# Parallel Fast Multipole

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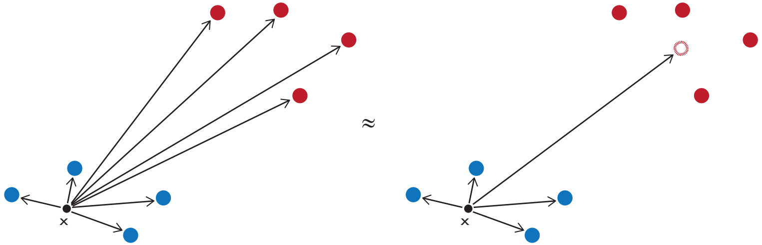
- Separate sum into **near-field** and **far-field** contributions
  - Take *aggregate* effect of far-field charges



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- Use **Taylor series expansion** of $\phi(x - x_j)$ for small $\delta = \dfrac{x_j - x^*}{x^* - x}$:

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\phi(x_j - x) &= \phi\left((x^* - x)\left(1 + \delta\right)\right) = \phi(x^* - x)\phi(1 + \delta) \\
&\approx \phi(x^* - x)\left[\sum_{k=0}^{p} \frac{\phi^{(k)}(1)}{k!}\delta^k + O(\delta^{p+1})\right] \\
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- Accuracy depends on choice of $x^*$ and thus size of $\delta = \dfrac{x_j - x^*}{x^* - x}$

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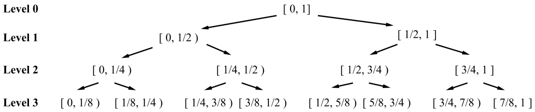

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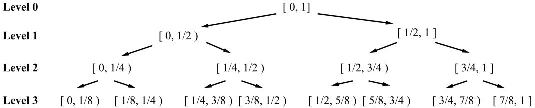


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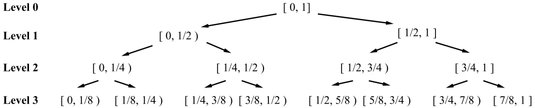


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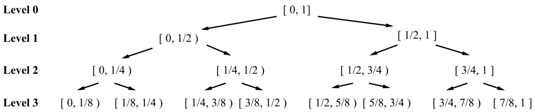


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- For a point $x$: far-field components added at increasingly *coarse* levels

$$u(x) = \sum_{\ell=1}^{O(\log N)} \sum_{k=0}^{p} w_{\ell,m(\ell),k} S_k \left( x^*_{\ell,m(\ell)} - x \right)$$

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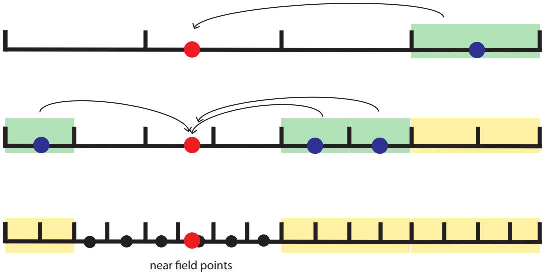

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    - *Challenge:* Minimizing communication between cores

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- What did we expect? Is there some unavoidable overhead?