditions at large radii.*

This gives us the strategy of finding the properties and the location of the termination shock. Given the value of $T_\infty < T_{crit}$, it can be translated into S_2, the value of entropy per baryon post-shock. At the same time, the pre-shock part of the solution is exactly known: it given by the expansion into empty space. The shock should be placed in whatever position in this solution that generates the necessary value of entropy S_2.

Let us illustrate this in the limit of strong shock.

2.3 Strong shock

When $T_\infty \ll T_{crit}$ the shock occurs at a large radius (in the limit $T_\infty \to 0$, we expect $R_{shock} \to \infty$). By this point, the preshock part has been expanding in the supersonic regime for a while. This means most of its energy got converted from thermal to kinetic, *i.e.*, $v^2/2 \to I$. In this limit, the preshock pressure P_1 is small. Let us see what this does to the shock parameters.

Noticing that specific enthalpy for our radiation-dominated pressure can be rewritten as $h = TS/m_N = 4P/\rho$, we can write the last Hugoniot condition as

$$\frac{v_1^2}{2} + \frac{4P_1}{\rho_1} = \frac{v_2^2}{2} + \frac{4P_2}{\rho_2}. \tag{23}$$

We can combine this with the Hugoniot condition on momentum conservation to eliminate P_2,

$$\frac{\rho_2 v_1^2}{2} + 4P_1\frac{\rho_2}{\rho_1} = \frac{\rho_2 v_2^2}{2} + 4(\rho_1 v_1^2 + P_1 - \rho_2 v_2^2), \tag{24}$$

$$\frac{\rho_2 v_1^2}{2} - 4\rho_1 v_1^2 + \frac{7\rho_2 v_2^2}{2} = 4P_1\left(1 - \frac{\rho_2}{\rho_1}\right). \tag{25}$$

In the limit we are considering, the r.h.s. of the last equation vanishes. Finally, using mass conservation, $\rho_1 v_1 = \rho_2 v_2$ allows us to also eliminate v_2, giving

$$\frac{\rho_2 v_1^2}{2} - 4\rho_1 v_1^2 + \frac{7\rho_1^2 v_1^2}{2\rho_2} \simeq 0. \tag{26}$$

The ratio $x = \rho_1/\rho_2$ satisfies a quadratic equation $7x^2 - 8x + 1 \simeq 0$, which has roots $x_1 \simeq 1$ and $x_2 \simeq 1/7$. Thus, we obtain the famous result for the so-called strong shock limit: the density jump is given by a factor of 7 (and the velocity accordingly decreases by 1/7).

Almost all of the pre-shock specific kinetic energy, $(v_1^2 - v_1^2/49)/2 = 24/49 v_1^2$ is converted to specific enthalpy. In term of post-shock pressure, this implies

$$P_2 \simeq \frac{6\rho_2 v_1^2}{49} \simeq \frac{6\rho_1 v_1^2}{7}. \tag{27}$$

The post-shock temperature follows,

$$T_2 \simeq \left(\frac{18\rho_1 v_1^2}{7a}\right)^{1/4}, \tag{28}$$

and entropy per baryon can be obtained from specific enthalpy as follows:

$$S_2 \simeq \frac{h_2 m_N}{T_2} \simeq \frac{24 m_N v_1^2}{49}\left(\frac{18\rho_1 v_1^2}{7a}\right)^{-1/4} \simeq 0.3868 m_N a^{1/4}\frac{v_1^{3/2}}{\rho_1^{1/4}}. \tag{29}$$

To locate the shock, one simply needs to find on the supersonic curve expanding into empty space a position with ρ_1 and v_1 such that the above equation reproduces the desired value of S_2, which, in turn is found directly from the far boundary condition on T_∞. We get

$$\frac{I}{T_\infty} \simeq 0.3868 a^{1/4}\frac{v_1^{3/2}}{\rho_1^{1/4}}. \tag{30}$$

In the approximation that the terminal velocity of the supersonic wind was reached pre-shock, we have $v_1^2 = 2I$, yielding

$$\frac{I}{T_\infty} \simeq 0.3868 a^{1/4}\frac{(2I)^{3/4}}{\rho_1^{1/4}}, \tag{31}$$

$$\rho_1^{1/4} \simeq 0.650 a^{1/4}\frac{T_\infty}{I^{1/4}}, \tag{32}$$

$$\rho_1 \simeq 0.179 a\frac{T_\infty^4}{I} \simeq 0.537\frac{P_\infty}{I} \tag{33}$$

To locate the shock, one looks up the location of density ρ_1 obtained according to this formula on the profile expanding into empty space. Clearly, lower final pressures mean the shock occurs farther away and conversely the shock moves closer as P_∞ increases, as was anticipated at the outset.

2.4 General shocks

As P_∞ increases and the shock moves closer in, the strong-shock assumptions of the last subsection begin to break down. At this point, the most direct way of finding the shock is by tackling the problem numerically, as described below.

Let us, however, begin by making several qualitative observations. We already saw that, as the far pressure is increased, the shock keeps moving closer to the PNS. When the far conditions approach their critical values, the additional entropy that needs to be generated at the shock gets smaller and smaller. This means the density jump across the shock should decrease.

Eventually, as $T_\infty \to T_{crit}$, the density ratio (Mach number) of the shock should approach one, so the shock gets weaker and weaker. But we already know this limiting solution, it is given by the critical curve.

Let us now see this quantitatively. We again begin with the Hugoniot conditions. Momentum conservation reads

$$\rho_1 v_1^2 + \frac{T_1 S_1}{4m_N}\rho_1 \quad = \quad \rho_2 v_2^2 + \frac{T_2 S_2}{4m_N}\rho_2, \tag{34}$$

where we used $P = TS\rho/4m_N$. Our strategy is, once again, to get an equation for ρ_1/ρ_2 while eliminating all other post-shock quantities. For example, v_2 is eliminated using mass conservation, $v_2 = \rho_1 v_1/\rho_2$. Thus gives

$$\frac{\rho_1}{\rho_2} v_1^2 + \frac{T_1 S_1}{4m_N}\frac{\rho_1}{\rho_2} \quad = \quad \frac{\rho_1^2}{\rho_2^2} v_1^2 + \frac{T_2 S_2}{4m_N}. \tag{35}$$

Next, $T_2 S_2/m_N$ is specific enthalpy and can be expressed using the Hugoniot condition for energy conservation:

$$\frac{\rho_1}{\rho_2} v_1^2 + \frac{T_1 S_1}{4m_N}\frac{\rho_1}{\rho_2} \quad = \quad \frac{\rho_1^2}{\rho_2^2} v_1^2 + \frac{1}{4}\left(\frac{v_1^2}{2} + \frac{T_1 S_1}{m_N} - \frac{\rho_1^2}{\rho_2^2}\frac{v_1^2}{2}\right). \tag{36}$$

We once again get a quadratic equation for ρ_1/ρ_2, which has solutions $\rho_1/\rho_2 = 1$ and

$$\frac{\rho_1}{\rho_2} = \frac{2T_1 S_1 + m_N v_1^2}{7m_N v_1^2} = \frac{6v_{s1}^2 + v_1^2}{7v_1^2}. \tag{37}$$

This is a general expression for the density jump at the shock front in systems with radiation pressure, in terms of the pre-shock sound speed v_{s1}. In the limit $v_{s1} \ll v_1$, the strong shock result, $\rho_1/\rho_2 \to 1/7$ is recovered. But now we are also able to consider the opposite limit, $v_1 \to v_s$. In this case, we explicitly see that $\rho_1/\rho_2 \to 1$.

What does this mean? We are once again able to match the far boundary condition using the shooting method. But this time, instead of shooting the initial velocity at the gain radius, we will be shooting the location of the termination shock R_s. Given any value of R_s (greater than the sonic point), we can take our transonic solution and apply a jump in the density—and the corresponding inverse jump in velocity— using only the quantities in the transonic solution according to Eq. (37). Once again, we get a lookup table relating R_s and T_∞ (or P_∞).

In summary, when $T_\infty \to 0$, the shock radius $R_s \to \infty$; the density jump across the shock is given by a factor 7. As $T_\infty \to T_{crit}$, the shock becomes weaker and moves closer to the sonic point. Eventually, the shock disappears at the sonic point and the profile becomes that of our critical solution. This restores the continuity that we conjectured earlier and makes the analogy with the phase transition dynamics complete. It also does more for us. We argue that the conditions, specifically the interplay between the heating rate and the far pressure, change in the course of the explosion. It may occur that they change in such a way that the termination shock weakens. In this case, it will be seen to move inward, until disappearing at the sonic point. This dynamical behavior could be mistaken for a reverse shock. A careful analysis of the simulation would be required to distinguish the two physical scenarios.