All Sorts of Quicksorts!

An investigation into multi-pivot quicksorts

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Introduction

- $O(n \log(n))$ Average Case Run Time

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- In place algorithm
- Picking Pivots
- Partitioning Data
- Recurse to a smaller sub-array
- Use Insertion Sort for a small sub-array

Classic Quicksort

- $2n \log n - 1.51n + O(\log(n))$ Comparisons

[Wild and Nebel(2012)]

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- $2n \log n - 1.51n + O(\log(n))$ Comparisons
- $0.33n \log n - 0.58n + O(\log(n))$ Swaps

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Dual Pivot Quicksort with Optimal Partitioning

- $1.8n \log n + O(n)$ Comparisons

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- $0.33n \log n + O(n)$ Swaps
- Two Pivots
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  - Uses the information to see who to compare to first
- Otherwise very similar to standard Dual Pivot Quicksort

[Aumüller and Dietzfelbinger(2013)]

Yaroslavskiy's Dual Pivot Quicksort

- $1.9n \log n - 2.46n + O(\log(n)$ Comparisons

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- $1.9n \log n - 2.46n + O(\log(n)$ Comparisons
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Kushagra-Ortiz-Qiao-Munro Tri-Pivot Quicksort

- $1.846n \log n + O(n)$ Comparisons

[Kushagra et al.(2013)Kushagra, López-Ortiz, Qiao, and Munro]

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- $1.846n \log n + O(n)$ Comparisons
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M-Pivot Quicksort

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- Partition each segment on at a time
- $M + 1$ Recursive Calls

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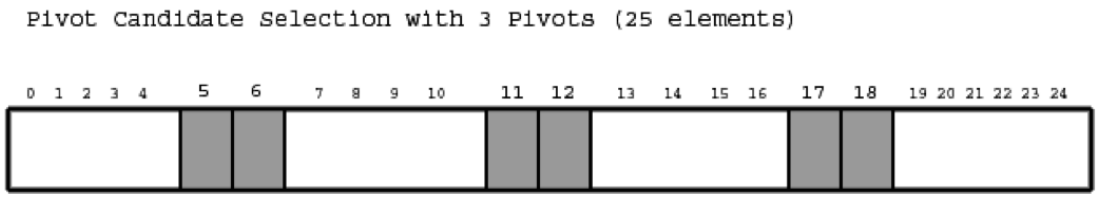
Example M-Pivot Selection and Partitioning



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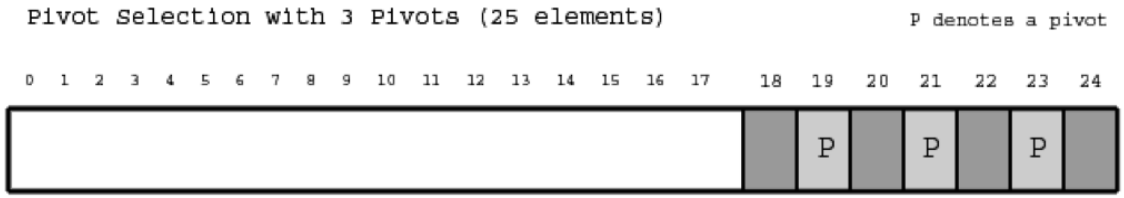

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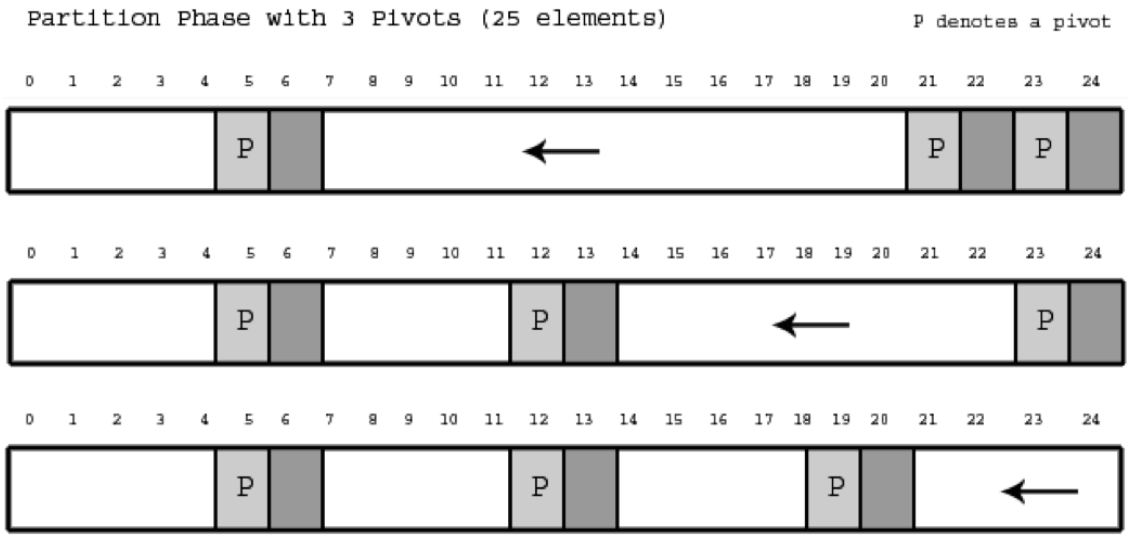

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Heap Optimized M-Pivot Quicksort

- Enforce the Heap Property to the array before any computation

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  - Adds only $O(n)$ run time at each call

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Heap Optimized M-Pivot Quicksort

- Enforce the Heap Property to the array before any computation
  - Adds only $O(n)$ run time at each call
  - Improves the sort to optimize pivot selection

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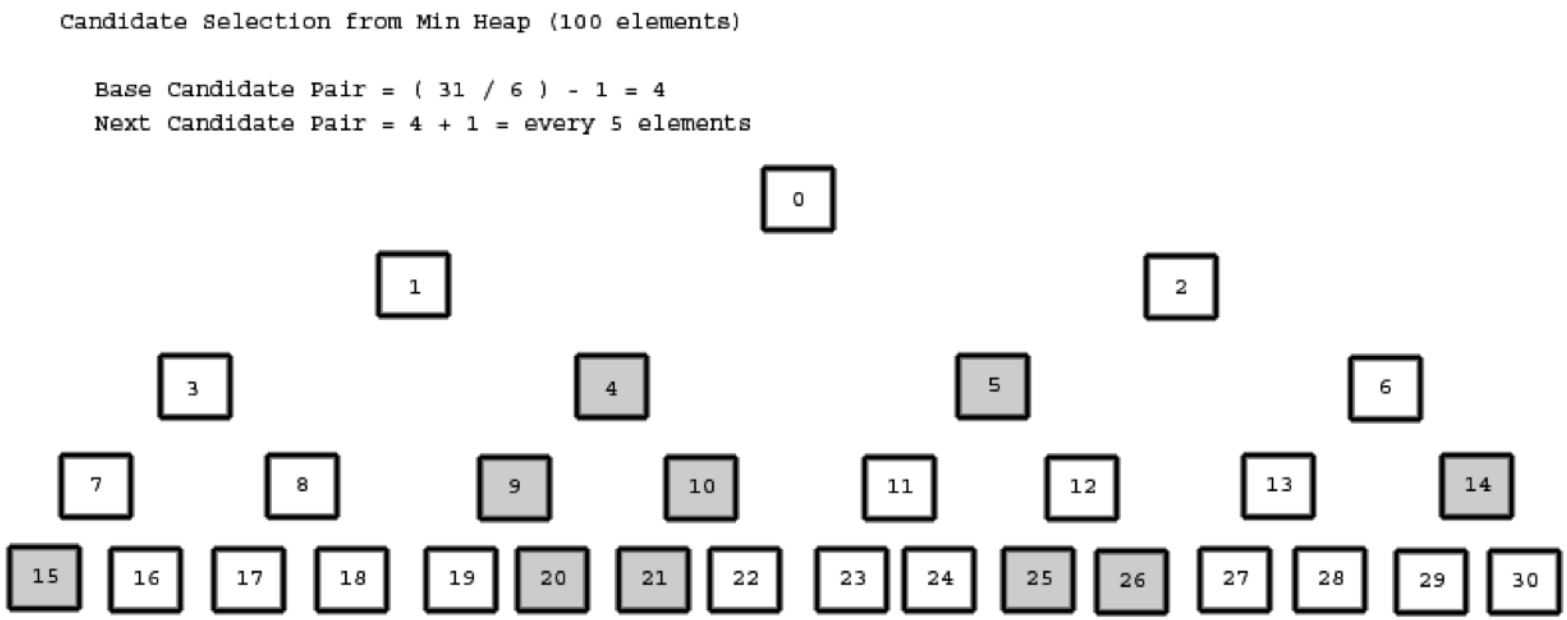
Example M-Pivot Heap Optimized Pivot Selection



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Theoretical Average Case Run Time

Sort Method	Comparisons
Classic	$2n \log n - 1.51n + O(\log(n))$
Dual Pivot	$2.13n \log n - 2.57n + O(\log(n))$
Optimal Dual Pivot	$1.8n \log n + O(n)$
Three Pivot	$1.846n \log n + O(n)$
Yaroslavskiy	$1.9n \log n - 2.46n + O(\log(n)$
M Pivot	$O(n \log n)$

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Theoretical Average Case Run Time

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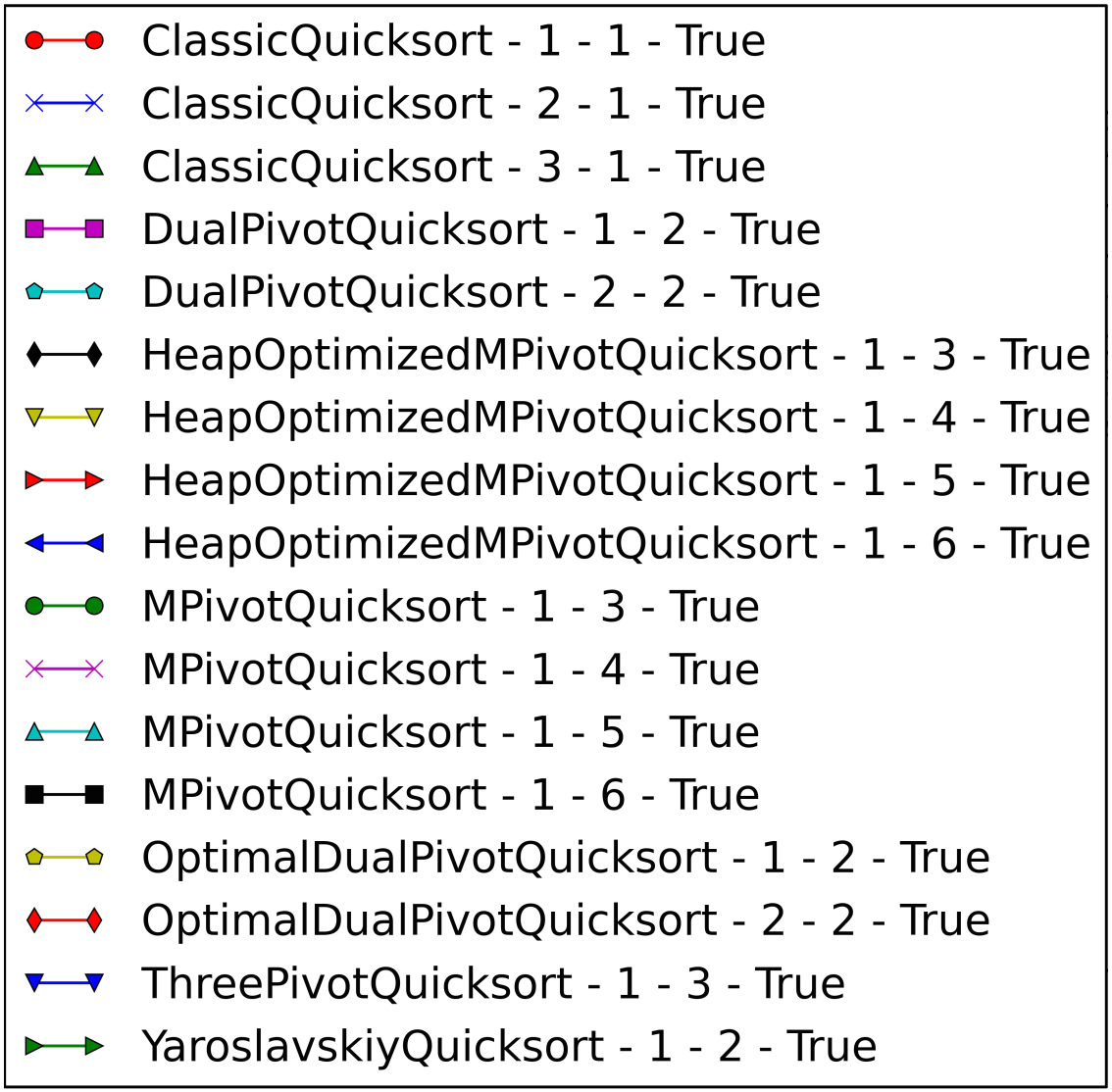
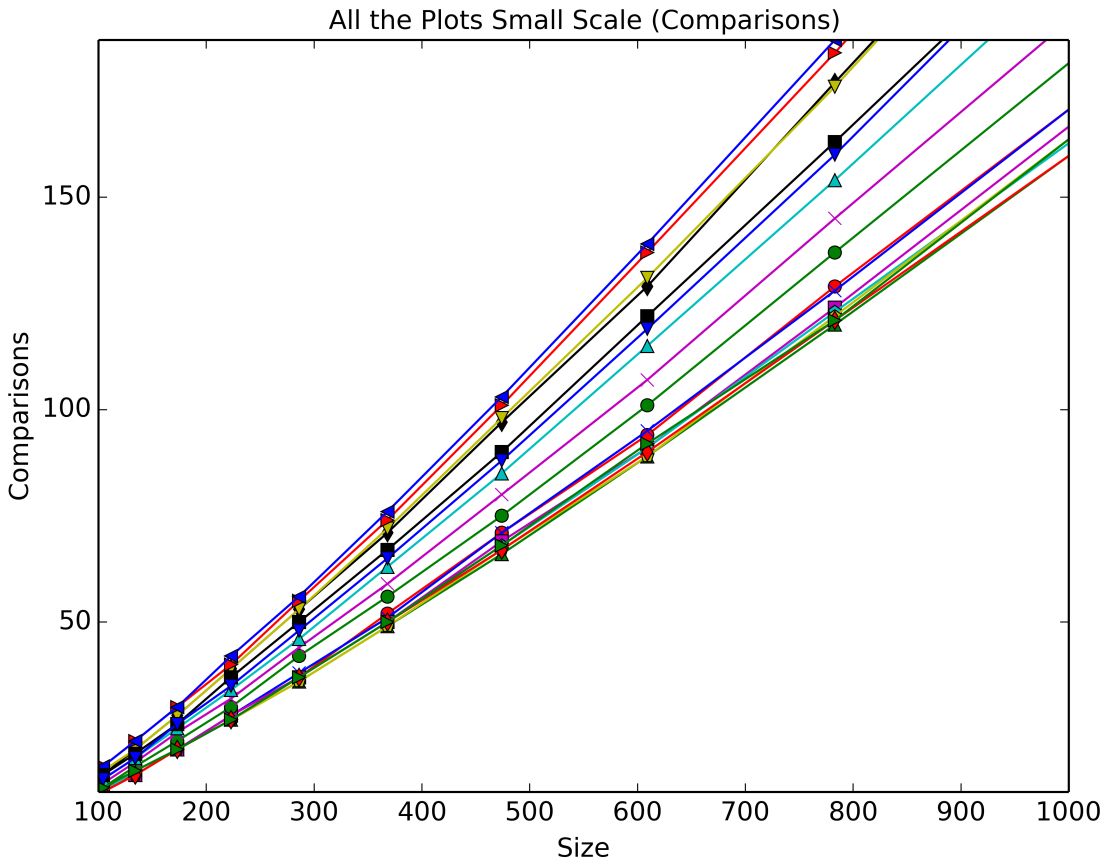
Legend



Mass Comparison Small Scale



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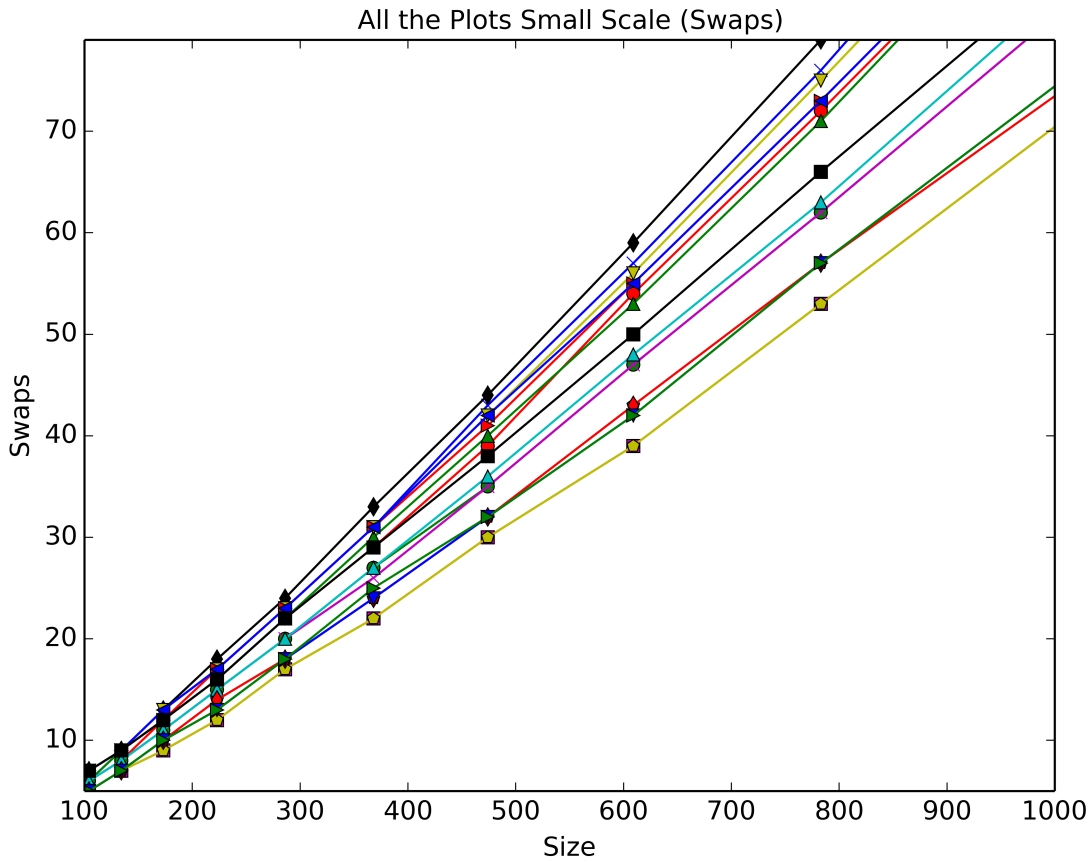
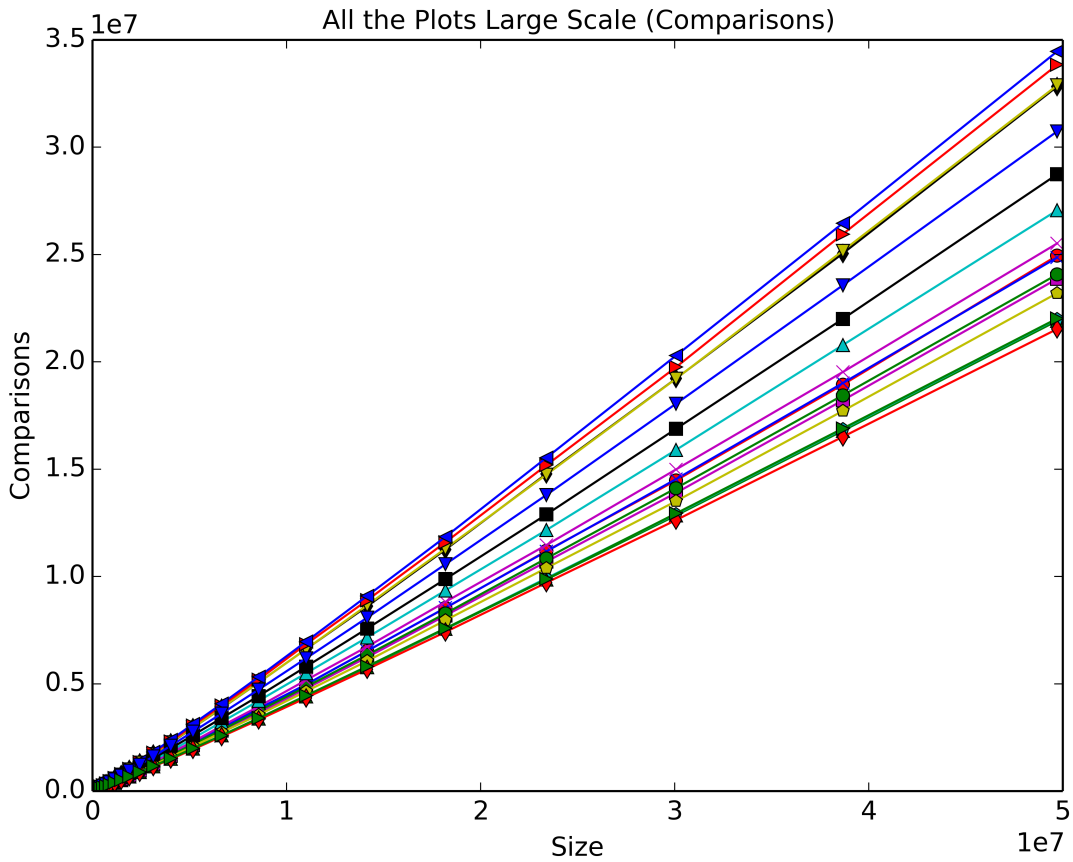


Mass Comparison Large Scale



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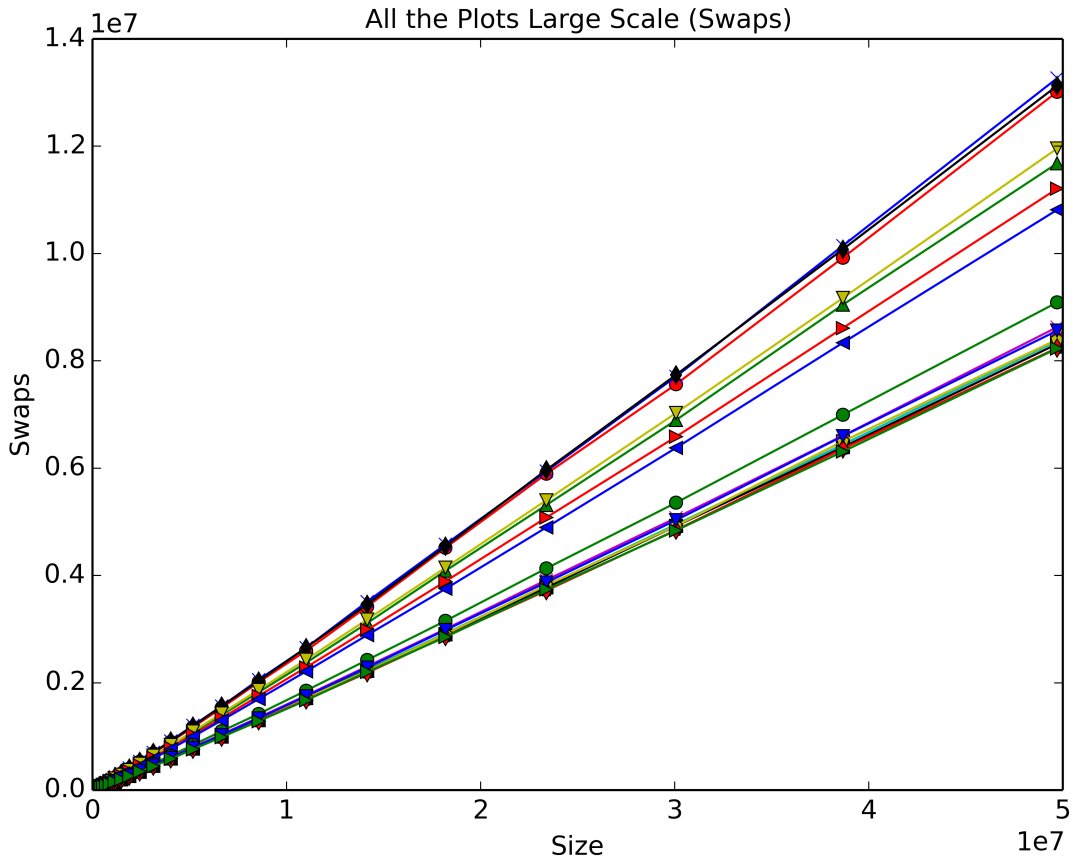
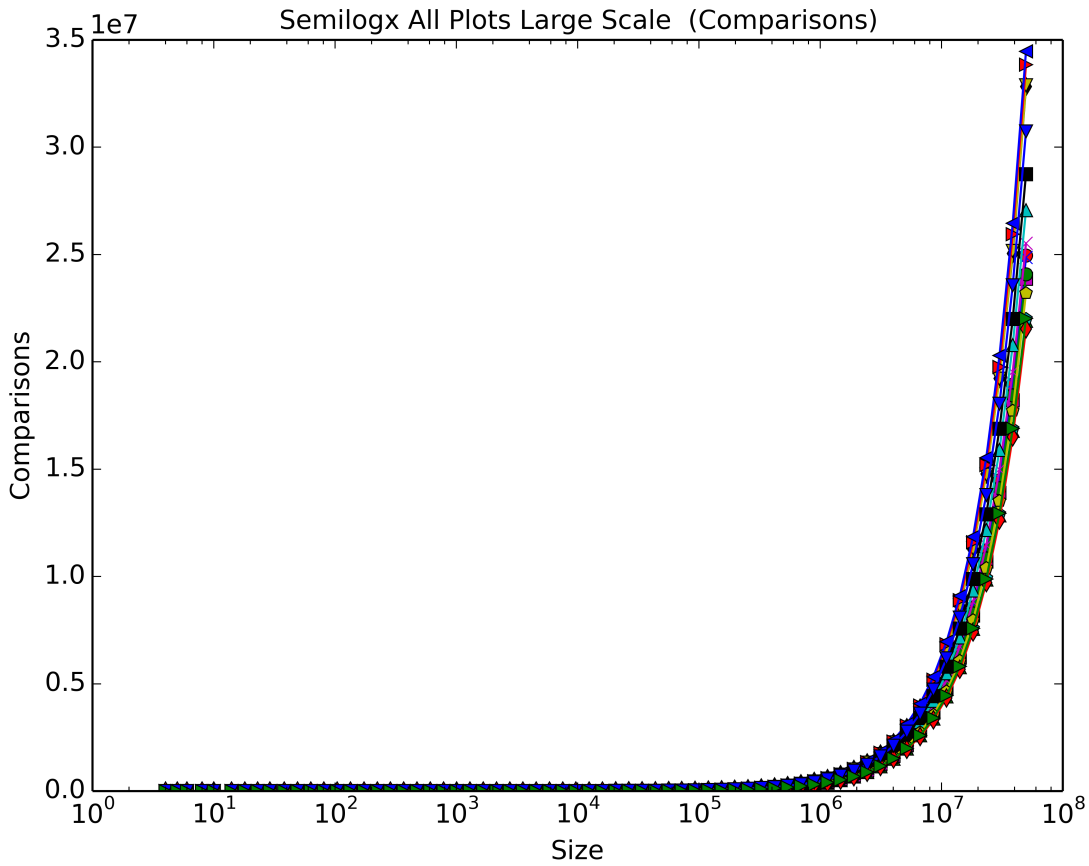


Mass Comparison Semi Log x



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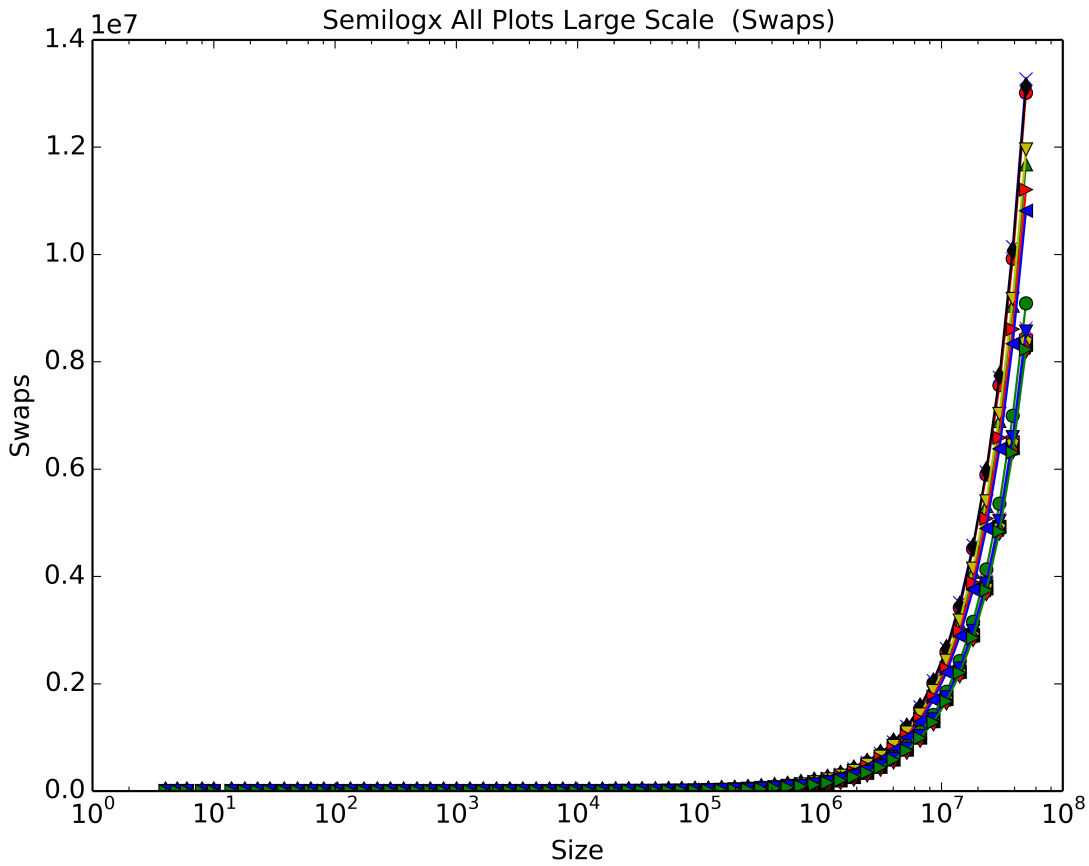

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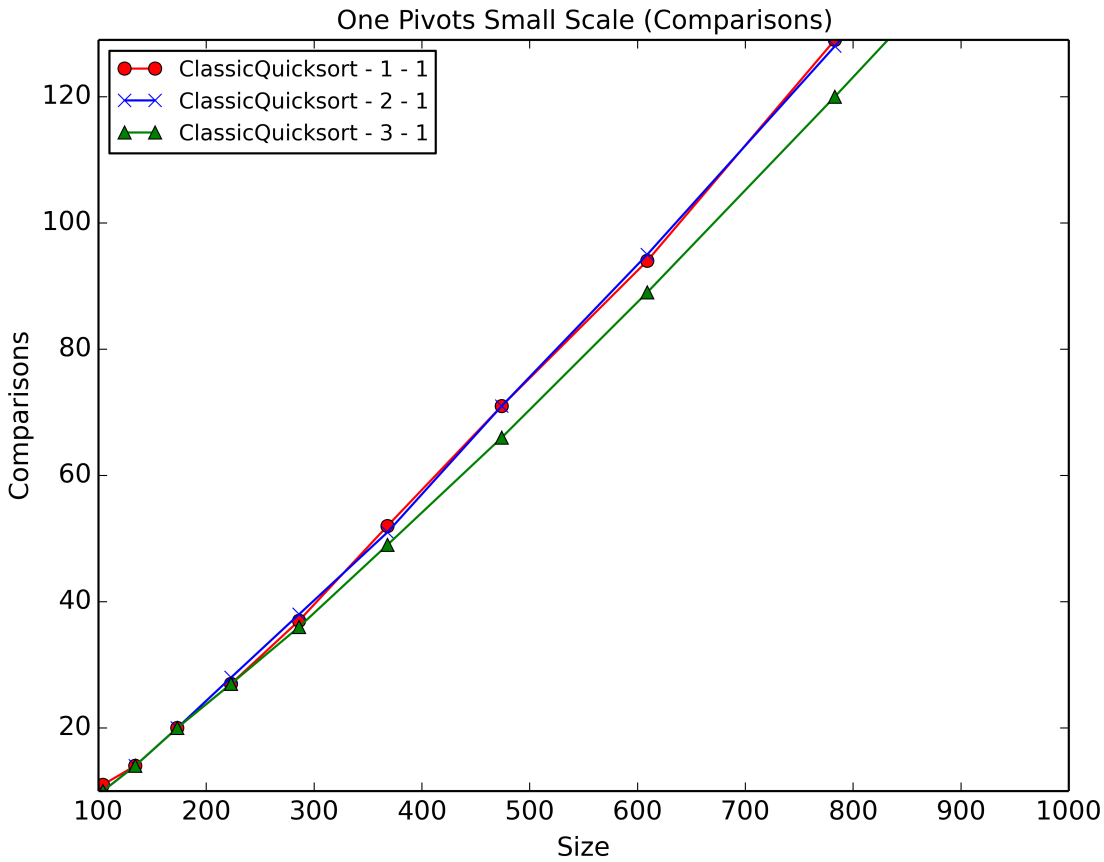

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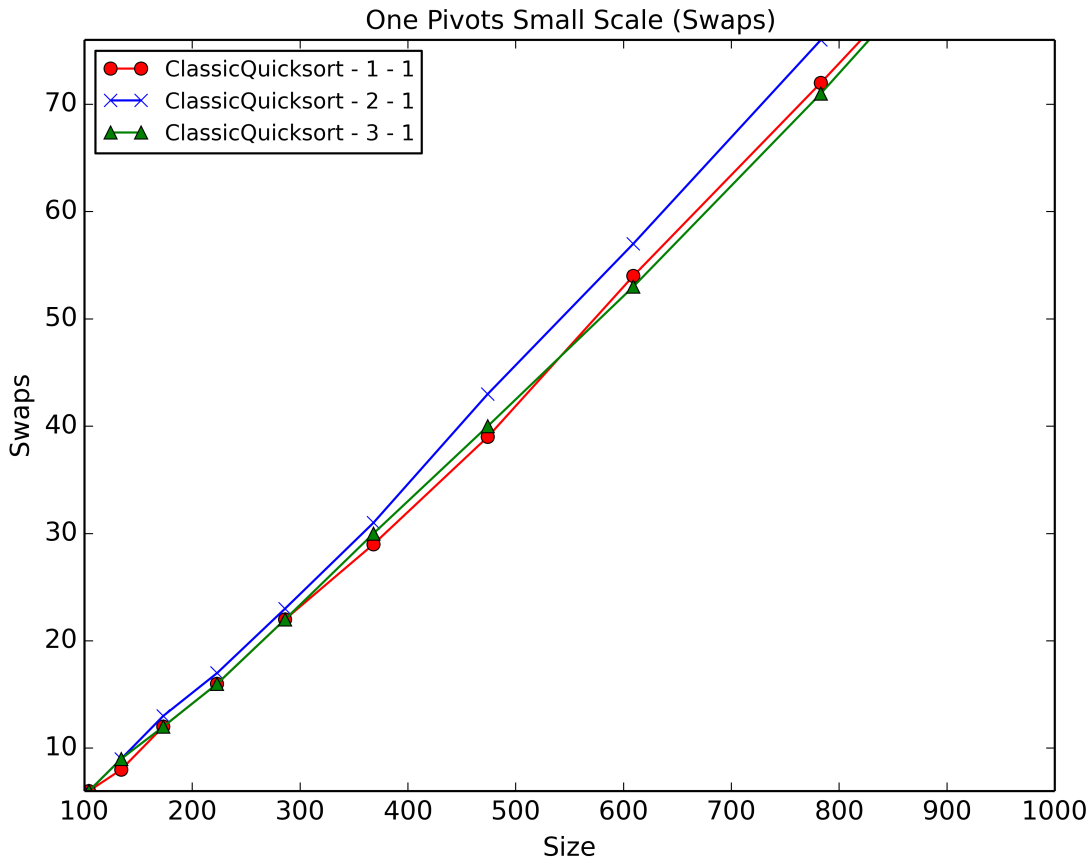

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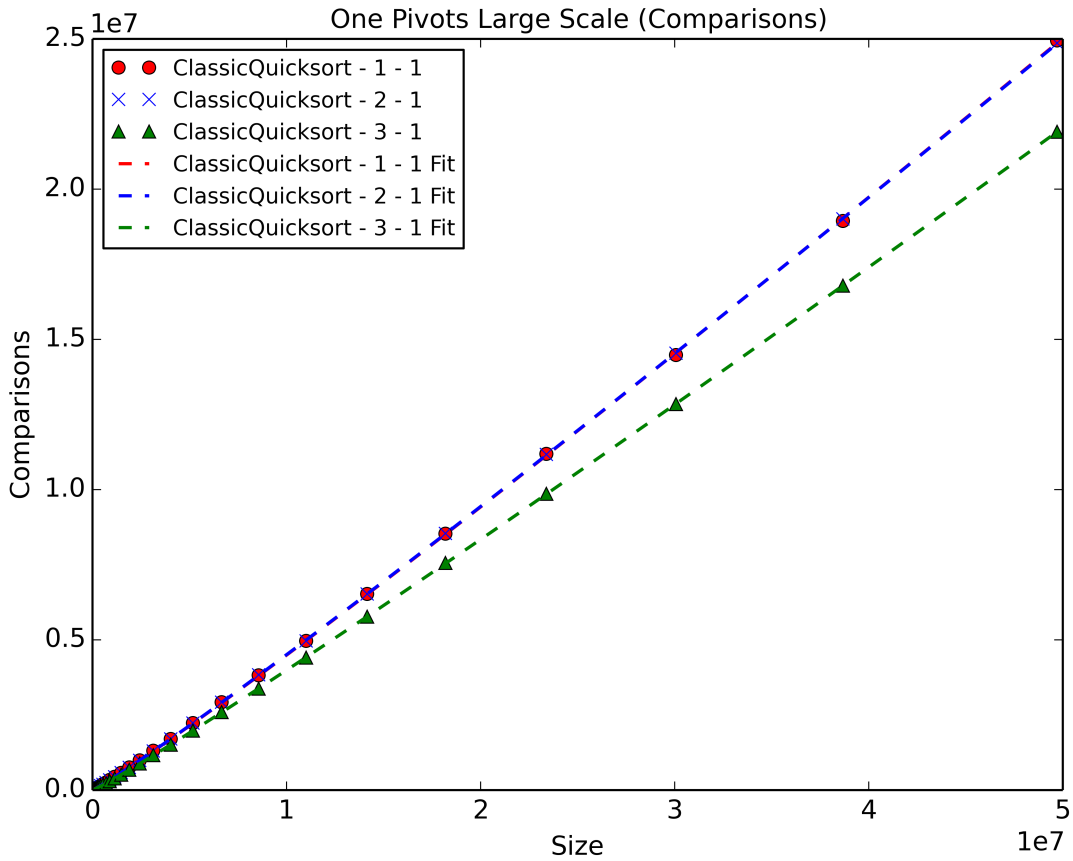

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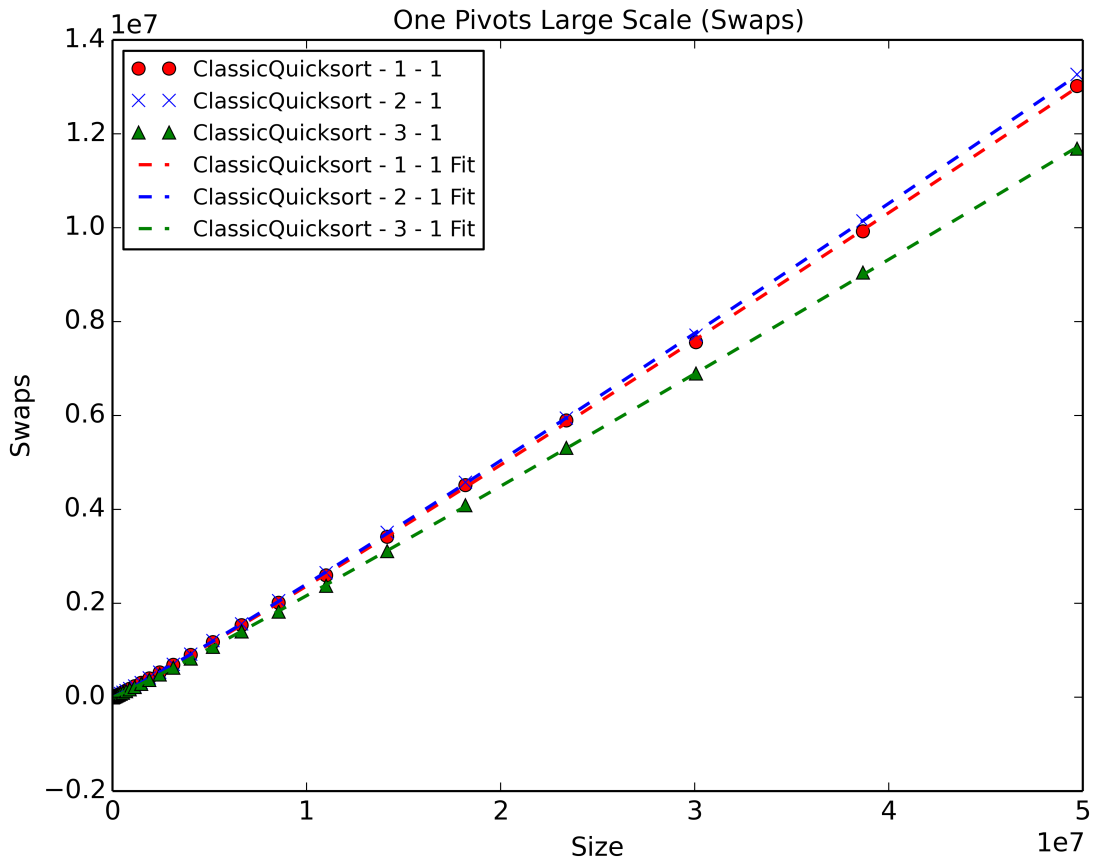
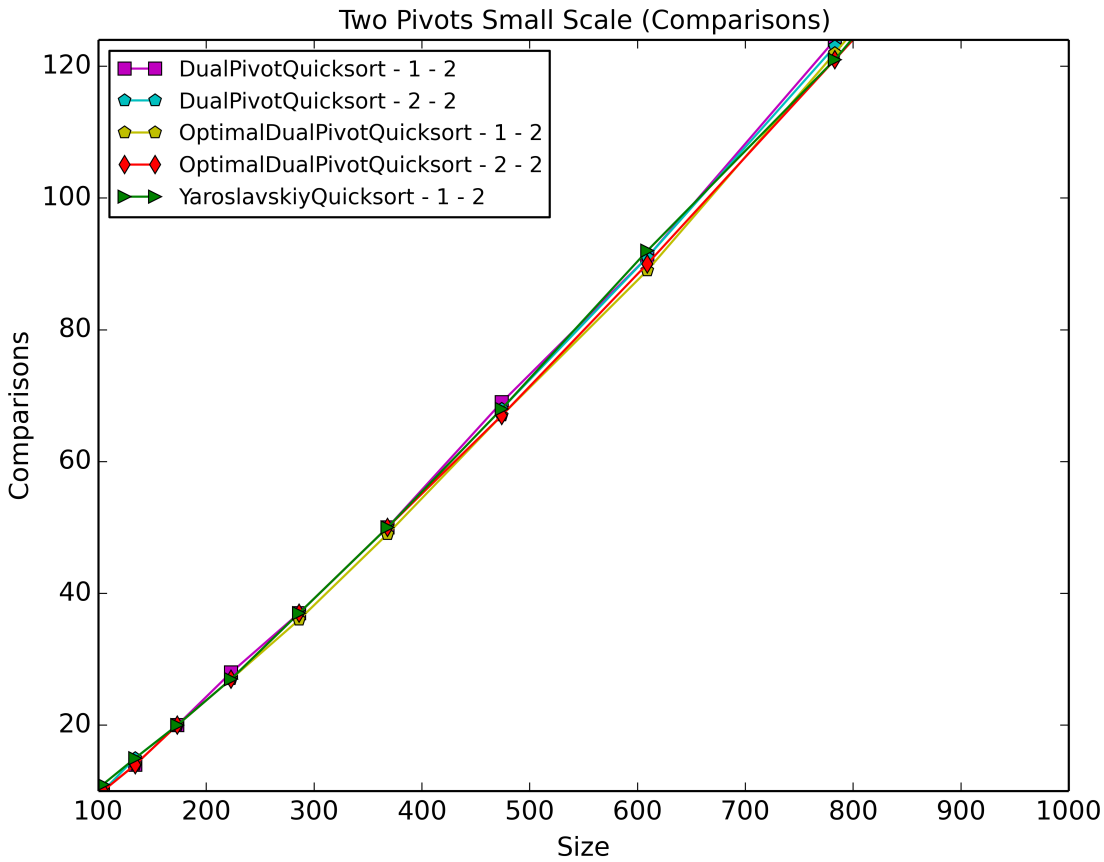


Two Pivot Comparison Small Scale



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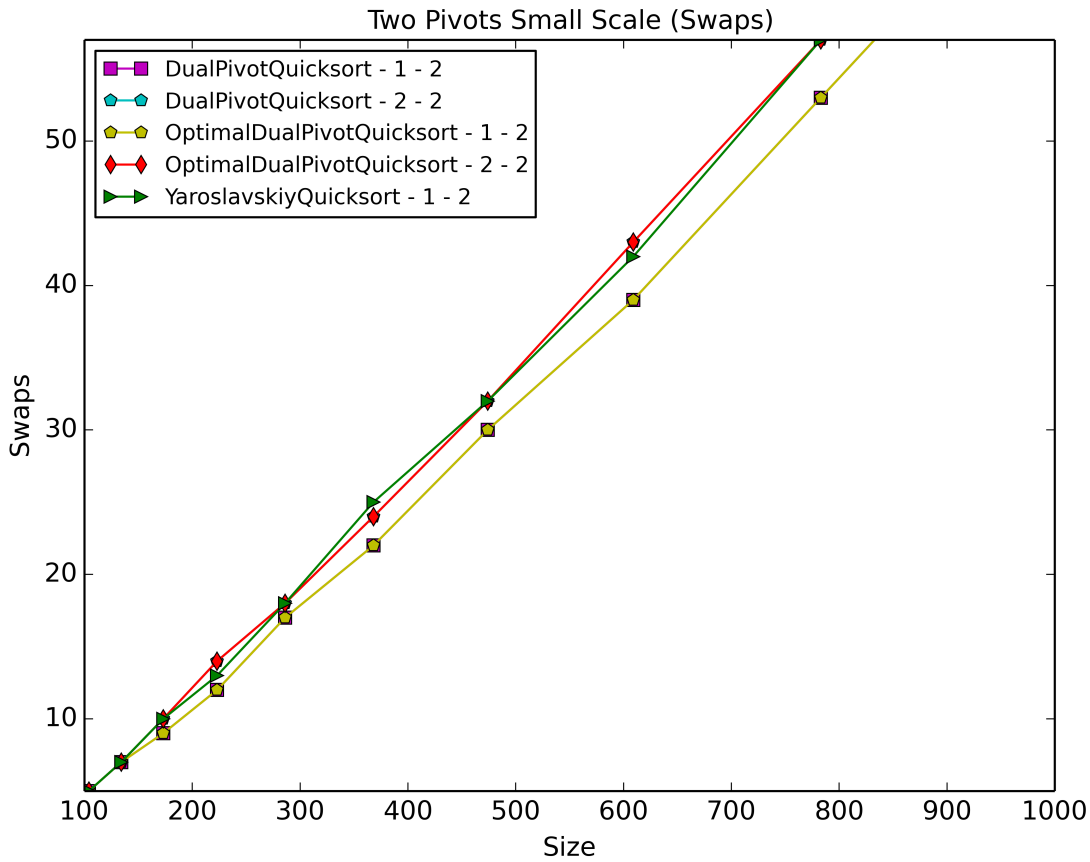

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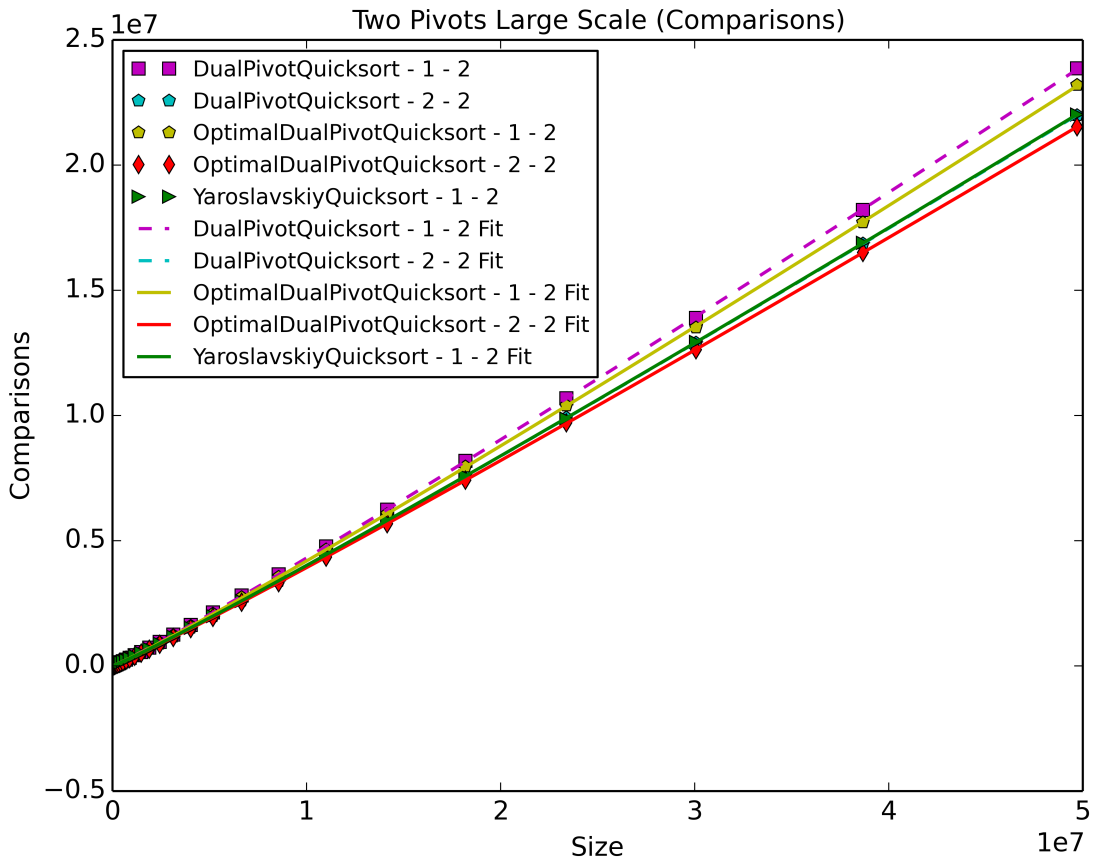

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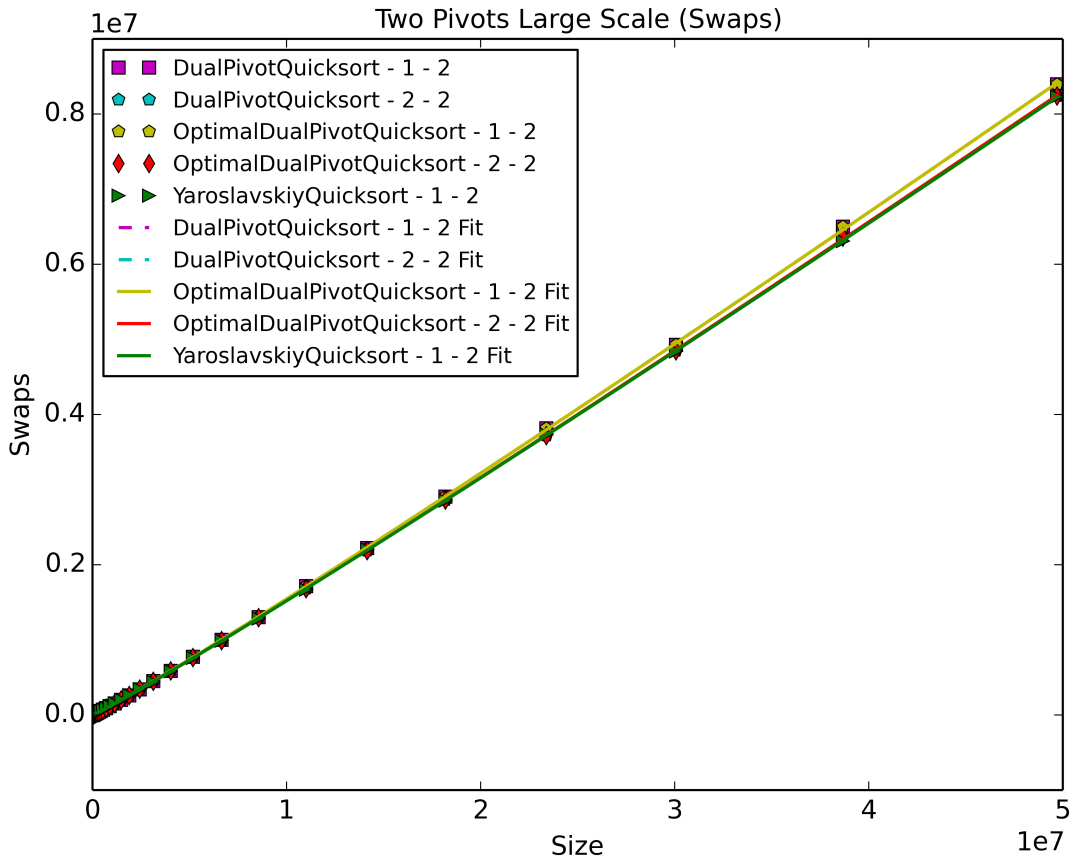
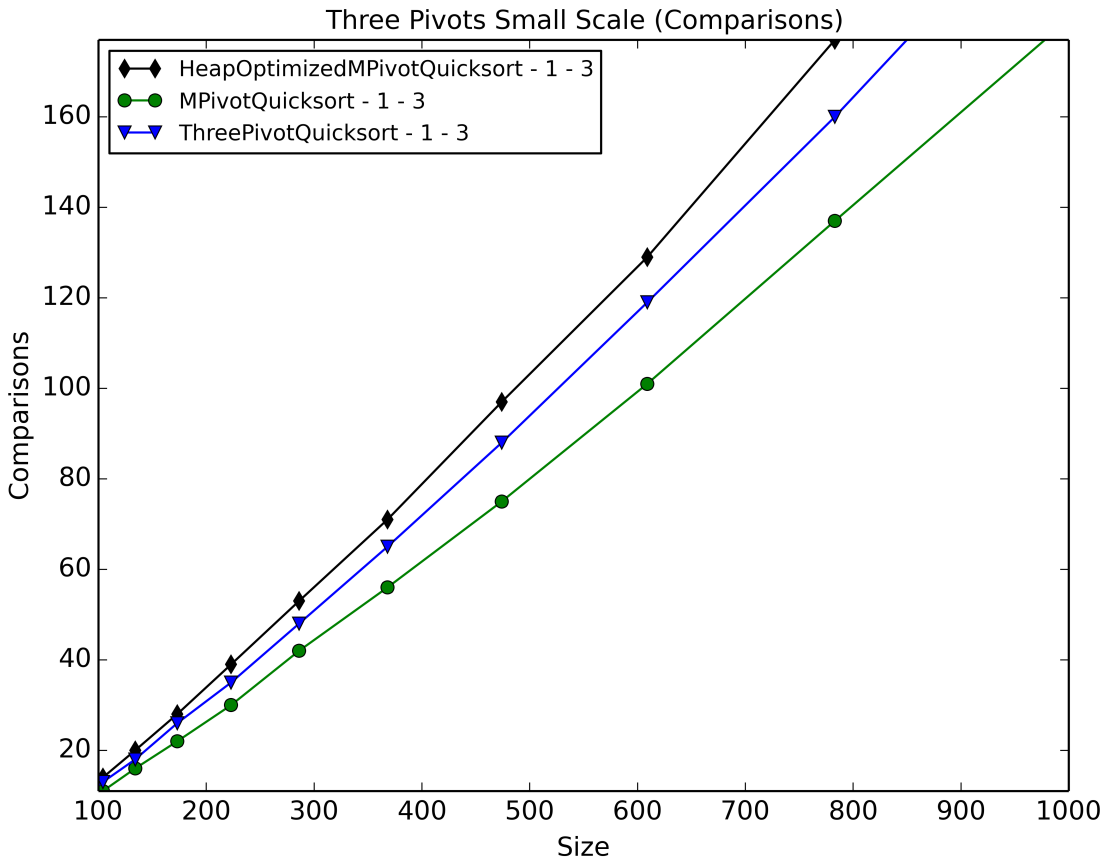


Three Pivot Comparison Small Scale



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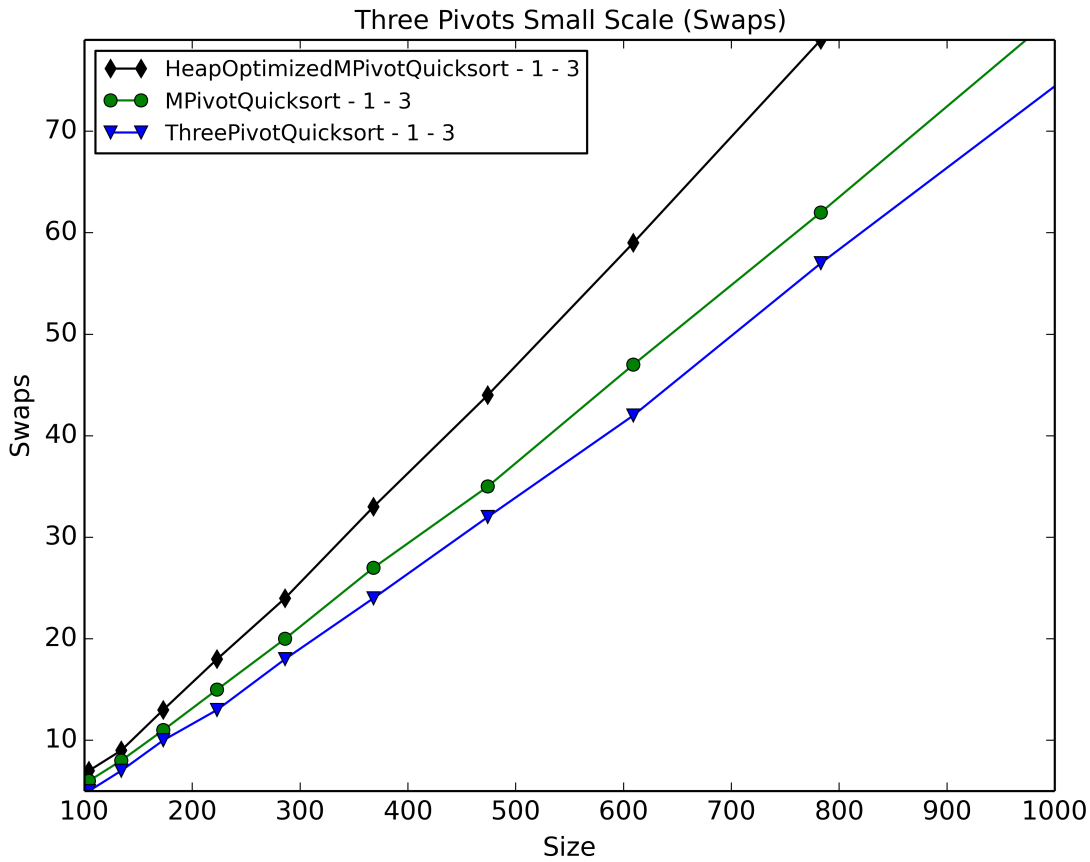

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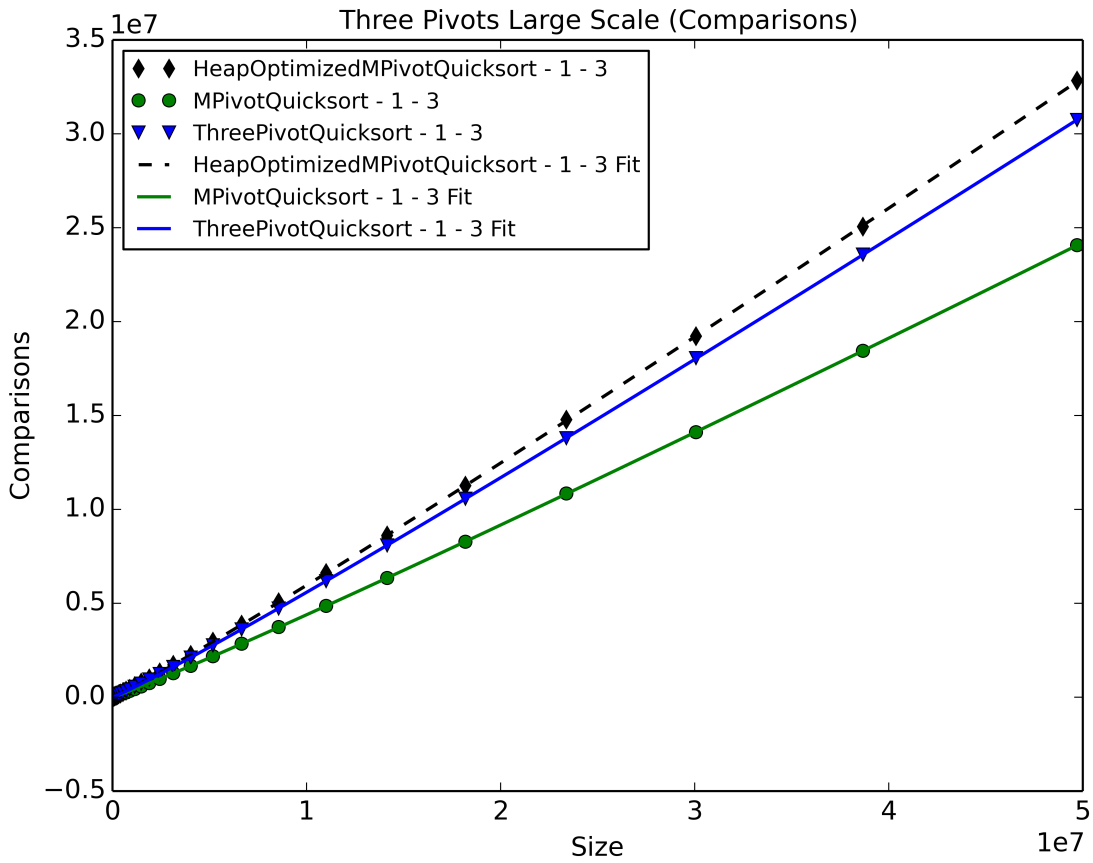

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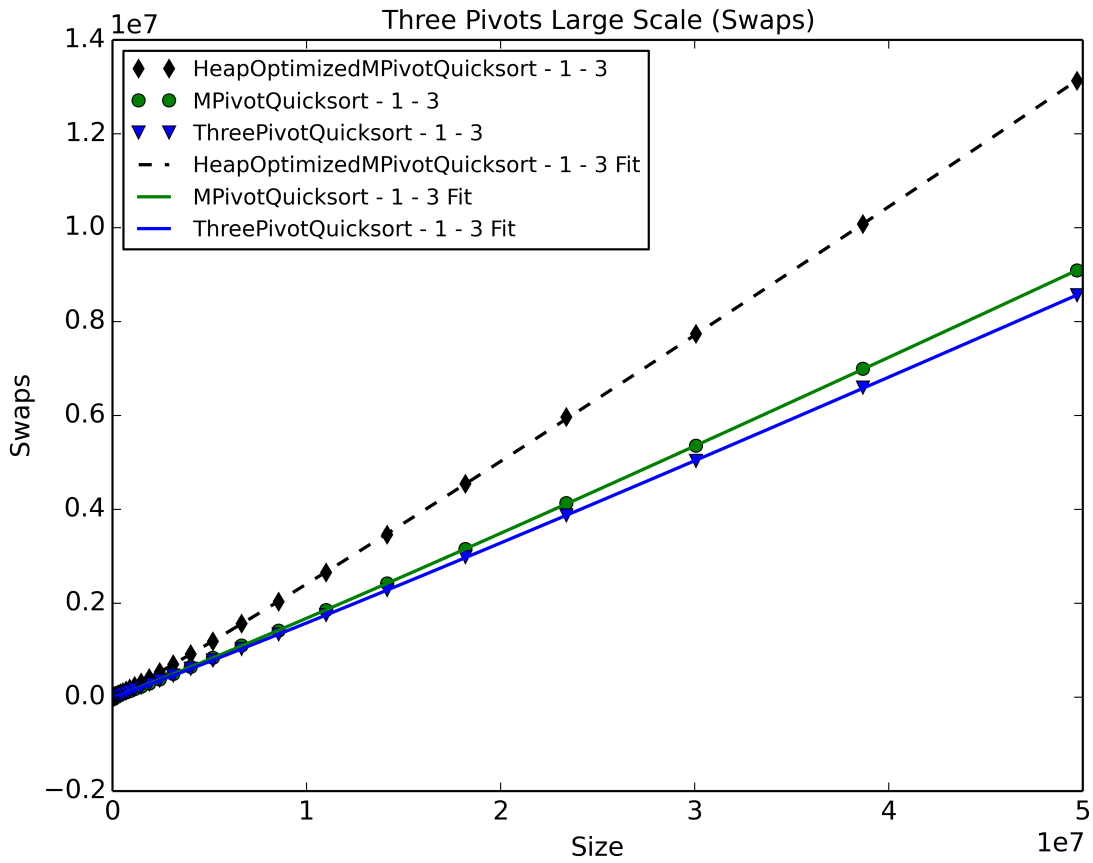
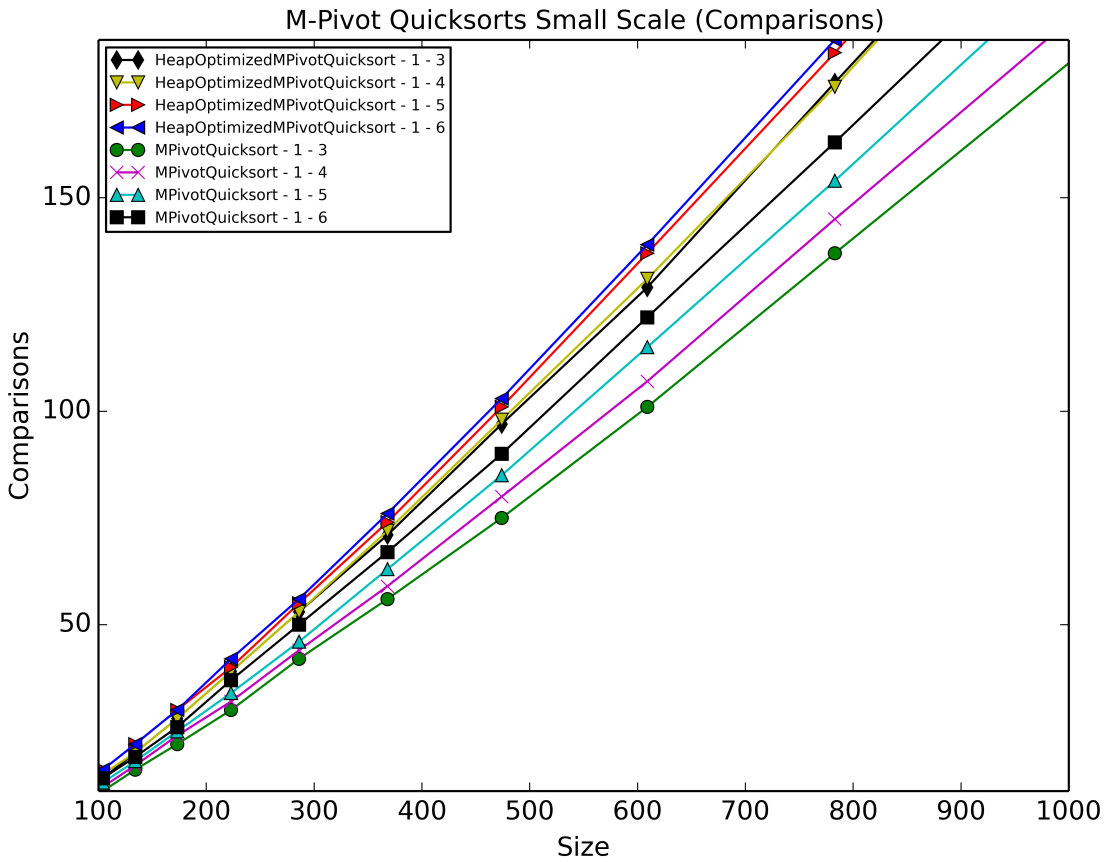


M Pivot Comparison Small Scale



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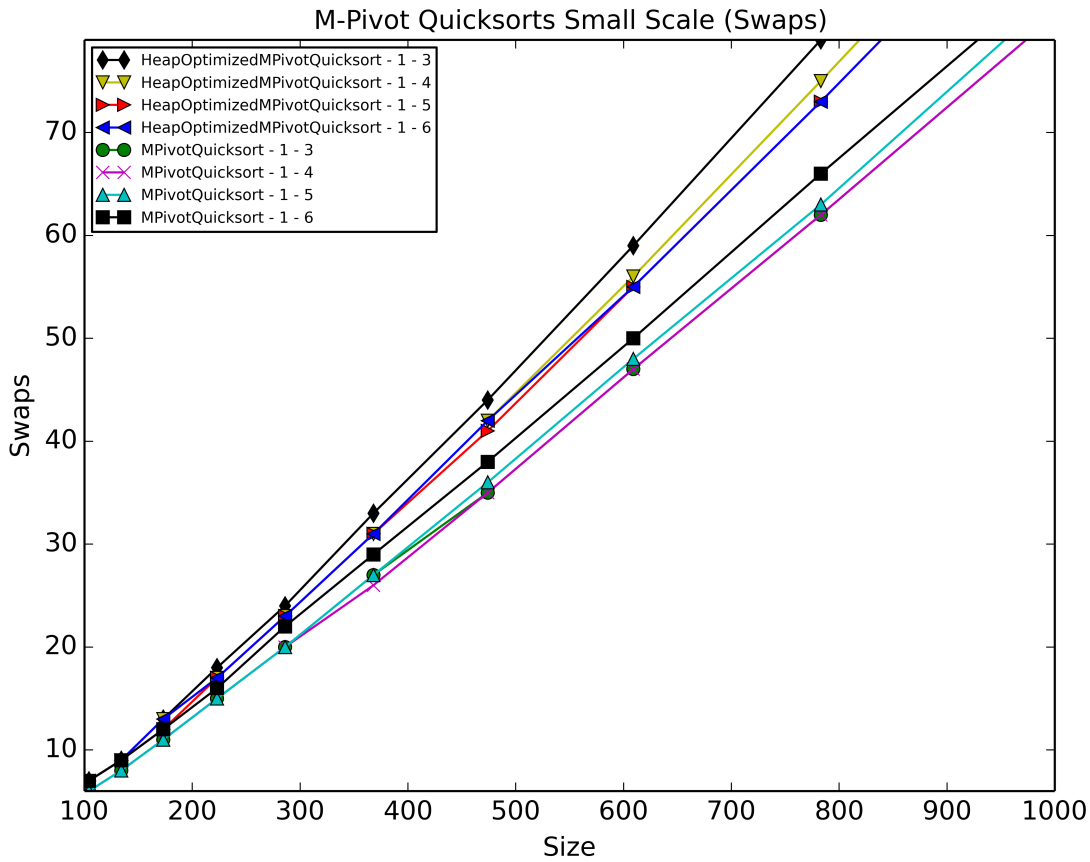

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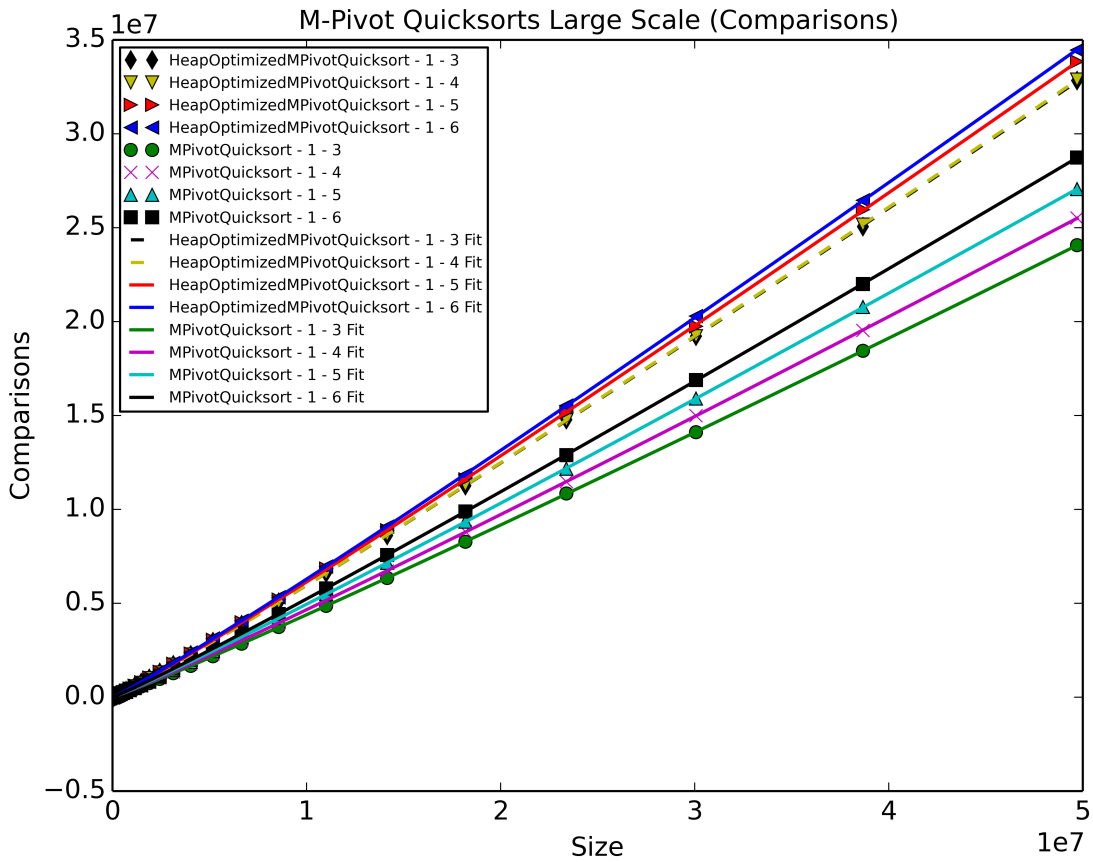
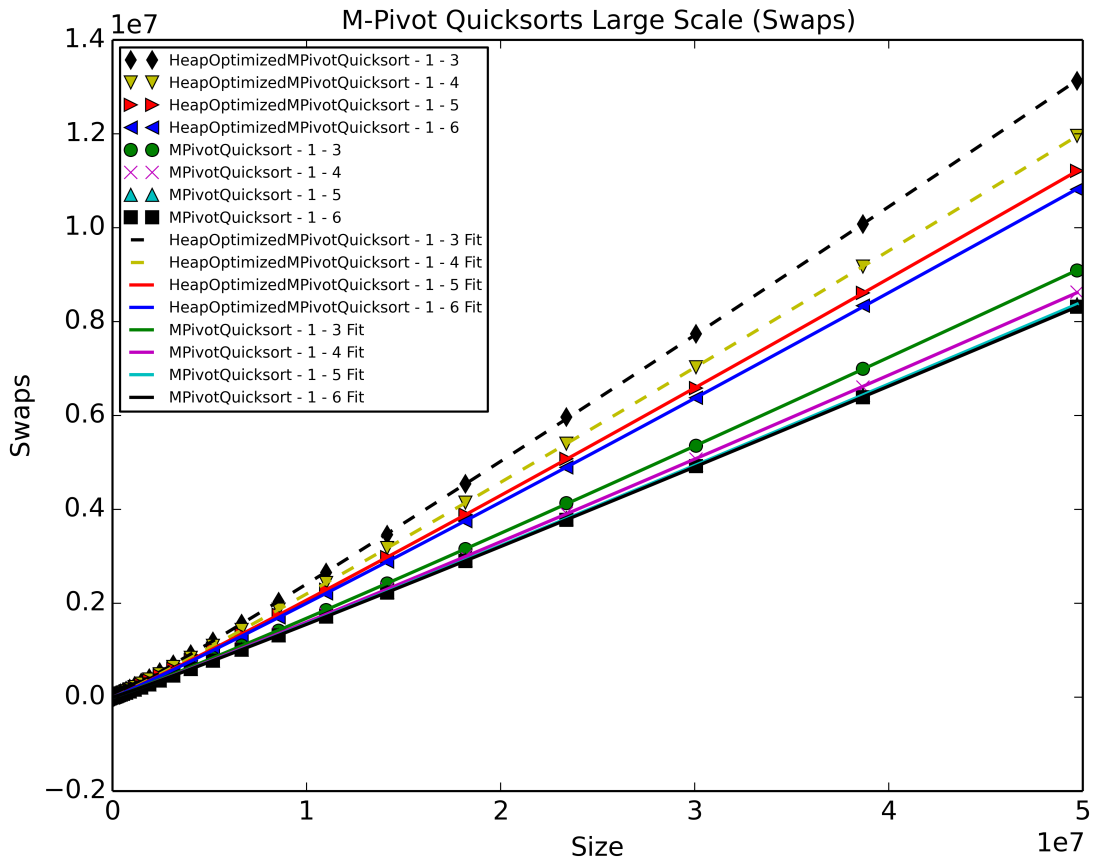


M Pivot Comparison Large Scale



Fit Coefficients of $An\log(n) + Bn + C\log(n)$

Sort Method	Comparisons	Swaps
Classic - 1 - 1	0.02219	0.01060
Classic - 2 - 1	0.02126	**0.01110**
Classic - 3 - 1	0.01799	0.00828
Dual Pivot - 1 - 2	0.02109	0.00636
Dual Pivot - 2 - 2	0.01787	0.00603
Optimal Dual Pivot - 1 - 2	0.02044	0.00636
Optimal Dual Pivot - 2 - 2	**0.01754**	0.00603
Three Pivot - 1 - 3	0.02595	0.00616
Yaroslavskiy - 1 - 2	0.01811	0.00584

Fit Coefficients of $An\log(n) + Bn + C\log(n)$

Sort Method	Comparisons	Swaps
Heap M Pivot - 1 - 3	0.02755	0.00999
Heap M Pivot - 1 - 4	0.02782	0.00885
Heap M Pivot - 1 - 5	**0.02903**	0.00809
Heap M Pivot - 1 - 6	0.02801	0.00769
M Pivot - 1 - 3	0.01955	0.00640
M Pivot - 1 - 4	0.02039	0.00594
M Pivot - 1 - 5	0.02136	0.00532
M Pivot - 1 - 6	0.02369	**0.00524**

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Questions?