Pre-Calculus 110 Curriculum

Implemented September 2012

Acknowledgements

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Curriculum Overview for Grades 10-12 Mathematics

BACKGROUND AND RATIONALE

Mathematics curriculum is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society.

It is essential the mathematics curriculum reflects current research in mathematics instruction. To achieve this goal, *The Common Curriculum Framework for Grades 10–12 Mathematics: Western and Northern Canadian Protocol* has been adopted as the basis for a revised mathematics curriculum in New Brunswick. The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators and others.

The framework identifies beliefs about mathematics, general and specific student outcomes, and achievement indicators agreed upon by the seven jurisdictions. This document is based on both national and international research by the WNCP and the NCTM.

There is an emphasis in the New Brunswick curriculum on particular key concepts at each grade which will result in greater depth of understanding and ultimately stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

The intent of this document is to clearly communicate high expectations for students in mathematics education to all education partners. Because of the emphasis placed on key concepts at each grade level, time needs to be taken to ensure mastery of these concepts. *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge (NCTM Principles and Standards, 2000).*

BELIEFS ABOUT STUDENTS AND MATHEMATICS LEARNING

The New Brunswick Mathematics Curriculum is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice. These beliefs include:
- mathematics learning is an active and constructive process;
- learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates;
- learning is most likely to occur when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking and that nurtures positive attitudes and sustained effort; and
- learning is most effective when standards of expectation are made clear with on-going assessment and feedback.

Students are curious, active learners with individual interests, abilities and needs. They come to classrooms with varying knowledge, life experiences and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and aspirations.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best developed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding students benefit from working with a variety of materials, tools and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial and symbolic representations of mathematics. The learning environment should value, respect and address all students' experiences and ways of thinking, so that students are comfortable taking intellectual risks, asking questions and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals for Mathematically Literate Students

The main goals of mathematics education are to prepare students to:
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- commit themselves to lifelong learning
- become mathematically literate adults, using mathematics to contribute to society.

Students who have met these goals will:
- gain understanding and appreciation of the contributions of mathematics as a science, philosophy and art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through:

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history.

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations and engage in reflective practices.

Teachers, students and parents need to recognize the relationship between the affective and cognitive domains, and attempt to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success, and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Diverse Cultural Perspectives

Students come from a diversity of cultures, have a diversity of experiences and attend schools in a variety of settings including urban, rural and isolated communities. To address the diversity of knowledge, cultures, communication styles, skills, attitudes, experiences and learning styles of students, a variety of teaching and assessment strategies are required in the classroom. These strategies must go beyond the incidental inclusion of topics and objects unique to a particular culture.

For many First Nations students, studies have shown a more holistic worldview of the environment in which they live (Banks and Banks 1993). This means that students look for connections and learn best when mathematics is contextualized and not taught as discrete components. Traditionally in Indigenous culture, learning takes place through active participation and little emphasis is placed on the written word. Oral communication along with practical applications and experiences are important to student learning and understanding. It is important that teachers understand and respond to both verbal and non-verbal cues to optimize student learning and mathematical understandings.

Instructional strategies appropriate for a given cultural or other group may not apply to all students from that group, and may apply to students beyond that group. Teaching for diversity will support higher achievement in mathematics for all students.

Adapting to the Needs of All Learners

Teachers must adapt instruction to accommodate differences in student development as they enter school and as they progress, but they must also avoid gender and cultural biases. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom. The reality of individual student differences must not be ignored when making instructional decisions.

As well, teachers must understand and design instruction to accommodate differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Designing classroom activities to support a variety of learning styles must also be reflected in assessment strategies.

Universal Design for Learning

The New Brunswick Department of Education and Early Childhood Development's definition of inclusion states that every child has the right to expect that his or her learning outcomes, instruction, assessment, interventions, accommodations, modifications, supports, adaptations, additional resources and learning environment will be designed to respect his or her learning style, needs and strengths.

Universal Design for Learning is a "…framework for guiding educational practice that provides flexibility in the ways information is presented, in the ways students respond or demonstrate knowledge and skills, and in the ways students are engaged." It also "...reduces barriers in instruction, provides appropriate accommodations, supports, and challenges, and maintains high achievement expectations for all students, including students with disabilities and students who are limited English proficient" (CAST, 2011).

In an effort to build on the established practice of differentiation in education, the Department of Education and Early Childhood Development supports *Universal Design for Learning* for all students. New Brunswick curricula are created with universal design for learning principles in mind. Outcomes are written so that students may access and represent their learning in a variety of ways, through a variety of modes. Three tenets of universal design inform the design of this curriculum. Teachers are encouraged to follow these principles as they plan and evaluate learning experiences for their students:

- **Multiple means of representation:** provide diverse learners options for acquiring information and knowledge
- **Multiple means of action and expression:** provide learners options for demonstrating what they know
- **Multiple means of engagement:** tap into learners' interests, offer appropriate challenges, and increase motivation

For further information on *Universal Design for Learning*, view online information at http://www.cast.org/.

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in literacy, science, social studies, music, art, and physical education.

NATURE OF MATHEMATICS

Mathematics is one way of trying to understand, interpret and describe our world. There are a number of components that define the nature of mathematics and these are woven throughout this document. These components include: **change**, **constancy**, **number sense**, **patterns**, **relationships**, **spatial sense** and **uncertainty**.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as:
• skip counting by 2s, starting from 4
• an arithmetic sequence, with first term 4 and a common difference of 2
• a linear function with a discrete domain
(Steen, 1990, p. 184).

Students need to learn that new concepts of mathematics as well as changes to already learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms stability, conservation, equilibrium, steady state and symmetry (AAAS–Benchmarks, 1993, p. 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include:
• the area of a rectangular region is the same regardless of the methods used to determine the solution
• the sum of the interior angles of any triangle is 180°
• the theoretical probability of flipping a coin and getting heads is 0.5.

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems involving constant rates of change, lines with constant slope, direct variation situations or the angle sums of polygons.

Many important properties in mathematics do not change when conditions change. Examples of constancy include:
• the conservation of equality in solving equations
• the sum of the interior angles of any triangle
• the theoretical probability of an event.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education, 2000, p. 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and when students use benchmarks and referents. This results in students who are computationally fluent and flexible with numbers and who have intuition about numbers. Evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing mathematically rich tasks that allow students to make connections.

Patterns

Mathematics is about recognizing, describing and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the
equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

ASSESSMENT

Ongoing, interactive assessment (*formative assessment*) is essential to effective teaching and learning. Research has shown that formative assessment practices produce significant and often substantial learning gains, close achievement gaps and build students' ability to learn new skills (Black & William, 1998, OECD, 2006). Student involvement in assessment promotes learning. Interactive assessment, and encouraging self-assessment, allows students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes:
- providing clear goals, targets and learning outcomes
- using exemplars, rubrics and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning (Davies, 2000)

Formative assessment practices act as the scaffolding for learning which, only then, can be measured through summative assessment. *Summative assessment,* or assessment *of* learning, tracks student progress, informs instructional programming and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning and produce achievement gains.

Student assessment should:
- align with curriculum outcomes
- use clear and helpful criteria
- promote student involvement in learning mathematics during and after the assessment experience
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction
  (adapted from: NCTM, Mathematics Assessment: A practical handbook, 2001, p.22)

Work Samples
- math journals
- portfolios
- drawings, charts, tables and graphs
- individual and classroom assessment
- pencil-and-paper tests

Rubrics
- constructed response
- generic rubrics
- task-specific rubrics
- questioning

Surveys
- attitude
- interest
- parent questionnaires

Assessing Mathematics Development in a Balanced Manner

Observations
- planned (formal)
- unplanned (informal)
- read aloud (literature with math focus)
- shared and guided math activities
- performance tasks
- individual conferences
- anecdotal records
- checklists
- interactive activities

Self-Assessment
- personal reflection and evaluation

Math Conferences
- individual
- group
- teacher-initiated
- child-initiated

CONCEPTUAL FRAMEWORK FOR 10-12 MATHEMATICS

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.

GRADE	10	11	12
TOPICS			
The topics of study vary in the courses for grades 10–12 mathematics. Topics in the pathways include: • *Algebra* • *Financial Mathematics* • *Geometry* • *Logical Reasoning* • *Mathematics Research Project* • *Measurement* • *Number* • *Permutations, Combinations and Binomial Theorem* • *Probability* • *Relations and Functions* • *Statistics* • *Trigonometry*	**GENERAL OUTCOMES** **SPECIFIC OUTCOMES** **ACHIEVEMENT INDICATORS**		

NATURE OF MATHEMATICS

Change
Constancy
Number Sense
Patterns
Relationships
Spatial Sense
Uncertainty

MATHEMATICAL PROCESSES:
Communication, Connections, Mental Mathematics and Estimation,
Problem Solving, Reasoning, Technology, Visualization

MATHEMATICAL PROCESSES

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics. Students are expected to:
- communicate in order to learn and express their understanding of mathematics (Communications: C)
- develop and apply new mathematical knowledge through problem solving (Problem Solving: PS)
- connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines (Connections: CN)
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation: ME)
- select and use technologies as tools for learning and solving problems (Technology: T)
- develop visualization skills to assist in processing information, making connections and solving problems (Visualization: V).
- develop mathematical reasoning (Reasoning: R)

The New Brunswick Curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning.

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics.

Communication is important in clarifying, reinforcing and modifying ideas, knowledge, attitudes and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology.

Communication can help students make connections among concrete, pictorial, symbolic, verbal, written and mental representations of mathematical ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type, *How would you...?* or *How could you ...?,* the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learnings in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative and innovative solutions. Creating an environment where students openly look for, and engage in, finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students can begin to view mathematics as useful, relevant and integrated.

Learning mathematics within contexts and making connections relevant to learners can validate past experiences, and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"*Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding… Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*" (Caine and Caine, 1991, p. 5).

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external memory aids.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental mathematics" (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "*become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving*" (Rubenstein, 2001).

Mental mathematics "*provides a cornerstone for all estimation processes offering a variety of alternate algorithms and non-standard techniques for finding answers*" (Hope, 1988).

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students

need to know how, when and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Technology [T]

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures and solve problems.

Calculators and computers can be used to:
• explore and demonstrate mathematical relationships and patterns
• organize and display data
• generate and test inductive conjectures
• extrapolate and interpolate
• assist with calculation procedures as part of solving problems
• increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
• reinforce the learning of basic facts
• develop personal procedures for mathematical operations
• model situations
• develop number and spatial sense.

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong, 1993, p. 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills.
Measurement sense includes the ability to determine when to measure and when to estimate and involves knowledge of several estimation strategies (Shaw and Cliatt, 1989, p. 150).

Visualization is fostered through the use of concrete materials, technology and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking.

Questions that challenge students to think, analyze and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, *Why do you believe that's true/correct?* or *What would happen if ….*

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

ESSENTIAL GRADUATION LEARNINGS

Graduates from the public schools of Atlantic Canada will be able to demonstrate knowledge, skills, and attitudes in the following essential graduation learnings. These learnings are supported through the outcomes described in this curriculum document.

Aesthetic Expression
Graduates will be able to respond with critical awareness to various forms of the arts and be able to express themselves through the arts.

Citizenship
Graduates will be able to assess social, cultural, economic, and environmental interdependence in a local and global context.

Communication
Graduates will be able to use the listening, viewing, speaking, reading and writing modes of language(s) as well as mathematical and scientific concepts and symbols to think, learn, and communicate effectively.

Personal Development
Graduates will be able to continue to learn and to pursue an active, healthy lifestyle.

Problem Solving
Graduates will be able to use the strategies and processes needed to solve a wide variety of problems, including those requiring language, mathematical, and scientific concepts.

Technological Competence
Graduates will be able to use a variety of technologies, demonstrate an understanding of technological applications, and apply appropriate technologies for solving problems

PATHWAYS AND TOPICS

The Common Curriculum Framework for Grades 10–12 Mathematics on which the New Brunswick Grades 10-12 Mathematics curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics*. In New Brunswick all Grade 10 students share a common curriculum covered in two courses: *Geometry, Measurement and Finance 10* and *Number, Relations and Functions 10*.   Starting in Grade 11, three pathways are available: *Finance and Workplace*, *Foundations of Mathematics*, and *Pre-Calculus*.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. Students are encouraged to cross pathways to follow their interests and to keep their options open. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all three pathways are to provide prerequisite attitudes, knowledge, skills and understandings for specific post-secondary programs or direct entry into the work force. All three pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of each pathway has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* and on consultations with mathematics teachers.

Financial and Workplace Mathematics
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, statistics and probability.

Foundations of Mathematics
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of theoretical calculus. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, statistics and probability.

Pre-calculus
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical

calculus. Students develop a function tool kit including quadratic, polynomial, absolute value, radical, rational, exponential, logarithmic and trigonometric functions. They also explore systems of equations and inequalities, degrees and radians, the unit circle, identities, limits, derivatives of functions and their applications, and integrals.

Outcomes and Achievement Indicators

The New Brunswick Curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes and achievement indicators.

General Curriculum Outcomes (GCO) are overarching statements about what students are expected to learn in each course.

Specific Curriculum Outcomes (SCO) are statements that identify the specific knowledge, skills and understanding that student are required to attain by the end of a given course. The word *including* indicates that any ensuing items must be addressed to fully meet the learning outcome. The phrase *such as* indicates that the ensuing items are provided for clarification and are not requirements that must be addressed to fully meet the learning outcome. The word *and* used in an outcome indicates that both ideas must be addressed to fully meet the learning outcome, although not necessarily at the same time or in the same question.

Achievement indicators are samples of how students may demonstrate their achievement of the goals of a specific outcome. The range of samples provided is meant to reflect the scope of the specific outcome. The word *and* used in an achievement indicator implies that both ideas should be addressed at the same time or in the same question.

Instructional Focus

Each pathway in *The Common Curriculum Framework for Grades 10–12 Mathematics* is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful. Teachers should consider the following points when planning for instruction and assessment.
- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches, and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning, assessment as learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.

Mathematics K-9

Grade 10
- *2 x 90 hr courses; required to pass both*
- *May be taken in any order or in the same semester*

Geometry, Measurement and Finance 10

Number, Relations and Functions 10

Grade 11
- *3 x 90 hr courses offered in 3 pathways*
- *Students are required to pass at least one of "Financial and Workplace Mathematics 11" or "Foundations of Mathematics 11".*
- *Pre-requisite Grade 10 course(s) must be passed before taking Grade 11 courses.*

Financial and Workplace Mathematics 110

Pre-requisite: Geometry, Measurement and Finance 10

Foundations of Mathematics 110

Pre-requisites: Geometry, Measurement and Finance 1 and Number, Relations and Functions 10

Pre-Calculus 110

Pre-requisite or Co-requisite: Foundations of Mathematics 110

Grade 12
- *5 x 90 hr courses offered in 3 pathways*
- *Pre-requisite Grade 11 or Grade 12 course must be passed before taking Grade 12 courses.*

Financial and Workplace Mathematics 120

Pre-requisite: Financial and Workplace Mathematics 110 or Foundations of Mathematics 110

Foundations of Mathematics 120

Pre-requisite: Foundations of Mathematics 110

Pre-Calculus A 120

Pre-requisite: Pre-Calculus 110

Pre-Calculus B 120

Pre-requisite or Co-requisite: Pre-Calculus A 120

Calculus 120

Pre-requisites: Pre-Calculus A 120 and Pre-Calculus B 120

SUMMARY

The Conceptual Framework for Grades 10–12 Mathematics describes the nature of mathematics, the mathematical processes, the pathways and topics, and the role of outcomes and achievement indicators in grades 10–12 mathematics. Activities that take place in the mathematics classroom should be based on a problem-solving approach that incorporates the mathematical processes and leads students to an understanding of the nature of mathematics.

CURRICULUM DOCUMENT FORMAT

This guide presents the mathematics curriculum by grade level so that a teacher may readily view the scope of the outcomes which students are expected to meet during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings at a particular grade level are part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes (GCOs).

The heading of each page gives the General Curriculum Outcome (GCO), and Specific Curriculum Outcome (SCO). The key for the mathematical processes follows. A Scope and Sequence is then provided which relates the SCO to previous and next grade SCO's. For each SCO, Elaboration, Achievement Indicators, Suggested Instructional Strategies, and Suggested Activities for Instruction and Assessment are provided. For each section, the *Guiding Questions* should be considered.

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Mathematical Processes

[C] Communication Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Math **[T]** and Estimation

Scope and Sequence

Previous Grade or Course SCO's	Current Grade SCO	Following Grade or Course SCO's

Elaboration

Describes the "big ideas" to be learned and how they relate to work in previous Grades

Guiding Questions:
• What do I want my students to learn?
• What do I want my students to understand and be able to do?

Achievement Indicators

Describes observable indicators of whether students have met the specific outcome

Guiding Questions:
• *What evidence will I look for to know that learning has occurred?*
• *What should students demonstrate to show their understanding of the mathematical concepts and skills?*

GCO: General Curriculum Outcome
SCO: Specific Curriculum Outcome

Suggested Instructional Strategies

General approach and strategies suggested for teaching this outcome

Guiding Questions
• What learning opportunities and experiences should I provide to promote learning of the outcomes and permit students to demonstrate their learning?
• What teaching strategies and resources should I use?
• How will I meet the diverse learning needs of my students?

Suggested Activities for Instruction and Assessment

Some suggestions of specific activities and questions that can be used for both instruction and assessment.

Guiding Questions
• What are the most appropriate methods and activities for assessing student learning?
• How will I align my assessment strategies with my teaching strategies?

Guiding Questions
• What conclusions can be made from assessment information?
• How effective have instructional approaches been?
• What are the next steps in instruction?

Pre-Calculus 110

Specific Curriculum Outcomes

SCO	**AN1: Demonstrate an understanding of the absolute value of real numbers.** [R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Algebra and Number

AN1: Demonstrate an understanding of the absolute value of real numbers.

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
AN2: Demonstrate an understanding of irrational numbers by representing, identifying, simplifying and ordering irrational numbers. *(NRF10)*	**AN1:** Demonstrate an understanding of the absolute value of real numbers.	**RF8:** Assemble a function toolkit. *(PC12B)*

ELABORATION

This will be the students first introduction to the concept of **absolute value**. Students should be introduced to the concept of absolute value as the distance of any real number from 0 on a number line. Absolute value can also be referred to as the magnitude of the number.

For the general case, both $-a$ (negative a) and $+a$ (positive a) are a units from zero so have an absolute value of a, represented symbolically as $|-a| = |a| = a,\ where\ a \in R$. For example: (-5) is 5 units from 0, as is $(+5)$, so both have an absolute value of 5. Symbolically this is represented as $|-5| = |5| = 5$. Also, the square root of a square is always positive, $\sqrt{x^2} = |x|$, and is called the principal square root.

The distance between two points on a number line, a and b, can be found by taking the absolute value of the difference between a and b, where $|a - b| = |b - a|$ and $a\ and\ b \in R$.

Absolute value signs are considered as brackets for order of operations.

ACHIEVEMENT INDICATORS

- Determine the distance of two real numbers in the form $\pm\, a, a \in R$, from 0 on a number line, and relate this to the absolute value of a $(|a|)$.

- Determine the absolute value of a positive or negative real number.

- Explain, using examples, how distance between two points on a number line can be expressed in terms of absolute value.

- Determine the absolute value of a numerical expression.

- Compare and order the absolute values of real numbers in a given set.

SCO   **AN1: Demonstrate an understanding of the absolute value of real numbers.** [R, V]
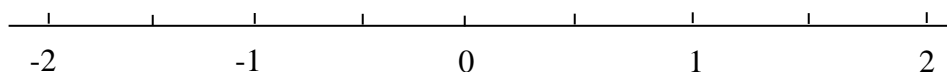
Suggested Instructional Strategies

- Have students place positive and negative numbers on a number line and determine the distance each number is from 0. This distance represents the absolute value.

- Have students order real numbers in various forms (decimal, fractions, integers, mixed numbers) including absolute values.

- Students could follow a particular stock (i.e. price of gold) and look for the volatility (how active) of the stock. Stock prices can be found online on the Toronto Stock Exchange.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Place the following numbers on the number line below:

i) $A(0.7)$, $B(-1.4)$, $C\left(-\frac{2}{3}\right)$, $D\left(-2\frac{1}{6}\right)$, $E\left(1\frac{1}{5}\right)$

-2		-1	0	1	2

Answer:

$-2\frac{1}{6}$	-1.4	$-\frac{2}{3}$		0.7	$1\frac{1}{5}$
-2		-1	0	1	2

ii) Determine the absolute value of each number.   *Answer*: 0.7, 1.4, $\frac{2}{3}$, $2\frac{1}{6}$, $1\frac{1}{5}$

iii) Determine the distance between B and E.   *Answer*: $|B - E| = \left|-1.4 - 1\frac{1}{5}\right| = 2.6$

iv) Determine the distance between C and D.   *Answer*: $\left|-\frac{2}{3} - \left(-2\frac{1}{6}\right)\right| = 1\frac{1}{2}$

Q Determine the value of: $7|0.4 - 5| + |(-2)^3|$ *Answer*: 40.2

Q Order the following from least to greatest:

-2.5, $\left|\frac{-16}{7}\right|$, -2, $\left|-2\frac{6}{9}\right|$, $|-2.09|$, $-2\frac{3}{5}$ *Answer*: $-2\frac{3}{5}, -2.5, -2, |-2.09|, \left|\frac{-16}{7}\right|, \left|-2\frac{6}{9}\right|$

SCO	**AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.** [CN, ME, PS, R, T]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

Scope and Sequence of Outcomes:

| Grade Ten	Grade Eleven	Grade Twelve
AN2: Demonstrate an understanding of irrational numbers by representing, identifying, simplifying and ordering irrational numbers. *(NRF10)* **AN3:** Demonstrate an understanding of powers with integral and rational exponents. *(NRF10)*	**AN2:** Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.	**RF5:** Graph and analyze radical functions (limited to functions involving one radical). *(PC12B)*

ELABORATION

In grade 10 students have represented, identified, simplified and ordered radical expressions as well as expressing entire radicals as mixed radicals and vice versa. In this outcome students will define the **principal square root** and define radicands as ≥ 0, , In this outcome, students will practice performing arithmetic operations (adding, subtracting, multiplying and dividing) on radical expressions. This will include **numerical radicands** and be extended to **variable radicands**. They will also learn how to **rationalize a denominator**.

To simplify radical expressions, adding or subtracting radicals can only be done when the terms have the same radicand. For a numerical example: $\sqrt{3} - 2\sqrt{5} + 4\sqrt{3} + 3\sqrt{5} = 5\sqrt{3} + \sqrt{5}$. As an example with variable radicands: $3\sqrt{x^5} - 2x^2\sqrt{x} = 3x^2\sqrt{x} - 2x^2\sqrt{x} = x^2\sqrt{x}$.

The square root of any real number is positive and is called the **principal square root**. It is expressed symbolically as $\sqrt{x^2} = |x|$. If the square root is already present in the equation, then the solution is the principal square root. However, if in solving $x^2 = 49$, the square root is introduced into the equation, you must consider both the positive and negative solutions, 7 and -7.

To illustrate this, $y = \sqrt{x}$ is shown below as a solid curve, and only includes positive values. This will be this first time that students have seen $y = \sqrt{x}$ graphed. Point $(49,7)$ is shown as the intersection of $y = \sqrt{x}$ and $y = 7$. If negative points such as $(49, -7)$ were included, shown below as the intersection of $y = -\sqrt{x}$ and $y = -7$, the graph would no longer be a function (no longer pass the vertical line test).
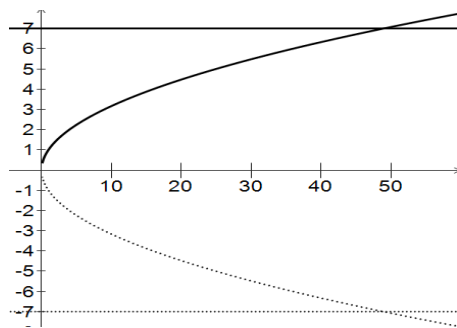
SCO   **AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.** [CN, ME, PS, R, T]

For radical expressions that are rational, the radicand must be greater than or equal to zero. When variables are part of the radicand, they must be defined to ensure this is true. For example, for $\sqrt{x-2}$, the radicand, $x-2$, must be equal to or greater than zero, so solutions are restricted to $x \geq 2$. For $x + 2\sqrt{x^2(x-1)} = 9$ solutions are restricted to $x \geq 1$.

If a radical appears in the denominator, the expression should be rationalized to produce a rational denominator. This is done by multiplying both the numerator and denominator by the radical in the denominator. In effect the expression is multiplied by 1, and therefore the value of the expression is unchanged. For example, the expression below, which has a monomial denominator, is rationalized as follows: $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3}$

If the denominator is a binomial, the **conjugate** of the binomial must be used to rationalize the denominator. The conjugate of a binomial has the alternate operation, subtraction or addition, between the terms. For example, the conjugate of $(2a - 3)$ is $(2a + 3)$. Students have seen the conjugate in grade 10 in relation to the difference of squares. To rationalize a binomial denominator, the numerator and denominator are both multiplied by the conjugate of the denominator, multiplying by the equivalent of 1 and thereby leaving the value of the expression unchanged. For example $\frac{3}{\sqrt{5}-1} = \frac{3}{\sqrt{5}-1}\left(\frac{\sqrt{5}+1}{\sqrt{5}+1}\right) = \frac{3\sqrt{5}+3}{4}$

ACHIEVEMENT INDICATORS

- Compare and order radical expressions with numerical radicands in a given set.

- Express an entire radical, as a mixed radical.

- Express a mixed radical an entire radical.

- Perform one or more operations to simplify radical expressions with numerical or variable radicands (maximum index of 2).

- Rationalize the denominator of a rational expression with monomial or binomial denominators.

- Describe the relationship between rationalizing a binomial denominator of a rational expression and the product of the factors of a difference of squares expression.

- Explain, using examples, that $(-x)^2 = x^2$, $\sqrt{x^2} = |x|$, and $\sqrt{x^2} \neq \pm x$ $(e.g., \sqrt{9} \neq \pm 3)$

- Identify the values of the variable for which a given radical expression is defined.

- Solve both equality and inequality problems that involve radical expressions.

- Solve a problem that involves radical expressions.

SCO   **AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.** [CN, ME, PS, R, T]

Suggested Instructional Strategies

- Provide each student with a radical expression written on a card and have them arrange themselves in a line in order of value.

- Provide students with blank cards and a list of entire radicals. On each one card have them write the entire radical, and on another write the corresponding mixed radical. Use these cards in matching games such as "memory", "go fish", and "find your match".

- When identifying the possible values of a variable for a given radical expression, have students use graphing technology to graph the expressions and investigate where the corresponding function is defined.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Order from least to greatest: $\sqrt{5},\ 2\sqrt{3}, -\sqrt{3},\ \sqrt{2^2},\ \frac{\sqrt{9}}{3}$

Q Match the following equivalent expressions:

A. $\sqrt{a^9}$ i. $\sqrt{98}$ *Answers*: A = iv, B = ii, C = i, D = v, E = iii

B. $\sqrt{32t^7}$ ii. $4t^3\sqrt{2t}$

C. $7\sqrt{2}$ iii. $\dfrac{2}{\sqrt{5}}$

D. $\sqrt{300}$ iv. $a^4\sqrt{a}$

E. $\dfrac{2\sqrt{5}}{5}$ v. $10\sqrt{3}$

Q Write each of the following radicals in the alternate form: if entire, write as mixed, if mixed, write as entire.

a) $2\sqrt{a+1}$ b) $\sqrt{2d^2}$ c) $\sqrt{90t^5}$ d) $5x\sqrt{6x}$ e) $3\sqrt[3]{y^2-2}$

Answers: a) $\sqrt{4a+4}$ b) $d\sqrt{2}, d \geq 0$ c) $3t^2\sqrt{10t}$ d) $\sqrt{150x^3}$ e) $\sqrt[3]{27y^2-54}$

Q Simplify the following:

a) $\sqrt[3]{54x^5y^{12}}$ b) $\sqrt[4]{50\,000}$ c) $\sqrt{32t^2d^4}$ d) $\sqrt{1125x^9y^5}$

Answers: a) $3xy^4\sqrt[3]{2x^2}$ b) $10\sqrt[4]{5}$ c) $16td^2\sqrt{2}, t \geq 0$ d) $15x^4y^2\sqrt{5xy}$

Q Simplify the following expressions:

a) $\sqrt{3}(2\sqrt{6}-4\sqrt{5})$ b) $\dfrac{-12\sqrt{22}}{4\sqrt{11}}$ c) $5\sqrt{12}-2\sqrt{3}+\sqrt{20}-2\sqrt{125}$ d) $(4\sqrt{2}+3)(\sqrt{7}-5\sqrt{14})$

Answers: a) $6\sqrt{2}-4\sqrt{15}$ b) $-3\sqrt{2}$ c) $8\sqrt{3}-8\sqrt{5}$ d) $-11\sqrt{14}-37\sqrt{7}$

Q Determine the restrictions on the values for x in the following radical expressions.

a) $\sqrt{2x+1}$ b) $\dfrac{3}{\sqrt{x-9}}$ c) $\dfrac{4-\sqrt{6x}}{8}$ *Answers*: a) $x \geq -\frac{1}{2}$ b) $x > 9$ c) $x \geq 0$

SCO **AN2: Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.** [CN, ME, PS, R, T]

Q Philip determines the restrictions on the values for x in the radical expression $\sqrt{3 - 7x}$ as
follows:
$$3 - 7x > 0$$
$$-7x > -3$$
$$x > \frac{3}{7}$$

a) Explain and correct the error(s).
b) Explain why radical expressions with variables in the radicand have restrictions.

Answers: a) 1) should be $\geq$ as $\sqrt{0}$ is permitted 2) dividing by $-$'ve reverses the sign to $\leq$
b) the radicand cannot be a negative number (in the Real number system)

Q Simplify each expression:

a) $\dfrac{\sqrt{12x^2}}{\sqrt{3x} + 2}$ b) $\dfrac{4\sqrt{5}}{3\sqrt{2n}}$ c) $\dfrac{11}{\sqrt{5} + 3}$

Answers: $a)$ $\dfrac{6x\sqrt{x} - 4x\sqrt{3}}{3x - 4}, x \geq 0$ $b)$ $\dfrac{4\sqrt{10n}}{6n}$ $c)$ $\dfrac{33 - 11\sqrt{5}}{4}$

SCO   **AN3: Solve problems that involve radical equations (limited to square roots).** [C, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

AN3: Solve problems that involve radical equations (limited to square roots).

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
AN2: Demonstrate an understanding of irrational numbers by representing, identifying, simplifying and ordering irrational numbers. *(NRF10)* **AN3:** Demonstrate an understanding of powers with integral and rational exponents. *(NRF10)* **AN4:** Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. *(NRF10)*	**AN3:** Solve problems that involve radical equations (limited to square roots).	**RF5:** Graph and analyze radical functions (limited to functions involving one radical). *(PC12B)*

ELABORATION

In the last outcome students worked with radical expressions. In this outcome they will use these skills to solve problems involving radical equations, ensuring that solutions to radical equations are restricted to those that satisfy the original equation: the radicand must be ≥ 0, and the solution(s) must be verified in the original equation.

To solve a radical equation, if a radical is already present it must first be isolated on one side, and then both sides are squared. If this results in a quadratic equation it can be solved using factoring techniques, or by using the quadratic formula as explored in *Foundations of Mathematics 110*.

All solutions must meet both the restriction that the radicand be ≥ 0, and a**lso must** satisfy the original equation, verified by substituting the solution back into the original equation. Because radical equations are solved by squaring both sides, answers are sometimes found that will not verify. These are called **extraneous roots**. They occur because only the principal square root can be used in the verification process. This is shown in the following example

Solve for x	Verify $x = 5$	Verify $x = 17$
$x + 2\sqrt{(x-1)} = 9$ (restriction $x \geq 1$)	$x + 2\sqrt{x-1} = 9$	$x + 2\sqrt{x-1} = 9$
$2\sqrt{(x-1)} = 9 - x$	*Left Side*	*Left Side*
$\left(2\sqrt{x-1}\right)^2 = (9-x)^2$	$5 + 2\sqrt{5-1}$	$17 + 2\sqrt{17-1}$
$4x - 4 = 81 - 18x + x^2$	$= 5 + 2(2)$	$= 17 + 2(4)$
$0 = (x-5)(x-17)$	$= 9$	$= 25$
$x = \{5, 17\}$	*Right Side*	*Right Side*
Both $5 \; and \; 17 \geq 1$ so both	$= 9$	$= 9$
roots satisfy the restriction.	*Left side $=$ Right side*	*Left side $\neq$ Right side*
	$\therefore 5 \; is \; a \; valid \; solution$	$\therefore 17 \; is \; an \; extraneous \; root$

SCO   **AN3: Solve problems that involve radical equations (limited to square roots).**   [C, PS, R]

Extraneous roots should not be confused with answers that don't make sense within the context of a word problem. For example:

The depth of a particular submarine can be expressed as a function of time. The formula is $d = t^2 + 9t - 36$. When will the submarine surface next? Solving for t with $d = 0$ (depth is 0 at surface):

$0 = t^2 + 9t - 36$
$0 = (t - 3)(t + 12)$
$t = \{3, -12\}$

Even though verification shows -12 to be valid, a time of negative 12 seconds is not possible so this answer is incorrect.

ACHIEVEMENT INDICATORS

- Determine any restrictions on values for the variable in a radical equation.

- Determine the roots of a radical equation algebraically, and explain the process used to solve the equation.

- Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation.

- Explain why some roots determined in solving a radical equation algebraically are extraneous.

- Solve problems by modeling a situation using a radical equation

Suggested Instructional Strategies

- When introducing the restriction in radical expressions, that the radicand cannot be negative, move from simpler to more complicated algebraic expressions. For example:

  for $\sqrt{x}$, $x \geq 0$ for $\sqrt{x-2}$, $x \geq 2$ for $\sqrt{5x+17}$, $x \geq -3\frac{2}{5}$ for $\sqrt{(-2x-19)}$, $x \leq -9\frac{1}{2}$

- Challenge students with a variety of questions including solving for a variable in an equation with only one radical, solving for a variable in an equation with two radicals, and solving word problems involving a radical.

- As an extension, use a graphics calculator (or a table of values) to graph extraneous roots. For example: for $x + 3 = \sqrt{x+2} + 7$, two roots are found, $x = \{2, 7\}$, but validating shows that 2 is an extraneous root. To illustrate that $x = 7$ is the correct solution, graph $y = x + 3$ and $y = \sqrt{x+2} + 7$ to show their intersection at $x = 7$. The extraneous root at $x = 2$ does not intersect with $y = \sqrt{x+2} + 7$, but can only be found at the intersection of $y = x + 3$ and $y = -\sqrt{x+2} + 7$.

SCO   **AN3: Solve problems that involve radical equations (limited to square roots).**   [C, PS, R]
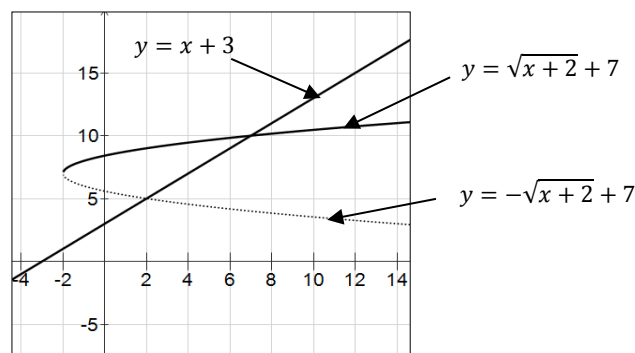
Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Solve for x.

     a) $4\sqrt{x+1} - 5 = 7$ b) $\sqrt{3x} + 4 = \sqrt{2x+1} + 5$

Answers:
a) restriction $x \geq -1$, $x = 8$, meets the restriction   substituting: $LS = RS = 7 \therefore x = 8$ is a valid solution
b) restrictions $x \geq 0$, $x \geq -\frac{1}{2}$
  solving for x, $x = 0$, $x = 12$
  when $x = 0$, $LS = 4$ and $RS = 6$, $LS \neq RS \therefore x = 0$ is an extraneous root
  when $x = 12$, $LS = RS = 10$, $\therefore 12$ is a valid solution

Q The period (T) of a pendulum can be approximated by the formula $T \sim 2\pi\sqrt{\dfrac{l}{g}}$, where l is the length of the pendulum and g is the gravitational constant. What is the gravitational constant on the moon, knowing that a pendulum of 2.00 meters has a period of 7.02 seconds?

Answer: $7.02 \ sec \cong 2\pi\sqrt{\dfrac{2m}{g}}$ $\therefore$ $g \cong 1.6 \ m/sec$ $\therefore$ *The grav. const. of the moon $\cong 1.6 \ m/sec$*

(Note to teacher: You are solving for "g". In this equation, physicists actually only use the magnitude of the gravitational acceleration and ignore the negative sign that would indicate the fact that it is a "downward" acceleration. With this, we don't need to worry about the restriction where the radicand is negative but we will have to consider the restriction where g is not equal to zero because it is a denominator).

SCO	**AN4: Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, ME, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

AN4: Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
AN4: Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. *(NRF10)* **AN5:** Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. *(NRF10)*	**AN4:** Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).	**RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). *(PC12B)*

ELABORATION

For the next few outcomes students will be drawing on many of the skills developed in previous courses to express rational expressions in equivalent forms. They will build on their understanding of operations on polynomials as well as make use of their factoring skills of polynomials to simplify rational expressions.

This outcome focuses on developing and comparing strategies for writing equivalent forms of rational expressions that will include those with variables in the numerator and/or denominator.

Values in rational expressions that result in the denominator equalling zero, or non-permissible values, will be identified. These **non-permissible values** are specific values that the solution must not equal, for the solution to be valid.

ACHIEVEMENT INDICATORS

- Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.

- Explain why a given value is non-permissible for a given rational expression.

- Determine the non-permissible values for a rational expression.

- Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.

- Simplify a rational expression.

- Explain why the non-permissible values of a given rational expression and its simplified form are the same.

- Identify and correct errors in a simplification of a rational expression, and explain the reasoning.

SCO   **AN4: Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, ME, R]

Suggested Instructional Strategies

- Review equivalent fractions and relate these skills to writing equivalent rational expressions. Begin with multiplying both numerator and denominator by the same factor, working up to multiplying both numerator and denominator by the same algebraic expression.

  For example: $\frac{1}{3} \times \left(\frac{2}{2}\right) = \frac{2}{6}$ to $\frac{x+3}{5} \times \left(\frac{3}{3}\right) = \frac{3x+9}{15}$ to $\frac{x}{2x-5} \times \frac{(x+1)}{(x+1)} = \frac{x^2+x}{2x^2-3x-5}$

- Review factoring of polynomials as a basis for learning to simplify rational expressions and to determining non-permissible values for variables, so that the denominator will not equal zero.

  For example:

  To solve the following, use factoring to simplify and determine non permissible values.

  $$\frac{4x+12}{2x^2+7x+3} = \frac{4(x+3)}{(2x+1)(x+3)} = \frac{4}{2x+1} \quad x \neq -\frac{1}{2}, -3$$

  Note: both $-\frac{1}{2}$ *and* -3 *are non* $-$ *permissible,*

  even though -3 *gives a rational answer in the simplified expression* $\frac{4}{2x+1}$.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q State the operation and quantity to apply to the numerator and denominator of the first expression to obtain the equivalent second expression.

a) $\frac{3s}{2q}, \frac{3s^2q}{2sq^2}$ b) $\frac{-3(x+3)}{x^2-9}, \frac{-3}{(x-3)}, \ x \neq 3$ c) $\frac{2x-2}{x^2+1}, \frac{2(x^2-x)}{x^3+x}$ d) $\frac{1}{s^2-1}, \frac{s^2+1}{s^4-1}$

Answers: a) $\times \frac{sq}{sq}$ b) $\div \frac{(x+3)}{(x+3)}$ c) $\times \frac{x}{x}$ d) $\times \frac{s^2+1}{s^2-1}$

Q State the non-permissible values and simplify the following rational expressions:

a) $\frac{-2m^3n}{6m^2n^4}$ b) $\frac{2x^2+5x-7}{x^2-2x-3}$ c) $\frac{12x^2-54x+24}{3x^2-3x-36}$

d) $\frac{5x-20}{12-3x}$ e) $\frac{x^2-36}{3x^2-16x-12}$

Answers: a) $\frac{-1m}{3n^3}$ $m \neq 0, n \neq 0$ b) $\frac{(2x+7)(x-1)}{(x-3)(x+1)}$ $x \neq 3, -1$ c) $\frac{2(2x-1)}{x+3}$ $x \neq 4, -3$

d) $\frac{-5}{3}$ $x \neq 4$ e) $\frac{x+6}{3x+2}$ $x \neq -\frac{2}{3}, 6$

Q Sally made a mistake while simplifying the following rational expression. Determine the error(s) and provide the correct solution:

$$\frac{4x^2+8x-140}{25-x^2} = \frac{4(x^2+2x-35)}{25-x^2} = \frac{4(x+7)(x-5)}{(5+x)(5-x)} = \frac{4(x+7)}{5+x}$$

Answer: $\frac{4x^2+8x-140}{25-x^2} = \frac{4(x+7)(x-5)}{-1(x+5)(x-5)} = \frac{-4(x+7)}{x+5}$

Note: Even simplifying incorrectly, would yield the correct non $-$ *permissable values,* $x \neq \pm 5$

Act As students come into class, hand them a rational expression written on a card, and have them find a classmate with a rational expression with at least one identical non-permissible value.

SCO	**AN5: Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, ME, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

AN5: Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
AN4: Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. *(NRF10)* **AN5:** Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. *(NRF10)*	**AN5:** Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).	**RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). *(PC12B)*

ELABORATION

For this outcome students will apply skills learned in previous grades to rational expressions, as related to operations on rational numbers. This will include the sum or difference of rational expressions with and without the same denominator, and the product and quotient of rational expressions. Students will simplify expressions that involve two or more operations, and factor polynomials.

The student's understanding of lowest common multiple (LCM), covered in *Number, Relations and Functions 10*, will be extended to polynomials to find common denominators for addition and subtraction of rational expressions.

Non-permissible values will be determined for denominators of rational expressions, ensuring that denominators do not equal zero. In the case of division, all expressions within the divisor must also be considered when stating non-permissible values, as the numerator of the divisor ends up in the denominator. For example:

$$\frac{2x-4}{x^2+9x+20} \div \frac{x^2+x-6}{x^2+7x+12} = \frac{2(x-2)}{(x+5)(x+4)} \times \frac{(x+4)(x+3)}{(x+3)(x-2)} = \frac{2}{x+5} \qquad x \neq -5, -4, -3, 2$$

ACHIEVEMENT INDICATORS

- Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.

- Determine the non-permissible values when performing operations on rational expressions.

- Determine, in simplified form, the sum or difference of rational expressions with the same denominator.

- Determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors.

- Determine, in simplified form, the product or quotient of rational expressions.

- Simplify an expression that involves two or more operations on rational expressions.

SCO **AN5: Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, ME, R]

Suggested Instructional Strategies

- Review operations with fractions, and factoring of polynomials to build new knowledge on a strong foundation as student develop skills for simplifying rational expressions.

- Explore with students the connection between finding a common denominator with rational numbers to finding a common denominator with rational expressions.

Example 1: The two denominators do not have any common factors therefore the LCM will be the product of the two.

Rational Numbers

$$\frac{4}{5} + \frac{3}{2}$$

$$= \frac{8}{10} + \frac{15}{10}$$

$$= \frac{23}{10}$$

Rational Expressions

$$\frac{5}{x+1} + \frac{3}{x+4}$$

$$= \frac{5}{(x+1)}\frac{(x+4)}{(x+4)} + \frac{3}{(x+4)}\frac{(x+1)}{(x+1)}$$

$$= \frac{(5x+20)+(3x+3)}{(x+1)(x+4)}$$

$$= \frac{8x+23}{(x+1)(x+4)}$$

Example 2: The two denominators have a common factor therefore the LCM will not be the product of the two denominators, but the product of the common factor and the remaining uncommon factors.

Rational Numbers

$$LCM = 2 \times (3 \times 5)$$

$$\frac{5}{6} + \frac{3}{10}$$

$$= \frac{5}{3 \times 2} + \frac{3}{5 \times 2}$$

$$= \frac{5}{3 \times 2} \times \frac{5}{5} + \frac{3}{5 \times 2} \times \frac{3}{3}$$

$$= \frac{25}{30} + \frac{9}{30}$$

$$= \frac{34}{30}$$

Rational Expressions

$$LCM = (x+2) \times [(x-1)(x-2)]$$

$$\frac{5}{x^2+x-2} + \frac{2}{x^2-4}$$

$$= \frac{5}{(x+2)(x-1)} + \frac{2}{(x+2)(x-2)}$$

$$= \frac{5}{(x+2)(x-1)}\frac{(x-2)}{(x-2)} + \frac{2}{(x+2)(x-2)}\frac{(x-1)}{(x-1)}$$

$$= \frac{(5x-10)+(2x-2)}{(x+2)(x-1)(x-2)}$$

$$= \frac{7x-12}{(x+2)(x-1)(x-2)}$$

SCO   **AN5: Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, ME, R]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Simplify the following rational expressions and determine the non-permissible values.

a) $\dfrac{5}{4m} + \dfrac{1}{3m}$

b) $\dfrac{6}{x+5} + \dfrac{x}{x+5}$

c) $\dfrac{2}{(x-2)(x+3)} - \dfrac{5}{x+3}$

d) $\dfrac{x-1}{x^2-x-2} - \dfrac{2x+4}{x^2-7x+10}$

e) $\dfrac{3t^2}{-2y} \div \dfrac{6t^3}{4y^2}$

f) $\dfrac{x^2+7x}{x^2-1} \times \dfrac{x^2+3x+2}{x^2+14x+49}$

g) $\dfrac{x^2+4x+4}{4-x^2} \times \dfrac{2x-4}{x^2-3x-10} \div \dfrac{4x+16}{x^2-25}$

h) $\dfrac{x-3}{x+2} + \dfrac{8x+4}{x^2+2x-35} \times \dfrac{x^2+4x-21}{2x^2-5x-3}$

Answers:

a) $\dfrac{19}{12m}$ $m \neq 0$

b) $\dfrac{6+x}{x+5}$ $x \neq -5$

c) $\dfrac{-5x+12}{(x-2)(x+3)}$ $x \neq -3, 2$

d) $\dfrac{-x^2-12x+1}{(x-2)(x+1)(x-5)}$ $x \neq -1, 2, 5$

e) $\dfrac{-y}{t}$ $t, y \neq 0$

f) $\dfrac{x(x+2)}{(x-1)(x+7)}$ $x \neq \pm 1, -7$

g) $\dfrac{-(x+5)}{2(x-4)}$ $x \neq \pm 2, \pm 5, 4$

h) $\dfrac{x^2-4x+23}{(x+2)(x-5)}$ $x \neq -2, -7, 5, 3, -\dfrac{1}{2}$

SCO   **AN6: Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, PS, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

AN6: Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
AN4: Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. *(NRF10)* **AN5:** Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. *(NRF10)*	**AN6:** Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).	**RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials) *(PC12B)*

ELABORATION

This outcome extends the work with rational expressions in AN5, to working with rational equations. It is intended that the rational equations be limited to those that can be simplified to linear and quadratic equations.

To solve rational equations, the non-permissible values are first determined, and then the denominator is eliminated by multiplying all terms by the lowest common denominator. The resulting linear or quadratic equation can then be solved.

For example, to solve $\frac{m}{m+1} - \frac{3m}{m^2-1} = 4$

Step 1: Factor to determine the non-permissible values.

$$\frac{m}{m+1} - \frac{3m}{(m+1)(m-1)} = 4 \quad m \neq -1, 1 \; (non-permissible \; values)$$

Step 2: Determine, and then multiply each term by the lowest common denominator.

$$\frac{m}{m+1}(m+1)(m-1) \; - \; \frac{3m}{(m+1)(m-1)}(m+1)(m-1) = 4(m+1)(m-1)$$

Step 3: Simplify and solve for the variable.

$$m(m-1) - 3m = 4(m+1)(m-1)$$
$$m^2 - 1 - 3m = 4m^2 - 4$$
$$3m^2 + 4m - 4 = 0$$
$$(3m-2)(m+2) = 0$$

$$\therefore m = \frac{2}{3} \quad or \quad m = -2$$

Neither solution is non-permissible as determined earlier, so both solutions are valid.

For some word problems, students will need to be familiar with the relationship between distance, speed and time $(d = vt)$.

SCO **AN6: Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).** [C, PS, R]

ACHIEVEMENT INDICATORS

- Determine the non-permissible values for the variable in a rational equation.

- Determine the solution to a rational equation algebraically, and explain the process used to solve the equation.

- Explain why a value obtained in solving a rational equation may not be a solution of the equation.

- Solve problems by modeling a situation using a rational equation.

Suggested Instructional Strategies

- Many word problems requiring the use of rational equations will deal with questions that involve moving, such as moving: with or against current, with a head or tail wind, or with an increase or decrease in speed. These situations can be conceptualized with the use of video clips found on sites such as YouTube e.g., Dan Meyer walking up and down an escalator http://blog.mrmeyer.com/?p=7649

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Solve the following rational equations:

a) $\dfrac{3}{a+5} = \dfrac{2}{a+4}$

b) $y - \dfrac{3}{2} = \dfrac{7}{y}$

c) $\dfrac{x}{x-2} + \dfrac{1}{x-4} = \dfrac{2}{x^2-6x+8}$

Answers: $a) \; a = -2 \; (a \neq -5, -4)$ $b) \; y = -2 \; or \; \dfrac{7}{2} \; (y \neq 0)$

$c) \; x = 4 \; or - 1, \; but \; x \neq 4, 2 \; \therefore \; answer \; is \; x = -1$

Q A family on vacation leaves Edmundston, NB and travels 286 km to Fredericton, NB. The family stops for a quick lunch and continues for 133 km on to Sussex, NB along the scenic route at a speed of 15 km/h less than the first part of the trip. If the total driving time took 4 hours, how fast were they travelling on the scenic route?

(Answer: $t = \dfrac{d}{v}$ $Ed'ton - F'ton \; (t_1) = \dfrac{286 \; km}{x}$, $F'ton - Sussex \; (t_2) = \dfrac{133 \; km}{x - 15 \; km}$

$t_1 + t_2 = 4 \; hours \quad \therefore \dfrac{286}{x} + \dfrac{133}{x - 15} = 4, \quad x \neq 0, x \neq 15$

This simplifies to $4x^2 - 479x + 4290 = 0$

Using quadratic formula $x = 110 \; or \; 9.75$

Since x is the faster of the two speeds, and the second speed is $15 \; km/hr \; less, x \neq 9.75.$

Therefore $x = 100 \; km/hr$ and speed on the scenic routs is $110 - 15 = 95 \; km/hr.)$

SCO: **T1. Demonstrate an understanding of angles in standard position [0°to 360°].** [R, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Trigonometry

**T1: Demonstrate an understanding of angles in standard position [0°to 360°].**

Scope and Sequence of Outcomes:

| Grade Ten	Grade Eleven	Grade Twelve
G5: Demonstrate an understanding of angles, including acute, right, obtuse, straight and reflex, by: drawing, replicating and constructing, bisecting, and solving problems. *(GMF 10)*	**T1:** Demonstrate an understanding of angles in standard position [$0°to\ 360°$].	**T1:** Demonstrate an understanding of angles in standard position, expressed in degrees and radians *(PC 12A)*

ELABORATION

A **rotation angle** is formed by rotating an **initial arm** through an angle, θ, about a fixed point called a vertex, to a terminal position called the **terminal arm**.
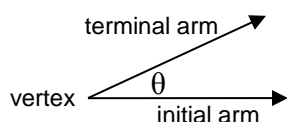
With respect to the coordinate axes, a rotation angle is in **standard position** if the initial arm is on the positive x-axis and the vertex is at the origin.   A positive angle results from a counter-clockwise rotation, and a negative angle results from a clockwise rotation.

The **reference angle** is the positive acute angle that can represent an angle of any measure. Any angle on the coordinate axis has a reference angle between 0° and 90°. The reference angle is always the smallest angle that you can make from the terminal arm with the x-axis.

The x-axis and y-axis divide a plane into four **quadrants**.

Quadrant II
$90°\theta < \theta < 180°$

Quadrant I
$0° < \theta < 90°$

Quadrant III
$180° < \theta < 270°$
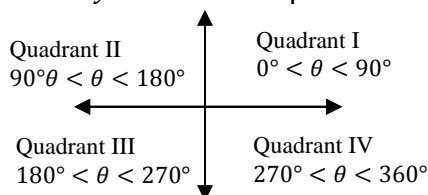
Quadrant IV
$270° < \theta < 360°$

Students should be given ample time to use and become confident in using a coordinate axis, determining reference angles, drawing angles in standard position, and locating the quadrant which contains the terminal arm. They should be able to use the terminology easily and effectively with reference to the coordinate axis.

Any angle on a coordinate axis between 90° and 360° is a reflection of a reference angle in the x or y axis (or both). Students should be able to illustrate this on a coordinate axis and determine the reference angle. For example the reference angle for 210° is 30°, which is a reflection in both the x and the y axis. This concept can be extended to any point (x, y) and its reflection in either the x and y axis (or both)

SCO: **T1. Demonstrate an understanding of angles in standard position [0°to 360°].** [R, V]

ACHIEVEMENT INDICATORS

- Sketch an angle in standard position, given the measure of the angle.

- Determine the reference angle for an angle in standard position.

- Explain, using examples, how to determine the angles from 0° to 360° that have the same reference angle as a given angle.

- Illustrate, using examples, that any angle from 90° to 360° is the reflection in the x-axis and/or the y-axis of its reference angle.

- Determine the quadrant in which a given angle in standard position terminates.

- Draw an angle in standard position given any point $P(x, y)$ on the terminal arm of the angle.

- Illustrate, using examples, that the points $P(x, y)$, $P(-x, y)$, $P(-x, -y)$ and $P(x, -y)$ are points on the terminal sides of angles in standard position that have the same reference angle.

Suggested Instructional Strategies

- To determine the reference angle using reflections in the x and y-axis, have students accurately draw an angle between 90° and 360° on a coordinate grid, then fold the paper on either the x or y axis (or both) until they can determine the reference angle, using a protractor. A pencil should be used so that the graphite will transfer with each fold.

- This same process (paper folding) can be used to illustrate the reflection of point (x, y) in all quadrants and determine that the reference angle formed as a reflection of the point on a terminal arm is the same.

SCO: **T1. Demonstrate an understanding of angles in standard position [0°to 360°].** [R, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Sketch a $310°$ angle.

a) Determine which quadrant contains the terminal arm.

b) Find its reference angle.

c) Find two other angles that have the same reference angle.

Answer: *a*) *quadrant IV*   *b*) $50°$ *c*) $230°, 130°$

Q Mark point $(-5, -2)$ on a coordinate grid.

a) Name the quadrant which contains this point.

b) Find the reference angle for the angle formed by joining this point to the origin.

c) Find two other points that have the same reference angle.

Answers: *a*) *Qudrant III*   *b*) $\tan^{-1}\left(\frac{2}{5}\right) = 22°$ *c*) $(-5,2), (5,-2), (5,2)$

Act Have students explore angles and their reference angles through the following activity.

Step 1 Draw an x and y axis on a sheet of paper, using a ruler.

Step 2 Draw an angle with a terminal arm of $135°$.

Step 3 Using a pencil, shade from the terminal arm to the x axis.

Step 4 Fold the loose leaf on the y axis and press firmly to transfer the lead.

Step 5 Unfold the paper and measure the angle between the initial arm and the terminal arm to give the reference angle (round to the nearest degree).

Extension: Have students repeat this with any angle between $90°$ and $360°$ to determine the reference angle. Students should look for patterns that would help to determine the reference angle without drawing.

Note: This same activity can be repeated for the reflection of any point (x, y).

SCO: **T2: Solve problems, using the three primary trigonometric ratios for angles from 0°
to 360° in standard position.** [C, ME, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

T2: Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

Scope and Sequence of Outcomes:

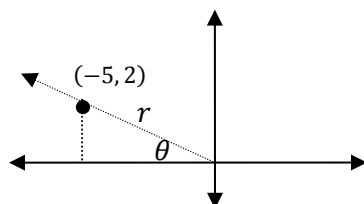| Grade Ten	Grade Eleven	Grade Twelve
G2: Demonstrate an understanding of the Pythagorean theorem by: identifying situations that involve right triangles, verifying the formula, applying the formula, and solving problems. *(GMF10)* **G3:** Demonstrate an understanding of primary trigonometric ratios (sine, cosine, tangent) by: applying similarity to right triangles, generalizing patterns from similar right triangles, applying the primary trigonometric ratios, and solving problems. *(GMF10)* **RF8:** Solve problems that involve the distance between two points and the midpoint of a line segment. *(NRF10)*	**T2:** Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T2:** Develop and apply the equation of the unit circle. *(PC 12A)* **T3:** Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees. *(PC12A)*

ELABORATION

In Grade 10 students solved problems using the Pythagorean theorem, the three primary trigonometric ratios, and the distance formula. This outcome extends the students' prior knowledge of trigonometric ratios to points located on the coordinate plane. They will solve problems for angles from 0° to 360°. Angles greater than 360° will be addressed in a later course.

Students will identify the x and y coordinates of a point on the terminal arm of an angle θ, as the length of sides adjacent to, and opposite angle θ for a given triangle. They will use the Pythagorean theorem or distance formula, and the x and y coordinates to determine r, the length of the terminal arm to that point. Using the x, y and r values they will then be able to determine the exact value of the sine, cosine or tangent for a given angle, θ. For example:

$(x, y) = (-5, 2)$

$\sqrt{(-5)^2 + (2)^2} = \sqrt{29} = r$ $\qquad$ $\sin\theta = \frac{2}{\sqrt{29}} \left(or \frac{2\sqrt{29}}{29}\right)$ $\quad$ $\cos\theta = \frac{-5}{\sqrt{29}} \left(or \frac{-5\sqrt{29}}{29}\right)$ $\quad$ $tan\theta = \frac{2}{-5}$


Establishing a link between trigonometric ratios and the coordinate plane will be foundational to an understanding of the unit circle, which will be covered in a later course.

Students will determine the exact values of the trigonometric value for each of the **special angles,** 30°, 45°, *and* 60° and the special cases of 0°, 90°, 180°, 270°, 360°, without the use of technology.
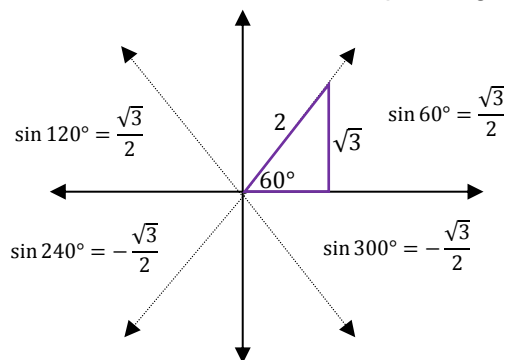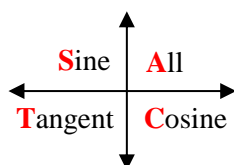
SCO: **T2: Solve problems, using the three primary trigonometric ratios for angles from 0°
to 360° in standard position.** [C, ME, PS, R, T, V]

Students will develop an understanding that the absolute value of the sine, cosine or tangent will be the same for all angles that share the same reference angle.

For example, a reference angle of $60°$ in quadrant I, creates a triangle to the point $(1, \sqrt{3})$, with a hypotenuse of 2. This same triangle can be formed at $120°, 240°, 300°$, which have the same reference angle. Because the lengths of all three sides are the same (or proportional), the absolute value of the sine, cosine or tangent will be the same for all four angles.
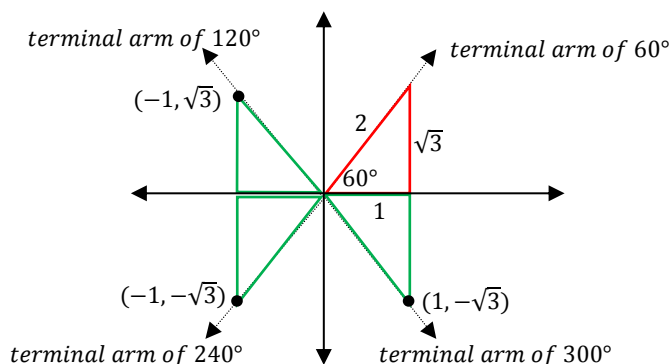
Depending upon the quadrant in which the angle is located, the value will have a positive (if named) or negative (otherwise) sign according to the **CAST rule** summarized in the first diagram, an example of which is shown for the sine of $60°$ and its corresponding angles in the second diagram:

SCO: **T2: Solve problems, using the three primary trigonometric ratios for angles from 0°
to 360° in standard position.** [C, ME, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Determine, using the Pythagorean theorem or the distance formula, the distance from the origin to a point $P(x, y)$ on the terminal arm of an angle.

- Determine the value of $\sin\theta$, $\cos\theta$, $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ .

- Determine, without the use of technology, value of $\sin\theta$, $\cos\theta$, $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ , where $\theta = 0°, 90°, 180°, 270°\ or\ 360°$.

- Determine the sign of a given trigonometric ratio for a given angle, without the use of technology, and explain.

- Solve, for all values of θ, an equation of the form $sin\ \theta = a$ or $cos\ \theta = a$, where $-1 \le a \le 1$, and an equation of the form $\tan \theta = a$, where a is a real number.

- Determine the exact value of the sine, cosine or tangent of a given angle with a reference angle of $30°, 45°\ or\ 60°$.

- Describe patterns in and among the values of the sine, cosine and tangent ratios for angles from $0°\ to\ 360°$.

- Sketch a diagram to represent a problem.

- Solve a contextual problem, using trigonometric ratios.

Suggested Instructional Strategies

- Have students use a coordinate axis to determine the value of $\sin\theta,\ \cos\theta, and\ \tan\theta$, given any point (x, y) on the terminal arm, by dropping a perpendicular from the y coordinate to the x axis and determining the hypotenuse, r. Have them use the triangle created to determine the *sine*, *cosine* and *tangent* of the angle θ. Have students choose another point (x_1, y_1) on the same terminal arm and repeat the process.

  Students should quickly recognize that the *sine*, *cosine* and *tangent* will be the same regardless of where on the terminal arm the point is located. Repeat this activity with several values of θ, until students develop the pattern:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x}$$

SCO: **T2: Solve problems, using the three primary trigonometric ratios for angles from 0°
to 360° in standard position.** [C, ME, PS, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Have students develop the CAST rule by developing a chart of values for
$sine, cosine\ and\ tangent\ for\ 30°, 45°, and\ 60°$ and for the angles for which these are
reference angles.

**Q** Solve for all values of θ when $\cos\theta = \frac{-\sqrt{3}}{2}$, $0 \le \theta \le 360°$

Answer: $\theta = 30°, \cos -'ve\ in\ quadrants\ II\ and\ III$ $\therefore \theta = 150°, 210°$

Q Sketch the reference angles that would correspond to $\tan\theta = -\frac{3}{5}$.

Q Explain how you can use reference angles to determine the trig ratios of any angle, θ .

Q Explain why there are only two angles between 0˚ and 360˚ that have the same cosine ratio.

Q Radar can locate aircraft hundreds of km away. A radar screen is divided into pixels which
can be translated into x and y coordinates (1 pixel represents 1 km). If an aircraft is at an
angle of $75°$ with the horizontal and a distance of $180\ km$ from the tower located at (0,0),
then which pixel (x and y coordinates) on the monitor would represent the position of the
aircraft relative to the tower?
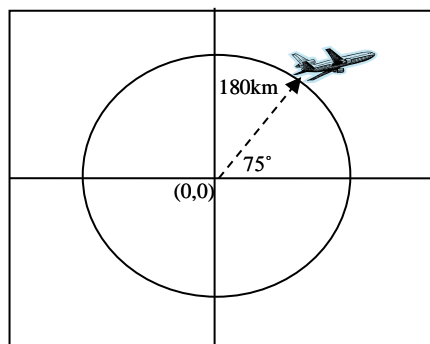
Answer: $\cos 75° = \frac{x}{180km}$, $\sin 75° = \frac{y}{180\ km}$

$(x, y) = (46.6, 173.9)$

SCO **RF1: Factor polynomial expressions of the form:**
- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
- $a\big(f(x)\big)^2 + b\big(f(x)\big) + c,\ a \neq 0$
- $a^2\big(f(x)\big)^2 - b^2\big(g(y)\big)^2,\ a \neq 0,\ b \neq 0$

where a, b, and c are rational numbers
[CN, ME, R]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

Relations and Functions

RF1: Factor polynomial expressions of the form: $ax^2 + bx + c,\ a \neq 0;\ a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0;\ a\big(f(x)\big)^2 + b\big(f(x)\big) + c,\ a \neq 0;\ a^2\big(f(x)\big)^2 - b^2\big(g(y)\big)^2,\ a \neq 0,\ b \neq 0;\ $ **where $a, b,$ and c are rational numbers**.

Scope and Sequence of Outcomes:

Grade Ten/Eleven	Grade Eleven	Grade Twelve
AN1: Demonstrate an understanding of factors of whole numbers by determining the prime factors, greatest common factor, least common multiple, square root, and cube root. *(NRF)* **AN4:** Demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially and symbolically. *(NRF10)* **AN5:** Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically. *(NRF10)* **RF9:** Represent a linear function, using function notation. *(NRF10)* **RF2:** Demonstrate an understanding of the characteristics of quadratic functions, including: vertex, intercepts, domain and range, axis of symmetry. (Found11)	**RF1:** Factor polynomial expressions of the form: $ax^2 + bx + c, a \neq 0$ $a^2x^2 - b^2y^2,\ a \neq 0, b \neq 0$ $a\big(f(x)\big)^2 + b\big(f(x)\big) + c, a \neq 0$ $a^2\big(f(x)\big)^2 - b^2\big(g(y)\big)^2, a \neq 0, b \neq 0$ where a, b, and c are rational numbers.	**T5:** Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. *[(PC12A)* **T6:** Prove trigonometric identities, using reciprocal identities, quotient identities, Pythagorean identities, sum or difference identities (restricted to sine, cosine and tangent), double-angle identities (restricted to sine, cosine and tangent). *(PC12A)* **RF3:** Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients). *(PC12B)* **RF4:** Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). (PC12B) **RF5:** Graph and analyze radical functions (limited to functions involving one radical). *(PC12B)* **RF6:** Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions. *(PC12B)* **RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). *(PC12B)*

ELABORATION

Students have factored polynomial expressions in *Number Relations and Functions 10* as summarized in the scope and sequence shown above. Trinomial factoring included removing the common factor, factoring by inspection, modelling with algebra tiles, and identifying perfect squares and the difference of squares. In *Foundations of Mathematics 110* students added partial factoring to their repertoire, and then practiced all of these methods in the context of solving quadratic equations.

SCO **RF1: Factor polynomial expressions of the form:**
- $ax^2 + bx + c, \ a \neq 0$
- $a^2x^2 - b^2y^2, \ a \neq 0, \ b \neq 0$
- $a\big(f(x)\big)^2 + b\big(f(x)\big) + c, \ a \neq 0$
- $a^2\big(f(x)\big)^2 - b^2\big(g(y)\big)^2, \ a \neq 0, \ b \neq 0$

where $a, b,$ and c are rational numbers
[CN, ME, R]

For this outcome, students will extend their knowledge to build a tool kit of strategies for factoring polynomial expressions in the form $ax^2 + bx + c$ where a is a rational number other than $zero$, and a, b, and c are rational numbers. New factoring strategies will include:

- decomposition in which factors of the coefficients a and c are found, that also sum to b to decompose the middle term and find common factors e.g.,

 $3x^2 - x - 4$
 $= 3x^2 - 4x + 3x - 4$
 $= x(3x - 4) + 1(3x - 4)$
 $= (3x - 4)(x + 1)$

- substituting for x^2 e.g.,
 to solve $x^4 - 13x^2 + 36,$ substitute $s = x^2$.

   $s^2 - 13s + 36$
 $= s^2 - 9s - 4s + 36$
 $= s(s - 9) - 4(s - 9)$
 $= (s - 4)(s - 9)$

 Substituting back:
   $(x^2 - 4)(x^2 - 9)$
 $= (x - 2)(x + 2)(x - 3)(x + 3)$

- substituting for an expression e.g.,
 to solve $(x - 5)^2 + 9(x - 5) + 18,$ substitute $s = (x - 5)$,

   $s^2 + 9s + 18$
 $= s^2 + 3s + 6s + 18$
 $= (s + 3)(s + 6)$

 Substituting back:
 $\big((x - 5) + 3\big)\big((x - 5) + 6\big)$
 $= (x - 2)(x + 1)$

Students can be shown the *Factor Theorem* to test whether or not a given binomial is a factor of a trinomial, ensuring that they are able to explain why this theorem works.

The *Factor Theorem* states that, if the solution for the variable of the binomial, substituted into the trinomial expression, makes that expression equal to zero, then the solution is the root of the expression and the binomial factor is a factor of the trinomial e.g.,

$(x - 2)$ is a factor of $x^4 - 10x^2 + 24$ $\ \boldsymbol{iff^*}\ f(2) = 0$

$f(2) = 2^4 - 10(2^2) + 24 = 16 - 40 + 24 = 0$

$\therefore (x - 2)$ is a factor, and 2 is a root of the expression. $\ \boldsymbol{iff} = if\ and\ only\ if$

SCO   **RF1: Factor polynomial expressions of the form:**
- $ax^2 + bx + c, \ a \neq 0$
- $a^2x^2 - b^2y^2, \ a \neq 0, \ b \neq 0$
- $a(f(x))^2 + b(f(x)) + c, \ a \neq 0$
- $a^2(f(x))^2 - b^2(g(y))^2, \ a \neq 0, \ b \neq 0$

where $a, b,$ **and** c **are rational numbers**
[CN, ME, R]

ACHIEVEMENT INDICATORS

- Factor a given polynomial expression that requires the identification of common factors.

- Determine whether a given binomial is a factor for a given polynomial expression, and explain why or why not.

- Factor a given polynomial expression of the form:

  $ax^2 + bx + c, \ a \neq 0$

  $a^2x^2 - b^2y^2, \ a \neq 0, \ b \neq 0$

- Factor a given polynomial expression that has a quadratic pattern, including:

  $a(f(x))^2 + b(f(x)) + c, \ a \neq 0$

  $a^2(f(x))^2 - b^2(g(y))^2, \ a \neq 0, \ b \neq 0$

Suggested Instructional Strategies

- There are numerous strategies used for factoring, depending upon preference and the structure or the expression. Students should be given ample time to explore the wide range of expression types and the tool kit of strategies they will require including ones not usually considered (e.g. factoring using a box method can be found at: http://www.regentsprep.org/Regents/math/algtrig/ATV1/LgroupingBox.htm)

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Factor the following:

a) $x^2 + 12x + 20$

b) $x^2 + 6x - 16$

c) $x^2 + 5x - 14$

d) $x^2 - 10x + 24$

e) $8x^2 + 33x + 4$

f) $6x^2 - 47x + 15$

g) $5x^2 - 3x - 8$

h) $x^2 - 16$

i) $16x^2 - 25y^2$

j) $4x^2 - 36y^2$

k) $\frac{1}{25}x^2 - 2\frac{1}{4}y^2$

l) $x^4 + 7x^2 + 12$

m) $sin^2(x) + 2sin(x) - 3$

n) $(2x - 5)^2 + 9(2x - 5) + 18$

o) $3^{2x} + 5(3^x) + 6$

p) $9x^6 - y^4$

q) $16x^{16} - 1$

r) $144(3x + 5)^6 - 9(2y - 8)^{10}$

s) $sin^2(x) - 36y^2$

Answers: a) $(x + 2)(x + 10)$ b)$(x - 2)(x + 8)$ c) $(x - 2)(x + 7)$ d) $(x - 4)(x - 6)$ e) $(8x + 1)(x + 4)$
    f) $(3x - 1)(2x - 15)$ g) $(5x - 8)((x + 1)$ h) $(x + 4)(x - 4)$ i) $)4x - 5y)(4x + 5y)$
    j) $(2x - 6y)$ k) $\left(\frac{1}{5}x - \frac{3}{2}y\right)\left(\frac{1}{5}x + \frac{3}{2}y\right)$ l) $(x^2 + 3)(x^2 + 4)$ m) $(sin x - 1)(sin x + 3)$ n) $2(x - 1)(2x + 1)$
    o) $(3^x + 2)(3^x + 3)$ p) $(3x^3 - y^2)(3x^3 + y^2)$ q) $(4x^3 + 1)(2x^4 + 1)(2x^4 - 1)$
    r) $[12(3x + 5)^3 + 3(2y - 8)^5][12(3x + 5)^3 - 3(2y - 8)^5]$ s) $(sin x - 6y)(sin x + 6y)$

SCO **RF1: Factor polynomial expressions of the form:**
- $ax^2 + bx + c, \ a \neq 0$
- $a^2x^2 - b^2y^2, \ a \neq 0, \ b \neq 0$
- $a(f(x))^2 + b(f(x)) + c, \ a \neq 0$
- $a^2(f(x))^2 - b^2(g(y))^2, \ a \neq 0, \ b \neq 0$

where a, b, and c are rational numbers
[CN, ME, R]

Q Use the factor theorem:
a) *Is $(x + 2)$ a factor of $x^2 + 7x + 10$?*
b) *Is $(x - 1)$ a factor of $3x^2 + 2x - 5$?*
c) *Is $(3x - 2)$ a factor of $3x^2 - x - 2$?*
d) *Is $(3x + 2)$ a factor of $3x^2 - x - 2$?*

Answer: a) $(-2)^2 + 7(-2) + 10 = 0 \ \therefore a \ factor$
b) $3(1)^2 + 2(1) - 5 = 0 \ \therefore a \ factor$
c) $3\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) - 2 \neq 0 \ \therefore not \ a \ factor$
d) $3\left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right) - 2 = 0 \ \therefore a \ factor$

Q Factor the following:
a) $x^2 + 7x + 12$
b) $x^2 - x - 42$
c) $x^2 - 8x - 20$
d) $3x^2 + 9x - 12$
e) $x^2 - 4$
f) $m^2 - 16$
g) $25x^2 - 49$
h) $5x^2 - 45y^2$
i) $4x^2 + 11x + 7$
j) $3x^2 - 11x + 6$
k) $x^4 - 16x^2 + 60$
l) $(x^2 + 2x)^2 - 11(x^2 + 2x) + 24$

Answers: a) $(x + 3)(x + 4)$ b) $(x - 7)(x + 6)$ c) $(x - 10)(x + 2)$ d) $3(x + 4)(x - 1)$
e)$(x - 2)(x + 2)$ f) $(m - 4)(m + 4)$ g) $(5x - 7)(5x + 7)$ h) $5(x - 3y)(x + 3y)$
i) $(4x + 7)(x + 1)$ j) $(3x - 2)(x - 3)$ k) $(x^2 - 6)(x^2 - 10)$ l) $(x + 4)(x - 2)(x + 3)(x - 1)$

Q Factor the following:
a) $x^2 - 6x + 8$
b) $a^2 - 81$
c) $3b^3 + 12b^2 - 15b$
d) $2x^2 - 5x - 7$
e) $7x^4 - 252x^2$
f) $m^2 + 12m + 35$
g) $-3r^2 - r + 10$
h) $4x^3 - 64x$
i) $5t^2 - 13t + 6$
j) $4n^2 - 15n + 9$

Answers: a) $(x - 4)(x - 2)$ b) $(a - 9)(a + 9)$ c) $3b(b + 5)(b - 1)$ d) $(2x - 7)(x + 1)$
e) $7x^2(x - 6)(x + 6)$ f) $(m + 7)(m + 5)$ g) $1(3r - 5)(r + 2)$ h) $4x(x - 4)(x + 4)$
i) $(5t - 3)(t - 2)$ j) $(4n - 3)(n - 3)$

SCO	**RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.** [C, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

Scope and Sequence of Outcomes:

Grade Ten/Eleven	Grade Eleven	Grade Twelve
RF1: Interpret and explain the relationships among data, graphs and situations. *(NRF10)* **RF2:** Demonstrate an understanding of relations and functions (NRF10) **RF4:** Describe and represent linear relations using words, ordered pairs, tables of values, graphs and equations. *(NRF10)* **RF9:** Represent a linear function, using function notation. *(NRF10)* **RF2:** Demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. *(FM11)*	**RF2:** Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.	**RF2:** Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations *(PC12A)* **RF3:** Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations *(PC12A)* **RF4:** Apply translations and stretches to the graphs and equations of functions *(PC12A)* **RF5:** Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, line $y =$ x. *(PC12A)* **RF8:** Assemble a function toolkit. *(PC12B)*

ELABORATION

In this course *AN1,* which introduces absolute value, should be covered prior to this outcome. For this outcome, students will graph the absolute value function using a table of values for $y = |f(x)|$. Students will discover that for all values of $f(x)$ less than 0, the y-value of $|f(x)|$ is $-f(x)$; and for all values of $f(x)$ greater than or equal to 0, the y-value of $|f(x)|$ is $f(x)$.

Because there are two distinct rules for each interval $(x < 0; x \geq 0)$ the function $y = |x|$ is known as a **piecewise function** defined as a function composed of two or more separate functions or pieces, each with its own specific domain, that combine to define the overall function. $y = |x|$ *is defined as the piecewise function* $y = \begin{cases} x, & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$ When solving questions involving absolute value both parts of the piecewise function must be examined individually.

When graphing, students will graph the line $y = f(x)$. The x-intercept of this line is the same as the x-intercept of the absolute value function because the absolute value of zero is still zero. This point (representing the x-intercept) is an **invariant point** defined as a point that remains unchanged when a transformation is applied to it.

Students will look at various absolute value functions including quadratics. They will be expected to state the **domain** and **range** of the function. The domain is defined as the set of all possible values of x. The range is defined as the set of all possible values of y. For $y = |x|$, the domain is $\{x | x \in R\}$, read as the set of all x, such that x belongs to the set of Real numbers. For $y = |x|$, the range is $\{y | y \geq 0, y \in R\}$, read as: y such that y is greater than or equal to zero, y belongs to the set of Real numbers.

SCO   **RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.** [C, PS, R, T, V]

Students will explore why a function such as $|2x + 3| = -11$ cannot exist, and why there is no way to take the absolute value of an expression to give a negative value.

ACHIEVEMENT INDICATORS

- Create a table of values for $= |f(x)|$, given a table of values for $y = f(x)$.

- Generalize a rule for writing absolute value functions in piecewise notation.

- Sketch the graph of $= |f(x)|$; state the intercepts, domain and range; and explain the strategy used.

- Solve an absolute value equation graphically, with or without technology.

- Solve, algebraically, an equation with a single absolute value, and verify the solution.

- Explain why the absolute value equation $|f(x)| < 0$ has no solution.

- Determine and correct errors in a solution to an absolute value equation.

- Solve a problem that involves an absolute value function.

Suggested Instructional Strategies

- Have student explore graphs of absolute functions with a graphics calculator. Encourage them to predict graphs before creating the graph.

- Give students a table of values for $y = f(x)$ and then have them fill in values for $|f(x)|$.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q The following table of values is given for $y = f(x)$. Fill in the corresponding values for $y = |f(x)|$.

x	$f(x)$	$|f(x)|$	Answers:
-3	32		32
-2	12		12
-1	-2		2
0	-10		10
1	-12		12
2	8		8
3	2		2

Q Write $y = |3x + 12|$ as a piecewise function.

$Answer: y = \begin{cases} 3x + 12, if\ x \geq -4 \\ -3x - 12, if\ x < -4 \end{cases}$

Q Write $y = |x^2 - 3x - 10|$ as a piecewise function.

$Answer: y = \begin{cases} x^2 - 3x - 10, if\ x \leq -2\ or\ x \geq 5 \\ -x^2 + 3x + 10, if -2 < x < 5 \end{cases}$

SCO **RF2: Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.** [C, PS, R, T, V]

Q Graph $y = |-7x + 4|$

Q Graph $y = |-x^2 + 7x - 6|$

Q Find the y-intercepts for the following functions:

a) $y = \left|\frac{1}{2}x + 5\right|$ 　　　　　b) $y = |6x^2 + 8x - 21|$

Answer: a) $(0,5)$　b) $(0,21)$

Q What is the domain and range of the following absolute value functions?

a) $y = |0.25x + 8|$ 　　　　　b) $y = |x^2 + 3x - 40|$

Answer: $x \in R, y \geq 0$

Q Solve this equation graphically. $|4x - 9| = 3x + 2$

Q Solve the previous question algebraically.

$$|4x - 9| = \begin{cases} 4x - 9, for\ x \geq 2\frac{1}{4} \\ -4x + 9, for\ x < 2\frac{1}{4} \end{cases}$$

Answer: $x = 1\ and\ x = 11$

Q Explain why $|x + 9| = -2$ has no solution.

Q <u>Correct any errors</u> in the following solution.

Solve $|-12\ x + 6| = 18$

$$|-12x + 6| = \begin{cases} -12x + 6, for\ x \geq 2 \\ 12x - 6, for\ x \leq 2 \end{cases}$$

Case 1:   ***Case 2:***

$-12x + 6 = 18$ $12x - 6 = 18$

$-12x = 12$ $12\ x = 24$

$x = -1$ $x = 2$

This is an extraneous root because it is not $x \geq 2$ This root satisfies the condition $x \leq 2$

Answer:
Correct Solution
|-12 x + 6| = 18
$$|-12x + 6| = \begin{cases} -12x + 6, for\ \cancel{x \geq 2}\ x \leq 0.5 \\ 12x - 6, for\ \cancel{x \leq 2}\ x > 0.5 \end{cases}$$
<u>*Case 1:*</u>   <u>*Case 2:*</u>
$-12x + 6 = 18$ $12x - 6 = 18$
$-12x = 12$ $12\ x = 24$
$x = -1$ $x = 2$
~~This is an extraneous root because it is not $x \geq 2$~~   This root satisfies the condition ~~$x \leq 2$~~ $x > 0.5$
This now satisfies the condition $x \leq 0.5$

There are two valid answers, -1 and 2.

SCO   **RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts**.
[CN, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts.

Scope and Sequence of Outcomes:

Grade Ten/Eleven	Grade Eleven	Grade Twelve
RF2: Demonstrate an understanding of relations and functions *(NRF10)* **RF5:** Determine the characteristics of the graphs of linear relations, including the: intercepts, slope, domain, range. *(NRF10)* **RF6:** Relate linear relations expressed in: slope-intercept form $(y = mx + b)$, general form $(Ax + By + C = 0)$, slope-point form $(y - y_1 = m(x - x_1))$ *(NRF10)* **RF7:** Determine the equation of a linear relation, given a graph, a point and the slope, two points, a point and the equation of a parallel or perpendicular line, and a scatterplot *(NRF10)* **RF9:** Represent a linear function, using function notation. *(NRF10)* **RF2:** Demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, axis of symmetry. *(FM11)*	**RF3:** Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts.	**RF2:** Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations *(PC12A)* **RF3:** Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations *(PC12A)* **RF4:** Apply translations and stretches to the graphs and equations of functions *(PC12A)* **RF5:** Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, line $y = x$. *(PC12A)* **RF4:** Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5). *(PC12B)* **RF5:** Graph and analyze radical functions (limited to functions involving one radical). *(PC12B)* **RF6:** Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). *(PC12B)* **RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). *(PC12B)* **RF8:** Assemble a function toolkit. *(PC12B)*

ELABORATION

Students will be familiar with quadratic functions in standard form, $= ax^2 + bx + c$, and how to determine vertex, intercepts, domain and range, and axis of symmetry from this form. The focus of this outcome will be on quadratic functions written in vertex form, $= a(x - p)^2 + q$, and on how the values of a, p and q indicate changes to the graph of the parabola $y = x^2$.

Students will be given quadratic function in vertex form to analyze and will not be required to convert from general to vertex form by completing the square, which will be covered in **RF4**. However, in order to sketch graphs accurately they will need to expand from vertex to general form, to determine the x-intercepts.

The value of a indicates the opening direction of the parabola. If a is positive the parabola opens upward, if negative it opens downward. The value of a also affects the **vertical stretch** or how quickly the y values increase in relation to the x values. When $|a| < 1$, the y values increase or decrease more slowly than they would for $y = x^2$ and so the graph appears wider. When $|a| > 1$, the y values increase or decrease more quickly and so the parabola appears narrower than the graph of $y = x^2$.

SCO   **RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts**.
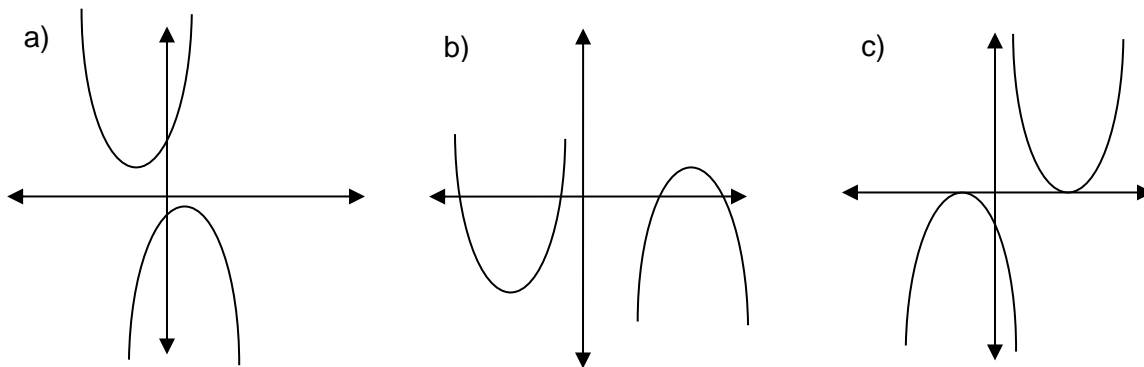   [CN, R, T, V]

The value of p, indicates the magnitude of the **horizontal translation** of $y = x^2$. The sign before p in the equation indicates the direction of the translation: if it is $+$ the parabola is shifted p units to the left; if it is $-$, the parabola is shifted p units to the right. For example, for $y = 3(x + 2)^2 + 6$, the parabola is shifted 2 units to the left.

The value of q indicates the **vertical translation** of $y = x^2$. When $q < 0$ the parabola is shifted down q units. When $q > 0$ the parabola is shifted up q units.

If you compare $y = ax^2$ with $= a(x - p)^2 + q$, the vertex $(0,0)$ of $y = ax^2$ is shifted horizontally p units and vertically q units, so the vertex of $y = a(x - p)^2 + q$ will be (p, q).

The number of x-intercepts for the quadratic function is determined from the values of a and q.

a) If a and q are both positive, or both negative, the parabola will either open upward with a vertex above the x axis, or will open downward with a vertex below the x axis, and there will be zero x intercepts.

b) If a and q have opposite signs, the parabola will open upwards with a vertex below the x axis, or will open downward with the vertex above the x axis, and there will be two x-intercepts.

c) If a is either positive or negative, but $q = 0$ the vertex will sit right on the x axis and there will be two equal x intercepts (of the same value).

SCO **RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts**.
[CN, R, T, V]

ACHIEVEMENT INDICATORS

- Explain why a function given in the form $y = a(x - p)^2 + q$ is a quadratic function.

- Compare the graphs of a set of functions of the form $y = ax^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of a.

- Compare the graphs of a set of functions of the form $y = x^2 + q$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of q.

- Compare the graphs of a set of functions of the form $y = (x - p)^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of p.

- Determine the coordinates of the vertex for a quadratic function of the form $y = a(x - p)^2 + q$, and verify with or without technology.

- Generalize, using inductive reasoning, a rule for determining the coordinates of the vertex for the quadratic functions of the form $y = a(x - p)^2 + q$.

- Sketch the graph of $y = a(x - p)^2 + q$, using transformations, and identity the vertex, domain and range, direction of opening, axis of symmetry and x- and y-intercepts.

- Explain, using examples, how the values of a and q may be used to determine whether a quadratic function has zero, one or two x-intercepts.

- Write a quadratic function in the form $y = a(x - p)^2 + q$ for a given graph or a set of characteristics of a graph.

Suggested Instructional Strategies

- Use technology (graphing software) to show the effects of the stretch factor *a* on the parabola $(y = x^2)$ for many positive and negative values of *a*. Repeat the same process separately for the horizontal shift *p* and the vertical shift *q*. Have students discuss how the shape, position and direction of the parabola are altered when these parameters are changed.

- Using technology, explore the number of *x*-intercepts in reference to the values of *a* and *q*.

GCO: Relations and Functions (RF): Develop algebraic and graphical
reasoning through the study of relations.

GRADE 11

SCO **RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts.**
[CN, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Act Give each student a card with a different equation in the form of $y = a(x - p)^2 + q$ and have students graph the function on a small grid. Place students in groups of 6 to 8 to review their graphs and correct any errors. Have students shuffle the cards of graphs and equations, and exchange cards with another group who will, in turn, match equations and graphs.

Variation on the above: Have students give the pile of graphs to the other group and have them write the equation. They would then be given the pile of original equations for the graphs they were given in order to correct their work.

Q For each of the quadratic functions below, sketch the graph and determine: vertex, domain and range, direction of opening, axis of symmetry, x and y intercepts.

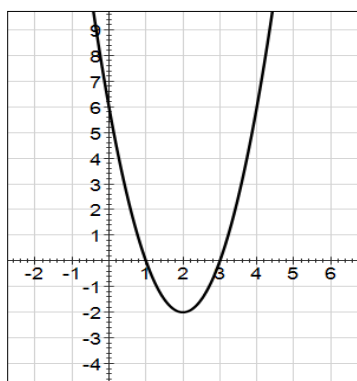   a) $y = 3(x + 1)^2 - 12$

   b) $y = -x^2 - 1$

   c) $y = \frac{1}{2}(x + 2)^2$
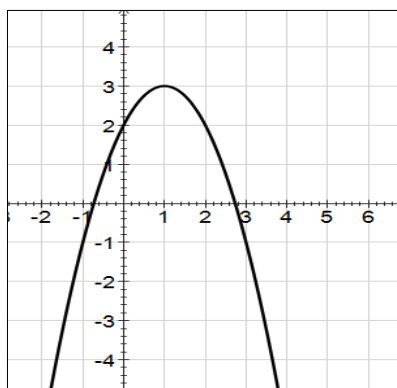
   d) $y = -2(x - 4)^2 - 3$

Answers	vertex	domain	range	opening	Axis of symmetry	x-int	y-int
$y = 3(x + 1)^2 - 12$	$(-1, -12)$	$\{x \mid x \in R\}$	$\{y \mid y \geq -12, y \in R\}$	up	$x = -1$	$(-3, 0)(1, 0)$	$(0, -9)$
$y = -x^2 - 1$	$(0, -1)$	$\{x \mid x \in R\}$	$\{y \mid y \leq -1, y \in R\}$	down	$x = 0$	no intercepts	$(0, -1)$
$y = \frac{1}{2}(x + 2)^2$	$(-2, 0)$	$\{x \mid x \in R\}$	$\{y \mid y \geq 0, y \in R\}$	up	$x = -2$	$(-2, 0)$	$(0, 2)$
$y = -2(x - 4)^2 - 3$	$(4, -3)$	$\{x \mid x \in R\}$	$\{y \mid y \leq -3, y \in R\}$	down	$x = 4$	no intercepts	$(0, -35)$

Q For each of the quadratic functions graphed below, write an equation in the form of $y = a(x - p)^2 + q$.
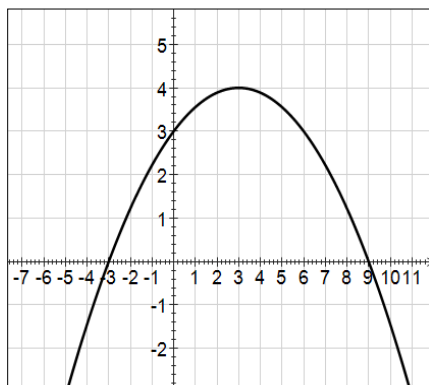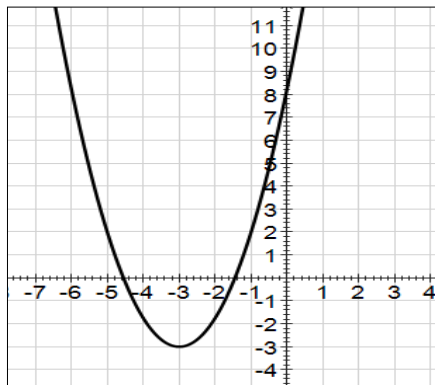
a)

b)

SCO **RF3: Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts.**
[CN, R, T, V]

c)

d)

Answers: *a) vertex $(2, -2)$, $x - int$ $(1,0), (3,0)$, $y = 2(x - 2)^2 - 2$*
b) vertex $(1,3)$, pt $(3, -1)$, $y = -(x - 1)^2 + 3$
c) vertex $(3, 4)$, $y - int$ $(0,3)$, $y = -\frac{1}{9}(x - 3)^2 + 4$
d) vertex $(-3, -3)$, pt $(-1, 2)$, $y = \frac{5}{4}(x + 3)^2 - 3$

Q From the information given, write the equation in the form of $y = a(x - p)^2 + q$, for each of the following quadratic functions.

a) vertex $(5, -7)$; $a = 2$

b) vertex $(0, 3)$; $a = -\frac{1}{4}$

c) parabola opens downward but keeps its original shape (as for $y = x^2$); vertex $(-2, 3)$

Answers: a) $y = 2(x - 5)^2 + 7$ b) $y = -\frac{1}{4}x^2 + 3$ c) $y = -(x + 2)^2 + 3$

SCO   **RF4: Analyze quadratic functions of the form** $y = ax^2 + bx + c$ **to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.**
[CN, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF4: Analyze quadratic functions of the form $y = ax^2 + bx + c$ **to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.**

Scope and Sequence of Outcomes:

Grade Ten	Grade Eleven	Grade Twelve
RF1: Interpret and explain the relationships among data, graphs and situations. *(NRF10)* **RF2:** Demonstrate an understanding of relations and functions *(NRF10)* **RF5:** Determine the characteristics of the graphs of linear relations including the intercepts, slope, domain and range. *(NRF10)* **RF6:** Relate linear relations expressed in slope-intercept form $(y = mx + b)$, general form $(Ax + By + C = 0)$, and slope-point form $(y - y_1) = m(x - x_1)$, to their graphs. *(NRF10)* **RF7:** Determine the equation of a linear relation, given a graph, a point and the slope, two points, a point and the equation of a parallel or perpendicular line, and a scatterplot *(NRF10)* **RF9:** Represent a linear function, using function notation. *(NRF10)*	**RF4:** Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.	**RF2:** Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations *(PC12A)* **RF3:** Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations *(PC12A)* **RF4:** Apply translations and stretches to the graphs and equations of functions *(PC12A)* **RF5:** Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, line y=x. *(PC12A)* **RF4:** Graph and analyze polynomial functions (limited to polynomial functions of degree ≤5). *(PC12B)* **RF5:** Graph and analyze radical functions (limited to functions involving one radical). *(PC12B)* **RF6:** Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). *(PC12B)* **RF7:** Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials). *(PC12B)* **RF8:** Assemble a function toolkit. *(PC12B)*

ELABORATION

For this outcome students will be required to express quadratic functions in the form $y = ax^2 + bx + c$ in the form $y = a(x - p)^2 + q$. This will require learning to complete the square, which they have not seen before. However, they should be familiar with squaring binomials, and can be encouraged to identify patterns between the binomial and the resulting trinomial. For example: for $(x + 5)^2 = x^2 + 10x + 25$, the middle and the last term of the trinomial are related to the second term in the binomial: $2x5 = 10$, $5^2 = 25$.

SCO **RF4: Analyze quadratic functions of the form** $y = ax^2 + bx + c$ **to identify**
characteristics of the corresponding graph, including: vertex, domain and range,
direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.
[CN, PS, R, T, V]

Using algebra tiles:

Students should be given the opportunity to complete the square by using algebra tiles which will lead to a greater understanding of the concept of completing the square. Also, for some learners and for some questions, algebra tiles may be the most efficient method for completing the square.

To complete the square using algebra tiles from the form $y = ax^2 + bx + c$, the first two terms of the trinomial are arranged to form the beginning of a square that can then be completed by filling in with unit tiles in the bottom right corner.

For example: for $y = x^2 - 4x - 2$, one x^2 algebra tile, and four $-x$ tiles are arranged as shown below. The square is completed by adding the missing $+4$ unit tiles in the bottom right corner. These must be $+'ve$ as a product of the $-x$ tiles. To maintain equivalency, four -4 unit tiles must also be added to the -2 unit tiles already present.
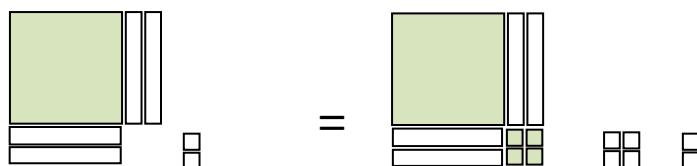
The dimensions of the square gives the factor $(x - 2)$ and the remaining tiles (-6) represent the vertical translation. Therefore: $x^2 - 4x - 2 = (x - 2)^2 - 6$

Algebraically:

To complete the square from the form $y = ax^2 + bx + c$, the second term of the trinomial is halved and then squared to determine the third term needed to make a perfect square. This value is added and then subtracted to maintain equivalency. The perfect square is factored and the remaining terms are combined.

For example, for $y = x^2 - 4x - 2$, half of (-4) is (-2), and (-2) squared is 4, so 4 is added and then subtracted, the perfect square is factored to $(x - 2)^2$, and the remaining terms are combined:
$$y = x^2 - 4x - 2$$
$$y = x^2 - 4x + 4 - 4 - 2$$
$$y = (x - 2)^2 - 6$$

Once the equation is in the form, $y = a(x - p)^2 + q$, the x value for the vertex and the axis of symmetry can be identified as $x = p$ and the y value can be identified as $y = q$. For the example above, the vertex (p, q) is $(2, -6)$, and the axis of symmetry is $x = 2$.

If the coefficient of x^2 is not 1, in $Step$ 1 the coefficient must be factored from the **first two terms** before completing the square. In $Step$ 2 the square is completed within the brackets as shown above. In $Step$ 3, the last term in the bracket is multiplied by the coefficient factor, and then removed from the brackets. For example, to complete the square for $y = 2x^2 - 16x + 25$:

SCO **RF4: Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.**
[CN, PS, R, T, V]

$Step\ 1$	$y = 2(x^2 - 8x) + 25$	2 is factored from the first two terms
$Step\ 2$	$y = 2(x^2 - 8x + 16 - 16) + 25$	(-4) is squared, 16 is added and then subtracted.
$Step\ 3$	$y = 2(x^2 - 8x + 16) - \mathbf{32} + 25$	$(-16) \times 2 = -32$.
$Step\ 4$	$y = 2(x - 4)^2 - 7$	The perfect square is factored and the remaining terms are combined.

ACHIEVEMENT INDICATORS

- Explain the reasoning for the process of completing the square as shown in a given example.

- Write a quadratic function given in the form $y = ax^2 + bx + c$, in the form $y = a(x - p)^2 + q$ by completing the square.

- Identify, explain and correct errors in an example of completing the square.

- Determine the characteristics of a quadratic function, given in the form $y = ax^2 + bx + c$, by changing to vertex form $y = a(x - p)^2 + q$.

- Sketch the graph of a quadratic function, given in the form $y = ax^2 + bx + c$, by changing to vertex form $y = a(x - p)^2 + q$.

- Verify, with or without technology, that a quadratic function in the form $y = ax^2 + bx + c$ represents the same function as a given quadratic function in the form $= a(x - p)^2 + q$.

- Write a quadratic function that models a given situation, and explain any assumptions made.

- Solve a problem, with or without technology, by analyzing a quadratic function.

Suggested Instructional Strategies

- Make use of algebra tiles to demonstrate the process of completing the square.

- From the core resource, the Chapter 3 Study Quiz is an excellent formative assessment tool (individual or whole class),
  http://www.mcgrawhill.ca/school/learningcentres/course/view.php?id=9780070738737
  (Pre-Calculus 11> Student Centre> Chapter 3 > Chapter 3 Study Quiz.

- Use technology to demonstrate that a quadratic function in the form $y = ax^2 + bx + c$ will represent the same function when written in the form $y = a(x - p)^2 + q$.

- Use technology, to write a quadratic equation which models the trajectory of a projectile. An example of this activity can be viewed on Youtube e.g., Dan Meyer on Real-World Math, throwing a basketball in a hoop.

SCO **RF4: Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including: vertex, domain and range, direction of opening, axis of symmetry, x- and y-intercepts; and to solve problems.** [CN, PS, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Write each of the following quadratic functions in the form $y = a(x - p)^2 + q$

   a) $y = x^2 - 10x + 19$

   b) $y = 3x^2 + 18x + 22$

   c) $y = -2x^2 - 8x - 23$

   d) $y = -\frac{2}{3}x^2 + 12x - 31$

Answers:
a) $y = (x - 5)^2 - 6$ b) $y = 3(x + 3)^2 - 5$ c) $y = -2(x + 2)^2 - 15$ d) $y = -\frac{2}{3}(x - 9)^2 + 23$

Q Using the answers for a) and b) in the previous question, determine the characteristics of the quadratic functions and sketch the graphs.

Answers:
a) $y = (x - 5)^2 - 6$ *vertex* $(5, -6)$, *domain* $\{x \in R\}$, *range* $\{y \geq -6 | y \in R\}$, *opening up,*
   axis of symmetry $x = 5$, *x int* $(7.45, 0), (2.55, 0)$, *y int* (0.19)
b) $y = 3(x + 3)^2 - 5$ *vertex* $(-3, -5)$, *domain* $\{x \in R\}$, *range* $\{y \geq -5 | y \in R\}$, *opening up,*
   axis of symmetry $x = -3$, *x int* $(-4.3, 0), (-1.7, 0)$, *y int* $(0, 22)$

Q A football is kicked into the air and follows the path $h = -5x^2 + 20x$, where x is time in seconds and h is the height in metres.

   a) Determine the characteristics of the function and sketch the graph.
   b) What is the maximum height attained by the football?
   c) How long does the ball stay in the air?

Answers:
a) $y = -5(x - 2)^2 + 20$ *x int* $(0, 0), (4, 0)$, *y int* $(0, 0)$, *vertex* $(2, 20)$, *opens down*
b) *maximum height* $= 20\ m$ c) *4 seconds*

SCO: **RF5: Solve problems that involve quadratic equations.** [C, CN, PS, R, T, V]

| **[C]** Communication | **[PS]** Problem Solving | **[CN]** Connections | **[ME]** Mental Math |
| **[T]** Technology | **[V]** Visualization | **[R]** Reasoning | and Estimation |

RF5: Solve problems that involve quadratic equations.

Scope and Sequence of Outcomes:

Grade Ten/Eleven	Grade Eleven	Grade Twelve
AN1: Demonstrate an understanding of factors of whole numbers by determining the prime factors, greatest common factor, least common multiple, square root, and cube root. *(NRF10)* **AN5:** Demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially and symbolically*. (NRF10)* **RF1:** Interpret and explain the relationships among data, graphs and situations. *(NRF10)* **RF5:** Determine the characteristics of the graphs of linear relations, including the: intercepts; slope; domain; and range. *(NRF10)* **RF9:** Represent a linear function, using function notation. *(NRF10)* **RF1:** Model and solve problems that involve systems of linear inequalities in two variables *(FM 11)*	**RF5:** Solve problems that involve quadratic equations.	**T5:** Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians *(PC12A)* **RF10:** Solve problems that involve exponential and logarithmic equations. *(PC12A)*

ELABORATION

In the *Foundations of Mathematics 11*, the students studied the characteristics of quadratic functions including vertex, intercepts, domain, range and axis of symmetry. They used a variety of methods to solve quadratic equations, including the Quadratic Formula.

For this outcome, students will explore the relationship between roots, x-intercepts, and zeros in more detail. They will derive the Quadratic Formula using deductive reasoning and then identify and use the value of the discriminant ($b^2 - 4ac$) to determine the type of roots and how this relates to the graph of the function.

If $b^2 - 4ac > 0$, there are 2 distinct real roots, and 2 x-intercepts.
If $b^2 - 4ac = 0$, there are 2 real equal roots, and 1 x-intercept.
If $b^2 - 4ac < 0$, the roots are imaginary (no real roots), and there are no x-intercepts.

When the discriminant is negative, there will be no real roots. These roots can be expressed using the mathematical notation $\sqrt{-1} = i$. For example $\sqrt{-15} = \sqrt{15}i$.

Having added completing the square, and using the discriminant to their toolkit, students will practice solving quadratic equations in a variety of ways, as appropriate to the situation and problem to be solved.

SCO: **RF5: Solve problems that involve quadratic equations.** [C, CN, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Explain, using examples, the relationship among the roots of a quadratic equation, the zeros of the corresponding quadratic function and the *x*-intercepts of the graph of the quadratic function.

- Derive the quadratic formula, using deductive reasoning.

- Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by using strategies such as:
  - determining square roots
  - factoring
  - completing the square
  - applying the quadratic formula
  - graphing its corresponding function

- Select a method for solving a quadratic equation, justify the choice, and verify the solution.

- Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one or no real roots; and relate the number of zeros to the graph of the corresponding quadratic function.

- Identify and correct errors in a solution to a quadratic equation.

- Solve a problem by determining and/or analyzing a quadratic equation.

Suggested Instructional Strategies

- Allow students a significant amount of time to explore the most efficient methods to use to solve various quadratic equations in the form $ax^2 + bx + c = 0$. Let students generate some general rules to guide them in approaching new questions. For example:

  - graphing may be best if graphing calculators are available;
  - factoring may be appropriate if numbers are simple squares;
  - completing the square may be used if factoring is difficult but fractions or decimals are not involved;
  - the Quadratic Formula may need to be used if numbers are fractions or decimals.

> SCO: **RF5: Solve problems that involve quadratic equations.** [C, CN, PS, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q For each of the following, explain the relationship between the roots of the equation, the zeros of the corresponding function and the x-intercepts of that function.

a) $f(x) = x^2 + 11x + 30$
b) $f(x) = x^2 - 12x + 36$
c) $f(x) = 4x^2 + 2x + 3$

Answers:
a) $f(x) = (x + 5)(x + 6)$, $x = -5$, $x = -6$, *two distinct real roots, two x intercepts*
b) $f(x) = (x - 6)^2$, $x = 6$, *two real equal roots, one x intercept*
c) $x = \dfrac{-1 \pm \sqrt{11}i}{4}$, *two imaginary roots, no x intercepts*

Q Show how the quadratic formula can be derived based on the idea of completing the squares.

Answer:

$ax^2 + bx + c = 0$

Solve for x

$a\left(x^2 + \dfrac{b}{a}x\right) + c = 0$ Factor a from the first two terms

$a\left(x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} - \dfrac{b^2}{4a^2}\right) + c = 0$ Add zero in side the bracket, using half the second coefficient squared

this will create a perfect square which we can factor later.

$a\left(x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2}\right) + c - \dfrac{b^2}{4a} = 0$ Remove the negative from the bracket, remember to multiply by a

$a\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a}$ Factor the perfect square and move all other terms to the other side of equation

$\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$ Divide both sides by a

$x + \dfrac{b}{2a} = \pm\dfrac{\sqrt{b^2 - 4ac}}{2a}$ Take the square roots of both sides, this introduces the $\pm$

$x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$ Isolate the x

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Group fractions together

Q Solve the following using the method indicated:

a) $x^2 - \dfrac{4}{49} = 0$ square root
b) $2x^2 + x - 6 = 0$ factoring
c) $6x^2 - 30x - 58.5 = 0$ completing the square
d) $4x^2 - 18x - 10$ quadratic formula
e) $-x^2 + 5x + 20$ graphically

Answers: a) $x = \pm\dfrac{2}{7}$ b) $(2x - 3)(x + 2) = 0$, $x = \dfrac{3}{2}$ *or* -2

SCO: **RF5: Solve problems that involve quadratic equations.** [C, CN, PS, R, T, V]

$c)\ (x - 2.5)^2 - 16 = 0 \quad x = 6.5\ or - 1.5 \quad d)\ x = \frac{18 \pm \sqrt{484}}{8} \quad x = 5, or -\frac{1}{2} \quad e)\ x\ int\ (-2.62, 0)(7.62, 0)$

Act Provide students with a variety of equations to evaluate and, working in groups of two or three, have each group determine which methods are most appropriate and/or efficient for solving the quadratic equation. Have them present to each other and explain their reasoning.
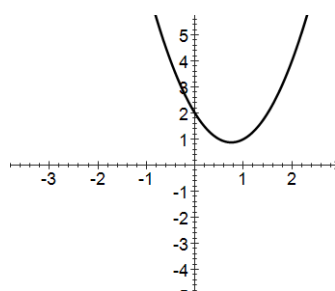
Q Sketch graphs for quadratic functions for which the discriminant is $less\ than\ 0$, $equal\ to\ 0$, and $greater\ than\ 0$.

Answer:
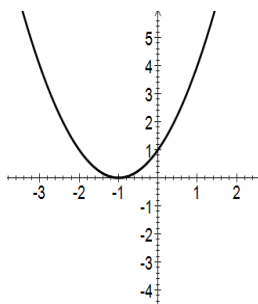$3\ cases\ of\ the\ discriminant, b^2 - 4ac$

$b^2 - 4ac < 0$
2 distinct Imaginary roots
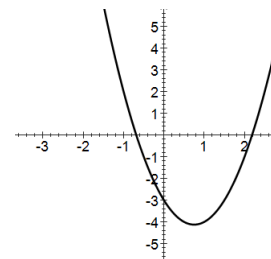quadratic does not touch x-axis
no Real zeros

$b^2 - 4ac = 0$
2 equal Real roots
quadratic touches x-axis once
does not cross x-axis

$b^2 - 4ac > 0$
2 distinct Real roots
quadratic crosses x-axis in two places

Q Write quadratic equations for each of the following.

a) The discriminant is $less\ than\ 0$
b) The discriminant is $equal\ to\ 0$
c) The discriminant is $greater\ than\ 0$

Many possible answers:
a) $f(x) = 9x^2 - 9x + 3, \quad f(x) = 6x^2 - 4x + 7, \quad f(x) = 3x^2 + 1, \quad f(x) = 7x^2 + 2x + 1$
b) $f(x) = 2x^2 + 10x + 12.5, \quad f(x) = -x^2 + 4x - 4, \quad f(x) = -9x^2 - 6x - 1, \quad f(x) = 4x^2 + 4x + 1$
c) $f(x) = -6x^2 - 2x + 4, \quad f(x) = 2x^2 - 8x - 10, \quad f(x) = 10x^2 + 9x - 7, \quad f(x) = -2x^2 + x + 6$

SCO: **RF6: Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.**
[CN, PS, R, T, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF6: Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.

Scope and Sequence of Outcomes:

| Grade Ten	Grade Eleven	Grade Twelve
RF9: Represent a linear function, using function notation. *(NRF10)* **RF10:** Solve problems that involve systems of linear equations in two variables, graphically and algebraically *(NRF10)*	**RF6:** Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.	**RF6:** Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions) *(PC12B)*

ELABORATION

In grade 10 students solved systems of linear equations in two variables, graphically and algebraically. In this course this will be extended to solving systems of **linear-quadratic** equations and **quadratic-quadratic** equations. This involves finding the intersection points between a line and a parabola (linear-quadratic) or between two parabolas (quadratic-quadratic).

Algebraically, linear-quadratic systems can be solved by substitution, in which one of the equations is rearranged to solve for y and substituted back into the other equation to solve for the x value at the point the two functions intersect. This x value is then used in either equation to find the y-coordinate for the intersection point. In some situations, elimination may be used.
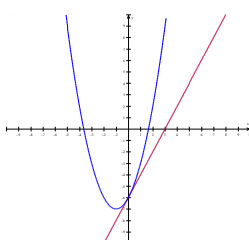
Algebraically, quadratic-quadratic systems can be solved by eliminating the variable that is not squared, so that the equation that remains after addition or subtraction is quadratic in one variable. Once one variable is found, it is then substituted back into either equation to find the other coordinate for the intersection point(s)

Graphing technology can also be used to solve systems of equations, by using the intersection function to identify the points of intersection. Once found, the solutions should be verified algebraically. The points of intersection of a system of equations will be solutions that will work for both equations.
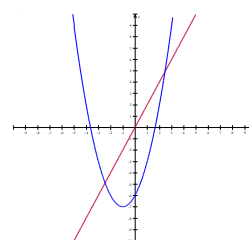
A system of linear-quadratic and quadratic-quadratic equations can have zero, one or two solutions, depending on if the graphs do not intersect, touch at one point, or cross at two points, respectively. A system will have infinite solutions if the equations are equivalent.
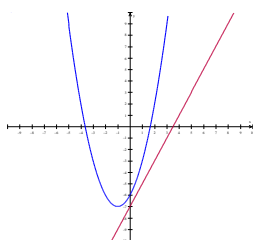
0 solutions   1 solution   2 solutions

SCO: **RF6: Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.**
[CN, PS, R, T, V]

ACHIEVEMENT INDICATORS

- Model a situation, using a system of linear-quadratic or quadratic-quadratic equations.

- Relate a system of linear-quadratic or quadratic-quadratic equations to the context of a given problem.

- Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations graphically, with technology.

- Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations algebraically.

- Explain the meaning of the point(s) of intersection of a system of linear-quadratic or quadratic-quadratic equations.

- Explain, using examples, why a system of linear-quadratic or quadratic-quadratic equations may have zero, one, two or an infinite number of solutions.

- Solve a problem that involves a system of linear-quadratic or quadratic-quadratic equations, and explain the strategy used.

Suggested Instructional Strategies

- A quick review of solving systems of linear equations using substitution, elimination and graphing should be used to introduce this unit.

- It is intended that students will use graphing technology to verify their graphical solutions to the problems in this unit. Tables of values can be used to determine algebraic solutions to these systems. Use common x values to create the tables for the two equations (this can be done using graphing technology) and have students look for common y values.

> SCO: **RF6: Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.**
> [CN, PS, R, T, V]

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q State the number of possible solutions to each system. Include graphs to support your answers.

   a) A quadratic equation and a horizontal line.
   b) Two quadratics with positive coefficients on x^2
   c) A quadratic and a line with negative slope
   d) A quadratic and a vertical line.

Answers: *a*) $0, 1$ *or* 2 *times*   *b*) $0, 1, 2$ *or infinite if quadratics equivalent*
                c) $0, 1,$ *or* 2 *times*   *d*) *once*

Q From the base of a hill with a constant slope, Steve hits a golf ball as hard as he can up the hill. The system of equations below could model this situation:

$$h = -0.18d^2 + 3.6d$$
$$h = \frac{1}{2}d$$

  a) Solve the system.

  b) If d represents horizontal distance, and h represents vertical height, interpret the points of intersection in the context of the problem.

Answer:
One point of intersection represents the starting point of the ball $(d, h) = (0,0)$,
the other is the location on the hill where the ball lands $(d, h) = (17.2, 8.6)$,

Q Solve the following system of equations.

$$y = 2x^2 - 2x - 3$$
$$y = -x^2 - 2x - 3$$

Answer: $(0, -3)$

Q Solve the following system of equations.

$$3y = -5x^2 - x - 3$$
$$-2y = 2x^2 - x + 4$$

Answer: $\left(\frac{3}{4}, -2\frac{3}{16}\right), (-2, -7)$

SCO   **RF7. Solve problems that involve quadratic inequalities in two variables.**
       [C,PS,T,V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF7: Solve problems that involve linear and quadratic inequalities in two variables.

Scope and Sequence of Outcomes:

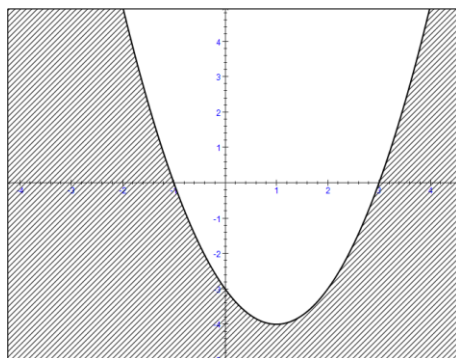| Grade Ten	Grade Eleven	Grade Twelve
RF6: Relate linear relations expressed in; slope intercepts form ($y = mx + b$); general form ($ax + by + c = 0$); and slope point form ($y - y_1) = m(x - x_1)$ to their graphs *(NRF10)*	**RF7:** Solve problems that involve quadratic inequalities in two variables.	
RF9: Represent a linear function, using function notation. *(NRF10)*		
RF10: Solve problems that involve systems of linear equations in two variables, graphically and algebraically. *(NRF10)*		

ELABORATION

In *Mathematics 9* students have solved linear inequalities, and in *Foundations of Mathematics 11* they have solved systems of linear inequalities. Students have learned that the boundary for a linear inequality is found by solving the inequality as an equation. They will apply what they have learned for linear inequalities to quadratic inequalities.

For this outcome **test points** will be used to determine which region on the graph satisfies a quadratic inequality. A test point can be any coordinate above or below the boundary. If the inequality holds true, the test point is included as part of the solution.

The **solution region** is the set of points that satisfy a quadratic inequality, also known as the solution set. Students will use set notation to describe the solution set: $\{x | a \leq x \leq b, x \in R\}$ in which a and b are numerical values. For quadratic inequalities there may be more than one region.

For example: for $y \leq x^2 - 2x - 3$, the quadratic function $y = x^2 - 2x - 3$, will mark the boundary, and will be a solid line. The solution set will include all points below and including that function.

GCO: Relations and Functions (RF): Develop algebraic and graphical
reasoning through the study of relations.

GRADE 11

SCO **RF7. Solve problems that involve quadratic inequalities in two variables.**
[C,PS,T,V]

ACHIEVEMENT INDICATORS

- Explain, using examples, how test points can be used to determine the solution region that satisfies an inequality.

- Explain, using examples, when a solid or broken line should be used in the solution for an inequality.

- Sketch, with or without technology, the graph of a quadratic inequality.

- Solve a problem that involves a quadratic inequality.

Suggested Instructional Strategies

- For this outcome students should begin with a review of linear inequalities, and then move on to quadratic inequalities, first graphing the curve of the corresponding equation as a solid ($\leq, \geq$) or broken line ($<, >$). The solution region should first be determined visually, and then confirmed by using a test point above or below the curve. If the inequality is true for the test point, then it is part of the solution region. This is an opportunity for students to make strong connections between algebraic and graphical solutions.

## Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Graph $3x - 2y < 12$ and determine if $(-2, 4)$ is part of the solution. Is the boundary line solid or dashed?

Answer: $(-2, 4)$ *is part of the solution. The boundary line is dashed,* $<$.

Q Graph $x^2 - 4x - 5 \geq y$ and determine if $(-2, 4)$ is part of the solution. Is the boundary line solid or dashed? Explain how you know.

Answer: $(-2, 4)$ *is part of the solution. The boundary line is solid,* $\geq$.

Q The Hugh John Flemming Bridge in Hartland is supported by several parabolic arches. The function $h = -0.03d^2 + 0.84d - 0.08$, approximates the curve of one of the arches, where h represents the height above where the arch meets the vertical pier, and d represents the horizontal distance from the bottom left edge of the arch to the other end of the arch, both in metres.
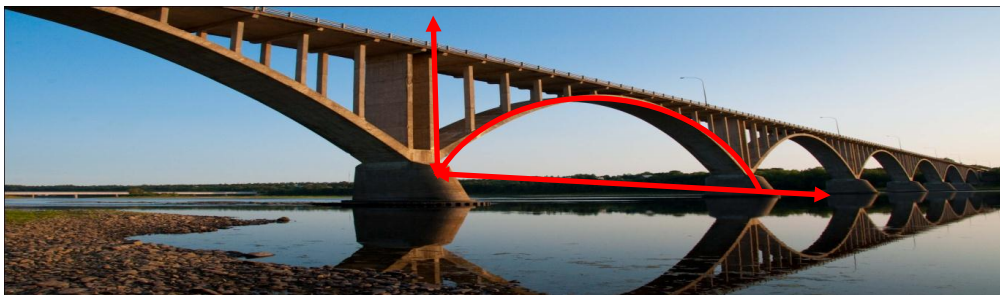
Write the inequality that describes the area under the curve. Is $(4.5, 2.3)$ part of the solution set? Explain.

Answer: $h < -0.03d^2 + 0.84d - 0.08$
Yes, $(4.5, 2.3)$ *is part of the solution because it satisfies the inequality statement when subsitituted in.*

SCO **RF8: Solve problems that involve quadratic inequalities in one variable.**
    [CN, PS, V]

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Math
[T] Technology	**[V]** Visualization	**[R]** Reasoning	and Estimation

RF8: Solve problems that involve quadratic inequalities in one variable.

Scope and Sequence of Outcomes:

| Grade Ten	Grade Eleven	Grade Twelve
RF9: Represent a linear function, using function notation. (NRF10)	**RF8:** Solve problems that involve quadratic inequalities in one variable.	

ELABORATION

This outcome follows from the previous outcome, with a numerical value being substituted for y. Given a value for y, the solution will be the values for x which satisfy the inequality. For example, for $x^2 - x - 2 > 0$, the solution will be the set of values of x for which $y > 0$.

There are different methods used to solve quadratic inequalities in one variable, and students should be given the opportunity to practice the various methods to gain a deeper understanding of the concept and to find a method with which they are comfortable. Methods include:
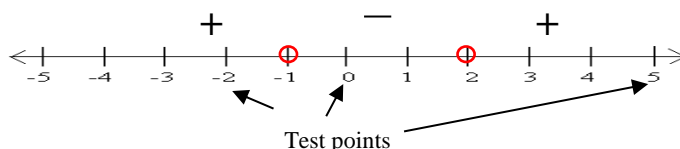
Graphing Method $(x - 2)(x + 1) > 0$

The roots of the corresponding equation, $x^2 - x - 2 = 0,$ are $x = 2 \ or \ x = -1$, and are called the **critical values** for the quadratic inequality
Visually students can see that the function, $y = x^2 - x - 2$, is greater than zero when $x < -1 \ or \ x > 2$
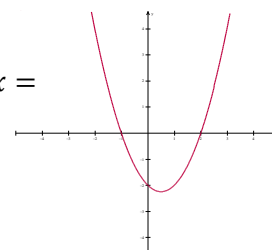
 For $x^2 - x - 2 > 0$ the solution is: $x < -1 \ or \ x > 2, x \in R$

Roots and Test Points $(x - 2)(x + 1) > 0$

The roots of the corresponding equation, $x = 2 \ or \ x = -1$, define the boundary for the inequality, and three different regions. These critical values can be shown on a number line as open circles to indicate that these roots are not included in the solution, and a test point from each region can be substituted into the inequality. If the value satisfies the inequality, that region is part of the solution set. All points in a region will have the same sign so one test point is sufficient to establish the entire region as the solution.

Test points

$(-2 - 2)(-2 + 1) > 0$	$(0 - 2)(0 + 1) > 0$	$(5 - 2)(5 + 1) > 0$
$(-4)(-1) > 0$	$(-2)(1) \not> 0$	$(3)(6) > 0$
$4 > 0$	$-2 \not> 0$	$18 > 0$

The points in the outside regions $(-2 \ and \ 5)$ satisfy the inequality (give an answer > 0), but the point in the centre region (0) doesn't. Therefore the solution set lies in the outside regions: $x < -1 \ or \ x > 2, x \in R$.

SCO	**RF8: Solve problems that involve quadratic inequalities in one variable.** [CN, PS, V]

ACHIEVEMENT INDICATORS

- Determine the solution of a quadratic inequality in one variable, using strategies such as case analysis, graphing, or roots and test points; and explain the strategy used.

- Represent and solve a problem that involves a quadratic inequality in one variable.

- Interpret the solution to a problem that involves a quadratic inequality in one variable.

Suggested Instructional Strategies

- Students should learn how to solve quadratic inequalities using their graphing calculators. This provides a quick check for their algebraic methods.

Suggested Questions (Q) and Activities (Act) for Instruction and Assessment

Q Solve using two methods: $2x^2 - 5x - 7 \leq 0$.

Answer: *1) by graphing using Quad Formula, x intercepts* $(3.5, 0), (-1, 0)$
    2) using test points to determine when ≤ 0

Q A chef determines that she can use the formula $-m^2 + 5 \geq P$ to estimate when the price of flour will be P dollars per kilogram or less in m months from the present.

   a) When will flour be available for $2/kg$ or less?
   b) Explain why some of the values of m that satisfy the inequality do not make sense in this context.
   c) Write and solve the inequality if the flour is $1/kg$ or less?

Answer: $a) 0 \leq m \leq \sqrt{3}$ $b) m \geq 0$ *as it represents time* $c) 0 \leq m \leq 2$

Q   A fish jumps from the water and follows the trajectory described by the equation $d = -4.9t^2 + 4t$ where d is distance above the water in feet, and t is time in seconds. Graph the function and determine the time interval when the fish is above the water.

Answer: $\{t | 0 < t < 0.82\}$

SUMMARY OF CURRICULUM OUTCOMES

Pre-Calculus 110

[Mathematical Processes
[C] Communication **[PS]** Problem Solving **[CN]** Connections **[ME]** Mental Mathematics
[T] Technology   **[V]** Visualization   **[R]** Reasoning   and Estimation

Algebra and Number

General Outcome: Develop algebraic reasoning and number sense.

Specific Outcomes
AN1: Demonstrate an understanding of the absolute value of real numbers. [R, V]
AN2: Solve problems that involve operations on radicals and radical expressions with numerical and
   variable radicands. [CN, ME, PS, R, T]
AN3. Solve problems that involve radical equations (limited to square roots). [C, PS, R]
AN4: Determine equivalent forms of rational expressions (limited to numerators and denominators that are
   monomials, binomials or trinomials). [C, ME, R]
AN5: Perform operations on rational expressions (limited to numerators and denominators that are
   monomials, binomials or trinomials). [CN, ME, R]
AN6: Solve problems that involve rational equations (limited to numerators and denominators that are
   monomials, binomials or trinomials). [C, PS, R]

Trigonometry

General Outcome: Develop trigonometric reasoning.

Specific Outcomes
T1. Demonstrate an understanding of angles in standard position $[0° \, to \, 360°]$. [R, V]
T2. Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard
   position. [C, ME, PS, R, T, V]

Relations and Functions

General Outcome: Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes
RF1. Factor polynomial expressions of the form:
- $ax^2 + bc + c, \ a \neq 0$
- $a^2x^2 - b^2y^2, \ a \neq 0, \ b \neq 0$
- $a\big(f(x)\big)^2 + b\big(f(x)\big) + c, \ a \neq 0$
- $a^2\big(f(x)\big)^2 - b^2\big(g(y)\big)^2, \ a \neq 0, \ b \neq 0$
   where a, b and c are rational numbers [CN, ME, R]
RF2. Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve
   problems. [C, PS, R, T, V]
RF3. Analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the vertex, domain and
   range, direction of opening, axis of symmetry, *x*- and *y*-intercepts. [CN, R, T, V]
RF4. Analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the
   corresponding graph, including vertex, domain and range, direction of opening, axis of symmetry, *x*-
   and *y*-intercepts, and to solve problems. [CN, PS, R, T, V]
RF5. Solve problems that involve quadratic equations. [C, CN, PS, R, T, V]
RF6. Solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-
   quadratic equations in two variables. [CN, PS, R, T, V]
RF7. Solve problems that involve linear and quadratic inequalities in two variables. [C, PS, T, V]
RF8. Solve problems that involve quadratic inequalities in one variable. [CN, PS, V]

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