

MEASUREMENT OF MOMENT OF INERTIA

The aim of this measurement is to determine the moment of inertia of the rotor of an electric motor.

1. General relations

Rotating motion and moment of inertia

Let us consider the case when a body of mass m moves on a circular path with acceleration \mathbf{a}_t due to the tangential component \mathbf{F}_t of force \mathbf{F} . By virtue of Newton's second law we have

$$\mathbf{F} = m \cdot \mathbf{a} ,$$

or in scalar form

$$F_t = m \cdot a_t .$$

Let us multiply both sides by the radius \mathbf{r} of the circular path:

$$r \cdot F_t = r \cdot m \cdot a_t$$

Using the relation $a_t = r\varepsilon$ (in which ε is the angular acceleration) and denoting the product $r F_t$ by moment M we obtain

$$M = m \cdot r^2 \cdot \varepsilon .$$

The right side multiplier mr^2 is the **moment of inertia** Θ [kg m²]. Thus **Newton's second law for rotary motion is**

$$M = \Theta \cdot \varepsilon .$$

The moment of inertia $\Theta = mr^2$ depends on the rotating masses, as well on the distances of the masses from the centre of rotation. In **Fig. 1** depicts a solid disk (a disk of constant thickness).

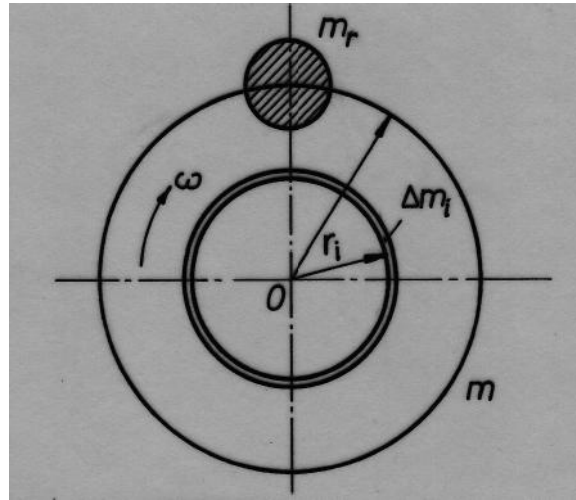


Fig. 1

The mass \mathbf{m} is in different distances from the \mathbf{O} axis of rotation. If we cut a mass $\Delta \mathbf{m}_i$, on \mathbf{r}_i radius, the moment of inertia of this thin ring will be

$$\Delta \theta_i = \Delta m_i \cdot r_i^2.$$

By summing up these thin rings over the whole radius we obtain the moment of inertia of the disc as

$$\theta = \sum_i \Delta \theta_i = \sum_i \Delta m_i \cdot r_i^2$$

To determine the moment of inertia in such way, we must know the distribution of mass along the radius, i.e. the function, from which one can see what Δm_i mass belongs to radius r_i .

For example, let us calculate the moment of inertia of a disc of constant density and width b . Let us divide the radius to N pieces, with $r_i = R/N$, $i = 1 \dots N$. Then we have

$$\Delta m_i = A_i b \rho = \underbrace{2 r_i \pi}_{\text{circumference}} \underbrace{\Delta r}_{\text{width}} \underbrace{b}_{\text{thickness}} \rho = 2 i \frac{R}{N} \pi \frac{R}{N} b \rho$$

Thus the moment of inertia is

$$\begin{aligned} \theta &= \sum_{i=1}^N \Delta m_i \cdot r_i^2 = \sum_{i=1}^N 2 i \frac{R}{N} \pi \frac{R}{N} b \rho i^2 \frac{R^2}{N^2} = 2 \pi b \rho \frac{R^4}{N^4} \sum_{i=1}^N i^3 \\ &= 2 \pi b \rho \frac{R^4}{N^4} \frac{1}{4} N^2 (1 + N)^2 \approx \pi b \rho R^4 \frac{1}{2} = R^2 \pi b \rho \frac{R^2}{2} = \frac{1}{2} m R^2 \end{aligned}$$

For practical purposes, we introduce the concept of **reduced mass: m_r** . The moment of inertia of a body is expressed with the moment of inertia of the mass being on a single radius times a reduction factor λ :

$$\theta = \sum_i \Delta m_i \cdot r_i^2 = \lambda m r^2.$$

The reduction factor λ is 0.5 in the case of a solid disc (as seen above). The value of reducing factor depends on the form.

Physical pendulum

Making use of the well-known relation of the mathematical pendulum let us examine the period of oscillation of such a rigid body which - due to gravitational force - can turn round a fixed horizontal axle. On the left-hand side of **Fig.2** the mathematical pendulum and on the right-hand side the physical pendulum can be seen.

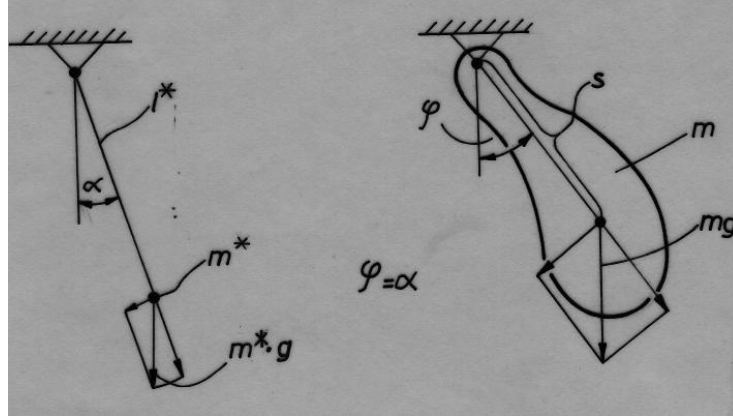


Fig. 2

By continuously measuring the motion of the physical pendulum let us change the length of the mathematical pendulum until the two pendulums swing together. If we denote with s the distance of the centre of gravity of the physical pendulum from the point of suspension, its moment of inertia to the point of suspension with Θ , the mass with m , the angular acceleration of the physical pendulum ϵ_{ph} will be

$$\epsilon_{ph} = -\frac{M}{\theta} = -\frac{m \cdot g \cdot s \cdot \sin \phi}{\theta}$$

(The negative sign means the moment effecting in the direction opposite to the deflection.)

The ϵ_{mat} angular acceleration of the mathematical pendulum deflected with $\alpha = \phi$ and swinging together with the physical pendulum:

$$\epsilon_{mat} = -\frac{M}{\theta} = -\frac{m^* \cdot g \cdot l^* \cdot \sin \phi}{m^* \cdot l^{*2}} = -g \frac{\sin \phi}{l^*}$$

From the equality of the two angular accelerations ($\epsilon_{ph} = \epsilon_{mat}$) we can get the reduced length of that mathematical pendulum which swings together with the physical pendulum.

We may write

$$\frac{m \cdot g \cdot s \cdot \sin \alpha}{\theta} = \frac{g \cdot \sin \phi}{l^*}$$

From this the reduced length

$$l^* = \frac{\theta}{m \cdot s}$$

And with this the period of oscillation of the physical pendulum becomes

$$T = 2 \cdot \pi \cdot \sqrt{\frac{l^*}{g}} = 2 \cdot \pi \cdot \sqrt{\frac{\theta}{m \cdot g \cdot s}}$$

2. Determination of moment of inertia with measurement of the period of oscillation

Description of the measurement technique

This method is based on the measurement **T** period of oscillation of a physical pendulum. According to **Fig.3** we turn our rotor into a physical pendulum by mounting a cylinder (of uniform mass-distribution) to the rotating part at a distance of **d** from the rotation axis. This physical pendulum built in a measuring apparatus can be seen in **Fig 4**.

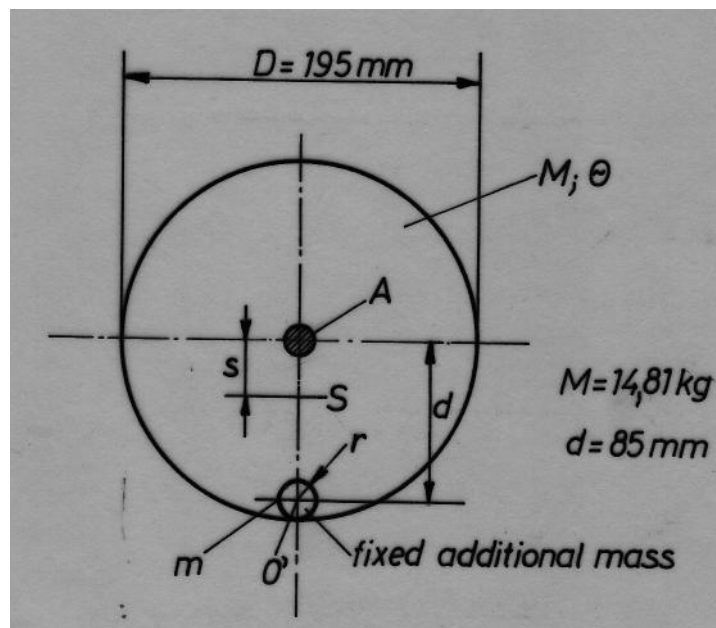


Fig. 3

The period of oscillation of the physical pendulum is:

$$T = 2 \cdot \pi \cdot \sqrt{\frac{\theta_A}{(M + m) \cdot g \cdot s}},$$

where Θ_A is the moment of inertia of the swinging (**M+m**) mass to the **A** axis which is composed of the moment of inertia of the body with mass **M** and the additional **m** mass fixed on it.

It can be written that $\Theta_A = \Theta_{MA} + \Theta_{mA} = \Theta + \Theta_{mA}$. The moment of inertia Θ_{mA} of the fixed additional mass (a cylinder of mass **m** and with radius **r**) consists of the moment of inertia $1/2mr^2$ to the **O'** fixing spot and of the term taking into regard the effect of removal:

$$\theta_{mA} = \frac{1}{2} \cdot m \cdot r^2 + m \cdot d^2$$

With this the moment of inertia of the physical pendulum to the suspension point **A**:

$$\theta_A = \theta + \frac{1}{2} \cdot m \cdot r^2 + m \cdot d^2$$

The torque equilibrium at point A is

$$M \cdot g \cdot 0 + m \cdot g \cdot d = (M + m) \cdot g \cdot s.$$

Substituting into the formula of period of oscillation we get:

$$T = 2 \cdot \pi \cdot \sqrt{\frac{\theta + \frac{1}{2} \cdot m \cdot r^2 + m \cdot d^2}{m \cdot g \cdot d}}$$

So we can get Θ from the relation:

$$\theta = \left(\frac{T}{2 \cdot \pi} \right)^2 \cdot m \cdot g \cdot d - m \cdot \left(\frac{r^2}{2} + d^2 \right)$$

The formula of the period of oscillation gives a precise value only for little angle deflections. The deflection of the pendulum should not be greater than 10° during the measurements.

The measurement

In the equipment seen in **Fig 4.**, we mount a mass m (marked by 2) to the mass M (marked by 1). Then we deviate this physical pendulum till the mark (10°) scratched to the plate (4) and then we let it go to swing freely. The plate cuts the way of the light and through the photoelectric impulse sender (3) it starts and after a whole swinging it stops the electronic chronometer (5). Knowing the **T** period of oscillation the moment of inertia can be calculated.

During the measurement we note the periods of oscillation, than taking the arithmetic mean of the time values we calculate the value of the moment of inertia with the relation:

$$\theta = \left(\frac{T}{2 \cdot \pi} \right)^2 \cdot m \cdot g \cdot d - m \cdot \left(\frac{r^2}{2} + d^2 \right)$$

where the arithmetic mean of the measured periods of oscillation:

$$T = \frac{1}{N} (T_1 + T_2 + \dots + T_N)$$

in which N is the number of swingings, $T_1, T_2 \dots T_N$ are the periods of oscillation. The periods of oscillation can be measured with stopwatch, too. The stopwatch continuously measures the time, so the sum of the $T_1 + T_2 + \dots + T_N$ values. Also here we have to count the N numbers of the swingings. The mean time **T** can be calculated as above. The radius r and mass m of the fixed additional mass of cylinder should be noted during the measurement. The distance of fixing $d = 85$ mm, knowing Θ the factor of reduction can be calculated from

$$\lambda = \frac{\theta}{M \cdot R^2}$$

with $M = 14.81$ kg, $R = 0.0975$ m.

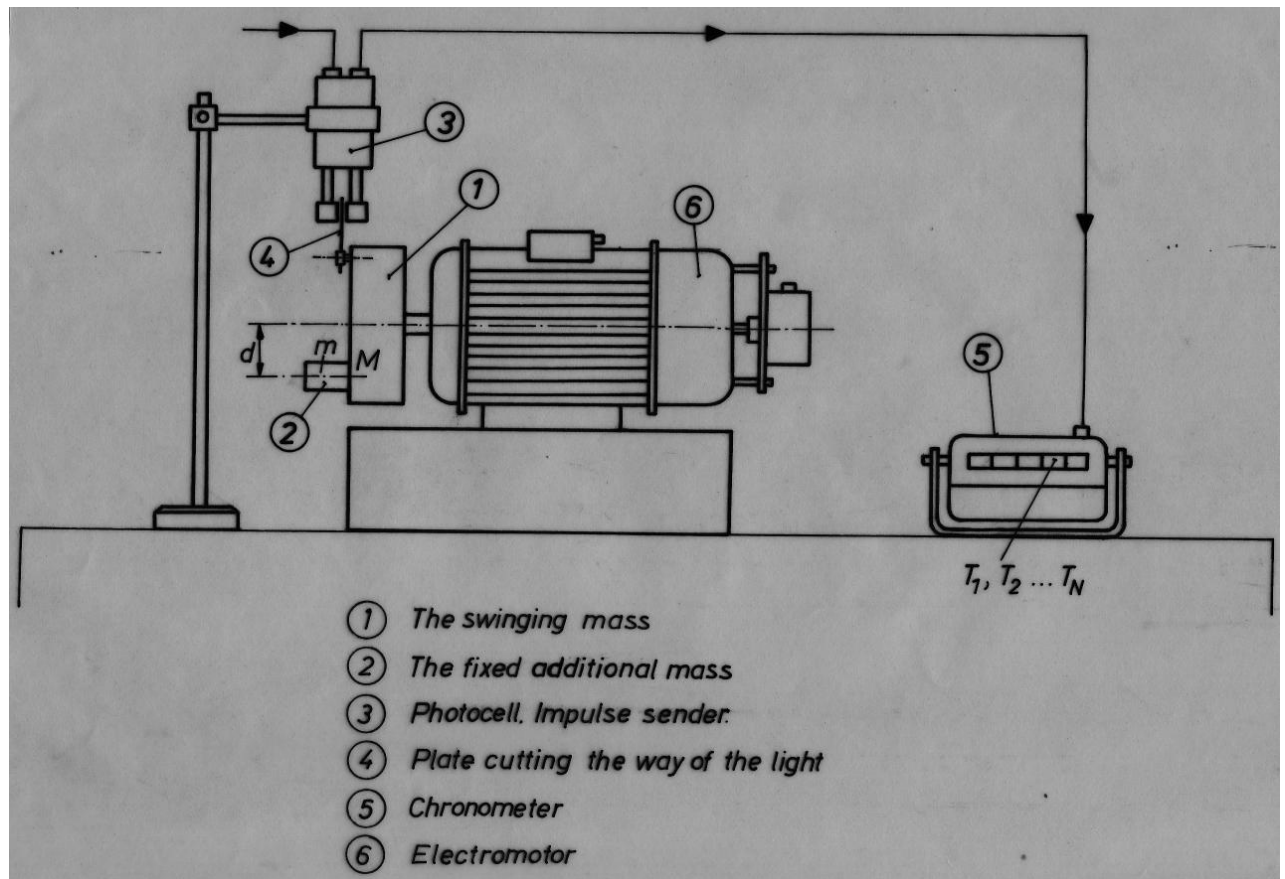


Fig. 4.

3. Preparation questions

1. Make a sketch of a mathematical and a physical pendulum and define the quantities influencing their period of swinging.
2. Explain the train of thought with which the moment of inertia will be measured.
3. Define reduced mass and give the reduction factor for a homogenous disc.
4. Make a sketch of the test rig on which the measurement will be carried out. Give a short description of the main parts.