Parallel Multipole Methods

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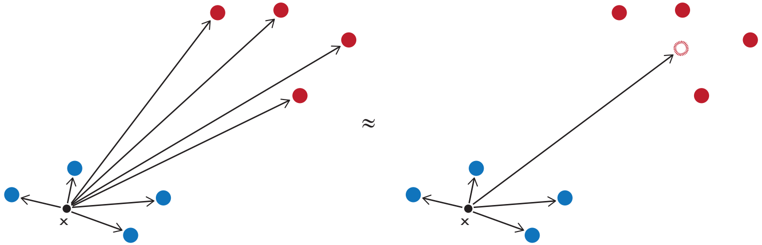
  - Take *aggregate* effect of far-field charges via **Taylor series expansion**



Barnes-Hut Tree Code: Taylor Expansion

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- Use **Taylor series expansion** of ϕ for small $\delta = \dfrac{x_j - x^*}{x^* - y}$:

$$\begin{aligned}
\phi(x_j - y) &= \phi(x^* - y)\phi(1 + \delta) \\
&\approx \phi(x^* - y)\left[\sum_{m=0}^{p} \frac{\phi^{(m)}(1)}{m!}\delta^m + O(\delta^{p+1})\right] \\
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- For a set of points x_j, potential contribution can be approximated:

$$\sum_{j\in\text{far-field}} q_j\phi(x_j - y) \approx \sum_{m=0}^{p} \underbrace{\left[\sum_{j\in\text{far-field}} q_j a_m(x_j - x^*)\right]}_{\text{weight } w} S_m(x^* - y) + O(\delta^{p+1})$$

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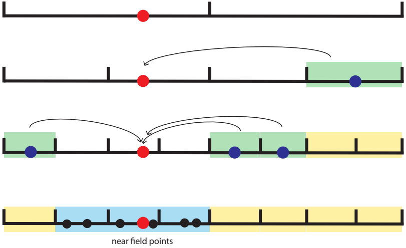

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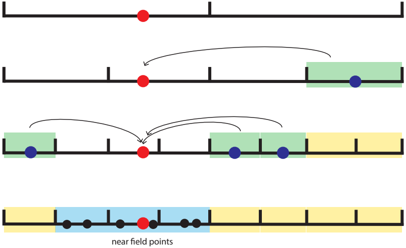

- Then for a point x, the potential is approximately

$$u(x) = \sum_{\ell=1}^{O(\log N)} \sum_{m=0}^{p} w_{\ell,k(\ell),m} S_m \left(x^*_{\ell,k(\ell)} - x \right) + O\left(2^{-p}\right)$$

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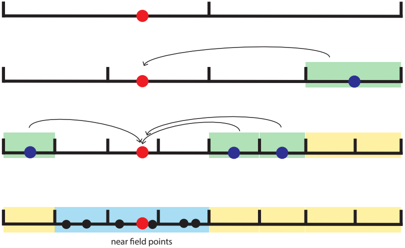

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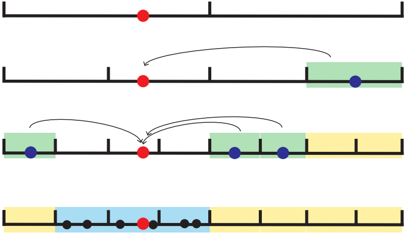

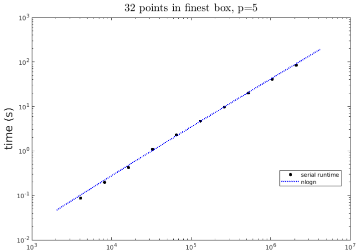

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  - *Potential Problem:* Nested parallelism

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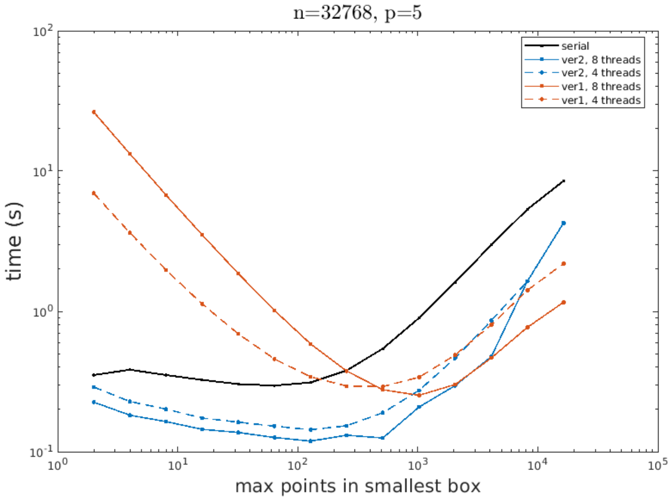

`crackle1`: Intel Xeon E5630 (2.53 GHz)

Results: Strong Scaling

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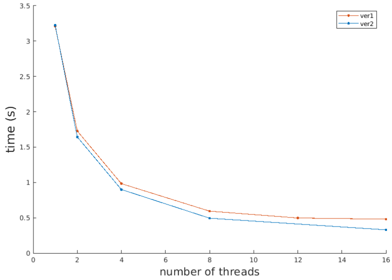
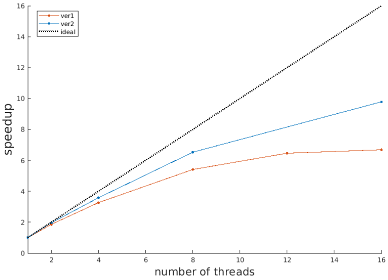
$n = 32768$, max points in box 2048, and $p = 5$





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Results: Weak Scaling

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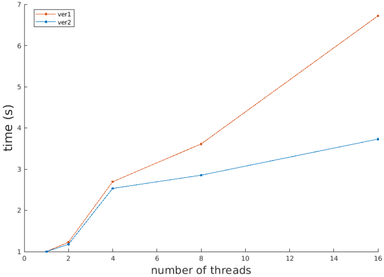
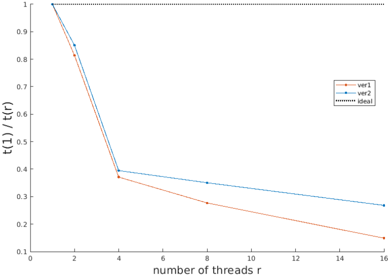
$n = 8192 \cdot r$, max points in box $= 2048$, and $p = 5$





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References

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