



# Pre-Calculus B 120

## Curriculum

Implemented September 2013

New Brunswick Department of Education and  
Early Childhood Development  
2015

## Acknowledgements

The New Brunswick Department of Education and Early Childhood Development gratefully acknowledges the contributions of the following groups and individuals toward the development of the *New Brunswick Pre-Calculus B 120 Curriculum Guide*:

- The Western and Northern Canadian Protocol (WNCP) for Collaboration in Education: *The Common Curriculum Framework for Grade 10-12 Mathematics*, January 2008. *New Brunswick Pre-Calculus B 120* curriculum is based on Outcomes and Achievement Indicators from *WNCP Pre-Calculus Grade 12*. Reproduced (and/or adapted) by permission. All rights reserved.
- Newfoundland and Labrador Department of Education, Prince Edward Island Department of Education and Early Childhood Development.
- The NB High School Mathematics Curriculum Development Advisory Committee of Bev Amos, Roddie Duguay, Suzanne Gaskin, Nicole Giberson, Karen Glynn, Beverlee Gonzales, Ron Manuel, Stacey Leger, Jane Pearson, Elaine Sherrard, Alyssa Sankey (UNB), Mahin Salmani (UNB), Maureen Tingley (UNB), Guohua Yan (UNB).
- The NB Grade 12 Curriculum Development Writing Team of Carolyn Campbell, Mary Clarke, Gail Coates, Megan Crosby, Richard Cuming, Geordie Doak, Nancy Everett, Ryan Hachey, Nancy Hodnett, Wendy Hudon, Wendy Johnson, Julie Jones, Andrea Linton, Brad Lynch, Erin MacDougall, Sheridan Mawhinney, Chris McLaughlin, Nick Munn, Yvan Pelletier, Parise Plourde, Tony Smith, Anne Spinney, Glen Spurrell.
- The NB Grade 12 Limits Resource Development Committee of Lori Brophy, Carolyn Campbell, Steve MacMillan, Anne Spinney, and Glen Spurrell.
- Martha McClure, Learning Specialist, 9-12 Mathematics and Science, NB Department of Education
- The Mathematics Coordinators, Numeracy Leads, and Mathematics teachers of New Brunswick who provided invaluable input and feedback throughout the development and implementation of this document.

# Table of Contents

Curriculum Overview for Grades 10-12 Mathematics .....	1
<b>BACKGROUND AND RATIONALE</b> .....	1
<b>BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING</b> .....	2
<i>Goals for Mathematically Literate Students</i> .....	2
<i>Opportunities for Success</i> .....	3
<i>Diverse Cultural Perspectives</i> .....	3
<i>Adapting to the Needs of All Learners</i> .....	4
<i>Universal Design for Learning</i> .....	4
<i>Connections across the Curriculum</i> .....	4
<b>NATURE OF MATHEMATICS</b> .....	5
<i>Change</i> .....	5
<i>Constancy</i> .....	5
<i>Patterns</i> .....	6
<i>Relationships</i> .....	6
<i>Spatial Sense</i> .....	6
<i>Uncertainty</i> .....	7
<b>ASSESSMENT</b> .....	7
<b>CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS</b> .....	9
<i>Communication [C]</i> .....	10
<i>Problem Solving [PS]</i> .....	10
<i>Connections [CN]</i> .....	11
<i>Mental Mathematics and Estimation [ME]</i> .....	11
<i>Technology [T]</i> .....	12
<i>Visualization [V]</i> .....	12
<i>Reasoning [R]</i> .....	13
<b>ESSENTIAL GRADUATION LEARNINGS</b> .....	14
<b>PATHWAYS AND TOPICS</b> .....	15
<i>Goals of Pathways</i> .....	15
<i>Design of Pathways</i> .....	15
<i>Instructional Focus</i> .....	16
<b>SUMMARY</b> .....	17
<b>CURRICULUM DOCUMENT FORMAT</b> .....	18

## Specific Curriculum Outcomes ..... 19

### Relations and Functions .....20

- RF1: Analyze arithmetic sequences and series to solve problems. ....20
- RF2: Analyze geometric sequences and series to solve problems. ....24
- RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2  
(limited to polynomials of degree  $\leq 5$  with integral coefficients).....28
- RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ). ....31
- RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). ....35
- RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). ....39
- RF7: Demonstrate an understanding of operations on, and compositions of functions. ....43
- RF8: Assemble a function tool kit comparing various types of functions and compositions of them. ....51

### Permutations, Combinations and Binomial Theorem .....59

- PCB1: Apply the fundamental counting principle to solve problems. ....59
- PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems. ....62
- PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. ....66
- PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). ....70

### Limits .....74

- L1: Determine the limit of a function at a point both graphically and analytically. ....74
- L2: Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation. ....78
- L3: Analyze the continuity of a function. ....80
- L4: Explore limits which involve infinity .....82

## SUMMARY OF CURRICULUM OUTCOMES ..... 85

## REFERENCES ..... 86

### Limits Resource Supplement.....87

#### ***L1. Determine the limit of a function at a point both graphically and analytically. ....90***

- L1 Exercises .....91

- L1 Solutions .....93

#### ***L2. Explore one-sided limits graphically and analytically.....94***

- L2 Exercises .....96

- L2 Solutions .....97

#### ***L3. Analyze the continuity of a function.....99***

- L3 Exercises .....103

- L3 Solutions .....105

#### ***L4 Explore limits which involve infinity.....106***

- L4 Exercises .....111

- L4 Solutions .....113

# Curriculum Overview for Grades 10-12 Mathematics

## BACKGROUND AND RATIONALE

Mathematics curriculum is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society.

It is essential the mathematics curriculum reflects current research in mathematics instruction. To achieve this goal, *The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol* has been adopted as the basis for a revised mathematics curriculum in New Brunswick. The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators and others.

The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP and the NCTM.

There is an emphasis in the New Brunswick curriculum on particular key concepts at each Grade which will result in greater depth of understanding and ultimately stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early Grades to ensure students develop a solid foundation in numeracy.

The intent of this document is to clearly communicate high expectations for students in mathematics education to all education partners. Because of the emphasis placed on key concepts at each Grade level, time needs to be taken to ensure mastery of these concepts. *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM Principles and Standards, 2000).*

## **BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING**

The New Brunswick Mathematics Curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice. These beliefs include:

- mathematics learning is an active and constructive process;
- learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates;
- learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking and that nurtures positive attitudes and sustained effort; and
- learning is most effective when standards of expectation are made clear with on-going assessment and feedback.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and aspirations.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best developed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial and symbolic representations of mathematics. The learning environment should value, respect and address all students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

### ***Goals for Mathematically Literate Students***

The main goals of mathematics education are to prepare students to:

- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:

- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

### ***Opportunities for Success***

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success, and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

### ***Diverse Cultural Perspectives***

Students come from a diversity of cultures, have a diversity of experiences and attend schools in a variety of settings including urban, rural and isolated communities. To address the diversity of knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of students, a variety of teaching and assessment strategies are required in the classroom. These strategies must go beyond the incidental inclusion of topics and objects unique to a particular culture.

For many First Nations students, studies have shown a more holistic worldview of the environment in which they live (Banks and Banks 1993). This means that students look for connections and learn best when mathematics is contextualized and not taught as discrete components. Traditionally in Indigenous culture, learning takes place through active participation and little emphasis is placed on the written word. Oral communication along with practical applications and experiences are important to student learning and understanding. It is important that teachers understand and respond to both verbal and non-verbal cues to optimize student learning and mathematical understandings.

Instructional strategies appropriate for a given cultural or other group may not apply to all students from that group, and may apply to students beyond that group. Teaching for diversity will support higher achievement in mathematics for all students.

## ***Adapting to the Needs of All Learners***

Teachers must adapt instruction to accommodate differences in student development as they enter school and as they progress, but they must also avoid gender and cultural biases. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom. The reality of individual student differences must not be ignored when making instructional decisions.

As well, teachers must understand and design instruction to accommodate differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Designing classroom activities to support a variety of learning styles must also be reflected in assessment strategies.

## ***Universal Design for Learning***

The New Brunswick Department of Education and Early Childhood Development's definition of inclusion states that every child has the right to expect that his or her learning outcomes, instruction, assessment, interventions, accommodations, modifications, supports, adaptations, additional resources and learning environment will be designed to respect his or her learning style, needs and strengths.

Universal Design for Learning is a "...framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged." It also "...reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient" (CAST, 2011).

In an effort to build on the established practice of differentiation in education, the Department of Education and Early Childhood Development supports *Universal Design for Learning* for all students. New Brunswick curricula are created with universal design for learning principles in mind. Outcomes are written so that students may access and represent their learning in a variety of ways, through a variety of modes. Three tenets of universal design inform the design of this curriculum. Teachers are encouraged to follow these principles as they plan and evaluate learning experiences for their students:

- **Multiple means of representation:** provide diverse learners options for acquiring information and knowledge
- **Multiple means of action and expression:** provide learners options for demonstrating what they know
- **Multiple means of engagement:** tap into learners' interests, offer appropriate challenges, and increase motivation

For further information on *Universal Design for Learning*, view online information at <http://www.cast.org/>.

## ***Connections across the Curriculum***

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, and physical education.



## NATURE OF MATHEMATICS

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this document. These components include: **change**, **constancy**, **number sense**, **patterns**, **relationships**, **spatial sense** and **uncertainty**.

### *Change*

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, ... can be described as:

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Students need to learn that new concepts of mathematics as well as changes to already learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

### *Constancy*

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include:

- the area of a rectangular region is the same regardless of the methods used to determine the solution
- the sum of the interior angles of any triangle is  $180^\circ$
- the theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Many important properties in mathematics do not change when conditions change. Examples of constancy include:

- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- the theoretical probability of an event.

## ***Number Sense***

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p. 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. Evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing mathematically rich tasks that allow students to make connections.

## ***Patterns***

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

## ***Relationships***

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

## ***Spatial Sense***

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations

### ***Uncertainty***

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

## **ASSESSMENT**

Ongoing, interactive assessment (*formative assessment*) is essential to effective teaching and learning. Research has shown that formative assessment practices produce significant and often substantial learning gains, close achievement gaps and build students' ability to learn new skills (Black & William, 1998, OECD, 2006). Student involvement in assessment promotes learning. Interactive assessment, and encouraging self-assessment, allows students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes:

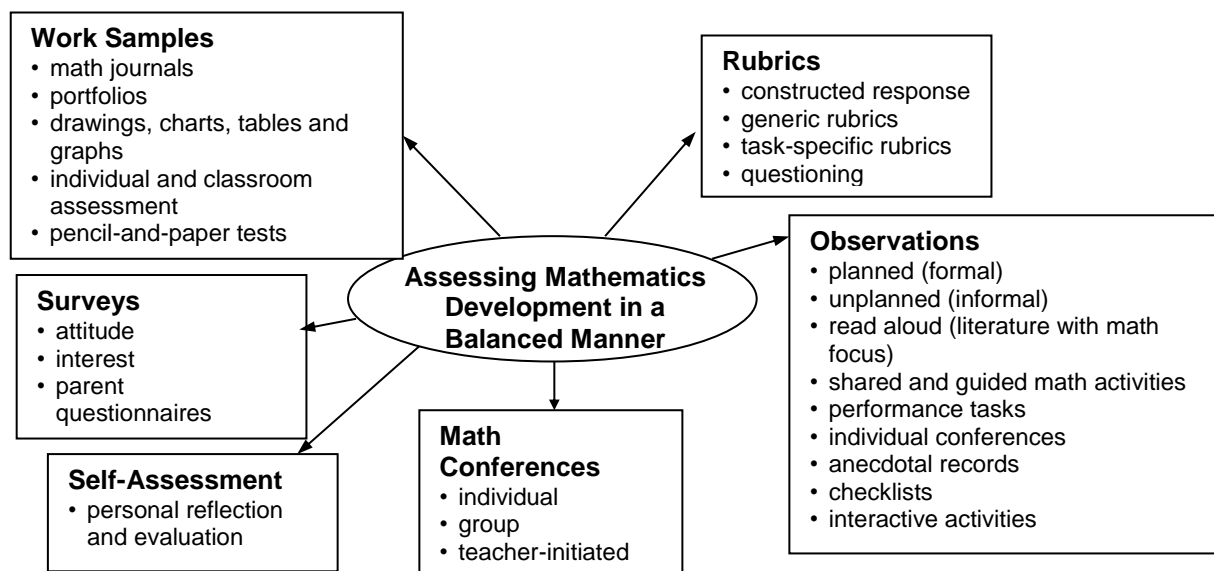
- providing clear goals, targets and learning outcomes
- using exemplars, rubrics and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies, 2000)

Formative assessment practices act as the scaffolding for learning which, only then, can be measured through summative assessment. *Summative assessment*, or assessment of learning, tracks student progress, informs instructional programming and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning and produce achievement gains.

Student assessment should:

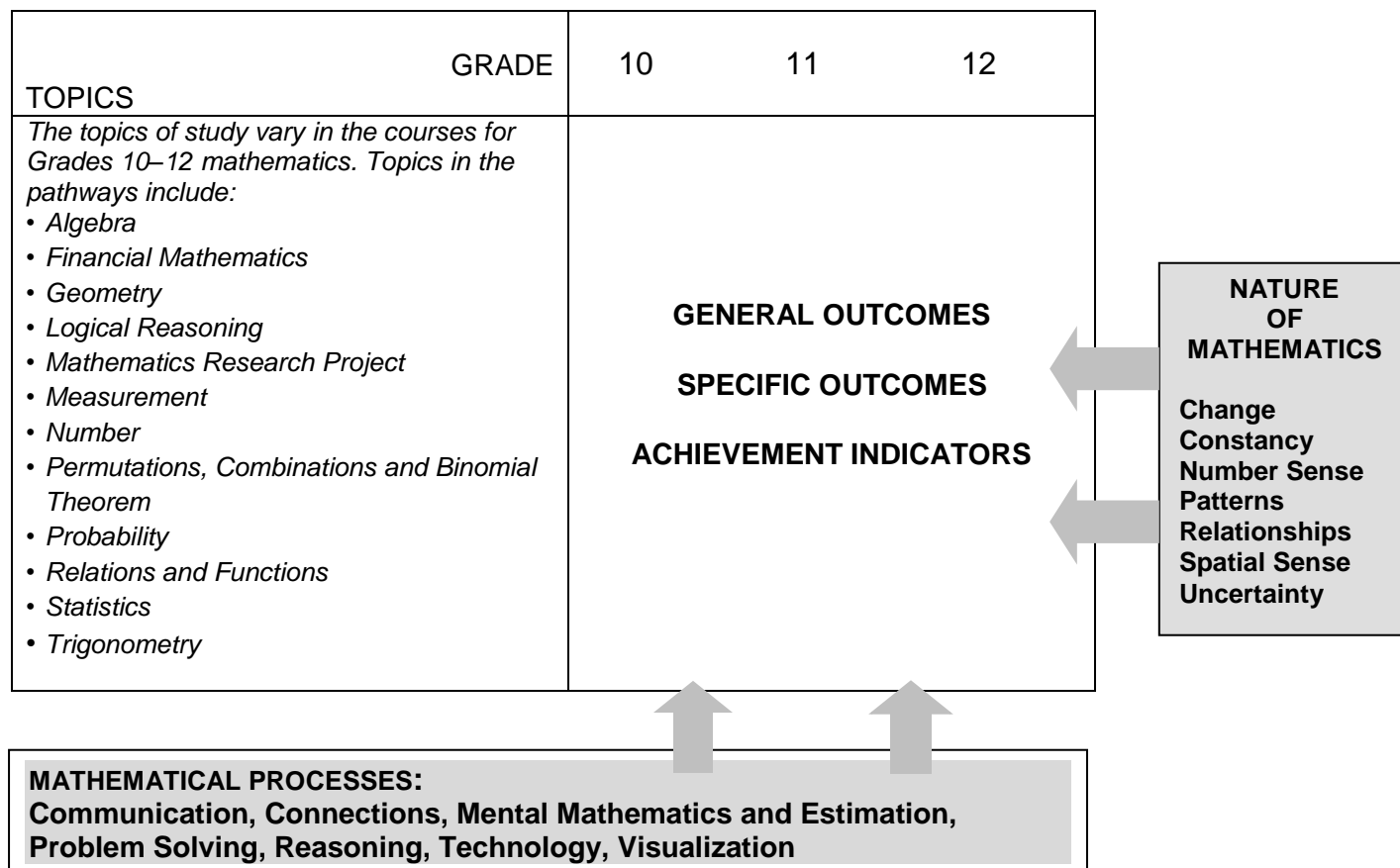
- align with curriculum outcomes
- use clear and helpful criteria
- promote student involvement in learning mathematics during and after the assessment experience
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction

(adapted from: NCTM, Mathematics Assessment: A practical handbook, 2001, p.22)



## CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



## MATHEMATICAL PROCESSES

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to:

- communicate in order to learn and express their understanding of mathematics (Communications: C)
- develop and apply new mathematical knowledge through problem solving (Problem Solving: PS)
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines (Connections: CN)
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation: ME)
- select and use technologies as tools for learning and solving problems (Technology: T)
- develop visualization skills to assist in processing information, making connections and solving problems (Visualization: V).
- develop mathematical reasoning (Reasoning: R)

The New Brunswick Curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

### ***Communication [C]***

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, knowledge, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication can help students make connections among concrete, pictorial, symbolic, verbal, written and mental representations of mathematical ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

### ***Problem Solving [PS]***

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all Grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type, *How would you...?* or *How could you ...?*, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learnings in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly look for, and engage in, finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

### **Connections [CN]**

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences, and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

*“Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding... Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching”* (Caine and Caine, 1991, p. 5).

### **Mental Mathematics and Estimation [ME]**

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

*“Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental mathematics”* (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics *“become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving”* (Rubenstein, 2001).

Mental mathematics *“provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers”* (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

### ***Technology [T]***

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Calculators and computers can be used to:

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense.

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all Grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

### ***Visualization [V]***

Visualization “involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world” (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills.

Measurement sense includes the ability to determine when to measure and when to estimate and involves knowledge of several estimation strategies (Shaw and Cliatt, 1989, p. 150).



Visualization is fostered through the use of concrete materials, technology and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence and fluency.

### ***Reasoning [R]***

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking.

Questions that challenge students to think, analyze and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, *Why do you believe that's true/correct?* or *What would happen if ....*

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

## ESSENTIAL GRADUATION LEARNINGS

Graduates from the public schools of Atlantic Canada will be able to demonstrate knowledge, skills, and attitudes in the following essential graduation learnings. These learnings are supported through the outcomes described in this curriculum document.

### ***Aesthetic Expression***

Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

### ***Citizenship***

Graduates will be able to assess social, cultural, economic, and environmental interdependence in a local and global context.

### ***Communication***

Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) as well as mathematical and scientific concepts and symbols to think, learn, and communicate effectively.

### ***Personal Development***

Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

### ***Problem Solving***

Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language, mathematical, and scientific concepts.

### ***Technological Competence***

Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems

## PATHWAYS AND TOPICS

*The Common Curriculum Framework for Grades 10–12 Mathematics* on which the New Brunswick Grades 10-12 Mathematics curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics*. In New Brunswick all Grade 10 students share a common curriculum covered in two courses: *Geometry, Measurement and Finance 10* and *Number, Relations and Functions 10*. Starting in Grade 11, three pathways are available: *Finance and Workplace*, *Foundations of Mathematics*, and *Pre-Calculus*.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. Students are encouraged to cross pathways to follow their interests and to keep their options open. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

### **Goals of Pathways**

The goals of all three pathways are to provide prerequisite attitudes, knowledge, skills and understandings for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

### **Design of Pathways**

Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of each pathway has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* and on consultations with mathematics teachers.

### **Financial and Workplace Mathematics**

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, statistics and probability.

### **Foundations of Mathematics**

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, statistics and probability.

## **Pre-calculus**

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Students develop of a function tool kit including quadratic, polynomial, absolute value, radical, rational, exponential, logarithmic and trigonometric functions. They also explore systems of equations and inequalities, degrees and radians, the unit circle, identities, limits, derivatives of functions and their applications, and integrals.

## ***Outcomes and Achievement Indicators***

The New Brunswick Curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes and achievement indicators.

**General Curriculum Outcomes** (GCO) are overarching statements about what students are expected to learn in each strand/sub-strand. The general curriculum outcome for each strand/sub-strand is the same throughout the pathway.

**Specific Curriculum Outcomes** (SCO) are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as required for a given Grade.

**Achievement indicators** are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome. In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for clarification and are not requirements that must be addressed to fully meet the learning outcome. The word *and* used in an outcome indicates that both ideas must be addressed to fully meet the learning outcome, although not necessarily at the same time or in the same question.

## ***Instructional Focus***

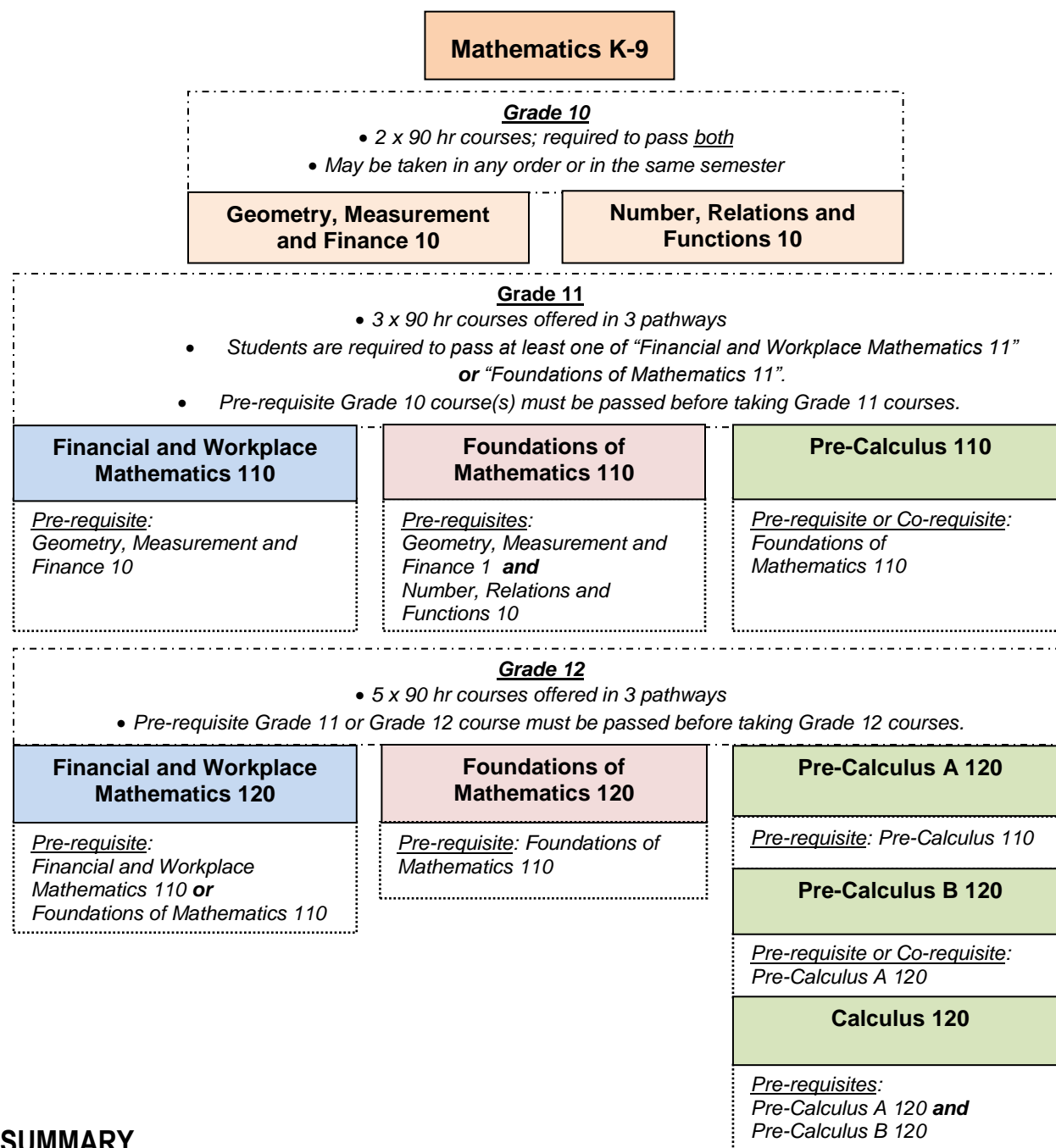
Each pathway in *The Common Curriculum Framework for Grades 10–12 Mathematics* is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful. Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches, and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning, assessment as learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

## Pathways and Courses

The graphic below summarizes the pathways and courses offered.



## SUMMARY

The Conceptual Framework for Grades 10–12 Mathematics describes the nature of mathematics, the mathematical processes, the pathways and topics, and the role of outcomes and achievement indicators in grades 10–12 mathematics. Activities that take place in the mathematics classroom should be based on a problem-solving approach that incorporates the mathematical processes and leads students to an understanding of the nature of mathematics.

## CURRICULUM DOCUMENT FORMAT

This guide presents the mathematics curriculum by Grade level so that a teacher may readily view the scope of the outcomes which students are expected to meet during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings at a particular Grade level are part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The heading of each page gives the General Curriculum Outcome (GCO), and Specific Curriculum Outcome (SCO). The key for the mathematical processes follows. A Scope and Sequence is then provided which relates the SCO to previous and next Grade SCO's. For each SCO, Elaboration, Achievement Indicators, Suggested Instructional Strategies, and Suggested Activities for Instruction and Assessment are provided. For each section, the *Guiding Questions* should be considered.

GCO: General Curriculum Outcome SCO: Specific Curriculum Outcome		
Mathematical Processes		
[C] Communication Technology	[PS] Problem Solving [V] Visualization	[CN] Connections [R] Reasoning
[ME] Mental Math [T] and Estimation		
<b>Scope and Sequence</b>		
Previous Grade or Course SCO's	Current Grade SCO	Following Grade or Course SCO's
<b>Elaboration</b>		
Describes the "big ideas" to be learned and how they relate to work in previous Grades		
<b>Guiding Questions:</b>		
<ul style="list-style-type: none"> <li>• <i>What do I want my students to learn?</i></li> <li>• <i>What do I want my students to understand and be able to do?</i></li> </ul>		
<b>Achievement Indicators</b>		
Describes observable indicators of whether students have met the specific outcome		
<b>Guiding Questions:</b>		
<ul style="list-style-type: none"> <li>• <i>What evidence will I look for to know that learning has occurred?</i></li> <li>• <i>What should students demonstrate to show their understanding of the mathematical concepts and skills?</i></li> </ul>		

GCO: General Curriculum Outcome SCO: Specific Curriculum Outcome
<b>Suggested Instructional Strategies</b>
General approach and strategies suggested for teaching this outcome
<b>Guiding Questions</b>
<ul style="list-style-type: none"> <li>• <i>What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?</i></li> <li>• <i>What teaching strategies and resources should I use?</i></li> <li>• <i>How will I meet the diverse learning needs of my students?</i></li> </ul>
<b>Suggested Activities for Instruction and Assessment</b>
Some suggestions of specific activities and questions that can be used for both instruction and assessment.
<b>Guiding Questions</b>
<ul style="list-style-type: none"> <li>• <i>What are the most appropriate methods and activities for assessing student learning?</i></li> <li>• <i>How will I align my assessment strategies with my teaching strategies?</i></li> </ul>
<b>Guiding Questions</b>
<ul style="list-style-type: none"> <li>• <i>What conclusions can be made from assessment information?</i></li> <li>• <i>How effective have instructional approaches been?</i></li> <li>• <i>What are the next steps in instruction?</i></li> </ul>

# *Pre-Calculus B 120*

## *Specific Curriculum Outcomes*

SCO: RF1: Analyze arithmetic sequences and series to solve problems. [CN, PS, R, T]			
[C] Communication [T] Technology	[PS] Problem Solving [V] Visualization	[CN] Connections [R] Reasoning	[ME] Mental Math and Estimation

## Relations and Functions

RF1: Analyze arithmetic sequences and series to solve problems.

### Scope and Sequence of Outcomes:

Grade 10	Pre-Calculus B 120
<b>RF4:</b> Describe and represent linear relations, using words; ordered pairs; tables of values; graphs; and equations. (NRF10) <b>RF5:</b> Determine the characteristics of the graphs of linear relations, including the: intercepts, slope, domain and range. (NRF10)	<b>RF1:</b> Analyze arithmetic sequences and series to solve problems.

### ELABORATION

Students have previously studied linear relations and this outcome connects that learning to **arithmetic sequences**, a sequence in which the difference between consecutive terms is constant. It is important for students to recognize that the graphs of sequences are discrete rather than continuous, and that the first elements of the ordered pairs are restricted to the natural numbers ( $N$ ).

Students will examine sequences to develop an algorithm for describing them. They will apply this generalization (e.g.,  $t_n = t_1 + (n - 1)d, n \in N$ ) for arithmetic sequences.

$t_n$  is the **general term** or the  $n^{th}$  term of the sequence

$t_1$  the first term of the sequence,  $t_2$  the second term,  $t_3$  the third term, etc.

$n$  term number (or number of terms)

$d$  **common difference** is the difference between successive terms in an arithmetic sequence. For example;  $t_{10} = t_1 + 9d$

When the terms of an arithmetic sequence are re-expressed as a sum, the resulting expression e.g., is called an **arithmetic series**. Students will express these series in this expanded form, and also using **sigma notation**:

For example,  $5 + 10 + 15 + 20 + 25 + 30$  is an **arithmetic series**, expressed in **sigma notation** as:  $\sum_{k=1}^6 5k$ .

It is important to note the effect of brackets.

For example, for  $\sum_{k=1}^6 5k + 1$  the 1 is added at the end of the series.

$$\sum_{k=1}^6 5k + 1 = 5 + 10 + 15 + 20 + 25 + 30 + 1 = 106.$$

However, if brackets are included, the sum is different.

$$\sum_{k=1}^6 (5k + 1) = 6 + 11 + 16 + 21 + 26 + 31 = 111.$$

Students will derive and apply algorithms for determining the sums of arithmetic series. The formulas  $S_n = \frac{n}{2} [2t_1 + (n - 1)d]$  or  $S_n = \frac{n}{2} [t_1 + t_n]$  will be developed where  $S_n$  is the sum of the first  $n$  terms, and  $n \in N$ . Students will use the formulas they have derived to solve problems that involve arithmetic sequences and series. *Note: In many resources  $t_1 = a$ .*



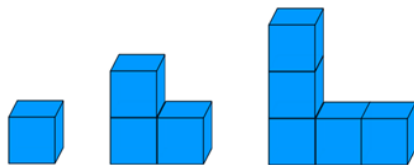
**SCO: RF1: Analyze arithmetic sequences and series to solve problems.** [CN, PS, R, T]

### **ACHIEVEMENT INDICATORS**

- Define an arithmetic sequence and arithmetic series.
- Provide and justify an example of an arithmetic sequence.
- Derive a rule for determining the general term of an arithmetic sequence.
- Describe the relationship between arithmetic sequences and linear functions.
- Determine  $t_1, d, n$ , or  $t_n$  in a problem that involves an arithmetic sequence.
- Use sigma notation to define the sum of  $n$  terms of an arithmetic series.
- Derive and use a rule for determining the sum of  $n$  terms of an arithmetic series.
- Use sigma notation to define the sum of  $n$  terms of an arithmetic series.
- Determine  $t_1, d, n$ , or  $S_n$  in a problem that involves an arithmetic series.
- Solve problems that involve arithmetic sequences or series.

### **Suggested Instructional Strategies**

- Students should plot the ordered pairs  $(n, t_n)$  of various arithmetic sequences to discover that the discrete points appear linear. Ask students to relate the slope of the perceived line to the common difference of the arithmetic sequence.
- Have students use multi-link cubes to build various shape progressions where the number of blocks used would form an arithmetic sequence. Students should then determine the general term, the common difference, and how many blocks would be needed for the 10<sup>th</sup> progression. Also, have students calculate terms of a sequence that keeps track of the sum of blocks used ( $n \leq 10$ ), as an illustration that not all sequences are arithmetic series.

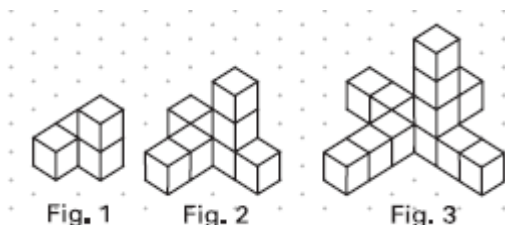


- Have students use technology (graphing calculators, excel, autograph etc.) to determine terms in a sequence.

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Act** List the numbers from 1 to 100 (1, 2, 3, 4, ..., 97, 98, 99, 100) on the board and have the students find the sum. (This is a lead-in to deriving the  $S_n$  formula and some students may find the solution this way without formalizing what they are doing. See p.22 in the Pre-Calculus 11 text to link this exercise to Gauss)

**Act** Ask students to build a sequence of figures using interlocking cubes like the shapes shown below:



Have students describe the pattern using ordered pairs, and graph the sequence. e.g., for this sequence students should make the connection that the common difference of 5 is the slope of the perceived line.

Some students will develop well-defined algorithms which lead to non-arithmetic sequences (e.g. number of blocks require to build squares or cubes with side lengths 1, 2, 3, ...). Have students plot points  $(n, t_n)$ , and help them understand that arithmetic sequences are those that give points on a straight line. Other sequences are not arithmetic.

**Act** Explain why the sum of the first  $n$  natural numbers  $1 + 2 + 3 + 4 + \dots + n$  is given by

$$\frac{n(n+1)}{2}. \text{ Write this in sigma notation and then use this to help you find } \sum_{k=1}^{50} 5k.$$

**Q** For each of the following arithmetic sequences:

- (i) write the next three terms.
- (ii) find the general term.
- (iii) find  $t_{47}$ .

- a) 2, 5, 8, ...      b) 7, 5, 3, ...      c)  $a, a + 2b, a + 4b, \dots$

*Answer:* a) i) 11, 14, 17    ii)  $t_n = 3n - 1$     iii)  $t_{47} = 146$   
 b) i) 1, -1, -3    ii)  $t_n = -2n + 9$     iii)  $t_{47} = 85$   
 c) i)  $a + 6b, a + 8b, a + 10b$     ii)  $t_n = 2bn + a - 2b$     iii)  $t_{47} = a + 92b$

**Q** Determine the first term, the common difference and the general term for each of the following arithmetic sequences.

- a)  $t_{10} = 29$  and  $t_{14} = 41$       b)  $t_9 = -6$  and  $t_{12} = -12$

*Answer:* a)  $d = 3$      $a = 2$      $t_n = 3n - 1$     b)  $d = -2$      $a = 10$      $t_n = -2n + 12$

**Q** How many terms are in each of the following arithmetic sequences?

- a) 3, 7, 11, ..., 39      b)  $5a - 3b, 4a - 2b, 3a - b, \dots, -5a + 7b$

*Answer:* a)  $39 = 3 + 4(n - 1)$      $n = 10$     b)  $-5a + 7b = 5a - 3b + (-a + b)(n - 1)$      $n = 11$

**SCO: RF1: Analyze arithmetic sequences and series to solve problems.** [CN, PS, R, T]

**Q** Expand and evaluate the following:  $\sum_{k=1}^5 (3k - 2)$

*Answer: 25*

**Q** Write the series,  $12 + 5 - 2 - 9 - 16 - 23$  using sigma notation.

*Answer:*

$$\sum_{n=1}^6 (-7n + 19)$$

**Q** How many terms are in the arithmetic series  $3 + 8 + 13 + \dots + 248$ ? Evaluate the sum.

$$\begin{aligned} \text{Answer: } t_1 &= 3 \quad d = 5, \quad t_n = t_1 + (n-1)(d) \\ 5n - 2 &= 248 \quad n = 50 \quad \therefore 248 \text{ is the } 50^{\text{th}} \text{ term} \\ S_{50} &= \frac{50}{2}[2(3) + (50-1)(5)] = 1375 \end{aligned}$$

**Q** Using examples, show that

$$\text{a) } \sum_{k=1}^n 5(k+1) = 5 \sum_{k=1}^n (k+1) \qquad \text{b) } \sum_{k=m}^n 7k = \sum_{k=1}^n 7k - \sum_{k=1}^{m-1} 7k$$

*Answer: a) let  $n = 4$   $LS = RS = 70$       b) Let  $m = 2$  and  $n = 4$   $LS = RS = 63$*

**Q** In a movie theatre there are 10 seats in the first row. The next 6 rows increase by 2 seats each and the remaining rows increase by 3 seats. If there are 11 rows in the theatre, how many people can be seated at one time?

$$\begin{aligned} \text{Answer: } t_1 \rightarrow t_7 \quad d &= 2 \quad S_7 = \frac{7}{2}[2(10) + (7-1)(2)] = 112 \quad t_7 = 10 + (7-1)(2) = 22 \\ t_7 \rightarrow t_{11} \quad d &= 3 \quad t_8 \rightarrow t_{11} = 25, 28, 31, 34 \quad S_{8 \rightarrow 11} = 118 \\ \text{Total \# seats} &= 112 + 118 = 230 \text{ seats} \end{aligned}$$

**Q** The Mill in Doaktown has a pile of logs with 25 in the bottom row, 24 logs in the next row, 23 logs in the next row, and so on. If there are 12 logs in the top row, how many logs are there in the pile?

$$\text{Answer: } a = 25 \quad d = -1 \quad t_n = 25 - (n-1) \quad n = 14 \quad S_{14} = \frac{14}{2}(25 + 12) = 259$$

**SCO: RF2: Analyze geometric sequences and series to solve problems.** [PS, R, T]

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

**RF2: Analyze geometric sequences and series to solve problems.**

**Scope and Sequence of Outcomes:**

Pre-Calculus A 120	Pre-Calculus B 120
<b>RF9:</b> Graph and analyze exponential and logarithmic functions. <b>RF11:</b> Solve problems that involve exponential and logarithmic equations.	<b>RF2:</b> Analyze geometric sequences and series to solve problems.

**ELABORATION**

In *Pre-Calculus A 120*, students have studied exponential growth and decay. Students will now connect exponential relationships with **geometric sequences**, in which the ratio between successive terms is a constant. It is important for students to recognize that the graphs of sequences are discrete, rather than continuous and that the first elements of the ordered pairs are restricted to the natural numbers ( $N$ ).

Students will examine sequences to develop algorithms for describing them. They will apply this generalization,  $t_n = t_1 r^{n-1}$ ,  $n \in N$  for geometric sequences.

$t_n$  – is the general term or the  $n$ th term of the sequence

$t_1$  – the first term of the sequence,  $t_2$  – second term, etc. ( $a$  is used to represent  $t_1$  in many textbooks)

$n$  – term number (or number of terms)

**$r$  – common ratio** is the ratio between successive terms in an geometric sequence and can be determined as:  $r = \frac{t_n}{t_{n-1}}$

It is important to highlight that geometric sequences with negative  $r$  values are not exponential in nature, but rather oscillating functions.

When the terms of a geometric sequence are re-expressed as a sum, the resulting expression is called a **geometric series** in expanded form. Students will express these series in expanded form and also using **sigma notation**:

For example, the geometric series  $5 + 25 + 125 + 625$  can be expressed as:

$$S_4 = \sum_{k=1}^4 5 \cdot 5^{k-1} \text{ or } \sum_{k=1}^4 5^k.$$

Students will derive and apply algorithms for determining the sums of geometric series.

The formulas  $S_n = \frac{t_1(1-r^n)}{1-r}$  or  $S_n = \frac{t_1(r^n-1)}{r-1}$  will be developed where  $S_n$  is the sum of the first  $n$  terms, and  $n \in N$ . Students will use the formulas they have derived to solve problems that involve geometric sequences and series.

Students will determine whether or not an infinite geometric series is **convergent**. A series with an infinite number of terms, is called convergent if the sequence of partial sums approaches a fixed value. A series with an infinite number of terms, in which the

sequence of partial sums does not approach a fixed value is called **divergent**. The concept of limit will be explored later in this course.

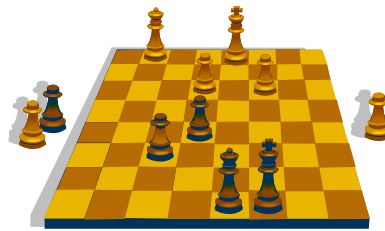
Once students are able to recognize when a geometric series is convergent ( $-1 < r < 1$  or  $|r| < 1$ ), they will use inductive reasoning to develop the algorithm  $s_{\infty} = \frac{t_1}{1-r}$ , which determines the sum of an infinite geometric series.

### **ACHIEVEMENT INDICATORS**

- Define a geometric sequence and geometric series.
- Provide and justify an example of a geometric sequence.
- Derive a rule for determining the general term of a geometric sequence.
- Determine  $t_1, r, n$  or  $t_n$  in a problem that involves a geometric sequence.
- Use sigma notation to define the sum of  $n$  terms of a geometric series.
- Derive a rule for determining the sum of  $n$  terms of a geometric series.
- Determine  $t_1, r, n$  or  $S_n$  in a problem that involves a geometric series.
- Determine whether a geometric series is convergent or divergent.
- Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.
- Solve problems that involve a geometric sequence or series.

### **Suggested Instructional Strategies**

- For a convergent series have the students use a motion sensor to measure the vertical distance (height up and down) of a bouncing ball to determine the total distance the ball travels before it comes to rest.
- To illustrate geometric series go to <http://www.singularitysymposium.com/exponential-growth.html> for the famous Egyptian legend of how the tradition of serving Paal Paysam to visiting pilgrims started after a game of chess between the local king and the lord Krishna. In this legend the king promises to pay lord Krishna “just a few grains of rice”. He is to pay a single grain of rice for the first chess square and double the number of grains each time for each square on the chess board - 64 squares in all. For the 64<sup>th</sup> square the king would have had to pay more than 18,000,000,000,000,000 grains of rice, about 210 *billion tons* or enough to cover India a metre deep with rice.



**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Act** Lead students in an investigation of the Koch Snowflake.

- An equilateral triangle, side length  $a$  is cut out of paper (Figure 1).
- Next, 3 equilateral triangles, each side measuring  $\frac{1}{3}a$  are cut out and added to the middle of each side of the first triangle (Figure 2).
- 12 equilateral triangles, each side measuring  $\frac{1}{9}a$ , are then added to the middle of the sides (Figure 3).
- Figure 4 shows the result of adding 48 equilateral triangles, each side measuring  $\frac{1}{27}a$ , to the previous figure.
- Assume that this procedure can be repeated indefinitely. Find the perimeter of each figure, and a general form to describe building this figure.



Figure 1



Figure 2



Figure 3



Figure 4

Answer: Fig. 1  $P_1 = 3a$  Fig. 2  $P_2 = 12\left(\frac{1}{3}a\right) = 4a = \frac{4}{3}P_1$

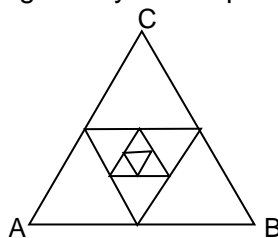
Fig. 3  $P_3 = 48\left(\frac{1}{9}a\right) = \frac{16}{3}a = \frac{4}{3}P_2$  Fig. 4  $P_4 = 4 \times 48\left(\frac{1}{27}a\right) = \frac{4}{3}P_3$

General form  $P_n = 3a\left(\frac{4}{3}\right)^{n-1}$

- Q** An initial investment of \$500 earns 2% interest each year, and the interest is reinvested. What is the value of the investment after 1 year? 2 years?  $n$  years?

Answer:  $t_n = 500(1.02)^{n-1}$   $t_1 = \text{initial amount}$   $t_2 = \$510$   $t_3 = \$520$

- Q** The midpoints of the sides of an equilateral triangle form new triangles. If  $AB$  is  $2m$ , find the total area of the triangles given by the sequence. If the pattern continues indefinitely, what is the area?



Answer: Area of  $\triangle ABC = \frac{1}{2}b \times h = \sqrt{3}$

Geometric Series  $\sqrt{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} + \dots$   $r = \frac{1}{4}$   $S_\infty = \frac{t_1}{1-r} = \frac{2\sqrt{3}}{3}$

- Q** Find  $t_7$  for the following geometric sequence:  $\frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$

Answer:  $t_1 = \frac{2}{3}$   $r = \frac{1}{3}$   $t_n = t_1(r)^{n-1}$   $t_7 = \frac{2}{3}\left(\frac{1}{3}\right)^{7-1} = \frac{2}{3^7}$

**Q** Find the number of terms in the following geometric sequence:  $16, -8, 4, \dots, \frac{1}{4}$

*Answer:*  $a = 16 \quad r = -\frac{1}{2} \quad t_{last} = \frac{1}{4} \quad \therefore \frac{1}{4} = 16\left(-\frac{1}{2}\right)^{n-1} \quad \frac{1}{2^6} = \frac{(-1)^{n-1}}{2^{n-1}} \quad n = 7$

**Q** Find the missing term for the geometric sequence that has:  $t_3 = \frac{1}{9}, t_7 = 9, t_4 = ?$

*Answer:*  $t_3 = \frac{1}{9} \quad t_3 \times r^4 = t_7 = 9 \quad \therefore r = 3 \quad t_4 = \frac{1}{9} \times 3 = \frac{1}{3}$

**Q** Write the series  $3 - 6 + 12 - 24 + 48 - 96 + 192$  using sigma notation.

*Answer:*  $r = -2 \quad t_1 = 3 \quad \sum_{n=1}^7 [3(-2)^{n-1}]$

**Q** The sum of the first five terms of a geometric series is 186 and the sum of the first six terms is 378. If the fourth term is 48, find:

- a) the first term
- b) the common ratio
- c) the tenth term
- d) the sum of the first ten terms

*Answer:*

$t_n = t_1(r)^{n-1} \quad S_6 - S_5 = t_6 = 192 = t_1 r^5 \quad t_4 = 48 = t_1 r^3 \quad r^2 = \frac{192}{48} = 4 \quad \therefore r = 2$

a)  $t_1 = 6$     b)  $r = 2$     c) 3072    d) 6138

**Q** a)  $\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \dots\right\}$     b)  $\left\{2, \frac{20}{9}, \frac{200}{81}, \frac{2000}{729}, \dots\right\}$

- i. Determine if the the geometric sequence is divergent or convergent.
- ii. If the sequence is convergent, what value does  $t_n$  approach?
- iii. Express each sequence as a series and determine the sum, if possible.

*Answer:*

a)  $r = \frac{2}{3}$     Since  $\left|\frac{2}{3}\right| < 1$  the sequence is convergent and approaches 0.     $S_\infty = \frac{t_1}{1-r} = 6$

b)  $r = \frac{10}{9}$     Since  $\left|\frac{10}{9}\right| > 1$  the sequence is divergent.

**SCO: RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree  $\leq 5$  with integral coefficients).** [C, CN, ME]

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

**RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree  $\leq 5$  with integral coefficients).**

### Scope and Sequence of Outcomes:

Pre-Calculus 110	Pre-Calculus B 120
<b>RF1:</b> Factor polynomial expressions of the form: <ul style="list-style-type: none"> <li><math>ax^2+bc+c, a \neq 0</math></li> <li><math>a^2x^2 - b^2y^2, a \neq 0, b \neq 0</math></li> <li><math>a(f(x))^2 + b(f(x))^2 + c, a \neq 0</math></li> <li><math>a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0</math></li> </ul> where $a, b$ and $c$ are rational numbers	<b>RF3:</b> Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree $\leq 5$ with integral coefficients).

### ELABORATION

This outcome extends understanding to allow students to explain connections between the methods they are learning and the underlying mathematics. Previously, students learned that the linear factors of a quadratic expression correspond to the zeros (roots) of the corresponding quadratic function. In this outcome they work with polynomials of degree  $\leq 5$ .

Various methods can be used to factor polynomials including **synthetic division**, the **factor theorem**, and the **integral zero theorem**. Finding the factors of polynomials allows students to sketch graphs of the polynomial functions.

Students are introduced to dividing polynomials  $P(x)$  by a linear binomial  $(x-a)$  using long division, to determine remainders and verify that the remainder  $R$  is the value of the polynomial at  $P(a)$ . **Synthetic division**, is a method of polynomial long division involving a binomial divisor that uses only the coefficient of the terms, and is introduced as an alternative to long division.

The **factor theorem** can be used for factoring polynomials with rational roots. It states that a polynomial  $P(x)$  has a factor  $(x-a)$  if and only if  $P(a) = 0$ .

The **integral zero theorem** states that if  $(x-a)$  is a factor of a polynomial function  $P(x)$  with integral coefficients, then  $a$  is a factor of the constant term of  $P(x)$ .

When the leading coefficient of a polynomial is 1, the possible test values for a rational root,  $a$ , are the factors of the constant. For example:

For  $P(x) = x^3 - 2x^2 + 3x - 6$ , the possible test values for  $a$ , are:  $a = \pm 1, \pm 2, \pm 3, \pm 6$ . The value of  $a$  that makes  $P(a) = 0$  is  $a = +2$ , so  $(x-2)$  is a factor of  $P(x)$ .

If the quotient can be factored, the remaining factor(s) can be determined by using long division or synthetic division or other factoring methods.

When the leading coefficient of a polynomial does not equal 1, the possible test values for a rational root  $a$  are the factors of the constant, and the factors of the constant



**SCO: RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree  $\leq 5$  with integral coefficients). [C, CN, ME]**

divided by the factors of the leading coefficient. This is known as the Rational Root Theorem. For example:

For  $P(x) = 2x^4 + 3x^3 - x^2 - 3x - 1$  the possible test values for  $a$ , are:  $a = \pm 1, \pm \frac{1}{2}$ . The values of  $a$  that make  $P(a) = 0$  are  $a = -1, -1, -\frac{1}{2}, +1$ , so factors of  $P(x)$  are  $(x + 1)$ ,  $(x + 1)$ ,  $(x + \frac{1}{2})$  and  $(x - 1)$ . The leading factor  $c$  must be chosen so that the coefficient of  $x^4$  is 2. Therefore  $c = 2$ , and

$$P(x) = 2(x + 1)(x + 1)\left(x + \frac{1}{2}\right)(x - 1)$$

$$P(x) = (x + 1)(x + 1)(2)\left(x + \frac{1}{2}\right)(x - 1)$$

$$P(x) = (x + 1)(x + 1)(2x + 1)(x - 1)$$

Factoring by decomposition and the difference of perfect squares should be reviewed before developing knowledge of other specialized factoring techniques such as:

**sum & difference of cubes,**

$$a^3x^3 + d^3 = (ax + d)(a^2x^2 - adx + d^2) \text{ and } (a^3x^3 - d^3 = (ax - d)(a^2x^2 + adx + d^2);$$

**factoring by grouping,**

$$x^3 - 2x^2 - 16x + 32 = x^2(x - 2) - 16(x - 2) = (x - 2)(x^2 - 16) = (x - 2)(x - 4)(x + 4); \text{ ; and}$$

**factoring quartics as trinomials,**

$$x^4 - 5x^2 - 4 = (x^2)^2 - 5x^2 - 4.$$

### **ACHIEVEMENT INDICATORS**

- Factor polynomials using the specialized factoring techniques: sum & difference of cubes, factor by grouping, and factoring quartics as trinomials
- Explain how long division of a polynomial expression by a binomial expression of the form  $x - a, a \in I$ , is related to synthetic division.
- Divide a polynomial expression by a binomial expression of the form  $x - a, a \in I$ , using long division or synthetic division.
- Explain the relationship between the linear factors of a polynomial expression and the zeros of the corresponding polynomial function.
- Explain the relationship between the remainder when a polynomial expression is divided by  $x - a, a \in I$ , and the value of the polynomial expression at  $x = a$  (remainder theorem).
- Explain and apply the factor theorem to express a polynomial expression as a product of factors.
- Recognize when the specialized factoring methods are more efficient than factoring using the factor theorem and synthetic division.

**SCO: RF3: Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree  $\leq 5$  with integral coefficients). [C, CN, ME]**

### **Suggested Instructional Strategies**

- Using graphing software or graphing calculators, have students find the zeros of several polynomial functions (cubics, quartics, and quintics) and use these values to determine the linear factors of the corresponding polynomial. Students can multiply their factors to verify.
- Most students prefer using synthetic division over long division. However, to understand the method, and be able to recall it in future courses, students should be encouraged to continue to use long division. The mathematics behind synthetic division and its relationship to long divisions should be explained, along with the limitation of the synthetic method.

### **Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Act** Make a set of cards with factorable polynomials and a set with possible factors. Hand each student a card when they enter the classroom. Have them find their partner by matching the factor cards with the appropriate polynomial card. This will be their seat partner for the day. This could also be played as a Memory game or with a set of cards with possible zeros.

**Q** Explain why  $(x - 5)(2x + 3)(x + 1) = 0$  produces zeros at  $+5, -\frac{3}{2}, -1$ .

*Answer: If any one of the three factors equals 0, the product will be zero and the answer c.*

**Q** Expand:

- a)  $(x - 3)(x + 1)$       b)  $(x + 1)(x - 1)(2x - 4)$

*Answers: a)  $x^2 - 2x - 3$       b)  $2x^3 - 4x^2 - 2x + 4$*

**Q** Factor:

- a)  $2x^2 + 4x - 6$       b)  $16x^2 - 81$       c)  $x^3 + 2x^2 - x - 2$

*Answers: a)  $2(x - 1)(x + 3)$       b)  $(4x - 9)(4x + 9)$       c)  $(2x^2 - 1)(x^2 + 3)$*

**Q** Factor the following:

- a)  $8x^3 - 125$       b)  $x^3 - 2x^2 - 16x + 32$       c)  $2x^4 + 5x^2 - 3$   
 d)  $8x^5 - 40x^4 + 32x^3 - x^2 + 5x - 4$       e)  $64x^3 + 125$       f)  $3x^4 - 192x$   
 g)  $4x^4 - 37x^2 + 9$       h)  $x^5 - 5x^4 - 10x^3 + 50x^2 + 9x - 45$

*Answers: a)  $(2x - 5)(4x^2 + 10x + 25)$       b)  $(x - 4)(x + 4)(x - 2)$*

*c)  $(2x^2 - 1)(x^2 + 3)$       d)  $(2x - 1)(4x^2 + 2x + 1)(x - 4)(x - 1)$*

*e)  $(4x + 5)(16x^2 - 20x + 25)$       f)  $3x(x - 4)(x^2 + 4x + 16)$*

*g)  $(2x - 1)(2x + 1)(x - 3)(x + 3)$       h)  $(x - 3)(x + 3)(x - 1)(x + 1)(x - 5)$*

**Q** What is the remainder when  $f(x) = x^3 + 2x^2 - 5x + 4$  is divided by  $(x - 2)$ , and how is it related to  $f(2)$ ?

*Answer: The remainder is 10 which equals  $f(2)$ .*

**Q** Use the factor theorem to factor the following expression completely:

$x^3 + 2x^2 - 5x - 6$ .

*Answer:  $(x - 2)(x + 3)(x + 1)$*

**SCO: RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ). [C, CN, T, V]**

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math and Estimation

**RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ )**

**Scope and Sequence of Outcomes:**

Foundations 110/ Pre-Calculus 110	Pre-Calc A 120	Pre-Calc B 120
<p><b>RF2:</b> Demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. (Found11)</p> <p><b>RF1:</b> Factor polynomial expressions of the form:</p> <ul style="list-style-type: none"> <li><math>ax^2+bx+c, a \neq 0</math></li> <li><math>a^2x^2 - b^2y^2, a \neq 0, b \neq 0</math></li> <li><math>a(f(x))^2 + b(f(x))^2 + c, a \neq 0</math></li> <li><math>a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0</math></li> </ul> <p>where <math>a, b</math> and <math>c</math> are rational numbers. (PC11)</p>	<p><b>RF1:</b> Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.</p> <p><b>RF2:</b> Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.</p> <p><b>RF3:</b> Apply translations and stretches to the graphs and equations of functions.</p>	<p><b>RF4:</b> Graph and analyze polynomial functions (limited to polynomial functions of degree <math>\leq 5</math>)</p>

**ELABORATION**

In previous grades, students graphed polynomial functions of degree 1 (linear) and degree 2 (quadratic).

For this outcome, students will look at the graphs of polynomial functions of higher degrees up to and including degree 5.

A **polynomial function** is a function consisting of two or more terms with only one variable. Within each term the variable is raised to a whole number power and is multiplied by a constant. In general, a polynomial with real coefficients can be represented as:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

$a_n x^n$  is the **leading term**

$a_n$  is the **leading coefficient**

$n$  is the **degree of the polynomial**

Students will explore how a polynomial function's degree,  $x$ -intercepts,  $y$ -intercept and the sign of its leading coefficient affect the shape of its graph. They will also look at how the **multiplicity of zeros**, which is the number of times a zero ( $x$ -intercept) occurs, impacts on the graph. Students should be able to sketch the basic shape of a graph, showing information learned from the sign of  $a_n$ , the degree of the polynomial, and known rational roots. This is just a sketch as local maximum and minimum points are not known. Students should also understand that there may be other, irrational, real roots (and associated crossings of the  $x$  axis) and also imaginary roots.

Students will be expected to solve contextual problems involving polynomial functions using two different methods. One method is to graph the equation that is created and to estimate the solution from this graph. The other method is to create an equation and to solve it algebraically using factoring methods.

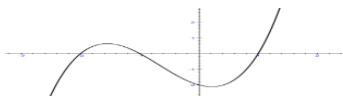
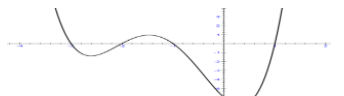
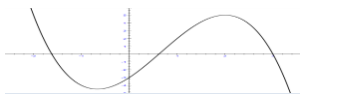

**SCO: RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ). [C, CN, T, V]**

### **ACHIEVEMENT INDICATORS**

- Identify the polynomial functions in a set of functions, and explain the reasoning.
- Explain the role of the constant term and leading coefficient in the equation of a polynomial function with respect to the graph of the function.
- Generalize rules for graphing polynomial functions of odd or even degree.
- Explain the relationship between:
  - the zeros of a polynomial function
  - the roots of the corresponding polynomial equation
  - the x-intercepts of the graph of the polynomial function.
- Explain how the multiplicity of a zero of a polynomial function affects the graph.
- Sketch, with or without technology, the graph of a polynomial function.
- Solve a problem by modeling a given situation with a polynomial function and analyzing the graph of the function.
- Given a table or graph of a polynomial function, determine a possible equation.

### **Suggested Instructional Strategies**

- Give students polynomial functions in factored form and ask them to graph the functions (Math Modeling Book 4).
- Students can develop their own chart or graphic organizer, like the one shown below, to summarize the characteristics of odd and even degree polynomial functions.

Odd Degree Polynomial Functions	Even Degree Polynomial Functions
<ul style="list-style-type: none"> <li>• When <math>a_n</math> is positive, the graph starts in the 3<sup>rd</sup> quadrant and ends in the 1<sup>st</sup> quadrant</li> </ul> 	<ul style="list-style-type: none"> <li>• When <math>a_n</math> is positive, the graph starts in the 2<sup>nd</sup> quadrant and ends in the 1<sup>st</sup> quadrant</li> </ul> 
<ul style="list-style-type: none"> <li>• When <math>a_n</math> is negative, the graph starts in the 2<sup>nd</sup> quadrant and ends in the 4<sup>th</sup> quadrant</li> </ul> 	<ul style="list-style-type: none"> <li>• When <math>a_n</math> is negative, the graph starts in the 3<sup>rd</sup> quadrant and ends in the 4<sup>th</sup> quadrant</li> </ul> 

- To challenge students, this outcome can be extended to writing an equation for a polynomial function from a graph or a description of a graph.
- Students should assemble a polynomial toolkit, with fairly accurate sketches of specific polynomials that illustrate each of these cases with examples such as:
  - $P(x) = x^3 + 2x^2 + x + 2$ , which has only one real root,
  - $P(x) = 2x^3 + x^2 - 6x - 3$ , which has three real roots, one of which is rational.
- To challenge students ask them to solve polynomial inequalities using the graph of the corresponding function (*Math Modeling, Book 4*)

SCO: RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ).  
[C, CN, T, V]

**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q** Graph each of the following polynomial functions, clearly indicating  $x$ -intercepts,  $y$ -intercept, and the end behaviours:

a)  $y = (x - 2)(x + 3)(x - 1)$

b)  $y = (x + 1)(x - 4)(2x + 5)(x - 3)$

c)  $f(x) = -x(x - 1)(x + 3)$

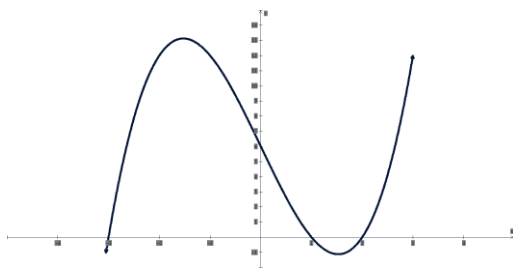
d)  $g(x) = \frac{1}{3}(x^2 - 16)(x^2 - 6x)$

e)  $y = -\frac{1}{3}(x - 2)^2(x + 2)^3$

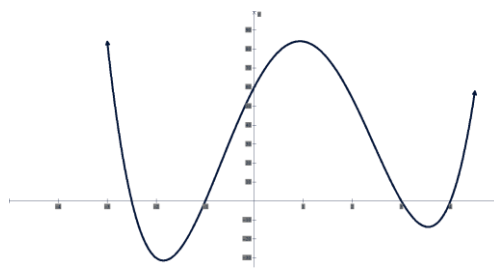
f)  $y = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$

*Answers:*

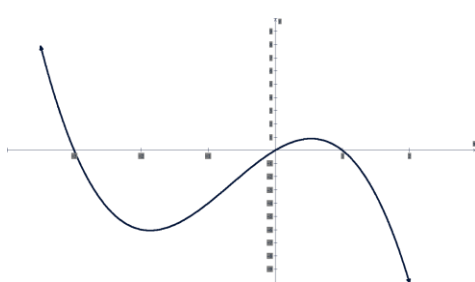
a)



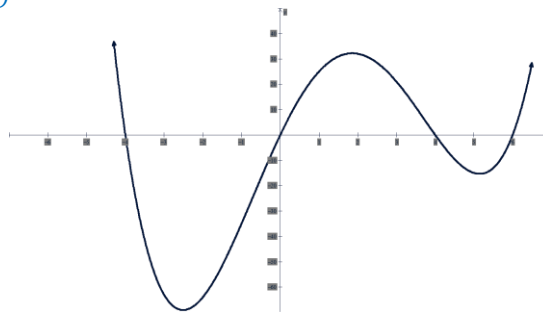
b)



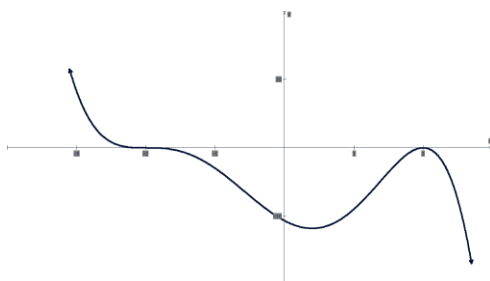
c)



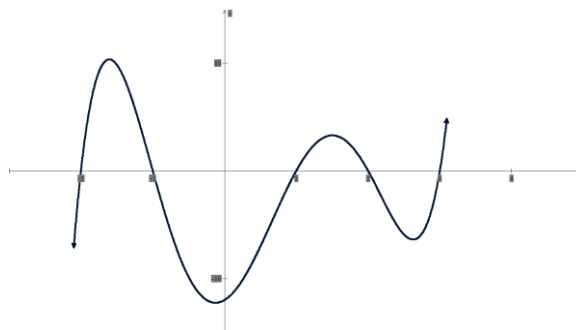
d)



e)



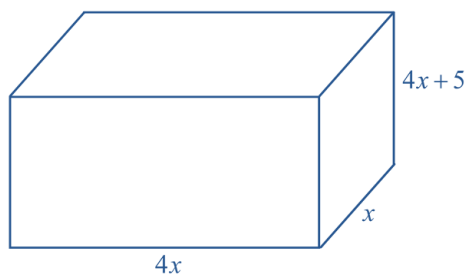
f)



SCO: RF4: Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ).  
[C, CN, T, V]

- Q** A rectangular shipping container has a volume of  $2500 \text{ cm}^3$ . The container is 4 times as wide as it is deep, and 5 cm taller than it is wide. What are the dimensions of the container?

*Answer:*



$$(4x)(x)(4x + 5) = 2500$$

$$16x^3 + 20x^2 = 2500$$

$$4x^3 + 5x^2 - 625 = 0$$

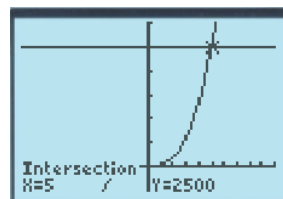
*The depth is 5 cm, the width is 20 cm, and the height is 25 cm. Alternatively, find the first root ( $x = 5$ ) by substitution.*

```

Plot1 Plot2 Plot3
Y1= (4X)(X)(4X+5)
Y2= 2500
Y3=
Y4=
Y5=
Y6=
    
```

```

WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-500
Ymax=3000
Yscl=500
Xres=1
    
```



- Q** Explain how the number of zeros affect the graph of a polynomial function.

*Answer: If the number of zeroes is equal to the degree of the function, then the graph will cross over the x axis that number of times. If the number of zeroes is less than the degree of the function, then at least one of the roots will be of multiplicity 2 or higher. If the multiplicity of a root is even, then the graph will just touch the x axis at that value. If the multiplicity of a root is odd, then the graph will cross over the x axis at that value.*

**SCO RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]**

**[C]** Communication  
**[TI]** Technology

**[PS]** Problem Solving  
**[VI]** Visualization

**[CN]** Connections  
**[RI]** Reasoning

**[ME]** Mental Math  
and Estimation

**RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).**

**Scope and Sequence of Outcomes:**

<b>Founds 11/ Pre-Calc 11</b>	<b>Pre-Calc B 120</b>
<p><b>RF2:</b> Demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. (Found11)</p> <p><b>RF1:</b> Factor polynomial expressions of the form:</p> <ul style="list-style-type: none"> <li><math>ax^2+bx+c, a \neq 0</math></li> <li><math>a^2x^2 - b^2y^2, a \neq 0, b \neq 0</math></li> <li><math>a(f(x))^2+b(f(x))^2+c, a \neq 0</math></li> <li><math>a^2(f(x))^2-b^2(g(y))^2, a \neq 0, b \neq 0</math></li> </ul> <p>where <math>a, b</math> and <math>c</math> are rational numbers. (PC11)</p> <p><b>RF3:</b> Analyze quadratic functions of the form <math>y = a(x - p)^2 + q</math> and determine the vertex, domain and range, direction of opening, axis of symmetry, <math>x</math>- and <math>y</math>- intercepts. (PC11)</p> <p><b>RF4:</b> Analyze quadratic functions of the form: <math>y = ax^2 + bx + c</math> to identify characteristics of the corresponding graph, including vertex, domain and range, directions of opening, axis of symmetry, <math>x</math>- and <math>y</math>-intercepts, and to solve problems. (PC11)</p>	<p><b>RF5:</b> Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).</p>

**ELABORATION**

Students have learned to graph linear and quadratic functions, determining the vertex, domain and range,  $x$ - and  $y$ - intercepts, and the direction of opening (if applicable). For this outcome, students will build on this knowledge, first graphing the original linear or quadratic functions, and then their reciprocal functions.

If  $a$  satisfies  $f(a) = 1$  (or  $f(a) = -1$ ) then  $1/f(a)$  will take the same value. So that the point  $(a, 1)$  (or  $(a, -1)$ ) lies on the graph of  $y = f(x)$  and on the graph of  $y = 1/f(x)$ . Such a point is called invariant under the transformation “take the reciprocal of the function”.

For quadratic functions, the  $x$ -intercepts of the original function locate the vertical asymptotes of the reciprocal function. Also, if the vertex of the original quadratic function is above or below the  $x$ -axis, the local max/min of the reciprocal function will also be above or below the  $x$ -axis. If the original function is entirely above the  $x$ -axis, the reciprocal function will also be above the  $x$ -axis. Likewise, if the original function is entirely below the  $x$ -axis, the reciprocal will be below the  $x$ -axis.

The **horizontal asymptote** for every reciprocal of linear, quadratic, and cubic functions will always be  $y = 0$ . The reason is: for all polynomial functions  $y = f(x)$ , as  $x$  becomes arbitrarily large in absolute value, so does  $y$ . Therefore  $1/y$  becomes closer and closer to 0 as  $|x|$  gets larger and larger.

SCO RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]

### ACHIEVEMENT INDICATORS

- Compare the graph of  $y = \frac{1}{f(x)}$  to the graph of  $y = f(x)$ .
- Identify, given a function  $y = f(x)$ , values of  $x$  for which  $y = \frac{1}{f(x)}$  is not a permissible value. For each value of  $x$ , decide whether or not the graph has a vertical asymptote.
- Graph, with and without technology,  $y = \frac{1}{f(x)}$ , given a function  $y = f(x)$  and explain the strategies used.

### Suggested Instructional Strategies

- Before the students begin the study of reciprocal graphs, it is recommended that students review factoring, strategies for graphing linear and quadratic functions, and the definition of a reciprocal.
- To introduce the idea of reciprocal functions, a table of values can be used to compare the same  $x$  values to see how the  $y$ -values change. For example:

$x$	$y = x$	$y = \frac{1}{x}$
-10	-10	$-\frac{1}{10}$
$-\frac{1}{2}$	$-\frac{1}{2}$	-2
0	0	<i>undefined</i>
$\frac{1}{2}$	$\frac{1}{2}$	2
10	10	$\frac{1}{10}$

- When graphing by hand,  $x$  and  $y$  intercepts, vertex, invariant points, and vertical and horizontal asymptotes should be clearly indicated.
- Using graphing technology allows students to model and analyze functions more quickly and to explore more complex functions.
- As an **extension**, students can sketch the graphs  $y = \sec x$ ,  $y = \csc x$  and  $y = \cot x$ . This will provide a connection to the trigonometry they worked on in *Pre-Calculus A 120*. (*This is a good way to meet the needs of students who are in need of being challenged past the level of the curriculum to keep their interest*)



SCO RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Q** For the function  $f(x) = \frac{1}{3}x - 1$ , sketch each of the following on the same axes:

- $y = f(x)$
- $y = f^{-1}(x)$ , the inverse function  
(recall from Pre-Calculus A,  $f(f^{-1}(x)) = x$  and  $f^{-1}(f(x)) = x$ ).
- $y = 1/f(x)$ , the reciprocal function.

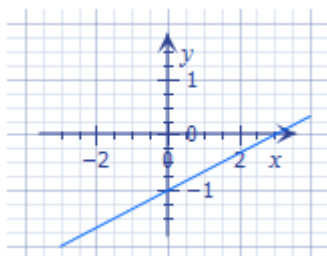
Label the three graphs clearly and compare them in point form.

Answers:

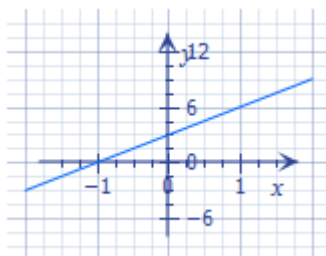
a)  $y = \frac{1}{3}x - 1$

b)  $f^{-1}(x) = 3x + 3$

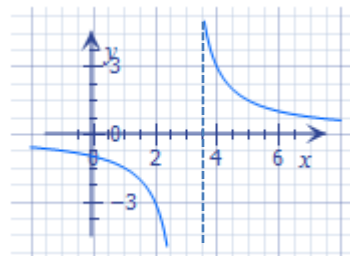
c)  $y = \frac{1}{\frac{1}{3}x - 1}$



linear function  
x-intercept at  $x = 3$   
y-intercept at  $y = -1$   
slope  $\frac{1}{3}$   
 $D \{x|x \in R\}$   
 $R \{y|y \in R\}$



linear function  
x-intercept at  $x = -1$   
y-intercept at  $y = 3$   
slope 3  
 $D \{x|x \in R\}$   
 $R \{y|y \in R\}$



rational function  
no intercepts  
vertical asymptote at  $x = 3$   
horizontal asymptote at  $y = 0$   
 $D \{x|x \neq 3, x \in R\}$   
 $R \{y|y \neq 0, y \in R\}$

**Q** Compare the graphs of:

- $y = 2x$  and  $y = \frac{1}{2x}$ .
- $y = x + 3$  and  $y = \frac{1}{x+3}$ .

Answers:

a)  $y = 2x$  is linear, x-int = 0, y-int = 0,  $D\{x|x \in R\}$   $R\{y|y \in R\}$

$y = \frac{1}{2x}$  is not linear, vert asymp  $x = 0$ , horiz asymp  $y = 0$ ,  $D\{x|x \neq 0, x \in R\}$   $R\{y|y \neq 0, y \in R\}$

b)  $y = x + 3$  is linear, x-int = -3, y-int = 3,  $D\{x|x \in R\}$   $R\{y|y \in R\}$

$y = \frac{1}{x+3}$  is not linear, vert asymp  $x = -3$ , horiz asymp  $y = 0$ ,  $D\{x|x \neq -3, x \in R\}$   $R\{y|y \neq 0, y \in R\}$

**Q** Compare the graphs of:

- $f(x) = x^2$  and  $g(x) = \frac{1}{x^2}$
- $f(x) = x^2 - 4$  and  $g(x) = \frac{1}{x^2 - 4}$
- $f(x) = x^2 + 4$  and  $g(x) = \frac{1}{x^2 + 4}$

Answers:

a)  $f(x) = x^2$  x-int = 0, y-int = 0,  $D\{x|x \in R\}$   $R\{y|y \geq 0, y \in R\}$ , dec  $x \in (-\infty, 0)$ , inc  $x \in (0, \infty)$

$g(x) = \frac{1}{x^2}$  vert asymp  $x = 0$ , horiz asymp  $y = 0$ ,  $D\{x|x \neq 0, x \in R\}$   $R\{y|y \neq 0, y \in R\}$ ,  
increasing  $x \in (-\infty, 0)$ , decreasing  $x \in (0, \infty)$

b)  $f(x) = x^2 - 4$  x-int =  $\pm 2$ , y-int = -4,  $D\{x|x \in R\}$   $R\{y|y \geq -4, y \in R\}$

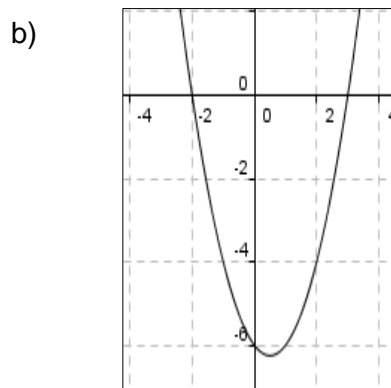
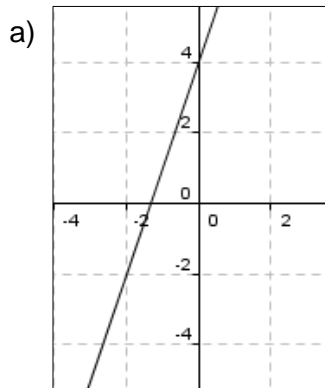
$y = \frac{1}{x^2 - 4}$  vert asymp  $y = \pm 2$  horiz asymp  $y = 0$  y-

int =  $-\frac{1}{4}$   $D\{x|x \neq \pm 2, x \in R\}$   $R\{y|y > 0, y \leq -\frac{1}{4}, y \in R\}$

increasing  $x \in (-\infty, -2)$   $y > 0$ , inc  $x \in (-2, 0)$   $y < 0$ , decreasing  $x \in (0, 2)$   $y < 0$ , dec  $x \in (2, \infty)$   $y > 0$

SCO RF5: Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]

**Q** Sketch the graph of the reciprocal function for the following:



Answers: a)  $y$ -intercept  $= \frac{5}{3}$ , asymptotes  $x = -\frac{4}{3}$ ,  $y = 0$   
 b)  $y$ -intercept  $= -\frac{1}{6}$ , asymptotes  $x = -2$  and  $2\frac{1}{2}$ ,  $y = 0$

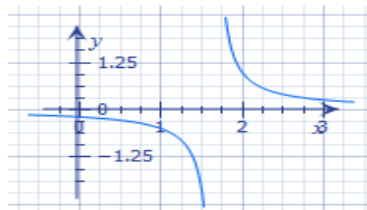
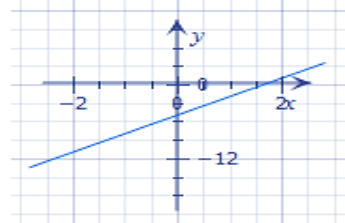
**Q** For each function:

- i.  $g(x) = 3x - 5$
- ii.  $h(x) = x^2 + 2x - 3$

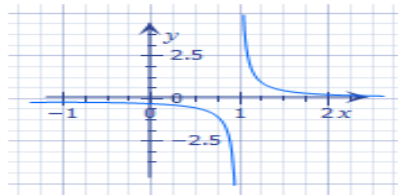
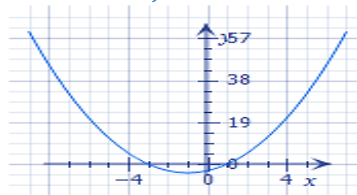
- a) State the zeros ( $x$ -intercepts) and the  $y$ -intercept and other critical points such as the vertex where applicable.
- b) Write an equation for the reciprocal function.
- c) State the non-permissible values (restrictions on the domain) for the reciprocal function.
- d) State the invariant points.
- e) Sketch each function and its reciprocal function.

Answers:

i)  $g(x) = 3x - 5$  a)  $x$ -int  $= \frac{5}{3}$ ,  $y$ -int  $= -5$  b)  $y = \frac{1}{3x-5}$  c)  $x \neq \frac{5}{3}$  d)  $(2, 1)$   $(\frac{4}{3}, -1)$  e)



ii)  $h(x) = x^2 + 2x - 3$  a)  $x$ -int  $= 1, -3$ ,  $y$ -int  $= -3$  b)  $y = \frac{1}{x^2+2x-3}$  c)  $x \neq 1, -3$   
 d)  $(-1 + \sqrt{5}, 1)$   $(-1 - \sqrt{5}, 1)$   $(-1 + 3, -1)$   $(-1 - \sqrt{3}, -1)$   
 e)



**SCO: RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]**

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

**RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).**

**Scope and Sequence of Outcomes:**

Found 110/ Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
<p><b>RF2:</b> Demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. (Found11)</p> <p><b>AN4:</b> Determine equivalent forms of rational expressions (PC11)</p> <p><b>AN5:</b> Perform operations on rational expressions (PC11)</p> <p><b>AN6:</b> Solve problems that involve rational equations (PC11)</p> <p><b>RF1:</b> Factor polynomial expressions of the form:</p> <ul style="list-style-type: none"> <li><math>ax^2+bc+c, a \neq 0</math></li> <li><math>a^2x^2 - b^2y^2, a \neq 0, b \neq 0</math></li> <li><math>a(f(x))^2 + b(f(x))^2 + c, a \neq 0</math></li> <li><math>a^2(f(x))^2 - b^2(g(y))^2, a \neq 0, b \neq 0</math></li> </ul> <p>where <math>a, b</math> and <math>c</math> are rational numbers. (PC11)</p> <p><b>RF3:</b> Analyze quadratic functions of the form <math>y = a(x - p)^2 + q</math> and determine the vertex, domain and range, direction of opening, axis of symmetry, <math>x</math>- and <math>y</math>- intercepts. (PC11)</p> <p><b>RF4:</b> Analyze quadratic functions of the form: <math>y = ax^2 + bx + c</math> to identify characteristics of the corresponding graph, including vertex, domain and range, directions of opening, axis of symmetry, <math>x</math>- and <math>y</math>-intercepts, and to solve problems. (PC11)</p>	<p><b>RF1:</b> Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.</p> <p><b>RF2:</b> Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.</p> <p><b>RF3:</b> Apply translations and stretches to the graphs and equations of functions.</p>	<p><b>RF6:</b> Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).</p>

**ELABORATION**

In *Foundations of Mathematics 11* students will have been introduced to quadratic functions. In *Pre-Calculus 110* students will have developed this understanding further and would have also worked with rational expressions and equations. In *Pre-Calculus A 120* they will have explored in detail the effect of transformations on graphs of functions and their related equations.

This outcome will build on this previous work, as well as build on the previous outcome on reciprocal functions (RF5) in which non-permissible values were explored. The focus now shifts to rational functions.

Students will be introduced to characteristics of rational functions including non-permissible values (**vertical asymptotes** or **points of discontinuity**); zeroes and behaviour near these values; and **end behaviour**, which may include **horizontal asymptotes**. Students will learn to graph these functions using a table of values, technology and/or vertical and horizontal transformations.

Graphs may intersect the horizontal asymptote but will not intersect the vertical asymptote. The horizontal asymptote will reflect the end behaviour of the graph whereas

**SCO: RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]**

the vertical asymptotes will indicate non-permissible value for the given function. Students should, at this point, be introduced to the term **limit** (limits will be developed further in L1 of this course). A limit is the  $y$ -value of the function that is approached as  $x$  approaches a certain value. This would be helpful in determining the behaviour of the function around vertical asymptotes or points of discontinuity as well as end behaviour of functions.

Students should understand and compare rational functions presented in different forms: equation, graph or word problem, and for each be able to determine where the non-permissible values are located and which type they represent, either points of discontinuity or vertical asymptotes.

Using technology would be helpful here to allow students to analyze more functions in a short period of time.

### **ACHIEVEMENT INDICATORS**

- Graph, with or without technology, a rational function.
- Analyze the graphs of a set of rational functions to identify common characteristics.
- Explain the behaviour of the graph of a rational function for values of the variable near a non-permissible value.
- Determine if the graph of a rational function will have a vertical asymptote or a hole for a non-permissible value.
- Determine when horizontal asymptotes occur and when they don't occur.
- Match a set of rational functions to their graphs, and explain the reasoning.
- Describe the relationship between the roots of a rational equation and the  $x$ -intercepts of the graph of the corresponding rational function.
- Determine, graphically, an approximate solution of a rational equation.

### **Suggested Instructional Strategies**

- To develop students' understanding remind students of transformations of linear, quadratic, radical, trigonometric and absolute value functions that they have already studied.

Have them compare how the values of  $a$ ,  $h$ , and  $k$  in the quadratic  $y = a(x - h)^2 + k$ , affect the original graph of  $y = x^2$ , and how these values in  $y = \frac{a}{(x-h)} + k$ , affect the original graph of  $y = \frac{1}{x}$  (including the transformation of the asymptotes). Also, have them examine the effect of the value of  $b$  in the equations  $y = \frac{a}{b(x-h)} + k$  and  $y = \frac{a}{b(x+h)^2} + k$ .

Students should then be ready to try a linear divided by a linear example and be able to present in the above form to aid them in graphing.

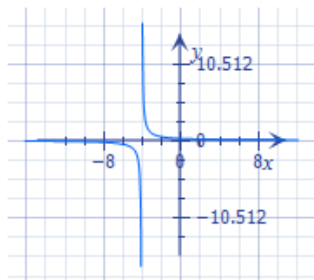
SCO: RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]

- A chart of characteristics would be essential for understanding, including: non-permissible values and behaviour near them, domain, range, intercepts, horizontal and vertical asymptotes.
- Students should add to their function tool kit a collection of carefully drawn graphs for functions with specific values of  $a$ ,  $h$  and  $k$ .

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Q** Sketch the graph of the function  $y = \frac{1}{x+4}$ , showing all important characteristics.

*Answers: non – permissible value (vertical asymptote)  $x = -4$ ,  $D\{x|x \neq -4, x \in R\}$   $R\{y|y \neq 0, y \in R\}$ , no  $x$ -intercepts,  $y$ -int  $(0, \frac{1}{4})$ , horizontal asymp  $y = 0$*



**Q** For the two rational functions shown below, complete the characteristics chart.

	$y = \frac{1}{x^2 - 4x + 4}$	$y = \frac{-1}{(x+3)^2} + 2$
<b>Characteristics</b>		
Non-Permissible value(s)	<i>Answers: <math>x = 2</math></i>	$x = 3$
Feature at the non-permissible value(s) (Point of discontinuity or vertical asymptote)	<i>vertical asymptote</i>	<i>vertical asymptote</i>
Behaviour near non-permissible value(s)	<i>As <math>x</math> approaches 2, <math>y</math> becomes very large</i>	<i><math>y</math> approaches <math>-\infty</math>, becoming increasing smaller</i>
End Behaviour	<i>As <math> x </math> gets larger, <math>y</math> approaches 0</i>	<i>as <math> x </math> gets larger, <math>y</math> approaches 2</i>
Domain	$\{x x \neq 2, x \in R\}$	$\{x x \neq -3, x \in R\}$
Range	$\{y y > 0, y \in R\}$	$\{y y < 2, y \in R\}$
Intercepts ( State as Points)	$(0, \frac{1}{4})$	$(0, \frac{17}{9})$ $(-2.3, 0)$ $(-3.7, 0)$
Equation of vertical asymptote	$x = 2$	$x = 3$
Equation of horizontal asymptote	$y = 0$	$y = 2$

SCO: RF6: Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]

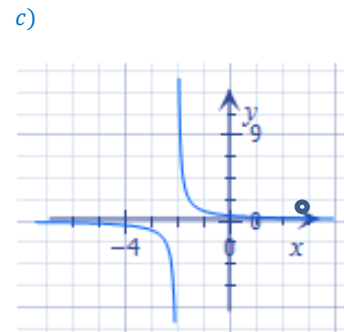
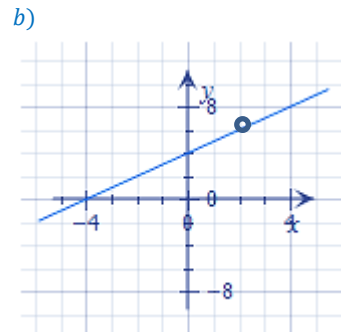
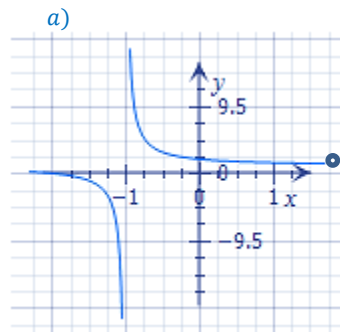
**Q** For each of the following functions, draw a labelled graph and list its characteristics.

a)  $y = \frac{x^2-4}{x^2-x-2}$

b)  $y = \frac{x^2+2x-8}{x-2}$

c)  $y = \frac{x-3}{x^2-x-6}$

Answers:



$$f(x) = \frac{(x+2)(x-2)}{(x+1)(x-2)} = \frac{x+2}{x+1}, x \neq 2$$

$x$ -int  $(-2, 0)$   
 $y$ -int  $(0, 2)$   
 H.A.  $y = 1$   
 V.A.  $x = -1$   
 point of disc  $(2, \frac{4}{3})$   
 $D\{x|x \neq -1, x \neq -2, x \in \mathbb{R}\}$   
 $R\{y|y \neq 1, y \neq \frac{4}{3}, y \in \mathbb{R}\}$

$$f(x) = \frac{(x+4)(x-2)}{(x-2)} = x + 4, x \neq 2$$

$x$ -int  $(-4, 0)$   
 $y$ -int  $(0, 4)$   
 H.A. none  
 V.A. none  
 point of disc  $(2, 6)$   
 $D\{x|x \neq 2, x \in \mathbb{R}\}$   
 $R\{y|y \neq 6, y \in \mathbb{R}\}$

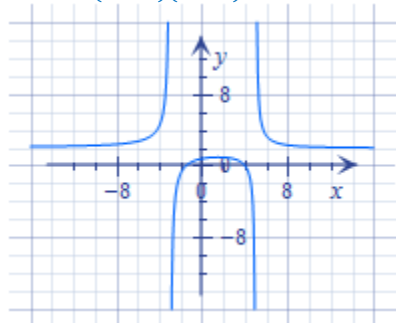
$$f(x) = \frac{(x-3)}{(x-3)(x+2)} = \frac{1}{x+2}, x \neq 3$$

no  $x$ -int  
 $y$ -int  $(0, \frac{1}{2})$   
 H.A.  $y = 0$   
 V.A.  $x = -2$   
 point of disc  $(3, \frac{1}{5})$   
 $D\{x|x \neq -2, x \neq 3, x \in \mathbb{R}\}$   
 $R\{y|y \neq 0, y \neq \frac{1}{5}, y \in \mathbb{R}\}$

**Q** Sketch the graph of the function  $f(x) = \frac{2x^2-5x-12}{x^2-2x-15}$ , and list its characteristics.

Answer:

$$f(x) = \frac{(2x+3)(x-4)}{(x-5)(x+3)}$$



Non – permissible values  
 Feature at non – permissible value  
 Behaviour near non – permissible values

End behaviour  
 Domain  
 Range

Intercepts  
 Equation of vertical asymptotes  
 Equation of horizontal asymptote

$x \neq 5, x \neq -3$   
 vertical asymptote  
 as  $x$  approaches 5,  $|y|$  gets larger  
 as  $x$  approaches -3,  $|y|$  gets larger  
 as  $|x|$  gets large,  $y$  approaches 2  
 $\{x|x \neq 5, x \neq -3, x \in \mathbb{R}\}$   
 Because the function crosses  $y = 2$   
 students will not be able to accurately determine the range.  
 $(-\frac{3}{2}, 0)$   $(4, 0)$   $(0, \frac{12}{15})$   
 $x = 5, x = -3$   
 $y = 2$

**SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions.** [CN, R, T, V]

**RF7: Demonstrate an understanding of operations on, and compositions of functions.**

**Scope and Sequence of Outcomes:**

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
<b>AN5:</b> Perform operations on Rational Expressions (limited to numerators and denominators that are monomials, binomials or trinomials).		<b>RF7:</b> Demonstrate an understanding of operations on, and compositions of functions.

**ELABORATION**

Students have experience with graphical representations of linear, absolute value and quadratic functions, as well as practice determining the domain and range of each function. They have also been exposed to non-permissible values (restrictions) in terms of rational expressions which will serve as a foundation for the introduction of asymptotes in sketching graphs of combinations of functions (quotients).

Students' facility with functions now becomes increasingly more sophisticated as they learn to reason about new functions derived from familiar ones via the combination or composition of functions. For example, a profit relationship can be viewed as a combination of functions by subtracting the functions that represent revenue and expenses. Students should explore visually the effects upon the domains and ranges of combination situations such as: *linear*  $\pm$  *linear*, *linear*  $\pm$  *quadratic*, *quadratic*  $\pm$  *quadratic*, and *linear*  $\times$  *linear*. Students will have already worked with the product of two linear functions which form a quadratic function.

Students will be expected to sketch graphs of composition of functions using graphs of linear, quadratic, etc. In order to sketch the new graphs, students will have to find the value of the dependent variable  $y$  for each function for a certain value of the independent variable  $x$ , and repeat for multiple values. Vertical asymptotes and open circles (where the function does not exist) will be explored by stating non-permissible values with the quotients of functions. Horizontal asymptotes do not need to be addressed at this time.

As students write equations of functions that are sums, differences, products, or quotients of two functions, they will be required to simplify their answers. For quotient functions, students will need to identify non-permissible values.

When determining the domain and range of a function representing the sum, difference, product or quotient of two functions, the domain will be the domain common to both of the original functions. The domain of the quotient function is further restricted by excluding values where the denominator would equal zero. The range will be determined by the graph.

Sometimes students will encounter a function within another function, where both functions are needed to answer a question or analyze a problem. The domain of the second function has to connect to the first function. The symbol  $f(g(x))$  is a composition of the two functions,  $f$  and  $g$ . Students should understand that the

**SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions.** [CN, R, T, V]

composition  $f(g(x))$  gives the final outcome when the independent value is substituted into the inner function  $g$ , and its value  $g(x)$  is then substituted into the outer function  $f$ . The range of  $g(x)$  becomes the domain of  $f(x)$ . For example:

Given  $f(x) = 4(x-3)^2$  and  $g(x) = -0.5(x-5)^2 + 8$ , evaluate  $f(g(2))$ :

$$g(x) = -0.5(x-5)^2 + 8$$

$$g(2) = -0.5(2-5)^2 + 8$$

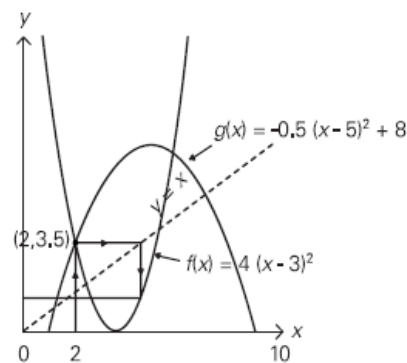
$$g(2) = 3.5$$

$$f(3.5) = 4(3.5-3)^2$$

$$f(3.5) = 1$$

Students can also visualize the composition of functions using graphs. Students should be able to explain each step of the process. For example:

- to evaluate  $g(2)$ , trace a vertical line from  $x = 2$  to  $g$
- trace a horizontal line to  $y = x$  changing the  $y$ -value 3.5 to an  $x$ -value of 3.5 ( $y = x$ )
- trace a vertical line from this point to intersect the graph  $f$  (other parabola)
- find the  $y$ -value of this point by tracing a horizontal line to the  $y$ -axis



Students will be required to write a function as a composition of two or more functions. This will require them to identify patterns in the function where they can insert an algebraic expression inside the given function. There will be many potential solutions to these types of questions. For example, the following function can be written as a composition of two or more functions:

$$h(x) = (x+3)^2 - x + 5$$

$$h(x) = (x+3)^2 - (x+3-3) + 5$$

$$h(x) = (x+3)^2 - (x+3) + 3 + 5$$

$$h(x) = (x+3)^2 - (x+3) + 8$$

The linear function  $f(x) = x + 3$  is in the quadratic function  $g(x) = x^2 - x + 8$ .



**SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions.** [CN, R, T, V]

### **ACHIEVEMENT INDICATORS**

- Sketch the graph of a function that is the sum, difference, product, or quotient of two functions, given their graphs.
- Write the equation of a function that is the sum, difference, product, or quotient of two or more functions, given their equations.
- Determine the domain and range of a function that is the sum, difference, product, or quotient of two functions.
- Write a function  $h(x)$  as the sum, difference, product, or quotient of two or more functions.
- Determine the value of the composition of functions when evaluated at a point, including:  $f(f(a))$ ,  $f(g(a))$ ,  $g(f(a))$ ,  $(f \circ g)(x)$ ,  $(g \circ f)(x)$ ,  $(f \circ f)(x)$ ,  $(g \circ g)(x)$ .
- Determine, given the equations of two functions  $f(x)$  and  $g(x)$ , the equation of the composite function:  $f(f(x))$ ,  $f(g(x))$ ,  $g(f(x))$ ,  $(f \circ g)(x)$ ,  $(g \circ f)(x)$ ,  $(f \circ f)(x)$ ,  $(g \circ g)(x)$ .
- Sketch, given the equations of two functions  $f(x)$  and  $g(x)$ , the graph of the composite function:  $f(f(x))$ ,  $f(g(x))$ ,  $g(f(x))$ ,  $(f \circ g)(x)$ ,  $(g \circ f)(x)$ ,  $(f \circ f)(x)$ ,  $(g \circ g)(x)$ .
- Write a function  $h(x)$  as a composition of two or more functions.
- Write a function  $h(x)$  by combining two or more functions through operations on, and compositions of, functions.

### **Suggested Instructional Strategies**

- Give the students a graph of two linear functions  $f(x)$  and  $g(x)$ . Then have them use a table of values to sketch the graphs of the sum, difference, product, and quotient of the two functions and their domain and range.
- Use questions from the resource, *Math Modeling*, Book 4, Chapter 2.

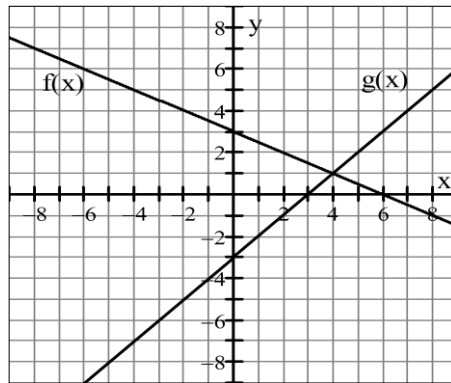
SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions. [CN, R, T, V]

**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q** Using the graph of two linear functions  $f(x)$  and  $g(x)$ , shown below, use a table of values to sketch the graphs of the sum, difference, product and quotient of the two functions.

- Sketch the function  $h(x) = f(x) + g(x)$
- Sketch the function  $j(x) = f(x) - g(x)$
- Sketch the function  $k(x) = f(x) \times g(x)$
- Sketch the function  $l(x) = f(x) \div g(x)$

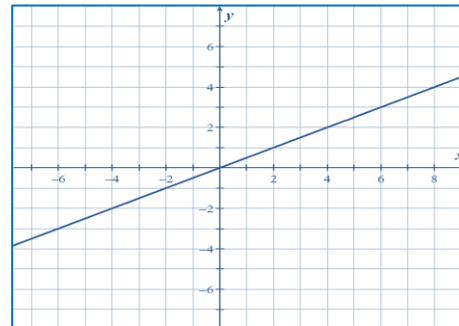
Determine the domain and range for each of the functions, a) to d).



Answers:

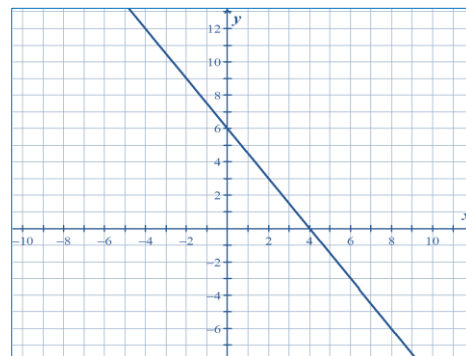
a)  $D\{x|x \in R\}$   $R\{y|y \in R\}$

$x$	$f(x)$	$g(x)$	$h(x) = f(x) + g(x)$
-4	5	-7	-2
-2	4	-5	-1
0	3	-3	0
2	2	-1	1
4	1	1	2
6	0	3	3
8	-1	5	4



b)  $D\{x|x \in R\}$   $R\{y|y \in R\}$

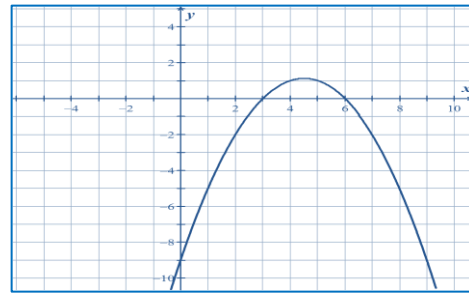
$x$	$f(x)$	$g(x)$	$h(x) = f(x) - g(x)$
-4	5	-7	12
-2	4	-5	9
0	3	-3	6
2	2	-1	3
4	1	1	0
6	0	3	-3
8	-1	5	-6



SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions. [CN, R, T, V]

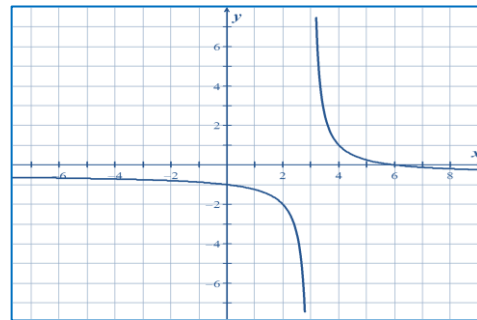
c)  $D\{x|x \in R\} \quad R\{y|y \leq \frac{9}{8}, y \in R\}$

$x$	$f(x)$	$g(x)$	$h(x) = f(x) \times g(x)$
-4	5	-7	-35
-2	4	-5	-20
0	3	-3	-9
2	2	-1	-2
4	1	1	1
6	0	3	0
8	-1	5	-5

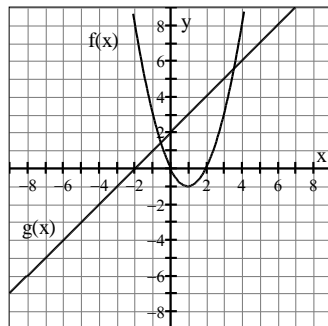


d)  $D\{x|x \neq 3, x \in R\} \quad R\{y|y \neq -\frac{1}{2}, y \in R\}$

$x$	$f(x)$	$g(x)$	$h(x) = f(x) \div g(x)$
-4	5	-7	$-\frac{5}{7}$
-2	4	-5	$-\frac{4}{5}$
0	3	-3	-1
2	2	-1	-2
4	1	1	1
6	0	3	0
8	-1	5	$-\frac{1}{5}$

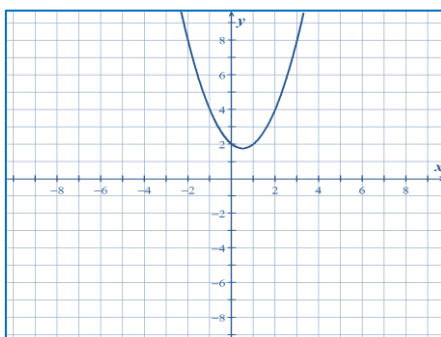


**Q** The graph below shows two functions  $f(x)$  and  $g(x)$ . Sketch the graphs of the a) sum, b) difference, c) product, and d) quotient of these two functions. Determine the domain and range for each of these combined functions.

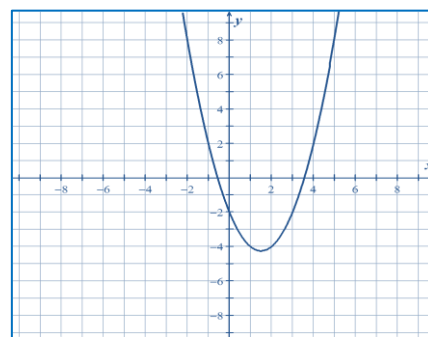


Answers:

a)  $D\{x|x \in R\} \quad R\{y|y \geq 1.75, y \in R\}$

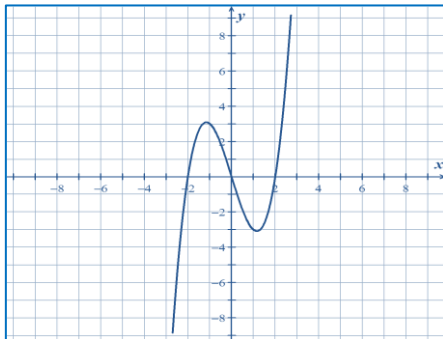


b)  $D\{x|x \in R\} \quad R\{y|y \geq -4.25, y \in R\}$

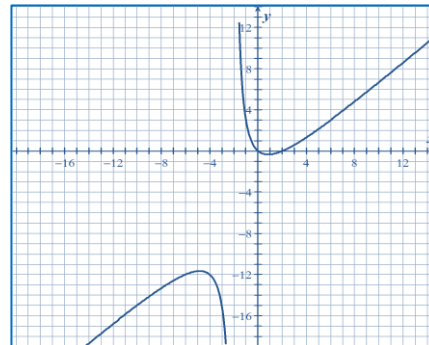


SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions. [CN, R, T, V]

c)  $D\{x|x \in R\}$   $R\{y|y \in R\}$



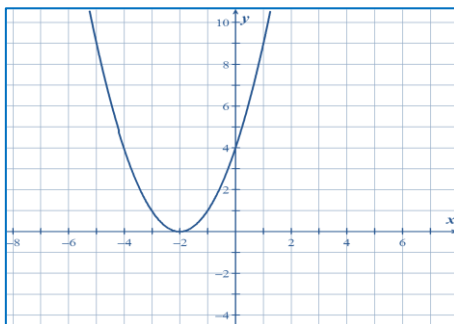
d)  $D\{x|x \neq 2, x \in R\}$   $R\{y|y \leq -11.657 \cup y \geq -0.343, y \in R\}$



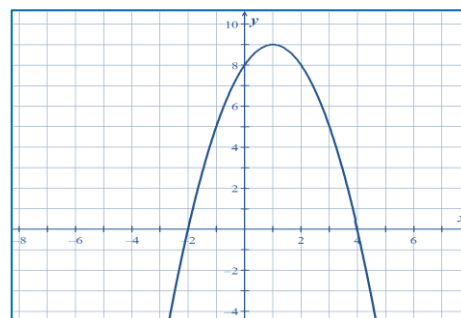
**Q** Given  $f(x) = 3x + 6$  and  $g(x) = x^2 + x - 2$ , determine a)  $(f + g)(x)$ , b)  $(f - g)(x)$ , c)  $(f \cdot g)(x)$ , and d)  $\left(\frac{f}{g}\right)(x)$ . Determine the domain and the range for the combined functions. (Use technology to sketch the graphs).

Answers:

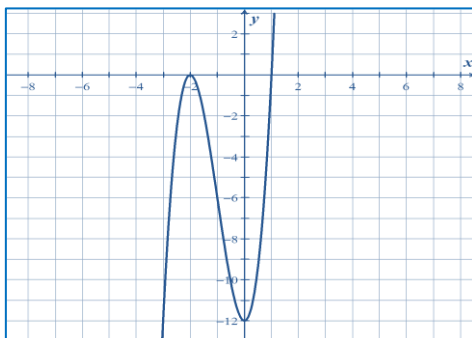
a)  $(f + g)(x) = x^2 + 4x + 4 = (x + 2)^2$   
 $D\{x|x \in R\}$   $R\{y|y \geq 0, y \in R\}$



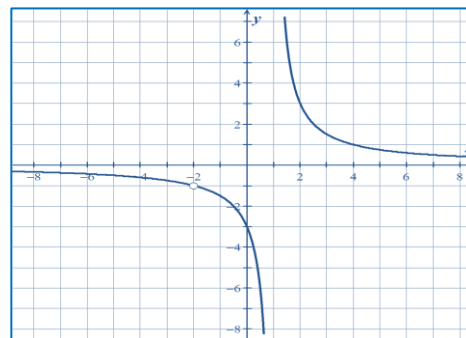
b)  $(f - g)(x) = 8 + 2x - x^2 = (4 - x)(2 + x)$   
 $\{x|x \in R\}$   $\{y|y \leq 9, y \in R\}$



c)  $(f \cdot g)(x) = 3(x + 2)^2(x - 1)$   
 $D\{x|x \in R\}$   $R\{y|y \in R\}$



d)  $(f \div g)(x) = \frac{3(x + 2)}{(x + 2)(x - 1)}$   
 $\{x|x \neq -2, 1, x \in R\}$   $\{y|y \neq -1, 0, y \in R\}$



SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions. [CN, R, T, V]

**Q** You need to save \$1500 to contribute to the cost of tuition in the fall, and are considering mowing lawns for the summer. You would like to buy an electric mower with rechargeable batteries, and your parents are asking for a flat rate of \$100 for the summer to pay for the electricity for recharging the batteries each night. The cost of the mower is \$499 plus 13% tax and you plan to charge \$10 to cut a lawn.

- a) Set up a chart with four columns: number of lawns cut, cost, income, profit.
- b) Determine the equation for each column in relation to the number of lawns cut.
- c) Graph the three equations on the same graph.
  - i. What does the  $x$  represent?
  - ii. What does the  $y$  represent?
  - iii. What do the slopes represent?
  - iv. Which lines appear parallel? Are they parallel? Explain.
  - v. How many lawns would you need to mow to break even?
  - vi. How many lawns would you need to mow to cover your costs and clear \$1500?
  - vii. On average, how many lawns would you need to cut per week? Is this a reasonable expectation?
  - viii. Do you think this would be a good business venture?
  - ix. Is there anything you would do differently?

Answers:

a) and b)

# Lawns Cut	Cost	Income	Profit
$x$	$f(x) = 100 + 499(1.13) = 663.87$	$g(x) = 10x$	$h(x) = 10x - 663.87$

c) i)  $x$  represent the number of lawns ii)  $y = f(x) = \text{cost}$   $y = g(x) = \text{income}$   $y = h(x) = \text{profit}$

iii) slope represents the unit change

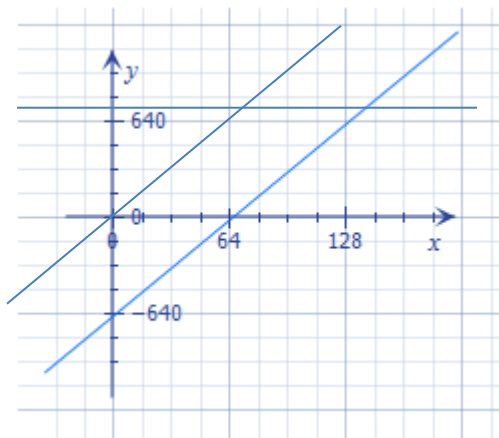
$f(x)$ : slope is zero (no change, flat rate)  $g(x)$ : slope is 10 (\$10 per lawn)  $h(x)$ : slope is 10 (\$10 per lawn)

iv)  $g(x)$  and  $h(x)$  both have the same slope, are parallel

v) to break even, cost = income  $10x = 663.87$   $x = 67$  lawns

vi) to clear \$1500, profit = \$1500  $10x - 663.87 = 1500$   $x = 217$  lawns

vii) lawns per week, assume 16 weeks  $217 \div 16 \cong 14$  lawns per week



SCO: RF7: Demonstrate an understanding of operations on, and compositions of functions. [CN, R, T, V]

**Q** If  $g(x) = 2x - 7$  and  $(x) = \frac{1}{x}$ , determine the function composition for  $f(g(x))$ , and  $g(f(x))$ .

Answers:  $f(g(x)) = \frac{1}{(2x-7)}$       $g(f(x)) = 2\left(\frac{1}{x}\right) - 7$

**Q** If  $f(x) = 5x - 1$  and  $f(g(x)) = 5x^2 + 10x + 4$ , find  $g(x)$ .

Answer:  $g(x) = x^2 + 2x + 1$

**Q** For  $f(x) = \sqrt{x+2}$  and  $g(x) = x^2 - 6x + 7$ , evaluate each of the compositions of functions below.

- a)  $f(g(0))$      b)  $(f \circ g)(3x)$      c)  $g(f(-2))$   
 d)  $g(f(7))$      e)  $g(g(5))$      f)  $(g \circ g)(1)$

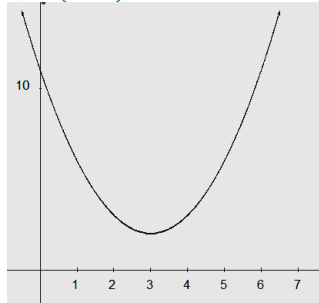
Answers: a)  $f(g(0)) = 3$     b)  $(f \circ g)(3x) = 3x - 3$     c)  $g(f(-2)) = 7$     d)  $g(f(7)) = -2$   
 e)  $g(g(5)) = -1$     f)  $(g \circ g)(1) = -1$

**Q** For  $f(x) = x^2 + 2$  and  $g(x) = x - 3$ , sketch the graphs of each of the following:

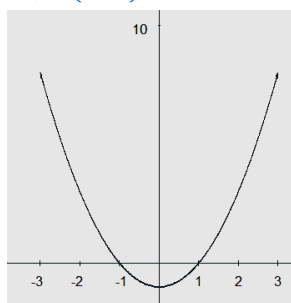
- a)  $f(g(x))$      b)  $(g \circ f)(x)$      c)  $g(g(x))$

Answers:

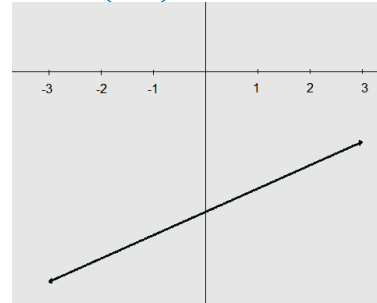
a)  $f(g(x)) = (x-3)^2 + 2$



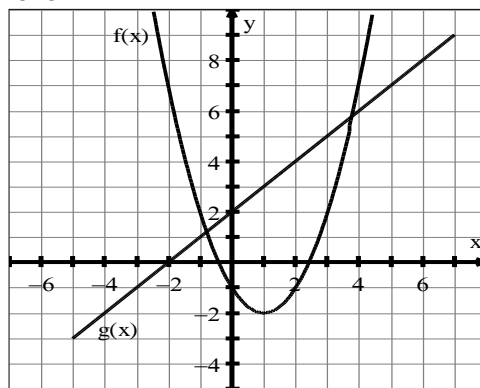
b)  $g(f(x)) = (x^2 + 2) - 3$



c)  $g(g(x)) = (x-3) - 3 = x - 6$



**Q** From the graph below, determine the value of each of the composition of functions that follow.



- a)  $f(g(1))$      b)  $(f \circ g)(-1)$      c)  $(g \circ f)(3)$      d)  $g(g(2))$

Answers: a) 2    b) -2    c) 4    d) 6

**SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them.** [C, CN, T, V, R]

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

**RF8: Assemble a function tool kit comparing various types of functions and compositions of them.**

**Scope and Sequence of Outcomes:**

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
<p><b>RF2:</b> Graph and analyze absolute value functions</p> <p><b>RF3:</b> Analyse quadratic functions of the form <math>y = a(x - p)^2 + q</math> and determine the</p> <p><b>RF4:</b> Analyse quadratic functions of the form <math>y = ax^2 + bx + c</math> to identify characteristics of the corresponding graph, including the vertex, domain and range, direction of opening, axis of symmetry, <math>x</math>- and <math>y</math>-intercepts, and to solve problems.</p> <p><b>RF5:</b> Solve problems that involve quadratic equations.</p>	<p><b>RF1.</b> Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.</p> <p><b>RF2.</b> Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.</p> <p><b>RF3.</b> Apply translations and stretches to the graphs and equations of functions</p> <p><b>RF4.</b> Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the <math>x</math>-axis, <math>y</math>-axis, line <math>y = x</math>.</p> <p><b>RF5.</b> Demonstrate an understanding of inverses of relations.</p> <p><b>RF6.</b> Graph and analyze radical functions (limited to functions involving one radical).</p> <p><b>RF7.</b> Demonstrate an understanding of exponential functions.</p> <p><b>RF8.</b> Demonstrate an understanding of logarithms.</p> <p><b>RF9.</b> Graph and analyze exponential and logarithmic functions.</p> <p><b>RF10.</b> Demonstrate an understanding of the product, quotient and power laws of logarithms.</p> <p><b>RF11.</b> Solve problems that involve exponential and logarithmic equations.</p>	<p><b>RF8:</b> Assemble a function tool kit comparing various types of functions and compositions of them.</p>

**ELABORATION**

Students have studied the properties and sketched the corresponding graphs for linear, absolute value, polynomial, radical, trigonometric, logarithmic, exponential, reciprocal and rational functions. Some functions have been compared, but students will now consolidate their learning as they develop a **Function Tool-Kit**. Included will be the ability to sketch compositions  $f(g(x))$  or  $f \circ g$  of these functions and discuss their properties.

Development of a Function Tool-Kit will be foundational for the development of concepts taught in Calculus. Students will learn the shapes of the graphs of various functions and discover that they will all change in the same way through transformations, including translations, reflections, and stretches. Graphing calculators and/or graphing software will be used to enhance understanding.

**SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]**

All functions can be written as the vertex form of a quadratic. The only change will be in the process that is indicated for  $(x - h)$ .

For example, when  $x$  is transformed to  $x - h$ , a quadratic function can be written as  $y = a(x - h)^2 + k$ , an exponential function as  $y = a^{b(x-h)} + k$ , and a sine function as  $y = a \sin b(x - h) + k$ .

Students will assemble a “toolkit” of graphs that illustrates the effects of a variety of specific values of  $a, b, h, k$ .

As each new function is explored it will be added to the student’s toolkit. Graphs will be sketched when possible, or created with technology. For each graph students will determine the domain and range, and will identify all key points including maxima/minima, intercepts and any horizontal or vertical asymptotes.

When sketched, the functions can show only the  $x$ - and  $y$ -axis, and the approximate position of the  $x$ -intercepts. In order to find the exact  $x$ -intercepts of the functions the solution methods for various types of equations will need to be reviewed. More exact graphs can then be drawn on graph paper.

In this course students will also sketch composite functions,  $h(x) = f(g(x))$ , using their knowledge of domain and range, intercepts and asymptotes for both  $f(x)$  and  $g(x)$ .

Graphing of a composite function,  $h(x) = f(g(x))$ , should be done in steps. The graph of  $g(x)$  should be sketched first and then the domain and characteristics of  $f(x)$  should be considered to complete the graph of the composite. Technology can be used to check answers. Students should be able to explain their solutions.

### **ACHIEVEMENT INDICATORS**

- Provide examples of linear, absolute value, polynomial, trigonometric, radical, logarithmic, exponential reciprocal and radical functions
- Discuss the domain and range of each of the above functions.
- Form compositions and where appropriate, inverses of these functions.
- Compare the functions in the toolkit, considering domain, range, asymptotes, periodicity, and basic shapes of their graphs.
- Graph compositions of each of the above functions.



SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]

### **Suggested Instructional Strategies**

- Graphing calculators or graphing software can be used to quickly review the shapes and transformations of the various types of functions. Key points,  $x$ - and  $y$ -intercepts, asymptotes, and domain and range can also be more readily investigated and discussed using technology.
- Students should be shown many examples, and families of curves should be discussed.
- Asymptotes should lead to an informal introduction to limits.
- Students can work in pairs so that they can discuss the graphs and their characteristics or can work individually and then check their graphs with other students.

### **Suggested Questions (Q) and Activities (Act)**

*Note: provide grids for each question.*

- Q** (i) Use the Function Tool Kit to draw a rough sketch of each of the following functions.  
 (ii) Show the nature of key point(s) on each sketch.  
 (iii) State the domain and range of each.

a)  $y = \frac{1}{2}(x + 4)^2 + 3$

b)  $y = -2(x - 5)^3 - 3$

c)  $y = -3\sqrt{x + 4} - 3$

d)  $y = 3|x - 2| + 4$

e)  $y = \frac{3}{x-2} + 5$

f)  $y = -3(x - 4)^2 + 6$

g)  $y = 4^{x-2} + 3$

h)  $y = \log_{10}(x - 3) - 4$

i)  $y = \frac{-2}{x+4} - 3$

j)  $y = \frac{-1}{2}\sqrt{x} - 3$

k)  $y = \frac{1}{(x+3)^2} - 2$

l)  $y = \ln(x - 6) + 3$

m)  $y = \frac{-1}{3}|x + 4| - 2$

n)  $y = e^{x+3} - 2$

o)  $y = \sqrt{-x} + 2$

p)  $y = \frac{-1}{(x-3)^2} + 4$

q)  $y = \frac{2}{\sqrt{x}} + 4$

r)  $y = |(x - 2)^2 - 3|$

s)  $y = \left| \frac{1}{x+3} \right| - 2$

t)  $y = |\log_5(x - 2)|$

u)  $y = 2 \sin\left(x - \frac{\pi}{4}\right) + 1$

v)  $y = -3 \cos\left(x + \frac{\pi}{3}\right) - 4$

SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]

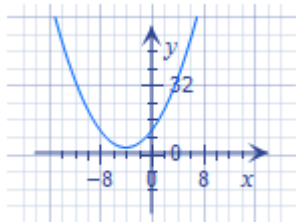
Answers:

a)  $y = \frac{1}{2}(x+4)^2 + 3$

vertex  $(-4, 3)$

$D\{x|x \in R\}$

$R\{y|y \geq 3, y \in R\}$

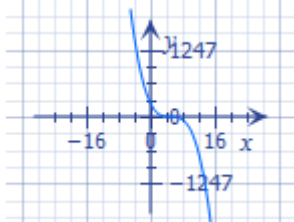


b)  $y = -2(x-5)^3 - 3$

KP  $(5, -3)$

$D\{x|x \in R\}$

$R\{y|y \in R\}$

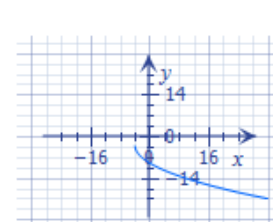


c)  $y = -3\sqrt{x+4} - 3$

KP  $(-4, -3)$

$D\{x|x \geq -4, x \in R\}$

$R\{y|y \leq -3, y \in R\}$

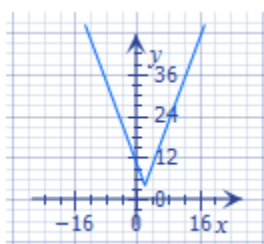


d)  $y = 3|x-2| + 4$

KP  $(2, 4)$

$D\{x|x \in R\}$

$R\{y|y \geq 4, y \in R\}$

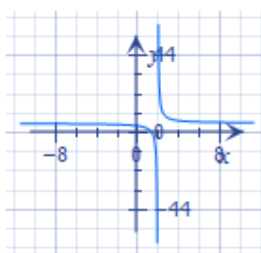


e)  $y = \frac{3}{x-2} + 5$

VA  $x = 2$ , HA  $y = 5$

$D\{x|x \neq 2, x \in R\}$

$R\{y|y \neq 5, y \in R\}$

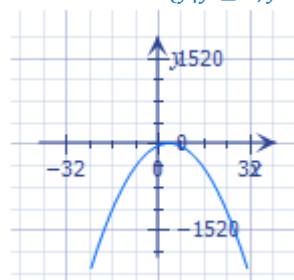


f)  $y = -3(x-4)^2 + 6$

KP  $(4, 6)$

$D\{x|x \in R\}$

$R\{y|y \leq 6, y \in R\}$

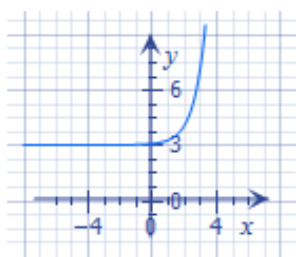


g)  $y = 4^{x-2} + 3$

$(x, y) \rightarrow (x+2, y+3)$

$D\{x|x \in R\}$

$R\{y|y > 3, y \in R\}$

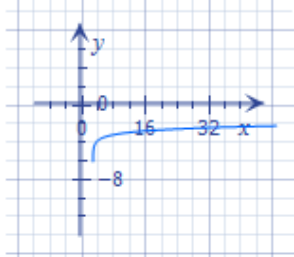


h)  $y = \log_{10}(x-3) - 4$

$(x, y) \rightarrow (x+3, y-4)$

$D\{x|x > 3, x \in R\}$

$R\{y|y \in R\}$

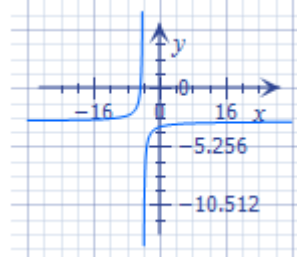


i)  $y = \frac{-2}{x+4} - 3$

VA  $x = -4$ , HA  $y = -3$

$D\{x|x \neq -4, x \in R\}$

$R\{y|y \neq -3, y \in R\}$



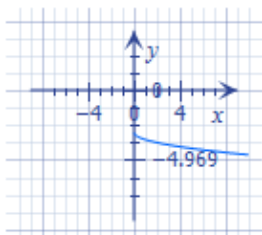
SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]

j)  $y = \frac{-1}{2}\sqrt{x} - 3$

KP(0, -3)

$D\{x|x \geq 0, x \in R\}$

$R\{y|y \leq -3, y \in R\}$

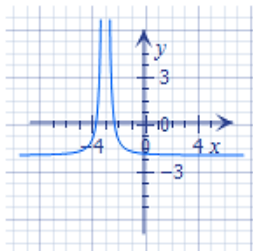


k)  $y = \frac{1}{(x+3)^2} - 2$

VA  $x = -3$ , HA  $y = -2$

$D\{x|x \neq -3, x \in R\}$

$R\{y|y > -2, y \in R\}$



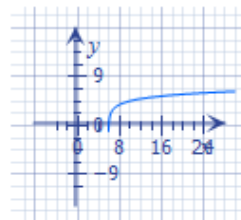
l)  $y = \ln(x - 6) + 3$

$(x, y) \rightarrow (x + 6, y + 3)$

$(1, 0) \rightarrow (7, 3)$  VA  $x = 6$

$D\{x|x \neq 6, x \in R\}$

$R\{y|y \neq 3, y \in R\}$

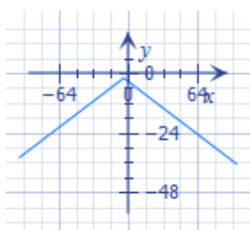


m)  $y = \frac{-1}{3}|x + 4| - 2$

KP(-4, -2)

$D\{x|x \in R\}$

$R\{y|y \leq -2, y \in R\}$



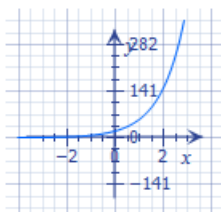
n)  $y = e^{x+3} - 2$

$(x, y) \rightarrow (x - 3, y - 2)$

$(0, 1) \rightarrow (-3, -1)$

$D\{x|x \in R\}$

$R\{y|y > -2, y \in R\}$

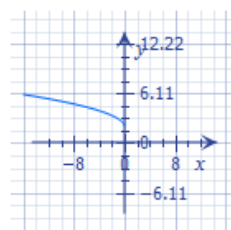


o)  $y = \sqrt{-x} + 2$

KP(0, 2)

$D\{x|x \leq 0, x \in R\}$

$R\{y|y \geq 2, y \in R\}$

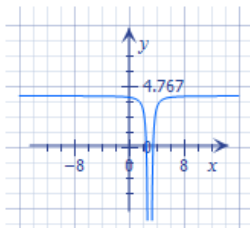


p)  $y = \frac{-1}{(x-3)^2} + 4$

VA  $x = 3$ , HA  $y = 4$

$D\{x|x \neq 3, x \in R\}$

$R\{y|y < 4, y \in R\}$

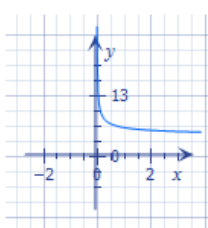


q)  $y = \frac{2}{\sqrt{x}} + 4$

VA  $x = 0$ , HA  $y = 4$

$D\{x|x > 0, x \in R\}$

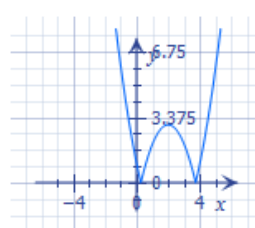
$R\{y|y > 4, y \in R\}$



r)  $y = |(x - 2)^2 - 3|$

$D\{x|x \in R\}$

$R\{y|y \geq 0, y \in R\}$



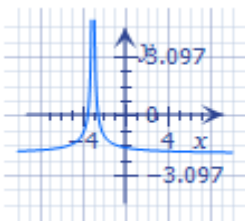
**SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]**

s)  $y = \left| \frac{1}{x+3} \right| - 2$

$VA \ x = -3, HA \ y = 0$

$D\{x|x \neq -3, x \in R\}$

$R\{y|y > -2, y \in R\}$



t)  $y = |\log_5(x - 2)|$

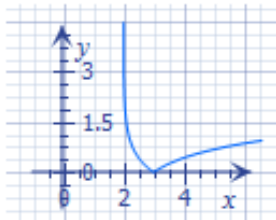
$(x, y) \rightarrow (x + 2, y)$

$(1, 0) \rightarrow (3, 0)$

$VA \ x = 2$

$D\{x|x > 2, x \in R\}$

$R\{y|y \geq 0, y \in R\}$

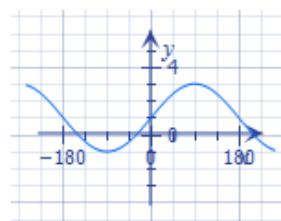


u)  $y = 2 \sin\left(x - \frac{\pi}{4}\right) + 1$

$SA \ y = 1$

$D\{x|x \in R\}$

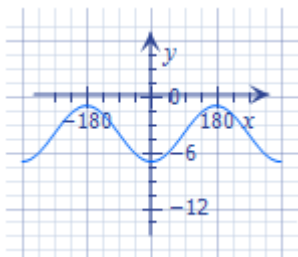
$R\{y|-1 \leq y \leq 3, y \in R\}$



v)  $y = -3 \cos\left(x + \frac{\pi}{3}\right) - 4$

$D\{x|x \in R\}$

$R\{y|-7 \leq y \leq -1, y \in R\}$



**SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them.** [C, CN, T, V, R]

- Q** (i) Find the  $x$ - and  $y$ -intercepts  
 (ii) Sketch the graph of each of the following.  
 (iii) State the domain and range.  
 (iv) Where appropriate, indicate points of discontinuity, asymptotes (vertical and horizontal) and period.
- a)  $y = -3(x - 2)^2 + 9$

c)  $y = -2\sqrt{x - 3} + 6$

e)  $y = 3^{x+1} + 4$

g)  $y = \frac{5}{x-5} - 2$

i)  $y = -(x - 2)^3 - 1$

k)  $y = 2^{x-1} - 10$

m)  $y = \left| \cos\left(x + \frac{\pi}{6}\right) \right|$

b)  $y = 2|x + 3| + 6$

d)  $y = -(x - 4)^3 + 1$

f)  $y = \ln(x - 4) + 3$

h)  $y = \frac{-2}{(x-4)^3} + 5$

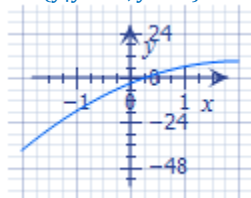
j)  $y = e^{x+3} + 9$

l)  $y = \log_2(x - 3) - 4$

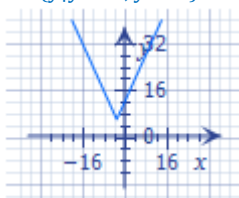
n)  $y = \log(\sin x)$

**Answers:**

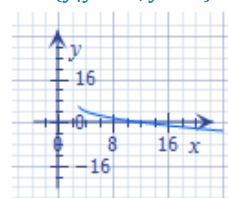
a)  $y = -3(x - 2)^2 + 9$   
 $x$  int (3.73,0)(0.27,0)  
 $y$  int (0, -3)  
 $D\{x|x \in \mathbb{R}\}$   
 $R\{y|y \leq 9, y \in \mathbb{R}\}$



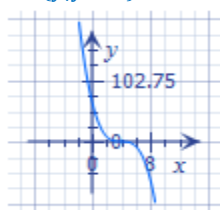
b)  $y = 2|x + 3| + 6$   
 KP (-3,6)  
 no  $x$  int,  $y$  int (0,12)  
 $D\{x|x \in \mathbb{R}\}$   
 $R\{y|y \geq 6, y \in \mathbb{R}\}$



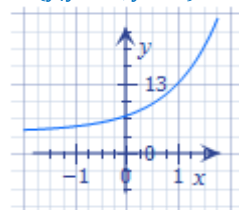
c)  $y = -2\sqrt{x - 3} + 6$   
 KP (3,6)  
 $x$  int (12,0), no  $y$  int  
 $D\{x|x \geq 3, x \in \mathbb{R}\}$   
 $R\{y|y \leq 6, y \in \mathbb{R}\}$



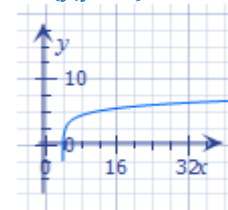
d)  $y = -(x - 4)^3 + 1$   
 KP (4,1)  
 $x$  int (5,0),  $y$  int (0,65)  
 $D\{x|x \in \mathbb{R}\}$   
 $R\{y|y \in \mathbb{R}\}$



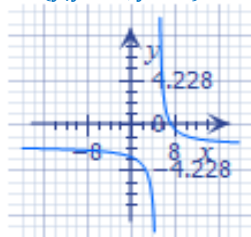
e)  $y = 3^{x+1} + 4$   
 HA  $y = 4$   
 no  $x$  int,  $y$  int (0,7)  
 $D\{x|x \in \mathbb{R}\}$   
 $R\{y|y \geq 4, y \in \mathbb{R}\}$



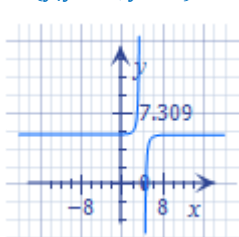
f)  $y = \ln(x - 4) + 3$   
 VA  $x = 4$   
 $x$  int ( $e^{-3} + 4, 0$ ), no  $y$  int  
 $D\{x|x > 4, x \in \mathbb{R}\}$   
 $R\{y|y \in \mathbb{R}\}$



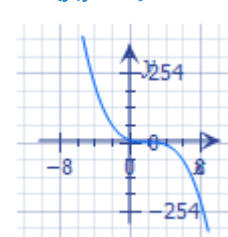
g)  $y = \frac{5}{x-5} - 2$   
 VA  $x = 5$ , HA  $y = -2$   
 $x$  int (7.5,0),  $y$  int (0, -3)  
 $D\{x|x \neq 5, x \in \mathbb{R}\}$   
 $R\{y|y \neq -2, y \in \mathbb{R}\}$



h)  $y = -\frac{2}{(x-4)^3} + 5$   
 VA  $x = 4$ , HA  $y = 5$   
 $x$  int ( $\sqrt[3]{\frac{2}{5}} + 4, 0$ ),  $y$  int ( $0, 5\frac{1}{32}$ )  
 $D\{x|x \neq 4, x \in \mathbb{R}\}$   
 $R\{y|y \neq 5, y \in \mathbb{R}\}$



i)  $y = -(x - 2)^3 - 1$   
 KP (2, -1)  
 $x$  int (1,0),  $y$  int (0,7)  
 $D\{x|x \in \mathbb{R}\}$   
 $R\{y|y \in \mathbb{R}\}$



**SCO: RF8: Assemble a function toolkit comparing various types of functions and compositions of them.** [C, CN, T, V, R]

j)  $y = e^{x+3} + 9$

HA  $y = 9$

no x int

y int  $(0, e^3 + 9 \cong 29)$

$D\{x|x \in \mathbb{R}\}$

$R\{y|y > 9, y \in \mathbb{R}\}$



k)  $y = 2^{x-1} - 10$

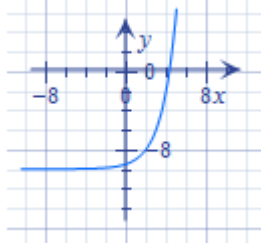
HA  $y = -10$

x int  $(\frac{\log 10}{\log 2} + 1 \cong 4.3, 0)$

y int  $(0, -9\frac{1}{2})$

$D\{x|x \in \mathbb{R}\}$

$R\{y|y > -10, y \in \mathbb{R}\}$



l)  $y = \log_2(x - 3) - 4$

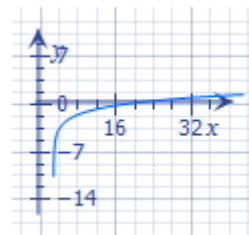
VA  $x = 3$

x int  $(19, 0)$

no y int

$D\{x|x > 3, x \in \mathbb{R}\}$

$R\{y|y \in \mathbb{R}\}$



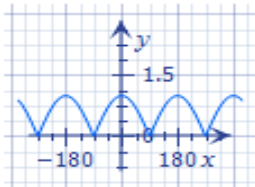
m)  $y = |\cos(x + \frac{\pi}{6})|$

x int  $(\frac{\pi}{3} + \pi n(n \in \mathbb{I}), 0)$

y int  $(0, \frac{\sqrt{3}}{2})$

$D\{x|x \in \mathbb{R}\}$

$R\{y|0 \leq y \leq 1, y \in \mathbb{R}\}$



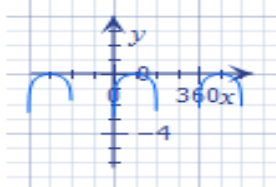
n)  $y = \log(\sin x)$

x int  $(\frac{\pi}{2} + 2\pi n(n \in \mathbb{I}), 0)$

no y int

$D\{x|x \neq 0, x \neq 0 + 2\pi n(n \in \mathbb{I}), x \neq \pi, x \neq \pi + 2\pi n(n \in \mathbb{I}), x \in \mathbb{R}\}$

$R\{y|y \leq 0, y \in \mathbb{R}\}$



**SCO: PCB1: Apply the fundamental counting principle to solve problems. [C, PS, R, V]**

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

## Permutations, Combinations and Binomial Theorem

**PCB1: Apply the fundamental counting principle to solve problems.**

### Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
	<b>PCB1:</b> Apply the fundamental counting principle to solve problems.

### ELABORATION

For this outcome, students will use graphic organizers, such as tree diagrams, to visualize and calculate sample space. From patterns observed, they will formulate an understanding of the **Fundamental Counting Principle**.

This **Fundamental Counting Principle**, also known as the **Multiplication Principle**, states that if one event has  $m$  possible outcomes and a second independent event has  $n$  possible outcomes, there will be a total of  $m \times n$  possible outcomes for these two outcomes occurring together. This enables finding the number of outcomes without listing and counting each one.

### ACHIEVEMENT INDICATORS

- Count the total number of possible choices that can be made, using graphic organizers such as lists and tree diagrams.
- Explain, using examples, why the total number of possible choices is found by multiplying rather than adding the number of ways the individual choices can be made.
- Solve a contextual problem by applying the fundamental counting principle.

### Suggested Instructional Strategies

- Begin instruction with having students construct tree diagrams to represent different outfits (tops, pants, socks, shoes) or different meal combinations served in a restaurant (appetizers, main course, desserts, beverages), and then count the total number of different arrangements/combinations in each situation. Ask students to write a journal entry summarizing any shortcuts they observe as to how they can calculate the total number of outcomes. Verify the results obtained by applying the *Fundamental Counting Principle* to each situation.

**SCO: PCB1: Apply the fundamental counting principle to solve problems. [C, PS, R, V]**

**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

- Q** A restaurant offers “select-your-own” sundaes choosing one item from each of three categories:

**Ice Cream**

vanilla  
strawberry  
chocolate

**Sauce**

chocolate  
caramel  
mint

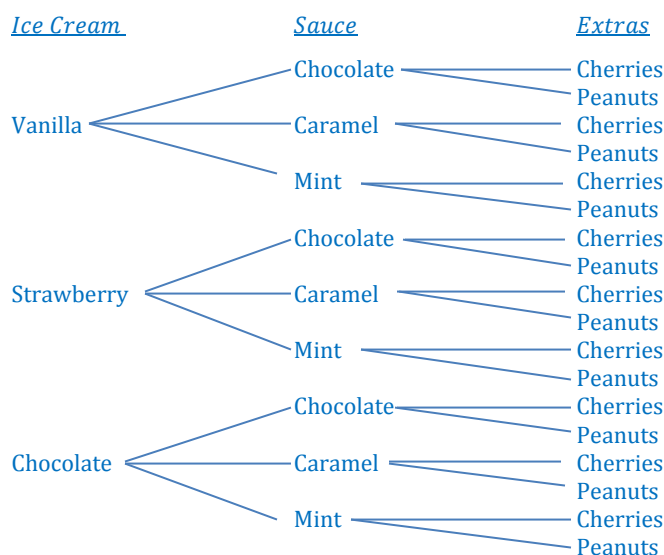
**Extras**

cherries  
peanuts

- a) Using a tree diagram, list all possible desserts that can be ordered.  
b) Find the number of ways to make a sundae using the Fundamental Counting Principle and compare your answer to your answer in part a).

*Answer:*

*a) 18 different sundaes*

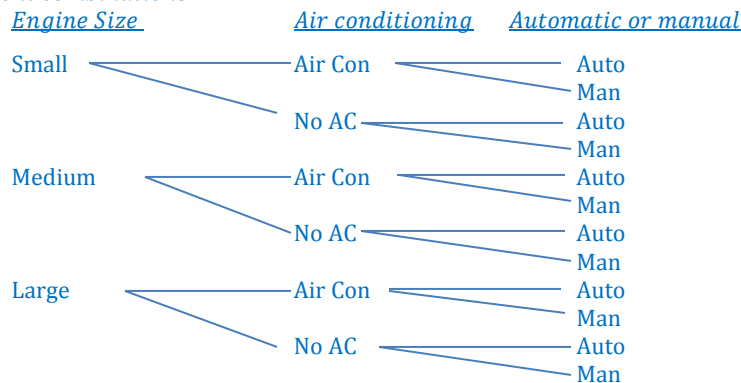


*b)  $3 \times 3 \times 2 = 18$  unique sundaes*

- Q** A certain model car can be ordered with one of three engine sizes, with or without air conditioning, and with automatic or manual transmission.

- a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.  
b) Calculate the number of different ways this model care can be ordered.

*Answer: a) 12 different combinations*



*b)  $3 \times 2 \times 2 = 12$  unique combinations*



**SCO: PCB1: Apply the fundamental counting principle to solve problems. [C, PS, R, V]**

- Q** In a restaurant there are 4 kinds of soup, 12 different entrees, 6 desserts, and 3 kinds of drinks. How many different four-course meals can a patron choose from?

*Answer:  $4 \times 12 \times 6 \times 3 = 864$  unique combinations*

- Q** a) How many different four digit numbers can you create from the digits 2, 5, 7 and 9?  
b) How many different numbers in total could you create using the digits 0 to 9?

*Answers: a)  $4 \times 4 \times 4 \times 4 = 256$       b)  $10 \times 10 \times 10 \times 10 = 10\,000$*

SCO: PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems. [C, PS, R, V]

[C] Communication  
[T] Technology

[PS] Problem Solving  
[V] Visualization

[CN] Connections  
[R] Reasoning

[ME] Mental Math  
and Estimation

PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems.

### Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
	PCB2: Determine the number of permutations of $n$ elements taken $r$ at a time to solve problems.

### ELABORATION

As students create tree diagrams and determine the number of possible outcomes using the fundamental counting principle, they should learn to recognize and use  **$n!$  ( $n$  factorial)** to represent the number of ways to arrange  $n$  distinct objects.

**$n$  factorial** or  **$n!$**  is the product of all natural numbers less than or equal to  $n$ .

In general,  $n! = n(n-1)(n-2)\dots(3)(2)(1)$ , where  $n \in \mathbb{N}$  and  $0! = 1$ .

e.g.,  $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

When order matters, factorials can be used to find the number of possible arrangements or **permutations** for a given number of people or objects.

For example,  $3!$  will give the number of permutations for three people standing in a line. There are three people to choose from for the first position, two people left to choose from for the second position in the line, and only one person to choose for the end of the line, or  $3! = 3 \times 2 \times 1 = 6$  possible arrangements.

Factorials are also used when there is a limited number of positions for a greater number of people or objects, and the order matters. However, in this case the number of permutations is determined with reference to both the total number of elements and the number of positions available.

For example, if 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> prizes are to be awarded to a group of 8 trumpeters, there is a choice of 8 people for the 1<sup>st</sup> prize, 7 people for the 2<sup>nd</sup> prize, and 6 people for the 3<sup>rd</sup> prize. Therefore there are  $8 \times 7 \times 6 = 336$  ways to award 3 prizes to 8 people (assuming no difference between performances!), or more formally stated, there are 336 permutations of 8 objects taken 3 at a time.

This is expressed  ${}_8P_3 = 8 \times 7 \times 6$ .

It is equivalent to  ${}_8P_3 = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$ .

The general form is  ${}_nP_r$  for  $n$  objects taken  $r$  at a time, so  ${}_nP_r = \frac{n!}{(n-r)!}$

**SCO: PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems. [C, PS, R, V]**

Another situation is when some objects of a set are the same. In this case there are fewer permutations because some arrangements are identical. To avoid counting the permutations of letters more than once, the number of permutations will be equal to

$\frac{n!}{n_1! n_2! n_3! \dots n_r!}$  when  $n$  is the total number of objects, and  $n_1, n_2, \dots, n_r$  are numbers of identical objects, with  $n = n_1 + n_2 + \dots + n_r$ . For example, for the word BANANA there is one B, ( $n_1 = 1$ ), two N's ( $n_2 = 2$ ) and three A's ( $n_3 = 3$ ), so the total number of permutations of the letters for the word BANANA will be  $P = \frac{6!}{1!2!3!} = 60$ .

### **ACHIEVEMENT INDICATORS**

- Count, using graphic organizers such as lists and tree diagrams, the number of ways of arranging the elements of a set in a row.
- Determine, in factorial notation, the number of permutations of  $n$  different elements taken  $n$  at a time to solve a problem.
- Determine, using a variety of strategies, the number of permutations of  $n$  different elements taken  $r$  at a time to solve a problem.
- Explain why  $n$  must be greater than or equal to  $r$  in the notation  ${}_nP_r$ .
- Solve an equation that involves  ${}_nP_r$  notation, such as  ${}_nP_2 = 30$ .
- Explain, using examples, the effect on the total number of permutations when two or more elements are identical.

### **Suggested Instructional Strategies**

- Using their prior knowledge, allow students to work in groups to solve a simple problem similar to the following:

Adam, Marie, and Brian line up at a banking machine. In how many different ways could they order themselves?

Using a systematic list students might come up with the following solution:  
 $AMB, ABM, MBA, MAB, BAM, BMA$ .

Repeat with a similar type question, asking students to look for patterns which involve the use of factorials. Have them generalize the formula to apply for  $n$  objects selected  $r$  at a time.

SCO: PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems. [C, PS, R, V]

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Q** Which of the following will produce the number of greatest magnitude? (Use estimation first and then calculate each value.) Which will produce the smallest? How could you predict this without working out all the answers?

- a)  $5!$       b)  $9!$       c)  $\frac{16!}{10!}$       d)  $3!4$   
 e)  $\frac{8!}{3!}$       f)  $\frac{10!}{4!}$       g)  $\frac{100!}{57!}$       h)  $4! - 3!$       i)  $\frac{12!}{7!3!}$

Pick three of the above expressions and create a problem in which these symbols would be used in the solution.

*Answers: smallest to greatest*

*h) 18   d) 24   a) 120   e) 6720   i) 15 840   f) 151 200   b) 362 880   c) 5 765 760   g)  $2.3 \times 10^{81}$*

**Q** Write each as a ratio of factorials.

- a)  $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$   
 b)  $35 \times 34 \times 33 \times 32 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$   
 c)  $14 \times 13 \times 8 \times 7 \times 3 \times 2 \times 1$   
 d)  $\frac{25 \times 24 \times 23 \times 5 \times 4 \times 3 \times 2 \times 1}{9 \times 8}$

*Answers: a)  $7!$    b)  $\frac{35!6!}{31!}$    c)  $\frac{14!8!3!}{12!6!}$    d)  $\frac{25!7!5!}{22!9!}$*

**Q** Consider the word COMPUTER and the ways you can arrange its letters using each letter only once.

- a) One possible permutation is PUTMERO C. Write five other possible permutations.  
 b) Use factorial notation to represent the total number of permutations possible.  
 Write a written explanation to explain why your expression makes sense.

*Answers:*

*a) answers will vary*

*b) There are 8 letters for the 1st choice, 7 for the 2nd choice etc. thus  $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$*

**Q** Determine the number of ways four different graduation scholarships can be awarded to 30 students under each of the following conditions:

- a) No student may receive more than one scholarship.  
 b) Any student may receive any number of scholarships.

*Answers:*

*a)  $30 \times 29 \times 28 \times 27 = 657\,720$  or  ${}_{30}P_4$*

*b)  $30 \times 30 \times 30 \times 30 = 810\,000$*

**Q** Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random.

- a) Assuming all players can play each position on the field, in how many ways could the team be fielded?  
 b) If Martha must be the catcher, in how many ways could the team be fielded?

*Answers: a)  $9! = 362\,880$    b)  $8! = 40\,320$*

**SCO: PCB2: Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems.** [C, PS, R, V]

- Q** License plates are made using three letters followed by three digits.
- How many different license plates can be made if repetitions are not allowed?
  - How many different license plates can be made if all of them have to start with an  $N$ ?
  - How many different license plates can be made if all of them start with  $B$  and end with a 5?

*Answers:*

- a)  $26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11\,232\,000$   
 b)  $1 \times 26 \times 26 \times 10 \times 10 \times 10 = 67\,600$   
 c)  $1 \times 26 \times 26 \times 10 \times 10 \times 1 = 67\,600$

- Q** A student has five textbooks, Math, Chemistry, Physics, English and History to place on the bookshelf in her room.
- In how many ways can the textbooks be arranged on the shelf?
  - If the Math textbook is always placed at the end of the row because it is used the most, in how many ways can the textbooks be arranged on the shelf?
  - If the Math textbook is on the student's desk, in how many ways can the four remaining textbooks be arranged on the shelf?
  - Compare your answers from parts c) and d) and explain your conclusions.

*Answers:*

- a)  $5 \times 4 \times 3 \times 2 \times 1 = 120$    b)  $4 \times 3 \times 2 \times 1 \times 1 = 24$    c)  $4 \times 3 \times 2 \times 1 = 24$   
 d) *Whether the book is on the shelf or on the desk there are still only 4 choices for the 1st book, 3 choices for the 2nd book, 2 choices for the 3rd book and 1 choice for the 4th book.*

- Q** a) How many arrangements can you make of the letters in the word "CALCULUS"?  
 b) How many of these arrangements will start with the letter "S"?  
 c) How many of them will end with the letters "AS"?

*Answers:*   a)  $\frac{8!}{2!2!2!} = 5040$    b)  $\frac{1 \times 7!}{2!2!2!} = 630$    c)  $\frac{6! \times 1 \times 1}{2!2!2!} = 90$

- Q** Four girls and three boys line up for a yearbook photo of the Student Council.
- In how many different ways could the students line up?
  - If the three boys stand together, how many different ways could they line up?
  - If a girl must be at each end of the line, how many different ways can they line up?

*Answers:*   a)  $7! = 5040$    b)  $3!5! = 720$    c)  $4 \times 5! \times 3 = 1440$

- Q** Solve the following:

- ${}_nP_2 = 56$
- ${}_5P_r = 20$

*Answers:*

a)  $\frac{n!}{(n-2)!} = 56!$

$n(n-1) = 56$

$(n-8)(n+7) = 0$

$n = 8$  or  ~~$n = 7$~~

b)  $\frac{5!}{(5-r)!} = 20$

$(5 \times 4)(5-r)! = 5!$

$(5-r)! = 3!$

$r = 2$

SCO: PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. [C, PS, R, V]

[C] Communication  
[T] Technology

[PS] Problem Solving  
[V] Visualization

[CN] Connections  
[R] Reasoning

[ME] Mental Math  
and Estimation

PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems.

### Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
	PCB3: Determine the number of combinations of $n$ different elements taken $r$ at a time to solve problems.

### ELABORATION

For this outcome, students will investigate **combinations**, where the order of the selection is not important.

For example, if a committee of 3 people is selected from a group of 5 individuals, it is not important in what order they are named, the committee still includes the same 3 people. However, the number of permutations of choosing  $ABC$  from the larger set of 5 or  ${}_5P_3$ , will include all 3! permutations of  $ABC$ . To eliminate counting each permutation as unique, the total number of possible permutations must be divided by 3!. Thus the number of combinations, denoted as  ${}_nC_r$ , or  $\binom{n}{r}$  or  $n$  choose  $r$ , would be  ${}_nC_r = \frac{{}_nP_r}{r!}$  which simplifies to  $\frac{n!}{r!(n-r)!}$ . For this example  ${}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = 10$ , so there are 10 possible ways to select 3 people from a group of 5.

In the case of selecting two groups from the same larger group, the number of possible combinations will combine both groups.

For example, if a committee of 4 people and a committee of 3 people are selected from a group of 10 people, and no person is assigned to both committees the combinations for each committee will be determined first.

For the first committee of 4 from 10 people,  ${}_{10}C_4 = \binom{10}{4} = 210$ , so there are 210 ways to form the committee. There are just 6 people left, so for the second committee of 3 from 6 people,  ${}_6C_3 = \binom{6}{3} = 20$ , and there are 20 ways to form this committee. Combining the two committees, there are  $210 \times 20 = 4200$  ways to form both committees from 10 people. Selecting the smaller committee first will yield the same result.

Combinations are sometimes used along with other counting techniques.

For example, a 17 member student council at the high school consists of 9 girls and 8 boys, and one of the committees has 4 council members, and must have 2 girls and 2 boys. There are  ${}_9C_2 = \frac{9!}{(2!)(7!)} = 36$  ways of selecting the 2 girls, and  ${}_8C_2 = \frac{8!}{(2!)(6!)} = 28$  ways of selecting 2 boys.

**SCO: PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. [C, PS, R, V]**

Because the committee must include 2 girls and 2 boys, there are  $36 \times 28 = 1008$  ways of forming the committee. If the 4 committee members are selected at random, there are  ${}_{17}C_4 = \frac{17!}{4!(13!)} = 2380$  possible combinations.

### **ACHIEVEMENT INDICATORS**

- Explain, using examples, the difference between a permutation and a combination.
- Determine the number of ways that a subset of  $k$  elements can be selected from a set of  $n$  different elements.
- Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve a problem.
- Explain why  $n$  must be greater than or equal to  $r$  in the notation  ${}_nC_r$  or  $\binom{n}{r}$ .
- Explain, using examples, why  ${}_nC_r = {}_nC_{n-r}$  or  $\binom{n}{r} = \binom{n}{n-r}$ .
- Solve an equation that involves  ${}_nC_r$  or  $\binom{n}{r}$  notation, such as  ${}_nC_2 = 15$  or  $\binom{n}{2} = 15$ .

### **Suggested Instructional Strategies**

- Group work is effective here. Have groups of students use systematic lists, and to look for patterns as they solve problems. Having worked previously with permutations and the formula involving factorials, now extend the formula. Have them work towards generalizing the formula to apply for  $n$  objects selected  $r$  at a time.
- When students are comfortable determining the number of combinations, introduce the use of combinations to determine wanted probabilities. This is an application of the use of combinations.

SCO: PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. [C, PS, R, V]

### **Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q** If a coin is flipped five times, how many ways could you get four heads and one tail?

*Answer:*  ${}_5C_4 = 5$

**Q** A scratch-and-win game has nine prize boxes. You are allowed to reveal three pictures. If the three pictures show identical objects, you win the object.

- a) If the ticket has four pictures of an Ipad and six other pictures, each showing a different object, how many ways are there to win?
- b) If the ticket has three pictures of an Ipad, four pictures of a computer game and two other pictures, how many ways are there to win?

*Answer:*

a)  ${}_4C_3 = 4$  ways to pick 3 Ipads

b) 1 way to win an Ipad, and 4 ways to win computer games = 5 ways to win

**Q** A coin purse contains 20 quarters and 10 dimes. It is shaken until two coins drop out.

- a) In how many ways can two quarters fall out?
- b) In how many ways can three dimes fall out?
- c) In how many ways can one of each type of coin fall out?

*Answer:*

a)  ${}_{20}C_2 = 190$  ways    b) none since only two coins drop out

c)  ${}_{20}C_1(\text{quarters}) \times {}_{10}C_1(\text{dimes}) = 200$  ways

**Q** A committee of six is chosen from a group of five males and six females. In how many ways can each of the following committees be formed?

- a) Two males and four females are chosen.
- b) All females are chosen.

*Answers:* a)  ${}_5C_2 \times {}_6C_4 = 150$  ways    b)  ${}_6C_6 = 1$  way

**Q** A golf bag contains 16 white balls, 10 yellow balls, and 8 green balls. Glen removes four balls at random, one for each of the four people about to tee off.

- a) In how many ways can he remove four yellow balls?
- b) In how many ways can he remove two white balls and two yellow balls?

*Answers:* a)  ${}_{10}C_4 = 210$  ways    b)  ${}_{16}C_2 \times {}_{10}C_2 = 5400$  ways

**Q** You have 8 summer t-shirts. How many ways can you pick the five you will pack to take on vacation?

*Answer:*  ${}_8C_5 = 56$  ways

**Q** There are 10 books on a shelf. In how many ways can you choose 6 books from the shelf?

*Answer:*  ${}_{10}C_6 = 210$  ways



**SCO: PCB3: Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. [C, PS, R, V]**

- Q** a) In how many ways can a 12 member organizing committee for the school prom be chosen from 30 students?  
 b) Is there another size of committee that could be chosen the same number of ways? Explain.

*Answers: a)  ${}_{30}C_{12} = 286\,493\,225$  ways b) Yes, an 18 person committee would be the same*

**Q** Solve the following.

a)  ${}_nC_4 = 126$

b)  ${}_nC_2 = {}_nC_6$

*Answers:*

a)  ${}_nC_4 = 126$

$n(n-1)(n-2)(n-3) = 126(4!)$  or guess  $\rightarrow$  4 consecutive #'s whose product is 3024

$(n^2 - n)(n^2 - 5n + 6) = 3024$

$n^4 - 6n^3 + 11n^2 - 6n - 3024 = 0$

~~$n = -6$~~  or  $n = 9$  (must be  $\in \mathbb{N}$ )

b)  ${}_nC_2 = {}_nC_6$  or  $\frac{n!}{(n-2)!2!} = \frac{n!}{(n-6)!6!}$

$r = 2, n - r = 6$  so  $n = 8$   $(n-2)!2! = (n-6)!6!$

$\frac{(n-2)!}{(n-6)!} = \frac{6!}{2!}$

$(n-2)(n-3)(n-4)(n-5) = 360$

$n = 8$

**SCO: PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]**

**[C]** Communication  
**[T]** Technology

**[PS]** Problem Solving  
**[V]** Visualization

**[CN]** Connections  
**[R]** Reasoning

**[ME]** Mental Math  
and Estimation

**PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).**

### Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
	<b>PCB4:</b> Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

### ELABORATION

This outcome will introduce new material to students, building on their knowledge of multiplying binomials. For  $(a + b)(c + d) = ac + bc + ad + bd$ , students should notice that each term in the expansion has one factor from  $(a + b)$  and one factor from  $(c + d)$ . So, for example,  $ac$  has two factors  $a$  and  $c$ . The  $a$  is from  $(a + b)$  and the  $c$  is from  $(c + d)$ . Thus the number of terms in the expansion is four since there are two choices from  $(a + b)$  and two choices from  $(c + d)$ . Students should also notice that since there are two factors  $(a + b)$ , and  $(c + d)$  there are two factors in each term of the expansion.

The product of one binomial and itself will also follow the same pattern,  $(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy$ , but the multiplication would be completed by collecting the like terms and using exponents,  $x^2 + 2xy + y^2$ .

Students should explore patterns as they expand  $(x + y)^n$  to develop an understanding of Pascal's triangle and the binomial theorem, building on their knowledge of combinations. For example:

for  $(x + y)^5 = (x + y)(x + y) \dots (x + y) \rightarrow xxxxx + xxxxy + \dots + yyyyy$ . Each term is made up of five factors and uses exponents, as in  $x^a y^b$  where  $a + b = 5$ . So for term 1,  $xxxxx \rightarrow x^5 \rightarrow x^5 y^0 \rightarrow 5 + 0 = 5$ . Term 2,  $xxxxy \rightarrow x^4 y \rightarrow x^4 y^1 \rightarrow 4 + 1 = 5$ . Term 3,  $xxxyy \rightarrow x^3 y^2 \rightarrow 3 + 2 = 5$  etc.

Combinations can be used to determine the coefficients for each term by determining the number of ways different terms occur. For example:

the number of times  $x^5$  or  $y^5$  occurs, can be determined as  ${}_5C_5 = 1$ . The number of times  $x^4$  or  $y^4$  occurs would be  ${}_5C_4 = 5$ , and the number of times  $x^3$  or  $y^3$  occurs is  ${}_5C_3 = 10$ . Students should then link the 5<sup>th</sup> row of Pascal's triangle, 1 5 10 10 5 1, to the binomial theorem.

Students should examine the pattern changes in the signs between terms when  $(x - y)^5$  is expanded. Because the second term in the expression could be considered negative  $(-y)$ , then the terms in the expansion that have odd numbers of  $y$ -factors will be negative.

Students should be aware that when  $x$  and  $y$  are replaced with terms that have exponents or coefficients, for every  $x$ - and  $y$ -factor, there is now a new factor.

For example, for  $(x + y)^3$  when  $x$  is replaced with  $x^2$ , and  $y$  is replaced with  $-3y$  the expansion becomes:

$$(x^2 - 3y)^3 = \binom{3}{3}(x^2)^3(-3y)^0 + \binom{3}{2}(x^2)^2(-3y)^1 + \binom{3}{1}(x^2)^1(-3y)^2 + \binom{3}{0}(x^2)^0(-3y)^3$$

**SCO: PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]**

### **ACHIEVEMENT INDICATORS**

- Explain the patterns found in the expanded form of  $(x + y)^n$ ,  $n \leq 4$  and  $n \in N$ , by multiplying  $n$  factors of  $(x + y)$ .
- Explain how to determine the subsequent row in Pascal's triangle, given any row.
- Relate the coefficients of the terms in the expansion of  $(x + y)^n$  to the  $(n + 1)$  row in Pascal's triangle.
- Explain, using examples, how the coefficients of the terms in the expansion of  $(x + y)^n$  are determined by combinations.
- Expand, using the binomial theorem,  $(x + y)^n$ .
- Determine a specific term in the expansion of  $(x + y)^n$ .

### **Suggested Instructional Strategies**

- For the binomial  $(x + y)$ , have students find simplified expressions for  $(x + y)^2$ ,  $(x + y)^3$ ,  $(x + y)^4$ , etc., look for patterns in their coefficients, and find a connection between the expansion power and that same row in the Pascal's Triangle with respect to the coefficient values. ( $(x + y)^0 = 1$  is the top row (row 0). (Extra support can be found at the Khan Academy website)

SCO: PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Q** What is the coefficient of the  $x^4y^2$  term in each of the following?

- a)  $(x + y)^6$
- b)  $(x - 2y)^6$
- c)  $(2x + y)^6$
- d)  $(3x - 2y)^6$

Answers: a) 15 b) 60 c) 240 d) 4860

**Q** When examining the terms from left to right, find the specified term in each expansion.

- a) 10th in  $(x - y)^{12}$
- b) 20th in  $(2x - 1)^{19}$
- c) 8th in  $(a + b)^{10}$
- d) 2nd in  $(x^3 - 5)^7$
- e) 3rd in  $(1 - 2x)^9$
- f) 15th in  $(1 + a^2)^{24}$

Answers: a)  $-220x^3y^9$  b)  $-1$  c)  $120a^3b^7$  d)  $-35x^{18}$  e)  $144x^2$  f)  $1\,961\,256\,a^{28}$

**Q** Find a decimal approximation for  $1.05^8$  by writing it as a binomial. Do you need to write all the terms to get close to the correct answer? Explain.

Answer:

$$1.05^8 = 1 + 8(1)(0.005) + 28(0.0025) + (0.000125) + 70(0.00000625) \dots$$

$$= 1 + 0.4 + 0.07 + 0.007 + \dots$$

$= 1.477$  You do not need all of the terms to get close to the correct answer because by the 4th ter, the numbers are so small, they have virtually no affect on the sum.

**Q** When expanding  $(a^2 - 2b)^5$ , Wally gets confused about the exponents in his answer. Write a paragraph to Wally to help him remember how to record the exponents on this expansion.

Answer: Wally should first expand the binomials  $(x + 5)^5$ . He needs to remember that for the 1st term, the exponent on the  $x$  is 5 and the exponent of the  $y$  is zero. For each subsequent term the exponent of  $x$  will decrease by 1 and the exponenet of  $y$  will increase by 1, until he reaches the final term  $x^0y^5$ . Wally should then substitute  $a^2$  for all of his  $x$  values and substitute  $-2b$  for all of his  $y$  values. He should then follow his exponent lasws and order of operations to simplify.

**Q** Henrietta is expanding  $(3a - 2b^2)^3$ . In her work below, explain what she is doing when going from step 2 to step 3. Is her work correct? Explain. What should she do to complete her work?

step 1: $a$	$ab$	$ab$	$b$
step 2: $a^3$	$a^2b$	$ab^2$	$b^3$
step 3: $(3a)^3$	$(3a)^2(2b^2)$	$(3a)^3(2b^2)^2$	$(2b^2)^3$

Answer: In step 3, she is including the coefficients with the variable. She has to include the negative with the  $(2b^2)$ . The next step is to use combinations  ${}_3C_0(3a)^3 + {}_3C_1(3a)^2(-2b^2) + {}_3C_2(3a)^1(-2b^2)^2 + {}_3C_3(-2b^2)^3$ . To complete her work she will simplify to:  $27a^3 - 54a^2b^2 - 8b^6$ .

**SCO: PCB4: Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]**

- Q** Betty Lou missed math class today. Helen phoned her at night to tell her about how combinations are helpful when expanding binomials. Write a paragraph or two about what Helen would have told her.

*Answer: Combinations relate to each term in Pascal's triangle and relate to expanding binomials. In a binomial expansion, combinations are used that relate to the exponent e.g.,  $(a + b)^4 = {}^4C_0 a^4 + {}^4C_1 a^3b + {}^4C_2 a^2b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$ . Then the terms  $a$  and  $b$  are included. The first one,  $a$ , starts at the value of the exponent, and the degree decreases for each term. The second term,  $b$ , starts at the second term of the expansion and the degree increases for each term. Addition signs are placed between the terms:  ${}^4C_0 a^4 + {}^4C_1 a^3b + {}^4C_2 a^2b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4$ . The terms are then simplified:  $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ .*

*As an enrichment activity*

- Q** a) Find the sum of the elements in each row, for the first six rows of Pascal's Triangle.  
 b) Find the number of subsets in a 1-, 2-, 3-, 4-, and 5- element set.  
 c) How are parts (a) and (b) related?  
 d) How many elements are there in an  $n$ -element set?

*Answer: a) the sums are 1, 2, 4, 8, 16, 32, 64, for each of the first six rows respectively.  
 b) 1, 3, 7, 15, and 31 subsets in a 1, 2, 3, 4, and 5 element set respectively.  
 c) The number of subsets in an  $n$  element set is one less than the sum of row  $n$  of Pascal's Triangle.  
 d)  $2^n - 1$*

**SCO: L1:** Determine the limit of a function at a point both graphically and analytically. [C, PS, T, V]

[C] Communication  
[T] Technology

[PS] Problem Solving  
[V] Visualization

[CN] Connections  
[R] Reasoning

[ME] Mental Math  
and Estimation

## Limits

**L1: Determine the limit of a function at a point both graphically and analytically.**

### Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
	<b>L1:</b> Determine the limit of a function at a point both graphically and analytically.

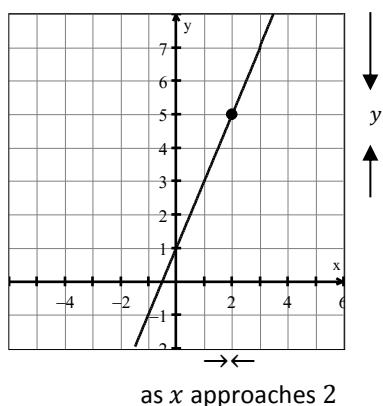
### ELABORATION

Students will need a function toolkit to begin their study of Calculus and **limits**. Students should begin with a graphical approach to analyze the behaviour of a function as  $x$  approaches a certain value. The limit indicates the range value that is approached as the value of the domain approaches a given number. Limits can be evaluated by substitution of the domain value that is being approached while following the **Laws of Limits**.

For example, if  $f(x) = 2x + 1$ , then:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (2x + 1) = \lim_{x \rightarrow 2} (2(2) + 1) = \lim_{x \rightarrow 2} (5) = 5$$

The closer that  $x$  gets to the value of 2, the closer the  $y$  value gets to 5.



#### *Approaching 2 from the left*

$x < 2$	$f(x)$
1.5	4.0
1.9	4.8
1.99	4.98
1.999	4.998
1.9999	4.9998

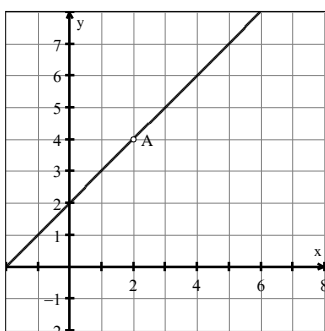
#### *Approaching 2 from the right*

$x > 2$	$f(x)$
1.5	4.0
1.9	4.8
1.99	4.98
1.999	4.998
1.9999	4.9998

If dealing with a **discontinuous function**, substitution may yield either  $\frac{0}{0}$  or  $\frac{\text{non-zero value}}{0}$ . The solution often corresponds to a “hole” in the graph, shown below as the point  $A(2,4)$ , on the graph of the function:  $f(x) = \frac{(x^2-4)}{(x-2)}$ . This is referred to as the **removable discontinuity** (point of discontinuity) when evaluating the limit.

**SCO: L1:** Determine the limit of a function at a point both graphically and analytically.  
PS, T, V]

[C,



For  $f(x) = \frac{(x^2-4)}{(x-2)}$ , substituting  $x = 2$  into the numerator and denominator yields the undefined symbol  $\frac{0}{0}$ , so  $x = 2$  is not in the domain of  $f$ .

To find point A:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{(x - 2)} = (x + 2)$$

By removing the discontinuity of the function,  $\frac{(x-2)}{(x-2)}$ , we can evaluate  $f(x) = (x + 2)$  for  $x = 2$  to identify the point A(2,4) as the removable discontinuity. We are not “cancelling”  $\frac{(x-2)}{(x-2)}$ , but removing it for evaluation purposes.

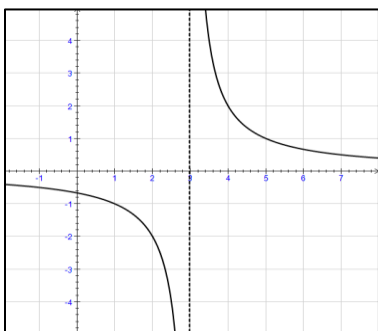
This is written as:

$$\lim_{x \rightarrow 2} \frac{(x^2 - 4)}{(x - 2)} = 4$$

The domain of this function is  $\{x \in (-\infty, 2) \cup (2, \infty)\}$ .

For some functions, when substituting a specific value for  $x$  yields  $\frac{\text{non-zero value}}{0}$  or  $\frac{0}{0}$ , there is a vertical asymptote at the excluded  $x$  value. We say that the limit from each side is  $\infty$  or  $-\infty$ , depending on the values of  $y = f(x)$  as it approaches the vertical asymptote. The limiting infinite value often “changes” sign across the asymptote as shown in the example shown below.

The following figure shows the graph of  $y = f(x)$  for the function  $f(x) = \frac{2}{x-3}$ . There is no limit for  $f(x)$  as  $x$  approaches 3.



Students should explore limits with all functions in their function toolkit.

**SCO: L1:** Determine the limit of a function at a point both graphically and analytically.  
PS, T, V]

[C,

**ACHIEVEMENT INDICATORS**

- Demonstrate an understanding of the concept of limit at a point, and the notation to express the limit of a function  $f(x)$  as  $x$  approaches  $a$  as  $\lim_{x \rightarrow a} f(x) = L$
- Evaluate the limit of a function at a point graphically, analytically and using tables of values.
- Distinguish between the limit of a function as  $x$  approaches  $a$  and the  $y$ -value of the function at  $x = a$
- Use the properties of limits to evaluate a limit by substitution of the  $x$ -value into  $f(x)$ , where possible.
- Explore undefined limits (limits that do not exist (DNE)).
- Explore limits that arise, when substituting yields an indeterminate form.

**Suggested Instructional Strategies**

- Students can be introduced to the concept of limits by using a concrete approach to help them gain an understanding of the true meaning of this term. This could be achieved by the use of graphs of functions and breaking down the tables of values. Pencil and paper techniques as well as the use of technology such as graphing calculators or computer aided systems and software packages could be used.
- One way to introduce limits (both finite and infinite) is to refer back to sequences. Have students look at values of  $t_n$  as  $n$  increases, for various arithmetic and geometric sequences. Develop the idea of “closer and closer... then guess the limit.”
- When students are ready to look at limit notation and begin solving limit problems analytically by substitution, begin with problems that yield a definite solution and compare to a picture of the graph such as any continuous function. Proceed to limits where substitution doesn't work, giving solutions with 0 as a denominator. The concept of infinity will be explored later, but could be introduced at this point.
- Review graphs from the function toolkit such as linear, quadratic, cubic, other polynomials, exponential, logarithmic, radical, and rational. Have students find limits at various points on these graphs. They could use technology to verify solutions by tracing or looking at the table of values.
- Use online sites such as: [www.calculus-help.com](http://www.calculus-help.com) as a supplementary resource.



**SCO: L1:** Determine the limit of a function at a point both graphically and analytically.  
PS, T, V]

[C,

### Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

**Act** To introduce this topic, have each student take a blank sheet of paper, and tear the sheet in half a number of times, always keeping one half and discarding the other half. Have them speculate on how many times they would tear the sheet before they could not proceed any farther. Follow this with a class discussion on how much paper they would have left at this point. What value would the amount of remaining paper approach? Would there always be some paper left? There will always be some paper remaining, but it will get so small after many tears that a student may have trouble holding and tearing it, and so there would be a limit to the amount of paper remaining. Lead the discussion to the concept of a limit.

- Q** i) Which of these limits exist and are finite?  
ii) What are the limiting values?

a)  $\lim_{x \rightarrow 1} x^2 - 4x + 3$

b)  $\lim_{x \rightarrow 5} \frac{2x + 10}{x^2 - 25}$

c)  $\lim_{x \rightarrow 1} \frac{\sqrt{x+3} - 2}{x - 1}$

d)  $\lim_{x \rightarrow 4} \frac{\frac{1}{x^2} - \frac{1}{16}}{x - 4}$

e)  $\lim_{x \rightarrow 1} \frac{3x^2 + 2x}{x^2 - 1}$

*Answer:*

a) exists and is finite   b) does not exist (DNE)   c) exists and is finite   d) exists and is finite   e) DNE

**Act** Find the limit of a given function and describe what is happening on the graph at that point. (Note to teacher: A limit question at various levels of difficulty can be used here to help differentiate your lesson.)

**Act** As a class, provide a function and have various students explain what is happening at different values of  $x$  in the domain. Students may use a graph in their explanation.

SCO: L2: Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation. [C, T, V, R]

[C] Communication  
[T] Technology

[PS] Problem Solving  
[V] Visualization

[CN] Connections  
[R] Reasoning

[ME] Mental Math  
and Estimation

L2: Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation.

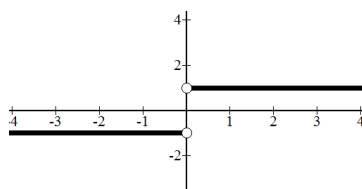
### Scope and Sequence of Outcomes:

Grade Eleven	Pre-Calc B 120
	L2: Explore and analyze left and right hand limits as $x$ approaches a certain value using correct notation.

### ELABORATION

In this outcome students explore the one-sided limits of a function. For example, for the function:  $f(x) = \frac{|x|}{x}$

$\lim_{x \rightarrow 0} f(x)$   
is shown as:



Considering only the values of  $x$  that are to the right or left of a number, you would be looking at a **one-sided limit**.

The **right-hand limit**, includes all values on the right side of the target number. In the example shown above, the target number is 0 for  $f(x) = 1$ . The right-hand limit is written:

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

The left-hand limit, includes all values to the left of the target number. For the example shown above, the target number is 0 for  $f(x) = -1$ . The left-hand limit is written:

$$\lim_{x \rightarrow 0^-} f(x) = -1$$

A function  $f(x)$  has a limit as  $x$  approaches  $c$ , if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal. This can be written as:

If

$$\lim_{x \rightarrow c^-} f(x) = L$$

and

$$\lim_{x \rightarrow c^+} f(x) = L$$

then

$$\lim_{x \rightarrow c} f(x) = L$$

A **piecewise function** is one in which different formulas are used to define it on different parts of its domain. For example,  $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

SCO: L2: Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation. [C, T, V, R]

### **ACHIEVEMENT INDICATORS**

- Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation.
- Evaluate the left and right hand limits of piecewise functions.

### **Suggested Instructional Strategies**

- The following websites provide explanatory tutorials on one sided limits:

<http://www.rootmath.org/calculus/continuity-and-one-sided-limits>

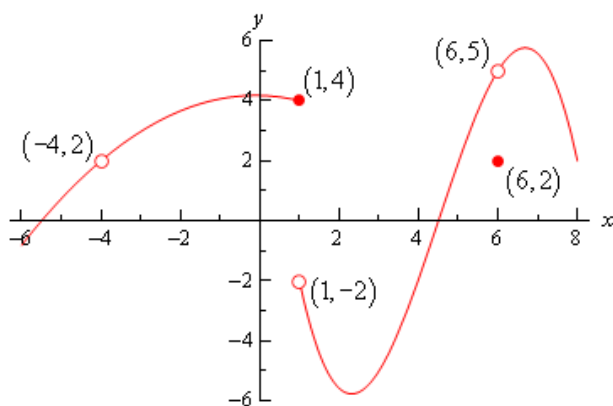
<http://www.youtube.com/watch?v=JXVUyk2JOU4>

<http://www.calculus-help.com/tutorials>

<http://ocw.mit.edu/courses/mathematics/18-01-single-variable-calculus-fall-2006/video-lectures/>

### **Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q:** Given the graph of  $f(x)$ , find the following limits:



a)  $\lim_{x \rightarrow -4^-} f(x)$

b)  $\lim_{x \rightarrow -4^+} f(x)$

c)  $\lim_{x \rightarrow -4} f(x)$

d)  $\lim_{x \rightarrow 1^-} f(x)$

e)  $\lim_{x \rightarrow 1^+} f(x)$

f)  $\lim_{x \rightarrow 1} f(x)$

Answers: a)  $f(x) = 2$  b)  $f(x) = 2$  c)  $f(x) = 2$  d)  $f(x) = 4$  e)  $f(x) = -2$  f)  $f(x) = DNE$

**Q** i) Sketch the following piecewise functions.

ii) Does the following limit exist?

$$\lim_{x \rightarrow 1} f(x)$$

a)  $f(x) = \begin{cases} x - 3, & x < 1, x \in R \\ 2 - x, & x \geq 1, x \in R \end{cases}$

Answer:

$$\lim_{x \rightarrow 1^-} f(x) = -2, \lim_{x \rightarrow 1^+} f(x) = 1, \lim_{x \rightarrow 1} f(x) = DNE$$

b)  $y = \begin{cases} x + 2, & x > 1, x \in R \\ x^2, & x \leq 1, x \in R \end{cases}$

Answer:

$$\lim_{x \rightarrow 1^-} f(x) = 1, \lim_{x \rightarrow 1^+} f(x) = 3, \lim_{x \rightarrow 1} f(x) = DNE$$

SCO: L3: Analyze the continuity of a function. [C, T, V, R]

[C] Communication  
[T] Technology[PS] Problem Solving  
[V] Visualization[CN] Connections  
[R] Reasoning[ME] Mental Math  
and Estimation

L3: Analyze the continuity of a function.

**Scope and Sequence of Outcomes:**

Grade Eleven	Pre-Calc B 120
	L3: Analyze the continuity of a function.

**ELABORATION**

Previously in this course, students looked at discrete versus continuous data. They are familiar with the definition of continuity but this approach to the topic will be new.

A continuous function is one whose graph can be drawn without lifting your pencil in a single stroke. There are three types of discontinuity:

1. Removable (hole)
2. Infinite (asymptotes)
3. Jump (gaps)

In this outcome, students will explore the concept of continuity using limits. A function  $y = f(x)$  is continuous at the point  $(a, f(a))$  if:

1.  $f(a)$  exists meaning the domain of the function includes an interval that contains the point  $a$ .
2.  $\lim_{x \rightarrow a} f(x)$  exists, meaning the left and right hand limit as  $x$  approaches  $a$  must be the same.
3.  $\lim_{x \rightarrow a} f(x) = f(a)$

For piece-wise functions, it is necessary to use one-sided limits to evaluate the potential point(s) of discontinuity.

For continuous functions, limits can be found by substitution.

**ACHIEVEMENT INDICATORS**

- Demonstrate an understanding of the definition of continuity.
- Determine if a function is continuous at a given point,  $x = a$ .
- Determine if a function is continuous on a given open or closed interval.
- Explore properties of continuous functions.
- Apply the Intermediate Value Theorem to continuous intervals of a function.

SCO: L3: Analyze the continuity of a function. [C, T, V, R]

**Suggested Instructional Strategies**

- Websites such as the Khan Academy or the following, can be used.  
<http://www.calculus-help.com/continuity/>  
<http://ocw.mit.edu/courses/mathematics/18-01-single-variable-calculus-fall-2006/video-lectures/>
- Students should be introduced to the formal definition of continuity. Explanations should begin with reference to many different graphs, and should work up to an analytical approach.
- This topic is challenging for many students to understand, so sufficient time should be provided for students to work with all three types of discontinuity.

**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q** Prove that each of the following functions is continuous or discontinuous at the given value. If discontinuous, describe the type of discontinuity.

a)  $f(x) = \frac{x^2-25}{x-5}$  at  $x = 5$

b)  $f(x) = \begin{cases} \frac{x^2-7x+10}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$  at  $x = 2$

c)  $f(x) = \frac{x^2+4x+3}{x-1}$  at  $x = 1$

d)  $f(x) = \begin{cases} \frac{\sqrt{x}-4}{x-16}, & x \neq 16 \\ \frac{1}{8}, & x = 16 \end{cases}$  at  $x = 16$

e)  $f(x) = \frac{x^2-3x-10}{x^2-4x-5}$  at  $x = 5, -1$

**Answers:**

a) *pt does not exist, discontinuous at 5, hole in graph (removable discontinuity)*

b) *discontinuous (removable)*

c) *discontinuous at  $x = 1$ , vertical asymptote (infinite discontinuity)*

d) *continuous*

e) *removable discontinuity at  $x = 5$ , pt does not exist (infinity discontinuity)*

**Q** For the following, find any points of discontinuity. State the type of discontinuity and explain your conclusions. If the discontinuity is removable, state how the graph could become continuous at the point.

a)  $f(x) = \begin{cases} x-5, & x \leq -1 \\ x^3, & -1 < x \leq 2 \\ 3x+2, & x > 2 \end{cases}$

b)  $f(x) = \begin{cases} \frac{x^3-8}{x-2}, & x \neq 2 \\ 10, & x = 2 \end{cases}$

c)  $f(x) = \frac{3}{x-4}$

d)  $f(x) = \frac{x^4-81}{x+3}$

**Answers:**

a) *jump continuity at  $x = -1$ , continuous at  $x = 2$*

b) *removable discontinuity at  $x = 2$ , the graph could be continuous by changing the second part of the piecewise function to  $f(x) = 12$*

c) *vertical asymptote at  $x = 4$ , infinite discontinuity at  $x = 4$*

d) *removable discontinuity at  $x = -3$ . you could change the function to  $f(x) = (x-3)(x^2+9)$*

**Q** Find the value of  $b$  so that the following function is continuous. Justify your answer.

$$f(x) = \begin{cases} 4x-5, & x \leq 3 \\ bx+3, & x > 3 \end{cases}$$

**Answer:**  $b = \frac{4}{3}$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x)$$

SCO: L4: Explore limits which involve infinity [C, T, V, R]

[C] Communication  
[T] Technology[PS] Problem Solving  
[V] Visualization[CN] Connections  
[R] Reasoning[ME] Mental Math  
and Estimation

L4: Explore limits which involve infinity

**Scope and Sequence of Outcomes:**

Grade Eleven	Pre-Calc B 120
	L4: Explore limits which involve infinity

**ELABORATION**

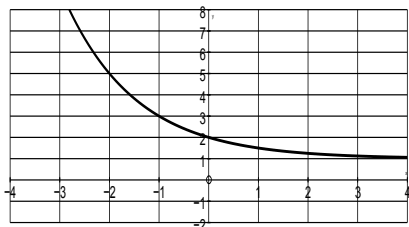
The symbol  $\infty$  is used to represent infinity. However, this is not a number, but a concept of a numerical quantity that gets larger and larger without bound. Infinity can also be represented negatively as  $-\infty$ , which indicates that the numerical quantity is getting smaller and smaller, or larger negatively, without bound. Students should have discussed this in their exploration of infinite sequences and series, end behaviour, and vertical asymptotes for various functions.

Limits involving infinity will occur either when the domain approaches infinity or when the domain approaches a vertical asymptote.

When the domain of a function approaches infinity it is called end behaviour. This is called the limit of the function as  $x$  approaches infinity. The domain will:

1) approach a real number producing a horizontal asymptote;

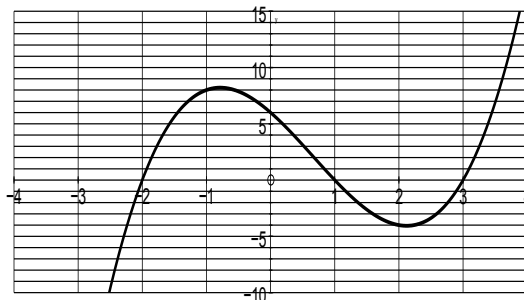
$$\lim_{x \rightarrow \infty} \left( \frac{1}{2} \right)^x + 1 = 1$$



2) approach infinity where the range of the function increases or decreases without bound;

$$\lim_{x \rightarrow \infty} (x-1)(x+2)(x-3) \text{ increases without bound}$$

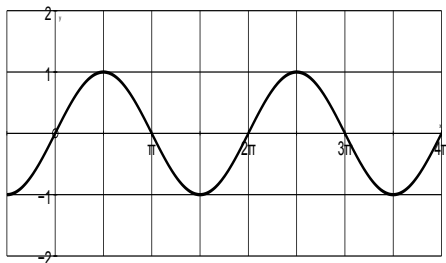
$$\lim_{x \rightarrow -\infty} (x-1)(x+2)(x-3) \text{ decreases without bound}$$



SCO: L4: Explore limits which involve infinity [C, T, V, R]

3) the limit does not exist as for various trigonometric functions.

$\lim_{x \rightarrow \infty} \sin(x)$  oscillates between 1 and -1



When a function approaches a vertical asymptote, the range of the function approaches infinity. This is the limit of  $f(x)$  as  $x$  approaches the vertical asymptote (in the positive or negative direction).

Texts disagree as to whether infinity can be expressed as an answer to a limit. Infinity is not an actual number, but rather it is the concept of a value getting larger and larger. However, the definition of a limit states that the limit is a number  $L$ . Some texts will accept infinity as an answer to a limit, but other texts say the limit does not exist (DNE)

All limits involving infinity should be solved using the properties of limits which include:

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow c^-} \frac{1}{x - c} = -\infty$$

$$\lim_{x \rightarrow c^+} \frac{1}{x - c} = \infty$$

### **ACHIEVEMENT INDICATORS**

- Understand the definitions and the properties of limits at infinity.
- Use limits as  $x \rightarrow \infty$  to find horizontal asymptotes of the function.
- Calculate limits of rational functions involving infinity.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \pm \infty, \quad \text{where } g(x) \neq 0$$

- Explore trigonometric graphs, such as sine and cosine, to show limits at infinity as an oscillating function.

SCO: L4: Explore limits which involve infinity [C, T, V, R]

**Suggested Instructional Strategies**

- It is worthwhile to have students use very large numbers to show how the limit of functions of the form  $y = \frac{c}{x}$  actually approaches 0. If the function has a vertical asymptote (VA), students should input values that are closer and closer to the value of the VA, to see that the limit approaches infinity.
- Provide students with graphs of various functions and/or the corresponding function characteristics which involve domain/range values that approach infinity. Students would be given a card and required to find their corresponding card so that the function card matches the characteristic card.

**Suggested Questions (Q) and Activities (Act) for Instruction and Assessment**

**Q** Determine the horizontal asymptote of the following functions.

a)  $y = \frac{3x^2 - 2x + 4}{18x^2 + x}$

b)  $y = \frac{7x^2 - 4x + 5}{2x^3 - 9x^2 + 3}$

Answers: a)  $y = \frac{1}{6}$  b)  $y = 0$

**Q** Evaluate the following:

a)  $\lim_{x \rightarrow \infty} \left( \frac{5x + 1}{2 - 3x} \right)$

b)  $\lim_{x \rightarrow \infty} \left( \frac{x^2 - 4x + 5}{x^3} \right)$

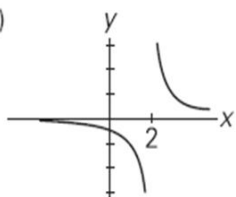
c)  $\lim_{x \rightarrow 3} \left( \frac{4}{x - 3} \right)$

Answers: a)  $-\frac{5}{3}$  b) 0 c) DNE

**Q** Match the following graphs to the corresponding equation and then determine which graph(s) have:

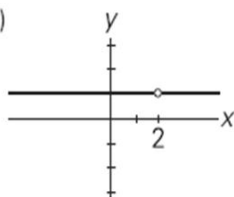
- limits that approach infinity
- horizontal asymptotes (write the equation of the asymptote)
- right hand and left hand limits that are different or equal.

1)



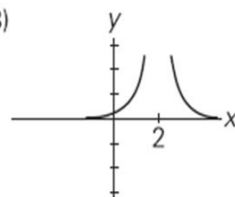
a)  $y = \frac{1}{(x-2)^2}$

2)



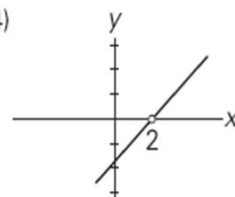
b)  $y = \frac{1}{x-2}$

3)



c)  $y = \frac{(x-2)^2}{x-2}$

4)



d)  $y = \frac{x-2}{x-2}$

Answers:

1)b VA  $x = 2$ , HA  $y = 0$

2)d no HA, equal

3)a HA  $y = 0$ , equal left and right hand limits, VA  $x = 2$

4)c no HA, equal



## SUMMARY OF CURRICULUM OUTCOMES

### **Pre-Calculus B 120**

[C] Communication, [PS] Problem Solving, [CN] Connections, [R] Reasoning,  
[ME] Mental Mathematics and Estimation, [T] Technology [V] Visualization

### **Relations and Functions**

**General Outcome:** Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

**RF1.** Analyze arithmetic sequences and series to solve problems. [CN, PS, T, R]

**RF2.** Analyze geometric sequences and series to solve problems. [PS, T, R]

**RF3.** Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree  $\leq 5$  with integral coefficients). [C, CN, ME]

**RF4.** Graph and analyze polynomial functions (limited to polynomial functions of degree  $\leq 5$ ). [C, CN, T, V]

**RF5.** Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions [CN, T, V, R]

**RF6.** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). [CN, R, T, V]

**RF7.** Demonstrate an understanding of operations on, and compositions of, functions. [CN, R, T, V]

**RF8.** Assemble a function toolkit comparing various types of functions and compositions of them. [C, CN, T, V, R]

### **Permutations, Combinations and Binomial Theorem**

**General Outcome:** Develop algebraic and numeric reasoning that involves combinatorics.

Specific Outcomes

**PCB1.** Apply the fundamental counting principle to solve problems. [C, PS, V, R]

**PCB2.** Determine the number of permutations of  $n$  elements taken  $r$  at a time to solve problems. [C, PS, V, R]

**PCB3.** Determine the number of combinations of  $n$  different elements taken  $r$  at a time to solve problems. [C, PS, V, R]

**PCB4.** Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers). [CN, R, V]

### **Limits**

**General Outcome:** Develop an understanding of Limits

Specific Outcomes

**L1.** Determine the limit of a function at a point both graphically and analytically. [C, PS, T, V]

**L2.** Explore one-sided limits graphically and analytically. [C, T, V, R]

**L3.** Analyze the continuity of a function. [C, T, V, R]

**L4.** Explore limits which involve infinity. [C, T, V, R]

## REFERENCES

- Alberta Education, System Improvement Group. *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings*. Edmonton, AB:
- Alberta Education, 2006. Available at [http://www.wncp.ca/math/report\\_2006.pdf](http://www.wncp.ca/math/report_2006.pdf) (Accessed September 20, 2007).
- Armstrong, Thomas. *7 Kinds of Smart: Identifying and Developing Your Many Intelligences*. New York, NY: Plume, 1993.
- Banks, J. A. and C. A. M. Banks. *Multicultural Education: Issues and Perspectives*. 2nd ed. Boston, MA: Allyn and Bacon, 1993.
- British Columbia Ministry of Education. *The Primary Program: A Framework for Teaching*. Victoria, BC: British Columbia Ministry of Education, 2000.
- Caine, Renate Nummela and Geoffrey Caine. *Making Connections: Teaching and the Human Brain*. Alexandria, VA: Association for Supervision and Curriculum Development, 1991.
- Hope, Jack A. et al. *Mental Math in the Primary Grades*. Palo Alto, CA: Dale Seymour Publications, 1988.
- McAskill, B. et al. *WNCP Mathematics Research Project: Final Report*. Victoria, BC: Holdfast Consultants Inc., 2004. Available at [http://www.wncp.ca/math/Final\\_Report.pdf](http://www.wncp.ca/math/Final_Report.pdf) (Accessed September 20, 2007).
- National Council of Teachers of Mathematics. *Computation, Calculators, and Common Sense: A Position of the National Council of Teachers of Mathematics*. May 2005. [http://www.nctm.org/uploadedFiles/About\\_NCTM/Position\\_Statements/computation.pdf](http://www.nctm.org/uploadedFiles/About_NCTM/Position_Statements/computation.pdf) (Accessed September 20, 2007).
- Rubenstein, Rheta N. "Mental Mathematics beyond the Middle School: Why? What? How?" *Mathematics Teacher* 94, 6 (September 2001), pp. 442–446.
- Shaw, J. M. and M. J. P. Cliatt. "Developing Measurement Sense." In P. R. Trafton (ed.), *New Directions for Elementary School Mathematics: 1989 Yearbook* (Reston, VA: National Council of Teachers of Mathematics, 1989), pp. 149–155.
- Steen, L. A. *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, DC: Mathematical Sciences Education Board, National Research Council, 1990.
- Western and Northern Canadian Protocol for Collaboration in Education. *The Common Curriculum Framework for K–9 Mathematics* May 2006. <http://www.wncp.ca/english/subjectarea/mathematics/ccf.aspx>
- Western and Northern Canadian Protocol for Collaboration in Education. *The Common Curriculum Framework for Grades 10-12 Mathematics* January 2008. <http://www.wncp.ca/english/subjectarea/mathematics/ccf.aspx>

# Limits Resource Supplement

for

Pre-Calculus B 120

The NB Department of Education and Early Childhood Development

gratefully acknowledges the work done by the *Limits Resource Development Committee* of  
Lori Brophy, Carolyn Campbell, Steve MacMillan, Anne Spinney, and Glen Spurrell,  
to produce this resource for use by New Brunswick Teachers.

November 2013

# Table of Contents

<b>Introduction .....</b>	<b>89</b>
<b>L1. Determine the limit of a function at a point both graphically and analytically. ....</b>	<b>90</b>
L1 Exercises .....	91
L1 Solutions.....	93
<b>L2. Explore one-sided limits graphically and analytically. ....</b>	<b>94</b>
L2 Exercises .....	96
L2 Solutions.....	97
<b>L3. Analyze the continuity of a function .....</b>	<b>99</b>
L3 Exercises .....	103
L3 Solutions.....	105
<b>L4 Explore limits which involve infinity .....</b>	<b>106</b>
L4 Exercises .....	111
L4 Solutions.....	113

# Introduction

This resource has been produced by New Brunswick teachers to supplement the core text resource, *Pre-Calculus 12* (MHR 2012) in which the topic of limits is not covered.

The concept of a limit is essential to the development and understanding of Calculus. Limits are used in the definition of the continuity of a function and are used to develop the concepts of instantaneous rates of change (slope of a tangent, derivative) and the area under a curve (integrals). These concepts have many applications in Science, Social Science, and Business.

The ideas that would develop into Calculus began with the Ancient Greeks. Zeno developed several paradoxes that could not be explained at the time. One of these involved reaching a destination by going half the remaining distance in each step. He said that you would never be able to reach the destination because you would always have half the remaining distance to travel. Archimedes explored a similar idea, using what was called the process of exhaustion to find the area of a circle. He placed regular polygons with increasing number of sides inside a circle and determined their areas. As the number of sides increased the area of the polygon approached the area of the circle. Although both of these mathematicians were talking about the concept we now refer to as limits, the definition was not formalized until the early 1800's. Many scholars, including Cavalieri, Fermat, Descartes, Barrow, Newton and Leibnitz developed the concepts of Calculus in the intervening years but the formal definition of a limit was not given until Cauchy (1789 – 1857) used limits to define continuity. The following websites have more on the history of calculus:

<http://www.uiowa.edu/~c22m025c/history.html>

<http://www.mscs.dal.ca/~kgardner/History.html> , [http://faculty.unlv.edu/bellomo/Math714/Notes/11\\_Calculus.pdf](http://faculty.unlv.edu/bellomo/Math714/Notes/11_Calculus.pdf)

<http://www.andrewsaladino.com/calculus/history.html> .

Students may have previously heard the term 'limit' in relation to horizontal asymptotes (exponential, reciprocal, and rational functions) and vertical asymptotes (logarithmic, reciprocal, and rational functions). The ideas would also have been discussed in conjunction with infinite sequences and series. Students will need to be shown the notation,

$$\lim_{x \rightarrow a} f(x) = L$$

and how it is used to describe asymptotes, continuity, sequences and series. The approach in the four limit outcomes in *Pre-Calculus B 120* is in more depth than that in previous courses. In this course, students will not only evaluate limits but will also apply their understanding of these concepts to better understand the behavior of functions. Specifically, limits will be used to describe how a function behaves near a specific point but not at that point. The value of the limit does not depend on the value of the function. Limits will also be used to help determine whether or not a function is continuous and determine its end behavior.

Set Builder Notation	Interval Notation
$\{x   -2 \leq x \leq 4\}$	$[-2, 4]$
$\{x   -2 < x \leq 4\}$	$(-2, 4]$
$\{x   -2 \leq x < 4\}$	$[-2, 4)$
$\{x   -2 < x < 4\}$	$(-2, 4)$
$\{x   x \leq 4\}$	$(-\infty, 4]$
$\{x   x < 4\}$	$(-\infty, 4)$
$\{x   x \geq 4\}$	$[4, \infty)$
$\{x   x > 4\}$	$(4, \infty)$

Please note that as students begin Calculus they will see set builder notation and interval notation used interchangeably. They should be allowed to use either type of notation in their solutions. In Interval Notation, to indicate that one of the endpoints is to be excluded from the set, the corresponding square bracket can be replaced with a parenthesis (in another convention, instead of the parenthesis, the square bracket is reversed). Examples of both notations are shown in the chart. Real numbers are assumed unless a different set of numbers is stated.

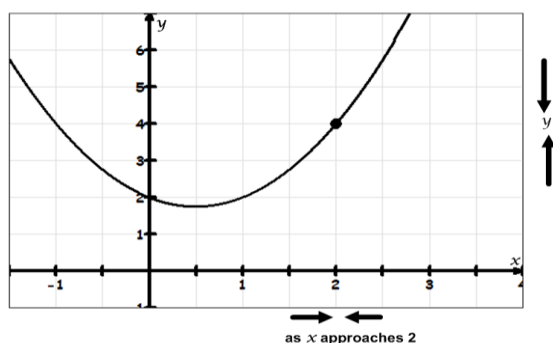
L1: Determine the limit of a function at a point both graphically and analytically.

## L1. Determine the limit of a function at a point both graphically and analytically.

### Achievement Indicators

- Demonstrate an understanding of the concept of limit at a point, and the notation to express the limit of a function  $f(x)$  as  $x$  approaches  $a$  as  $\lim_{x \rightarrow a} f(x) = L$
- Evaluate the limit of a function at a point graphically, analytically and using tables of values.
- Distinguish between the limit of a function as  $x$  approaches  $a$  and the  $y$ -value of the function at  $x = a$ .
- Use the properties of limits to evaluate a limit by substitution of the  $x$ -value into  $f(x)$ , where possible.
- Explore undefined limits (limits that do not exist (DNE)).
- Explore limits that arise, when substituting yields an indeterminate form.

Limits describe how functions behave as the independent variable approaches a given value. For example, the graph and **Table 1** for the function  $f(x) = x^2 - x + 2$ , are shown below. The values of  $f(x)$  get closer and closer to 4 as  $x$  gets closer and closer to 2.



**Table 1.1**

Approaching 2 from the left	
$x < 2$	$f(x)$
1.0	2
1.5	2.75
1.9	3.71
1.99	3.9701
1.999	3.997001

Approaching 2 from the right	
$x > 2$	$f(x)$
3.0	8
2.5	5.75
2.1	4.31
2.01	4.0301
2.0001	4.003001

### Properties of Limits

If the following limits both exist

$$\lim_{x \rightarrow a} f(x) \quad \lim_{x \rightarrow a} g(x)$$

and  $c$  is a constant, then the following properties hold:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} [c f(x)] = c \lim_{x \rightarrow a} f(x)$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$$

$$\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n \text{ if } n \text{ is a positive integer}$$

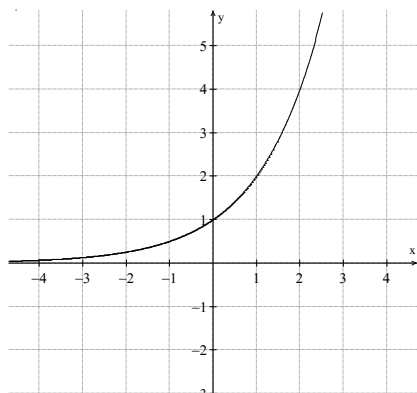
$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \text{ if the root on the right side exists}$$

L1: Determine the limit of a function at a point both graphically and analytically.

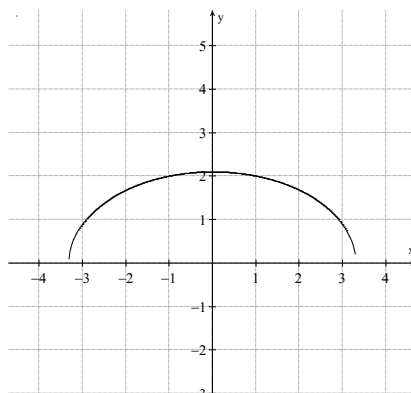
## L1 Exercises

1. Find the limit of each function shown.

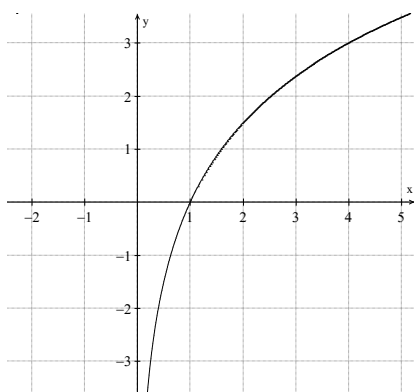
a)  $\lim_{x \rightarrow 1} f(x)$



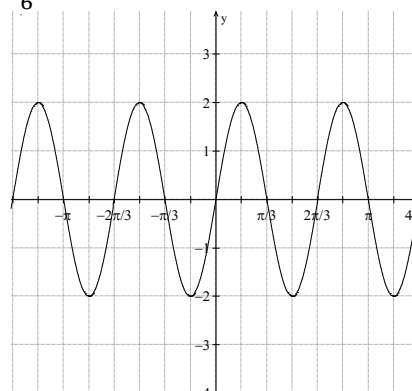
b)  $\lim_{x \rightarrow -1} f(x)$



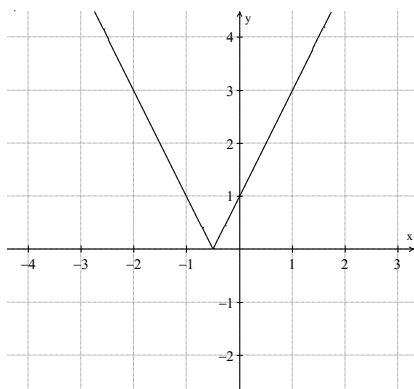
c)  $\lim_{x \rightarrow 1} f(x)$



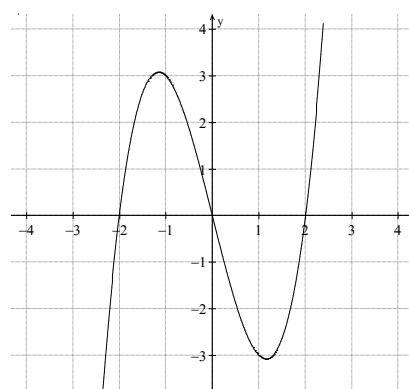
d)  $\lim_{x \rightarrow \frac{\pi}{6}} f(x)$



e)  $\lim_{x \rightarrow \frac{-3}{2}} f(x)$

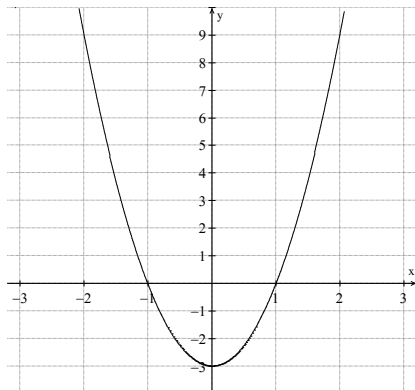


f)  $\lim_{x \rightarrow -1} f(x)$

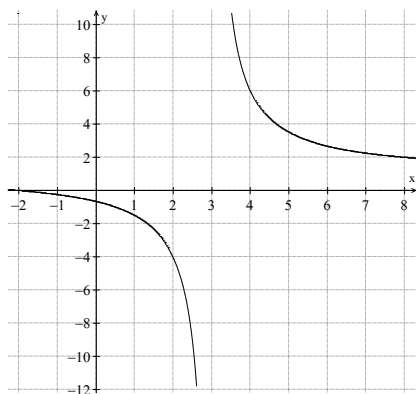


L1: Determine the limit of a function at a point both graphically and analytically.

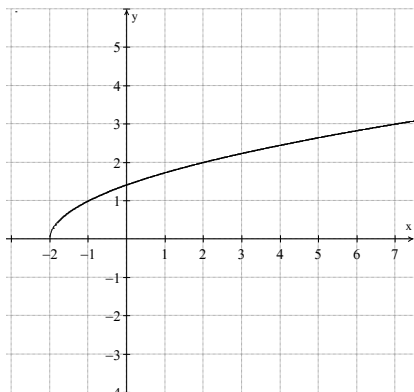
g)  $\lim_{x \rightarrow 2} f(x)$



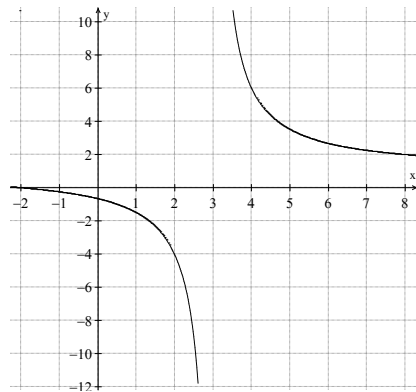
h)  $\lim_{x \rightarrow 4} f(x)$



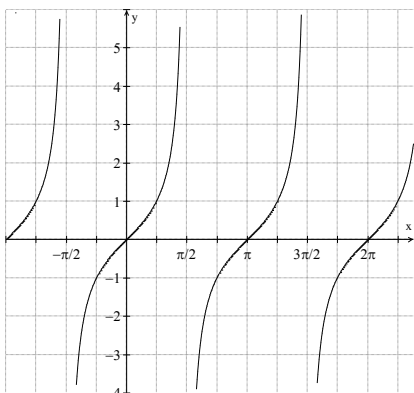
i)  $\lim_{x \rightarrow 2} f(x)$



j)  $\lim_{x \rightarrow 3} f(x)$



k)  $\lim_{x \rightarrow \pi/2} f(x)$



l)  $\lim_{x \rightarrow 3} f(x)$

$x$	$f(x)$
2	-1
2.5	-3
2.9	-19
2.99	-199
2.999	-1999
3	
3.001	2001
3.01	201
3.1	21
3.5	5
4	3



L1: Determine the limit of a function at a point both graphically and analytically.

2. Evaluate the following limits:

a)  $\lim_{x \rightarrow 4} \frac{x-4}{x^3-64}$

b)  $\lim_{x \rightarrow 7} \frac{\sqrt[5]{3-5x}}{(x-5)^3}$

c)  $\lim_{x \rightarrow -3} \frac{x+3}{3-\sqrt{x+12}}$

d)  $\lim_{x \rightarrow -3} \sqrt[3]{\frac{x-4}{6x^2+2}}$

e)  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2}-\frac{1}{9}}{x-3}$

f)  $\lim_{h \rightarrow 0} \frac{(-3+h)^2-9}{h}$

g)  $\lim_{h \rightarrow 0} \frac{(2+h)^3-8}{h}$

h)  $\lim_{x \rightarrow 0} \frac{\frac{1}{1+x}-1}{x}$

i)  $\lim_{x \rightarrow -1} \frac{2x+3}{3x+2}$

j)  $\lim_{x \rightarrow -8} \frac{x+8}{\sqrt[3]{x}+2}$

k)  $\lim_{x \rightarrow -1} \frac{8}{(3+x)\sqrt{3-x}}$

l)  $\lim_{x \rightarrow 9} \frac{x-9}{3-\sqrt{x}}$

m)  $\lim_{x \rightarrow 0} \frac{(3+x)^3-3^3}{(3+x)^2-3^2}$

n)  $\lim_{x \rightarrow 9} \frac{x^2-81}{3-\sqrt{x}}$

o)  $\lim_{x \rightarrow 16} \frac{2\sqrt{x}+x^{3/2}}{\sqrt[4]{x}+5}$

p)  $\lim_{x \rightarrow 2} \frac{x^2-4}{3x^2-7x+2}$

q)  $\lim_{x \rightarrow 7} \frac{\sqrt{x+2}-3}{x-7}$

r)  $\lim_{x \rightarrow 4} \left( \frac{8}{x^2-16} - \frac{1}{x-4} \right)$

s)  $\lim_{x \rightarrow 5} \frac{2x}{x-5}$

t)  $\lim_{x \rightarrow 0} \frac{x^3}{2x^3+3x^4}$

u)  $\lim_{x \rightarrow 2} \frac{x^2-2x}{2x^2-7x+6}$

v)  $\lim_{x \rightarrow \pi} \frac{1+\sin x}{\cos x}$

w)  $\lim_{x \rightarrow \frac{\pi}{4}} (\cos x) + (\sin x)$

x)  $\lim_{x \rightarrow \frac{3\pi}{2}} \frac{\sin x}{\cos x}$

## L1 Solutions

1. a) 2   b)  $\approx 2$    c) 0   d) 2   e) 2   f) 3   g) 9   h) 6   i) 2   j) DNE   k) DNE   l) DNE

2. a)  $\frac{1}{48}$    b)  $-\frac{1}{4}$    c) -6   d)  $-\frac{1}{2}$    e)  $-\frac{2}{27}$    f) -6   g) 12   h) -1

i) -1   j) 12   k) 2   l) -6   m)  $\frac{9}{2}$    n) -108   o)  $\frac{72}{7}$    p)  $\frac{4}{5}$    q)  $\frac{1}{6}$

r)  $-\frac{1}{8}$    s) DNE   t)  $\frac{1}{2}$    u) 2   v) -1   w)  $\sqrt{2}$    x) DNE

L2: Explore one-sided limits graphically and analytically.

## L2. Explore one-sided limits graphically and analytically.

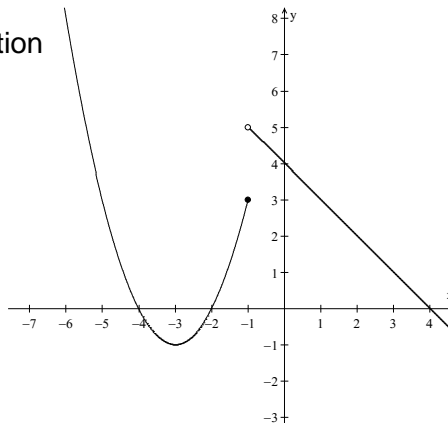
### Achievement Indicators

- Explore and analyze left and right hand limits as  $x$  approaches a certain value using correct notation.
- Evaluate the left and right hand limits of piecewise functions.

### Example 1

Consider the following piecewise defined function

$$\text{as, } f(x) = \begin{cases} x^2 + 6x + 8 & x \leq -1 \\ -x + 4 & x > -1 \end{cases}$$



What  $y$  value is the function approaching as  $x$  approaches  $-1$  from the left?

$$\text{Find: } \lim_{x \rightarrow -1^-} f(x)$$

(Note that the ‘ $-$ ’ superscript as an exponent on  $-1$ , means ‘from the left side’.)

Substituting the  $x$  value of  $-1$  into the function  $f(x) = x^2 + 6x + 8$  (since  $-1$  is in the domain of this function), gives a  $y$  value of 3.

$$\therefore \lim_{x \rightarrow -1^-} f(x) = 3$$

What  $y$ -value is the function approaching as  $x$  approaches  $-1$  from the right?

$$\text{Find: } \lim_{x \rightarrow -1^+} f(x)$$

(Note that the ‘ $+$ ’ superscript as an exponent on  $-1$ , means ‘from the right side’.)

The  $x$  value of  $-1$  cannot be substituted into the function  $f(x) = -x + 4$  because  $-1$  is not in the domain of this function. **Table 2**, shows that  $x$  gets closer and closer to  $-1$  from the right side, the  $y$ -values get closer and closer to 5. Since limits are used to determine the value a function **approaches** a value, we can conclude that the limit is 5.

**Table 2.1**

Approaching -1 from the right	
$x > -1$	$f(x)$
1.0	3
0.5	3.5
0	4
-0.05	4.5
-0.09	4.9

$$\therefore \lim_{x \rightarrow -1^+} f(x) = 5$$

L2: Explore one-sided limits graphically and analytically.

For each case, students can trace a finger along the graph (from each side) to see what  $y$  value the function approaches as the  $x$  value is approached. It is important to highlight that even though a function does not exist at a particular  $x$  value (empty circle), there is still a limit as shown in the example.

To find the limit for the piecewise function defined in the example, the limit for each piece has been determined as:

$$\lim_{x \rightarrow -1^-} f(x) = 3$$

$$\lim_{x \rightarrow -1^+} f(x) = 5$$

Since the limit from the left does not equal the limit from the right, then the general limit for this piecewise function does not exist (*page 51 of the Pre-Calculus B 120 curriculum document*).

### Example 2

Given  $f(x) = \begin{cases} -x^2 - 4x - 2 & x \leq 0 \\ (x-1)^3 - 1 & x > 0 \end{cases}$

- a) Sketch the graph of  $f(x)$ .
- b) Find the following limits, if they exist.

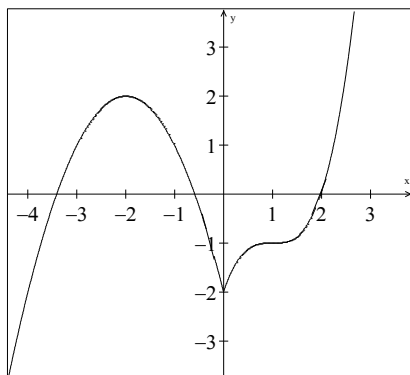
i)  $\lim_{x \rightarrow 0^-} f(x)$

ii)  $\lim_{x \rightarrow 0^+} f(x)$

iii)  $\lim_{x \rightarrow 0} f(x)$

**Solution:**

a)



b)

i)  $\lim_{x \rightarrow 0^-} f(x) = -2$

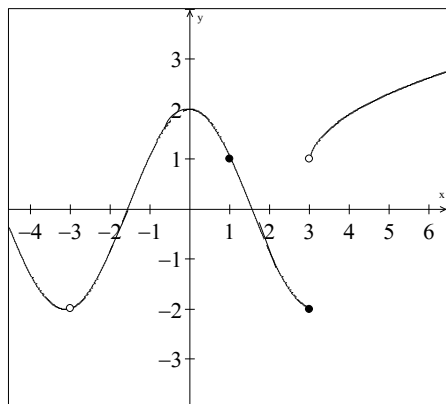
ii)  $\lim_{x \rightarrow 0^+} f(x) = -2$

iii) Since the limit from the left and from the right are equal then,  $\lim_{x \rightarrow 0} f(x) = -2$

L2: Explore one-sided limits graphically and analytically.

## L2 Exercises

1. Given the graph of  $f(x)$ , find the following limits.



- |                                     |                                     |                                   |
|-------------------------------------|-------------------------------------|-----------------------------------|
| a) $\lim_{x \rightarrow -3^-} f(x)$ | b) $\lim_{x \rightarrow -3^+} f(x)$ | c) $\lim_{x \rightarrow -3} f(x)$ |
| d) $\lim_{x \rightarrow 1^+} f(x)$  | e) $\lim_{x \rightarrow 1^-} f(x)$  | f) $\lim_{x \rightarrow 1} f(x)$  |
| g) $\lim_{x \rightarrow 3^-} f(x)$  | h) $\lim_{x \rightarrow 3^+} f(x)$  | i) $\lim_{x \rightarrow 3} f(x)$  |

2. Sketch each piecewise function below and find the limit at the indicated point. If the limit does not exist, provide an explanation.

a)  $f(x) = \begin{cases} 2, & x < 1 \\ 3, & x = 1 \\ x + 1, & x > 1 \end{cases}$

Find  $\lim_{x \rightarrow 1} f(x)$

b)  $f(x) = \begin{cases} 4 - x^2, & -2 < x \leq 2 \\ x - 2, & x > 2 \end{cases}$

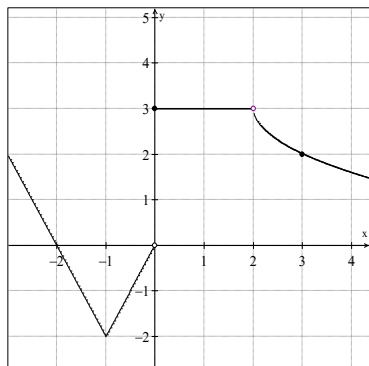
Find  $\lim_{x \rightarrow 2} f(x)$

c)  $f(x) = \begin{cases} |x + 2| + 1, & x < -1 \\ -x + 1, & -1 \leq x \leq 1 \\ x^2 - 2x + 2, & x > 1 \end{cases}$

Find  $\lim_{x \rightarrow 1} f(x)$

3. For the graph shown below

- a) Does the limit exist at  $x = 0$ ?  
 b) Does the limit exist at  $x = 2$ ?



4. The function  $f(t)$  is defined by  $f(t) = \begin{cases} 3t + b, & t < 1 \\ 2 - bt^2, & t \geq 1 \end{cases}$  where  $b$  is a constant.

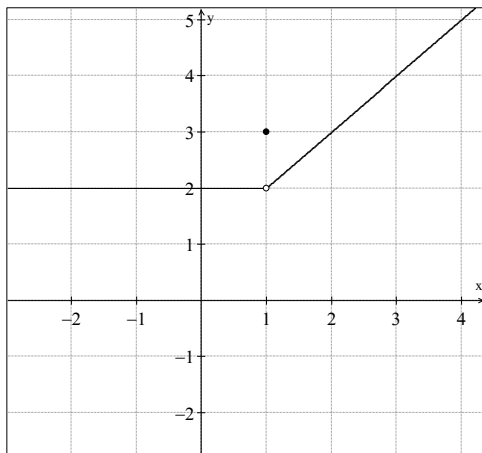
Compute  $\lim_{t \rightarrow 1^+} f(t)$  and  $\lim_{t \rightarrow 1^-} f(t)$  in terms of  $b$ .

L2: Explore one-sided limits graphically and analytically.

## L2 Solutions

1.
  - a)  $\lim_{x \rightarrow -3^-} f(x) = -2$
  - b)  $\lim_{x \rightarrow -3^+} f(x) = -2$
  - c)  $\lim_{x \rightarrow -3} f(x) = -2$
  - d)  $\lim_{x \rightarrow 1^+} f(x) = 1$
  - e)  $\lim_{x \rightarrow 1^-} f(x) = 1$
  - f)  $\lim_{x \rightarrow 1} f(x) = 1$
  - g)  $\lim_{x \rightarrow 3^-} f(x) = -2$
  - h)  $\lim_{x \rightarrow 3^+} f(x) = 1$
  - i)  $\lim_{x \rightarrow 3} f(x)$  does not exist

2. a)

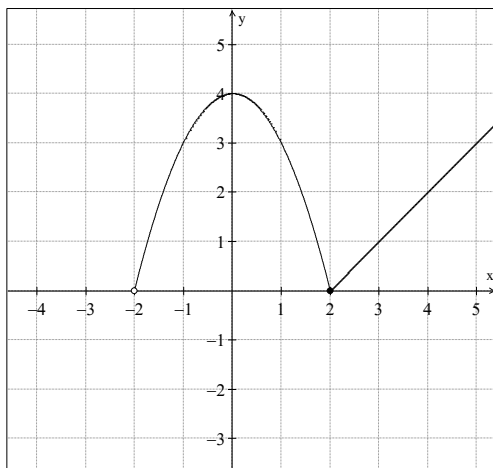


$$\lim_{x \rightarrow 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 2$$

Since the limits from the left and right of 1 are equal,

$$\lim_{x \rightarrow 1} f(x) = 2$$

b)



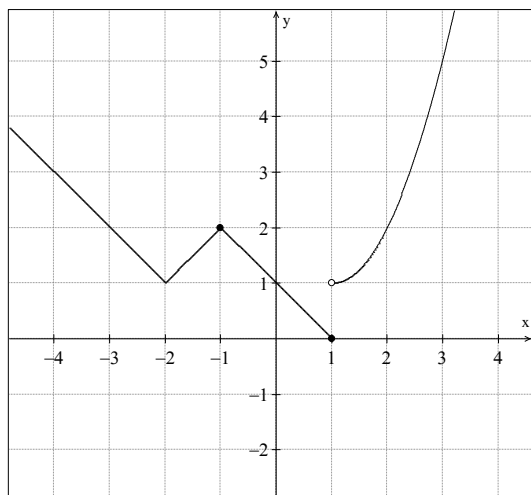
$$\lim_{x \rightarrow 2^-} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^+} f(x) = 0$$

Since the limits from the left and right of 2 are equal,

$$\lim_{x \rightarrow 2} f(x) = 0.$$

L2: Explore one-sided limits graphically and analytically.

c)



$$\lim_{x \rightarrow 1^-} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 1^+} f(x) = 1$$

Since the limit from the left and right of 1 are not equal,  
 $\lim_{x \rightarrow 1} f(x)$  does not exist.

3. a) The limit as  $x$  approaches 0 does not exist since the limit from the left does not equal the limit from the right.

$$\lim_{x \rightarrow 0^-} f(x) = 0 \quad \lim_{x \rightarrow 0^+} f(x) = 3 \quad \text{Therefore } \lim_{x \rightarrow 0} f(x) \text{ does not exist}$$

- b) The limit as  $x$  approaches 2 equals 3 since the limit from the left equals the limit from the right.

$$\lim_{x \rightarrow 2^-} f(x) = 3 \quad \lim_{x \rightarrow 2^+} f(x) = 3 \quad \text{Therefore } \lim_{x \rightarrow 2} f(x) = 3$$

4.  $\lim_{t \rightarrow 1^+} f(t) = 2 - b$       and       $\lim_{t \rightarrow 1^-} f(t) = 3 + b$

If the limit as  $x$  approaches 1 does exist, the value of  $b$  could be calculated. Students may want to return to this question to find the value of  $b$  that makes the function continuous after outcome L3 has been explored.

L3: Analyze the continuity of a function.

## L3. Analyze the continuity of a function

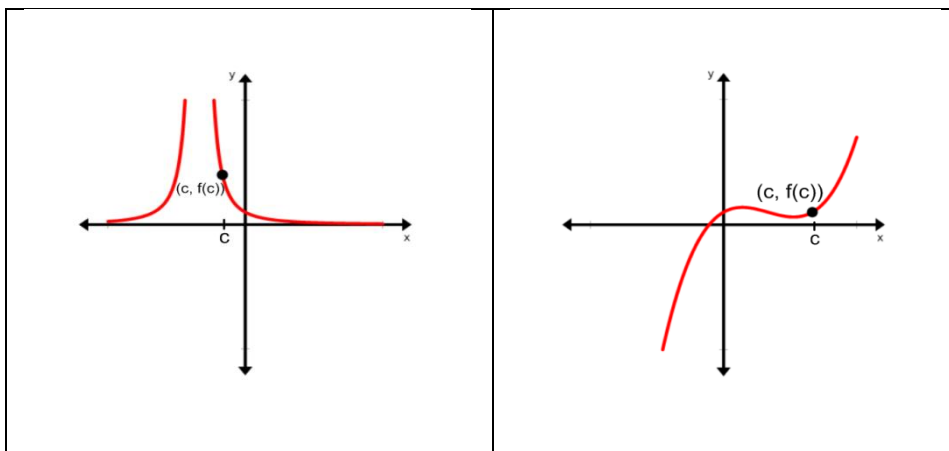
### Achievement Indicators

- Demonstrate an understanding of the definition of continuity.
- Determine if a function is continuous at a point,  $x = a$ .
- Determine if a function is continuous on a given open or closed interval.
- Explore properties of continuous functions.
- Apply the Intermediate Value Theorem to continuous intervals of a function.

### Continuity of a function

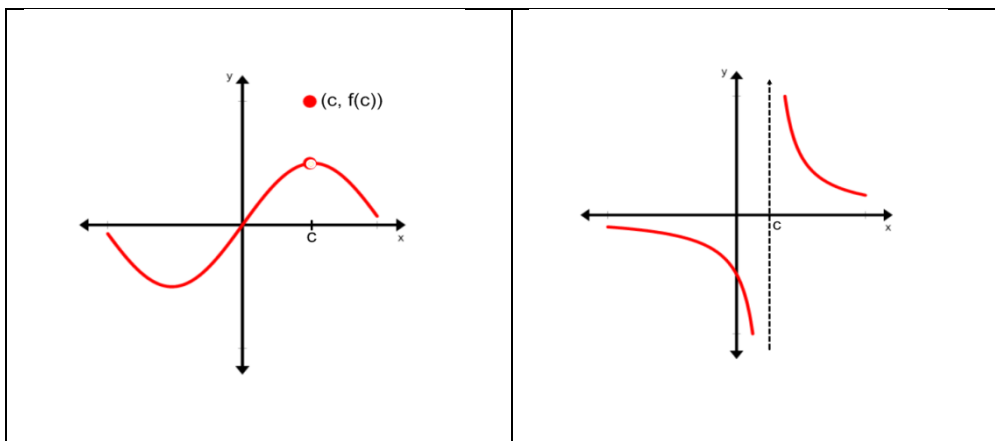
Continuity of a function at a given value can be determined by starting with any value of  $c$  in the domain of a function, and testing to see if it is possible to draw the graph of  $f(x)$  at, and near the point  $(c, f(c))$  without lifting a pencil from the paper. For example, in *Figure 3.1* below, each graph is a function  $f(x)$  that is continuous at  $x = c$ .

**Figure 3.1**



In *Figure 3.2* the graphs illustrate functions that are not continuous at the point  $x = c$ . It is not possible to trace the graph around the point  $(c, f(c))$  without lifting a pencil.

**Figure 3.2**



### L3: Analyze the continuity of a function.

#### Definition of Continuity

A function  $f$  is said to be continuous at  $x = c$  if the following conditions are true:

- I.  $f(c)$  is defined
- II.  $\lim_{x \rightarrow c} f(x)$  exists
- III.  $\lim_{x \rightarrow c} f(x) = f(c)$

If condition III is true, the first two conditions are implied.

#### **Example 1**

Show, by solving analytically, that the function  $f(x) = \frac{\sqrt{x^2+2x-1}}{x-3}$  is continuous at the point where  $x = 1$ .

#### Solution

$$\text{I. } f(1) = \frac{\sqrt{(1)^2 + 2(1) - 1}}{(1) - 3} = \frac{\sqrt{2}}{-2} = -\frac{\sqrt{2}}{2}$$

$$\text{II. } \lim_{x \rightarrow 1} \frac{\sqrt{x^2 + 2x - 1}}{x - 3} = \frac{\sqrt{(1)^2 + 2(1) - 1}}{(1) - 3} = -\frac{\sqrt{2}}{2}$$

Therefore,  $\lim_{x \rightarrow 1} f(x) = f(1)$ , and  $f$  is continuous at  $x = 1$ .

*Note: it is not necessary to check the right and left hand limits unless you are dealing with a point of discontinuity such as  $x = 3$  in this example.*

---

#### **Example 2**

For what values of  $x$ , if any, is the function  $f(x) = \frac{x^2-4}{x^2+3x+2}$  discontinuous?

#### Solution

To solve questions for discontinuity, it is necessary to find the values in the domain of the function,  $f$ , that are undefined. Since  $f(x)$  is a rational polynomial, these would be values in the domain which would cause the denominator to become 0, found by setting the denominator equal to 0 and solving for  $x$ .

$$x^2 + 3x + 2 = 0$$

$$(x + 2)(x + 1) = 0$$

Therefore, either  $x + 2 = 0$  or  $x + 1 = 0$ , and  $x = -2$  or  $x = -1$  would be the values in the domain where the function  $f(x)$  would be discontinuous.

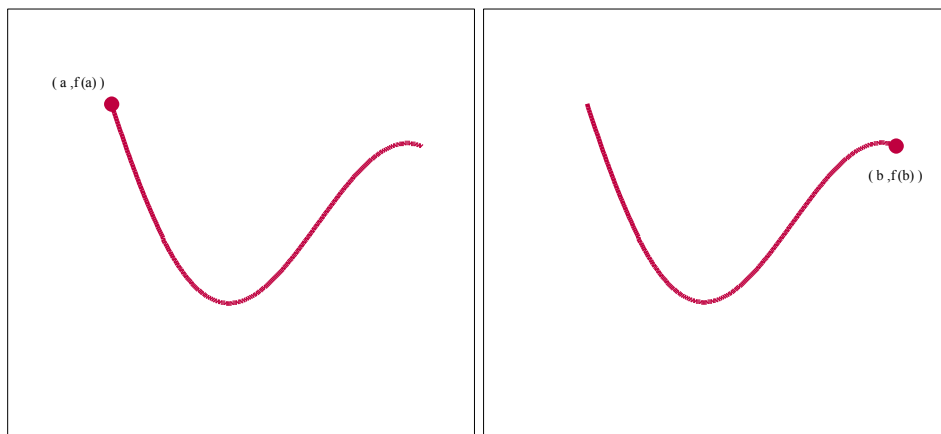


### L3: Analyze the continuity of a function.

#### Continuity on an Interval

Continuity can also be defined from the left or the right of an endpoint as in the *Figure 3.3* at the points  $(a, f(a))$  or  $(b, f(b))$ .

**Figure 3.3**



#### Continuity from the Left and Right

- I. A function  $f$  is continuous from the right at  $x = a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .
- II. A function  $f$  is continuous from the left at  $x = b$  if  $\lim_{x \rightarrow b^-} f(x) = f(b)$ .

#### Continuity on a open and closed interval

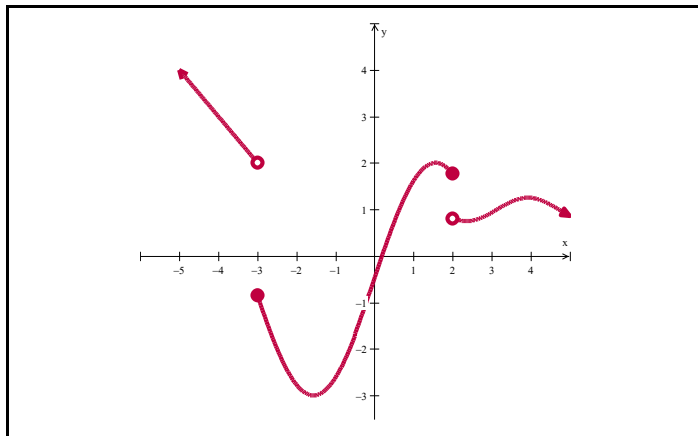
- I. A function is continuous on an open interval  $(a, b)$  if it is continuous at every point  $c$ , where  $c \in (a, b)$ .
- II. A function  $f$  is continuous on a closed interval  $[a, b]$  if the following conditions are satisfied:
  - $f$  is continuous on  $(a, b)$
  - $f$  is continuous from the right at  $a$
  - $f$  is continuous from the left at  $b$

L3: Analyze the continuity of a function.

### Example 3

The function  $f$ , shown in Figure 3.4 below, is discontinuous at  $x = -3$  and  $x = 2$ , but is continuous on each of the intervals  $(-\infty, -3) \cup [-3, 2] \cup (2, +\infty)$ .

Figure 3.4



### Intermediate Value Theorem

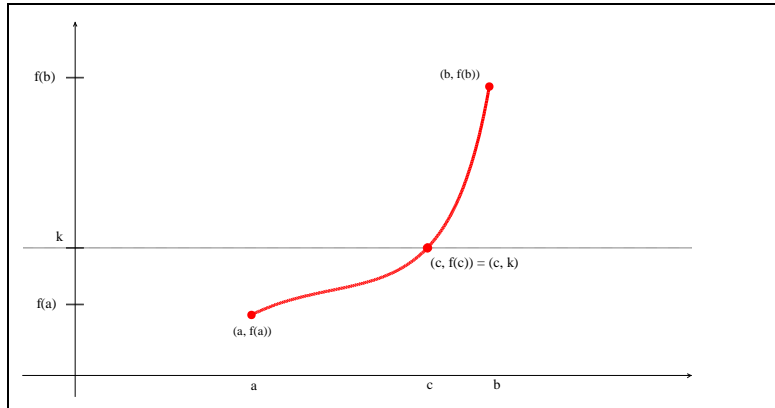
It is not necessary to prove the intermediate value theorem at this stage. However it can be introduced to students, using a simple geometric argument to support it.

#### Intermediate Value Theorem

If  $f$  is continuous on a closed interval  $[a, b]$  and  $k$  is any number between  $f(a)$  and  $f(b)$  inclusive, then there is at least one number  $c$  in the interval  $[a, b]$  such that  $f(c) = k$ .

In Figure 3.5 the function  $f$  is continuous on the interval  $[a, b]$  and a number  $k$  that is between  $f(a)$  and  $f(b)$ .

Figure 3.5

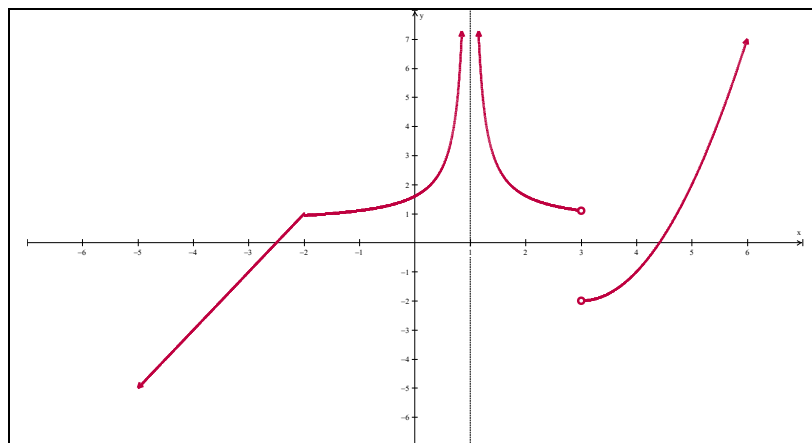


Because  $f$  is continuous on the interval, its graph can be drawn from  $(a, f(a))$  to  $(b, f(b))$  without lifting the pencil from the paper. As Figure 3.5 indicates there is no way to do this unless the function crosses the horizontal line at  $y = k$  at least once between  $x = a$  and  $x = b$ . The coordinates of a point where this happens is either  $(c, f(c))$  or  $(c, k)$ .

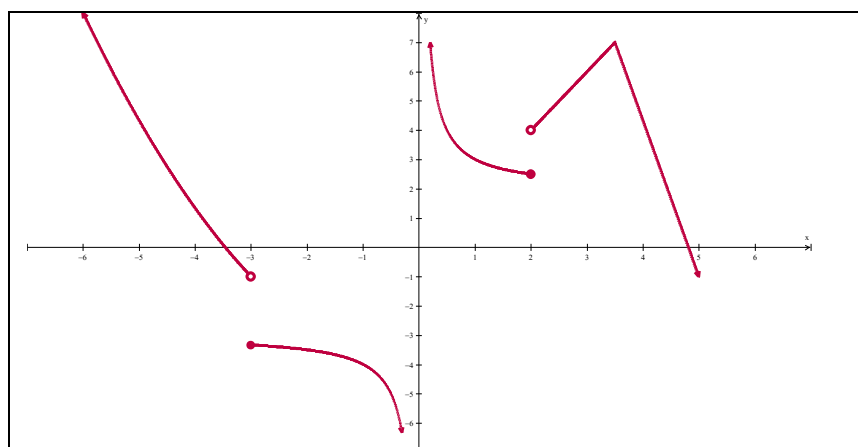
L3: Analyze the continuity of a function.

### L3 Exercises

1. Given the graph of the function  $f$  below, find all the values in the domain of  $f$  at which  $f$  is **not** continuous.



2. Find all values in the function below where it is defined but **not** continuous.



L3: Analyze the continuity of a function.

In Exercises 3 to 8, use the definition of continuity to show that the function is continuous at the given  $x$  value.

3.  $f(x) = x^2 + 3x + 5$  at  $x = 3$

4.  $g(x) = x(x^2 - 3x + 5)$  at  $x = 0$

5.  $f(x) = \frac{x^2 - 4}{(x^2 + 4x + 4)(x^2 + 2x + 1)}$  at  $x = 2$

6.  $g(x) = \frac{x + 3}{(x^2 - x - 1)(x^2 + 1)}$  at  $x = -2$

7.  $f(x) = \frac{x\sqrt{x}}{(x-4)^2}$  at  $x = 16$

8.  $g(x) = \frac{\sqrt{2-x^2}}{3x^2-1}$  at  $x = -1$

Explain why the functions in questions 9 to 12 are not continuous at the given number.

9.  $f(x) = \frac{1}{(x-3)^3}$  at  $x = 3$

10.  $g(x) = \frac{(x^2+4)}{(x^2-x-2)}$  at  $x = 2$

11.  $h(x) = \frac{x^2+4x+3}{x^2-x-2}$  at  $x = -1$

12.  $k(x) = \begin{cases} \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$  at  $x = 0$

13. Show that the function  $f(x) = \frac{x^4 - 5x^2 + 4}{x - 1}$  is not continuous on  $[-3, 3]$  but does satisfy the conclusion of the Intermediate Value Theorem (that is, if  $k$  is a number between  $f(-3)$  and  $f(3)$ , there is a number  $c$  between  $-3$  and  $3$  such that  $f(c) = k$ ).

*Hint: What can be said about  $f$  on the intervals  $[-3, -2]$  and  $[2, 3]$ ?*

14. For what value of  $b$  is the following function continuous at  $x = 3$ ?

$$f(x) = \begin{cases} bx + 4, & \text{if } x \leq 3 \\ bx^2 - 2, & \text{if } x > 3 \end{cases}$$

*Note: Question 4 in L2 can have an added question to find the value of  $b$  to make it continuous.+*

### L3: Analyze the continuity of a function.

## L3 Solutions

1.  $f(x)$  is not continuous at  $x = 1$  and  $x = 3$
2.  $f(x)$  is defined but not continuous at  $x = -3$  and  $x = 2$ . Note:  $x = 0$  is not continuous but also is not defined.
3.  $\lim_{x \rightarrow 3} x^2 + 3x + 5 = (3)^2 + 3(3) + 5 = 23$   
 $f(3) = (3)^2 + 3(3) + 5 = 23$   
Since  $f(3) = \lim_{x \rightarrow 3} f(x)$  then the function is continuous at  $x = 3$ .
4.  $\lim_{x \rightarrow 0} x(x^2 - 3x + 5) = (0)((0)^2 - 3(0) + 5) = 0$   
 $g(0) = (0)((0)^2 - 3(0) + 5) = 0$   
Since  $g(0) = \lim_{x \rightarrow 0} g(x)$  then the function is continuous at  $x = 0$ .
5.  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{(x^2 + 4x + 4)(x^2 + 2x + 1)} = \frac{(2)^2 - 4}{((2)^2 + 4(2) + 4)((2)^2 + 2(2) + 1)} = \frac{0}{144} = 0$   
 $f(2) = \frac{(2)^2 - 4}{((2)^2 + 4(2) + 4)((2)^2 + 2(2) + 1)} = \frac{0}{144} = 0$   
Since  $f(2) = \lim_{x \rightarrow 2} f(x)$  then the function is continuous at  $x = 2$ .
6. Since  $g(-2) = \lim_{x \rightarrow -2} g(x) = \frac{1}{25}$  then the function is continuous at  $x = -2$ .
7. Since  $f(16) = \lim_{x \rightarrow 16} f(x) = \frac{4}{9}$  then the function is continuous at  $x = 16$ .
8. Since  $g(-1) = \lim_{x \rightarrow -1} g(x) = \frac{1}{2}$  then the function is continuous at  $x = -1$ .
9.  $f(x)$  is not continuous at  $x = 3$  since the  $f(3)$  is undefined (denominator becomes 0 at  $x = 3$ ).
10.  $g(x)$  is not continuous at  $x = 2$  since the  $g(2)$  is undefined (denominator becomes 0 at  $x = 2$ ).
11.  $h(x)$  is not continuous at  $x = -1$  since the  $h(-1)$  is undefined (denominator becomes 0 at  $x = -1$ ).
12.  $\lim_{x \rightarrow 0} k(x) \neq k(0)$ , therefore not continuous.
13. Even though the function is not continuous at  $x = 1$ ,  $f(-3) = -10$  and  $f(-2) = 0$ , and the function is continuous on the interval  $[-3, -2]$ . As well,  $f(2) = 0$  and  $f(3) = 20$  and the function is continuous on the interval  $[2, 3]$ . Therefore, there would have to be a value of  $k \in [f(-3), f(3)]$  that would satisfy the Intermediate value theorem.
14.  $b = 1$

L4: Explore limits which involve infinity.

## L4 Explore limits which involve infinity

### Achievement Indicators

- Understand the definitions and the properties of limits at infinity.
- Use limits as  $x \rightarrow \infty$  to find horizontal asymptotes of the function.
- Calculate limits of rational functions involving infinity.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \pm \infty, \text{ where } g(x) \neq 0. \quad \square$$

- Explore trigonometric graphs, such as sine and cosine, to show limits at infinity as an oscillating graph.

### Vertical asymptotes

In evaluation of limits at infinity has been seen with reference to vertical asymptotes. When direct substitution of  $x = c$  yields zero in the denominator and  $x = c$  is not a removable point of discontinuity, the graph has a vertical asymptote at  $x = c$ .

The only option on either side of the vertical asymptote is for the graph to increase without bound (to positive infinity) or decrease without bound (to negative infinity). The limit at  $x = c$  does not exist.

If  $\lim_{x \rightarrow c^+} f(x) = \pm \infty$  or  $\lim_{x \rightarrow c^-} f(x) = \pm \infty$ , then there exists a vertical asymptote at  $x = c$ .

Once the vertical asymptote is identified, the behavior at, or at both sides of the vertical asymptote, can be examined.

### Example 1

The function  $f(x) = \frac{2}{x+5}$ , has a vertical asymptote at  $x = -5$ .

As shown in **Table 3**, as  $x$  approaches  $-5$  from the positive direction (from the right),  $f(x)$  gets (infinitely) bigger and bigger. As  $x$  approaches  $-5$  from the negative direction (from the left),  $f(x)$  gets (infinitely) smaller and smaller.

**Table 4.1**

$\lim_{x \rightarrow -5^+} \frac{2}{x+5} = \infty$		$\lim_{x \rightarrow -5^-} \frac{2}{x+5} = -\infty$	
$x$	$f(x)$	$x$	$f(x)$
-4.9	20	-5.1	-20
-4.99	200	-5.01	-200
-4.999	2000	-5.001	-2000
-4.9999	20 000	-5.0001	-20 000

L4: Explore limits which involve infinity.

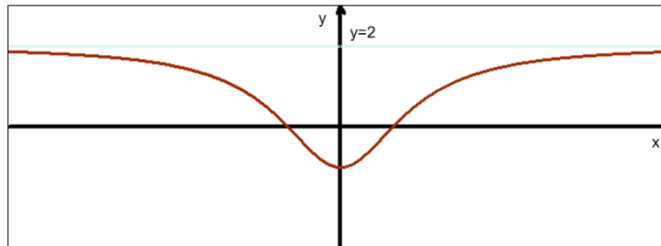
## Limits at Infinity

### Example 1

Investigate the behavior of the rational function  $f(x) = \frac{2x^2-2}{x^2+2}$

Table 4.2

$x$	$f(x)$
0	-1
$\pm 1$	0
$\pm 2$	1
$\pm 3$	1.454545
$\pm 5$	1.777777
$\pm 10$	1.941176
$\pm 100$	1.999400
$\pm 1000$	1.9999940



As  $x$  grows larger and larger the values of  $f(x)$  get closer and closer to 2.

Evaluating the behaviour of the function as  $x$  grows infinitely large in either the positive or negative direction, is also referred to as evaluating the limits at infinity as  $x$  approaches  $\pm\infty$ . This is an analysis of the end behavior of the function as either  $x \rightarrow -\infty$  or  $x \rightarrow \infty$ . The end behavior of a graph is very important. Limits at infinity must be one-sided;

$$\lim_{x \rightarrow \infty} f(x) \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x)$$

Some functions do not have end behaviour. For example,  $f(x) = \sqrt{x}$  does not have end behavior at  $-\infty$ , because the domain of the graph  $x \geq 0$ , so  $x$  cannot approach  $-\infty$ .

If a graph does have end behavior there are only a few things that can happen.

- The graph can increase without bound.

For example:

$$f(x) = x^2; \quad \lim_{x \rightarrow \infty} x^2 = \infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} x^2 = \infty$$

- The graph can decrease without bound.

For example:

$$f(x) = -x^2; \quad \lim_{x \rightarrow \infty} -x^2 = -\infty \quad \text{or} \quad \lim_{x \rightarrow -\infty} -x^2 = -\infty$$

- The graph can oscillate between two fixed values.

For example:

$$f(x) = \sin x; \quad \lim_{x \rightarrow \infty} f(x) = DNE$$

- The graph can taper off toward a specific finite  $y$ -value,  $L$ .

For example:

$$\text{If } \lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L \quad \text{then there exists a **horizontal asymptote** at } y = L$$

When taking limits at infinity, we're essentially looking for the existence or non-existence of any horizontal asymptotes.

#### L4: Explore limits which involve infinity.

Horizontal asymptotes are **not** discontinuities. A horizontal asymptote describes the end behavior of the graph. A graph can cross a horizontal asymptote any number of times as the limit only exists when  $x$  increases or decreases without bound. Horizontal asymptotes are a special characteristic of many rational functions (a polynomial over a polynomial,  $y = \frac{P(x)}{Q(x)}$  where  $Q(x) \neq 0$ ) and can be determined by examining the growth of the numerator in relation to the denominator.

There are **3 Cases** based on the comparison of the degrees of the leading terms of the numerator and denominator. The leading terms are not necessarily the ones written in front, but rather the terms with the largest power of  $x$ . The main technique for these is to divide each term in the numerator and denominator by the highest power of  $x$  in the denominator.

*Note: Careful consideration must be given to the forms  $\frac{\pm\infty}{\pm\infty}$  and  $\infty - \infty$  during computations of limits of infinity. Often it is incorrectly concluded that  $\frac{\pm\infty}{\pm\infty} = 1$ , or that  $\infty - \infty = 0$ . These are examples of indeterminate forms which can usually be algebraically manipulated to obtain a solution.*

#### Case 1

The degree of denominator is greater than the degree of the numerator.

In this case, for extremely large values of  $x$ , the denominator is getting larger faster than the numerator, so the fractions are getting smaller and approaching zero. All terms other than the term with the highest power are relatively small for very large  $x$ 's, thus they do not significantly contribute to the growth. In this case, there will be a horizontal asymptote at  $y = 0$ .

$$\lim_{x \rightarrow -\infty} \frac{4x^3 - 6x^2 + 8}{7x^5 + 6x^2 - 9x + 2} = \lim_{x \rightarrow -\infty} \frac{\frac{4x^3}{x^5} - \frac{6x^2}{x^5} + \frac{8}{x^5}}{\frac{7x^5}{x^5} + \frac{6x^2}{x^5} - \frac{9x}{x^5} + \frac{2}{x^5}} = \lim_{x \rightarrow -\infty} \frac{\frac{4}{x^2} - 0 + 0}{7 + 0 - 0 + 0} = \frac{0}{7} = 0$$

Between step 2 and 3 of the above example we are simplifying the limit process as we know that those terms will approach 0. It doesn't matter how much larger the denominator's degree is than the numerator's, only that it is. The greater the degree difference, the faster the values go to zero.

#### Case 2

The degree of the numerator and denominator are exactly the same.

In this case, the top and bottom are essentially growing at the same rate. Again the trailing terms are insignificantly small compared to the leading terms. This means that the fractions or ratios are approaching the ratio of the leading terms, or the leading terms' coefficients. In this case, there will always be an horizontal asymptote at  $y = \frac{\text{leading coefficient}}{\text{leading coefficient}}$ .

##### Example 1:

$$\lim_{x \rightarrow \infty} \frac{12x^4 - 6x^3 + 5}{5x^4 - x^2 + x + 2} = \lim_{x \rightarrow \infty} \frac{\frac{12x^4}{x^4} - \frac{6x^3}{x^4} + \frac{5}{x^4}}{\frac{5x^4}{x^4} - \frac{x^2}{x^4} + \frac{x}{x^4} + \frac{2}{x^4}} = \lim_{x \rightarrow \infty} \frac{12 - \frac{6}{x} + \frac{5}{x^4}}{5 - \frac{1}{x^2} + \frac{1}{x^3} + \frac{2}{x^4}} = \lim_{x \rightarrow \infty} \frac{12 - 0 + 0}{5 - 0 + 0 + 0} = \frac{12}{5}$$

Horizontal Asymptote at  $y = \frac{12}{5}$

##### Example 2:



L4: Explore limits which involve infinity.

$$\lim_{x \rightarrow -\infty} \frac{10x^2 + 7x}{3x^2 - 5x + 1} = \lim_{x \rightarrow -\infty} \frac{\frac{10x^2}{x^2} + \frac{7x}{x^2}}{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{10 + \frac{7}{x}}{3 - \frac{5}{x} + \frac{1}{x^2}} = \lim_{x \rightarrow -\infty} \frac{10 + 0}{3 - 0 + 0} = \frac{10}{3}$$

Horizontal Asymptote at  $y = \frac{10}{3}$

### Case 3

The degree of the numerator is greater than the degree of the denominator.

In this case, the numerator's is growing much faster than the denominator either in the positive or negative directions (increasing or decreasing without bound). The limit will not exist and there will be no horizontal asymptote. In Summary, if the numerator has higher degree than the denominator, then we will always have the function approaching  $+\infty$  or  $-\infty$  as the end behavior of our rational function.

$$\lim_{x \rightarrow \infty} \frac{3 - 4x^2}{5x - 1} = \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - \frac{4x^2}{x}}{\frac{5x}{x} - \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - 4x}{5 - \frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{0 - 4x}{5 - 0} = \frac{-4x}{5}$$

The limit approaches  $-\infty$ .

To determine what **type** of infinity (positive or negative), any representative huge  $x$ -value can be substituted into the leading terms to determine the sign, in the direction that is being approached.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{-5x^6 - 8x + 14}{8x^3 + 3x^2 + 2x - 7} & \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{-2x^7 - 4x^4 + 10}{-6x^3 - x^2 + x + 5} \\ &= \lim_{x \rightarrow \infty} \frac{-5(\text{positive})^6 - \text{does not matter}}{8(\text{positive})^3 + \text{does not matter}} &= \lim_{x \rightarrow -\infty} \frac{-2(\text{negative})^7 - \text{does not matter}}{-6(\text{negative})^3 - \text{does not matter}} \\ &= \lim_{x \rightarrow \infty} \frac{-5(\text{positive})}{8(\text{positive})} &= \lim_{x \rightarrow -\infty} \frac{-2(\text{negative})}{-6(\text{negative})} \\ &= -\infty &= \infty \end{aligned}$$

Sometimes, the degrees of the numerator and denominator are not as explicit. This often occurs when either the numerator or denominator are under a radical. For these types of problems, we can use a similar analysis.

Compute:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sqrt{2x^3 - x}}{3x^2 - 6} & < - - - - \text{this is an indeterminate form of type } \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{2x^3 - x}}{3x^2 - 6} \left( \frac{\sqrt{\frac{1}{x^4}}}{\frac{1}{x^2}} \right) \quad \text{Note: } \sqrt{\frac{1}{x^4}} = \frac{1}{x^2} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{2x^3}{x^4} - \frac{x}{x^4}}}{\frac{3x^2}{x^2} - \frac{6}{x^2}} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{2}{x} - \frac{1}{x^3}}}{3 - \frac{6}{x^2}} = \lim_{x \rightarrow \infty} \frac{\sqrt{0 - 0}}{3 - 0} = 0 \end{aligned}$$

Students should be aware at this point that when evaluating the end behavior of any function in their toolkit, they are actually finding the limit of that function as  $x$  approaches infinity.

Examples:

L4: Explore limits which involve infinity.

(students should verify their conclusions by sketching the graphs for each function)

$$\lim_{x \rightarrow +\infty} 3^x = \infty \quad \text{rationale; } 3^\infty = \infty$$

$$\lim_{x \rightarrow -\infty} \left(\frac{2}{7}\right)^x = \infty \quad \text{rationale; } \left(\frac{2}{7}\right)^{-\infty} = \left(\frac{7}{2}\right)^\infty = \infty$$

$$\lim_{x \rightarrow +\infty} \ln(x) = \infty \quad \text{rationale; As } x \text{ increases in value, } \ln(x) \text{ increases in value.}$$

$$\lim_{x \rightarrow +\infty} \sqrt{x-2} = \infty \quad \text{rationale; As } x \text{ increase in value, } \sqrt{x-2} \text{ increases in value.}$$

$$\lim_{x \rightarrow -\infty} |x| = \infty \quad \text{rationale; As } x \text{ becomes larger negatively, } |x| \text{ becomes larger positively.}$$

$$\lim_{x \rightarrow +\infty} (x-4)^5(2x-1)^2(3x+4) = \infty \quad \text{rationale; (large +)^5(large +)^2(large +) = } +\infty$$

$$\lim_{x \rightarrow -\infty} (x+3)^3(x+1)^2 = -\infty \quad \text{rationale; (large -)^3(large -)^2 = } -\infty$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x-4} = 0 \quad \text{rationale; As } x \text{ increases in value, } \frac{1}{x-4} \text{ becomes very small.}$$

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2-2x-3} = 0 \quad \text{rationale; As } x \text{ increases in value, } \frac{1}{x^2-2x-3} \text{ becomes very small.}$$

## L4 Exercises

1. Explain in your own words the meaning of each of the following:

a)  $\lim_{x \rightarrow -\infty} f(x) = 6$       b)  $\lim_{x \rightarrow \infty} f(x) = -9$       c)  $\lim_{x \rightarrow 4^+} f(x) = \infty$       d)  $\lim_{x \rightarrow 6^-} f(x) = -\infty$

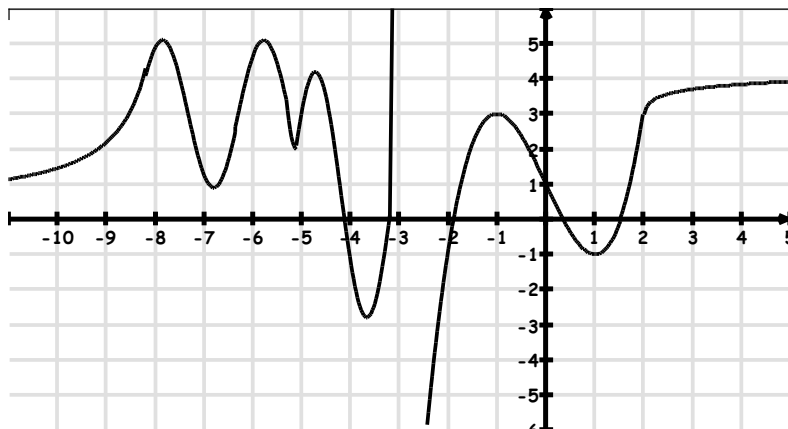
2. Can the graph of  $y = f(x)$  intersect :

- a) a vertical asymptote?  
b) a horizontal asymptote?

Explain your conclusions.

3. For the function whose graph is shown below, determine the following:

- a)  $\lim_{x \rightarrow \infty} f(x)$       b)  $\lim_{x \rightarrow -\infty} f(x)$       c)  $\lim_{x \rightarrow -3^-} f(x)$       d)  $\lim_{x \rightarrow -3^+} f(x)$   
e) State the equations of the vertical and horizontal asymptotes.



4. Evaluate  $\lim_{x \rightarrow \infty} \frac{x^2}{2^x}$  by using a table of values.

5. Evaluate the following limits, if possible.

a)  $\lim_{x \rightarrow \infty} \frac{5x^4 - 7x^3 + 7x^2 - 1}{3x^4 + 2x^3}$

b)  $\lim_{x \rightarrow -\infty} \frac{x^5 - x^2}{x^3 - 2x}$

c)  $\lim_{x \rightarrow \infty} \frac{9x^2 - x + 8}{2x^4 + x^3 - 7}$

d)  $\lim_{x \rightarrow -\infty} \frac{12x^2 - 6x^3 + 5x^4 + 9x^5}{3x^5 + 2x^4 - 4x^3 + 2x}$

e)  $\lim_{x \rightarrow \infty} \frac{5x^3 - 4x^2 - 5x}{4x^3 + 3x}$

f)  $\lim_{x \rightarrow \infty} \left(\frac{1}{8}\right)^x + \frac{x^3 - 4x^2 - 5x}{4x^3 + 3x} - 7$

g)  $\lim_{x \rightarrow \infty} \left(\frac{7}{3}\right)^{-x} + \frac{4x^2 - 5x}{2x^2 + 1} - 9$

h)  $\lim_{x \rightarrow -\infty} \left(\frac{1}{5}\right)^{2x}$

i)  $\lim_{x \rightarrow \infty} \frac{(-2x^2 - 3)(x + 1)}{2 - 5x^3}$

j)  $\lim_{x \rightarrow -\infty} 12 - \left(\frac{6}{5}\right)^x$

k)  $\lim_{x \rightarrow \infty} \frac{10x^2}{\sqrt{4x^4 + 1}}$

l)  $\lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2 - 2} - 3x}$

m)  $\lim_{x \rightarrow \infty} \frac{4 - \frac{3}{x}}{5x^2 + 1}$

n)  $\lim_{x \rightarrow \infty} \frac{6x}{2x - 1} - \frac{x + 5}{3x - 4}$

o)  $\lim_{x \rightarrow \infty} x - \sqrt{x^2 + 1}$

L4: Explore limits which involve infinity.

p)  $\lim_{x \rightarrow -\infty} \frac{|x - 8|}{x - 8}$

q)  $\lim_{x \rightarrow \infty} 2x - \sqrt{4x^2 + 6x}$

r)  $\lim_{x \rightarrow -\infty} (x - 3)^2(x + 1)^5$

s)  $\lim_{x \rightarrow -\infty} (2x - 3)^3(x + 1)^3$

6. Determine the horizontal and vertical asymptotes and any other points of discontinuity of the functions.

a)  $f(x) = \frac{x^2 + 8x - 20}{2x^2 + x - 6}$

b)  $f(x) = \frac{x^2 - 3x - 40}{x^3 + 8}$

c)  $f(x) = \frac{x^4 - 2x^3 - 63x^2}{x^2 - 10x + 16}$

d)  $f(x) = \frac{10x^3 - 18x}{x^3 - x^2 - 2x}$

e)  $f(x) = \frac{9 - x^2}{16 - 2x^3}$

f)  $f(x) = \frac{x - 7}{\sqrt{3x^2 + 10x + 8}}$

7. State the vertical asymptotes. Describe the behavior of  $f(x)$  on each side of the vertical asymptote(s):

a)  $f(x) = \frac{2}{x^2 - 9}$

b)  $f(x) = \frac{x^2 + x}{x + 5}$

c)  $f(x) = \tan x$

#### L4: Explore limits which involve infinity.

##### L4 Solutions

1. a) As  $x$  gets increasing larger in the negative direction, the value of the function approaches 6. The constant value of  $y = 6$  will not change.  
 b) As  $x$  gets increasing larger in the positive direction the value of the function approaches  $-9$ . The constant value of  $y = -9$  will not change.  
 c) As  $x$  approaches 4 from the right hand side, the value of the function gets increasingly larger, approaching positive infinity.  
 d) As  $x$  approaches 6 from the left hand side, the value of the function becomes increasingly larger, approaching negative infinity.

2. a) No, the function is undefined at a vertical asymptote.  
 b) Yes, a horizontal asymptote occurs for very large negative or positive values of  $x$ . Therefore, the graph may cross the asymptote when  $x$  is small in relation to  $\pm\infty$ .

3. a) 4  
 b) 1  
 c)  $\infty$   
 d)  $-\infty$   
 e)  $HA: y = 1$  and  $y = 4$   
 $VA: x = -3$

4.  $\lim_{x \rightarrow \infty} \frac{x^2}{2^x} = 0$

$x$	$f(x)$
3	1.125
5	0.78125
10	0.00001024
100	$7.8 \times 10^{-27}$
1000	$9.33 \times 10^{-296}$

5.

- a)  $\frac{5}{3}$       b)  $\infty$       c) 0      d) 3      e)  $\frac{5}{4}$       f)  $-\frac{27}{4}$       g)  $-\frac{1}{7}$   
 h)  $\infty$       i)  $\frac{2}{5}$       j) 12

- k) 5      l)  $-1$       m) 0      n)  $\frac{8}{3}$       o) 0      p)  $-1$       q)  $-\frac{3}{2}$       r)  $-\infty$       s)  $\infty$

6. a)  $HA: y = 1/2$      $VA: x = 3/2, x = -2$   
 b)  $HA: y = 0$      $VA: x = -2$ ,  
 c)  $HA: y = \text{none}$      $VA: x = 2, x = 8$   
 d)  $HA: y = 10$      $VA: x = -1$ ,     $x = 2$ ,    *Point of Discontinuity (0, 9)*  
 e)  $HA: y = 1/2$      $VA: x = 2$   
 f)  $HA: y = \frac{1}{\sqrt{3}}$      $VA: x = -2, x = \frac{-4}{3}$ . *The graph is undefined between the VA.*

L4: Explore limits which involve infinity.

7. a) VA:  $x = -3$ ;  $x = 3$

Behaviour:  $\lim_{x \rightarrow -3^-} \frac{2}{x^2 - 9} = +\infty$      $\lim_{x \rightarrow -3^+} \frac{2}{x^2 - 9} = -\infty$      $\lim_{x \rightarrow 3^-} \frac{2}{x^2 - 9} = -\infty$      $\lim_{x \rightarrow 3^+} \frac{2}{x^2 - 9} = +\infty$

b) VA:  $x = -5$

Behaviour:  $\lim_{x \rightarrow -5^-} \frac{x^2 + x}{x + 5} = -\infty$      $\lim_{x \rightarrow -5^+} \frac{x^2 + x}{x + 5} = +\infty$

c) VA:  $x = \frac{n\pi}{2}$ , where  $n$  is an odd integer.