Pre-Calculus A 120 Curriculum

Implemented September 2013

Acknowledgements

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Curriculum Overview for Grades 10-12 Mathematics

BACKGROUND AND RATIONALE

Mathematics curriculum is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society.

It is essential the mathematics curriculum reflects current research in mathematics instruction. To achieve this goal, *The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol* has been adopted as the basis for a revised mathematics curriculum in New Brunswick. The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators and others.

The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP and the NCTM.

There is an emphasis in the New Brunswick curriculum on particular key concepts at each grade which will result in greater depth of understanding and ultimately stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

The intent of this document is to clearly communicate high expectations for students in mathematics education to all education partners. Because of the emphasis placed on key concepts at each grade level, time needs to be taken to ensure mastery of these concepts. *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM Principles and Standards, 2000).*

BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING

The New Brunswick Mathematics Curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice. These beliefs include:
- mathematics learning is an active and constructive process;
- learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates;
- learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking and that nurtures positive attitudes and sustained effort; and
- learning is most effective when standards of expectation are made clear with on-going assessment and feedback.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and aspirations.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best developed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial and symbolic representations of mathematics. The learning environment should value, respect and address all students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals for Mathematically Literate Students

The main goals of mathematics education are to prepare students to:
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:
- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success, and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Diverse Cultural Perspectives

Students come from a diversity of cultures, have a diversity of experiences and attend schools in a variety of settings including urban, rural and isolated communities. To address the diversity of knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of students, a variety of teaching and assessment strategies are required in the classroom. These strategies must go beyond the incidental inclusion of topics and objects unique to a particular culture.

For many First Nations students, studies have shown a more holistic worldview of the environment in which they live (Banks and Banks 1993). This means that students look for connections and learn best when mathematics is contextualized and not taught as discrete components. Traditionally in Indigenous culture, learning takes place through active participation and little emphasis is placed on the written word. Oral communication along with practical applications and experiences are important to student learning and understanding. It is important that teachers understand and respond to both verbal and non-verbal cues to optimize student learning and mathematical understandings.

Instructional strategies appropriate for a given cultural or other group may not apply to all students from that group, and may apply to students beyond that group. Teaching for diversity will support higher achievement in mathematics for all students.

Adapting to the Needs of All Learners

Teachers must adapt instruction to accommodate differences in student development as they enter school and as they progress, but they must also avoid gender and cultural biases. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom. The reality of individual student differences must not be ignored when making instructional decisions.

As well, teachers must understand and design instruction to accommodate differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Designing classroom activities to support a variety of learning styles must also be reflected in assessment strategies.

Universal Design for Learning

The New Brunswick Department of Education and Early Childhood Development's definition of inclusion states that every child has the right to expect that his or her learning outcomes, instruction, assessment, interventions, accommodations, modifications, supports, adaptations, additional resources and learning environment will be designed to respect his or her learning style, needs and strengths.

Universal Design for Learning is a "…framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged." It also "...reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient" (CAST, 2011).

In an effort to build on the established practice of differentiation in education, the Department of Education and Early Childhood Development supports *Universal Design for Learning* for all students. New Brunswick curricula are created with universal design for learning principles in mind. Outcomes are written so that students may access and represent their learning in a variety of ways, through a variety of modes. Three tenets of universal design inform the design of this curriculum. Teachers are encouraged to follow these principles as they plan and evaluate learning experiences for their students:
- **Multiple means of representation:** provide diverse learners options for acquiring information and knowledge
- **Multiple means of action and expression:** provide learners options for demonstrating what they know
- **Multiple means of engagement:** tap into learners' interests, offer appropriate challenges, and increase motivation

For further information on *Universal Design for Learning*, view online information at
http://www.cast.org/.

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, and physical education.

NATURE OF MATHEMATICS

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this document. These components include: **change**, **constancy**, **number sense**, **patterns**, **relationships, spatial sense** and **uncertainty**.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12,… can be described as:
- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen, 1990, p. 184).

Students need to learn that new concepts of mathematics as well as changes to already learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include:
- the area of a rectangular region is the same regardless of the methods used to determine the solution
- the sum of the interior angles of any triangle is 180°
- the theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Many important properties in mathematics do not change when conditions change. Examples of constancy include:
- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- the theoretical probability of an event.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p. 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. Evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing mathematically rich tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

ASSESSMENT

Ongoing, interactive assessment (*formative assessment*) is essential to effective teaching and learning. Research has shown that formative assessment practices produce significant and often substantial learning gains, close achievement gaps and build students' ability to learn new skills (Black & William, 1998, OECD, 2006). Student involvement in assessment promotes learning. Interactive assessment, and encouraging self-assessment, allows students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes:
* providing clear goals, targets and learning outcomes
* using exemplars, rubrics and models to help clarify outcomes and identify important features of the work
* monitoring progress towards outcomes and providing feedback as necessary
* encouraging self-assessment
* fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies, 2000)

Formative assessment practices act as the scaffolding for learning which, only then, can be measured through summative assessment. *Summative assessment,* or assessment *of* learning, tracks student progress, informs instructional programming and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning and produce achievement gains.

Student assessment should:
- align with curriculum outcomes
- use clear and helpful criteria
- promote student involvement in learning mathematics during and after the assessment experience
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction
  (adapted from: NCTM, Mathematics Assessment: A practical handbook, 2001, p.22)

Work Samples
- math journals
- portfolios
- drawings, charts, tables and graphs
- individual and classroom

Rubrics
- constructed response
- generic rubrics
- task-specific rubrics
- questioning

Surveys
- attitude
- interest
- parent questionnaires

Assessing Mathematics Development in a Balanced Manner

Observations
- planned (formal)
- unplanned (informal)
- read aloud (literature with math focus)
- shared and guided math activities
- performance tasks
- individual conferences
- anecdotal records
- checklists
- interactive activities

Self-Assessment
- personal reflection and evaluation

Math Conferences
- individual
- group
- teacher-initiated
- child-initiated

CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.

GRADE TOPICS	10	11	12
The topics of study vary in the courses for grades 10–12 mathematics. Topics in the pathways include: • *Algebra* • *Financial Mathematics* • *Geometry* • *Logical Reasoning* • *Mathematics Research Project* • *Measurement* • *Number* • *Permutations, Combinations and Binomial Theorem* • *Probability* • *Relations and Functions* • *Statistics* • *Trigonometry*	**GENERAL OUTCOMES** **SPECIFIC OUTCOMES** **ACHIEVEMENT INDICATORS**		

NATURE OF MATHEMATICS

Change
Constancy
Number Sense
Patterns
Relationships
Spatial Sense
Uncertainty

MATHEMATICAL PROCESSES:
Communication, Connections, Mental Mathematics and Estimation,
Problem Solving, Reasoning, Technology, Visualization

MATHEMATICAL PROCESSES

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.
Students are expected to:
- communicate in order to learn and express their understanding of mathematics (Communications: C)
- develop and apply new mathematical knowledge through problem solving (Problem Solving: PS)
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines (Connections: CN)
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation: ME)
- select and use technologies as tools for learning and solving problems (Technology: T)
- develop visualization skills to assist in processing information, making connections and solving problems (Visualization: V).
- develop mathematical reasoning (Reasoning: R)

The New Brunswick Curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, knowledge, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication can help students make connections among concrete, pictorial, symbolic, verbal, written and mental representations of mathematical ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type, *How would you...?* or *How could you ...?,* the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learnings in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly look for, and engage in, finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences, and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"*Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding… Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*" (Caine and Caine, 1991, p. 5).

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental mathematics" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "*become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving*" (Rubenstein, 2001).

Mental mathematics "*provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers*" (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Technology [T]

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Calculators and computers can be used to:
- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense.

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate and involves knowledge of several estimation strategies (Shaw and Cliatt, 1989, p. 150).

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking.

Questions that challenge students to think, analyze and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, *Why do you believe that's true/correct?* or *What would happen if ….*

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

ESSENTIAL GRADUATION LEARNINGS

Graduates from the public schools of Atlantic Canada will be able to demonstrate knowledge, skills, and attitudes in the following essential graduation learnings. These learnings are supported through the outcomes described in this curriculum document.

Aesthetic Expression
Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship
Graduates will be able to assess social, cultural, economic, and environmental interdependence in a local and global context.

Communication
Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) as well as mathematical and scientific concepts and symbols to think, learn, and communicate effectively.

Personal Development
Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving
Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language, mathematical, and scientific concepts.

Technological Competence
Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems

PATHWAYS AND TOPICS

The Common Curriculum Framework for Grades 10–12 Mathematics on which the New Brunswick Grades 10-12 Mathematics curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics*. In New Brunswick all Grade 10 students share a common curriculum covered in two courses: *Geometry, Measurement and Finance 10* and *Number, Relations and Functions 10*. Starting in Grade 11, three pathways are available: *Finance and Workplace*, *Foundations of Mathematics*, and *Pre-Calculus*.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. Students are encouraged to cross pathways to follow their interests and to keep their options open. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all three pathways are to provide prerequisite attitudes, knowledge, skills and understandings for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of each pathway has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* and on consultations with mathematics teachers.

Financial and Workplace Mathematics
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, statistics and probability.

Foundations of Mathematics
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, statistics and probability.

Pre-calculus

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Students develop of a function tool kit including quadratic, polynomial, absolute value, radical, rational, exponential, logarithmic and trigonometric functions. They also explore systems of equations and inequalities, degrees and radians, the unit circle, identities, limits, derivatives of functions and their applications, and integrals.

Outcomes and Achievement Indicators

The New Brunswick Curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes and achievement indicators.

General Curriculum Outcomes (GCO) are overarching statements about what students are expected to learn in each strand/sub-strand. The general curriculum outcome for each strand/sub-strand is the same throughout the pathway.

Specific Curriculum Outcomes (SCO) are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as required for a given grade.

Achievement indicators are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome. In the specific outcomes, the word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for clarification and are not requirements that must be addressed to fully meet the learning outcome. The word *and* used in an outcome indicates that both ideas must be addressed to fully meet the learning outcome, although not necessarily at the same time or in the same question.

Instructional Focus

Each pathway in *The Common Curriculum Framework for Grades 10–12 Mathematics* is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful. Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches, and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning, assessment as learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.

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                          │   Mathematics K-9   │
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Grade 10
- *2 x 90 hr courses; required to pass both*
- *May be taken in any order or in the same semester*

Geometry, Measurement and Finance 10	Number, Relations and Functions 10

Grade 11
- *3 x 90 hr courses offered in 3 pathways*
- *Students are required to pass at least one of "Financial and Workplace Mathematics 11" or "Foundations of Mathematics 11".*
- *Pre-requisite Grade 10 course(s) must be passed before taking Grade 11 courses.*

Financial and Workplace Mathematics 110	**Foundations of Mathematics 110**	**Pre-Calculus 110**
Pre-requisite: Geometry, Measurement and Finance 10	*Pre-requisites:* Geometry, Measurement and Finance 1 **and** Number, Relations and Functions 10	*Pre-requisite or Co-requisite:* Foundations of Mathematics 110

Grade 12
- *5 x 90 hr courses offered in 3 pathways*
- *Pre-requisite Grade 11 or Grade 12 course must be passed before taking Grade 12 courses.*

Financial and Workplace Mathematics 120	**Foundations of Mathematics 120**	**Pre-Calculus A 120**
Pre-requisite: Financial and Workplace Mathematics 110 **or** Foundations of Mathematics 110	*Pre-requisite:* Foundations of Mathematics 110	*Pre-requisite: Pre-Calculus 110*
		Pre-Calculus B 120
		Pre-requisite or Co-requisite: Pre-Calculus A 120
		Calculus 120
		Pre-requisites: Pre-Calculus A 120 **and** Pre-Calculus B 120

SUMMARY

The Conceptual Framework for Grades 10–12 Mathematics describes the nature of mathematics, the mathematical processes, the pathways and topics, and the role of outcomes and achievement indicators in grades 10–12 mathematics. Activities that take place in the mathematics classroom should be based on a problem-solving approach that incorporates the mathematical processes and leads students to an understanding of the nature of mathematics.

CURRICULUM DOCUMENT FORMAT

This guide presents the mathematics curriculum by grade level so that a teacher may readily view the scope of the outcomes which students are expected to meet during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings at a particular grade level are part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The heading of each page gives the General Curriculum Outcome (GCO), and Specific Curriculum Outcome (SCO). The key for the mathematical processes follows. A <u>Scope and Sequence</u> is then provided which relates the SCO to previous and next grade SCO's. For each SCO, <u>Elaboration</u>, <u>Achievement Indicators</u>, <u>Suggested Instructional Strategies</u>, and <u>Suggested Activities for Instruction and Assessment</u> are provided. For each section, the *Guiding Questions* should be considered.

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Mathematical Processes

[C] Communication Technology	[PS] Problem Solving [V] Visualization	[CN] Connections [R] Reasoning	[ME] Mental Math [T] and Estimation

Scope and Sequence

Previous Grade or Course SCO's	Current Grade SCO	Following Grade or Course SCO's

Elaboration

Describes the "big ideas" to be learned and how they relate to work in previous Grades

Guiding Questions:
• What do I want my students to learn?
• What do I want my students to understand and be able to do?

Achievement Indicators

Describes observable indicators of whether students have met the specific outcome

Guiding Questions:
• *What evidence will I look for to know that learning has occurred?*
• *What should students demonstrate to show their understanding of the mathematical concepts and skills?*

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Suggested Instructional Strategies

General approach and strategies suggested for teaching this outcome

Guiding Questions
• What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?
• What teaching strategies and resources should I use?
• How will I meet the diverse learning needs of my students?

Suggested Activities for Instruction and Assessment

Some suggestions of specific activities and questions that can be used for both instruction and assessment.

Guiding Questions
• What are the most appropriate methods and activities for assessing student learning?
• How will I align my assessment strategies with my teaching strategies?

Guiding Questions
• What conclusions can be made from assessment information?
• How effective have instructional approaches been?
• What are the next steps in instruction?

Pre-Calculus A 120

Specific Curriculum Outcomes

SCO: **RF1: Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.**   [C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Relations and Functions

RF1: Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF1:** Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.	**RF4:** Graph and analyse polynomials functions (limited to polynomial functions of degree ≤5) **RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

Students have analyzed quadratic functions of the form, $y = a(x - p)^2 + q$ in *Pre-Calculus 110* and in the form $y = ax^2 + bx + c$ in *Foundations of Mathematics 110*. This outcome explores the effects of horizontal and vertical translations on other kinds of functions, through explorations of the effect of factors h and k on the graph of the function $y = f(x - h) + k$. Students should use p and h, and q and k interchangeably.

The value of h indicates the magnitude of the **horizontal translation** of $f(x)$. When $h > 0$, the function shifts to the right; and when $h < 0$, the function shifts to the left.

The value of k indicates the magnitude of the **vertical translation** of $f(x)$. When $k > 0$, the function shifts up; and when $k < 0$, the function shifts down.

SCO: **RF1: Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.**   [C, CN, R, V]

ACHIEVEMENT INDICATORS

- Compare the graphs of a set of equations of the form $y = f(x) + k$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of k.

- Compare the graphs of a set of functions of the form $y = f(x - h)$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of h.

- Compare the graphs of a set of equations of the form $y = f(x - h) + k$ to the graph of $y = f(x)$, and generalize, using inductive reasoning, a rule about the effects of h and k.

- Sketch the graph of $y = f(x) + k,\ y = f(x - h)$ or $y = f(x - h) + k$ for given values of h and k, given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

- Write the equation of a function whose graph is a vertical and/or horizontal translation of the graph of the function $y = f(x)$.

Suggested Instructional Strategies

- Direct students to internet sites such as http://www.purplemath.com/modules/fcntrans.htm for additional lessons on transforming functions.

- Use technology to explore the effect of h and k on the graph of a given equation, $y = f(x - h) + k$, to develop a rule about the effect of h and k. Repeat with a variety of different equations such as:

$y = x^2$	$y = x^2 - 5$	$y = (x - 5)^2$	$y = (x + 1)^2 + 4$
$y = |x|$	$y = |x + 6|$	$y = |x| - 4$	$y = |x + 3| - 5$

## Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Use the following graph to determine the translations that have been applied to the equation $y = f(x)$ to obtain the graph $y = g(x)$.
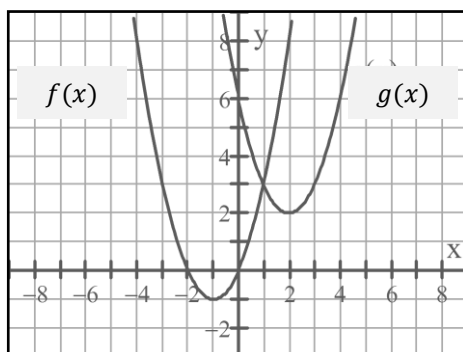
Answer: *horizontal translation* $(+3)$ *right* 3, *vertical translation* $(+3)$ *up* 3

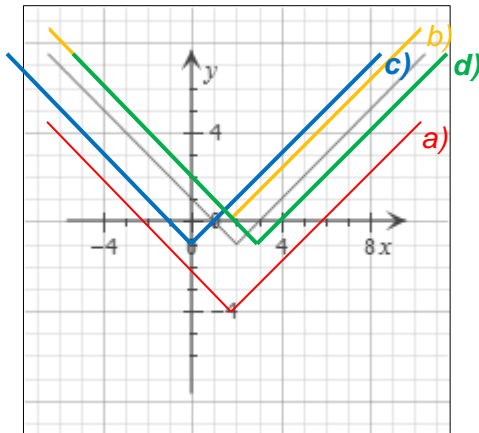SCO: **RF1: Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.**   [C, CN, R, V]

Q On the graph below use the graph of $y = f(x)$ to sketch the graph of each translation below:
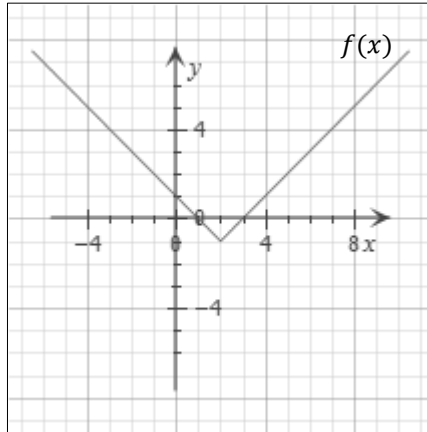
a) $y = f(x) - 3$ b) $y = f(x) + 1$ c) $y = f(x + 2)$ d) $y = f(x - 1)$

Answers: *a)* *b)* *c)* *d)*

SCO**: RF2: Demonstrate an understanding of the effects of horizontal and vertical
stretches on the graphs of functions and their related equations.**   [C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

**RF2: Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs
of functions and their related equations.**

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, *x*- and *y*-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, *x*- and *y*-intercepts; and to solve problems.	**RF2:** Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.	**RF4:** Graph and analyse polynomial functions (limited to polynomial functions of degree ≤5) **RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

For this outcome students will investigate the effect of a and b on the graph of the
function, $y = af(bx)$.

The value of a indicates the magnitude of the **vertical stretch** of $f(x)$ about the x-axis
by a factor of a. The reciprocal of the value of b indicates the magnitude of the
horizontal stretch of $f(x)$ about the y-axis by a factor of $\frac{1}{b}$.

If students have not completed the outcomes on trigonometric functions prior to this,
horizontal stretch will be a new concept for them. Horizontal stretches are difficult to see
unless a function has a period or a restricted domain. In some cases a horizontal stretch
may be viewed as a vertical stretch. For example if $y = (2x)^2$ and is half as wide, it can
be viewed as $y = 4x^2$ which is seen as four times as high.

ACHIEVEMENT INDICATORS

- Compare the graphs of a set of functions of the form $y = af(x)$ to the graph of
  $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of a.

- Compare the graphs of a set of functions of the form $y = f(bx)$ to the graph of
  $y = f(x)$, and generalize, using inductive reasoning, a rule about the effect of b.

- Compare the graphs of a set of functions of the form $y = af(bx)$, to the graph of
  $y = f(x)$, and generalize using inductive reasoning, a rule about the effects of a and
  b.

- Sketch the graph of $y = af(x), \ y = f(bx), \ y = af(bx)$, for given values of a and b,
  given a sketch of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

SCO**: RF2: Demonstrate an understanding of the effects of horizontal and vertical**
stretches on the graphs of functions and their related equations.   [C, CN, R, V]

- Write the equation of a function, given its graph which is a vertical and/or horizontal stretch of the graph of the function $y = f(x)$.

Suggested Instructional Strategies

- Direct students to internet sites such as http://www.purplemath.com/modules/fcntrans2.htm. for additional lessons on transforming functions.

- Use technology to explore the effect of various values of a and b on the graph of a given function, $y = af(bx)$, then develop a rule about the effects of a and b. Repeat with a variety of functions such as:

$$f(x) = |x| \qquad f(x) = 3|x| \qquad f(x) = \tfrac{1}{2}|x| \qquad f(x) = 5x^2$$

$$f(x) = \left(\tfrac{1}{3}x\right)^2 \qquad f(x) = \left|\tfrac{1}{4}x\right| \qquad f(x) = 2x^2 \qquad f(x) = 5|2x|$$
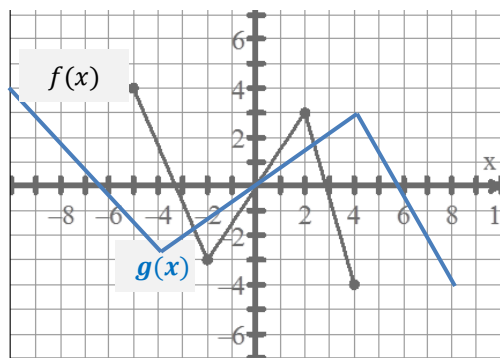
$$f(x) = \tfrac{1}{4}|3x| \qquad f(x) = \tfrac{1}{2}\left|\tfrac{1}{3}x\right|$$

SCO: RF2: Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.   [C, CN, R, V]

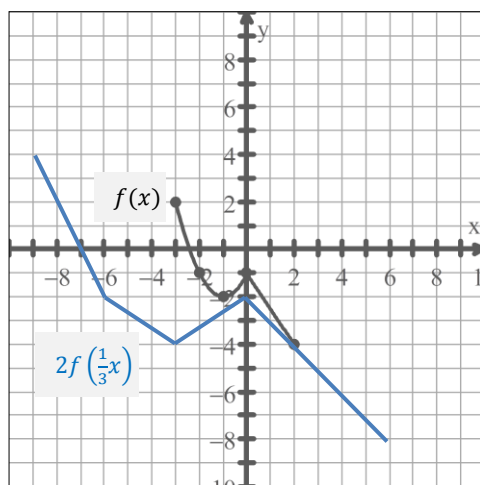## Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q For the following graph of $y = f(x)$ draw and label the graph of $y = g(x)$ that shows a horizontal stretch of 2.

Answer: *shown in blue on graph*

f(x)			g(x)	
x	y		x	y
−5	4		−10	4
−2	−3		−4	−3
2	3		4	3
4	−4		8	−4

Q From the graph of $y = f(x)$ below, draw the graph of $y = 2f\left(\frac{1}{3}x\right)$

Answer: *shown in blue on graph* $(x, y) -\!\!\rightarrow (3x, 2y)$

f(x)			$2f\left(\frac{1}{3}x\right)$	
x	y		x	y
−3	2		−9	4
−2	−1		−6	−2
−1	−2		−3	−4
0	−1		0	−2
2	−4		6	−8

SCO: **RF3: Apply translations and stretches to the graphs and equations of functions.**
     [C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF3: Apply translations and stretches to the graphs and equations of functions.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF3:** Apply translations and stretches to the graphs and equations of functions.	**RF4:** Graph and analyze polynomial functions (limited to polynomial functions of degree ≤5). **RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

This outcome will focus on using the notation $y = af\big(b(x - h)\big) + k$ to sketch graphs of translations and stretches of functions.

In some cases the horizontal stretch factor b, must first be factored from the binomial before indicating the horizontal translation.

For example:
$$y = \tfrac{1}{2}f(3x - 6) - 3 = \tfrac{1}{2}f\big(3(x - 2)\big) - 3$$
Horizontal translation$(-h)$: $+2$

Vertical translation$(+k)$: -3

Horizontal stretch$\left(\tfrac{1}{b}\right)$: $\tfrac{1}{3}$

Vertical stretch(a): $\tfrac{1}{2}$

When students sketch or analyze graphs of functions with translations and stretches, they will need to apply or **consider the stretch first**, and then the translation. Invariant points will need to be examined.

When students analyze a graph and a transformed graph, they must choose points which are located on both axes to determine the horizontal and vertical translations.

To determine the respective stretch factors, students will compare the distances between key points both vertically and horizontally.

SCO: **RF3: Apply translations and stretches to the graphs and equations of functions.**
     [C, CN, R, V]

ACHIEVEMENT INDICATORS

- Sketch the graph of the equation $y = af\big(b(x - h)\big) + k$ for given values of a, b, h and k, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

- Write the equation of a function, given its graph which is a translation and/or stretch of the graph of the function $y = f(x)$.

Suggested Instructional Strategies

- For extra practice and further explanation direct students to internet sites such as:
  http://www.purplemath.com/modules/fcntrans2.htm.
  http://www.mathsisfun.com/sets/function-transformations.html
  http://cims.nyu.edu/~kiryl/Precalculus/Section_2.5-
  Transformations%20of%20Functions/Transformations%20of%20Functions.pdf

- Use technology to explore the effect of a, b, h and k on the graph of a given equation, $y = af\big(b(x - h)\big) + k$, to develop a rule about the effect of a and b. For a few different equations, such as the ones shown below, have students identify $f(x)$, a, b, h, and k. Then have them sketch the graph of the given equation and the graph of $y = f(x)$ on the same set of axes.

$$y = \left|\tfrac{1}{4}(x + 2)\right| - 1 \qquad\qquad y = 5|3x| + 4 \qquad\qquad y = \tfrac{1}{2}|x - 1|$$
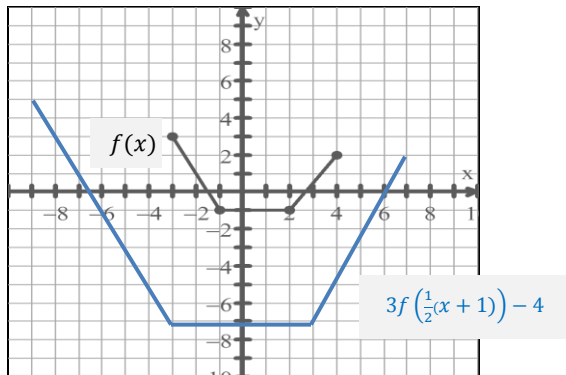
$$y = 2(x + 3)^2 \qquad\qquad y = \left(\tfrac{1}{3}x\right)^2 + 5 \qquad\qquad y = \left(\tfrac{1}{2}(x - 4)\right)^2 - 2$$

SCO: **RF3: Apply translations and stretches to the graphs and equations of functions.**
     [C, CN, R, V]

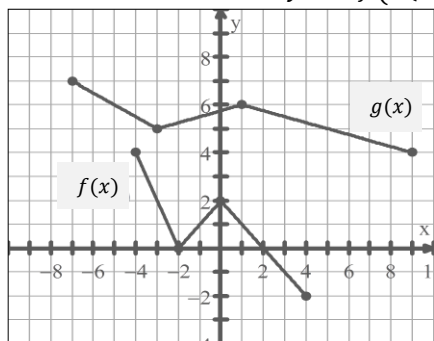## Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Use the graph of $y = f(x)$ below to draw the graph of $y = 3f\left(\frac{1}{2}(x+1)\right) - 4$

Answer: *shown in blue on graph* $(x, y) \longrightarrow (2x - 1, 3y - 4)$

\multicolumn{2}{c}{$f(x)$}		\multicolumn{2}{c}{$3f(\frac{1}{2}(x+1))$-4}
x	y	x	y
-3	3	-7	5
-1	-1	-3	-7
2	-1	3	-7
4	2	7	2

Q Determine what transformations were done to $f(x)$ to obtain the graph of (x) . Write your answer in the form of $y = af\left(b(x - h)\right) + k$.

Answer: $(x, y) \longrightarrow (2x + 1, \frac{1}{2}y + 5)$

\multicolumn{2}{c}{$f(x)$}		\multicolumn{2}{c}{$g(x) = \frac{1}{2}f\left(\frac{1}{2}(x-1)\right) + 5$}
x	y	x	y
-4	4	-7	7
-2	0	-3	5
0	2	1	6
4	-2	9	4

SCO: **RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.**
[C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, and x- and y-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF4:** Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.	**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

This outcome will focus on exploring reflections over the x-axis, y-axis, and over the line $y = x$. Reflections can be referred to as "reflection in" a line, or "reflection over" a line. Either convention may be used.

Students should analyze the relationship between the coordinates of an ordered pair and those of its given reflection and develop the following rules:
- A reflection over the x-axis will be represented as $y = -f(x)$, since the values for y will be additive inverses of each other.
- A reflection over the y-axis will be represented as $y = f(-x)$, since the values for x will be additive inverses of each other.
- A reflection over the line $y = x$ will be represented as $x = f(y)$, since the x and y values will be reversed. This is the inverse function, explored further in the next outcome. Technically the notation $x = f(y)$ requires drawing a new set of axes, with x values on the vertical axis and y values on the horizontal axis. Some of this confusion can be alleviated by introducing the notation f^{-1} as soon as possible.

The reflection of a function is not always a function. To remain a function, one x-value can correspond to only one y-value. This can be checked quickly by drawing a vertical line through the graph to determine if at any point, more than one point lies on the line (vertical line test). Students were introduced to this concept in Grade 10 when functions were introduced.

For example reflecting the graph of $y = x^2$ across the line $y = x$ will give the graph of $x = y^2$. A vertical line drawn anywhere through this graph crosses two points indicating that each x-value corresponds to two y-values. Therefore $x = y^2$ is not a function.

SCO: **RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.**
[C, CN, R, V]
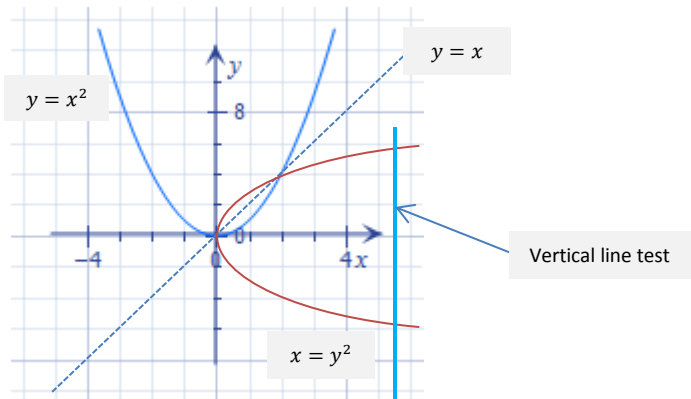
If the domain of $y = x^2$ is restricted to x values that are greater than or equal to zero, its inverse will be a function with each x-value corresponding to only one y-value.

The vertical line test, is extended in this outcome to the **horizontal line test** in which a horizontal line is drawn through the original function and if the line crosses more than one point, the function's inverse will fail a vertical line test and will not be a function.

Students should be exposed to multiple transformations applied to a single function. When sketching, order of applying these multiple transformations is important. The **reflection** should be applied first, then the **stretch,** and finally the **translation**. Similarly, when analyzing graphs the transformations should be considered in this same order.

ACHIEVEMENT INDICATORS

- Generalize the relationship between the coordinates of an ordered pair and the coordinates of the corresponding ordered pair that results from a reflection through the *x*-axis, the *y*-axis or the line $y = x$.

- Sketch the reflection of the graph of a function $y = f(x)$ through the x-axis, the y-axis, or the line $y = x$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$ is not given.

- Generalize, using inductive reasoning, and explain rules for the reflection of the graph of the function $y = f(x)$, through the x-axis, the y-axis or the line $y = x$.

- Sketch the graphs of the functions $y = -f(x)$, $y = f(-x)$ and $x = f(y)$, given the graph of the function $y = f(x)$, where the equation of $y = f(x)$, is not given.

- Write the equation of a function, given its graph which is a reflection of the graph of the function $y = f(x)$, through the *x*-axis, the *y*-axis, or the line $y = x$.

Suggested Instructional Strategies

- Provide graphs of various types of functions and have students run the horizontal line test on each to determine if the inverse will be a function or not. Have students find

SCO: **RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.**
[C, CN, R, V]

the inverse of the relation by interchanging the x- and y-coordinates of key points on the graph and run the vertical line test on the new graph to confirm their prediction.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

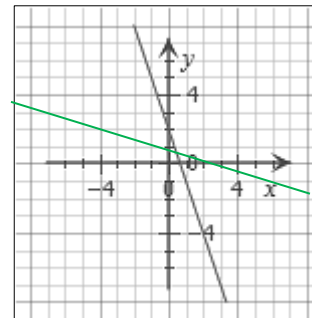**Q** Use the graph of $y = f(x)$ below to draw the graph of $y = -f(x)$.
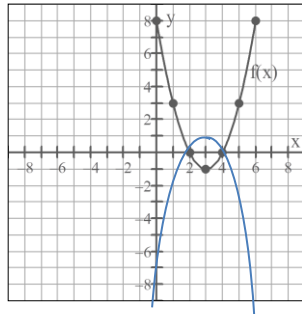
Answer: shown in blue on graph

$y = f(x)$		$y = -f(x)$	
x	y	x	y
0	8	0	−8
1	3	1	−3
2	0	2	0
3	−1	3	1
4	0	4	0
5	3	5	−3
6	8	6	−8

Q Sketch the graph of $y = 2 - 3x$, then sketch its reflection in the line $y = x$. Is the new graph a function? Find the equation of the new graph.

Answer: $y = f^{-1}(x)$ or
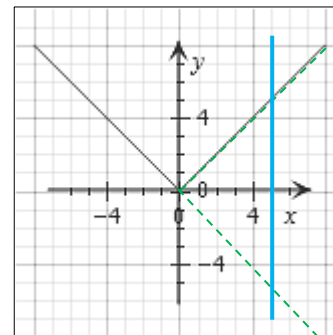*$x = 2 - 3y$ shown in green on graph*
The new graph is a function.

$y = f^{-1}(x)$	
x	y
2	0
-1	1

Q Sketch the graph of $y = |x|$, then sketch its reflection in the line $y = x$. Is the new graph a function? Find the equation for the new graph.
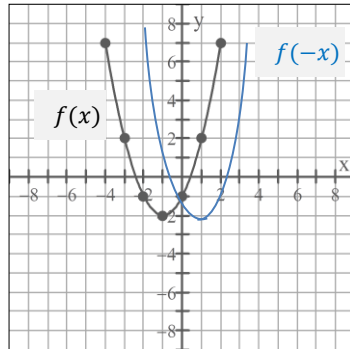
Answer:
$y = f^{-1}(x)$ or $x = |y|$
shown in green on graph
The new graph is not a function as it fails the vertical line test.

$y = f^{-1}(x)$	
x	y
0	0
4	4
4	-4

SCO: **RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.**
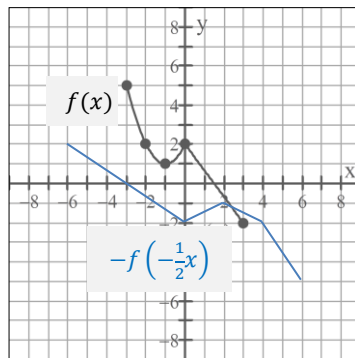[C, CN, R, V]

Q Use the graph of $y = f(x)$ below to draw the graph of $y = f(-x)$

Answer: shown in blue on graph

$y = f(x)$		$y = f(-x)$	
x	y	x	y
0	−1	0	−1
1	2	−1	2
2	7	−2	7
−1	−2	1	−2
−2	−1	2	−1
−3	2	3	2
−4	7	4	7

Q Use the graph of $y = f(x)$ below to draw the graph of $y = -f\left(-\frac{1}{2}x\right)$

Answer: shown in blue on graph
$(x, y) \longrightarrow (-2x, -y)$

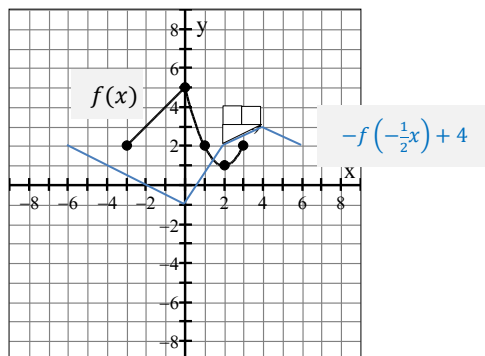| $y = f(x)$		$y = -f\left(-\frac{1}{2}x\right)$	
x	y	x	y
−3	5	6	−5
−2	2	4	−2
−1	1	2	−1
0	2	0	−2
3	−2	−6	2

Q Use the graph of $f(x)$ to draw the graph of $y = -f\left(\frac{1}{2}x\right) + 4$

Answer: shown in blue on graph
$(x, y) \longrightarrow (2x, -y + 4)$

$y = f(x)$		$y = -f\left(\frac{1}{2}x\right) + 4$	
x	y	x	y
−3	2	−6	2
0	5	0	−1
1	2	2	2
2	1	4	3
3	2	6	2

SCO: **RF4: Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$.**
[C, CN, R, V]

Act Determine the stretch factor between the pillars and arches shown.
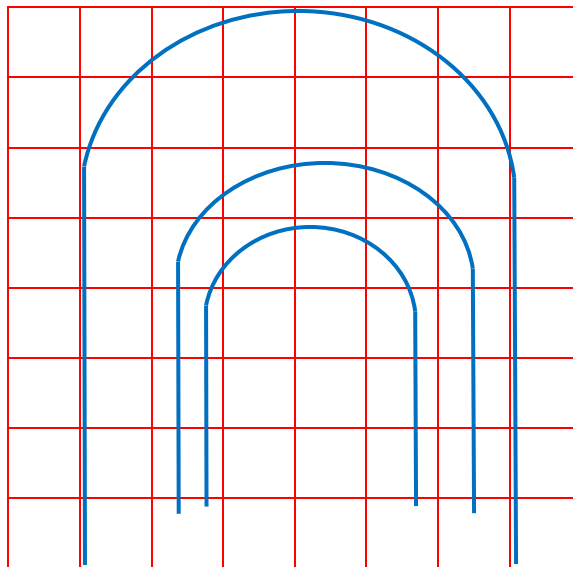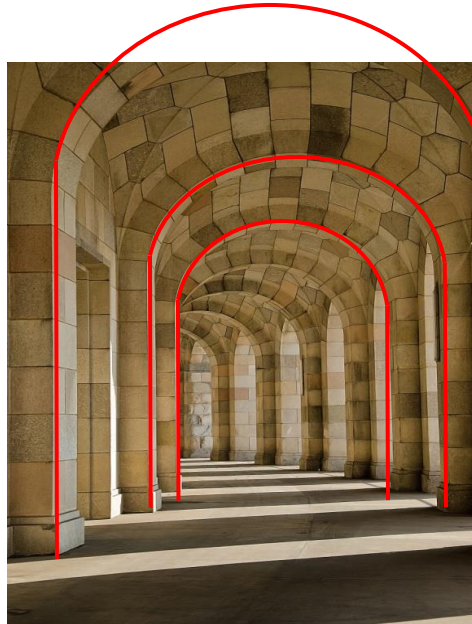
Answer: *Superimposing the arches on the grid, the largest (original) measures 8 units high, 6 units wide.*
Next largest is 5 units high, 4 units wide $(x, y) \longrightarrow \left(\frac{2}{3}x, \frac{5}{8}y \right)$
Smallest is 4 units high, 3 units wide $(x, y) \longrightarrow \left(\frac{1}{2}x, \frac{1}{2}y \right)$

SCO: **RF5:** Demonstrate an understanding of inverses of relations.   [C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF5: Demonstrate an understanding of inverses of relations.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF5:** Demonstrate an understanding of inverses of relations.	**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

When the graph of a function is reflected across the line $y = x$, the resulting graph is the inverse of the given function. The x and y values of the function and its inverse are interchanged.

The domain of the original graph becomes the range of the inverse and the range of the original graph becomes the domain of the inverse. For a function f with domain A and range B, the inverse function, if it exists, is denoted by f^{-1} and has domain B and range A. Also, f^{-1} maps y to x, if and only if, f maps x to y.

The inverse equation can be found by interchanging y and x: $y = x^2 \rightarrow x = y^2$. The inverse of $y = x^2$ is not a function because for each value of the independent variable (x), other than 0, there are two values for the dependent variable (y).

If a quadratic function is already graphed, its inverse relation can be graphed by plotting the ordered pairs with x and y values interchanged. This can be done by hand, or by using technology.

The inverse equation $y^2 = x$ can also be written as $y = \pm\sqrt{x}$ where $y = \sqrt{x}$ is the equation of the upper branch of its graph and $y = -\sqrt{x}$ is the equation of the lower branch. If the domain of the original graph, $y = x^2$, is restricted to either $x \geq 0 \ or \ x \leq 0$ then the inverse will be $y = \sqrt{x}$ or $y = -\sqrt{x}$ respectively and will be a function. It is important to note that this will be students' first introduction to radical functions. They will be studied in detail later in *Pre-Calculus A 120*.

SCO: **RF5:** Demonstrate an understanding of inverses of relations.   [C, CN, R, V]

ACHIEVEMENT INDICATORS

Note: Functions for which students should understand inverses (possibly with restricted domains) include linear, quadratic, radical and logarithmic/exponential functions with various bases.

- Explain how the graph of the line $y = x$ can be used to sketch the inverse of a relation.

- Explain how the transformation $(x, y) \rightarrow (y, x)$ can be used to sketch the inverse of a relation.

- Sketch the graph of the inverse relation, given the graph of a relation.

- Determine if a relation and its inverse are functions.

- Determine restrictions on the domain of a function in order for its inverse to be a function.

- Determine the equation and sketch the graph of the inverse relation, given the equation of a linear or quadratic relation.

- Explain the relationship between the domains and ranges of a relation and its inverse.

- Determine, algebraically or graphically, if two functions are inverses of each other.

- Determine the inverses of linear functions.

Suggested Instructional Strategies

- Have students explore the transformations of $f(x) = 2x + 1$, complete a table of values, then graph the function. Find the inverse of the function, complete a new table of values for the inverse, and then graph the inverse. Have the students reflect and respond on what has actually taken place.

- Repeat the above process with quadratic functions, $y = x^2$, which they have studied previously. Have them reflect this over the line $y = x$, and write the equation of the image. Some students will take $x = y^2$, solve for y and try to graph this using technology. Many will graph and expect to see a sideways parabola. Have them explain what is happening here.

SCO: **RF5:** Demonstrate an understanding of inverses of relations.   [C, CN, R, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment
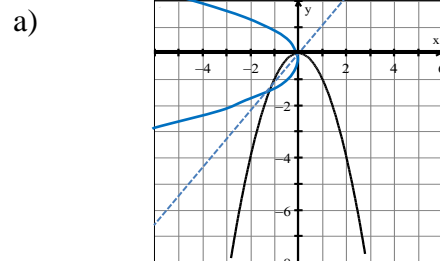
Q Find the equation of the inverse for each of the following.

a) $f(x) = 6x$ b) $f(x) = \frac{x-3}{2}$ c) $f(x) = \frac{1}{3}(x-4)$ d) $f(x) = (x+3)^2$

Answers: a) $f^{-1}(x) = \frac{1}{6}x$ *b)* $f^{-1}(x) = 2x+3$ *c)* $f^{-1}(x) = 3x+4$ *d)* $f^{-1}(x) = \pm\sqrt{x}-3$

Q For each of the following graphs:

   i. Draw its inverse.
  ii. Describe the domain and range of the function and its inverse.
 iii. Explain why the inverse is, or is not a function. If not, how would you restrict the
     domain of the original function in order to guarantee that the inverse relation is a
     function?
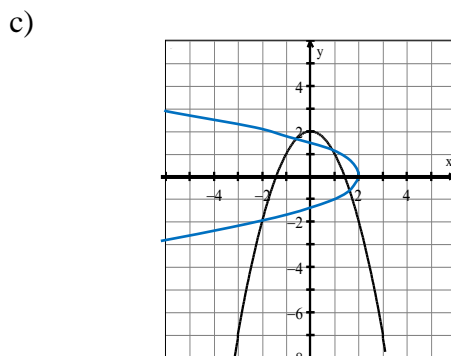 iv. State the equation for the inverse function.

a)

$f(x) = -x^2$

Answers: i) Inverse graph shown in blue
ii) Function: Domain $\{x|x \in R\}$ *Range* $\{y|y \leq 0, y \in R\}$
    Inverse: Domain $\{x|x \leq 0, x \in R\}$ *Range* $\{y|y \in R\}$
iii) Inverse is not a function as it fails the vertical line test
    and the original graph fails horizontal line test.
    The inverse would be a function if, in the original function
    $D = \{x \geq 0, x \in R\}$ *or* $D = \{x \leq 0, x \in R\}$
iv) $f^{-1}(x) = \pm\sqrt{-x}$

b)

$f(x) = \frac{1}{2}x^2$
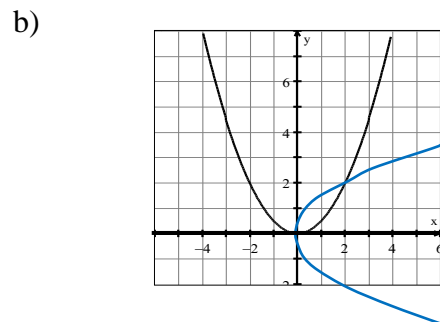
Answers: i) Inverse graph shown in blue
ii) Function: Domain $\{x|x \in R\}$ *Range* $\{y|y \geq 0, y \in R\}$
    Inverse: Domain $\{x|x \geq 0, x \in R\}$ *Range* $\{y|y \in R\}$
iii) Inverse is not a function as it fails the vertical line test
    and the original graph fails horizontal line test.
    The inverse would be a function if, in the original function
    $D = \{x \geq 0, x \in R\}$ *or* $D = \{x \leq 0, x \in R\}$
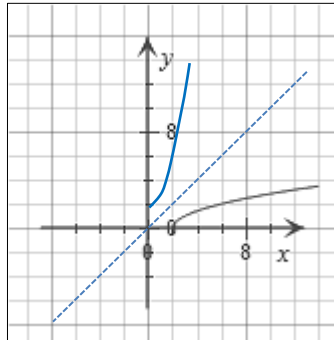*iv)* $f^{-1}(x) = \pm\sqrt{2x}$

c)

$f(x) = -x^2 + 2$

Answers: i) Inverse graph shown in blue
ii) Function: Domain $\{x|x \in R\}$ *Range* $\{y|y \leq 2, y \in R\}$
    Inverse: Domain $\{x|x \leq 2, x \in R\}$ *Range* $\{y|y \in R\}$
iii) Inverse is not a function as it fails the vertical line test
    and the original graph fails horizontal line test.
    The inverse would be a function if, in the original function
    $D = \{x \geq 0, x \in R\}$ *or* $D = \{x \leq 0, x \in R\}$
iv) $f^{-1}(x) = \pm\sqrt{-(x-2)}$

SCO: **RF5:** Demonstrate an understanding of inverses of relations.   [C, CN, R, V]

d)

$f(x) = \sqrt{x-2}$

Answers: *i*) *Inverse graph shown in blue*
ii) *Function*: *Domain* $\{x|x \geq 2, x \in R\}$ *Range* $\{y|\ y \geq 0, y \in R\}$
  Inverse: *Domain* $\{x|x \geq 0, x \in R\}$ *Range* $\{y|y \geq 2, y \in R\}$
iii) *Yes the inverse is a function* .
iv) $f^{-1}(x) = x^2 + 2$

e)
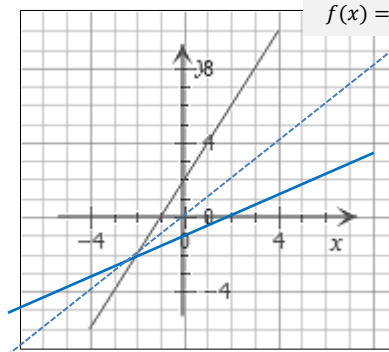
$f(x) = 2x + 2$

Answers: *i*) *Inverse graph shown in blue*
ii) *For both functions*: *Domain* $\{x|x \in R\}$ *Range* $\{y|\ y \in R\}$
iii) *Yes the inverse is a function* .
iv) $f^{-1} = \frac{1}{2}x + 2$

SCO: **RF6: Graph and analyze radical functions (limited to functions involving one radical).** [CN, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF6: Graph and analyze radical functions (limited to functions involving one radical).

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. **AN3:** Solve problems that involve radical equations (limited to square roots).	**RF6:** Graph and analyze radical functions (limited to functions involving one radical).	**RF7:** Demonstrate an understanding of operations on, and compositions of functions. **RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

Students have studied the properties of radicals to simplify radical expressions and solve radical equations. Students will be familiar with transformations from *Pre-Calculus 110* in relation to quadratic functions.

In outcome RF5 students learn how to take the inverse of a function with a focus on linear and quadratic functions. For this outcome students will be introduced to radical functions when the inverse of $y = x^2$ is split into two functions, $y = \pm\sqrt{x}$.

Students will graph radical functions. Their understanding of transformations of other functions, will be extended to graphs of radical functions. Using a table of values, they will apply transformations and sketch graphs.

Students will be introduced to the function $y = \sqrt{f(x)}$, limited to linear and quadratic functions, and will compare the domain and range to the domain and range of the function $y = f(x)$. For $y = \sqrt{f(x)}$ and $y = f(x)$ **invariant points** occur at $y = 0$ and $y = 1$.

Students will use the graphs of radical functions to determine approximate solutions to radical equations and to solve problems involving radicals.

Composite radical functions of the square root of trigonometric, polynomial and absolute value functions will be further developed in *Pre-Calculus B 120.*

SCO: **RF6: Graph and analyze radical functions (limited to functions involving one radical).**
[CN, R, T, V]

ACHIEVEMENT INDICATORS

- Sketch the graph of the function $= \sqrt{x}$, using a table of values, and state the domain and range.

- Sketch the graph of the equation $y = a\sqrt{b(x - h)} + k$ by applying tranfomations to the graph of the function $= \sqrt{x}$, and state the domain and range.

- Sketch the graph of the function $y = \sqrt{f(x)}$, given the graph of the function $y = f(x)$, and explain the strategies used.

- Compare the domain and range of the function $= \sqrt{f(x)}$, to the domain and range of the function $y = f(x)$, and explain why the domains and ranges may differ.

- Describe the relationship between the roots of a radical equation and the x-intercepts of the graph of the corresponding radical function.

- Determine, graphically, an approximate solution of a radical equation.

Suggested Instructional Strategies

- Have students apply transformations using a table of values to create graphs, and with mapping notation.

- To challenge some students this outcome can be extended to writing the equation of a radical function from a graph or a description of a graph.

- Have students use pipe cleaners or straws to model transformations of functions.

SCO: **RF6: Graph and analyze radical functions (limited to functions involving one radical).**
     [CN, R, T, V]

Questions (Q) and Activities (Act) for Instruction and Assessment

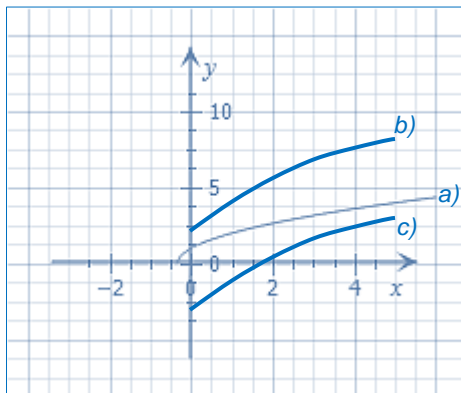**Q** Graph a) using a table of values. Then show how the graphs of b) and c) can be obtained from (a).

a) $y = \sqrt{3x + 1}$

b) $y = 2\sqrt{3x + 1}$

c) $y = 2\sqrt{3x + 1} - 5$

Answers:

a) $y = \sqrt{3x+1}$		b) $y = 2\sqrt{3x+1}$		c) $y = 2\sqrt{3x+1} - 5$	
x	y	x	$y \times 2$	x	$y \times 2 - 5$
0	1	0	2	0	-3
1	2	1	4	1	-1
2	$\sqrt{7} \cong 2.65$	2	$2\sqrt{7} \cong 5.29$	2	$2\sqrt{7} - 5 \cong 0.29$
3	$\sqrt{10} \cong 3.16$	3	$2\sqrt{10} \cong 6.32$	3	$2\sqrt{10} - 5 \cong 1.32$
4	$\sqrt{13} \cong 3.61$	4	$2\sqrt{13} \cong 7.21$	4	$2\sqrt{13} - 5 \cong 2.21$
5	4	5	8	5	3

Q For the above radical functions:

a) Use the graph to estimate the value of y when $x = 1.5, 3.5$.

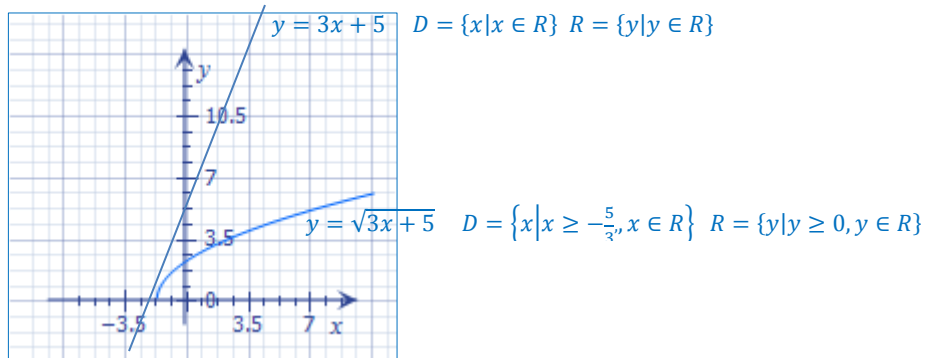b) Use the graph to estimate the value of x when $y = 5, 7, 8$.

SCO: **RF6: Graph and analyze radical functions (limited to functions involving one radical).**
[CN, R, T, V]

Q Complete the following.

a) Graph $f(x) = 3x + 5$

b) Write the domain and range of $f(x)$.

c) Using the graph of $f(x)$, graph $g(x) = \sqrt{3x + 5}$

d) Write the domain and range of $g(x)$.

e) Compare the domain and range of $f(x)$ and $g(x)$.

$$y = \sqrt{3x + 5}$$

Answers:

$y = 3x + 5$ $D = \{x | x \in R\}$ $R = \{y | y \in R\}$

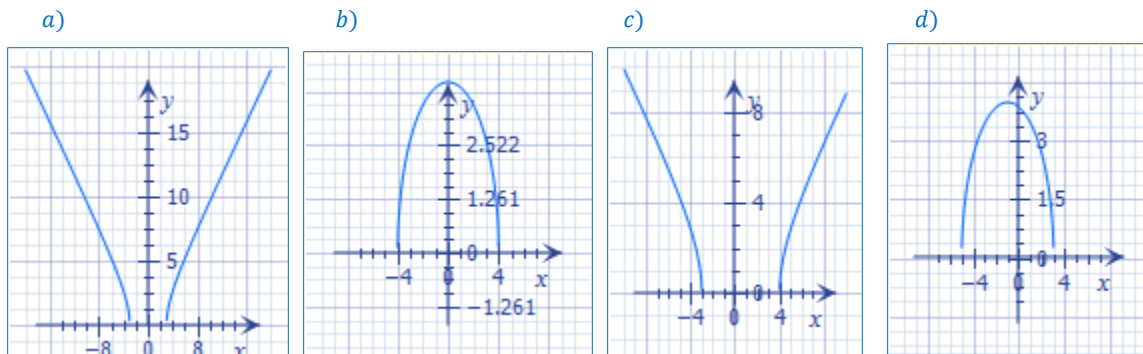$y = \sqrt{3x + 5}$ $D = \left\{x \middle| x \geq -\frac{5}{3}, x \in R\right\}$ $R = \{y | y \geq 0, y \in R\}$

Q Graph the following:

a) $y = \sqrt{x^2 - 9}$

b) $y = \sqrt{16 - x^2}$

c) $y = \sqrt{x^2 - x - 12}$

d) $y = \sqrt{15 - 2x - x^2}$

Answers:

a)

b)

c)

d)

SCO: **RF7:** Demonstrate an understanding of exponential functions.
　　　[C, CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF7: Demonstrate an understanding of exponential functions

Scope and Sequence of Outcomes:

| Grade 10 and Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
AN3: Demonstrate an understanding of powers with integral and rational components (NRF10). **RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts. (PC110) **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems. (PC110) **N4:** Demonstrate an understanding of simple and compound interest (GMF10).	**RF7:** Demonstrate an understanding of exponential functions.	**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

Exponential functions have the general form $y = c^x$ where $c > 0$. Students will have seen exponential functions before in science and math when describing situations such as population growth, compound interest, radioactive decay, and value depreciation. It is important that **exponent laws be reviewed** prior to working further with exponential functions.

Students will identify the characteristics of exponential graphs, which will include domain, range, horizontal asymptotes, x-intercept, and y-intercept. The importance and meaning of the horizontal asymptote for exponential graphs should be emphasized.

When $c > 1$, the exponential function models exponential growth e.g. $y = 2^x$ shown on graph A below. When $c < 1$, the exponential function models exponential decay e.g. $y = \left(\frac{1}{2}\right)^x$ shown on graph B, below.
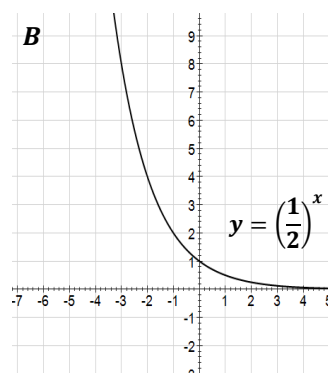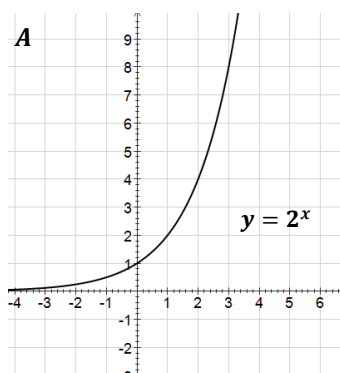
SCO: **RF7:** Demonstrate an understanding of exponential functions.
[C, CN, R, V]

For exponential functions of the form $y = c^x$, $c > 0$,
Domain	$\{x|x \in R\}$
Range	$\{y|y > 0\}$
y-intercept	$(0,1)$
Asymptote	x-axis

Students will explore ways that transformations affect graphs of exponential functions. They will develop the ability to sketch these graphs both with and without technology and learn to manipulate the exponential equation, $y = a(c)^{b(x-h)} + k$, to produce any given graph.

The following chart summarizes the effects of the parameters a, b, k and h, on the graphs of exponential functions. These parameters should be discussed in relation to previous work on function transformations.

Parameter	Transformation
a	• Vertical stretch about the x-axis by a factor of $|a|$ • For $a < 0$, reflection in the x-axis
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$ • For $b < 0$, reflection in the y-axis
k	• Vertical translation up or down by k units
h	• Horizontal translation left or right by h units

ACHIEVEMENT INDICATORS

- Sketch, with or without technology, a graph of an exponential function of the form $y = c^x$, $c > 0$.

- Identify the characteristics of the graph of an exponential function of the form $y = c^x, c > 0$, including the domain, range, horizontal asymptote and intercepts, and explain the significance of the horizontal asymptote.

- Sketch the graph of an exponential function by applying a set of transformations to the graph of $y = c^x$, $c > 0$, and state the characteristics of the graph.

SCO: **RF7:** Demonstrate an understanding of exponential functions.
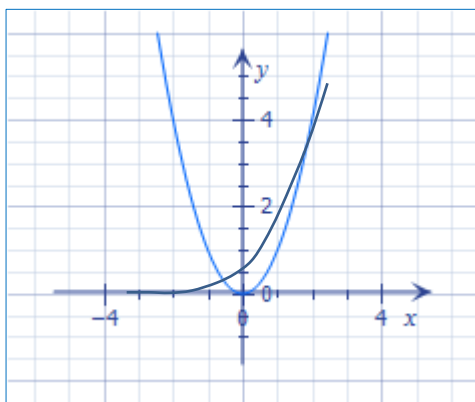[C, CN, R, V]

Suggested Instructional Strategies

- For exponential functions, have students explore real life examples, such as the exponential decrease of concentration of medication in the blood stream *(e.g. Investigation 5, p.127 Mathematical Modeling Book 3),* or the exponential increase in value of a collectable *(Investigation 7, p.134 Mathematical Modeling Book 3).*

- Have students use graphing technology to graph and make a table of values, to compare a variety of exponential equations such as: $y = 2^x$, $y = 2^x - 3$, $y = 2^x + 4$, $y = 2^{\frac{x}{3}}$, $y = 2^{2x}$

- In the current student text resource (MHR *Pre-Calculus 12*) there is an excellent investigation on p.346, "Investigate Transforming an Exponential Function".

- Starting with the base exponential equation $f(x) = ac^{b(x-h)} + k$, have students change one factor at a time, and see how each affects the graph. Students should do at least two manipulations of each (a, b, k and h), and record their results. Students should try negative values for a, b, k and h, to see the effect of a negative number for each of these variables. They should also explore functions with base e.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q On the same set of axes, sketch the graphs of $y = x^2$ and $y = 2^x$. Write a sentence describing any differences you notice between the two graphs.

Answers: $y = x^2$ *is a parabola* $y = 2^x$ *increases exponentionally by a factor of 2*

Q For points (x, y) on the graph of the function $y = 4^x$, find the missing values:
$(-2,\ y)$, $(-1,\ y)$, $(0,\ y)$, $(1,\ y)$, $(2,\ y)$, $(3,\ y)$, $(x,\ 1/8)$, $(x,\ 1/4)$, $(x,\ 1)$, $(x,\ 4)$, $(x,\ 1024)$.

Answers: $\left(-2,\frac{1}{16}\right), \left(-1,\frac{1}{4}\right), (0,1), (1,4), (2,16), (3,64), \left(-\frac{3}{2},\frac{1}{8}\right), \left(-1,\frac{1}{4}\right), (0,1), (1,4), (5,1024)$

SCO: **RF7:** Demonstrate an understanding of exponential functions.
[C, CN, R, V]

Q For each of the functions below, that are in the form $f(x) = a(c)^{b(x-h)} + k$, identify the transformations (parameters) $a, b, h,$ and k.

a) $f(x) = 3(2)^x - 4$ b) $h(x) = 5^{x+1} + 3$ c) $y = \left(\frac{1}{3}\right)^{2(x-1)}$

d) $f(x) = -\frac{1}{2}3^{(x+1)} + 7$ e) $y = 2.5(0.75)^{\frac{x-3}{5}} - 4$

Answers: a) $a = 3, b = 1, h = 0, k = -4$ b) $a = 1, b = 1, h = -1, k = 3$ c) $a = 1, b = 2, h = 1, k = 0$
d) $a = -\frac{1}{2}, b = 1, h = -1, k = 7$ e) $a = 2.5, b = \frac{1}{5}, h = 3, k = -4$

Q Identify the following characteristics of the graph of each function:
  i) the domain and range.
  ii) The y-intercept (to one decimal place)
  iii) The equation of the asymptote
  iv) The x-intercept (to one decimal place)

a) $y = 4(3)^{x-2} + 2$

b) $y = 2\left(\frac{1}{2}\right)^{x+3} - 3$

Answers: a) i) $\{x | x \in R\}$ $\{y | y > 2, y \in R\}$ ii) $2^{\frac{4}{9}} = 2.\overline{4}$ $(0,2.\overline{4})$ iii) $y = 2$ iv) None
b) i) $\{x | x \in R\}$ $\{y | y > -3, y \in R\}$ ii) $-2\frac{3}{4} = -2.\overline{8}$ $(0,2.\overline{8})$ iii) $y = -3$ iv) $(-3.6,0)$

Q Match each graph with the corresponding function:
  a) $y = 3^{2(x-1)} - 2$ b) $y = 2^{x-2} + 1$ c) $y = -4^{x+2}$ d) $y = -2(3)^{-x} - 5$

i)
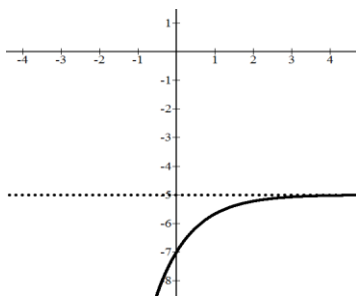
ii)

iii)

iv)

Answers: a) and ii)   b) and iv)   c) and iii)   d) and i)

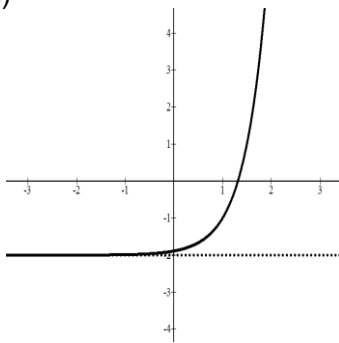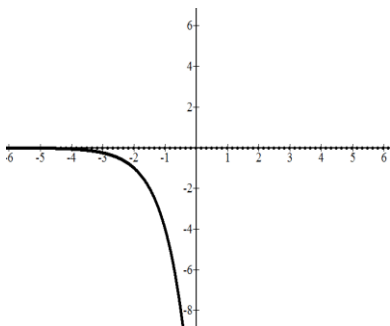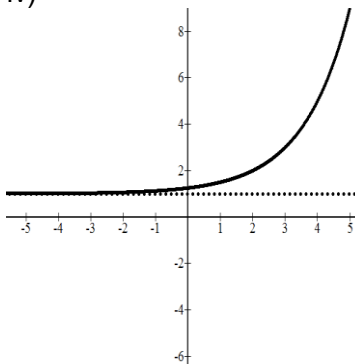SCO: **RF8: Demonstrate an understanding of logarithms.**
   [CN, ME, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

**RF8: Demonstrate an understanding of logarithms.**

Scope and Sequence of Outcomes:

| Grade 10	Pre-Calc A 120	Pre-Calc B 120
AN3: Demonstrate an understanding of powers with integral and rational components (NRF10) **N4:** Demonstrate an understanding of simple and compound interest (GMF10)	**RF8:** Demonstrate an understanding of logarithms.	**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

The inverse of an exponential function $y = c^x$ is $x = c^y$. This inverse is also a function and is called a **logarithmic function**, for which y is said to be the logarithm to base c of x, written as $y = \log_c x \ (c > 0, c \neq 1)$. These functions can be expressed in logarithmic or exponential form.

Logarithmic form

Exponential form

exponent

$$y = \log_c x \qquad\qquad x = c^y$$
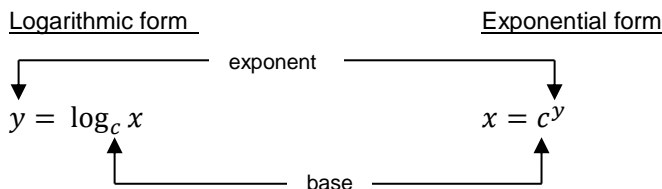
base

Students should see that a logarithm is an exponent on a given base. For example the $\log_3 9 = 2$ because 2 is the exponent for base 3 that would be equal to 9, $3^2 = 9$.

Students will compare the characteristics of logarithmic and exponential graphs, which will include domain, range, horizontal or vertical asymptotes, x-intercept, and y-intercept.

The graphs of the exponential function $y = 2^x$, and its inverse logarithmic function $y = \log_2 x$ are reflections of each other across the $y = x$ line, as shown below.
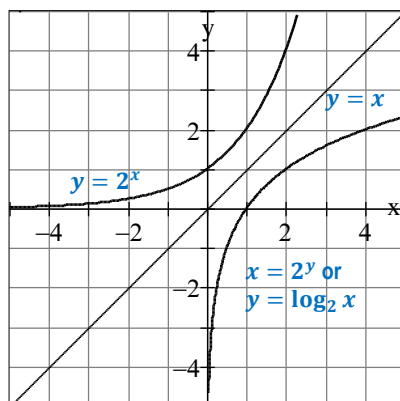
Point on the exponential curve $y = 2^x$	Corresponding point on the inverse function $y = \log_2 x$
$\left(-3, \frac{1}{8}\right)$	$\left(\frac{1}{8}, -3\right)$
$\left(-2, \frac{1}{4}\right)$	$\left(\frac{1}{4}, -2\right)$
$\left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -1\right)$
$(0, 1)$	$(1, 0)$
$(1, 2)$	$(2, 1)$
$(2, 4)$	$(4, 2)$
$(3, 8)$	$(8, 3)$

SCO: **RF8: Demonstrate an understanding of logarithms.**
　　[CN, ME, R]

The importance and meaning of the horizontal asymptote for exponential graphs, and the vertical asymptote for logarithmic graphs should be emphasized. The concept of limits can be introduced, but will be explored more formally in *Pre-Calculus B 120*.

The following chart summarizes characteristics of exponential and logarithmic functions.

	Exponential	Logarithmic
Function	$y = c^x, c > 0, c \neq 1$	$y = \log_c x, c > 0, c \neq 1$
Domain	$x \in R$	$x > 0$
Range	$y > 0$	$y \in R$
Intercept	$(0, 1)$	$(1, 0)$
Increasing	when $c > 1$	when $c > 1$
Decreasing	when $0 < c < 1$	when $0 < c < 1$
Asymptote	x-axis	y-axis

Common logarithms have a base of 10 and can be written with or without the base as $\log_{10} x$ or simply as $\log x$. Students should become very familiar with this convention.

Natural logarithms have a base of e and are written as $\log_e x = \ln x$. As this base is very important in future studies in Calculus and Science, students should gain experience with questions with base e. Base e is an irrational number equal to 2.71828...........

ACHIEVEMENT INDICATORS

- Demonstrate, graphically, that a logarithmic function and an exponential function with the same base are inverses of each other.

- Sketch, with or without technology, the graph of a logarithmic function of the form $y = \log_c x$, $c > 0, c \neq 1$.

- Identify the characteristics of the graph of a logarithmic function of the form $y = \log_c x$, $c > 0, c \neq 1$. including the domain, range, vertical asymptote and intercepts, and explain the significance of the vertical asymptote.

- Change an expression from logarithmic form to exponential form, and vice versa.

- Determine, without technology, the exact value of a logarithm, such as $\log_2 8$.

- Estimate the value of a logarithm, using benchmarks, and explain the reasoning; e.g. since $\log_2 8 = 3$ and $\log_2 16 = 4$, then $\log_2 9$ is between 3 and 4, but closer to 3 or approximately 3.1 or 3.2.

- Identify the characteristics of the graphs of $y = e^x$ and $y = \ln x$

- Determine the approximate values of natural logarithms or powers base e using a calculator.

SCO: **RF8: Demonstrate an understanding of logarithms.**
[CN, ME, R]

Suggested Instructional Strategies

- Time should be spent to formally define log, with calculations and graphs of inverse relationships such as $y = 2^x$ and $y = \log_2 x$, $y = 3^x$ and $y = \log_3 x$, $y = 10^x$ and $y = \log_{10} x$.

- The introduction to logarithms should be slow.

  1) Use examples to motivate discussion of big and small numbers. (Good source of ideas: http://www.vendian.org/envelope/).

  2) Ask questions which lead students to understand that the number of digits before the decimal, and the number of zeroes following the decimal indicate the size of numbers, and that this information is conveyed with exponential notation. Review scientific notation.

  3) Introduce the terminology and notation for logarithms. The relationship between logarithmic and exponential form should be explained, and students should practice converting from one form to the other.

  4) Have students explore logarithms with base 10: $\log_{10} 10 = 1$ because $10^1 = 10$; $\log_{10} 0.01 = -2$ because $10^{-2} = 0.01$; $\log_{10} 1 = 0$ because $10^0 = 1$ …..and so on.

  5) Have students explore logarithms with a smaller base, such as 2. Start with a review of powers of 2, leading to discussion of $\log_2$. Have students determine a series of logarithms: $\log_2 2 = 1$ because $2^1 = 2$; $\log_2 16 = 4$ because $2^4 = 16$; $\log_2 1 = 0$ because $2^0 = 1$; $\log_2 \frac{1}{8} = -3$ because $2^4 = 16$ …. and so on.

  6) Have students explore logarithims with base e. $\ln e = 1$ $because$ $\log_e e = 1$, $\ln e^5 = 5$ $because$ $\log_e e^5 = 5$, $\ln 1 = 0$ $because$ $e^0 = 1$, $\ln \sqrt{e} = \frac{1}{2}$ $because$ $\log_e e^{\frac{1}{2}} = \frac{1}{2}$ ….. and so on.

- Students should evaluate logarithmic expressions to develop benchmarks with whole number solutions. Once established, benchmarks should be used to find approximate values of other logarithmic expressions.

SCO: **RF8: Demonstrate an understanding of logarithms.**
[CN, ME, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q On the same set of axes, sketch the graphs of $y = 2^x$ and $y = \log_2 x$. Then sketch in the line $y = x$. Explain how you know that the two curves mirror images across the line $y = x$?

Answers: The two curves are inverses of one another, the x and y values are switched.
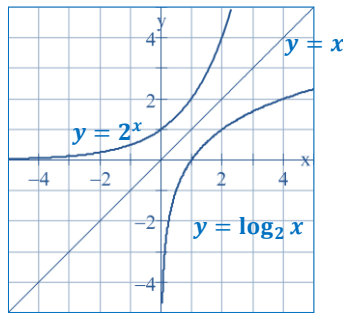
Q Simplify each of the following expressions:

a) $\log_{10} 10^3$ 　　b) $\log_{10} 10^{-1}$ 　　c) $\log_2 2^4$ 　　d) $\log_5 5^{-2}$

e) $\log_{10} 10^{0.5}$ 　　f) $\log_{10} 10^{-1.5}$ 　　g) $\log_2 2^{3.6}$ 　　h $\log_5 5^{-2.1}$

Answers:　a) 3　　b) −1　　c) 4　　d) −2,　　e) 0.5　　f) −1.5　　g) 3.6　　h) −2.1

Q Express each of the following in an alternate form, either exponential or logarithmic.

a) $\log_2 16 = 4$ 　　b) $\log_{27} 3 = \frac{1}{3}$ 　　c) $49^{\frac{1}{2}} = 7$ 　　d) $4^3 = 64$

Answers:　a) $2^4 = 16$　　b) $27^{\frac{1}{3}} = 3$　　c) $\log_{49} 7 = \frac{1}{2}$　　d) $\log_4 64 = 3$

Q Evaluate each of the following.

a) $\log_2 8$ 　　b) $\log_3 81$ 　　c) $\log_5 \sqrt{5}$

d) $\log_2 \left(\frac{1}{32}\right)$ 　　e) $\log_2 4^{25}$ 　　f) $\log_x x^{\frac{2}{3}}$

Answers:　a) 3　　b) 4　　c) $\frac{1}{2}$　　d) −5　　e) 50　　f) $\frac{2}{3}$

Q Write an expression that is equivalent to $y = 2 \log \left(\frac{4}{x}\right)$.

Answer:　$y = \log_{10} \left(\frac{4}{x}\right)^2$　or　$10^y = \left(\frac{4}{x}\right)^2$

Q Use benchmark logarithmic expressions to estimate $\log_2 11$.

Answer: $\log_2 8 = 3$,　$\log_2 16 = 4$,　11 is closer to 8　∴ $\log_2 11 \approx 3.4$

Q Evaluate the following:

a) $\ln e^5$ 　　b) $\ln e^{-4}$ 　　c) $\ln e^{10}$ 　　d) $\ln e^{-1}$

e) $\ln e$ 　　f) $4 \ln e$ 　　g) $2 \ln 1$ 　　h) $3 \ln e^2$

Answers: a) 5　　b) −4　　c) 10　　d) −1　　e) 1　　f) 4　　g) 0　　h) 6

SCO: **RF9: Graph and analyze logarithmic functions.** [C, CN, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF9: Graph and analyze logarithmic functions

Scope and Sequence of Outcomes:

Grade 10	Pre-Calc 110	Pre-Calc A 120	Pre-Calc B 120
AN3: Demonstrate an understanding of powers with integral and rational components (NRF10) **N4:** Demonstrate an understanding of simple and compound interest (GMF10)	**RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts. **RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF9:** Graph and analyze logarithmic functions	**RF2:** Analyze geometric sequences and series to solve problems. **RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

Students will explore ways that transformations affect graphs of logarithmic functions. They will develop the ability to sketch these graphs both with and without technology and learn to manipulate the logarithmic equation, $y = a\log_c\big(b(x - h)\big) + k$ to produce any given graph.

The following chart summarizes the effects of the parameters a, b, k and h, on the graphs of logarithmic functions. Students should realize that these effects are the same as they found for exponential functions.

Parameter	Transformation
a	• Vertical stretch about the x-axis by a factor of $|a|$ • For $a < 0$, reflection in the x-axis
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$ • For $b < 0$, reflection in the y-axis
k	• Vertical translation up or down by k units
h	• Horizontal translation left or right by h units

SCO: **RF9: Graph and analyze logarithmic functions.**   [C, CN, T, V]

ACHIEVEMENT INDICATORS

- Sketch the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, $c > 0, c \neq 1$ and state the characteristics of the graph.

- Sketch and state the characteristics of the graphs of exponential functions base e, and the natural logarithmic function, by applying a set of transformations to the graphs of $y = e^x$ and $y = \ln x$.

Suggested Instructional Strategies

- Starting with a logarithmic equation, $y = a \log_c \big(b(x - h)\big) + k$, have students change one factor at a time, and see how each affects the graph. Students should do at least two manipulations of each (a, b, k and h), and record their results. Students should try negative values for a, b, k and h, to see the effect of a negative number for each of these variables.

- Give students a logarithmic graph, and the stretch of that graph, and have them determine the equation for the stretch graph *(e.g. #6, p.330, Pre-Calculus 12).*

- Have students analyze the graphs of $y = e^x$ and $y = \ln x$ by creating graphs and a chart similar to those for the functions $y = 2^x$ and $\log_2 x$ shown in **RF8.**

SCO: **RF9: Graph and analyze logarithmic functions.**   [C, CN, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q For points (x, y) on the graph of the function $y = \log_4 x$. find the missing values:
$(x, -2)$, $(x, -1)$, $(x, 0)$, $(x, 1)$, $(x, 2)$, $(x, 3)$, $(\frac{1}{8}, y)$, $(\frac{1}{4}, y)$, $(1, y)$, $(4, y)$, $(64, y)$, $(1024, y)$.

Answers: $\left(\frac{1}{16}, -2\right), \left(\frac{1}{4}, -1\right), (1,0), (4,1), (16,2), (64,3), \left(\frac{1}{8}, -\frac{3}{2}\right), \left(\frac{1}{4}, -1\right), (1,0), (4,1), (64,3), (1024,5)$

Q For points (x, y) on the graph of the function $y = \log_{10} x$. find the missing values:
$(x, -2)$, $(x, -1)$, $(x, 0)$, $(x, 1)$, $(x, 2)$, $(x, 3)$,$(0.001, y)$, $(0.1, y)$, $(1, y)$, $(10, y)$, $(100, y)$, $(1000, y)$.

Answers: $\left(\frac{1}{100}, -2\right), \left(\frac{1}{10}, -1\right), (1,0), (10,1), (100,2), (1000,3), (0.001, -3), (0.1, -1), (1,0), (10,1), (100,2), (1000,3)$

Q Identify the following characteristics of the graph of each function:
   i) the domain and range.
   ii) The y-intercept (to one decimal place)
   iii) The equation of the asymptote
   iv) The x-intercept (to one decimal place)

a) $y = -2\log(x + 1)$

b) $y = \log_3\big(2(x + 3)\big)$

c) $y = -3\log(x - 2) - 4$

d) $y = \log(x + 3) + 1$

e) $y = 2(3)^{x-4} + 1$

f) $y = 3\left(\frac{1}{2}\right)^{x+1} - 5$

g) $y = \ln(x - 4) - 2$

h) $y = -\ln(x + 2) + 3$

Answers:
a) i) $\{x|x > -1, x \in R\}$ $\{y|y \in R\}$ ii) $(0,0)$ iii) $x = -1$ iv) $(0,0)$
b) i) $\{x|x > -3, x \in R\}$ $\{y|y \in R\}$ ii) $(0,1.6)$ iii) $x = -3$ iv) $\left(-\frac{5}{2}, 0\right)$
c) i) $\{x|x > 2, x \in R\}$ $\{y|y \in R\}$ ii) undefined, no y intercept   iii) $x = 2$ iv) $(2,0)$
d) i) $\{x|x > -3, x \in R\}$ $\{y|y \in R\}$ ii) $(0,1.5)$ iii) $x = -3$ iv) $(-2.9,0)$
e) i) $\{x|x \in R\}$ $\{y|y > 1, y \in R\}$ ii) $(0,1)$ iii) $y = 1$ iv)undefined, no x intercept
f) i) $\{x|x \in R\}$ $\{y|y > -5, y \in R\}$ ii) $(0, -3.5)$ iii) $y = -5$ iv) $(-1.7,0)$
g) i) $\{x|x > 4, x \in R\}$ $\{y|y \in R\}$ ii) undefined, no y intercept   iii) $x = 4$ iv) $(11.4,0)$
h) i) $\{x|x > -2, x \in R\}$ $\{y|y \in R\}$ ii) $(0,2.3)$ iii) $x = -2$ iv) $(18.1,0)$

SCO: **RF10: Demonstrate an understanding of the product, quotient and power laws of logarithms.**
    [C, CN, R, T]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF10: Demonstrate an understanding of the product, quotient and power laws of logarithms.

Scope and Sequence of Outcomes:

Pre-Calc A 120	Pre-Calc B 120
RF10: Demonstrate an understanding of the product, quotient and power laws of logarithms.	**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

The laws of logarithms allow us to work easily with logarithmic expressions and equations. Since logarithms are exponents, the laws of logarithms are related to the laws of exponents. Students should be able to explain this relationship with examples.

$\log_c(xy) = \log_c x + \log_c y$ *Product Law of Logarithms*

$\log_c\left(\frac{x}{y}\right) = \log_c(x) - \log_c(y)$ *Quotient Law of Logarithms*

$\log_c(x)^r = r \log_c(x)$ for any real number r.   *Power Law of Logarithms*

From what they know of solving logarithms, students should be given time to explore and develop the laws of logarithms. However, the main focus of this outcome is for students to become comfortable applying these laws to solving logarithmic expressions both with and without technology. These laws also apply to the natural logarithm, $\ln x$.

ACHIEVEMENT INDICATORS

- Develop and generalize the laws for logarithms, using numeric examples and exponent laws.

- Derive each law of logarithms.

- Determine, using the laws of logarithms, an equivalent expression for a logarithmic expression.

- Determine, with technology, the approximate value of a logarithmic expression such as $\log_2 9$.

SCO: **RF10:** **Demonstrate an understanding of the product, quotient and power laws of logarithms.**
[C, CN, R, T]

Suggested Instructional Strategies

- Have students practice different strategies to solve a given problem. For example, to solve: $3\log_8 x + \log_8 5 = \log_8 625$

Strategy 1
$3\log_8 x + \log_8 5 = \log_8 625$
$\log_8 x^3 + \log_8 5 = \log_8 625$
$\log_8 5x^3 = \log_8 625$
$5x^3 = 625$
$x^3 = 125$
$x = 5$

Strategy 2
$3\log_8 x + \log_8 5 = \log_8 625$
$\log_8 x^3 + \log_8 5 = \log_8 625$
$\log_8 5x^3 = \log_8 625$
$\log_8 5x^3 - \log_8 625 = 0$
$\log_8 \left(\frac{5x^3}{625}\right) = 0$
$\left(\frac{5x^3}{625}\right) = 8^0 = 1$
$5x^3 = 625$
$x^3 = 125$
$x = 5$

- Have students discuss the differences in the solution methods for:
$\log_2 x + \log_2 3 = \log_2 9$ and $\log_2 x + \log_2 3 = 9$.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Write each expression as a single logarithm statement:

a) $\log_3 x + 2\log_3 7$

b) $\log_5 x - \log_5 y$ c) $\log_2 5x + \log_2 7x$

d) $\log_3 54 - 4\log_3 2$

e) $\ln 10 + \ln 6 - \ln 15$

Answers: *a)* $\log_3 49x$ *b)* $\log_5 \left(\frac{x}{y}\right)$ *c)* $\log_2 35x^2$ *d)* $\log_3 \left(\frac{27}{8}\right)$ *e)* $\ln 4$

Q Expand each of the following:

a) $\log(abc)$ b) $3\log(6x)$ c) $2\ln\left(\frac{a}{b}\right)$

Answers: *a)* $\log a + \log b + \log c$ *b)* $3\log x + 3\log 6$ *c)* $2\ln a - 2\ln b$

Q Simplify each of the following:

a) $\log 3 + \log 7$ b) $\log x + \log(x + 2)$ c) $a\log(xz) + a\log(xy)$

d) $\ln 20 - \ln 10$ e) $2\ln e - 3\ln 1 + \ln e^6$

Answers: *a)* $\log 21$ *b)* $\log(x^2 + 2x)$ *c)* $a\log x^2 y^2$ *d)* $\ln 2$ *e)* $8\ln e$

SCO: **RF11: Solve problems that involve exponential and logarithmic equations.**
[C, CN, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF11: Solve problems that involve exponential and logarithmic equations.

Scope and Sequence of Outcomes:

Grade 10	Pre-Calc A 120	Pre-Calc B 120
AN3: Demonstrate an understanding of powers with integral and rational components (NRF10).	**RF11:** Solve problems that involve exponential and logarithmic equations.	**RF2:** Analyze geometric sequences and series to solve problems.
N4: Demonstrate an understanding of simple and compound interest (GMF10).		**RF8:** Assemble a function toolkit comparing various types of functions and compositions of them.

ELABORATION

To solve exponential and logarithmic equations a variety of methods are available including changing from one form to another.

For exponential equations it can be useful to express elements with like bases. For example:

$4^{x+1} = 8^{3x}$
$(2^2)^{x+1} = (2^3)^{3x}$
$2^{2x+2} = 2^{9x}$ Since both bases are the same, the exponents are equal to one another, and the bases can be excluded.
$2x + 2 = 9x$
$x = \frac{2}{7}$

Logarithms are helpful for solving exponential equations in which the bases cannot be made the same. For example:

$8(3^{2x}) = 568$
$3^{2x} = 71$
$\log 3^{2x} = \log 71$
$2x(\log 3) = \log 71$
$x = \frac{\log 71}{2\log 3}$
$x \approx 1.94$

When logarithmic expressions have the same base, the **arguments**, $(2x + 5)$ and 11, as in the following example, are equal to one another:

$\log_6(2x + 5) = \log_6 11$
$2x + 5 = 11$
$x = 3$

When logarithmic expressions in the form $x = a^{\log_a m}$ have the same base a, the base will be eliminated from the final answer. For example, to solve $x = 5^{\log_5 3}$, $x = 10^{\log 2^3}$, $x = 2^{\log_2 7^{-3}}$ and $x = e^{\ln 6}$ begin by expressing in logarithmic form:

$\log_5 x = \log_5 3$	$\log_{10} x = \log_{10} 2^{-3}$	$\log_2 x = \log_2 7^{-3}$	$\log_e x = \ln 6$
$x = 3$	$x = 2^{-3}$	$x = 7^{-3}$	$\ln x = \ln 6$
			$x = 6$

SCO: **RF11: Solve problems that involve exponential and logarithmic equations.**
    [C, CN, PS, R]

Both logarithmic and exponential equations can also be solved graphically, either by graphing as a single function and finding the x-intercept, or by graphing the functions that correspond to each side of the equal sign, and then identifying x at the point of intersection of the two graphs.

ACHIEVEMENT INDICATORS

- Determine the solution of an exponential equation in which the bases are powers of one another.

- Determine the solution of an exponential equation in which the bases are not powers of one another, using a variety of strategies.

- Determine the solution of a logarithmic equation, and verify the solution.

- Explain why a value obtained in solving a logarithmic equation may be extraneous.

- Solve a problem that involves exponential growth or decay.

- Solve a problem that involves the application of exponential equations to loans, mortgages and investments.

- Solve a problem that involves logarithmic scales, such as the Richter scale and the pH scale.

- Solve a problem by modeling a situation with an exponential or a logarithmic equation.

- Solve problems involving e^x and $\ln x$.

Suggested Instructional Strategies

- Provide opportunities for students to practice solving real world problems that involve exponential functions. Examples include problems involving depreciation, pH levels, the Richter scale, half-life of radioactive elements, compound interest.

- More questions involving base e and $\ln x$ can be found in $Math\ Modeling$ Book 4.

- Students may find the following mnemonic helpful:
  $B^e = n$ $\log_B n = e$ as   "Ben the log bunny"

SCO: **RF11: Solve problems that involve exponential and logarithmic equations.**
     [C, CN, PS, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Simplify each of the following expressions:

a) $10^{\log_{10} 0.01}$ b) $4^{\log_4 1/16}$ c) $2^{\log_2 8}$ d) $2^{\log_2 2^{-3}}$

e) $3^{\log_3 80}$ f) $10^{\log_{10} 0.071}$ g) $4^{\log_4 0.111}$ h) $2^{\log_2 10}$

i) $25^{\log_5 9}$ j) $3^{\log_9 16}$ k) $2^{\log_2 5 + \log_2 6}$ l) $5^{2\log_5 3 - \log_5 12}$

Answers:
a) 0.01 *b)* $\frac{1}{16}$ *c)* 8 *d)* 2^{-3} *e)* 80 *f)* 0.071 *g)* 0.111 *h)* 10 *i)* 81 *j)* 4 *k)* 30 *l)* $\frac{3}{4}$

Q . Evaluate.

a) $e^{\ln 5}$ b) $e^{\ln 10}$ c) $e^{2\ln 3}$ d) $e^{3\ln 2}$

e) $e^{-2\ln 4}$ f) $e^{\ln 10 - \ln 2}$ g) $e^{5\ln 2}$ h) $e^{-\ln 3}$

i) $e^{\ln 8 + \ln 7}$ j) $e^{4\ln 5 - 5\ln 4}$ k) $e^{2\ln 3 + \ln 5}$

Answers: *a)* 5 *b)* 10 *c)* 9 *d)* 8 *e)* $\frac{1}{16}$ *f)* 5 *g)* 32 *h)* $\frac{1}{3}$ *i)* 56 *j)* $\frac{625}{1024}$ *k)* 45

Q A start has been made to solving the equation shown below. Complete the solution. Solve using a different approach, and explain which approach you prefer and why.

$16(2^x) = 1024$
$2^x = 64$
$\log(2^x) = \log 64$
....
Answer: *To complete with this method:* $x\log 2 = \log 64$, $x = \log 64/\log 2 = 6$

     Alternate and more efficient method: $\frac{16(2^x)}{16} = \frac{1024}{16}$, $2^x = 64$, $2^x = 2^6$, $x = 6$

Q For which of the following equations are the y values equivalent?

a) $\log_{32} y = \frac{2}{5}$ b) $\log_{16} 2 = y$ c) $\log_y 81 = \frac{4}{3}$ d) $\log_8 y = \frac{2}{3}$
Answers: *a)* $y = 4$ *b)* $y = \frac{1}{4}$ *c)* $y = 27$ *d)* $y = 4$ $\therefore$ *a) and d) are equivalent*

Q The population of a small town is 125. If it is projected to double every 50 years, how long will it take for the town to reach a population of 8 000?
Answer: $y = a(c)^x$ $8000 = 125(2)^{\frac{x}{50}}$ $x = 300\ yrs.$

Q A cup of coffee contains approximately 100 mg of caffeine. When you drink the coffee, the caffeine is absorbed into the bloodstream, and is eventually metabolized by the body. Every 5 hours the amount of caffeine in the bloodstream is reduced by 50%. How many hours does it take for the caffeine to be reduced to 15 mg?
Answer: $y = a(c)^x$ $15 = 100\left(\frac{1}{2}\right)^{\frac{x}{5}}$ $x \approx 13.7$

Q If $\log x = q$ write $\frac{1}{3}\log x^5$ in terms of q.
Answer: $\frac{1}{3}\log x^5 = \frac{5}{3}q$

Q Determine the x-intercept for the graph of $y = b\log_c(ax)$.
Answer: *when* $y = 0$, $x = \frac{1}{a}$

SCO: **RF11: Solve problems that involve exponential and logarithmic equations.**
[C, CN, PS, R]

Q Arrange the following in order from smallest to largest:

$$\log 1000 \qquad \left(\frac{1}{2}\right)^{-3} \qquad 49^{\frac{1}{2}} \qquad 2^{-1}$$

Answer: $2^{-1} \qquad \log 1000 \qquad 49^{\frac{1}{2}} \qquad \left(\frac{1}{2}\right)^{-3}$

Q Solve for x.

a) $x^{\frac{3}{2}} + 2 = 29$ b) $2^{x+2} = 32^{2x-5}$

Answer: a) $x = 9$ b) $x = 3$

Q Solve for x algebraically.

$\log_2 x + \log_2(x - 7) = 3$

Answer: $(x - 8)(x + 1) = 0 \therefore x = 8$ and $\cancel{x = -1}$ extraneous root

Q i) Solve for x without a calculator.
   ii) Find the approximate solution for x with a calculator.

a) $e^x = 16$ b) $e^{3x} = 27$ c) $e^{2x} = 5$ d) $e^{5x} = 9$

e) $4e^{2x} = 5$ f) $e^{5-3x} = 2$ g) $e^{x+1} = 7$ h) $e^{2x-1} = 3$

i) $4 - 2e^x = -22$

Answers

a) $x = \ln 16 \approx 2.77$ b) $x = \frac{\ln 27}{3} \approx 1.099$ c) $x = \frac{\ln 5}{2} \approx 0.805$ d) $x = \ln \frac{\ln 9}{5} \approx 0.44$

e) $x = \frac{\ln\left(\frac{5}{4}\right)}{2} \approx 0.11$ f) $x = \frac{5 - \ln 2}{3} \approx 1.44$ g) $x = \ln 7 - 1 \approx 0.95$ h) $x = \frac{\ln 3 + 1}{2} \approx 1.05$

i) $x = \ln 13 \approx 2.56$

Q i) Solve for x without a calculator.
   ii) Find the approximate solution for x with a calculator.

a) $\ln x = -1$ b) $\ln x = \frac{1}{3}$ c) $\ln x = 5$

d) $\ln(2x - 1) = 3$ e) $\ln(x + 2) = 4$ f) $\ln\left(\frac{1}{x}\right) = 2$

g) $\ln x = \ln 5 + \ln 8$ h) $5 + 2\ln x = 6$ i) $\ln x^2 = 2\ln 4 - 4\ln 2$

j) $-5 + 2\ln 3x = 5$

Answers:

a) $x = \frac{1}{e} \approx 0.37$ b) $x = e^{\frac{1}{3}} \approx 1.40$ c) $x = e^5 \approx 148.4$ d) $x = \frac{e^3 + 1}{2} \approx 10.54$

e) $x = e^4 - 2 \approx 52.6$ f) $x = \frac{1}{e^2} \approx 0.14$ g) $x = 40$ h) $x = e^{\frac{1}{2}} \approx 1.65$

i) $x = \pm 1$ j) $x = \frac{e^5}{3} \approx 49.5$

Q Solve for x.

a) $e^{\ln(x+1)} = 5$ b) $e^{\ln 5x} = 10$ c) $e^{\ln(4x+2)} = 14$

d) $e^{\ln x^2 + \ln x} = 8$ e) $e^{\ln 2x} = 6$ f) $\log_{10} e^x = 1$

g) $\log_5 e^{2x} = 3$ h) $\ln(e^{2x-1}) = 5$ i) $\ln(e^{5x+2}) = 22$

Answers:

a) $x = 4$ b) $x = 2$ c) $x = 3$

d) $x = 2$ e) $x = 3$ f) $x = \ln 10$

g) $x = \frac{\ln 125}{2}$ h) $x = 3$ j) $x = 4$

SCO: **RF11: Solve problems that involve exponential and logarithmic equations.**
        [C, CN, PS, R]

Q Chemists define the acidity or alkalinity of a substance according to the formula $pH = -log[H^+]$ where $[H^+]$ is the hydrogen ion concentration, measured in moles per litre. Solutions with a pH value of less than 7 are acidic, greater than 7 are basic, and equal to 7 (water) are neutral.

a) Apple juice has a hydrogen ion concentration of $[H^+] = 0.0003$. Find the pH value and determine whether the juice is basic or acidic.

b) A test of ammonia shows the hydrogen ion concentration to be $[H^+] = 1.3 \times 10^{-9}$. Find the pH value and determine whether the ammonia is basic or acidic.

Answers: a) $pH = -\log(0.0003) = 3.52 \therefore acidic$ b) $pH = -\log(1.3 \times 10^{-9}) = 8.89 \therefore basic$

Q a) A solution with a pH of 5 is 1000 times more basic than one with a pH of _____.

b) A solution with a pH of 9 is 10 times more acidic than one with a pH of _____.

Answers: a) 2   b) 10

Q "Loudness" is measured in decibels (dB) calculated using the formula $dB = 10 \log(I \div I_0)$ where I_0 is the intensity of the threshold sound, or sound that can barely be perceived, and I is the intensity in terms of multiples of the intensity of threshold sound.

Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss, and a gunshot from a .22 rimfire rifle has an intensity of about $I = (2.5 \times 1013)I_0$. Calculate the decibels and determine if it makes sense to wear ear protection when at the rifle range.

Answer: $dB = 10 \log\left(\frac{2.5 \times 10^{13} I_0}{I_0}\right) = 133.98 \, dB \therefore You should wear ear protection.$

Q Earthquake intensity is measured on the Richter scale calculated as $R = log[I \div I_0]$, where I_0 is the threshold quake, or movement that can barely be detected, and the I is the intensity in terms of multiples of that threshold intensity.

A seismograph set up at home, indicates that there was an event while you were out that measured an intensity of $I = 989$. Given that a heavy truck rumbling by can give a Richter rating of 3 or 3.5, and moderate quakes have a Richter rating of 4 or more, what was likely the event that occurred while you were out?

Answer: $R = \log\left(\frac{989 \, I_0}{I_0}\right) \approx 3 \therefore It is more likely to be a truck rumbling by.$

Q The temperature , in °C of a cup of hot chocolate, t minutes after it is made, is given by the equation, $T(t) = 92e^{-0.06t}$.

a) Find the temperature of the hot chocolate 8 minutes after it is poured.
b) How long will it take the hot chocolate to cool to $50 \, °C$?

Answer:: a) $T(18) = 92e^{-0.06(8)} = 56.9°C$ b) $50°C = 92e^{-0.06t}$ $t = \frac{\ln\frac{50}{92}}{-0.06} \approx 10.3 \, minutes$

SCO:　**RF11: Solve problems that involve exponential and logarithmic equations.**
　　　[C, CN, PS, R]

Q The growth of a culture of bacteria can be modeled by the equation, $N(t) = N_0 e^{0.105t}$, where $N(t)$ is the number of bacteria after t hours and N_0 is the initial number of bacteria.

　a) If the culture has 300 bacteria initially, what is the estimated population in 12 hours?
　b) How much time will be required for it to double in population?

　Answers:　a)　$N_0 = 300$　$N(12) = 300e^{0.105(12)} \approx 1058\ bacteria$
　　　　　　b) $N = 600$　$600 = 300e^{0.105t}$　$t = \frac{\ln 2}{0.105} \approx 6.6\ hours$

Q　A radioactive element decays exponentially according to the formula $A = A_0 e^{-0.04463t}$, where A is the amount present after t days and A_0 is the initial amount. If the initial amount is $80g$,

　a) Find the amount remaining after 45 days.
　b) After how much time will the radioactive element decay to 20% of the initial amount?

　Answers:　a)　$A = 80e^{-0.04463(45)} \approx 10.74\ g$　　b)　$0.20 = e^{-0.04463t}$　$t = \frac{\ln 0.20}{-0.04463} \approx 36\ days$

SCO: **T1: Demonstrate an understanding of angles in standard position, expressed in degrees and radians.** [CN, ME, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Trigonometry

T1: Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120
T1: Demonstrate an understanding of angles in standard position [0° to 360°]	**T1:** Demonstrate an understanding of angles in standard position, expressed in degrees and radians.

ELABORATION

In *Pre-Calculus 110*, students identified measures of angles in degrees, and in standard position from 0° to 360°.

They will now extend this to define angles as **radians**, and determine the relationship between degrees and **radians**. They will determine the measure of all angles that are **coterminal** with angles in standard position.

Points will be defined in terms of **polar coordinates**, and in relation to the angle of rotation from the origin and the **arc length**.
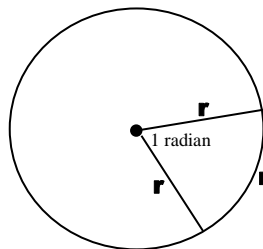
One **radian** is the measure of the central angle of a circle, subtended by an arc equal in length to the radius (r) of the circle.

One full rotation $= 360° = 2\pi \ radians$

One half a rotation $= 180° = \pi \ radians$

One third of a rotation $= 120° = \frac{2\pi}{3} \ radians$

One quarter of a rotation $= 90° = \frac{\pi}{2} \ radians$

If degrees are not indicated, **radians** are assumed. A **radian** is developed by dividing a unit of distance by itself, and as such, **radians** are unitless. For example, $\frac{\pi}{2}$ **radians** is usually expressed simply as $\frac{\pi}{2}$.

Students should develop how to convert from degrees to **radians**, or **radians** to degrees with proportional reasoning. They should understand that:

To convert **radians → degrees**: multiply by $\frac{180°}{\pi}$

To convert **degrees → radians**: multiply by $\frac{\pi}{180°}$

SCO: **T1: Demonstrate an understanding of angles in standard position, expressed in degrees and radians.** [CN, ME, R, V]

In science classes conversions are done using what is called the *Factor Label* method, in which units are positioned to ensure common units cancel out, leaving the appropriate unit. For example, when $60°$ is converted to **radian** measure, the degree units cancel, leaving **radians** only.

$$60° \times \frac{\pi\ radians}{180°} = \frac{\pi}{3}\ radians\ \textbf{ or }\ \frac{\pi}{3}$$

Students should also become proficient in making these angle conversions on a calculator.

Polar coordinates express the location of a point as (r, θ) where r is the distance from the origin to the point and θ is the angle from the positive x-axis to the point in **degrees** or **radians,** measured counter-clockwise. The points $(-r, \theta), (r, -\theta)$ and $(-r, -\theta)$ should also be discussed.
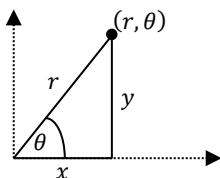
The distance r is also the hypotenuse of a right angle triangle relating to x and y coordinates, and relating θ to trigonometric ratios. In particular, $x^2 + y^2 = r^2$, $\cos\theta = x/r$ and $\sin\theta = y/r$. This is intended as an introduction to radius and point form, rather than as an introduction to complex numbers,

Arc Length is described by the formula $\boldsymbol{a = \theta r}$ as the relationship between **arc length**, (a), the **central angle** (θ) in **radians**, and the **radius** (r). The **arc length** and **radius** must be in the same units.
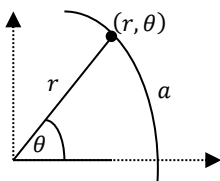
Coterminal Angles will also be explored. These are a set of that have the same terminal arms. By adding or subtracting full rotations repeatedly, $\pm 360°(n)$ for degrees and $\pm 2\pi(n)$ for **radians**, an infinite number of **coterminal angles** are created for a given angle.

For example, $60°$ is **coterminal** with $420°$ and $-300°$ $(60° \pm 360°)$, or in **radians**, $\frac{\pi}{3}$ is **coterminal** with $\frac{7\pi}{3}$ and $\frac{-5\pi}{3}$ $\left(\frac{\pi}{3} \pm 2\pi\right)$.

SCO: **T1: Demonstrate an understanding of angles in standard position, expressed in degrees and radians.** [CN, ME, R, V]

ACHIEVEMENT INDICATORS

- Sketch, in standard position, an angle (positive or negative) when the measure is given in degrees.

- Describe the relationship among different systems of angle measurement, with emphasis on radians and degrees.

- Sketch, in standard position, an angle with a measure of 1 radian.

- Sketch, in standard position, an angle with a measure expressed in the form $k\pi$ radians, where $k \in Q$.

- Express the measure of an angle in radians (exact value or decimal approximation), given its measure in degrees.

- Express the measure of an angle in degrees, given its measure in radians (exact value or decimal approximation).

- Determine the measures, in degrees or radians, of all angles in a given domain that are coterminal with a given angle in standard position.

- Determine the general form of the measures, in degrees or radians, of all angles that are coterminal with a given angle in standard position.

- Explain the relationship between the radian measure of an angle in standard position and the length of the arc cut on a circle of radius r, and solve problems based upon that relationship.

- Translate between polar and rectangular representations.
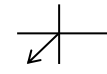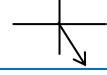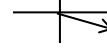
Suggested Instructional Strategies

- To have students visualize what $1\,radian$ looks like, make an equilateral triangle with three pipe cleaners where all angles are $60°$. By making the bottom side into an arc, the central angle will become a little smaller than $60°$ showing the students that $1\,radian$ is approximately $57.2958°$.

- Online sites can provide alternate explanations and extra practice questions for this outcome

- The *Pre-Calculus 12* Textbook does not include polar coordinates. However, this topic is covered in the resource, *Mathematical Modeling Book 4*, Section 5.2.

SCO: **T1: Demonstrate an understanding of angles in standard position, expressed in degrees and radians.** [CN, ME, R, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Sketch each of the following angles: $\frac{3\pi}{4}, \frac{13\pi}{4}, \frac{-5\pi}{6}, \frac{11\pi}{3}, \frac{8\pi}{9}, \frac{-25\pi}{12}$.

a) Determine which quadrant contains the terminal arm.
b) Name one positive and one negative coterminal angle in radians.
c) State the measure of the angle and its coterminal angles in degrees.

Answers:

| $\theta(rad)$	Sketch	$\theta(deg)$	Positive Coterminal		Negative Coterminal	
			radian	degree	radian	degree
$\frac{3\pi}{4}$	QII	$135°$	$\frac{11\pi}{4}$	$495°$	$\frac{-5\pi}{4}$	$-225°$
$\frac{13\pi}{4}$	QIII	$585°$	$\frac{5\pi}{4}$	$225°$	$\frac{-3\pi}{4}$	$-135°$
$\frac{-5\pi}{6}$	QIII	$-150°$	$\frac{7\pi}{6}$	$210°$	$\frac{-17\pi}{6}$	$-510°$
$\frac{11\pi}{3}$	QIV	$660°$	$\frac{5\pi}{3}$	$300°$	$\frac{-\pi}{3}$	$-60°$
$\frac{8\pi}{9}$	QII	$160°$	$\frac{26\pi}{9}$	$520°$	$\frac{-10\pi}{9}$	$-200°$
$\frac{-25\pi}{12}$	QIV	$-375°$	$\frac{23\pi}{12}$	$345°$	$\frac{-\pi}{12}$	$-15°$

Q Write the following in increasing order of magnitude. Explain your reasoning and show all work.

$3\ radians$ $190°$ $\frac{5\pi}{4}\ radians$ $58°$ $\frac{5\pi}{9}$

Answer: $58°\ <\ \frac{5\pi}{9}\ <\ 3\ radians\ <\ 190°\ <\ \frac{5\pi}{4}$

Q: Find the length of the arc.

Answer: $a = 7\pi\ cm\ \approx 22\ cm$
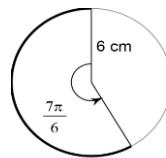
Q: Convert $A(5, -7)$ to polar coordinates.

Answer: $r^2 = (5)^2 + (-7)^2$ $r = \sqrt{74}$

$\tan\theta_r = \frac{7}{5}$ $\theta_r = 54°$ $\theta = 360° - 54° = 306°\ OR\ 5.34\ radians$

$\therefore\quad A(5, -7) \longrightarrow A(\sqrt{74}, 306°)\ OR\ A(\sqrt{74}, 5.34)$

Q: Convert each set of polar coordinates to rectangular coordinates.

a) $A(7, 135°)$ b) $B\left(2, -\frac{11\pi}{6}\right)$ c) $C\left(-4, \frac{2\pi}{3}\right)$

Answers: a) $A\left(\frac{-7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$ b) $B(\sqrt{3}, 1)$ c) $C(2, -2\sqrt{3})$

SCO: **T2: Develop and apply the equation of the unit circle.** [CN, R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120
T2: Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T2:** Develop and apply the equation of the unit circle.

ELABORATION

For this outcome students are introduced to the unit circle, with a radius of 1, and its centre at the origin $(0,0)$ on the Cartesian plane.
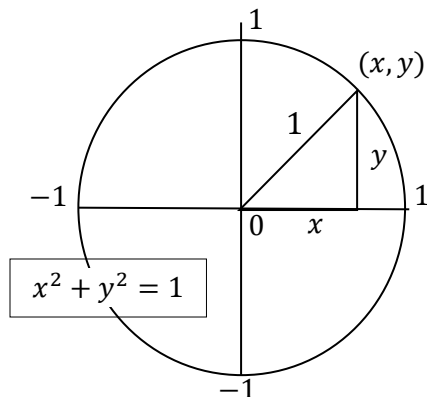
Dropping a perpendicular line from an (x, y) point on the circle to the x-axis creates a right angle triangle, with the origin.

Using the Pythagorean Theorem the equation of a unit circle is $x^2 + y^2 = 1$. For circles with a radius other than 1, the equation of the circle is $x^2 + y^2 = r^2$.

In *Pre-Calculus 11*, students learned the three primary trigonometric ratios on a coordinate plane, $sine\ \theta = \frac{y}{r}$, $cosine\ \theta = \frac{x}{r}$ and $tangent\ \theta = \frac{y}{x}$. As the unit circle has a radius of 1, $\sin \theta = \frac{y}{1} = y$ and $\cos \theta = \frac{x}{1} = x$, and each point (x, y) on the unit circle can be expressed in the form $(\cos \theta , \sin \theta)$

In this outcome students will extend their knowledge of trigonometric ratios to include the three reciprocal trigonometric ratios:

Cosecant(csc) is the reciprocal of the sine ratio: $csc\ \theta = \frac{r}{y}$

Secant(sec) is the reciprocal of the cosine ratio: $sec\ \theta = \frac{r}{x}$

Cotangent(cot) is the reciprocal of the tangent ratio: $cot\ \theta = \frac{x}{y}$

ACHIEVEMENT INDICATORS

- Derive the equation of the unit circle from the Pythagorean theorem.

- Describe the six trigonometric ratios, using a point $P(x, y)$ that is the intersection of the terminal arm of an angle and the unit circle.

- Generalize the equation of a circle with centre $(0, 0)$ and radius r.

SCO: **T2: Develop and apply the equation of the unit circle.**   [CN, R, V]

Suggested Instructional Strategies

- As a class, develop the equation of a circle from a circle with the center at the origin, and $radius = 1$. Then move on to a circle with $radius > 1$. Identify x, y and r values for various points on each circle, and then relate these values to the Pythagorean Theorem, and to the three primary trigonometric ratios.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Is the point $\left(\frac{2}{5}, -\frac{4}{5}\right)$ on the unit circle? How do you know?

Answer: $\left(\frac{2}{5}\right)^2 + \left(\frac{-4}{5}\right)^2 = \frac{20}{25} \neq 1 \;\therefore the\ point\ is\ not\ on\ the\ unit\ circle$

Q Point $P(-4,3)$, plotted on a coordinate plane, is on a circle centered at the origin $(0,0)$.

    a) Find the radius of the circle.
    b) Determine the equation for the circle.
    c) Identify additional points on the circle in the other three quadrants and on the x- and y- axes.
    d) Identify a point that is not on this circle.
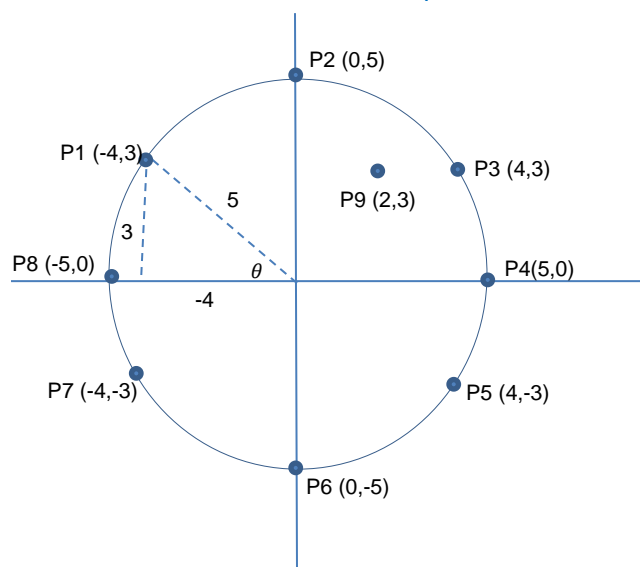   e) Determine the six trigonometric ratios for θ of the triangle formed at P(-4,3).

Answers:
a) $(-4)^2 + (3)^2 = 25 \quad \therefore r = 5$
b) $x^2 + y^2 = 5^2$
c) $P2 - P8,\ see\ diagram.\ Other\ points\ could\ be\ determined, but\ these\ points\ can\ be\ found\ quickly$
      $if\ a\ student\ understands\ the\ concept.$
d) $P9\ (2,3)$
e) For P(-4,3) $\quad \sin\theta = \frac{3}{5} \quad \cos\theta = \frac{-4}{5} \quad \tan\frac{-4}{3} \quad \csc\frac{5}{3} \quad \sec\frac{-5}{4} \quad \cot\frac{-3}{4}$

SCO: **T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.** [ME, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

Scope and Sequence of Outcomes:

| Pre-Calc 110	Pre-Calc A 120
T2: Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T3:** Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

ELABORATION

Students will develop patterns to determine, with technology, the approximate values for the $sine, cosine, tangent, cosecant, secant$ and $cotangent$ of any angle, using a scientific calculator. They will need to ensure that the mode is set to the correct mode; $degrees$ or $radians$.

Students will determine, using the unit circle or reference triangle, the exact values for the $sine, cosine, tangent, cosecant, secant$ and $cotangent$ of angles expressed in degrees that are multiples of $0°, 30°, 45°, 60°$ or $90°$, or for angles expressed in $radians$ that are multiples of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$, and explain the strategy. The unit circle can be derived from the special triangles using similar triangles.

Students will discover that angles will have a corresponding angle in each quadrant, with the same absolute value for each trigonometric ratio, by relating triangles formed by connecting a perpendicular line from the (x, y) point on the circle to the x-axis. They will also see that each ratio will be positive or negative depending on the quadrant in which they are found.

The following summarizes the sine and cosine values for these special angles.
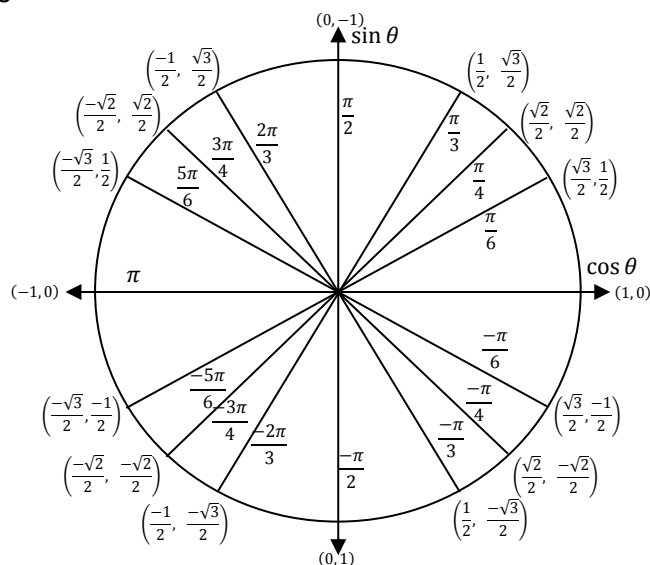
SCO: **T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.** [ME, PS, R, T, V]

Students only need to learn the quadrantal angles and the first quadrant angles. Values for the other three quadrants can be found using reference angles and the CAST rule.

For example: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Corresponding angles, $\frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$, have the same absolute value for the *cosine*, but are negative or positive depending upon the quadrant in which each sits.

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} \qquad\qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} \qquad\qquad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

The quadrant in which each ratio is positive is summarized by the CAST rule, first introduced in *Pre-Calculus 11*, and now extended to include the reciprocal trigonometric ratios.

Sine   **A**ll

Tangent   **C**osine
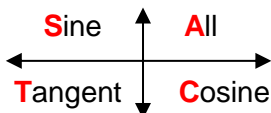
Students should understand that the CAST rule is just a quick way to remember in which quadrants x and y are positive or negative, because the signs of trig functions are determined by the signs of the x and y values used to calculate them.

Given a point on the terminal arm of an angle in standard position, students should be able to determine the trigonometric ratio, and the angle.

For this outcome students can be introduced to methods of rationalizing the denominator. Rationalizing the denominator provides a way of estimating a value such as $\frac{1}{\sqrt{2}}$. Rationalizing it to $\frac{\sqrt{2}}{2}$ makes estimation possible: $\sqrt{2}$ is between $\sqrt{1}$ and $\sqrt{4}$, or between 1 and 2, and closer to 1 or approximately 1.3. Therefore $\frac{\sqrt{2}}{2} \cong \frac{1.3}{2} = 0.65$, a good estimate of $\frac{1}{\sqrt{2}}$.

Although present day access to technology makes rationalizing the denominator unnecessary for estimating values such as $\frac{1}{\sqrt{2}}$, these rationalizing skills remain relevant as students progress in mathematics e.g., solving equations with radicals in the denominator, calculating inverses of complex numbers, and other techniques in abstract math.

SCO: **T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.** [ME, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Determine, with technology, the approximate value of a trigonometric ratio for any angle with a measure expressed in either *degrees* or *radians*.

- Determine, using the unit circle or a reference triangle, the exact value of a trigonometric ratio for angles expressed in *degrees* that are multiples of $0°, 30°, 45°, 60°$ or $90°$, or for angles expressed in *radians* that are multiples of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$, and explain the strategy.

- Determine, with or without technology, the measures, in *degrees* or *radians*, of the angles in a specified domain, given the value of a trigonometric ratio.

- Explain how to determine the exact values of the six trigonometric ratios, given the coordinates of a point on the terminal arm of an angle in standard position.

- Determine the measures of the angles in a specified domain in degrees or radians, given a point on the terminal arm of an angle in standard position.

- Determine the exact values of the other trigonometric ratios, given the value of one trigonometric ratio in a specified domain.

- Sketch a diagram to represent a problem that involves trigonometric ratios.

- Solve a problem, using trigonometric ratios.

SCO: **T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.** [ME, PS, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q: Complete the following table for a rotation of $P(1,0)$, centre $(0,0)$:

angle of rotation in *degrees*	angle of rotation in *radians*	Image of P (1, 0)		
		As (x, y)	As a decimal	Exact
45°				
		$(\cos 225°, \sin 225°)$		
			$(.707, -.707)$	
		$(\cos 270°, \sin 270°)$		
				$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
330°				
				$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Answers:

angle of rotation in *degrees*	angle of rotation in *radians*	Image of P (1, 0)		
		As (x, y)	As a decimal	Exact
45°	$\frac{\pi}{4}$	$(\cos 45°, \sin 45°)$	$(.707, .707)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
225°	$\frac{5\pi}{4}$	$(\cos 225°, \sin 225°)$	$(-.707, -.707)$	$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$
315°	$\frac{7\pi}{4}$	$(\cos 315°, \sin 315°)$	$(.707, -.707)$	$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$
270°	$\frac{3\pi}{2}$	$(\cos 270°, \sin 270°)$	$(0, -1)$	$(0, -1)$
30°	$\frac{\pi}{6}$	$(\cos 30°, \sin 30°)$	$(.866, .5)$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
330°	$\frac{11\pi}{6}$	$(\cos 330°, \sin 330°)$	$(.866, -.5)$	$\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$
145°	$\frac{3\pi}{4}$	$(\cos 145°, \sin 145°)$	$(-.707, .707)$	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Q Which coordinate could represent the value of each of the following:

a) $\cos 290°$ b) $\sin 180°$ c) $\cos 110°$
d) $\sin 225°$ e) $\cos(-70°)$ f) $\cos 420°$

Answers: *a) I   b) F   c) C   d) H   e) I   f) A*
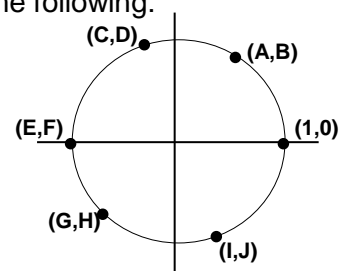
Q Calculate the exact values of the 5 other trigonometric ratios for

$\sec\theta = -\frac{13}{12}$, $\pi < \theta < \frac{3\pi}{2}$.

Answer: $x = -12$, $r = 13$, $y = -5$ $\therefore$ $\cos\frac{-12}{13}$ $\sin\frac{-5}{13}$ $\tan\frac{5}{12}$ $\csc\frac{-13}{5}$ $\cot\frac{12}{5}$

SCO: **T3: Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.** [ME, PS, R, T, V]

Q Evaluate the following expressions, stating the answer as an exact value.

a) $3 \sin 45° + \cos^2 30°$

b) $\sin \frac{5\pi}{4} \cos \left(-\frac{\pi}{6}\right)$

Answers: *a)* $\dfrac{6\sqrt{2} + 3}{4}$ *b)* $-\dfrac{\sqrt{6}}{4}$

Q Explain why $\frac{\sqrt{2}}{2}$ is called an exact value, while 0.707 is only approximate.

Answer: The value $\frac{\sqrt{2}}{2}$ is exact because $\sqrt{2}$ is an irrational number meaning that, as a decimal it continues on and never repeats. Therefore $\frac{\sqrt{2}}{2}$ cannot be expressed accurately as a decimal. The value 0.707 is the value of $\frac{\sqrt{2}}{2}$ after is has been rounded off at the tousandths place.
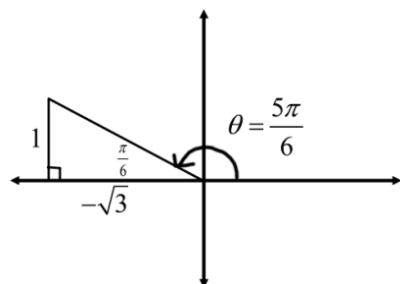
Q *(for a challenge)*

a) If $\cot \theta = -\sqrt{3}$, In which quadrants could the terminal arm be located?

b) Draw a labeled diagram to illustrate each case and calculate the value of θ in standard position.

c) Calculate $\sin \theta \cot \theta - \cos^2 \theta$ for each θ found.

Answer
a) Quadrants II and IV.
b) Case 1 (Quadrant II) Case 2 (Quadrant IV)

:

c) Case 1
$\sin \theta \cot \theta - \cos^2 \theta$

$= \left(\frac{1}{2}\right)\left(-\sqrt{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)^2$

$= -\frac{\sqrt{3}}{2} - \frac{3}{4}$
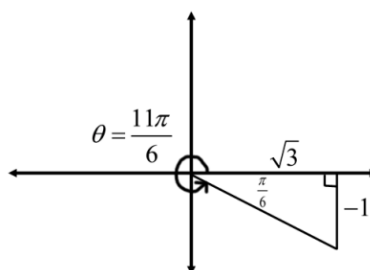
$= \frac{-2\sqrt{3} - 3}{4}$

Case 2
$\sin \theta \cot \theta - \cos^2 \theta$

$= \left(-\frac{1}{2}\right)\left(-\sqrt{3}\right) - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{\sqrt{3}}{2} - \frac{3}{4}$

$= \frac{2\sqrt{3} - 3}{4}$

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.**   [CN, PS, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

## Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120
T1: Demonstrate an understanding of angles in standard position [0° to 360°] **T2:** Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T4:** Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

ELABORATION

This outcome will introduce students to various characteristics of the graphs of trigonometric functions. These functions are **periodic**, meaning they repeat over a specific **period**. **Sinusoidal** curves oscillate repeatedly up and down from a centre line and can describe oscillating events such as the tides coming in and out over time, the height of a person on a Ferris Wheel going up and down, or the height of a point on a rolling object as it rolls along a horizontal distance.

The **sinusoidal axis** of a sine curve is the horizontal central axis of the curve, halfway between the maximum and minimum values. The **amplitude** is the maximum vertical distance of the graph of a sinusoidal function, above and below the sinusoidal axis of the curve. The graph of the function, $y = \sin x$ is shown below.
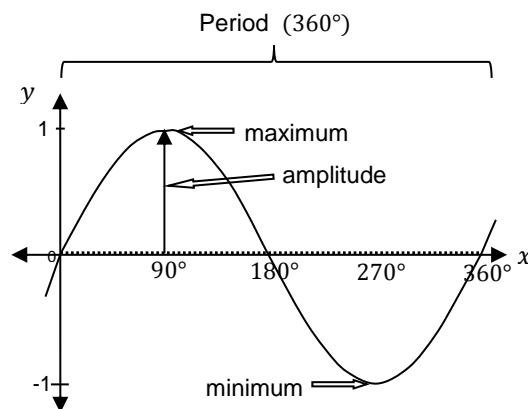
The x-axis is the sinusoidal axis; the graph is periodic; continuous; has a domain $\{x | x \in R\}$, and range $\{y | -1 \le y \le 1, y \in R\}$; has a maximum of $+1$, and a minimum of -1; has an amplitude of 1; a period of $360°$ or 2π; and a y-intercept of 0. The keypoints on the graph should be related to the sine values for the quandrantal angles on the unit circle.

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.**   [CN, PS, T, V]

Changing the value of a, b, c, or d in the sine function, $y = a \sin(b(x - c)) + d$, will result in a stretch or translation of the original $y = \sin x$ function.

a represents the amplitude, or the vertical stretch. $Amplitude = |a|$

b affects the length of the period, or the horizontal stretch. $Period = \frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$ Due to the periodic nature of trigonometric functions they are ideal for illustrating horizontal stretch.

c is the value of the horizontal translation, right if $c > 0$, or left if $c < 0$.

d is the value of the vertical translation up or down, and also shifts the sinusoidal axis, up if $d > 0$, or down if $d < 0$.

Trigonometric functions will provide a good review of the parameters which cause translation and stretches of functions. (RF1 to RF4)

ACHIEVEMENT INDICATORS

- Sketch, with or without technology, the graph of $y = \sin x$, $y = \cos x$ or $y = \tan x$.

- Determine the characteristics (amplitude, asymptotes, domain, period, range and zeros) of the graph of $y = \sin x$, $y = \cos x$ or $y = \tan x$.

- Determine how varying the value of a affects the graphs of $y = a \sin x$ and $y = a \cos x$.

- Determine how varying the value of b affects the graphs of $y = \sin bx$ and $y = \cos bx$.

- Determine how varying the value of c affects the graphs of $y = \sin (x + c)$ and $y = \cos(x + c)$.

- Determine how varying the value of d affects the graphs of $y = \sin x + d$ and $y = \cos x + d$.

- Sketch, without technology, graphs of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$, using transformations, and explain the strategies.

- Determine the characteristics (amplitude, asymptotes, domain, period, phase shift, range and zeros) of the graph of a trigonometric function of the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$.

- Determine the values of a, b, c and d for functions of the form $y = a \sin b(x - c) + d$ or or $y = a \cos b(x - c) + d$ that correspond to a given graph, and write the equation of the function.

- Determine a trigonometric function that models a real life situation to solve a problem.

- Explain how the characteristics of the graph of a trigonometric function relate to the conditions in a problem situation.

- Solve a problem by analyzing the graph of a trigonometric function.

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.** [CN, PS, T, V]

Suggested Instructional Strategies

- Have students create sinusoidal data sets by rolling and measuring objects, and recording their height and distance along a surface.

- Match cards with graphs and equations for both sine and cosine functions.

- Use interactive software, which allows the manipulation of the coefficients (a,b,c & d) from the equation to be demonstrated graphically. This is offered on various online sites such as: http://illuminations.nctm.org/ActivityDetail.aspx?ID=174 or http://www.intmath.com/trigonometric-graphs/1-graphs-sine-cosine-amplitude.php

- In your classroom, post a labeled sine and cosine graph with their corresponding equations.

- As students make connections between graphs and their equations, a good interactive summation tool can be used, such as the one found at: http://illuminations.nctm.org/ActivityDetail.aspx?ID=215

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q A Ferris wheel has a maximum height of $11m$, a radius of $5m$ (which allows a $1.0\ m$ clearance at the bottom) and rotates once every $40sec$. As the ride begins you are half way to the top of the wheel.

a) Sketch a graph that shows height above the ground as a function of time using a sine function.
b) What is the lowest you go as the wheel turns? Explain why this must be a positive number.
c) How high will you be after 2.5 minutes?

Answers: a)
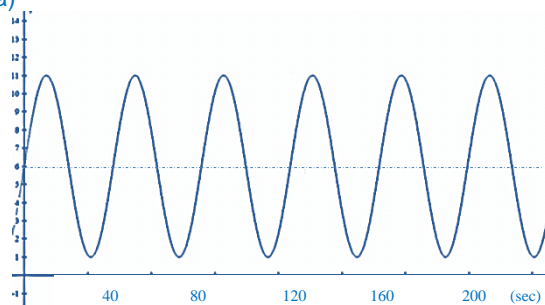
b) *the lowest you go as the wheel turns is* $1\ m$. *This must be positive, otherwise you would be going underground.*
c) *One cycle takes* $40s$ *which will bring you back to* 6 *metres high.* 2.5 *minutes or* $150s$ *or* 3.75 $40s$ *cycles which would be at the lowest point, or* $1\ m$. (*OR* $y = 5\sin 9x + 6$,   *so* $y = 5\sin(9*150s) + 6 = 1$)

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.** [CN, PS, T, V]

Q A spring oscillates in height according to the function $y = \left(\frac{1}{2}\right)sin(144(t-45)) + 5$ where t represents the time in seconds and y represents the height in metres.

a) What is the amplitude of the oscillation?
b) What is the period of oscillation?
c) What is the maximum height it reaches?

Answers: a) amplitude $= \frac{1}{2}$ *b) Period* $= \frac{360}{144} = 2.5\ s$ *c)* $5.5\ m$

Q Tommy has a tree swing near the river in his backyard. The swing is a single rope hanging from a tree branch. When Tommy swings, he goes back and forth across the shore of the river. One day his mother (who was taking an adult math course) decided to model his motion using her stopwatch. She finds that, after $2\ sec$, Tommy is at one end of his swing, $4\ m$ from the shoreline while over land. After $6\ sec$, he reaches the other end of his swing, $5.2\ m$ from the shoreline while over the water.

a) Sketch a graph of this sinusoidal function.

b) Write the equation expressing distance from the shore versus time.

c) Predict the distance when
   i) time is $6.8\ sec$
   ii) time is $15\ sec$
   iii) time is $30\ sec$

d) Where was Tommy when his mother started the watch?

Answers: a)

   b) $y = -4.6\sin 45x + 0.6$
   c) i) $4.3m\ (over\ water)$ ii) $3.9m\ (over\ water)$ iii) $5.3m\ (over\ water)$
   d) $0.6m\ (over\ water)$

Q Write a possible sinusoidal equation for the following graph.
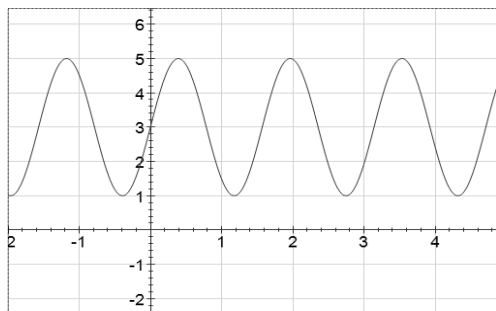

Answer: $a = 2$ $b = \frac{2\pi}{3}/\frac{3}{2}$ $c = 0$ $d = 3$ $\therefore\ y = 2\sin\left(\frac{4\pi}{3}x\right) + 3$

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.** [CN, PS, T, V]

Act Have students place a piece of masking tape on the bottom edge of an empty can. Roll the can along a metre stick a short distance on the floor, measuring the horizontal distance traveled and the height of the masking tape at various intervals during its rotation. Record at least 10 measurements within the first two revolutions. Have the students plot the coordinate pairs on a graph. They should draw connections between the diameter and circumference of the can and amplitude and period of the graph. Further explanation and follow-up questions can be found at: http://www.ed.gov.nl.ca/edu/k12/curriculum/documents/mathematics/math2204/3.pdf

Q a) Write the equation that represents the graph shown below, using $y = \sin\theta$ as the model, with θ measured in degrees.
b) Write the equation again using $y = \cos\theta$.
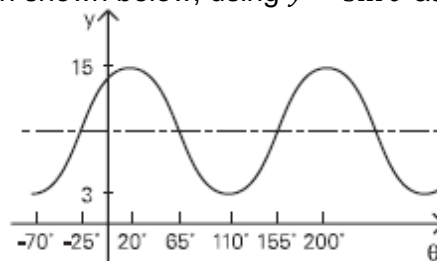
Answers: *a)* $y = 6\sin 2(\theta + 25°) + 9$
   *b)* $y = 6\cos 2(\theta - 20°) + 9$

Q a) Create a table of values for θ and y using $0 \le °\theta \le 360°$ and the equation $\frac{1}{2}(y - 1) = \cos\theta$.
b) Create a table of values using $0° \le \theta \le 360°$ and the equation $y - 1 = \cos\theta$.
c) Describe the relationship between the y-values in the two tables above.

Answers:

	θ	0°	90°	180°	270°	360°
a)	*y*	3	1	−1	1	3
b)	*y*	2	1	0	1	2

c)   *The second equation has an amp;itude of half the first equation.*

Q For the graph shown:
a) State the equation for the sinusoidal axis.
b) State the amplitude.
c) State the period, domain, and range.
d) Write the equation for this graph showing a reflection for y=sinθ in the x-axis.
e) Use the language of transformations to describe how this graph is related to y = sin θ.
f) Approximate the intercepts from the graph.
g) State the values for θ where the graph increases.
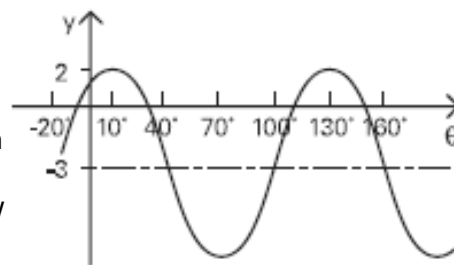
Answers: a) $y = -3$ *b)* $amp = 5$ *c)* $per = 120°$ $D\{x | x \in R\}$ $R\{y | -8 \le y \le 2, y \in R\}$
d) $y = -5\sin 3(x - 40°) - 3$
e) It is a reflection with a vertical stretch of 5, vertical translation of
               -3, *horizontal stretch of 3 and horizontal traslation of* 40°.
*f) x intercepts* $\approx$ −8° 32° 112° 152°
g) Increasing 0° to 12° *and* 72° to 132°

SCO: **T4: Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.** [CN, PS, T, V]

Q For the following graph:

a) Determine the period, amplitude and the equation of the sinusoidal axis for the following graph.
b) Write first a positive and then a negative sine function for the graph shown below.
c) Write first a positive and then a negative cosine function for the graph shown below.
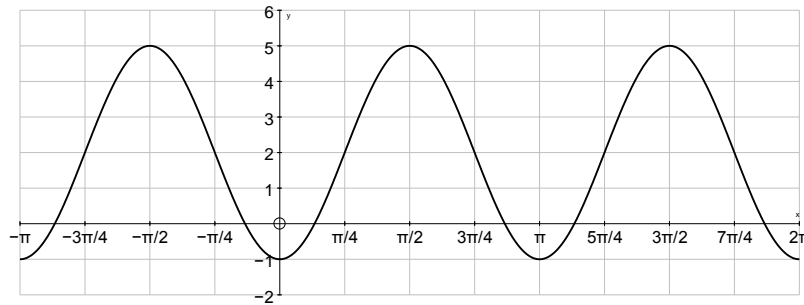
Answers: $a)$ $per = \pi$ $amp = 3$ $\sin axis$ $y = 2$

$b)$ $y = 3\sin 2\left(x - \frac{\pi}{4}\right) + 2$ $y = -3\sin 2\left(x + \frac{\pi}{4}\right) + 2$

$c)$ $y = 3\cos 2\left(x - \frac{\pi}{2}\right) + 2$ $y = -3\cos 2(x) + 2$

SCO: **T5: Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.**   [CN, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

T5: Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120
RF1: Factor polynomial expressions **RF5:** Solve problems that involve quadratic equations.	**T5:** Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

ELABORATION

When solving trigonometric equations algebraically, the unit circle can be used to determine exact values of θ.

Inverse trigonometric function keys on a calculator will provide approximate measures for θ, when given the values for $\sin\theta$, $\cos\theta$ or $\tan\theta$, where $\theta \in R$.

Solutions determined graphically should be compared to the algebraic solution. Students should understand that due to the periodic nature of trigonometric functions, multiple solutions will occur.

To solve a trigonometric equation involving $\csc\theta$, $\sec\theta$ or $\cot\theta$, students may need to work with the related reciprocal function(s).

ACHIEVEMENT INDICATORS

- Verify, with or without technology, that a given value is a solution to a trigonometric equation.

- Determine, algebraically, the solution of a trigonometric equation, stating the solution in exact form when possible.

- Determine, using technology, the approximate solution of a trigonometric equation in a restricted domain.

- Relate the general solution of a trigonometric equation to the zeros of the corresponding trigonometric function (restricted to sine and cosine functions).

- Determine, using technology, the general solution of a given trigonometric equation.

- Identify and correct errors in a solution for a trigonometric equation.

Suggested Instructional Strategies

- This topic is well covered in the previous resource, *Mathematical Modeling* Book 2, Section 4.1, published by Nelson.

SCO: **T5: Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.**   [CN, PS, R, T, V]

Questions (Q) and Activities (Act) for Instruction and Assessment

Q a) From the graph below, $(-2\pi \leq x \leq 2\pi)$, find the approximate values of x, when $y = 3$.

y=−cos3x + 2
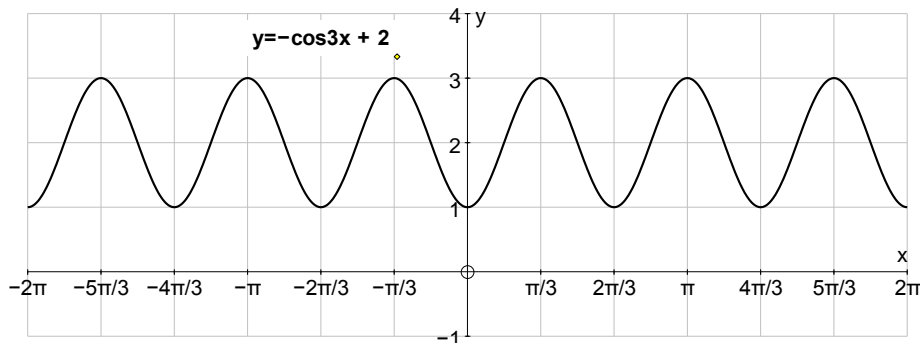
b) Confirm your findings by solving algebraically, $3 = -\cos 3x + 2$.

Answers:
a) when $y = 3$, $x = \frac{-5\pi}{3}$ $-\pi$ $\frac{-\pi}{3}$ $\frac{\pi}{3}$ π *and* $\frac{5\pi}{3}$
b) $x = \frac{\pi}{3} + \frac{2\pi}{3}k$ *When the interval is* $-2\pi \leq x \leq 2\pi$, *the solutions would include tha answers for a)*

Q A cork is bobbing up and down in the waves of the beautiful Bay of Fundy. Its height above and below normal water level with respect to time can be expressed by the sinusoidal function $y = 3 \sin 180x$, where y represents the height of the cork in centimetres, and x represents the time in seconds.

Using your graphing calculator, find all possible times for which the cork would be 2 *cm* above normal water level. Sketch the graph and explain.

Answer: When y = 2 $\frac{2}{3} = \sin 180x$ $180x = 41.8°$ *or* $138.2° + 360°k$ $x = 0.23°$ *or* $0.77° + 2k$
       These are the points at which the graph (and cork) is higher than 2 cm.

Q Solve each of the following trigonometric equations for $0° \leq \theta \leq 360°$.

a) $\sin^2 x - 2 \sin x - 3 = 0$ b) $2 \cos^2 x + 5 \cos x - 3 = 0$

   Answers: *a)* $(\sin x + 1)(\sin x - 3) = 0$ $\sin x = -1$ *or* $3(NA)$ $x = 270°$
                b) $(2 \sin x - 1)(\sin x + 3) = 0$ $\sin x = \frac{1}{2}$ *or* $-3(NA)$ $x = 30°$ *and* $150°$

Q Solve the following. State your answer in radian measure.

$$\sin \left(2x - \frac{\pi}{2} \right) = \frac{1}{2}$$

Answer: $x = \begin{cases} \frac{\pi}{3} + \pi n, n\epsilon I \\ \frac{2\pi}{3} + \pi n, n\epsilon I \end{cases}$

SCO: **T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).** [R, T, V]

[C] Communication	**[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Math
[T] Technology	**[V]** Visualization **[R]** Reasoning and Estimation

T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).

Scope and Sequence of Outcomes:

Pre-Calc 110	Pre-Calc A 120
	T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).

ELABORATION

In this outcome students will be introduced to proving **trigonometric identities**.

Students should develop a clear understanding of the difference between equations and identities. Equations can be solved for certain value(s) of a variable. For example, $2x^2 + 3 = 11$ is an equation. It has two solutions, -2 and 2, which can be verified by substitution back into the equation, remembering that a quadratic equation can have no more than two distinct roots. These are the only values for x that are valid.

An identity is any statement of equality that is true for all values of the variable x. For example, $(x + 1)^2 = x^2 + 2x + 1$. Any value of x can be used and this identity will hold true. Trigonometric identities are those that involve trigonometric ratios.

To prove identities, each side is worked through separately, until both the left side and the right side are shown to be the same. It is important that proofs are formal. Students need to clearly show all their processes as they develop their proofs.

It is important to stress to students that they cannot assume that the two expressions are equal before it is proven. For example, cross multiplying, squaring both sides of an identity, and comparing numerators from both sides of the identity are not permitted.

A common mistake is for students to drop the symbol for the angle as in sin versus $sin\,\theta$. It should be emphasized that the angle symbol is an important part of the identity and proof.

Non-permissible values of trigonometric identities occur when there is a denominator in the identity. The non-permissible values of any trigonometric identity can be determined by finding all values of θ that would cause a denominator of zero. These can be compared to finding the non-permissible values for rational expressions studied in *Pre-Calculus 110*.

For example, the trigonometric identity $\csc \theta = \frac{1}{\sin \theta}$ is not defined at $\sin \theta = 0$. This occurs at $0°$ and $180°$ and every subsequent rotation of $180°$. Therefore the non-

SCO: **T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).** [R, T, V]

permissible values for this identity in degrees would be $\theta \neq 0° + 180°n, n \in I$. In $radians$, $\theta \neq 0 + \pi n, n \in I$.

Students will use the following identities to prove trigonometric identities:

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad\qquad \sec x = \frac{1}{\cos x} \qquad\qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad\qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \implies \sin^2 \theta = 1 - \cos^2 \theta$$
$$\implies \cos^2 \theta = 1 - \sin^2 \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta \implies \tan^2 \theta = \sec^2 \theta - 1$$
$$\implies 1 = \sec^2 \theta - \tan^2 \theta$$
(This identity can be derived from $\sin^2 \theta + \cos^2 \theta = 1$ by dividing each term by $cos^2\theta$).

$$1 + \cot^2 \theta = \csc^2 \theta \implies 1 = \csc^2 \theta - \cot^2 \theta$$
$$\implies \cot^2 \theta = \csc^2 \theta - 1$$
(This identity can be derived from $\sin^2 \theta + \cos^2 = 1$ by dividing each term by $sin^2\theta$).

Sum & Difference Identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double Angle Identities

$$\sin 2A = 2 \sin A \cos A$$
(This identity can be derived from $sin(A + A)$.)

$$\cos 2A = \cos^2 A - \sin^2 A$$
(This identity can be derived from $cos(A + A)$.)

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

(This identity can be derived from $tan(A + A)$ or $\frac{\sin 2A}{\cos 2A}$.)

SCO: **T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).** [R, T, V]

ACHIEVEMENT INDICATORS

- Explain the difference between a trigonometric identity and a trigonometric equation.

- Verify a trigonometric identity numerically for a given value in either degrees or radians.

- Explain why verifying that the two sides of a trigonometric identity are equal for given values is insufficient to conclude that the identity is valid.

- Determine, graphically, the potential validity of a trigonometric identity, using technology.

- Determine the non-permissible values of a trigonometric identity.

- Prove, algebraically, that a trigonometric identity is valid.

- Determine, using the sum, difference and double-angle identities, the exact value of a trigonometric ratio for a given angle.

Suggested Instructional Strategies

- An activity that gives a deeper understanding of identities is Focus D (Trident Fish), from the previous resource, Mathematical Modeling Book 4, page 245. Also, the "Check Your Understanding" from page 247 may be used.

- To develop a deeper understanding of how compound angle identities and double angle identities are derived, refer to Investigation $7B$ on page 248 including the "Investigation" questions and the "Check Your Understanding" questions. *Focus E* on page 251 of Mathematical Modeling Book 4 is also helpful.

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Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Prove the following identities:

a) $\cos\theta\tan\theta = \sin\theta$

b) $\dfrac{\cot\theta}{\tan\theta} = \dfrac{1-\sin^2\theta}{1-\cos^2\theta}$

c) $\cot^2\theta = \dfrac{\cos^2\theta}{1-\cos^2\theta}$

d) $\dfrac{\csc x}{\sec x} = \cot x$

e) $(\sin x + \cos x)^2 = 1 + 2\sin x\cos x$

f) $2\sin^2\theta - 1 = \sin^2\theta - \cos^2\theta$

g) $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{1}{\sin^2 x\cos^2 x}$

h) $\cos^2 t = \sin^2 t + 2\cos^2 t - 1$

i) $\tan\theta = \tan^2\theta\cot\theta$

j) $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$

Answers:

	LS	RS
a)	$\cos\theta\tan\theta = \cos\theta\cdot\dfrac{\sin\theta}{\cos\theta} = \sin\theta$	$\sin\theta$
b)	$\dfrac{\cot\theta}{\tan\theta} = \dfrac{\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}} = \dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\cos\theta}{\sin\theta} = \dfrac{\cos^2\theta}{\sin^2\theta}$	$\dfrac{1-\sin^2\theta}{1-\cos^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta}$
c)	$\cot^2\theta$	$\dfrac{\cos^2\theta}{1-\cos^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$
d)	$\dfrac{\csc x}{\sec x} = \dfrac{\frac{1}{\sin x}}{\frac{1}{\cos x}} = \dfrac{\cos x}{\sin x} = \cot x$	$\cot x$
e)	$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x$ $= 1 + 2\sin x\cos x$	$1 + 2\sin x\cos x$
f)	$2\sin^2\theta - 1$	$\sin^2\theta - \cos^2\theta = \sin^2\theta - (1 - \sin^2\theta)$ $= 2\sin^2\theta - 1$
g)	$\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{\cos^2 x}{\sin^2 x\cos^2 x} + \dfrac{\sin^2 x}{\sin^2 x\cos^2 x}$ $= \dfrac{1}{\sin^2 x\cos^2 x}$	$\dfrac{1}{\sin^2 x\cos^2 x}$
h)	$\cos^2 t$	$\sin^2 t + 2\cos^2 t - 1$ $= 1 - \cos^2 t + 2\cos^2 t - 1$ $= \cos^2 t$
i)	$\tan\theta$	$\tan^2\theta\cot\theta = \dfrac{\sin^2\theta}{\cos^2\theta}\cdot\dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$
j)	$\sec^2\theta + \csc^2\theta = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} = \dfrac{1}{\sin^2\theta\cos^2\theta}$	$\sec^2\theta\csc^2\theta = \dfrac{1}{\cos^2\theta}\cdot\dfrac{1}{\sin^2\theta} = \dfrac{1}{\sin^2\theta\cos^2\theta}$

SCO: **T6: Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent).** [R, T, V]

Q Prove each identity:

a) $\dfrac{\cos 2\theta}{\cos \theta + \sin \theta} = \cos \theta - \sin \theta$

b) $\csc 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta} + \tan \theta$

c) $\dfrac{\sin 2x}{\cos x} + \dfrac{\cos 2x}{\sin x} = \csc x$

d) $\dfrac{\cot \alpha - \tan \alpha}{\sin \alpha \cos \alpha} = \csc^2 \alpha - \sec^2 \alpha$

e) $\dfrac{\sin x + \cos x \cot x}{\cot x} = \sec x$

f) $(1 + \tan x)^2 = \sec^2 x + 2\tan x$

Answers:

	LS	RS
a)	$\dfrac{\cos 2\theta}{\cos \theta + \sin \theta} = \dfrac{\cos^2 \theta - \sin^2 \theta}{\cos \theta + \sin \theta} = \dfrac{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)}{(\cos \theta + \sin \theta)}$ $= \cos \theta - \sin \theta$	$\cos \theta - \sin \theta$
b)	$\csc 2\theta$	$\dfrac{\cos 2\theta}{\sin 2\theta} + \tan \theta = \dfrac{1 - 2\sin^2 \theta}{2 \sin \theta \cos \theta} + \dfrac{\sin \theta}{\cos \theta}$ $= \dfrac{1 - 2\sin^2 \theta + 2\sin^2 \theta}{2 \sin \theta \cos \theta}$ $= \dfrac{1}{\sin 2\theta} = \csc 2\theta$
c)	$\dfrac{\sin 2x}{\cos x} + \dfrac{\cos 2x}{\sin x} = \dfrac{2 \sin x \cos x}{\cos x} + \dfrac{1 - 2\sin^2 x}{\sin x} = 2 \sin x +$ $\dfrac{1 - 2\sin^2 x}{\sin x} = \dfrac{2 \sin^2 x + 1 - 2\sin^2 x}{\sin x} = \dfrac{1}{\sin x} = \csc x$	$\csc x$
d)	$\dfrac{\cot \alpha - \tan \alpha}{\sin \alpha \cos \alpha} = \dfrac{\left(\frac{\cos \alpha}{\sin \alpha}\right) - \left(\frac{\sin \alpha}{\cos \alpha}\right)}{\sin \alpha \cos \alpha} = \dfrac{\frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha}}{\sin \alpha \cos \alpha}$ $= \dfrac{\cos^2 \alpha - \sin^2 \alpha}{\sin^2 \alpha \cos^2 \alpha} = \dfrac{1}{\sin^2 \alpha} - \dfrac{1}{\cos^2 \alpha} = \csc^2 \alpha - \sin^2 \alpha$	$\csc^2 \alpha - \sec^2 \alpha$
e)	$\dfrac{\sin x + \cos x \cot x}{\cot x} = \dfrac{\sin x}{\cot x} + \cos x$ $= \dfrac{\sin^2 x}{\cos x} + \cos x = \dfrac{\sin^2 x + \cos^2 x}{\cos x}$ $= \dfrac{1}{\cos x} = \sec x$	$\sec x$
f)	$(1 + \tan x)^2 = 1 + 2 \tan x + \tan^2 x$ $= 2 \tan x + \sec^2 x$	$\sec^2 x + 2 \tan x$

Q Find the exact value of:

a) $\cos \dfrac{7\pi}{12}$

b) $\cos 165°$

Answers:

a) $\cos \dfrac{7\pi}{12} = \cos \left(\dfrac{3\pi}{12} + \dfrac{4\pi}{12}\right) = \cos \dfrac{\pi}{4} \cos \dfrac{\pi}{3} - \sin \dfrac{\pi}{4} \sin \dfrac{\pi}{3} = \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right) - \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{6}}{4} = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

b) $\cos(165°) = -\cos 15°$
$= -\cos(45° - 30°) = -[\cos 45° \cos 30° \sin 45° \sin 30°]$
$= -\left[\left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right)\right] = \dfrac{-\sqrt{6} - \sqrt{2}}{4}$

SUMMARY OF CURRICULUM OUTCOMES

Pre-Calculus A 120

[C] Communication, **[PS]** Problem Solving, **[CN]** Connections, **[R]** Reasoning,
[ME] Mental Mathematics and Estimation, **[T]** Technology **[V]** Visualization

Relations and Functions

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

RF1. Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. [C, CN, R, V]

RF2. Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. [C, CN, R, V]

RF3. Apply translations and stretches to the graphs and equations of functions. [C, CN, R, V]

RF4. Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, line $y = x$. [C, CN, R, V]

RF5. Demonstrate an understanding of inverses of relations. [C, CN, R, V]

RF6. Graph and analyze radical functions (limited to functions involving one radical). [CN, R, T, V]

RF7. Demonstrate an understanding of exponential functions.

RF8. Demonstrate an understanding of logarithms. [CN, ME, R]

RF9. Graph and analyze exponential and logarithmic functions. [C, CN, T, V]

RF10. Demonstrate an understanding of the product, quotient and power laws of logarithms.[C, CN, R,T]

RF11. Solve problems that involve exponential and logarithmic equations**.** [C, CN, PS, R]

Trigonometry

General Outcome: Develop trigonometric reasoning.

Specific Outcomes

T1. Demonstrate an understanding of angles in standard position, expressed in degrees and radians.   [CN, ME, R, V]

T2. Develop and apply the equation of the unit circle. [CN, R, V]

T3. Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. [ME, PS, R, T, V]

T4. Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems. [CN, PS, T, V]

T5. Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. [CN, PS, R, T, V]

T6. Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent). [R, T, V]

REFERENCES

Alberta Education, System Improvement Group. *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings.* Edmonton, AB:

Alberta Education, 2006. Available at http://www.wncp.ca/math/report_2006.pdf (Accessed September 20, 2007).

Armstrong, Thomas. *7 Kinds of Smart: Identifying and Developing Your Many Intelligences.* New York, NY: Plume, 1993.

Banks, J. A. and C. A. M. Banks. *Multicultural Education: Issues and Perspectives.* 2nd ed. Boston, MA: Allyn and Bacon, 1993.

British Columbia Ministry of Education. *The Primary Program: A Framework for Teaching.* Victoria, BC: British Columbia Ministry of Education, 2000.

Caine, Renate Nummela and Geoffrey Caine. *Making Connections: Teaching and the Human Brain.* Alexandria, VA: Association for Supervision and Curriculum Development, 1991.

Hope, Jack A. et al. *Mental Math in the Primary Grades.* Palo Alto, CA: Dale Seymour Publications, 1988.

McAskill, B. et al. *WNCP Mathematics Research Project: Final Report.* Victoria, BC: Holdfast Consultants Inc., 2004. Available at http://www.wncp.ca/math/Final_Report.pdf (Accessed September 20, 2007).

National Council of Teachers of Mathematics. *Computation, Calculators, and Common Sense: A Position of the National Council of Teachers of Mathematics.* May 2005. http://www.nctm.org/uploadedFiles/About_NCTM/Position_Statements/computation.pdf (Accessed September 20, 2007).

Rubenstein, Rheta N. "Mental Mathematics beyond the Middle School: Why? What? How?" *Mathematics Teacher* 94, 6 (September 2001), pp. 442–446.

Shaw, J. M. and M. J. P. Cliatt. "Developing Measurement Sense." In P. R. Trafton (ed.), *New Directions for Elementary School Mathematics: 1989 Yearbook* (Reston, VA: National Council of Teachers of Mathematics, 1989), pp. 149–155.

Steen, L. A. *On the Shoulders of Giants: New Approaches to Numeracy.* Washington, DC: Mathematical Sciences Education Board, National Research Council, 1990.

Western and Northern Canadian Protocol for Collaboration in Education. *The Common Curriculum Framework for K–9 Mathematics* May 2006. http://www.wncp.ca/english/subjectarea/mathematics/ccf.aspx

Western and Northern Canadian Protocol for Collaboration in Education. *The Common Curriculum Framework for Grades 10-12 Mathematics* January 2008. http://www.wncp.ca/english/subjectarea/mathematics/ccf.aspx