$\boldsymbol{H}$(curl) finite element spaces

We saw previously the construction of the lowest-order **Raviart–Thomas $\boldsymbol{H}$(div)** finite element space. The construction in 2D (triangular meshes) was described explicitly, and the spaces for 3D tetrahedral meshes follows analogously.[1]

We will now consider the construction of $\boldsymbol{H}(\mathbf{curl})$ finite element spaces. We begin first with the 2D space, and then generalize to 3D.

In 2D, the curl of a vector field is scalar-valued, and is defined by

$$\nabla \times \boldsymbol{u} := \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}.$$

Note that the **2D curl** is equal to the **divergence** of the vector field rotated by 90 degrees, i.e.

$$\nabla \times \boldsymbol{u} = \nabla \cdot (\boldsymbol{u}^\perp) = \nabla \cdot \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}.$$

This suggests that (in 2D only!) $\boldsymbol{H}$(curl) finite elements can be obtained by rotating $\boldsymbol{H}$(div) elements.

1 Nédélec finite elements in 2D

Define the **lowest-order 2D local Nédélec space** by

$$\boldsymbol{N}_0 = \{\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x}^\perp : \boldsymbol{a} \in \mathbb{R}^2, b \in \mathbb{R}\},$$

i.e. for $\boldsymbol{u} \in \boldsymbol{N}_0$,

$$\boldsymbol{u} = \begin{pmatrix} a_1 - by \\ a_2 + bx \end{pmatrix}.$$

It is evident that $\boldsymbol{N}_0$ is the 90-degree rotation of $\boldsymbol{RT}_0$. We list some important properties of this space; these are the exact analogues of the properties of $\boldsymbol{RT}_0$, simply "rotated". Note that for $\boldsymbol{u} \in \boldsymbol{N}_0$,

$$\nabla \boldsymbol{u} := \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix},$$

which is skew-symmetric. Analogous to the case of RT elements, the curl of $\boldsymbol{u} \in \boldsymbol{N}_0$ is given by

$$\nabla \times \boldsymbol{u} = 2b.$$

[1] These spaces can also be defined for quadrilateral and hexahedral elements; the degrees of freedom are the same (integrals of normal components on edges/faces), but since there are 4 edges per quadrilateral (as opposed to 3 per triangle) and 6 faces per hexahedron (as opposed to 4 per tetrahedron), the local space has to be larger.

Therefore, if $\boldsymbol{u}$ is curl-free ("irrotational"), then $\boldsymbol{u}$ must be constant. Consider any affine line e defined by the equation $\boldsymbol{n} \cdot \boldsymbol{x} = c$ for some c, where $\boldsymbol{n}$ is the vector normal to e. Then, the tangent vector is given by $\boldsymbol{t} = \boldsymbol{n}^\perp$. Computing

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{t} &= \boldsymbol{a} \cdot \boldsymbol{t} + b(\boldsymbol{x}^\perp) \cdot (\boldsymbol{t}) \\
&= \boldsymbol{a} \cdot \boldsymbol{t} + b(x_1 t_2 - x_2 t_1) \\
&= \boldsymbol{a} \cdot \boldsymbol{t} + b(x_1 n_1 + x_2 n_2) \\
&= \boldsymbol{a} \cdot \boldsymbol{t} + bc,
\end{aligned}$$

which is a constant.

The natural degrees of freedom for $\boldsymbol{N}_0$ are given by **integrals of the tangential components** of $\boldsymbol{u}$,

$$c_e(\boldsymbol{u}) := \int_e (\boldsymbol{u} \cdot \boldsymbol{t}_e)\, ds,$$

for each edge e of κ, where $\boldsymbol{t}_e$ is the unit tangent vector. (The sign of the degrees of freedom in this definition depends on choosing an orientation for each edge). These degree of freedom are **unisolvent**; the argument is the same as in the case of $\boldsymbol{RT}_0$.

As usual, the degrees of freedom induce a basis. Consider the reference triangle $\hat{\kappa}$. Then, there exist basis functions $\hat{\boldsymbol{\phi}}_i$ satisfying $\hat{c}_i(\hat{\boldsymbol{\phi}}_j) = \delta_{ij}$, and $\text{span}\{\hat{\boldsymbol{\phi}}_i\} = \boldsymbol{N}_0(\hat{\kappa})$, where $\hat{c}_i$ are the degrees of freedom associated with the edges $\hat{e}$ of $\hat{\kappa}$. Given a **"physical triangle"** $\kappa = T(\hat{\kappa})$ (with affine map T), we wish to define basis functions $\boldsymbol{\phi}_i$ for $\boldsymbol{N}_0(\kappa)$ in terms of $\hat{\boldsymbol{\phi}}_i$ such that $c_i(\boldsymbol{\phi}_j) = \delta_{ij}$, where c_i is used to denote the degrees of freedom on the physical triangle. This means that we need to use the **tangential-preserving Piola transformation**.

Given $\boldsymbol{v} : \kappa \to \mathbb{R}^2$, we want to define $\hat{\boldsymbol{v}} : \hat{\kappa} \to \mathbb{R}^2$ such that

$$\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{t}\, ds = \int_{\partial\hat{\kappa}} \hat{w}\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{t}}\, ds.$$

Following an argument of the same structure as in the normal-preserving case, we see that $\hat{\boldsymbol{v}}$ must be defined by

$$\hat{\boldsymbol{v}} := J^T \boldsymbol{v} \circ T$$

and so the physical basis functions are defined in terms of the reference basis functions using the Piola transformation

$$\boxed{\boldsymbol{\phi} := J^{-T} \hat{\boldsymbol{\phi}} \circ T^{-1}}$$

Then lowest-order $\boldsymbol{H}(\text{curl})$-conforming Nédélec finite element space can then be defined by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{H}(\text{curl}, \Omega) : \boldsymbol{v}|_\kappa \in \boldsymbol{N}_0(\kappa)\}.$$

This space can be constructed by assigning to each edge e in the mesh (oriented appropriately) a global degree of freedom c_e.

1.1 The Nédélec interpolant

We now define the standard finite element interpolant, which is a projection onto the space $\boldsymbol{V}_h$. Note that the degrees of freedom c_e are well-defined for any $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega) = [C^1(\Omega)]^2$ (and can be extended to larger spaces as long as the tangential traces on the mesh skeleton are well-defined). Define $\mathcal{I}_h : \boldsymbol{C}^1(\Omega) \to \boldsymbol{V}_h$ by setting $\mathcal{I}_h \boldsymbol{v}$ to be the unique element $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that

$$\int_e \boldsymbol{v}_h \cdot \boldsymbol{t}\, ds = \int_e \boldsymbol{v} \cdot \boldsymbol{t}\, ds$$

for all edges e in the mesh. In other words,

$$\mathcal{I}_h \boldsymbol{v} := \sum_{e \in \Gamma} \left(\int_e \boldsymbol{v} \cdot \boldsymbol{t}\, ds \right) \boldsymbol{p}hi_e.$$

We recall the "curl" integration by parts formula.

$$\begin{aligned}
\int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{t}\, ds &= \int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{n}^{\perp}\, ds \\
&= -\int_{\partial\kappa} w\boldsymbol{v}^{\perp} \cdot \boldsymbol{n}\, ds \\
&= -\int_{\kappa} \nabla \cdot (w\boldsymbol{v}^{\perp})\, dx \\
&= -\int_{\kappa} \left(\nabla w \cdot \boldsymbol{v}^{\perp} + w \nabla \times \boldsymbol{v} \right) dx.
\end{aligned}$$

Rearranging,

$$\boxed{\int_{\kappa} w \nabla \times \boldsymbol{v}\, dx. = -\int_{\kappa} \nabla w \cdot \boldsymbol{v}^{\perp}\, dx - \int_{\partial\kappa} w\boldsymbol{v} \cdot \boldsymbol{t}\, ds.}$$

As before, let P_h denote the space of piecewise constants. Let $\boldsymbol{v}$ be arbitrary, and set $\boldsymbol{v}_h := \mathcal{I}_h \boldsymbol{v}$. For any $w_h \in P_h$,

$$\begin{aligned}
\int_{\Omega} w_h \nabla \times \boldsymbol{v}_h\, dx &= \sum_{\kappa} \int_{\kappa} w_h \nabla \times \boldsymbol{v}_h\, dx \\
&= \sum_{\kappa} \left(-\int_{\kappa} \nabla w_h \cdot \boldsymbol{v}_h^{\perp}\, dx - \int_{\partial\kappa} w_h \boldsymbol{v}_h \cdot \boldsymbol{t}\, ds \right) \\
&= \sum_{\kappa} - \int_{\partial\kappa} w_h \boldsymbol{v}_h \cdot \boldsymbol{t}\, ds && (w_h \text{ constant on } \kappa) \\
&= \sum_{\kappa} - \int_{\partial\kappa} w_h \boldsymbol{v} \cdot \boldsymbol{t}\, ds && \text{(interpolant)} \\
&= \sum_{\kappa} \int_{\kappa} w_h \nabla \times \boldsymbol{v}\, dx \\
&= \int_{\Omega} w_h \nabla \times \boldsymbol{v}_h\, dx.
\end{aligned}$$

Therefore, $\nabla \times \boldsymbol{v}_h = \Pi_0(\nabla \times \boldsymbol{v})$. This gives the **commuting diagram property**

$$\begin{array}{ccc} \boldsymbol{C}^1(\Omega) & \xrightarrow{\nabla\times} & C^0(\Omega) \\ \mathcal{I}_h \big\downarrow & & \big\downarrow \Pi_0 \\ \boldsymbol{V}_h & \xrightarrow{\nabla\times} & P_h \end{array}$$

1.2 Connection with H^1 elements

Let V_{H^1} denote the lowest-order H^1-conforming finite element space (continuous piecewise linears), and let $\mathcal{I}_{H^1}$ denote the associated interpolation operator (interpolation at mesh vertices). Let $\boldsymbol{V_N}$ denote the lowest-order Nédélec space and let $\mathcal{I}_{\boldsymbol{N}}$ denote the associated interpolation operator. We use the notation $V_{L^2} := P_h$. For any $v \in V_{H^1}$, we have that ∇v is constant, and so $\nabla(v|_\kappa) \in \boldsymbol{N}_0(\kappa)$. **Continuity** of v implies **tangential continuity** of ∇v, and so $\nabla v \in \boldsymbol{V_N}$. We claim the following diagram commutes.

$$\begin{array}{ccccc} C^2(\Omega) & \xrightarrow{\nabla} & \boldsymbol{C}^1(\Omega) & \xrightarrow{\nabla\times} & C^0(\Omega) \\ \big\downarrow \mathcal{I}_{H^1} & & \big\downarrow \mathcal{I}_{\boldsymbol{N}} & & \big\downarrow \Pi_0 \\ V_{H^1} & \xrightarrow{\nabla} & \boldsymbol{V_N} & \xrightarrow{\nabla\times} & V_{L^2} \end{array}$$

The right part of the diagram we have shown already. We prove the left part of the diagram. Let $v \in C^2(\Omega)$ be fixed. Let $v_h = \mathcal{I}_{H^1} \in V_{H^1}$. We wish to show that

$$\nabla v_h = \mathcal{I}_{\boldsymbol{N}} \nabla v.$$

For any edge $e \in \Gamma$, we have that (by the fundamental theorem of calculus)

$$\int_e \nabla v_h \cdot \boldsymbol{t}\, ds = [v_h]_{\partial e} = [v]_{\partial e} = \int_e \nabla v \cdot \boldsymbol{t}\, ds.$$

Therefore, the **Nédélec degrees of freedom** of ∇v_h and ∇v **are the same**, and so are the degrees of freedom of $\mathcal{I}_{\boldsymbol{N}} \nabla v$. Since the degrees of freedom are unisolvent and both $\nabla v_h, \mathcal{I}_{\boldsymbol{N}} \nabla v \in \boldsymbol{V_N}$, we have that they must be equal.

Essentially the same argument also gives the following diagram involving the Raviart–Thomas spaces.

$$\begin{array}{ccccc} C^2(\Omega) & \xrightarrow{\nabla^\perp} & \boldsymbol{C}^1(\Omega) & \xrightarrow{\nabla\cdot} & C^0(\Omega) \\ \big\downarrow \mathcal{I}_{H^1} & & \big\downarrow \mathcal{I}_{\boldsymbol{RT}} & & \big\downarrow \Pi_0 \\ V_{H^1} & \xrightarrow{\nabla^\perp} & \boldsymbol{V_{RT}} & \xrightarrow{\nabla\cdot} & V_{L^2} \end{array}$$

2 Nédélec finite elements in 3D

As opposed to the 2D case, where the Nédélecand Raviart–Thomas elements are just rotated versions of each other, in 3D the spaces are more distinct. In 3D, the Raviart–Thomas degrees of freedom are associated with **mesh faces**; the Nédélec degrees of freedom are associated with **mesh edges**. The local space is

$$\boldsymbol{N}_0 := \{\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + \boldsymbol{q}(x) : \boldsymbol{q} \in \mathbb{S}\}$$

where

$$\mathbb{S} := \operatorname{span}\left\{ \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}, \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}, \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \right\}.$$

Note that $\boldsymbol{q} \in \mathbb{S}$ has the property that $\boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0$. We see that in 3D, $\dim(\boldsymbol{N}_0) = 6$. The local degrees of freedom are

$$c_e(\boldsymbol{v}) := \int_e \boldsymbol{v} \cdot \boldsymbol{t}\, ds,$$

for each **edge** e in the mesh, where $\boldsymbol{t}$ is the unit tangent vector to that edge.

Many of the properties of $\boldsymbol{N}_0$ from the 2D case extend to the 3D case as well. The gradient of $\boldsymbol{v} \in \boldsymbol{N}_0$ is skew-symmetric. If $\nabla \times \boldsymbol{v} = 0$, then $\boldsymbol{v}$ is constant. This can be seen as follows: $\nabla \times \boldsymbol{v} = 0$ if and only if $\nabla \boldsymbol{v}$ is symmetric. Therefore, if $\nabla \times \boldsymbol{v} = 0$, then $\nabla \boldsymbol{v}$ is both symmetric and skew-symmetric, and hence zero; so $\boldsymbol{v}$ is constant. The tangential component of $\boldsymbol{v}$ is constant along any affine line. Unisolvence of the degrees of freedom is more complicated than in the 2D case, but still holds.

Defining the interpolation operators in the standard way, we can prove commutativity of the following diagram

$$\begin{tikzcd}
C^3(\Omega) \arrow[r, "\nabla"] \arrow[d, "\mathcal{I}_{H^1}"] & \boldsymbol{C}^2(\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\mathcal{I}_{\boldsymbol{N}}"] & \boldsymbol{C}^1(\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{I}_{\boldsymbol{RT}}"] & C^0(\Omega) \arrow[d, "\Pi_0"] \\
V_{H^1} \arrow[r, "\nabla"] & \boldsymbol{V}_{\boldsymbol{N}} \arrow[r, "\nabla\times"] & \boldsymbol{V}_{\boldsymbol{RT}} \arrow[r, "\nabla\cdot"] & V_{L^2}
\end{tikzcd}$$