**Lecture Note #9** **MTH653: Advanced Numerical Analysis**
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# $\boldsymbol{H}(\mathrm{div})$ finite element spaces

## 1 Finite Element Spaces

In order to construct finite element discretizations of problems like Darcy's equations and the curl-curl (Maxwell-type) problem, we must construct finite element spaces that are **subspaces** of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. Note that since $\boldsymbol{H}^1(\Omega)$ is a subspace of both $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, the standard vector-valued finite element spaces are also subspaces of $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$, but, for reasons that are perhaps not immediately obvious, these spaces are often not suitable for discretization of problems that are posed naturally in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$. For example, in Darcy's equations, taking the vector flux $\boldsymbol{u} \in \boldsymbol{V}_h \subseteq \boldsymbol{H}^1(\Omega)$ will lead to discretizations that are not inf-sup stable.

Finite element spaces are constructed by gluing together piecewise polynomial functions; these functions are piecewise-$C^\infty$, and so membership in $\boldsymbol{H}(\mathrm{div})$ or $\boldsymbol{H}(\mathbf{curl})$ is completely characterized by the continuity conditions listed above. For example, we would like to construct a finite-dimensional piecewise-polynomial subspace of $\boldsymbol{H}(\mathrm{div})$, which requires *normal continuity* but permits *tangential discontinuities*. This space can be thought of us somehow "in between" $H^1$ finite element spaces (which have full $C^0$ continuity) and $L^2$ (DG) finite element spaces (which have no continuity conditions).

### 1.1 Raviart–Thomas Finite Elements

Consider a triangle $\kappa$. Define the local lowest-order **Raviart–Thomas** space $\boldsymbol{RT}_0(\kappa)$ by

$$\boldsymbol{RT}_0(\kappa) := \left\{ \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^2, b \in \mathbb{R} \right\}.$$

In other words, every element of $\boldsymbol{RT}_0(\kappa)$ has the form

$$\boldsymbol{v}(\boldsymbol{x}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix}.$$

A simple computation shows that $\nabla \cdot \boldsymbol{v} = 2b$. From this, it is simple to see that $\nabla \cdot \boldsymbol{v} = 0$ implies that $\boldsymbol{v}$ is constant. Furthermore, the **normal component** of $\boldsymbol{v} \in \boldsymbol{RT}_0$ is **constant** along each face (edge) of $\kappa$. This can be seen as follows. The line containing the edge $e$ is defined by the equation $\boldsymbol{n} \cdot \boldsymbol{x} = c$, for some constant $c$, where $\boldsymbol{n}$ is the normal vector to $e$. Then, the normal component of $\boldsymbol{v}$ is $(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$, and

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= \boldsymbol{a} \cdot \boldsymbol{n} + b\boldsymbol{x} \cdot \boldsymbol{n} \\ &= \boldsymbol{a} \cdot \boldsymbol{n} + bc, \end{aligned}$$

which is a constant.

In order to create a finite element space on all of $\Omega$, we need to **"glue"** the local spaces $\boldsymbol{RT}_0(\kappa)$ together in such a way that the normal components match for every $e \in \Gamma$. The way that we do this is by assigning **degrees of freedom** to each face $e \in \Gamma$, such that the degrees of freedom on a single triangle uniquely determine $\boldsymbol{v}|_\kappa$. In the case of $H^1$ elements, the natural degrees of freedom were point values. In the case of $\boldsymbol{H}(\mathrm{div})$ elements, it is natural to work in terms of "moments", i.e. integrals of the normal components. Let $e_1, e_2, e_3$ denote the three edges of a triangle $\kappa$, and let $c_i$ denote the integrated normal component,

$$c_i := \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} (\boldsymbol{a} \cdot \boldsymbol{n} + bc) \, ds.$$

We claim that these degrees of freedom are **unisolvent**, i.e. the values $c_i$ uniquely determine every element of $\boldsymbol{RT}_0(\kappa)$.

First, we show uniqueness. Suppose that $c_i = 0$ for all $i$. Then,

$$\begin{aligned}
0 &= \sum_i c_i \\
&= \sum_i \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\
&= \int_{\partial\kappa} \boldsymbol{v} \cdot \boldsymbol{n} \, ds \\
&= \int_\kappa \nabla \cdot \boldsymbol{v} \, dx \\
&= \int_\kappa 2b \, dx \\
&= 2|\kappa| b.
\end{aligned}$$

So, $b = 0$, and $\boldsymbol{v} = \boldsymbol{a}$, a constant. Then,

$$c_i = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{e_i} \boldsymbol{a} \cdot \boldsymbol{n} = |e_i| \boldsymbol{a} \cdot \boldsymbol{n},$$

so $\boldsymbol{a}$ is normal to three linearly independent vectors in $\mathbb{R}^2$, and so $\boldsymbol{a} = 0$.

We have defined a map (the "degree of freedom" mapping)

$$C : \boldsymbol{RT}_0(\kappa) \to \mathbb{R}^3$$

defined by

$$C\boldsymbol{v} = (c_1, c_2, c_3).$$

This is a linear map between 3-dimensional spaces. The above argument shows that $C$ is injective, therefore it is a bijection, and every element of $\boldsymbol{RT}_0(\kappa)$ is uniquely determined by its degrees of freedom.

Following the standard construction, each local degree of freedom $c_i$ corresponds with a local basis function $\boldsymbol{\phi}_i$, satisfying $c_i(\boldsymbol{\phi}_j) = \delta_{ij}$ (denoting here $c_i$ as a linear functional). The standard procedure in finite elements is to transform the "physical element" $\kappa$ to the **reference element** $\hat{\kappa}$ through an affine transformation $T : \hat{\kappa} \mapsto \kappa$. For $H^1$ finite elements, we simply defined

$$\psi(\boldsymbol{x}) = \hat{\psi}(T^{-1}(\boldsymbol{x})).$$

This **does not work** for the Raviart–Thomas basis functions, because in general

$$c_i(\hat{\boldsymbol{\phi}}(T^{-1}(\boldsymbol{x}))) \neq \delta_{ij}.$$

The reason for this is that the normal vectors on $\hat{\kappa}$ are not mapped to normal vectors on $\kappa$ through the transformation $T$. To appropriately map functions in $\boldsymbol{H}(\text{div})$, we need a transformation that preserves normal vectors. Given $\boldsymbol{v} : \kappa \to \mathbb{R}^d$, we want to define $\hat{\boldsymbol{v}} : \hat{\kappa} \to \mathbb{R}^d$ such that

$$\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{n}\, ds = \int_{\partial\hat{\kappa}} \hat{w}\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}\, ds$$

for all scalar test functions $w$ (where $\hat{w} := w \circ T$). This can be achieved by ensuring that

$$\int_{\kappa} (\nabla \cdot \boldsymbol{v}) w\, dx = \int_{\hat{\kappa}} (\nabla \cdot \hat{\boldsymbol{v}}) \hat{w}\, dx \qquad \text{and} \qquad \int_{\kappa} \boldsymbol{v} \cdot \nabla w\, dx = \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx$$

since, if this holds, integrating by parts,

$$\begin{aligned}
\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{n}\, ds &= \int_{\kappa} (\nabla \cdot \boldsymbol{v}) w\, dx + \int_{\kappa} \boldsymbol{v} \cdot \nabla w\, dx \\
&= \int_{\hat{\kappa}} (\nabla \cdot \hat{\boldsymbol{v}}) \hat{w}\, dx + \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx \\
&= \int_{\partial\hat{\kappa}} \hat{w}\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}\, ds.
\end{aligned}$$

Therefore, we find $\hat{\boldsymbol{v}}$ such that

$$\int_{\kappa} \boldsymbol{v} \cdot \nabla w\, dx = \int_{\hat{\kappa}} \hat{\boldsymbol{v}} \cdot \nabla \hat{w}\, dx.$$

Let $J = D_T$ denote the Jacobian matrix of the element transformation $T$. Then, since $\hat{w} = w \circ T$, we have $\nabla \hat{w}(\hat{\boldsymbol{x}}) = (J^T \nabla w)(T(\hat{\boldsymbol{x}}))$, from which it follows that $\nabla w(T(\hat{\boldsymbol{x}})) = J^{-T} \nabla \hat{w}(\hat{\boldsymbol{x}})$. Therefore,

$$\begin{aligned}
\int_{\kappa} \boldsymbol{v} \cdot \nabla w\, dx &= \int_{\hat{\kappa}} \det(J) \boldsymbol{v}(T(\hat{\boldsymbol{x}})) \cdot \nabla w(T(\hat{\boldsymbol{x}}))\, d\hat{x} \\
&= \int_{\hat{\kappa}} \det(J) \boldsymbol{v}(T(\hat{\boldsymbol{x}})) \cdot J^{-T} \nabla \hat{w}(\hat{\boldsymbol{x}})\, d\hat{x} \\
&= \int_{\hat{\kappa}} \underbrace{\det(J) J^{-1} \boldsymbol{v}(T(\hat{\boldsymbol{x}}))}_{\hat{\boldsymbol{v}}} \cdot \nabla \hat{w}(\hat{\boldsymbol{x}})\, d\hat{x},
\end{aligned}$$

so $\hat{\boldsymbol{v}}$ is defined by

$$\hat{\boldsymbol{v}} := \det(J) J^{-1} \boldsymbol{v} \circ T. \tag{1}$$

Given this definition of $\hat{\boldsymbol{v}}$, we can show (omitting some tedious chain rule calculations) that

$$\nabla \cdot \hat{\boldsymbol{v}} = \det(J)(\nabla \cdot \boldsymbol{v}) \circ T.$$

From this, it follows that

$$\begin{aligned}\int_\kappa (\nabla \cdot \boldsymbol{v}) w \, dx &= \int_{\hat\kappa} \det(J)(\nabla \cdot \boldsymbol{v} \circ T) w \circ T \, dx \\ &= \int_{\hat\kappa} (\nabla \cdot \hat{\boldsymbol{v}}) \hat{w} \, dx,\end{aligned}$$

as desired, and so we obtain the desired relation

$$\int_{\partial\kappa} w \boldsymbol{v} \cdot \boldsymbol{n} \, ds = \int_{\partial\hat\kappa} \hat{w} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} \, ds.$$

The transformation from $\boldsymbol{v}$ to $\hat{\boldsymbol{v}}$ defined by (1) can easily be inverted to yield

$$\boxed{\boldsymbol{v} := \frac{1}{\det(J)} J \hat{\boldsymbol{v}} \circ T^{-1}} \tag{2}$$

This is called the (normal-preserving) **Piola transformation**. Using this transformation, the basis functions $\boldsymbol{\phi}$ on each triangle $\kappa \in \mathcal{T}$ can be defined in terms of the basis functions on the reference triangle $\hat\kappa$.

The lowest-order Raviart–Thomas finite element space $\boldsymbol{V}_h$ is defined by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) : \boldsymbol{v}|_\kappa \in \boldsymbol{RT}_0(\kappa)\}.$$

This space is constructed by assigning to each edge $e \in \Gamma$ a degree of freedom $c$. The global basis functions for this space are associated with each edge of the mesh, and are supported in the two triangles containing that edge. Note that although this space contains some piecewise linear functions, the local space is strictly smaller than the space of all linear functions.

### 1.1.1 The Raviart–Thomas Interpolant

We wish to define an **interpolation operator** $\mathcal{I}_h : [\boldsymbol{C}^1(\Omega)] \to \boldsymbol{V}_h$. In other words, given $\boldsymbol{v} \in [C^1(\Omega)]^d$, we want to define an "interpolant" $\boldsymbol{v}_h = \mathcal{I}_h(\boldsymbol{v}) \in \boldsymbol{V}_h$. The standard finite element interpolant uses the **degrees of freedom** of the space (considered as functionals) to define the interpolant. In other words, for any such $\boldsymbol{v}$, there is a unique element $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that

$$\int_e \boldsymbol{v}_h \cdot \boldsymbol{n} \, ds = \int_e \boldsymbol{v} \cdot \boldsymbol{n} \, ds$$

for all edges (faces) $e$ in the mesh. By the properties of the induced basis functions, $\boldsymbol{v}_h$ can be written explicitly as

$$\boldsymbol{v}_h = \sum_{e \in \Gamma} \left( \int_e \boldsymbol{v} \cdot \boldsymbol{n} \, ds \right) \boldsymbol{\phi}_e,$$

where $\boldsymbol{\phi}_e$ is the basis function associated with edge $e$. From this expression (and using the trace theorem), it is possible to show that

$$\|\mathcal{I}_h \boldsymbol{v}\|_{L^2(\Omega)} \lesssim \|\boldsymbol{v}\|_{H^1(\Omega)}$$

Let $\boldsymbol{v} \in [\boldsymbol{C}^1(\Omega)]^d$, and let $\boldsymbol{v}_h$ be its RT interpolant. Consider the space of piecewise constants,

$$P_h = \{p \in L^2(\Omega) : p|_\kappa \text{ is constant}\}.$$

Then, for any $w_h \in P_h$,

$$\begin{aligned}
\int_\Omega (\nabla \cdot \boldsymbol{v}_h)\, w_h \, dx &= \sum_\kappa \int_\kappa (\nabla \cdot \boldsymbol{v}_h)\, w_h \, dx \\
&= \sum_\kappa \left( - \int_\kappa \boldsymbol{v}_h \cdot \nabla w_h \, dx + \int_{\partial\kappa} \boldsymbol{v}_h \cdot \boldsymbol{n} w_h \, ds \right) && \text{(integration by parts)} \\
&= \sum_\kappa \int_{\partial\kappa} \boldsymbol{v}_h \cdot \boldsymbol{n} w_h \, ds && (w_h \text{ piecewise constant}) \\
&= \sum_\kappa \int_{\partial\kappa} \boldsymbol{v} \cdot \boldsymbol{n} w_h \, ds && \text{(interpolant)} \\
&= \int_\Omega (\nabla \cdot \boldsymbol{v})\, w_h \, dx
\end{aligned}$$

From this, we conclude that $\nabla \cdot \boldsymbol{v}_h$ is the $L^2$ **projection** of $\nabla \cdot \boldsymbol{v}$ onto the space of piecewise constants. The definition of the interpolant and subsequent arguments can be extended from $[\boldsymbol{C}^1(\Omega)]^d$ to all of $[H^1(\Omega)]^d$, resulting in the following commutative diagram

$$\begin{tikzcd}
{[H^1(\Omega)]^d} \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{I}_h"'] & L^2(\Omega) \arrow[d, "\Pi_0"] \\
\boldsymbol{V}_h \arrow[r, "\nabla\cdot"] & P_h
\end{tikzcd}$$

### 1.1.2 Application to Darcy's equations

Recall Darcy's equations

$$\begin{aligned}
\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\
\nabla \cdot \boldsymbol{u} &= g,
\end{aligned} \tag{3}$$

leading to the variational problem

$$\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}), \\
(\nabla \cdot \boldsymbol{u}, q) &= (g, q).
\end{aligned}$$

Here $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) =: \boldsymbol{V}$ and $p, q \in L^2(\Omega) =: P$. This has the general form of a saddle-point system,

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= F(\boldsymbol{v}), \\
b(\boldsymbol{u}, q) &= G(q),
\end{aligned}$$

with bilinear forms and linear functionals,

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v}) &:= (\boldsymbol{u},\boldsymbol{v}),\\
b(\boldsymbol{u},q) &:= -(\nabla\cdot\boldsymbol{u},q),\\
F(\boldsymbol{v}) &:= (\boldsymbol{f},\boldsymbol{v}),\\
G(q) &= (-g,q).
\end{aligned}$$

We proved (in MTH 652) that this variational problem is **well-posed** if

(C1) The bilinear forms and linear functionals are continuous (bounded)

(C2) $a(\cdot\,,\cdot)$ is coercive on $Z$, the nullspace of $b(\cdot\,,\cdot)$

(C3) The **inf–sup** condition

$$\inf_{q\in P}\sup_{\boldsymbol{v}\in\boldsymbol{V}}\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_P}\geq\beta \tag{4}$$

is satisfied for $\beta>0$.

Condition (C1) follows easily from the definitions of the forms. The set $Z$ is defined by

$$\begin{aligned}
Z &:= \{\boldsymbol{v}\in\boldsymbol{V} : b(\boldsymbol{v},q)=0 \text{ for all } q\in P\}\\
&= \{\boldsymbol{v}\in\boldsymbol{V} : \nabla\cdot\boldsymbol{v}=0\}.
\end{aligned}$$

Then, for $\boldsymbol{v}\in Z$,

$$\begin{aligned}
a(\boldsymbol{v},\boldsymbol{v}) &= (\boldsymbol{v},\boldsymbol{v})\\
&= \|\boldsymbol{v}\|^2_{L^2(\Omega)}\\
&= \|\boldsymbol{v}\|^2_{L^2(\Omega)}+\|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)^2}\\
&= \|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div},\Omega)},
\end{aligned}$$

since $\|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)^2}=0$. Therefore, $a(\cdot\,,\cdot)$ is coercive on $Z$ (with coercivity constant 1), proving condition (C2).

Condition (C3) follows from the fact that, for any $q\in L^2(\Omega)$, there exists $w\in H^2(\Omega)$ with $-\Delta w=q$ and $\|w\|_{H^2(\Omega)}\lesssim\|q\|_{L^2(Omega)}$. Defining $\boldsymbol{v}=\nabla w$, we have $\nabla\cdot\boldsymbol{v}=-q$ and $\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\lesssim\|q\|_{L^2(\Omega)}$. Then, given arbitrary $q\in L^2(\Omega)$, choosing $\boldsymbol{v}$ this way gives

$$\begin{aligned}
\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_P} = \frac{(-\nabla\cdot\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_P} &= \frac{(q,q)}{\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_P}\\
&= \frac{\|q\|_P^2}{\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_P}\geq\frac{\|q\|_P^2}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}\|q\|_P}\\
&\gtrsim\frac{\|q\|_P^2}{\|q\|_P\|q\|_P}=1.
\end{aligned}$$

Therefore, condition (C3) holds, and this problem is well-posed.

To obtain stable and convergent finite element approximations, we need to choose finite-dimensional subspaces $\boldsymbol{V}_h \subseteq \boldsymbol{v}$ and $P_h \subseteq P$ that satisfy the inf–sup condition at the discrete level. We have seen previously that the choice

$$\begin{aligned} \boldsymbol{V}_h &:= \{\text{continuous piecewise linears}\} \subseteq \boldsymbol{H}^1(\Omega), \\ P_h &:= \{\text{piecewise constants}\} \subseteq L^2(\Omega), \end{aligned}$$

**does not** satisfy the inf–sup condition. The argument is repeated here briefly. Note that $\dim(\boldsymbol{V}_h) = 2\times\#\text{non-essential vertices in the mesh} =: 2n_V$ and $\dim(P_h) = \#\text{triangles in the mesh} =: n_T$. The matrix representation $B$ has size $n_T \times 2n_V$. However, an angle counting argument shows that $2n_V < n_T$, and so $B$ has **more rows than columns**. Therefore, the rows are linearly dependent, and there exists some $q_h \in P_h$, $q_h \neq 0$ such that $\boldsymbol{q}_h^T B = 0$ (here $\boldsymbol{q}_h$ is the vector representation of $q_h$), i.e. $b(\boldsymbol{v}, q) = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}_h$, contradicting condition $(C3)$

The problem with the above choice of spaces is that the flux space $\boldsymbol{V}_h$ was **too small**. The infinite-dimensional variational problem was posed on $\boldsymbol{H}(\text{div}, \Omega)$, but we chose a finite element space that is a subspace of the strictly smaller space $\boldsymbol{H}^1(\Omega)$. Instead, we should choose a flux space that is in $\boldsymbol{H}(\text{div}, \Omega)$ but not in $\boldsymbol{H}^1(\Omega)$.

Now, consider the choice of spaces

$$\begin{aligned} \boldsymbol{V}_h &= \{\boldsymbol{v} \in \boldsymbol{H}(\text{div}, \Omega) : v|_\kappa \in \boldsymbol{RT}_0(\kappa)\}, \\ P_h &= \{p \in L^2(\Omega) : p|_\kappa \text{ is constant}\}. \end{aligned}$$

(We can identify $P_h$ as the lowest-order DG space). Given this choice of spaces, we seek to verify the **discrete inf–sup condition**,

$$\inf_{0\neq q_h \in P_h} \sup_{0\neq \boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} \geq \beta > 0.$$

Given $q_h \in P_h \subseteq L^2(\Omega)$, the proof of the inf–sup condition at the continuous level gives us $\boldsymbol{v} \in \boldsymbol{H}(\text{div}, \Omega)$ satisfying

$$\frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} \geq \beta.$$

By the **commuting diagram property**, we can define $\boldsymbol{v}_h := \mathcal{I}_h(\boldsymbol{v}) \in \boldsymbol{V}_h$ satisfying

$$\nabla \cdot \boldsymbol{v}_h = \Pi_0 \nabla \cdot \boldsymbol{v} = \Pi_0(-q_h).$$

This then gives

$$\begin{aligned} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} &= \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} \\ &= \frac{-(\nabla \cdot \boldsymbol{v}, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} \\ &= \frac{(q_h, q_h)}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}} \\ &= \frac{\|q_h\|_{P_h}^2}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h} \|q_h\|_{P_h}}. \end{aligned}$$

If we can bound $\|\boldsymbol{v}_h\|_{\boldsymbol{V}_h}$ in terms of $\|q\|_{P_h}$ (perhaps via bounds on $\boldsymbol{v}$), then the discrete inf–sup condition will follow.

**Lemma 1.** *Given $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, it holds that*

$$\|\mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + h\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}.$$

*and*

$$\|\mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.$$

*Proof.* Let $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ be given. We work on the reference element $\hat{\kappa}$. Considering an element $\kappa$ with $\kappa = T(\hat{\kappa})$, define $\hat{\boldsymbol{v}}$ on $\hat{\kappa}$ through the (inverse) Piola transformation

$$\hat{\boldsymbol{v}} := \det(J) J^{-1} \boldsymbol{v} \circ T$$

By the trace theorem,

$$\int_{\hat{e}} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} q \, ds \lesssim \|\hat{\boldsymbol{v}}\|_{\boldsymbol{H}^1(\Omega)}$$

where $\hat{e}$ is an edge of $\hat{\kappa}$, and $q \in L^2(\hat{e})$. Then, define

$$\hat{\boldsymbol{v}}_h := \sum_{i=1}^{3} v_i \hat{\boldsymbol{\phi}}_i,$$

where

$$v_i := \int_{\hat{e}_i} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} \, ds,$$

and $\hat{\boldsymbol{\phi}}_i$ are the associated dual basis functions. From this, it clearly follows that

$$\|\hat{\boldsymbol{v}}_h\|_{\boldsymbol{H}(\mathrm{div},\hat{\kappa})} \lesssim \|\hat{\boldsymbol{v}}\|_{\boldsymbol{H}^1(\hat{\kappa})}.$$

Consider now $\boldsymbol{v}_h$ defined on $\kappa = T(\hat{\kappa})$ using the normal-preserving Piola transformation,

$$\boldsymbol{v}_h := \frac{1}{\det(J)} J \hat{\boldsymbol{v}}_h \circ T^{-1}.$$

We can bound

$$\begin{aligned}
\|\boldsymbol{v}_h\|^2_{L^2(\kappa)} = \int_\kappa \|\boldsymbol{v}_h\|^2 \, dx &= \int_\kappa \frac{1}{\det(J)^2} \|J \hat{\boldsymbol{v}}_h \circ T^{-1}\|^2 \, dx \\
&= \int_{\hat{\kappa}} \frac{\det(J)}{\det(J)^2} \|J \hat{\boldsymbol{v}}_h\|^2 \, dx \\
&\leq \frac{\|J\|^2}{\det(J)} \int_{\hat{\kappa}} \|\hat{\boldsymbol{v}}_h\|^2 \, dx \\
&\lesssim \frac{\|J\|^2}{\det(J)} \|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{H}^1(\hat{\kappa})} \\
&= \frac{\|J\|^2}{\det(J)} \left( \|\hat{\boldsymbol{v}}\|^2_{\boldsymbol{L}^2(\hat{\kappa})} + \|\nabla \hat{\boldsymbol{v}}\|^2_{\boldsymbol{L}^2(\hat{\kappa})} \right) \\
&\leq \frac{\|J\|^2}{\det(J)} \left( \det(J) \|J^{-1}\|^2 \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)} + \det(J) \|J\|^2 \|J^{-1}\|^2 \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)} \right) \\
&= \|J\|^2 \|J^{-1}\|^2 \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)} + \|J\|^4 \|J^{-1}\|^2 \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)}.
\end{aligned}$$

Assuming a shape regular mesh (the minimum angle in the mesh is bounded below), then $\|J\| \sim h$ and $\|J^{-1}\| \sim h^{-1}$. In this case,

$$\|\boldsymbol{v}_h\|^2_{L^2(\kappa)} \lesssim \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)} + h^2\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\kappa)} \lesssim \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\kappa)}.$$

Furthermore, by the commuting diagram property,

$$\|\nabla\cdot \boldsymbol{v}_h\|^2_{L^2(\kappa)} = \|\Pi_0(\nabla\cdot\boldsymbol{v})\|^2_{L^2(\kappa)} \leq \|\nabla\cdot\boldsymbol{v}\|^2_{L^2(\kappa)} \leq \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1(\kappa)}.$$

From this, we conclude that

$$\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.$$

□

Concluding the proof of the inf–sup condition, using the above result, we have

$$\|\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \lesssim \|q\|_{L^2(\Omega)},$$

from which the result follows. Therefore, the spaces $\boldsymbol{RT}_0$–$\mathcal{P}_0$ are an **inf–sup stable pair** for Darcy's equations. Applying the general theory, we obtain convergence results

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}(\mathrm{div},\Omega)} &\leq C_1 \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + C_2 \inf_{q_h\in P_h}\|p-q_h\|_{L^2(\Omega)},\\ \|\boldsymbol{p}-\boldsymbol{p}_h\|_{L^2(\Omega)} &\leq C_3 \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \|\boldsymbol{u}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\mathrm{div},\Omega)} + C_4 \inf_{q_h\in P_h}\|p-q_h\|_{L^2(\Omega)}.\end{aligned}$$

We can do a bit better for the flux error. Galerkin orthogonality gives

$$a(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}, p-p_h) = 0, \tag{5}$$

$$b(\boldsymbol{u}-\boldsymbol{u}_h, q_h) = 0, \tag{6}$$

for all test functions $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$. Recall further that

$$b(\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}_h, q_h) = 0 \tag{7}$$

for all $q_h \in P_h$. Combining (6) and (7), we have

$$b(\boldsymbol{u}_h-\mathcal{I}_h\boldsymbol{u}, q_h) = 0$$

for all $q_h$, which implies that

$$\nabla\cdot\boldsymbol{u}_h = \nabla\cdot\mathcal{I}_h\boldsymbol{u}.$$

Choosing the test function $\boldsymbol{v}_h = \mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h$ in (5), we obtain

$$\begin{aligned}a(\boldsymbol{u}-\boldsymbol{u}_h,\mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h) + b(\mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h, p-p_h) &= (\boldsymbol{u}-\boldsymbol{u}_h,\mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h)\\ &= 0.\end{aligned}$$

Then,

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\boldsymbol{u}_h) &= (\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}+\mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h)\\ &= (\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}) + (\boldsymbol{u}-\boldsymbol{u}_h,\mathcal{I}_h\boldsymbol{u}-\boldsymbol{u}_h)\\ &\leq \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{u}-\mathcal{I}_h\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\end{aligned}$$

and so

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{u} - \mathcal{I}_h \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

Therefore, the $L^2$ error in $\boldsymbol{u}_h$ is bounded in terms of the **projection error** in $\boldsymbol{u}$ alone (the bound does not depend on $p$). The projection error can be estimated without too much difficulty using Lemma 1. On a given triangle $\kappa$, for any $\boldsymbol{v} \in \boldsymbol{H}^1(\kappa)$,

$$\|\boldsymbol{v} - \mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\kappa)} = \|\boldsymbol{v} - \boldsymbol{w} + \boldsymbol{w} - \mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\kappa)},$$

for any $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}_0(\kappa)$. Note further that, since $\boldsymbol{w} \in \boldsymbol{RT}_0(\kappa)$ and $\mathcal{I}_h$ is a projection, $\boldsymbol{w} = \mathcal{I}_h \boldsymbol{w}$, and so

$$\begin{aligned}
\|\boldsymbol{v} - \mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\kappa)} &= \|\boldsymbol{v} - \boldsymbol{w} - \mathcal{I}_h(\boldsymbol{v} - \boldsymbol{w})\|_{\boldsymbol{L}^2(\kappa)} \\
&= \|\boldsymbol{v} - \boldsymbol{w} - \mathcal{I}_h(\boldsymbol{v} - \boldsymbol{w})\|_{\boldsymbol{L}^2(\kappa)} \\
&\leq \|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^2(\kappa)} + \|\mathcal{I}_h(\boldsymbol{v} - \boldsymbol{w})\|_{\boldsymbol{L}^2(\kappa)} \\
&\lesssim \|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^2(\kappa)} + h\|\nabla(\boldsymbol{v} - \boldsymbol{w})\|_{\boldsymbol{L}^2(\kappa)},
\end{aligned}$$

and so the approximation properties of the projection $\mathcal{I}_h$ are reduced to the approximation properties of the space of piecewise constant functions (more generally, for $\boldsymbol{RT}_k$, this reduces to the approximation properties of $\boldsymbol{\mathcal{P}}_k$). A standard argument shows that

$$\inf_{\boldsymbol{w} \in \boldsymbol{\mathcal{P}_0}} \|\boldsymbol{v} - \boldsymbol{w}\|_{\boldsymbol{L}^2(\kappa)} \lesssim h\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}$$

and so

$$\|\boldsymbol{v} - \mathcal{I}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\kappa)} \lesssim h\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}.$$