Parallel Multipole Methods

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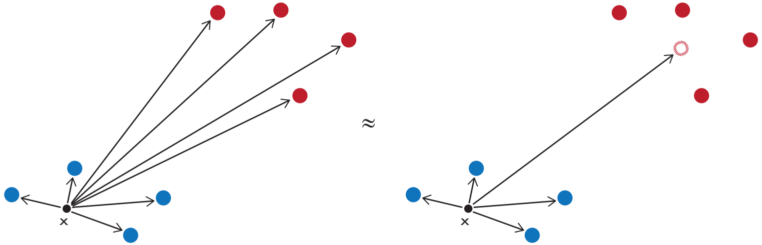
  - Take *aggregate* effect of far-field charges via **Taylor series expansion**



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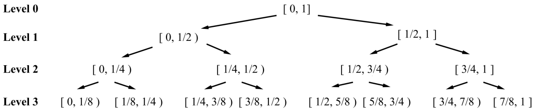

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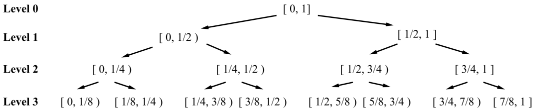

- For each $T_{\ell,k}$ cell, compute weights $w_{\ell,k,m} \longrightarrow$ total cost $O(N \log N)$

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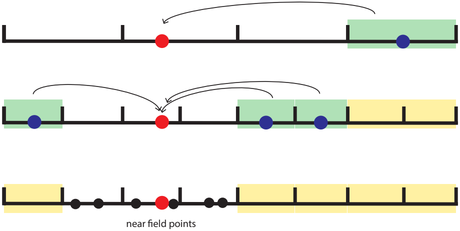

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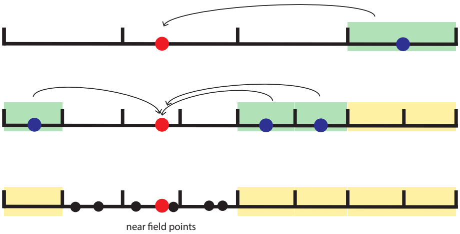

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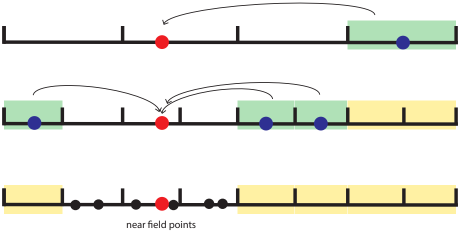

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- Total cost is $O(N \log N)$

32 points in finest box, p=5

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  - *Potential Problem:* Nested parallelism

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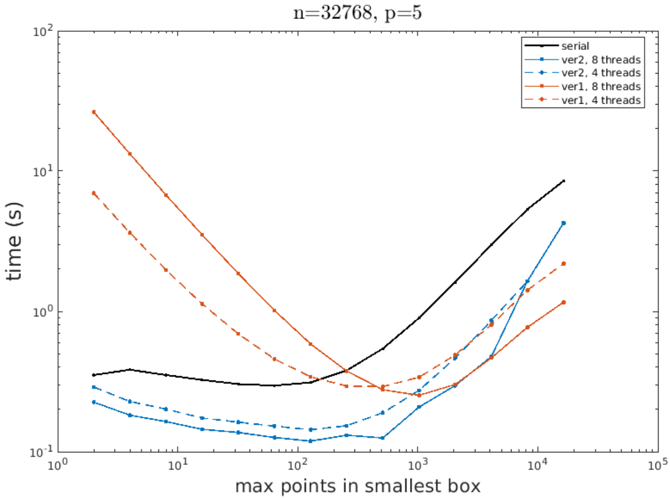

`crackle1`: Intel Xeon E5630 (2.53 GHz)

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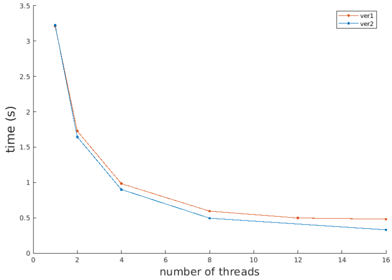
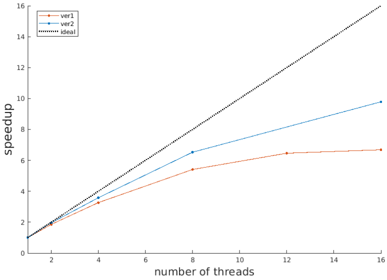
$n = 32768$, max points in box 2048, and $p = 5$





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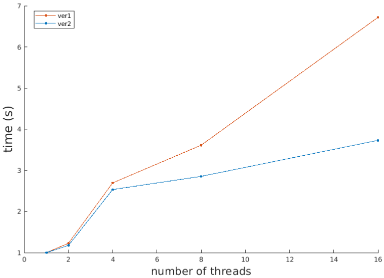
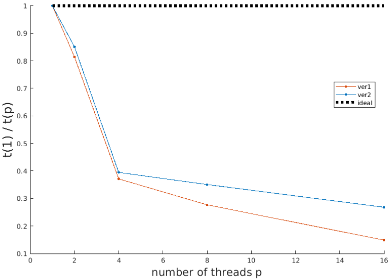
$n = 8192 \cdot r$, max points in box $= 2048$, and $p = 5$





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Supplemental Slides

Approximating far-field

- Use **Taylor series expansion** of $\phi(x - x_j)$ for small $\delta = \dfrac{x_j - x^*}{x^* - x}$:

$$\begin{aligned}
\phi(x_j - x) &= \phi\left((x^* - x)(1 + \delta)\right) = \phi(x^* - x)\phi(1 + \delta) \\
&\approx \phi(x^* - x)\left[\sum_{m=0}^{p} \frac{\phi^{(m)}(1)}{m!}\delta^m + O(\delta^{p+1})\right] \\
&= \sum_{m=0}^{p} a_m(x_j - x^*)S_m(x^* - x) + O(\delta^{p+1})
\end{aligned}$$

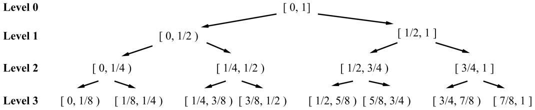
- Then potential from $x_j \in$ far-field is

$$\sum_{j \in \text{far-field}} q_j \phi(x - x_j) \approx \sum_{m=0}^{p}\left[\sum_{j \in \text{far-field}} q_j a_m(x_j - x^*)\right] S_m(x^* - x) + O(\delta^{p+1})$$

- Accuracy depends on choice of x^* and thus size of $\delta = \dfrac{x_j - x^*}{x^* - x}$

Tree Algorithm by Barnes and Hut, complexity $O(N \log N)$

- For N particles, partition $[0, 1]$ uniformly at $O(\log N)$ levels:



- Let $T_{\ell,k}$ be the cell at level ℓ with index $k = 1 : 2^\ell$ with center $x^*_{\ell,k}$.
- Compute weight at each cell, total cost $O(N \log N)$

$$w_{\ell,k,m} = \sum_{x_j \in T_{\ell,k}} q_j a_m(x_j - x^*_{\ell,k})$$

- For a point x: far-field components added at increasingly *coarse* levels

$$u(x) = \sum_{\ell=1}^{O(\log N)} \sum_{m=0}^{p} w_{\ell,k(\ell),m} S_m \left(x^*_{\ell,k(\ell)} - x \right)$$