

# **Final Paper**

## **An Investigation into Multi-Pivot Quicksort Algorithms**

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## 1 Abstract

The quicksort is a thoroughly studied sorting algorithm and is commonly among the first efficient sorts learned by students of computer science. Many variants of the quicksort have been proposed, from the classic quicksort introduced by Tony Hoare in 1961 [1] to Yaroslavskiy’s dual pivot quicksort introduced in 2009 and used by the Java 7 Standard Library [2]. Since Hoare’s first proposal, much research has gone into attempting to minimize the total number of swaps done by the sort, the total number of comparisons done by the sort and minimizing the worst case runtime. We aim to experimentally validate the swap and comparison count of several variants of the quicksort and compare the runtimes and various optimizations. Our results

SUMMARIZE THE RESULTS

## 2 Introduction

Sorting is a fundamental concept of computer science wherein a totally ordered multiset is modified such that the elements of the multiset are rearranged (permuted) in either non-decreasing or non-increasing order. A broad range of applications benefit from sorting such as organizing an MP3 library by song title to quickly identifying duplicates in a list to more advanced applications such as load balancing, data compression and computer graphics [3]. It is well known that all comparison based sorting algorithms are lower bound by  $\Omega(n \log n)$  comparisons [4] and quicksort is no exception to this rule. Interestingly, there are non-comparison based sorts such as the counting sort and the radix sort which take advantage of certain properties of the data set and get around the lower bound of comparison base sorts. A summary of space and time complexities can be found in Table 1.

The quicksort was first introduced by Tony Hoare in [1] and was quickly adopted by the field as a standard sorting algorithm alongside the mergesort [5] and several other sorting algorithms. Quicksorts simplicity, high average case performance, poor worst case performance and low use of additional memory has made it one of the most highly studied “efficient” sorting algorithms. van Emden [6] proposed an optimization to the quicksort in 1970 which resulted in a 15% improvement on the efficiency of the

Sort Method	Space	Average Case Time	Worst Case Time
Selection	$O(1)$	$O(n^2)$	$O(n^2)$
Insertion	$O(1)$	$O(n^2)$	$O(n^2)$
Merge	$O(n)$	$O(n \log(n))$	$O(n \log(n))$
Quicksort	$O(1)$	$O(n \log(n))$	$O(n^2)$
Radix	$O(n)$	$O(n * k)$	$O(n * k)$
Counting	$O(m)$	$O(n + m)$	$O(n + m)$

Table 1: Summary of space and time complexities of various sorts where  $n$  represents the number of elements,  $k$  represents the number of digits in the largest value and  $m$  represents the maximum value to be sorted.

quicksort. In 1971 Hoare and Foley [7] provided a formal proof of other correctness of the quicksort. In 1977, Sedgewick did a thorough analysis of quicksort [8] and then in 1978 Sedgewick again published a paper on the quicksort exploring various implementations and proposing several optimizations [9]. The academic interest in the quicksort has not slowed since the late 1970s; papers regarding the quicksort continue to be published for adapting the quicksort to GPUs [10], providing optimal pivot selection [11] and using more than one pivot [12].

In this paper we explore, compare and contrast the classic implementation of the quicksort against various multipivot quicksort implementations and their respective optimizations. We are particularly interested in comparing the classic quicksort against the standard dual pivot quicksort, Yaroslavskiy’s dual pivot quicksort[13], Aumüller and Dietzfelbinger’s optimized dual-pivot quicksort [14], Edmonson’s M-Pivot quicksort [12] and the three pivot quicksort introduced by Kushagra et al [2]. We will first introduce the various quicksorts by providing a brief overview of their behaviour, implementation and potential optimizations. Next we will discuss the experiments and provide their results. Following this we provide an analysis, discuss future work and conclude the paper.

### 3 Quicksort

The following descriptions will assume that the algorithms are being used to sort a list  $A$  in non-decreasing order where  $n$  denotes the number of elements in  $A$ . Additionally, all algorithms are assumed to perform the sort in place.

#### 3.1 Classic Quicksort

The classic quicksort, first introduced by Hoare [1] in 1962 is a recursive sorting algorithm which falls into the category of divide and conquer algorithms. All of

following variations of the quicksort have these qualities as well. Quicksort can be summarized as follows:

1. Select a pivot in the current range of elements to be sorted.
2. Partition the list such that all elements less than the pivot appear to the left of the pivot while all elements greater than or equal to the pivot appear to the right of the pivot.
3. Recursively call quicksort on the left and right halves of the currently partitioned range.

This naïve implementation does not handle it's worst case gracefully. The classic quicksort degrades to a time complexity of  $O(n^2)$  when the data being processed is in nearly sorted or already sorted order. Due to the simplicity of this implementation there are some minor changes which can be made to speed up the average case and aid the worst case runtimes. It was shown by Cook and Kim [15] that insertion sorts are the fastest sort for small lists and nearly sorted lists. This optimization does not make the quicksort asymptotically faster than it's naï variation in either the average or worst case but does provide a meaningful speedup. Cook and Kim also showed that a hybrid of insertion sort and quicksort outperforms either of the sorts used individually. A perfectly selected pivot (the median of the range) would ensure that the worst case space complexity is bound by  $O(n \log(n))$ . Perfect pivot selection cannot be achieved with an increase in time complexity; however, selecting pivot to be the median of first, middle and last elements is a good enough estimation of the true median of the range [9].

### 3.2 Yaroslavskiy Quicksort

### 3.3 Dual Pivot Quicksort

The dual pivot quicksort is a simplified version of Yaroslavskiy's quicksort and can be thought of as the classic quicksort with an extra pivot. Due to the extra pivot, this quicksort uses Yaroslavskiy's partitioning algorithm. The dual pivot quicksort suffers from the same worst-case space complexity as the classic quicksort and can be similarly optimized. Performing an insertion sort for small subarrays and selecting the pivot intelligently can both provide meaningful and measurable improvements. Similar to the classic quicksort, the best pivots to select are the first and third quartiles. These can be estimated by selecting the first and third quartiles of first, halfway between first and middle, middle, halfway between middle and last and last elements.

Sort Method	Comparisons
Classic	$2n \log n - 1.51n + O(\log(n))$
Dual Pivot	$2.13n \log n - 2.57n + O(\log(n))$
Optimal Dual Pivot	$1.8n \log n + O(n)$
Three Pivot	$1.846n \log n + O(n)$
Yaroslavskiy	$1.9n \log n - 2.46n + O(\log(n))$
M Pivot	$O(n \log n)$

Table 2: Summary table of theoretical comparisons.

Sort Method	Swaps
Classic	$0.33n \log n - 0.58n + O(\log(n))$
Dual Pivot	$0.8n \log n - 0.3n + O(\log(n))$
Optimal Dual Pivot	$0.33n \log n + O(n)$
Three Pivot	$0.615n \log n + O(n)$
Yaroslavskiy	$0.6n \log n + 0.08n + O(\log(n))$
M Pivot	$O(n \log n)$

Table 3: Summary table of theoretical swaps.

### 3.4 Optimal Dual Pivot Quicksort

### 3.5 Three Pivot Quicksort

### 3.6 M Pivot Quicksort

RAWR

#### 3.6.1 Testing

Just to test the subsection code. Test text : Recall from section 3.1 on page 3

### 3.7 Summary

## References

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Sort	Pivot Selection methods	Number of Pivots
Classic	3	1
Dual Pivot	2	2
Optimal Dual Pivot	2	2
Yaroslavskiy	1	2
Three Pivot	1	3
M-Pivot	1	3,4,5,6
Heap Optimized M-Pivot	1	3,4,5,6

Table 4: List of all the variation of quicksorts executed.

- [17] S. Wild and M. E. Nebel, “Average case analysis of java 7’s dual pivot quicksort,” in *Proceedings of the 20th Annual European Conference on Algorithms*, ESA’12, (Berlin, Heidelberg), pp. 825–836, Springer-Verlag, 2012. 14

## 4 Experiment

### 4.1 Overview

The experiments consisted of generating lists of integers ranging in size from four to fifty million. The integers were chosen uniformly at random between zero and  $10^{12}$ . Consecutive lists were  $9/7$  the size of the previous rounded down to the nearest integer. Each quicksort and each of its variations sorted the same list. The number of comparisons, swaps and time taken by each quicksort were recorded. A summary of the conditions can be found in Table 4 for a total of 17 conditions. A total of 69 iterations were run resulting in 109296 data points. During each iteration, the data gathering for each list size was done in parallel using four threads.

### 4.2 Platform

The experiments were run on a virtual machine with OSX 10.9 as the guest operating system and Windows 8.1 as the host operating system. The host operating system has 16 GB of ram and an i5-2500k CPU overclocked to 4.3 GHz. The guest OS had access to 8 GB of RAM and access to all 4 cores of the host CPU.

### 4.3 Results

Figure 1 provides the legend for all of the plots. Note that for the plots we overlay the function:

$$A \cdot n \log(n) + B \cdot n + C \log(n) \quad (1)$$

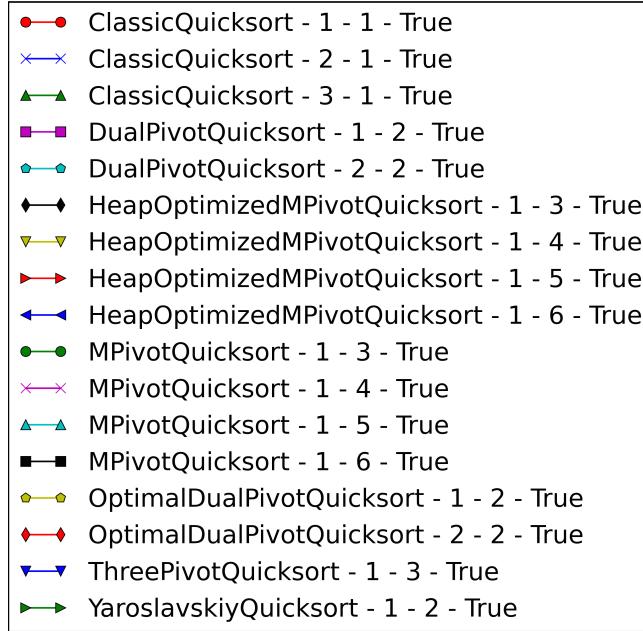


Figure 1: Plot legend

The nomenclature for the legend is as follows:

(Algorithm Name, Pivot Selection Method Index, Number of Pivots, is Insertion Sort used).

## 5 Analysis

### 5.1 Non-Linear Curve Fit

With the data that has collected we wish to be able to look at a whole data set and quantitatively compare to other data sets. From our introduction of the different quicksort algorithms, the asymptotic runtime of all of the had one of the following forms:

$$A \cdot n \log(n) + B \cdot n + O(\log(n)) \quad (2)$$

$$A \cdot n \log(n) + O(n) \quad (3)$$

$$O(n \log(n)) \quad (4)$$

In correspondence of this we will fit the data to the following equation:

$$A \cdot n \log(n) + B \cdot n + C \log(n) \quad (5)$$

With the function parameters  $A$ ,  $B$  and  $C$  where found using a non-linear curve fitting tool in the SciPy module in python. Qualitatively, the fit function that was picked

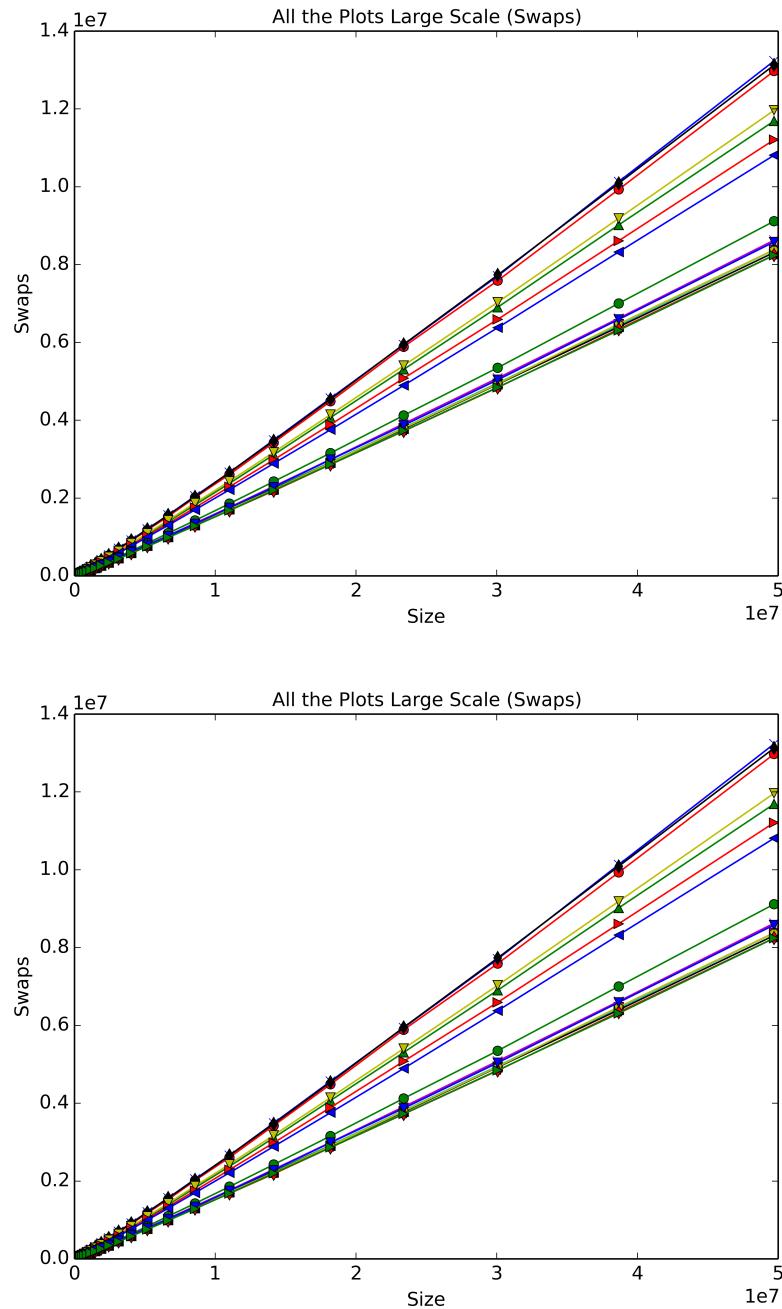


Figure 2: A plot of the data from all the sorting algorithm against the number of comparisons.

seems to result in good fits visually. Though the fit has less meaning do to the fact we don't have error bars on the data. A closer look at the reduced chi squared values

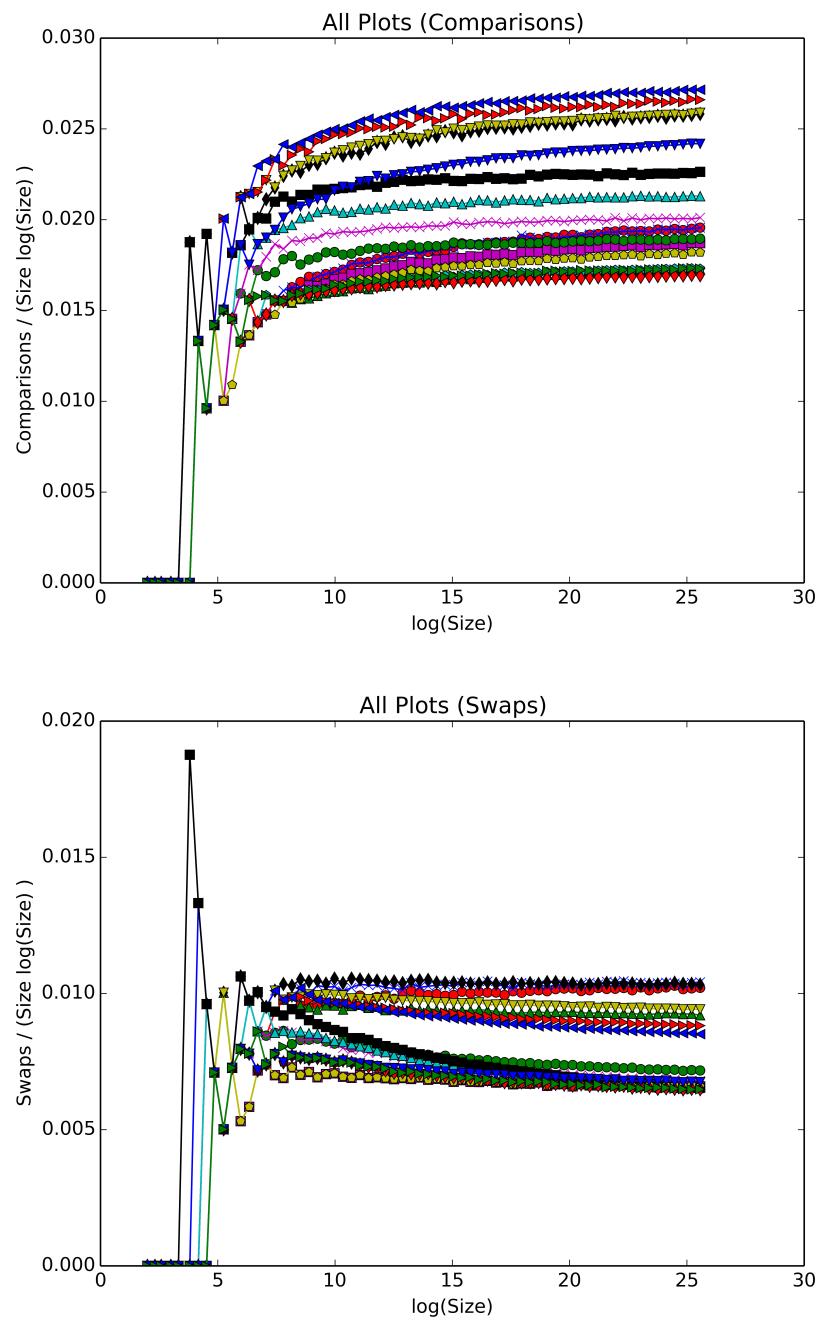


Figure 3: A plot of the data from all the sorting algorithm

of the fits leads us to believe that the fits are not good [16]. Thus we will treat the results from the best fit as qualitative.

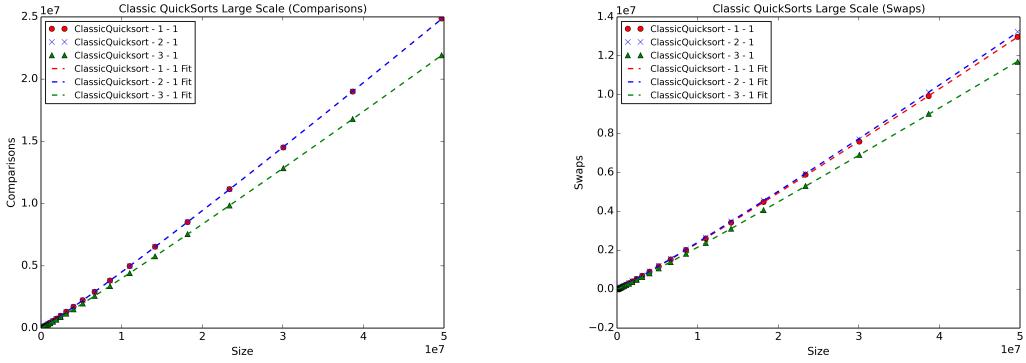


Figure 4: Swap and comparison counts for the classic quicksort.

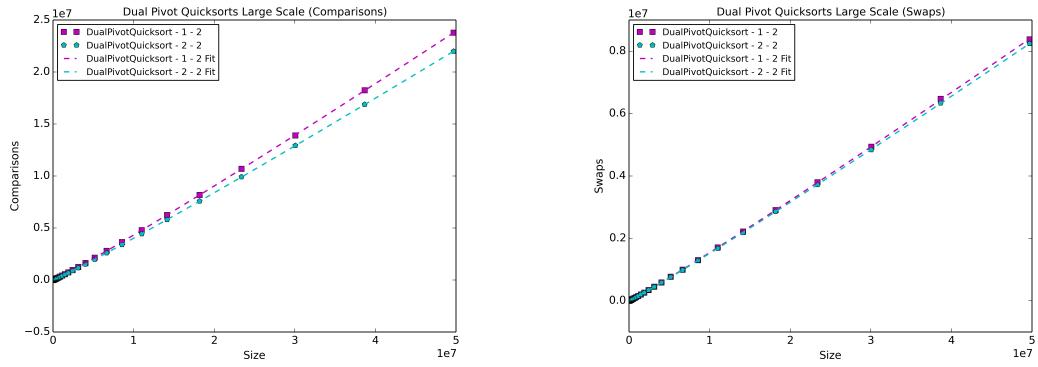


Figure 5: Swap and comparison counts for the dual pivot quicksort.

## 5.2 Effect of Pivot Selection

In Figure 4 we can see that picking the pivot as the median of the first, middle and last elements (green line) provides a significant decrease in comparisons and swaps compared to picking the first element (red line) or last element (blue line). Interestingly, we expect that picking any single element should provide no difference but we see that selecting the last element as the pivot is beneficial to picking the first element. We have no explanation for this.

## 6 Discussion

### 6.1 $n \log(n)$ Trend

A basic question to ask is if we are seeing the  $n \log(n)$  trend in all of the sort data we see. From figure 3 we see plots of  $y/n \log(n)$  (where  $y$  is either the number of comparisons or swaps). As  $n$  gets larger you can see that each curve is tending to a constant. Since

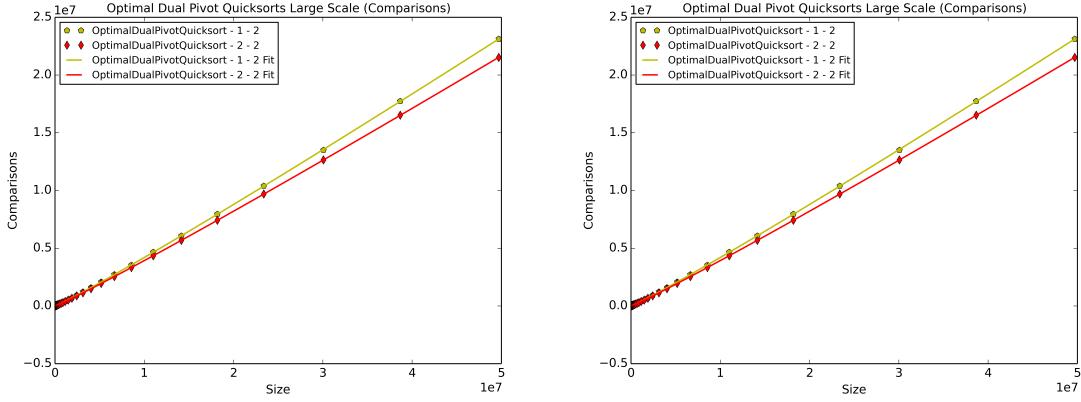


Figure 6: Swap and comparison counts for the optimal dual pivot quicksort.

the  $\frac{y}{n \log(n)}$  approaches a constant, this indicates that  $y$  is of  $O(n)$ . Which is consistent with the theory described in the intro section

## 6.2 Comparisons among Quicksorts

Preliminary observations on Figure 2 show that the optimal dual pivot quicksort had the lowest number of comparisons. This is as expected since it was designed to minimize the number of comparisons. Also the Yaroslavskiy and M-Pivot sort (for  $M = 3, 4, 5, 6$ ) have a minimal number of comparisons. In contrast, the number of comparisons for all of the heap optimized  $M$ -pivot sorts are poor. The three pivot quicksort lies within the neighborhood of the heap optimized  $M$ -pivot sorts. This is an unexpected behavior. When looking at the number of swaps three pivot quicksort does really well. But like in the case with the number of comparisons, the heap optimized  $M$ -pivot sort (for all values of  $M$ ) has a consistently larger number of swaps. The classic quicksort algorithm (regardless of pivot selection method) were in the same neighborhood as the heap optimized  $M$ -pivot sorts.

An observation from the two pivot quicksorts with pivot selection that selects the first and last element as the pivots don't do as well as the selection of the tertiles in the number of comparisons. This can be seen from figure 7. When looking at size vs. swaps in the two pivot there are no noteworthy differences.

In comparisons between all the  $M$ -pivot sort algorithms, the ones without the heap optimization always did better. This is due to an implementation error. In the original description of the algorithm the heapification of the array only happens on the very first call whereas in our implementation it occurs for every recursive call.

### 6.3 Fit Functions

From tables 5, 6, 7, 9, 10 and 11 the fit coefficients for the data collected is summarized based of equation 5. For the sake of asymptotic comparison between the algorithms we only focus on the fit parameter  $A$ . From the fit coefficients the optimal dual pivot quicksort by picking the pivot as a median of three elements is shown to be the best quick sort algorithm based off the number of comparisons. While the heap-optimized  $M$  pivot quicksort for  $M = 6$ , is the worst algorithm based off the number of comparisons. The classic quicksort algorithms by picking the median of three elements based off the number of swaps is terrible. The best algorithm for reducing the number of swaps is the basic  $M$ -pivot sort algorithm with  $M = 6$ . But the only algorithm that competitive runtimes based off both metrics (swaps and comparisons) is Yaroslavskiy dual pivot quicksort algorithm.

### 6.4 Future Work

While our tests only examined the performance of these quicksorts on lists of nearly unique numbers, we believe that examining their comparison and swap counts on different “kinds” of lists provides just as much, if not more, insight. This is especially true for concerns of their worst-case performance. Lists of numbers which can be modelled by a Gaussian or Poisson distribution may provide some insight into the behaviour of these quicksorts on real world data. Additionally, sorting lists in pseudo-sorted or sorted order will provide data for the worst-case performance of these sorts. Another potentially interesting may be one where elements of the list are chosen uniformly at random but where the range of the selected elements is less than the length of the list, thus guaranteeing value repetition.

Comparing these sorts against other sorts may also be of use. Other popular “efficient” sorts such as the merge sort or timsort can provide good points of reference for the performance of the quicksort. This can be especially useful if more “kinds” of lists are used as mentioned previously. We also believe that comparing the M-Pivot quicksort with  $M = 1$  and  $M = 2$  with the other one and two pivot sorts can result in potentially interesting graphs and analysis. We also believe that implementing the optimizations presented by Edmonson [12] on the quicksorts other than the M-Pivot Quicksort can prove useful in determining their true effectiveness.

## 7 Conclusion

We have implemented several variations of the quicksort and explored their efficiencies in regards to total number of swaps and comparisons. Our results show that Yaroslavskiy’s quicksort is the highest overall performer while the heap optimized  $M$ -pivot quicksorts performed the worse. Our results also validate the idea that intelligent pivot selection provides a measurable and significant effect on the overall

Sort Method	$A_{\text{comparisons}}$
Classic Quicksort - 1 - 1	$0.02151 \pm 0.00019$
Classic Quicksort - 2 - 1	$0.02135 \pm 0.00017$
Classic Quicksort - 3 - 1	$0.01807 \pm 0.00006$
DualPivot Quicksort - 1 - 2	$0.02014 \pm 0.00018$
DualPivot Quicksort - 2 - 2	$0.01772 \pm 0.00007$
Heap Optimized M-Pivot Quicksort - 1 - 3	$0.02778 \pm 0.00014$
Heap Optimized M-Pivot Quicksort - 1 - 4	$0.02788 \pm 0.00015$
Heap Optimized M-Pivot Quicksort - 1 - 5	$0.02842 \pm 0.00025$
Heap Optimized M-Pivot Quicksort - 1 - 6	$0.02869 \pm 0.00011$
M-Pivot Quicksort - 1 - 3	$0.01970 \pm 0.00009$
M-Pivot Quicksort - 1 - 4	$0.02076 \pm 0.00015$
M-Pivot Quicksort - 1 - 5	$0.02165 \pm 0.00010$
M-Pivot Quicksort - 1 - 6	$0.02386 \pm 0.00014$
Optimal Dual Pivot Quicksort - 1 - 2	$0.01959 \pm 0.00019$
Optimal Dual Pivot Quicksort - 2 - 2	$0.01744 \pm 0.00007$
Three Pivot Quicksort - 1 - 3	$0.02587 \pm 0.00009$
Yaroslavskiy Quicksort - 1 - 2	$0.01796 \pm 0.00010$

Table 5: Summary table coefficients of the non-linear fit for the parameter  $A$  on the comparison data.

performance of quicksorts. We find that for lists of nearly unique numbers, selecting a median of three and first and third quartiles of five provide good performance to developer time ratios. Our results also show that the optimal dual pivot quicksort performs better than Yaroslavskiy's quicksort in regards to total number of comparisons as claimed by [14] while the two sorts are nearly identical in swap count.

Overall, the  $M$ -Pivot quicksorts performed well. The  $M$ -Pivot quicksort with  $M = 3$  performed better than the three pivot quicksort of Kushagra et al. [2]. We also find that at small scales, the choice of quicksort makes little difference. Yaro better than we thought.  $M$  pivot surprisingly good. Partitioning is subtly difficult. Yaroslavs partition alg is amazing [14] [17].

## 8 Fit Coefficients

## 9 Misc Plots

Sort Method	$B_{\text{comparisons}}$
Classic Quicksort - 1 - 1	$-0.05025 \pm 0.00469$
Classic Quicksort - 2 - 1	$-0.04634 \pm 0.00428$
Classic Quicksort - 3 - 1	$-0.02126 \pm 0.00163$
DualPivot Quicksort - 1 - 2	$-0.03650 \pm 0.00465$
DualPivot Quicksort - 2 - 2	$-0.01045 \pm 0.00183$
Heap Optimized M-Pivot Quicksort - 1 - 3	$-0.05055 \pm 0.00355$
Heap Optimized M-Pivot Quicksort - 1 - 4	$-0.05152 \pm 0.00368$
Heap Optimized M-Pivot Quicksort - 1 - 5	$-0.04639 \pm 0.00626$
Heap Optimized M-Pivot Quicksort - 1 - 6	$-0.03889 \pm 0.00280$
M-Pivot Quicksort - 1 - 3	$-0.01917 \pm 0.00220$
M-Pivot Quicksort - 1 - 4	$-0.01716 \pm 0.00391$
M-Pivot Quicksort - 1 - 5	$-0.00902 \pm 0.00244$
M-Pivot Quicksort - 1 - 6	$-0.03206 \pm 0.00366$
Optimal Dual Pivot Quicksort - 1 - 2	$-0.03580 \pm 0.00490$
Optimal Dual Pivot Quicksort - 2 - 2	$-0.01266 \pm 0.00165$
Three Pivot Quicksort - 1 - 3	$-0.04282 \pm 0.00232$
Yaroslavskiy Quicksort - 1 - 2	$-0.01636 \pm 0.00251$

Table 6: Summary table coefficients of the non-linear fit for the parameter  $B$  on the comparison data.

Sort Method	$C_{\text{comparisons}}$
Classic Quicksort - 1 - 1	$121.34341 \pm 99.97322$
Classic Quicksort - 2 - 1	$78.33233 \pm 91.31275$
Classic Quicksort - 3 - 1	$22.52742 \pm 34.75034$
DualPivot Quicksort - 1 - 2	$52.22569 \pm 99.17037$
DualPivot Quicksort - 2 - 2	$-53.25104 \pm 39.07336$
Heap Optimized M-Pivot Quicksort - 1 - 3	$94.09226 \pm 75.71998$
Heap Optimized M-Pivot Quicksort - 1 - 4	$124.92512 \pm 78.46872$
Heap Optimized M-Pivot Quicksort - 1 - 5	$56.97527 \pm 133.52044$
Heap Optimized M-Pivot Quicksort - 1 - 6	$29.84293 \pm 59.67899$
M-Pivot Quicksort - 1 - 3	$51.60730 \pm 46.87649$
M-Pivot Quicksort - 1 - 4	$49.35746 \pm 83.31083$
M-Pivot Quicksort - 1 - 5	$4.76647 \pm 52.03322$
M-Pivot Quicksort - 1 - 6	$136.38815 \pm 77.96382$
Optimal Dual Pivot Quicksort - 1 - 2	$56.95127 \pm 104.39452$
Optimal Dual Pivot Quicksort - 2 - 2	$-26.20004 \pm 35.22097$
Three Pivot Quicksort - 1 - 3	$-17.79746 \pm 49.38183$
Yaroslavskiy Quicksort - 1 - 2	$0.73204 \pm 53.59187$

Table 7: Summary table coefficients of the non-linear fit for the parameter  $C$  on the comparison data.

Sort Method	Reduced $\chi^2$
Classic Quicksort - 1 - 1	72937905.931
Classic Quicksort - 2 - 1	60848264.801
Classic Quicksort - 3 - 1	8812566.923
DualPivot Quicksort - 1 - 2	71771741.054
DualPivot Quicksort - 2 - 2	11141543.015
Heap Optimized M-Pivot Quicksort - 1 - 3	41841106.311
Heap Optimized M-Pivot Quicksort - 1 - 4	44934160.832
Heap Optimized M-Pivot Quicksort - 1 - 5	130103260.047
Heap Optimized M-Pivot Quicksort - 1 - 6	25990338.417
M-Pivot Quicksort - 1 - 3	16035891.085
M-Pivot Quicksort - 1 - 4	50651081.263
M-Pivot Quicksort - 1 - 5	19755778.167
M-Pivot Quicksort - 1 - 6	44358052.229
Optimal Dual Pivot Quicksort - 1 - 2	79531658.368
Optimal Dual Pivot Quicksort - 2 - 2	9052794.315
Three Pivot Quicksort - 1 - 3	17796006.509
Yaroslavskiy Quicksort - 1 - 2	20981508.247

Table 8: Summary table of the reduces  $\chi^2$  of the fit for the comparison data.

Sort Method	$A_{\text{swap}}$
Classic Quicksort - 1 - 1	$0.01026 \pm 0.00017$
Classic Quicksort - 2 - 1	$0.01095 \pm 0.00016$
Classic Quicksort - 3 - 1	$0.00848 \pm 0.00012$
DualPivot Quicksort - 1 - 2	$0.00629 \pm 0.00010$
DualPivot Quicksort - 2 - 2	$0.00606 \pm 0.00006$
Heap Optimized M-Pivot Quicksort - 1 - 3	$0.01004 \pm 0.00009$
Heap Optimized M-Pivot Quicksort - 1 - 4	$0.00898 \pm 0.00004$
Heap Optimized M-Pivot Quicksort - 1 - 5	$0.00809 \pm 0.00004$
Heap Optimized M-Pivot Quicksort - 1 - 6	$0.00759 \pm 0.00005$
M-Pivot Quicksort - 1 - 3	$0.00672 \pm 0.00006$
M-Pivot Quicksort - 1 - 4	$0.00605 \pm 0.00003$
M-Pivot Quicksort - 1 - 5	$0.00535 \pm 0.00003$
M-Pivot Quicksort - 1 - 6	$0.00513 \pm 0.00003$
Optimal Dual Pivot Quicksort - 1 - 2	$0.00629 \pm 0.00010$
Optimal Dual Pivot Quicksort - 2 - 2	$0.00606 \pm 0.00006$
Three Pivot Quicksort - 1 - 3	$0.00635 \pm 0.00006$
Yaroslavskiy Quicksort - 1 - 2	$0.00586 \pm 0.00005$

Table 9: Summary table coefficients of the non-linear fit for the parameter  $A$  on the swap data.

Sort Method	$B_{\text{swap}}$
Classic Quicksort - 1 - 1	$-0.00132 \pm 0.00442$
Classic Quicksort - 2 - 1	$-0.01411 \pm 0.00417$
Classic Quicksort - 3 - 1	$0.01903 \pm 0.00298$
DualPivot Quicksort - 1 - 2	$0.00824 \pm 0.00264$
DualPivot Quicksort - 2 - 2	$0.01080 \pm 0.00153$
Heap Optimized M-Pivot Quicksort - 1 - 3	$0.00774 \pm 0.00233$
Heap Optimized M-Pivot Quicksort - 1 - 4	$0.01111 \pm 0.00097$
Heap Optimized M-Pivot Quicksort - 1 - 5	$0.01881 \pm 0.00113$
Heap Optimized M-Pivot Quicksort - 1 - 6	$0.02363 \pm 0.00129$
M-Pivot Quicksort - 1 - 3	$0.01160 \pm 0.00156$
M-Pivot Quicksort - 1 - 4	$0.01890 \pm 0.00069$
M-Pivot Quicksort - 1 - 5	$0.03171 \pm 0.00065$
M-Pivot Quicksort - 1 - 6	$0.03601 \pm 0.00077$
Optimal Dual Pivot Quicksort - 1 - 2	$0.00824 \pm 0.00264$
Optimal Dual Pivot Quicksort - 2 - 2	$0.01080 \pm 0.00153$
Three Pivot Quicksort - 1 - 3	$0.01030 \pm 0.00153$
Yaroslavskiy Quicksort - 1 - 2	$0.01592 \pm 0.00127$

Table 10: Summary table coefficients of the non-linear fit for the parameter  $B$  on the swap data.

Sort Method	$C_{\text{swap}}$
Classic Quicksort - 1 - 1	$-36.52075 \pm 94.28550$
Classic Quicksort - 2 - 1	$102.34036 \pm 88.94077$
Classic Quicksort - 3 - 1	$-103.21696 \pm 63.57056$
DualPivot Quicksort - 1 - 2	$-38.02577 \pm 56.38166$
DualPivot Quicksort - 2 - 2	$42.34411 \pm 32.52471$
Heap Optimized M-Pivot Quicksort - 1 - 3	$-53.81114 \pm 49.78070$
Heap Optimized M-Pivot Quicksort - 1 - 4	$-3.24127 \pm 20.75454$
Heap Optimized M-Pivot Quicksort - 1 - 5	$-16.07787 \pm 24.00667$
Heap Optimized M-Pivot Quicksort - 1 - 6	$-20.99394 \pm 27.47669$
M-Pivot Quicksort - 1 - 3	$41.76450 \pm 33.26955$
M-Pivot Quicksort - 1 - 4	$16.90493 \pm 14.70501$
M-Pivot Quicksort - 1 - 5	$-17.36329 \pm 13.95680$
M-Pivot Quicksort - 1 - 6	$-5.54593 \pm 16.46902$
Optimal Dual Pivot Quicksort - 1 - 2	$-38.02577 \pm 56.38166$
Optimal Dual Pivot Quicksort - 2 - 2	$42.34411 \pm 32.52471$
Three Pivot Quicksort - 1 - 3	$14.68107 \pm 32.72260$
Yaroslavskiy Quicksort - 1 - 2	$6.27071 \pm 27.13874$

Table 11: Summary table coefficients of the non-linear fit for the parameter  $C$  on the swap data.

Sort Method	Reduced $\chi^2$
Classic Quicksort - 1 - 1	64874821.504
Classic Quicksort - 2 - 1	57727853.077
Classic Quicksort - 3 - 1	29491583.122
DualPivot Quicksort - 1 - 2	23198666.416
DualPivot Quicksort - 2 - 2	7719893.458
Heap Optimized M-Pivot Quicksort - 1 - 3	18084500.975
Heap Optimized M-Pivot Quicksort - 1 - 4	3143341.422
Heap Optimized M-Pivot Quicksort - 1 - 5	4205854.111
Heap Optimized M-Pivot Quicksort - 1 - 6	5509544.029
M-Pivot Quicksort - 1 - 3	8077470.673
M-Pivot Quicksort - 1 - 4	1578050.446
M-Pivot Quicksort - 1 - 5	1421543.264
M-Pivot Quicksort - 1 - 6	1979357.053
Optimal Dual Pivot Quicksort - 1 - 2	23198666.416
Optimal Dual Pivot Quicksort - 2 - 2	7719893.458
Three Pivot Quicksort - 1 - 3	7814224.918
Yaroslavskiy Quicksort - 1 - 2	5375021.288

Table 12: Summary table of the reduces  $\chi^2$  of the fit for the swap data.

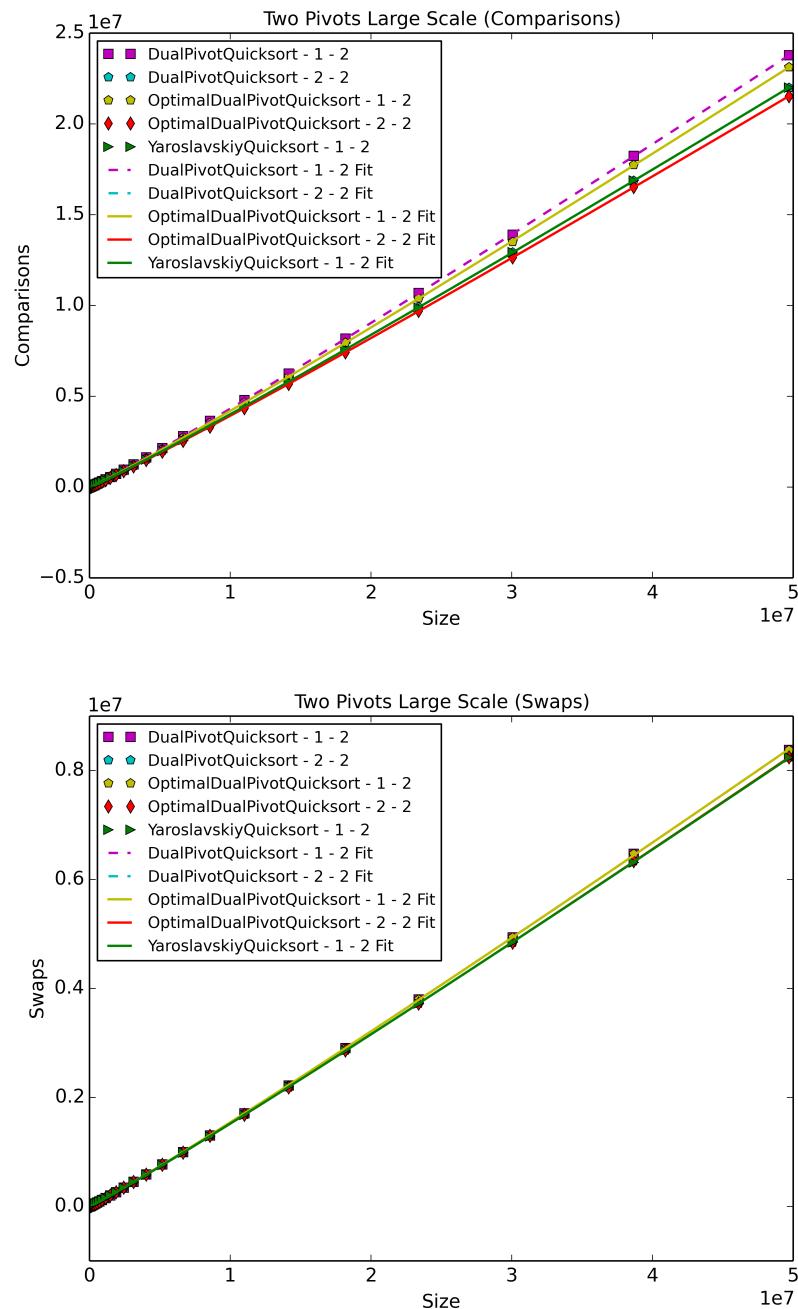


Figure 7: A plot of the data from all the sorting algorithm with two pivots

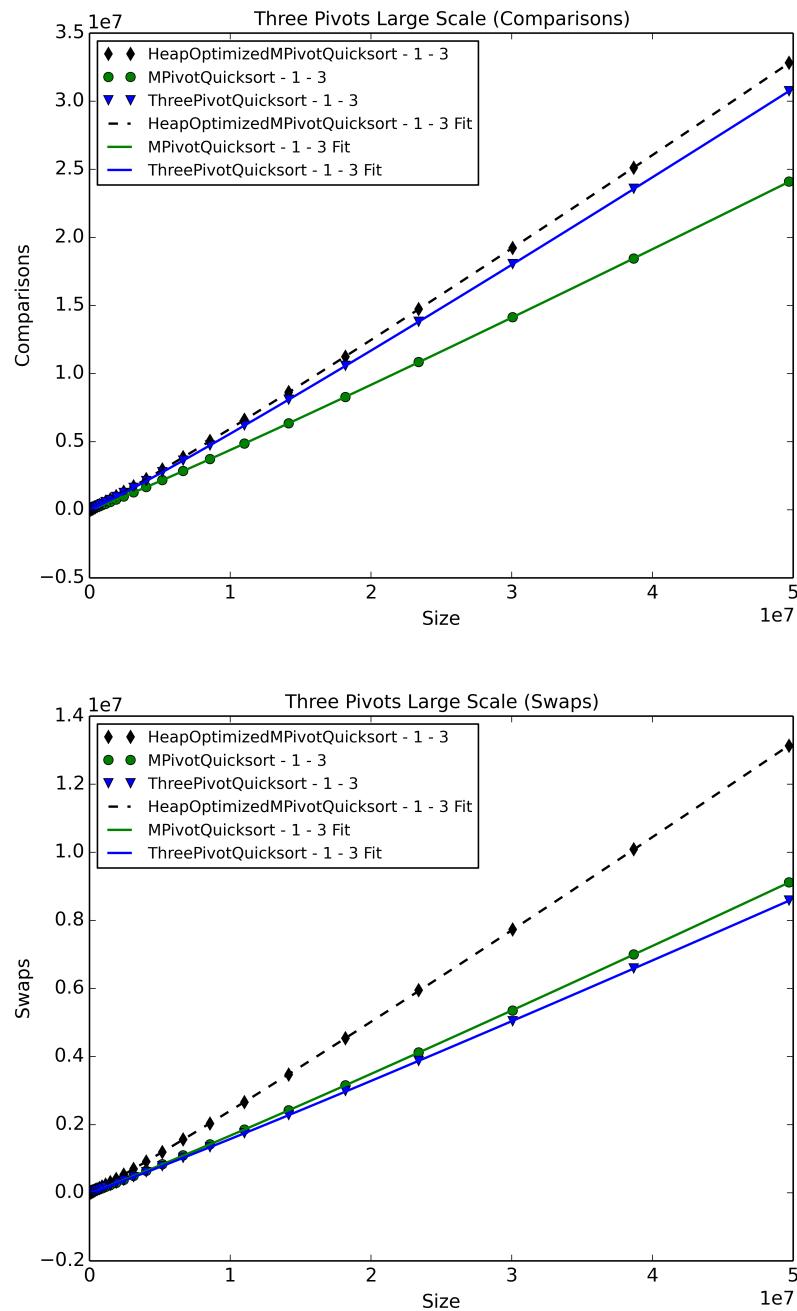


Figure 8: A plot of the data from all the sorting algorithm with three pivots

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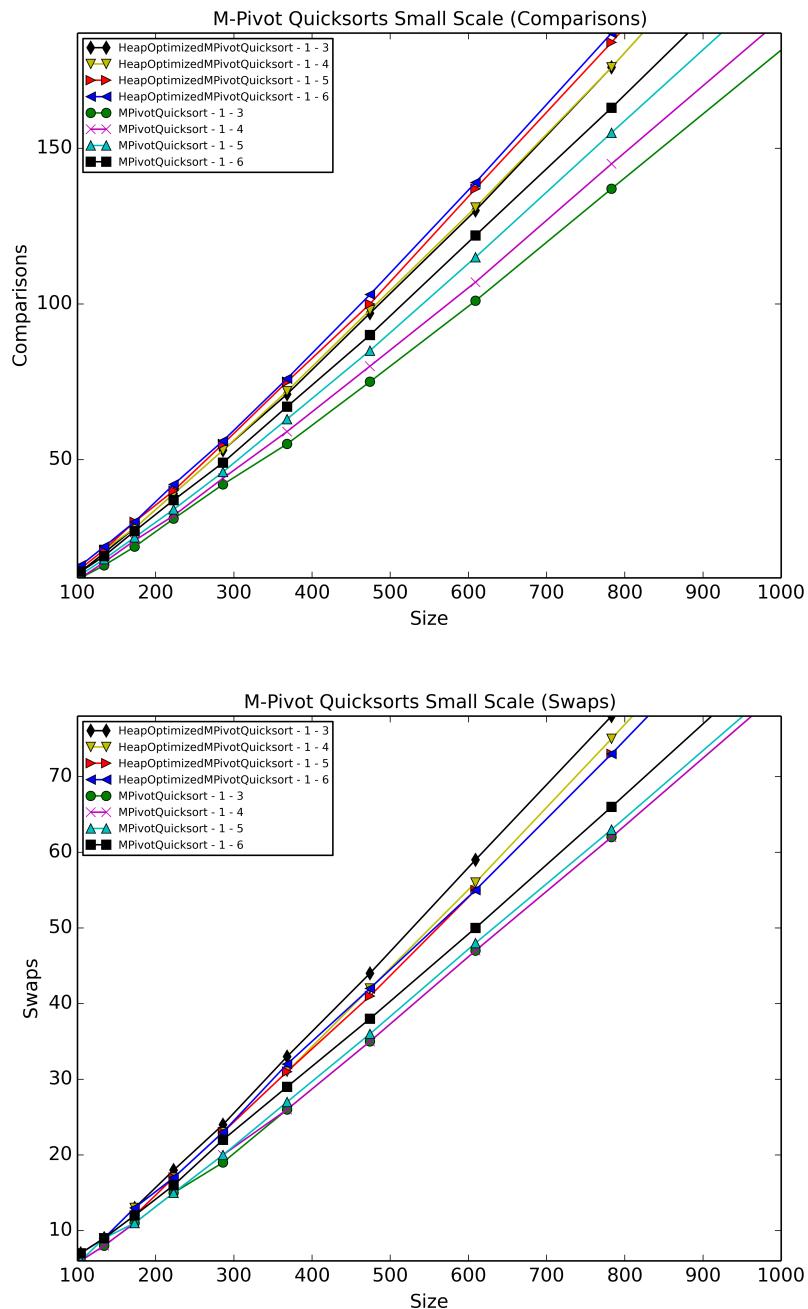


Figure 9: Data from all the version of M-Pivot Sort.

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