Numerical Methods Assignment № 5

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Problem 1

Part (a)

Formula for absolute error when v is approximating u is:

$$|u - v|$$

Formula for relative error when v is approximating u is:

$$\frac{|u - v|}{|u|}$$

Relative error is more meaningful especially if $|u| \gg 1$ as it doesn't change depending on the value of u.

Example:

u	v	Absolute Error	Relative Error
1	0.99	0.01	0.01
100	99	1	0.01

Part (b)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ... + \frac{h^n}{n!}f^n(x)$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + ... + \frac{(2h)^n}{n!}f^n(x)$$

$$2f(x+h) = 2f(x) + 2hf'(x) + 2\frac{h^2}{2!}f''(x) + ... + 2\frac{h^n}{n!}f^n(x)$$

$$2f(x+h) - f(x+2h) = f(x) + \frac{(2-2^2)}{2!}h^2 f''(x) + ... + \frac{(2-2^n)}{n!}h^n f^n(x)$$

$$h^2 f''(x) = f(x) - 2f(x+h) + f(x+2h) + \frac{(2-2^3)}{3!}h^3 f'''(x) + ... + \frac{(2-2^n)}{n!}h^n f^n(x)$$

$$f''(x) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} + \frac{(2-2^3)}{3!}hf'''(x) + ... + \frac{(2-2^n)}{n!}h^{n-2} f^n(x)$$

$$\left| f''(x) - \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} \right| \approx -h\left|f'''(x)\right|$$

So long as $f'''(x)$ is bounded, the discretization error is $\mathcal{O}(h)$.

Part (c)

$$(tan(x))'' = \frac{2sinx}{cos^3x}$$

$$f''(x) \approx \frac{tan(x) - 2tan(x+h) + tan(x+2h)}{h^2}$$

Actual error is:

$$\left| \frac{2sinx}{cos^3x} - \frac{tan(x) - 2tan(x+h) + tan(x+2h)}{h^2} \right|$$

Since expected error is $O(h)$, the graphic for difference between expected error and actual error is:

Listing 1: Python code to generate the graph below.

```
import matplotlib.pyplot as plt
import numpy as np

x = np.pi / 4
h = [1e-1,1e-2,1e-3,1e-4,1e-5,1e-6,1e-7,1e-8,1e-9,1e-10]

arr = []
for i in range(0,10):
    derivative = 2 * np.sin(x) / (np.cos(x) * np.cos(x) * np.cos(x))
    approx = (np.tan(x) - 2 * np.tan(x + h[i]) + np.tan(x + 2 * h[i])) / (h[i] * h[i])
    err = abs(derivative - approx)
    diff = err - h[i]
    arr.append(diff)
plt.plot(h,arr)
plt.xscale('log')
plt.yscale('log')
```

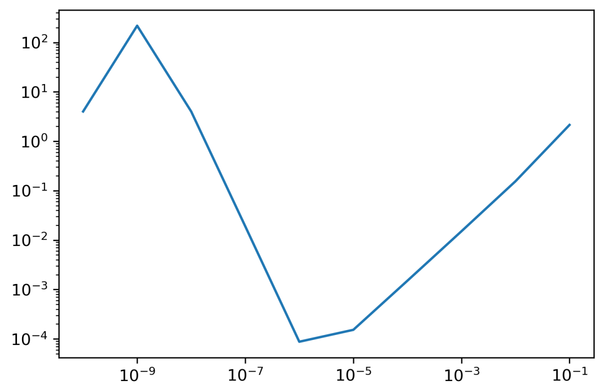

Figure 1: Difference between the expected error (O(h)) and the actual error

Part (d)

The actual error consists of discretization errors and roundoff errors. When h is relatively large, discretization error dominates roundoff errors but as h gets smaller roundoff error starts to dominate. For smaller h values actual error gets very large because of roundoff errors and as a result produces bigger errors than expected.

Part (e)

The problem is well conditioned because small perturbations in value of the function produces similar results near $\frac{\pi}{4}$.

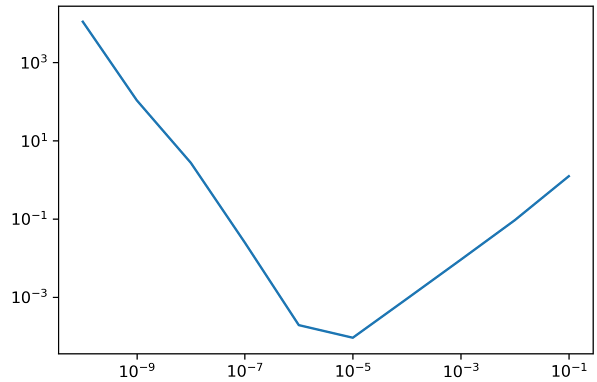

Figure 2: Actual error when $x = \frac{\pi}{4} - 0.1$

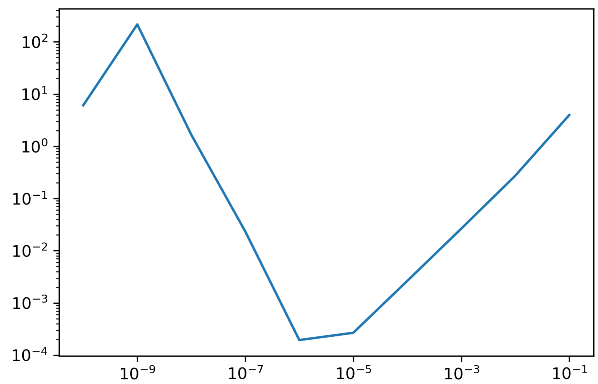

Figure 3: Actual error when $x = \frac{\pi}{4} + 0.1$

As we can see from the graphs, small perturbations in $\frac{\pi}{4}$ does not cause big changes in calculation.

Problem 2

Part (a)

0.03754 $\times$ 2 = 0.07508	0
0.07508 $\times$ 2 = 0.15016	0
0.15016 $\times$ 2 = 0.30032	0
0.30032 $\times$ 2 = 0.60064	0
0.60064 $\times$ 2 = 1.20128	1
0.20128 $\times$ 2 = 0.40256	0
0.40256 $\times$ 2 = 0.80512	0
0.80512 $\times$ 2 = 1.61024	1
0.61024 $\times$ 2 = 1.22048	1

Normalized floating point representation of the number is 1.0011×2^{-5}

Ten bit word for the number is 0101010011. First bit is 0 because number is positive, second bit is 1 because exponent is negative, next four bits are 0101 because exponent is 5, final four bits are 0011 because mantissa is 0011.

Part (b)

Decimal equivalent of the ten bit word is $(1 + \frac{1}{8} + \frac{1}{16}) \times 2^{-5} = 0.037109375$

Part (c)

Magnitude of the relative true error is:

$$\begin{aligned} |\epsilon_a| &= \left| \frac{0.03754 - 0.037109375}{0.03754} \right| \\ &= 0.01147097762 \end{aligned}$$

Machine epsilon η is:

$$\begin{aligned} \eta &= \frac{1}{2} \times 2^{-4} \\ &= 0.03125 \end{aligned}$$

$|\epsilon_a| < \eta$ is true.

Problem 3

Part (a)

Fixed points of a function are points where $h(x) = x$.

$$\begin{aligned} x^2 + \frac{4}{25} &= x \\ x^2 - x + \frac{4}{25} &= 0 \\ (x - \frac{1}{5})(x - \frac{4}{5}) &= 0 \end{aligned}$$

Points $(0.2, 0.2)$ and $(0.8, 0.8)$ are the fixed points.

Part (b)

Since $|h'(0.2)| = 0.4$ and there exists a constant $\rho < 1$ where $|h'(0.2)| < \rho$, the point $(0.2, 0.2)$ will converge if $-\infty < x_0 < 0.8$.
Since $|h'(0.8)| = 1.6$ and there is no constant $\rho < 1$ where $|h'(0.8)| < \rho$, the point $(0.8, 0.8)$ will diverge regardless of x_0.

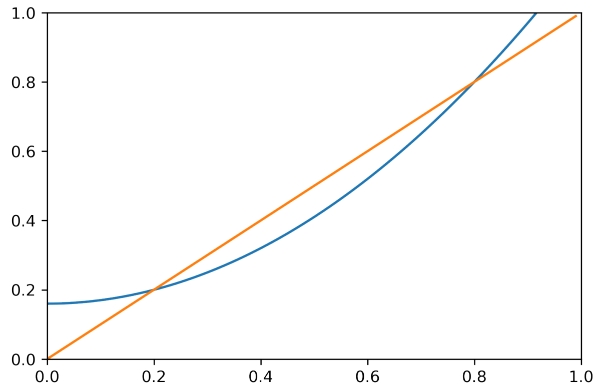

Figure 4: Graph of h(x)

Part (c)

$$\rho = |h'(0.2)|, \rho = 0.4$$

$$|x_k - x^*| \approx \rho|x_{k-1} - x| \approx ... \approx \rho^k|x_0 - x|$$

$$|x_k - x^*| \approx 0.1|x_0 - x|$$

$$\rho^k = 0.1$$

$$k log_{10} \rho \approx -1$$

$$\begin{aligned} k &= \lceil \frac{1}{-log_{10}\rho} \rceil \\ k &= 3 \end{aligned}$$

Part (d)

Newton's method is a special case of fixed point iteration where $f(x) = x - \frac{g(x)}{g'(x)}$. This relationship is useful because everything we know about fixed point iteration holds true for Newton's method. Newton's method converges fast but requires f to be differentiable.

Problem 4

If we denote first product as x_1, second product as x_2 and third product as x_3, the system of equations is as follows:

$$200x_1 + 600x_2 + 250x_3 = 55500$$
$$250x_1 + 500x_2 + 200x_3 = 51000$$
$$300x_1 + 350x_2 + 400x_3 = 51000$$

$$\begin{bmatrix} 200 & 600 & 250 & 55500 \\ 250 & 500 & 200 & 51000 \\ 300 & 350 & 400 & 51000 \end{bmatrix}$$

Multiply R_1 by $\frac{1}{200}$.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 250 & 500 & 200 & 51000 \\ 300 & 350 & 400 & 51000 \end{bmatrix}$$

Add $-250 \times R_1$ to R_2.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 0 & -250 & -\frac{225}{2} & -18375 \\ 300 & 350 & 400 & 51000 \end{bmatrix}$$

Add $-300 \times R_1$ to R_3.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 0 & -250 & -\frac{225}{2} & -18375 \\ 0 & -550 & 25 & -32250 \end{bmatrix}$$

Multiply R_2 by $-\frac{1}{250}$.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 0 & 1 & \frac{9}{20} & \frac{147}{2} \\ 0 & -550 & 25 & -32250 \end{bmatrix}$$

Add $550 \times R_2$ to R_3.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 0 & 1 & \frac{9}{20} & \frac{147}{2} \\ 0 & 0 & \frac{545}{2} & 8175 \end{bmatrix}$$

Multiply R_3 by $\frac{2}{545}$.

$$\begin{bmatrix} 1 & 3 & \frac{5}{4} & \frac{555}{2} \\ 0 & 1 & \frac{9}{20} & \frac{147}{2} \\ 0 & 0 & 1 & 30 \end{bmatrix}$$

Plugging in x_3 = 30 to second row gives us:

$$x_2 + \frac{9}{20} \cdot 30 = \frac{147}{2}$$
$$x_2 = 60$$

Plugging in x_3 = 30 and $x_2 = 60$ to first row gives us:

$$x_1 + 3 \cdot 60 + \frac{5}{4} \cdot 30 = \frac{555}{2}$$
$$x_1 = 60$$

Unit price of first product is \$ 60, unit price of the second product is \$60, unit price of the third product is \$ 30.

$$200 \cdot 60 + 600 \cdot 60 + 250 \cdot 30 = 55500$$
$$250 \cdot 60 + 500 \cdot 60 + 200 \cdot 30 = 51000$$
$$300 \cdot 60 + 350 \cdot 60 + 400 \cdot 30 = 51000$$

All equations are true for the values we have found.