IBM3103 – Mathematical Methods for Biological and Medical Engineering

Fall 2021

Instructor: Carlos A. Sing Long
Teaching Assistant: Carlos Castillo

Assignment #1

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Due on May 7, 2021

Problem #1: Orthogonal bases, feature extraction, and denoising (50pts)

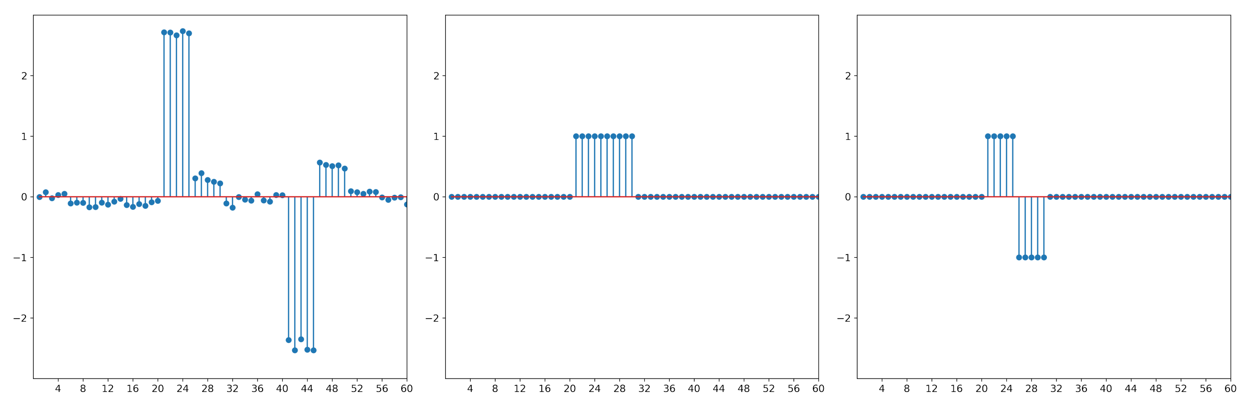
In this problem we consider a signal u that you observe during an hour at intervals of one minute. In other words, you measure the values $u(1), \ldots, u(60)$. You decide to represent these values on a vector

$$\boldsymbol{u} = \begin{bmatrix} u(1) \\ \vdots \\ u(60) \end{bmatrix}$$

in $\mathbb{R}^{60}$. The signal is shown in Fig. 1a and can be found in the file `signal.py`. To better understand the information contained in this signal you decide to approximate by taking linear combinations of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}$. An example of these vectors can be seen in Fig. 1b and Fig. 1c and you find them in the file `basis.py`. This file contains a 60×12 matrix `V` where each column represents a vector in the collection.

(a) Interpret each one of the elements of the collection $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}$ in terms of a **feature**. For example, do elements of the collection measure averages over certain time periods? Or changes in the value of the signal over different time periods?

(b) Are the elements of the collection $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}$ orthogonal? What is the Euclidean norm of each element of the collection? Are these vectors linearly independent? **Suggestion:** Use Python.



(A) (B) (C)

Figure 1. Signals used in Problem #1.

(c) Using Python, find the orthogonal projection of $\boldsymbol{u}$ onto the subspace

$$\mathbf{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}\}.$$

If we let $\boldsymbol{p}$ be the orthogonal projection, then plot the components of the vector $\boldsymbol{u}$ and the components of the vector $\boldsymbol{p}$ on the same figure. Comment on the differences between these two signals.

(d) Let $\alpha_1, \ldots, \alpha_{12}$ be scalars such that

$$\boldsymbol{p} = \alpha_1 \boldsymbol{v}_1 + \ldots + \alpha_{12} \boldsymbol{v}_{12}.$$

Plot the square of the magnitudes of the scalars $\alpha_1, \ldots, \alpha_{12}$. Which one of these values is larger? Deduce which term of the collection $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{12}$ contributes most to $\boldsymbol{u}$. Explain in your own words using your answer in (a) what feature of $\boldsymbol{u}$ is represented by the largest coefficient.

Problem #2: Christmas tree (50pts)

To impress everyone this Christmas you decide to animate your Christmas tree based on the 3D positions of the LED lights you use for decoration. To get the 3D coordinates of the LEDs, you take photos of your Christmas tree from $N = 100$ different angles $\theta_i \in [0, \pi)$. On each photo you write the coordinates of every one of the $M = 100$ lights. The data you collect looks like Fig. 2.

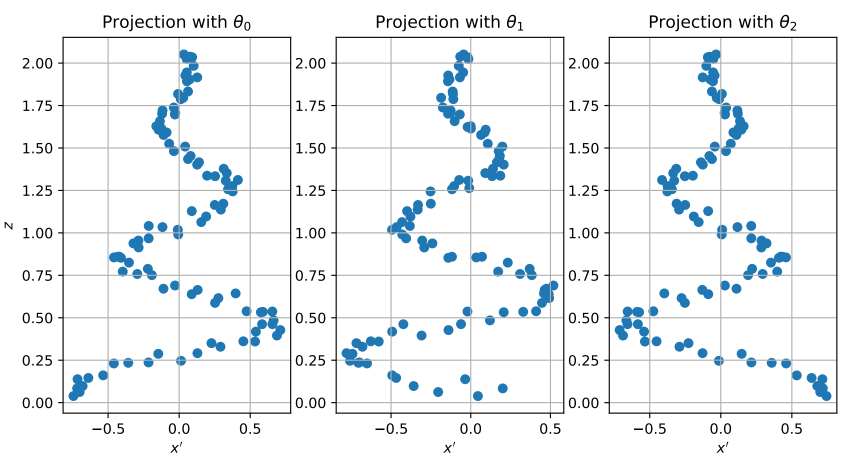


FIGURE 2. Christmas lights' positions from different angles. In this example only three angles are shown.

Your store these coordinates in the text files `/lights/0.txt`, ..., `/lights/99.txt`. The first line on the file represents the angle θ_i and the posterior lines are the coordinates x'_j and z'_j of the positions of the lights on each image as in Fig. 2. Here the index j denotes the j-th Christmas light.

(a) Find an expression relating the coordinates x'_j and z'_j of the lights on each image in terms of the "real world" positions in 3D space x_j, y_j, and z_j.

(b) Use the expression you found to formulate a linear system to find the original positions from the projections.

(c) Using Python, solve the aforementioned problem. Display the obtained positions using the function `plot_tree(x,y,z)`.

(d) Program a creative animation[1] in which you change the colors of the lights based on its 3D positions. As a start, try to color the lights using their distance to a moving plane. Save your animation as a GIF or a MP4.

[1] https://www.youtube.com/watch?v=v7eHTNm1YtU