Lecture 5

Deterministic chaos

Plan of the Lecture

When “deterministic” does not mean “predictable”

1 Physical (in)determinism

2-4 Deterministic chaos

5 Two related topics

PHYSICAL (IN)DETERMINISM

The simple story from physics

- “Classical” physics is deterministic in the sense of Laplace
  - Usually identified with “Newtonian” physics
  - Includes also Einstein’s relativity
- Quantum physics is not deterministic, it is intrinsically random

Begs many questions!

Classical → deterministic
Quantum → random

About physics

Correct as a description of the *theory*, but what about nature?

- How can you be sure that some phenomena are random?
- If they are, is it possible that everything is actually random?

Beyond physics

Does this imply anything at all beyond physics?

- Would physical determinism imply lack of free will? Conversely, could physical indeterminism be used as a "proof" of free will?

The conspiracy theater

"Maybe everything is an illusion and we are just acting in a theater play set up by…"

1. God (Bishop George Berkeley, 18^{th} century)
2. Robots who have taken control (The Matrix, 1999)
3. Aliens playing a computer game (the latest SciFi variation on the theme)
4. Nobody, the universe is just deterministic by itself.

Cf. also Plato's *Allegory of the cave*, Calderon's *Life is a dream* etc.

- Such a world view cannot be falsified
  - At least, not by physics, not even quantum
- Even if you uphold such a view, it makes sense to speak of randomness: as usual, we have to specify *for whom* the phenomenon is unpredictable.

Keep calm and random <u>for whom</u>?

A being…	**Cannot predict…**
In our universe, with limited computational power We are here	• Quantum phenomena • “Chaotic” classical phenomena
In our universe, with unlimited computational power (Laplace’s “intelligence”) We can imagine being here	Quantum phenomena
Outside our universe, in particular outside our time We can hardly conceive what it means to be here	It does not make sense to speak of “prediction”: it is passive “vision” or active “decision”.

Let’s say it again

- Physical (in)determinism = phenomena that can(not) be predicted with unlimited computational resources
  - The information regarding a deterministic phenomenon is already fully available
  - An indeterministic phenomenon creates new information
- Some deterministic phenomena may be unpredictable with limited resources
  - “Deterministic chaos”: this lecture
  - Analogy: security of RSA vs. security of OTP (Lecture 2)

Suggested Readings

Wikipedia pages:

- http://en.wikipedia.org/wiki/Determinism

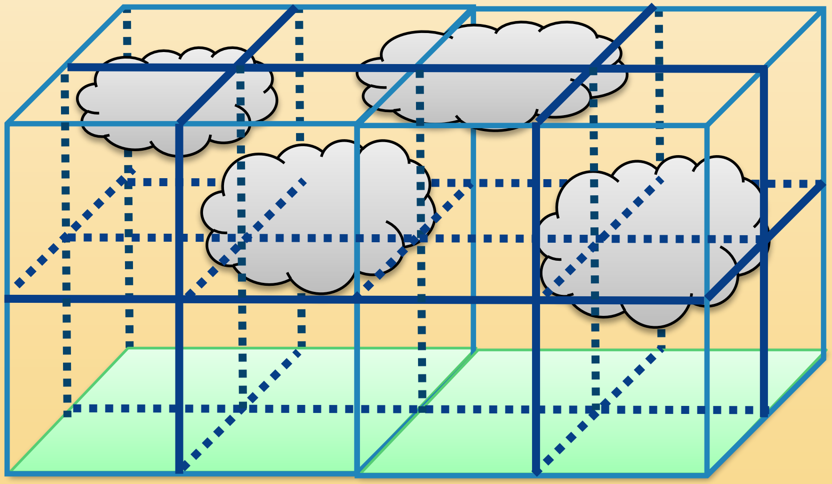
WEATHER FORECAST

Recipe for weather forecast

1. Preparation
   – Define a grid on a portion of the Earth.
   – Define the meaningful variables (pressure, temperature, humidity, wind…)
2. Specify the initial conditions in each point of the grid (i.e. the values of the variables *now*).
3. Use the suitable evolution equations (from fluid dynamics, thermodynamics…)

Computational complexity

More precision $\Rightarrow$ smaller mesh in the grid $\Rightarrow$ more computation
(assuming that the evolution equations are perfectly correct)



3D grid $\Rightarrow$ each time you double the precision, the number of cells grows by a factor eight:

$$\#(cells) \propto 8^n$$

with n = number of times precision is doubled.

What about the evolution equations?

An inescapable one is the Navier-Stokes equation from fluid dynamics:

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \right) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla}\mathbf{T} + \boldsymbol{f}$$

Density of the fluid

Speed of the fluid

Pressure on the fluid

Stress on the fluid

Forces on the fluid (e.g. gravity)

$(\rho, \boldsymbol{v})$ = variables that *describe* the fluid ("degrees of freedom")

All that *acts* on the fluid: need further modeling to say how they depend on $(\rho, \boldsymbol{v})$.

Does it have a solution at all? In general, open problem!

The Lorenz system of equations

- Navier-Stokes with some approximation
- "Incompressible" fluid
- An equation for heat transfer

A few further manipulations and assumptions

$$\frac{dX}{dt} = a(Y - X)$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - bZ$$

- Variables: only 3 real numbers (no more vector fields)
  - X related to velocity, Y and Z to variations of temperature
  - Remark: the original Lorenz model had 12 variables
- No derivatives in space, only in time ("ordinary" differential equations)
- Non-linear: terms XZ and XY.

Serendipity

- Edward Lorenz, around 1960: computer prediction of the weather with non-linear equations.
- Runs again the calculation for the same initial condition: after some time, the prediction changes completely: a bug in the computer?
- No, it's a real effect: he had not used the *same* initial conditions, but rounded to the 4th significant digit $\Rightarrow$ sensitivity to initial conditions
- 1963 paper; 1972 talk "butterfly effect". Explosion of interest only after 1975.

Simple, deterministic, and unpredictable

- The fact that the weather is hard to predict is a no-brainer: the system is too complex…
  - Same as “noise” last week
- Lorenz: even simple systems (3 variables!) can be chaotic.
  - In the literature on chaos: “chaotic” = variables under study; “random” = noise from other sources.
- By the way: weather forecast is quite an achievement ☺

Suggested Readings

A.E. Motter, D.K. Campbell, Chaos at fifty, Physics Today 66(5), 27-33 (2013)
Preprint available at http://arxiv.org/abs/1306.5777

Wikipedia pages:

- http://en.wikipedia.org/wiki/Weather_forecasting
- http://en.wikipedia.org/wiki/Lorenz_system

SIMPLE CHAOS

The “logistic map”

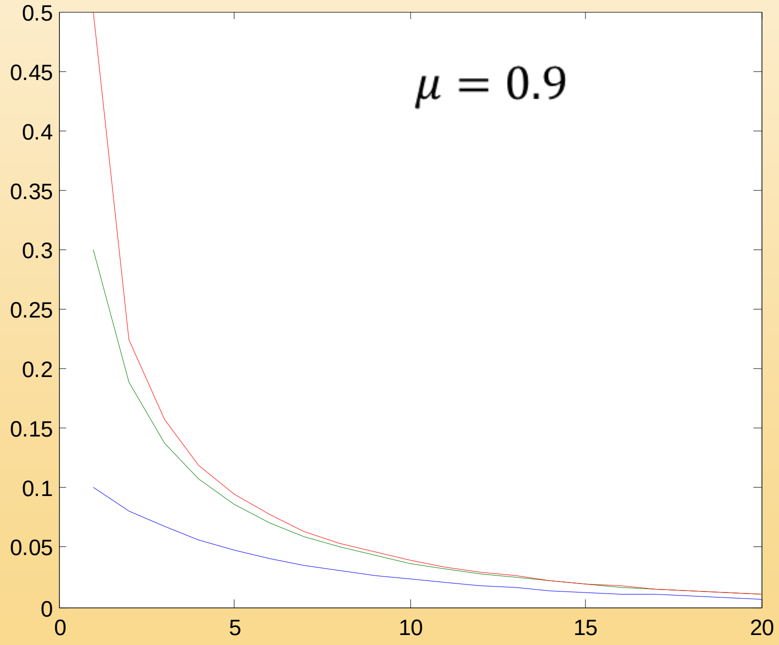
$$x_{k+1} = \mu\, x_k(1 - x_k)$$

$$0 \le x \le 1\,, 0 < \mu \le 4$$

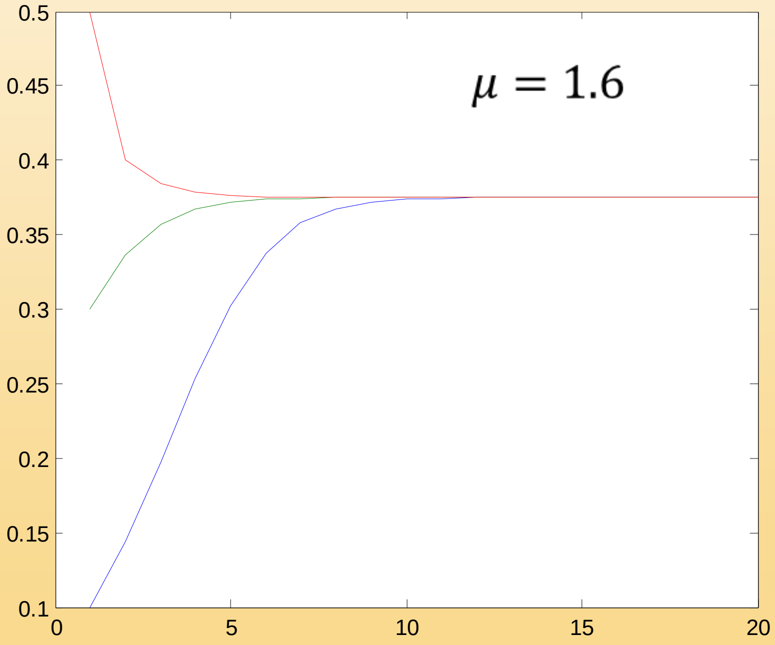
- Initial motivation: simple model for population evolution
- Nowadays known as the simplest example of chaotic map with rich behavior.

Non-chaotic behavior

$\mu < 1$: any initial population dies out

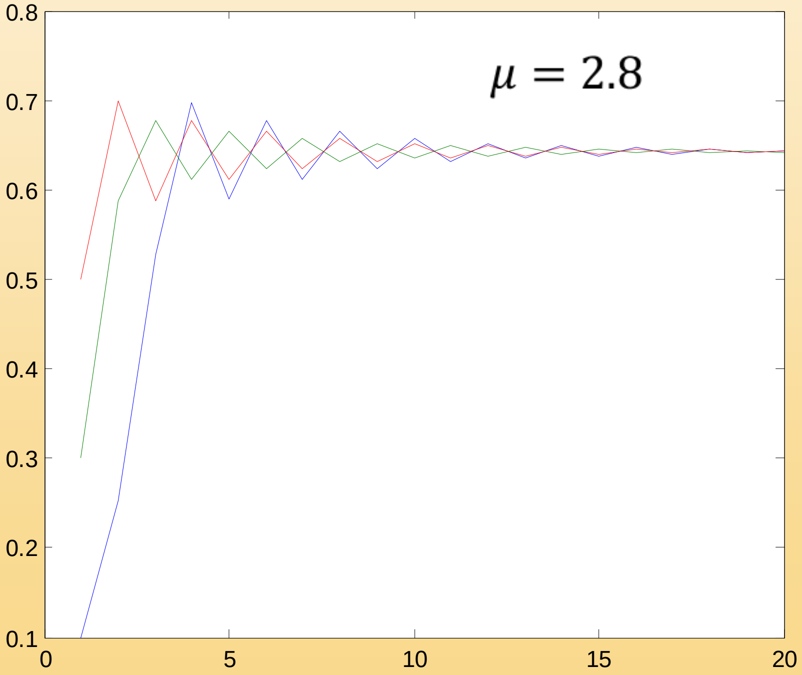


$1 \leq \mu < 2$: any initial population stabilizes to the same value $(\mu - 1)/\mu$

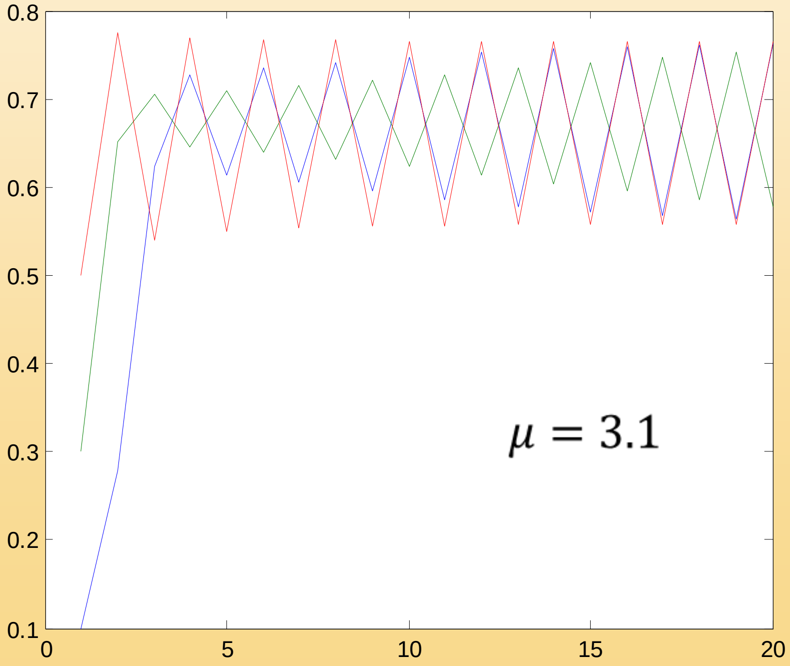


Getting interesting

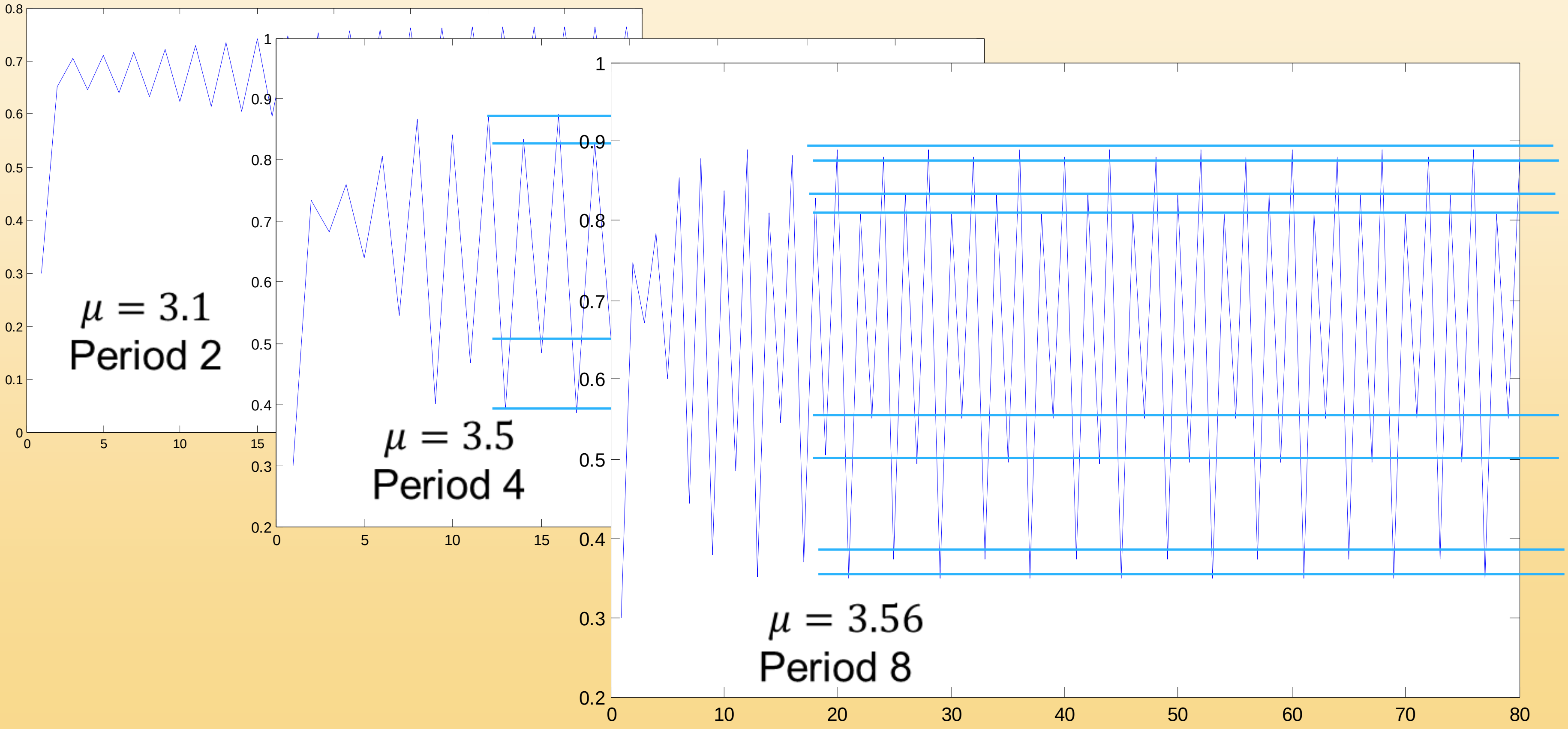
$2 \leq \mu < 3$: any initial population stabilizes to the same value $(\mu - 1)/\mu$ after oscillations



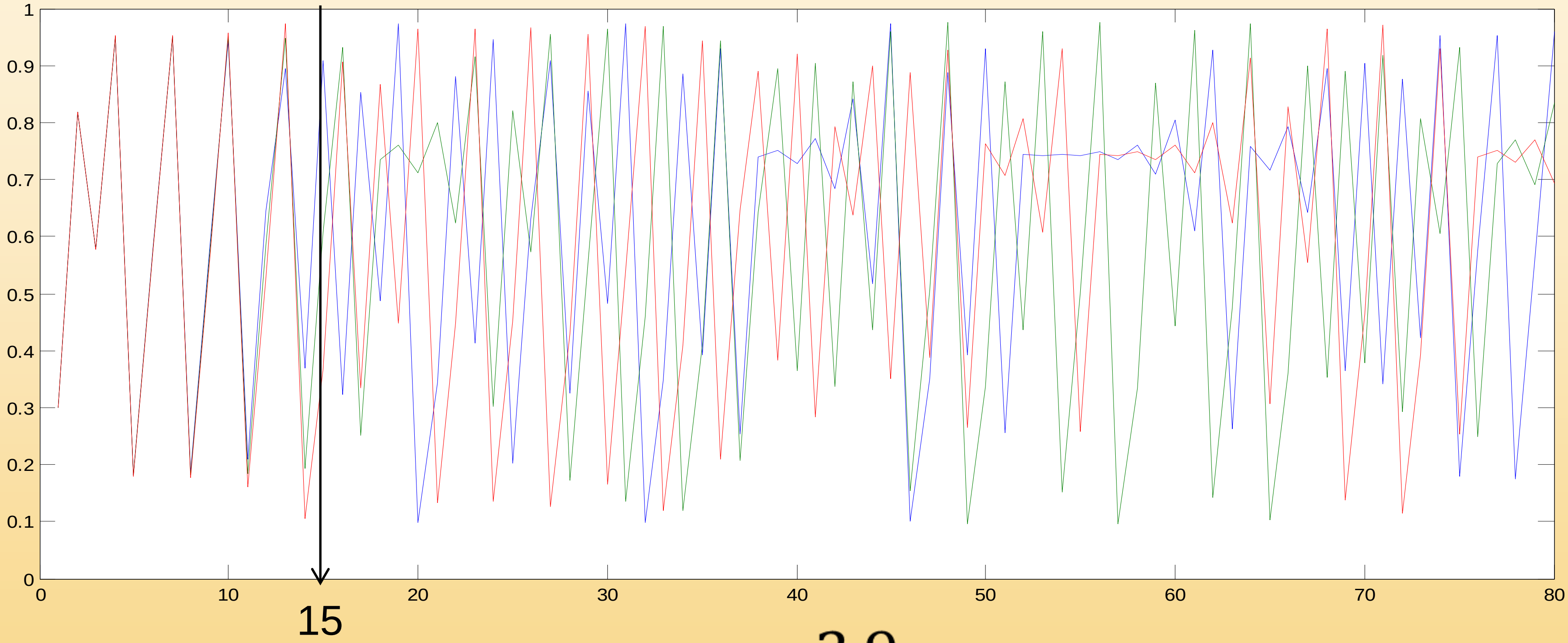
$3 \leq \mu < 1 + \sqrt{6}$: any initial population oscillates between two values: **bifurcation**



A cascade of bifurcations



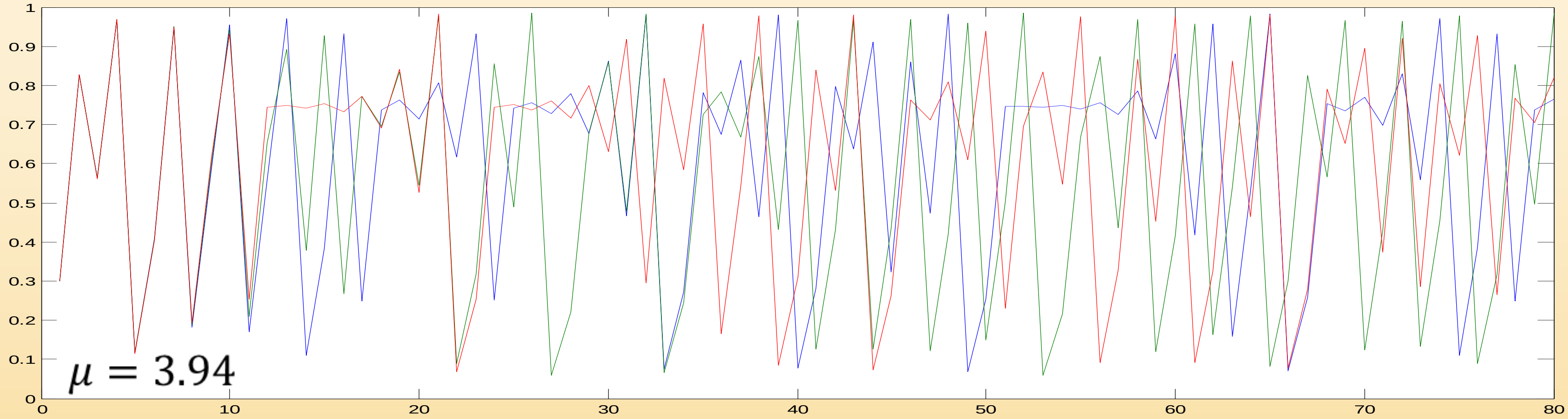
Chaotic behavior



$$\mu = 3.9$$

$x_1 = 0.2999$, $x_1 = 0.3000$, $x_1 = 0.3001$

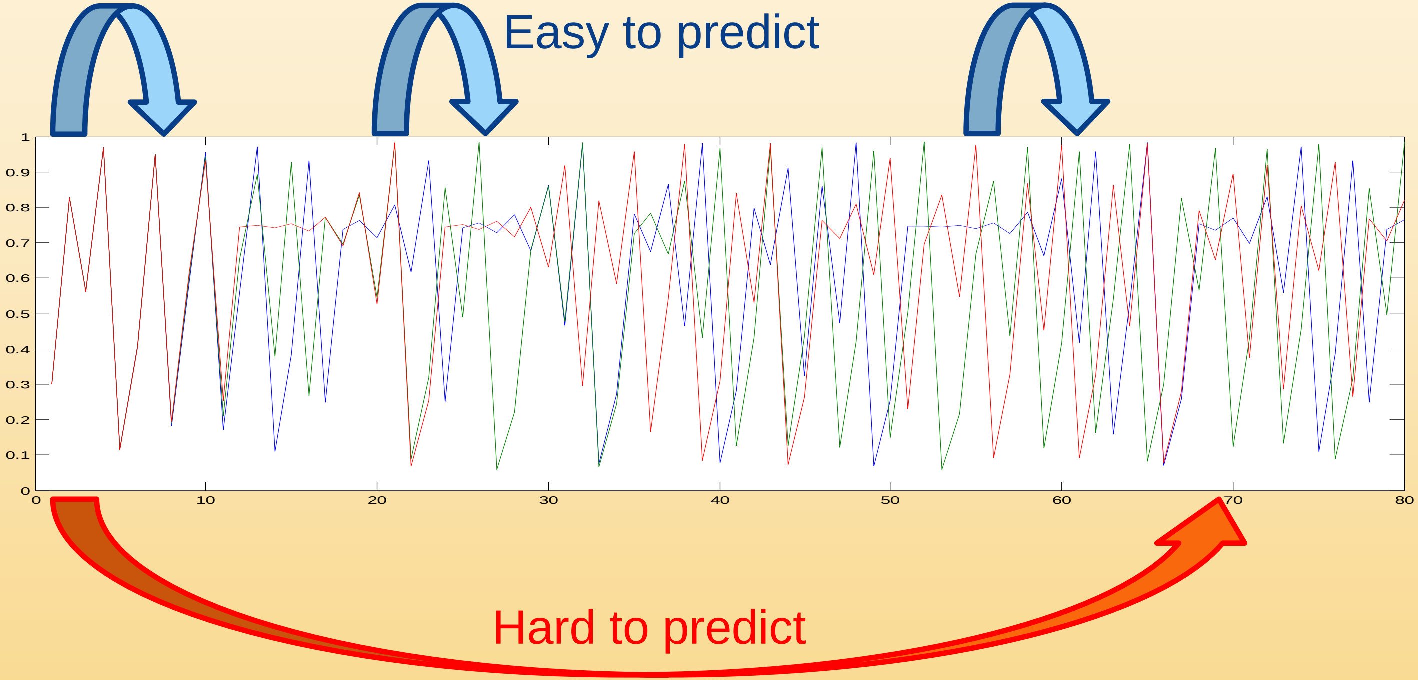
Definition of “chaos”



- Sensitivity to initial conditions
- “Mixing”: each evolution explores almost all possible values

Hard to predict *in the long run*

RNG? Not amazing…



Of course one could use this unpredictability, but it is slow: many iterations for just one effective step

Suggested Readings

Wikipedia pages:

- http://en.wikipedia.org/wiki/Logistic_map

MORE CHAOS

Before you browse…

Energy considerations

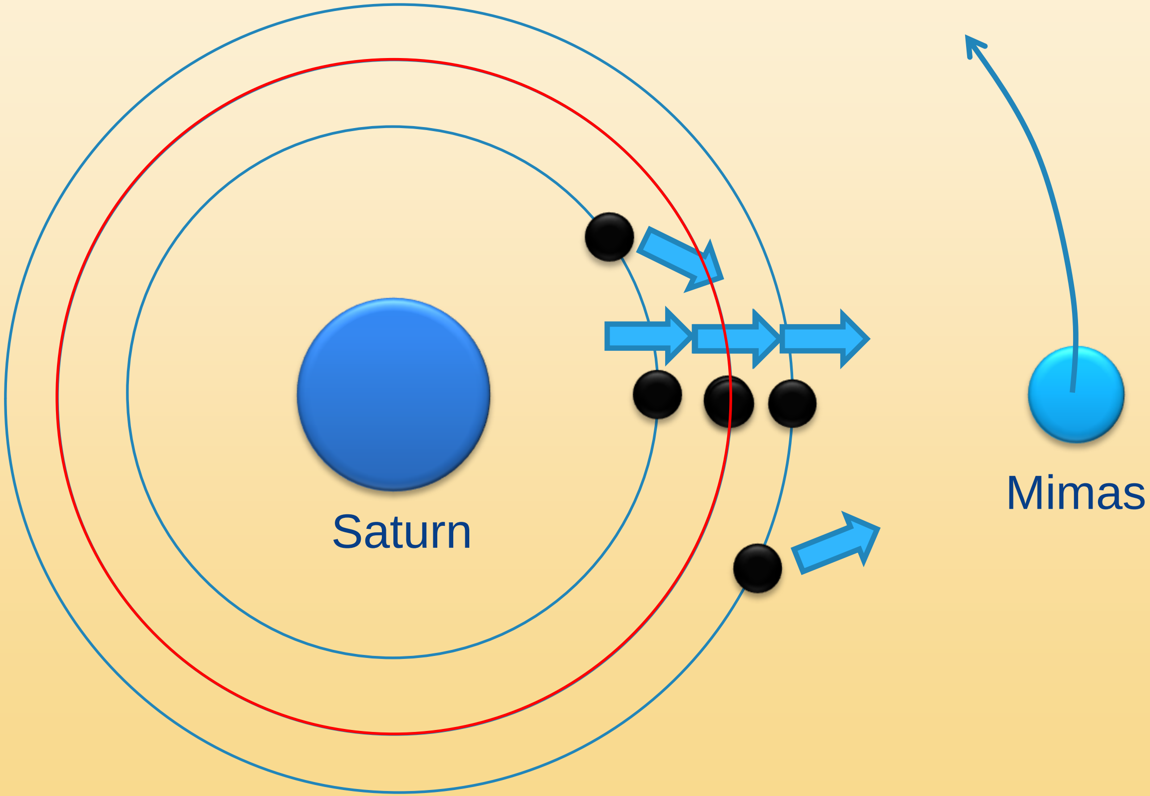
Closed ("conservative", "Hamiltonian") systems	**Open ("dissipative") systems**

• Completely isolated, their energy is conserved during the evolution.
• Theoretical idealization
• Examples
  – Most of high school mechanics: ideal springs and pendula, planets…
	• Exchange energy with other systems
	• The evolution goes toward an *attractor*
	• Examples
	  – Turbulence
	  – Lorenz
	  – logistic map [discrete]

Open systems

- Pure dissipation: the systems stops $\Rightarrow$ attractor = point.
- Dissipation & drive: the system may:
  - Oscillate with one or more frequencies $\Rightarrow$ attractor = cycle
  - Be chaotic $\Rightarrow$ “strange” attractor”
    - Requires at least 3 variables for continuous time
    - Example: Lorenz model
    - How can trajectories all go to an attractor, while showing sensitivity to initial conditions? “Fractal” geometry of the attractor.

Closed systems (1): three bodies

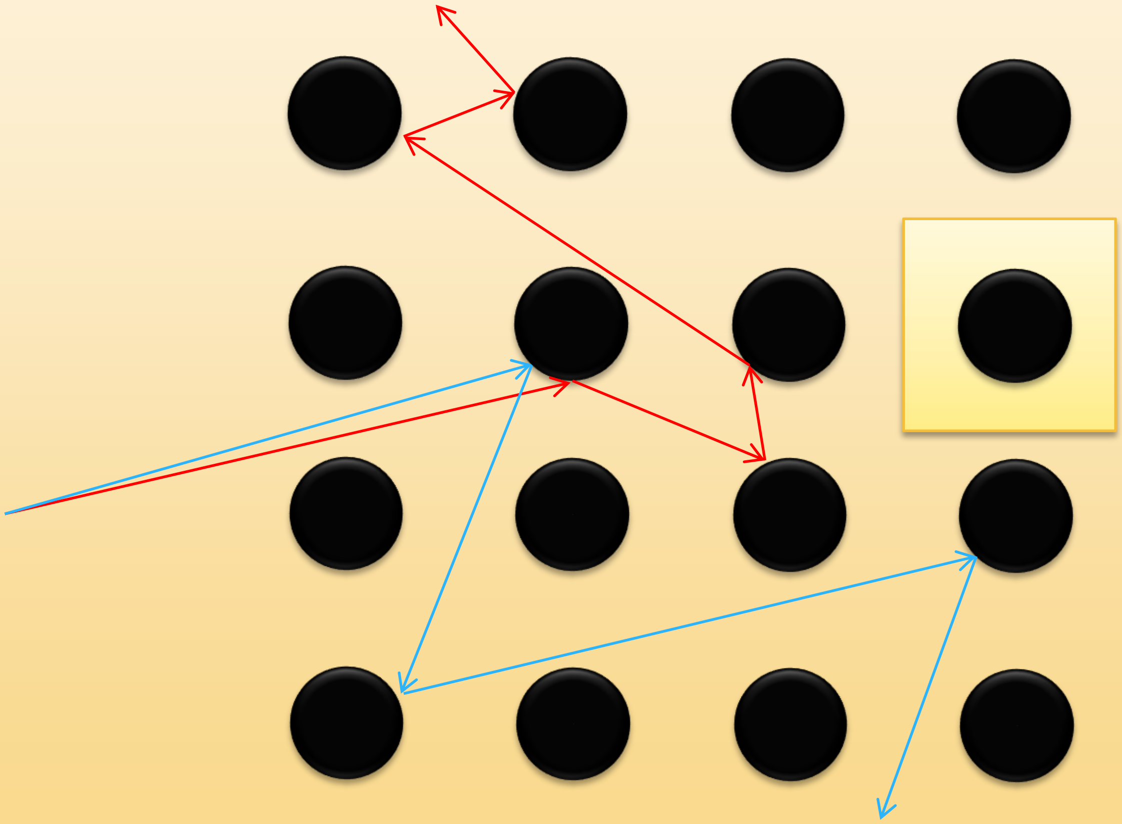


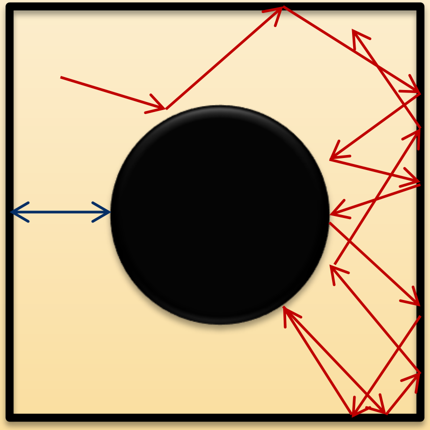
Kepler's law: the period depends on the distance as $T^2 \propto a^3$

The **resonant orbit** (2:1) becomes unstable under tidal forces $\Rightarrow$ chaotic motion $\Rightarrow$ matter expelled

Origin of the Cassini division and other (not all) features of Saturn's rings

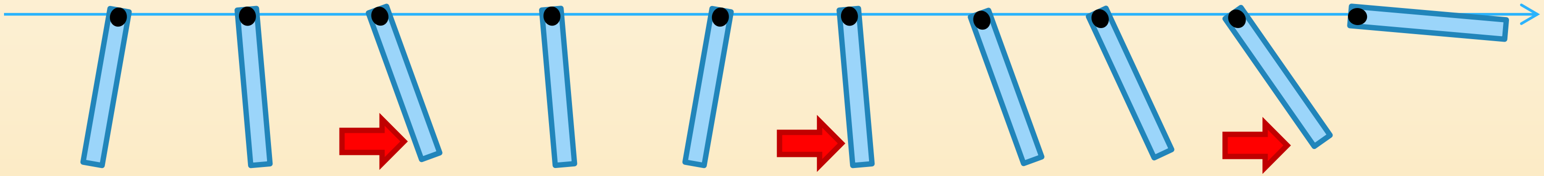
Closed systems (2): Sinai's billiard





Almost all trajectories end up filling the space ("ergodicity")

Closed systems (3): kicked rotor



Variables: p = angular momentum, θ = angle,

Continuous time evolution:
Kinetic energy $\frac{1}{2}p^2$
Potential energy $K\cos\theta\sum_{n=-\infty}^{+\infty}\delta(t-Tn)$

Discretized map at every kick
("standard map"):

$$p_{n+1} = p_n + K\sin\theta_n$$
$$\theta_{n+1} = \theta_n + p_{n+1}$$

Suggested Readings

Popular books:
J. Gleick, Chaos: Making a new science
http://en.wikipedia.org/wiki/Chaos:_Making_a_New_Science

Wikipedia pages:
- http://en.wikipedia.org/wiki/Dynamical_system
- http://en.wikipedia.org/wiki/Dynamical_billiards
- http://en.wikipedia.org/wiki/Chaos_theory
- http://en.wikipedia.org/wiki/Standard_map

FRACTALS & POWER LAWS

An innocent-looking map

$$z_{k+1} = z_k^2 + c$$

z, c complex

$$z_0$$
$$z_1 = z_0^2 + c$$
$$z_2 = (z_0^2 + c)^2 + c$$
$$\vdots$$

Reminder of complex numbers:

$z = x + iy$

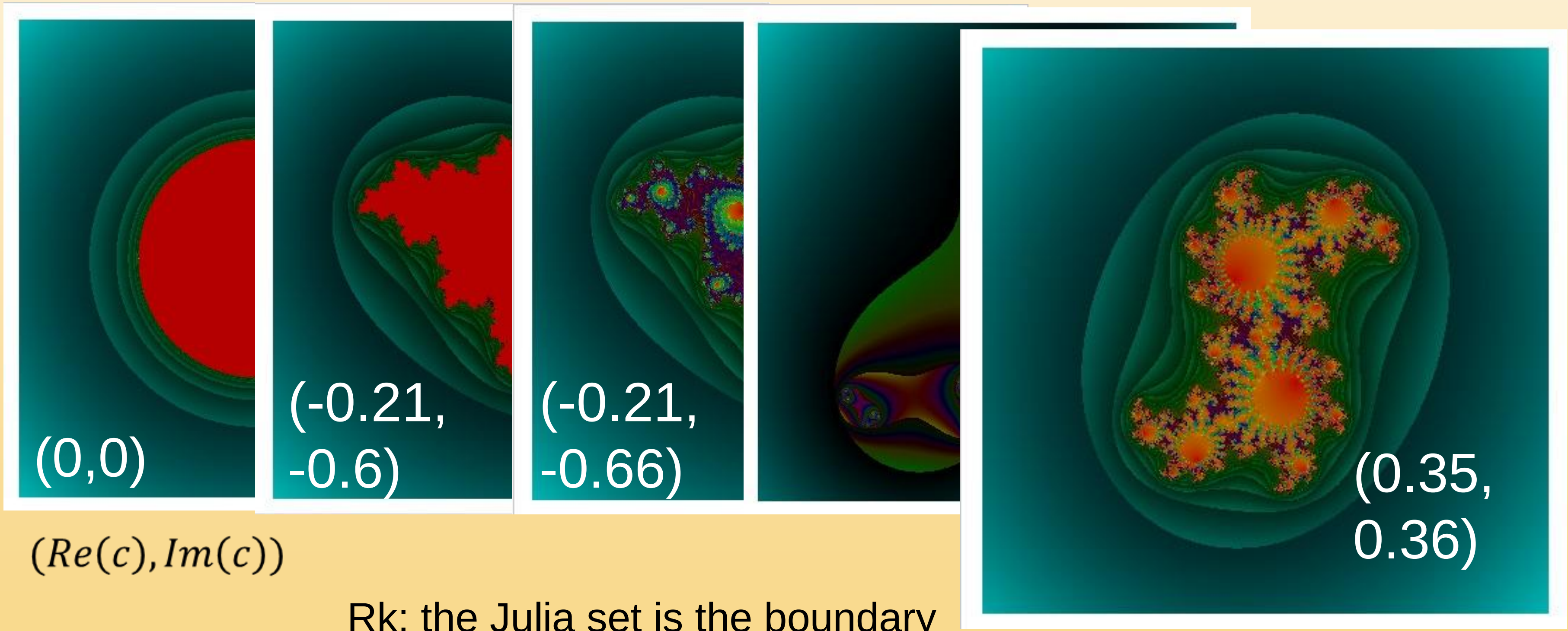
$\Rightarrow \; z^2 = (x^2 - y^2) + i\,2xy$

For most values of z_0 and c, $z_k \to \infty$ for $k \to \infty$

- For fixed c, which are the values of z_0 such that z_k escapes to infinity?
- For fixed z_0, which are the values of c such that z_k escapes to infinity?

Julia sets

For fixed c, which are the values of z_0 such that $z_k \to \infty$?

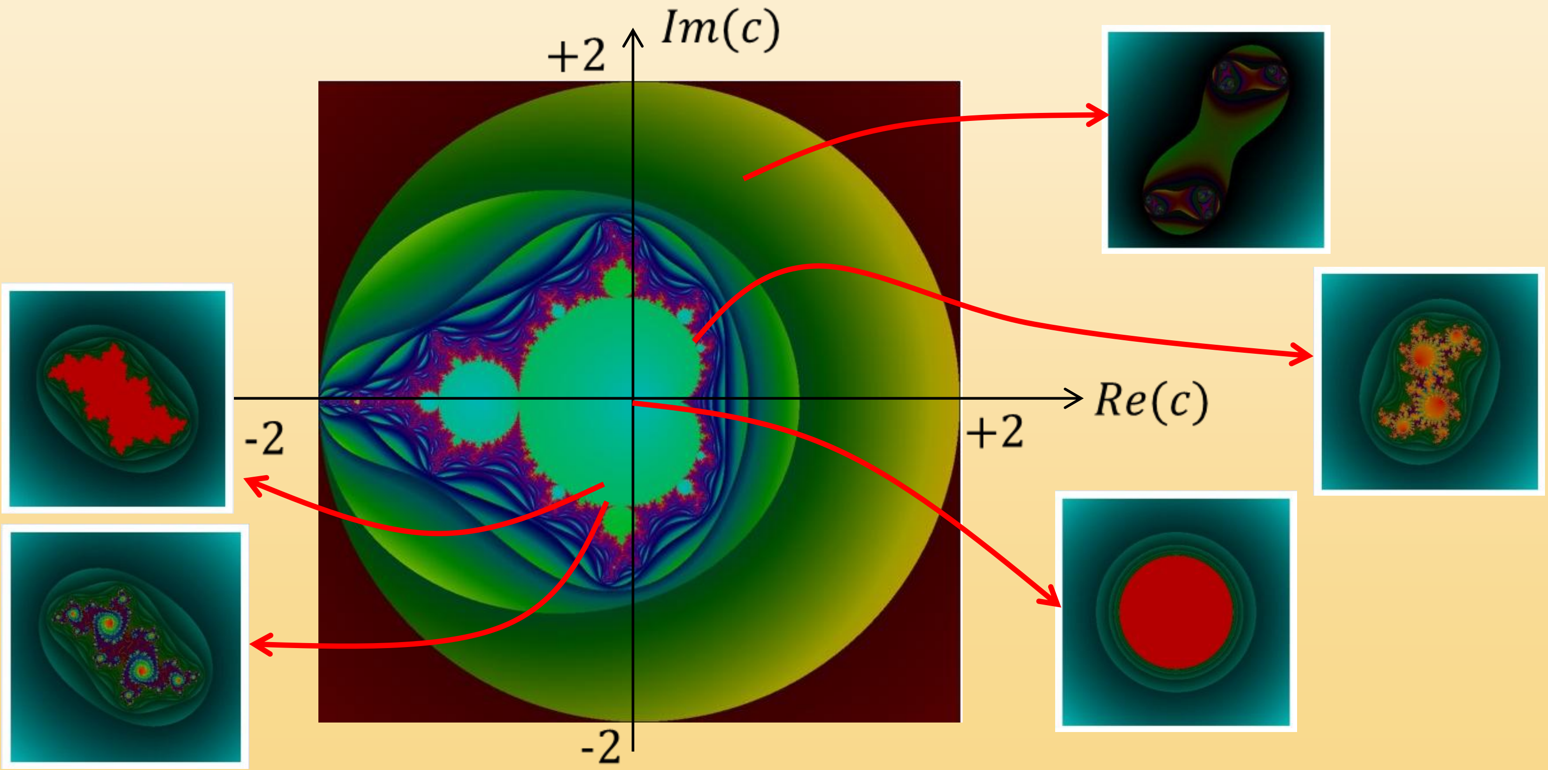


$(Re(c), Im(c))$

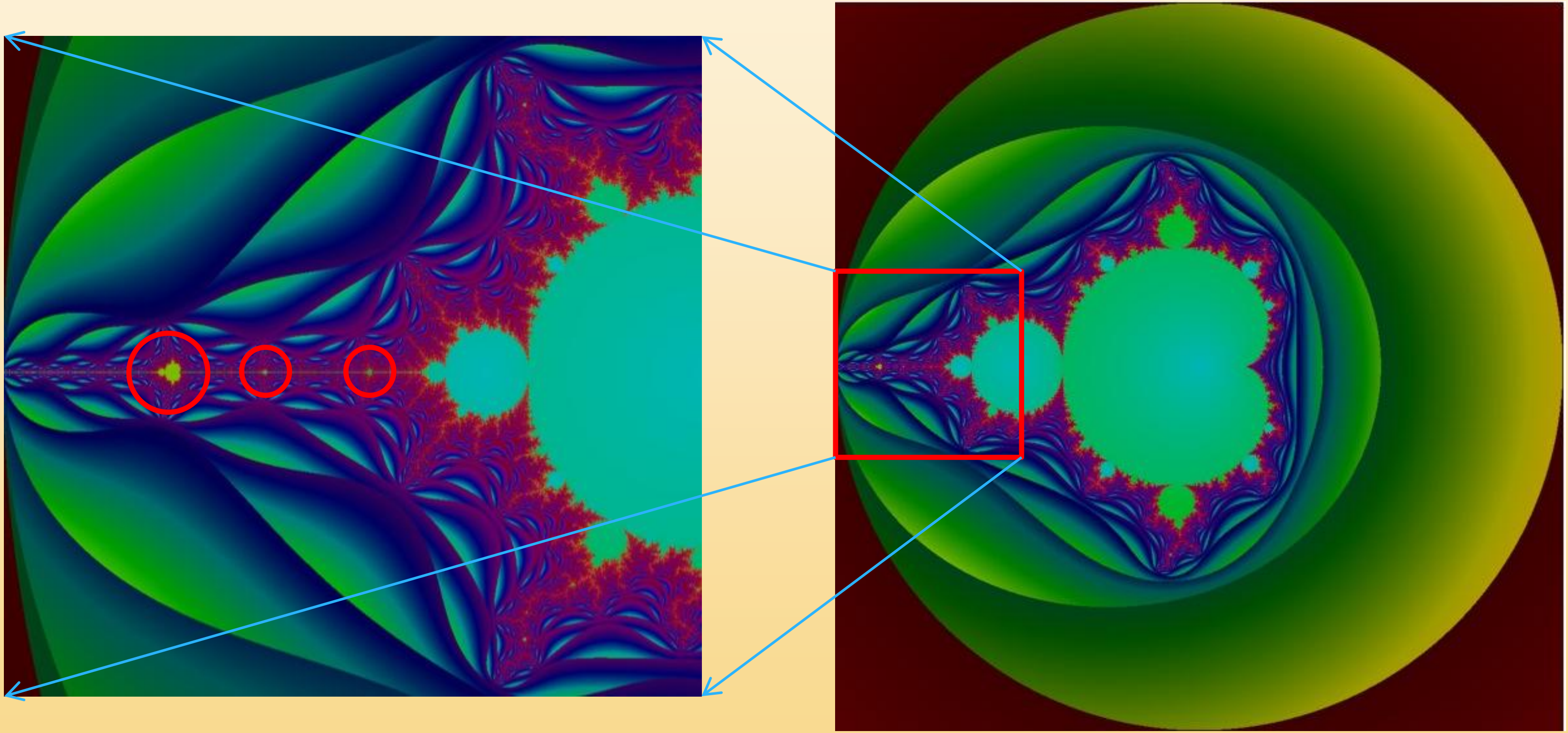
Rk: the Julia set is the boundary

Mandelbrot set

For fixed $z_0 = 0$, which are the values of c such that $z_k \to \infty$?



Self-similarity of Mandelbrot set



“Self-similarity”

- “It looks the same when you zoom in”
- Visual examples in nature
  - Stones ($\Rightarrow$ hard to estimate distances on mountains)
  - Coastlines
  - Galaxies and large structures in the universe
- Examples in mathematics
  - Fractals
  - Power laws

Power law

$$F(x) = C\, x^{\alpha}$$

Zoom: $x \to y = bx \qquad \Rightarrow F(y) \equiv F(bx) = C\, b^{\alpha} x^{\alpha} \;= C' x^{\alpha}$

Example: Gutenberg-Richter law

$$N(A) = N_{Tot}\, A^{-b}$$

N = number of earthquakes in a given region

A = amplitude of the seismic waves

Usually given as

$N = N_{Tot}\, 10^{-bM}$

with the magnitude

$M = \log_{10} A$

Earthquakes are not fair coins

Does nature choose earthquakes by tossing coins? Meaning:

- Earthquake of A=1 each time you toss a 0
- Earthquake of A=2 each time you toss a 00
- Earthquake of A=3 each time you toss a 000, etc.

NO, it does not!

Fair coin

$$R(n) = \frac{\#(n\,\mathbf{0})}{\#(m\,\mathbf{0})} = 2^{-n+m}$$

Gutenberg-Richter

$$R(n) = \frac{\#(A=n)}{\#(A=m)} = \left(\frac{\boldsymbol{n}}{\boldsymbol{m}}\right)^{-b}$$

Assuming an earthquake M=1 (A=10) happens every hour, a M=8 earthquake ($A=10^8$) happens…

- GR for b=1: every 10^7 hours i.e. every 1100 years.
- Fair coin: every 2^{10^8-10} hours (age of the universe: 13BY = 2^{46} hours).

Summary of Lecture 5

When "deterministic" does not mean "predictable"

- Physical determinism and indeterminism
- Deterministic systems can be unpredictable for limited precision and computational power
  - Complex systems, like the weather
  - Simple systems, like Lorenz model or logistic map
  - Even "closed" systems like many-planet motion
- Power laws: not everything is a fair coin

Suggested Readings

Mathematica codes used to generate Julia & Mandelbrot:
http://mathematica.stackexchange.com/questions/21714/why-is-this-mandelbrot-sets-implementation-infeasible-takes-a-massive-amount-o

Interview with Benoît Mandelbrot:
http://www.youtube.com/watch?v=Ehwy4Gq27uY

Wikipedia pages:
- http://en.wikipedia.org/wiki/Julia_set
- http://en.wikipedia.org/wiki/Mandelbrot_set
- http://en.wikipedia.org/wiki/Self_similarity
- http://en.wikipedia.org/wiki/Power_law