Special Edition # 47
April 2018

Fractions across the Curriculum

Our focus on fractions draws on the work of Dr. Catherine Bruce at Trent University. Dr. Bruce and her research team are the key architects of Ontario's Fractions Learning Pathways.

If you are reading this monograph in print and want to access the hyperlinks, go to www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/fractions_across_curriculum.html

How can I justify a heavy focus on fractions in my school? The answer is quite simple: The research shows fractions are difficult to teach and difficult to learn … and problems persist through to adulthood.

– Ontario School Administrator

As Behr, Harel, Post and Lesh noted over two decades ago, "learning fractions is probably one of the most serious obstacles to the mathematical maturation of children" (cited in Charalambous & Pitta-Pantazi, 2007, p. 293).

The challenges begin at an early age (Bruce & Flynn, 2011), and, over time, a lack of fractions understanding often forces students to memorize rules "to get by" because the foundations are not in place. And the rules, when not supported with conceptual understanding, become muddled.

Consider, for example, this fractions question:

$$\frac{1}{2} \times \frac{3}{4} = \_\_\_\_\_$$

In attempting to solve this question, we often hear things like:

- "Wait! Is that when I invert and multiply?"
- "I think it's $\frac{3}{8}$ but how can that be? Doesn't multiplying makes things bigger?"
- "I thought multiplication meant "groups of" but maybe it can mean something else? I wish I could just remember that rule!"

When thinking gets muddled like this, what is lacking is fraction sense. In the pages that follow, we'll identify what this is and share some important ideas for acquiring it. We'll also report on an Ontario school which took a cross-curricular approach to helping students build their own fraction sense.

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So what is fraction sense?

In practical terms, fraction sense helps us to understand that $\frac{28}{29}$ is very close to 1, and $\frac{9}{2}$ is greater than 4. It helps us to understand that the whole matters, and that $\frac{1}{2}$ of a large quantity could be more than $\frac{3}{4}$ of a smaller quantity. Fraction sense enables us to access a range of models and drawings to represent fractional amounts and even operations with fractions. And it recognizes that fractions are used in different contexts, all of which makes fractional thinking fairly complex.

Fractions can describe:

- the relationship between a part and a whole (e.g., one-fourth of a pan of brownies)
- the relationship between a part and a part (e.g., we need one-third as much concentrate as water to make orange juice)
- a division situation (e.g., when 2 bags of marbles are shared evenly by 5 children, each receives $\frac{2}{5}$ of the bags)
- when we change or "operate on" a quantity (e.g., shrinking a photo to $\frac{2}{3}$ of its height; increasing profits by $1\frac{1}{2}$ times)

In essence, a fraction represents a *number* and it is read as a number, so we say "two-thirds" and not "two over three." This number describes a relationship between *two numbers*.

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Let's consider this problem: we need to find out how long $\frac{3}{5}$ of a 2 metre ribbon is. First, we consider the two metres of ribbon as the whole, and then we partition (divide) that ribbon into 5 segments (to create fifths). This is what the bottom number (the **denominator**) tells us; it describes the unit (fifths) that we're counting. In this case, if we partition the 2 metre ribbon into 5 equal segments, each segment will be 40 cm (200 cm ÷ 5 = 40 cm). (See Figure 1.)

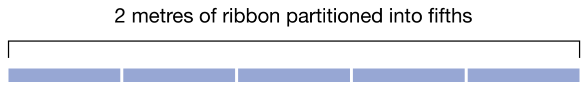


Figure 1. Two metres of ribbon equi-partitioned into five segments (fifths). Each segment would be 40 cm (200 cm ÷ 5 = 40 cm)

Once we have the unit, the top number (the **numerator**) counts or *enumerates* how many units we have. In this case we have 3 one-fifths. The numerator tells us that our amount is three times (multiplication) the length of our unit. If one-fifth of two metres (or 200 cm) is 40 cm, then 3 one-fifths is 120 cm. (See Figure 2.)

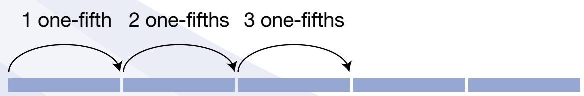


Figure 2. Three-fifths of a ribbon is three times one-fifth. If each one-fifth segment is 40 cm, then three segments, or three-fifths, is 120 cm.

There are other ways we could figure out how much $\frac{3}{5}$ of two metres of ribbon is. Fractions sense can help us here. For example, we could have:

- "changed the whole" from two metres to one metre
- found three-fifths of one metre by splitting it into fifths
- then doubled it for two metres

If one-fifth of a metre is 20 cm, then 3 one-fifths is 60 cm. Doubling 60 cm gives us three-fifths of two metres, or 120 cm. (See Figure 3.)

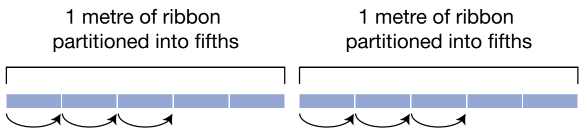


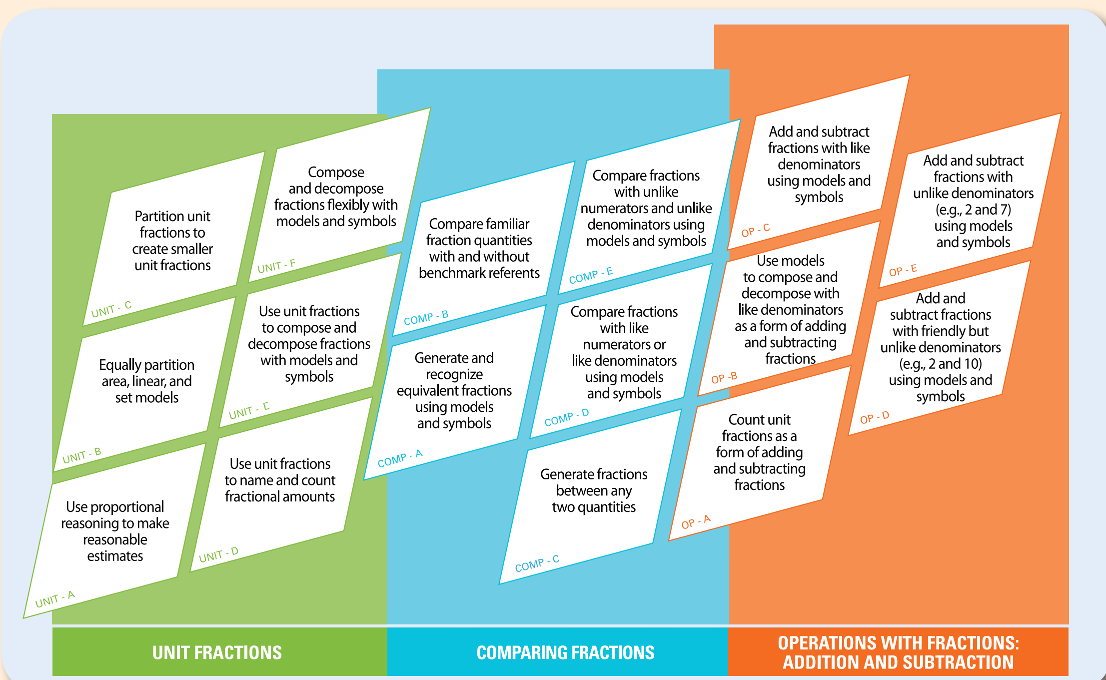
Figure 3. One-fifth of 1 metre is 20 cm. So 3 three segments, or three-fifths of 1 metre, is 60 cm. Two metres is twice as long (three-fifths x 2) so $\frac{3}{5}$ of 2 metres is 120 cm.

The drawing above also highlights other ways to think about this problem. We might notice that doubling three-fifths of 1 metre is the same as six-fifths of 1 metre. Or, if we consider the whole to be two metres, this diagram also shows six-tenths of 2 metres – which is the same as saying three-fifths of 2 metres, our original starting point. We say that three-fifths and six-tenths are equivalent fractions.

Perhaps your head is swimming now! You might want to read these last paragraphs again and work through the ideas yourself, slowly. The key idea here is that fractions always express a relationship to something else – a something else that can change! It's one of the things that can make fractional thinking complex. But as our fraction sense deepens, we become more flexible and adaptive in how we think about problems.

Fractions Learning Pathways

In 2011, a Trent University-Ministry of Education team embarked on a long-term research project to examine the foundations of the learning and teaching of fractions. The result of this ongoing research – Fractions Learning Pathways – is an interactive planning tool that includes a range of field-tested tasks for Grades 3–10. The collection of tasks follows a logical sequence that can be modified and/or adjusted to fit teacher and student needs. Video and photos are included to bring the learning to life. The tool also includes one-page summaries of key fractions ideas as well as samples of Ontario students' thinking.



Bruce, C., Flynn, T., & Yearley, S., (ongoing). Fractions Learning Pathways: www.fractionslearningpathways.ca

Seven Important Ideas when Teaching and Learning Fractions

1. Unit fractions are the key.

A unit fraction is a fraction with a numerator of 1. So $\frac{1}{4}$ and $\frac{1}{8}$ are both unit fractions but $\frac{3}{5}$ is not.

If you want students to understand fractions better, start by focusing on unit fractions. When we focus on the unit, we help students focus on one part of the fraction at a time. The numerator (top number) is simply a count of units. Any fraction can be thought of as a multiple of its unit fraction.

TIP: Have students count by unit fraction: 1 one-eighth, 2 one-eighths, etc. to emphasize the role of the unit. Practise "decomposing" fractions like $\frac{3}{4}$ into unit fractions (3 one-fourths; 1 one-half; and 1 one-fourth).

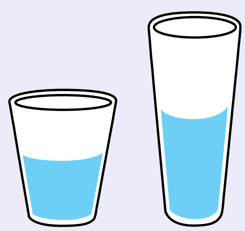
2. The whole matters. So do equal parts.

The whole might be an object, a collection or an amount but regardless of its makeup, a fraction describes *equal parts*. Contrary to what young children might ask for, there is no such thing as a "big half"!

TIP: Have students fold paper, partition an image or drawing or even create a measuring cup. As they do, have students pay attention to precision and remind them the parts must be equal. The act of partitioning reinforces an understanding of the relationship between the unit fraction and the whole.

Draw out that the size of a fraction depends on the whole it's related to. So it's possible for $\frac{1}{3}$ to be bigger than $\frac{1}{2}$ *if* the wholes are different.

But note that equal parts does not necessarily mean they're all the same shape. A half cup poured into a tall thin container will look different than if poured into a short wide container. Equal parts do not necessarily look alike.

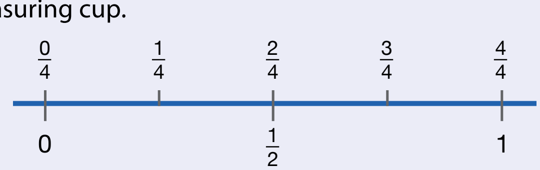


3. Number lines are a powerful tool to model fractions.

Number lines are a powerful mathematical thinking tool for solving problems and communicating ideas. They are particularly effective for helping students understand everything about fractions including equal partitions, comparing fractions, identifying equivalent fractions and operations with fractions. And unlike area models (rectangles, circles), there is only one dimension (length) to consider.

Number lines show a distance from zero. Fraction strips – whether student created or commercial – are a variation of the number line. So are tape measures and the scale on a measuring cup.

Double number lines can show two units at once and are great to show equivalence. A measuring tape that has both imperial and metric units is an example of a double number line.



TIP: Avoid using circles to model fractions – no more pizza scenarios! – since they are hard to partition evenly. Number lines have longevity across grades, mathematical number systems and a wide range of contexts.

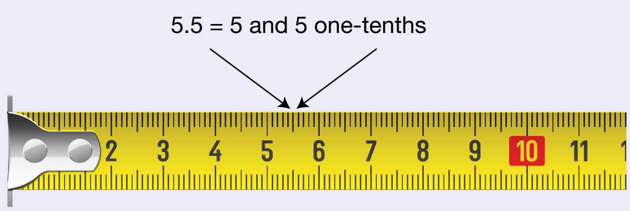
4. Decimals ARE fractions.

Many students have an over-reliance on converting fractions to decimals, yet with little understanding of the relationship or what a decimal means.

Like fractions, decimals have a unit; but unlike fractions, only the numerator is visible. In a decimal, the unit is hidden in the place value notation.

As a result, many students see 0.6, read it as "zero point six", but don't know 6 of what. They don't understand that 0.6 means 6 one-tenths and that 0.26 means 26 one-hundredths or 2 one-tenths and 6 one-hundredths.

TIP: Whenever possible, read decimals as fractions. This makes the unit explicit, connects decimals to fractions and strengthens understanding of place value.



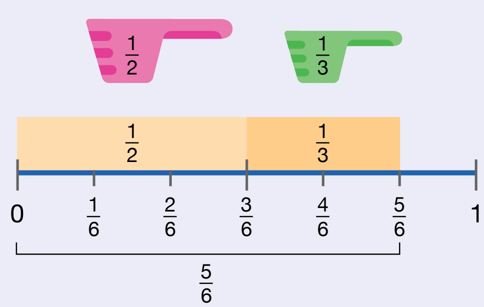
> If you want students to understand fractions better, start by focusing on unit fractions.

5. Adding and subtracting anything – including fractions – requires that we find the common unit.

Adding or subtracting any quantity – apples, oranges, fractions – requires that we count the same units. Five oranges and three oranges is eight oranges. And with fractions, 2-fifths and 4-fifths is 6-fifths. Even, for example, when we "just know" that $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$, it is only true because $\frac{1}{2}$ is equivalent to $\frac{2}{4}$.

When adding or subtracting fractional quantities with different units, we find equivalent forms of the fractions that have common *units*.

TIP: Some people call this "finding a common denominator." But to help students connect their knowledge of unit in other contexts (place value, money, measurement, life), we prefer "finding a common *unit*."



$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

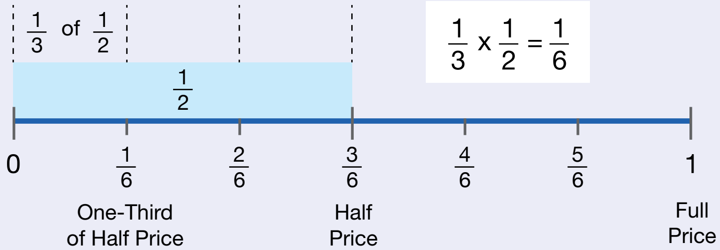
6. Multiplying fractions involves taking a part of a part.

When we speak about an athlete's production being a third of last year, or a half-price sales item being further reduced by one-third, we are multiplying fractions.

You can think of multiplying fractions as finding a part of a part. It involves partitioning something that has already been partitioned and sharing that space.

If we have a half-priced object – a portion – and need to find a third of that space, we could model it with a number line.

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$



One-third of half-price is the same as (or equals) one-sixth of the full price. We write $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

TIP: Notice how we consider different "wholes" at different times. Sometimes we're partitioning the full price; sometimes we're partitioning the half price. Visuals can help keep track of which whole we're considering.

7. Dividing fractions can answer how many of one fraction is in another fraction.

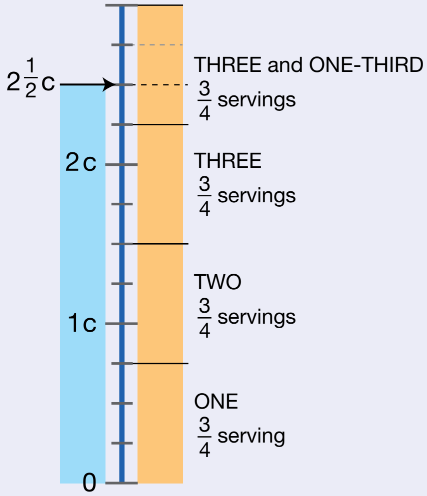
When we need to see how many $\frac{3}{4}$ cup servings are in a box holding $2\frac{1}{2}$ cups, we are dividing fractions. In this case, division tells us how many fractional pieces ($\frac{3}{4}$) will fit into an amount ($2\frac{1}{2}$). We can use repeated addition, and model it with a number line.

We can get 3 three-fourths cup servings out of this box and $\frac{1}{3}$ of another serving. Note that the remainder is a fraction of the $\frac{3}{4}$ cup serving – which is our whole.

Division can also tell us the size of each fractional piece if we know the number of pieces. Suppose we split a half box of cereal between three people. We write $\frac{1}{2} \div 3$. This means each person gets $\frac{1}{3}$ of the $\frac{1}{2}$ box or $\frac{1}{2} \times \frac{1}{3}$ – the same diagram as Note 6 for multiplication.

TIP: When calculating with fractions, estimating the answer is a great way to develop and test your fraction sense. Contrary to popular opinion, we notice that multiplying does not always "make bigger" and division does not always "make smaller!"

$$2\frac{1}{2} \div \frac{3}{4} = 3\frac{1}{3}$$



Common Fractions Challenges across the Curriculum

An Ontario Case Study

Because fractions are so central to future success and such a challenge for so many students, it is a prime candidate for a whole-school or grade-integration focus.

One Ontario high school chose to do a research project focused on teaching and learning fractions across multiple disciplines. They identified common goals and concepts for student learning, then planned and implemented a unified focus on required knowledge and key skills. The group looked for opportunities to provide students with distinct but overlapping opportunities to build and expand their fraction understanding.

The team consisted of a mathematics teacher, a food and nutrition teacher, a music teacher, a manufacturing teacher and two educational assistants. The team met formally four times (approximately 15 hours in total) per semester and informally between these meetings. An instructional coach was available for co-planning and co-teaching.

In their initial meeting, each of the teachers identified subject-specific concerns related to fractions that were preventing students from gaining a broad and deep understanding of their discipline. These needs and areas of intersection are captured in the next few pages and serve as examples of how fraction teaching and learning might be integrated across a school.

Following the summary of common fractions challenges in three subject areas – food and nutrition, manufacturing and music – we share with you two sample tasks that the educator team used to help students develop better fraction sense.

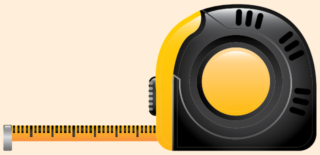
FOOD AND NUTRITION

Challenge: Students have trouble reading and understanding imperial measuring cups and spoons. They have difficulty, for example, using a one-third cup to generate recipe amounts for two-thirds or how a one-fourth cup could be used to find a one-eighth cup.

Underlying fraction ideas:

- We can equally partition or decompose any quantity and describe each part as a unit fraction (i.e., a fraction with 1 as a numerator such as $\frac{1}{4}$). Measuring cups and measuring spoons represent the results of different partitions. A one-fourth cup is the result of splitting one cup into four equal parts.
- We can recreate (or "compose") the whole by repeating ("iterating") the unit fraction amount and counting. For example, 4 one-fourth cups give us 1 whole cup.
- The same amount can be measured using different-sized unit fractions (e.g., 1 one-fourth cup fits into 2 one-eighth cups). We say that these are "equivalent fractions."
- Some measuring cups include a variety of different-sized unit fractions on the same cup (e.g., thirds, fourths, halves) and use a calibrated scale to replace the need to manually repeat (iterate) a unit fraction amount. Equivalent fractions – fractions with different units (denominators) but of equal amount – line up with each other. For example, 1 one-half cup is the same as 2 one-fourth cups.

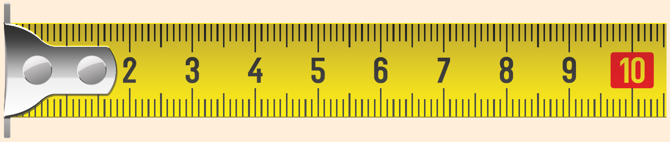
MANUFACTURING



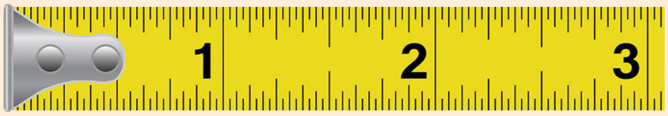
Challenge: Students struggle to measure and cut a board to a pre-determined length. They find it difficult to read a tape measure accurately.

Underlying fraction ideas:

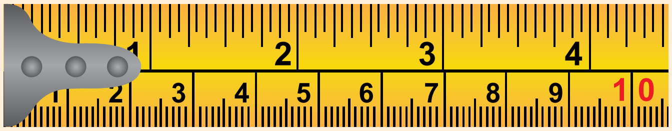
- We can equally partition a length and describe each part as a unit fraction (i.e., a fraction with 1 as a numerator such as $\frac{1}{4}$). Rulers and tape measures show these partitions with different line lengths.
- With metric tape measures, the different line lengths show centimeters, halves (5 millimeters) and tenths (millimetres). We can count the unit fractions and recreate (or compose) the whole. For example, 10 one-tenths (millimetres) make 1 centimetre and so do 2 one-halves.



- With imperial tape measures, the inch is segmented into sixteen (and sometimes even thirty-two) sections, with the lines becoming progressively longer as we move from thirty-seconds, sixteenths, eighths, fourths, halves and one inch. We can count the unit fractions and recreate (or "compose") the whole. For example, 8 one-eighths make 1 inch and so do 16 one-sixteenths.



- Sometimes both metric and imperial units are shown on the same tape measure. It is important to specify which unit we use.



- While all measurements are estimates – any unit can be further split into an infinite number of fractional parts! – we increase our precision by using smaller segments. Measuring to the nearest thirty-second is more precise than measuring to the nearest half. We choose how precise we need to be when we choose our unit.
- The same length can be measured using different-sized unit fractions (e.g., 1 one-quarter inch is the same length as 2 one-eighths of an inch and 4 one-sixteenths of an inch). We say that these measurements are "equivalent fractions."

MUSIC

Challenge: Students have trouble understanding the relationship between note values, including rests, and bars of music. The music teacher notices that students struggle with reading rhythms in music and wonders whether this is related to issues with fractions.

Underlying fraction ideas:

- To create rhythm, we partition (subdivide) an amount of time (a whole bar) into "beats." These beats are actually "unit fractions" – a fraction with 1 as its numerator, such as $\frac{1}{4}$.
- In a "four-beat bar," we can create 2 one-halves (called half notes), 4 one-fourths (called quarter notes, although mathematically it would be better to call these "fourth notes"!), 8 one-eighths (called eighth notes) or 16 one-sixteenths (called sixteenth notes). The same pattern follows for rests.
- The more partitions we make, the smaller the amount of time between the beats, meaning the beats happen faster. So eighth notes happen faster than "fourth" notes because the same amount of time is being split into eight sections rather than only four.
- Various combinations of unit fractions are possible as long as they add up to the amount of time in a whole bar.
- Partitioning a unit fraction (or beat) into smaller unit fractions (e.g., subdividing each fourth note to create 2 eighth notes) helps us see mathematical and musical relationships among:
    - ◊ the denominator (the type of note)
    - ◊ the number of parts in the whole quantity (how many fit into a bar)
    - ◊ the size of the part (the duration of that note)

Sample Tasks to Address Student Challenges

Although each of these challenges may seem unique to the particular subject area, the team learned that there was a common underlying problem: a fragile understanding of the *unit fraction*. Using resources from the Fractions Learning Pathways, they began by analyzing students' areas of need. They then used the Pathways to identify tasks to close gaps with students across subject areas, including the Unit Fractions Counting Game and Pretty Powerful Paper Folding, described next.

Sample Task:

The Unit Fractions Counting Game

The team learned that, to develop an understanding of unit fractions, students should have ongoing experience naming and counting unit fractions. In the Counting Game:

> Students "count up" using unit fractions. The students or teacher can choose any unit fraction (for example, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, etc.), and the teacher or students can set game rules such as: "When you get to one whole, stand up and state the quantity as both a fraction and as a whole." This game is similar to a well-known number game called BUZZ. For full instructions and examples of student thinking, including videos, click here.

The teachers modified the task to fit within their subject context so that in music students counted fourths and eighths, while in manufacturing the unit fractions included counting sixteenths, and food and nutrition extended this to counting thirds.

Why this helps: When thinking about part-whole relationships, we learn that one fourth is a 1 one-fourth unit of the whole. Two fourths are 2 one-fourth units. When we count these fourths, we use the language 1 one-fourth, 2 one-fourths, 3 one-fourths, 4 one-fourths, 5 one-fourths and so on. In this example, we are counting units that are fourths, and this allows us to count beyond one whole easily, such as 5 one-fourths. Unfortunately, we have had a tendency to shorthand these expressions in school mathematics to words like "one quarter" which is true for money, but may be confusing for other types of fractions situations that involve area and length – and we assume that students understand what we mean.

Sample Task:

Pretty Powerful Paper Folding

In order to help students see that a fraction is always relative to a whole, and to focus on the importance of equi-partitioning that whole (ensuring the segments are all the same amount), the team introduced students to the Pretty Powerful Paper Folding task. This task also formed a foundation for understanding equivalent fractions:

> Students fold colourful paper strips into equal parts that represent unit fractions and label the folded pieces with symbolic notation (e.g., $\frac{1}{2}$). The strips are powerful visual tools in that they allow students to see the relative size of fractional pieces, which allows them to compare familiar fractional quantities. For full instructions and examples of student thinking, including video, click here.

The teachers modified the task for their particular discipline. In food and nutrition, the paper was positioned in portrait format and the folds were made horizontally to simulate the scale on a measuring cup. In manufacturing, they turned strips of paper landscape and made vertical folds to simulate the linear nature of a tape measure. Students prioritized precision in the paper folding, and by extension, measurement and design.

To build understanding around equivalent fractions, food and nutrition students were asked, "What unit fractions could you use to make $\frac{3}{4}$ of a cup? How many different ways can you make $\frac{1}{2}$ of a cup?" This was extended to include combinations to make $1\frac{1}{4}$ cups and also included the relationship between cups and tablespoons (sixteenths).

In music, students were asked to identify the notes required to equal one $\frac{1}{2}$ note. To extend the learning, students could use the strips, divided into sixteenths, to model different possible rhythms (equivalent fractions) that total four beats (a whole).

Why this helps: Visual representations, such as paper strips, help students "see the math" and build mental models. They reinforce that a fraction expresses a relationship between the parts of something and its whole, so that if someone uses a differently sized whole then his or her one-third could be larger than another's half. The representation lets students see why as a fraction's denominator increases the size of the segment gets smaller – something that students typically struggle to understand. And it helps them understand the notion of equivalent fractions and see why two-fourths really is the same quantity as one-half. This contributes to the development of proportional reasoning, which is so important for mathematical literacy overall and for everyday mathematics in particular.

Making the Math Explicit

As the team continued to explore fractions connections across subject areas and identify ways to further support student learning in fractions, they added more activities with direct subject connections.

In manufacturing, the selection of the right tool for a task can be challenging. Prompted by a student's claim that he "could not find the $\frac{6}{8}$ wrench," the manufacturing teacher had students try ordering a set of sockets by their size. The teacher drew students' attention to equivalent fractions, and using fraction strips from paper folding, the students were then able to see that $\frac{6}{8}$ was in fact already represented by the $\frac{3}{4}$ socket. Students also started to appreciate the density of fractions (that there are an infinite amount of fractions between any two numbers) when noticing that the $\frac{7}{8}$ socket was slightly larger than the $\frac{3}{4}$ socket, while the $\frac{5}{8}$ socket was slightly smaller. With some guidance, students used the fraction strips to confirm that the $\frac{5}{8}$ socket was positioned between the $\frac{1}{2}$ inch socket and the $\frac{3}{4}$ inch socket. Students then had greater success ordering other sockets by size. These types of purposeful activities allowed students to connect fractions concepts learned in mathematics to their application in manufacturing.

Inspired by student thinking in the manufacturing class, the food and nutrition teacher decided to do a similar activity using measuring cups. Students explored capacity and equivalence between teaspoons, tablespoons and cup measures as they positioned the various measures on a vertical number line.

Through this process, these educators redefined their role in the teaching and learning of fractions. Although initially it was tempting to "hide the math" so that the students wouldn't know they were doing it ("It's like covering broccoli with cheese!"), by the end of the semester the teachers became more explicit about the connections. For example, the manufacturing teacher started a later lesson by stating that the task was going to focus on fractions and that fractions had many real-life applications, including design, cutting materials and selecting the correct tool size.

Putting It All Together

By now we know that students can deepen their understanding of fractions when the focus is on foundational ideas – whether in art, social science, history, manufacturing, or food and nutrition. The more connections that are made across subject areas, the greater the chance of solidifying understanding for students.

It is important to note that rather than requiring more time and resources, fractions teaching can be amplified through effective representations, the use of precise mathematics language and a grounding in unit fractions.

With a renewed focus on mathematics in Ontario, our goal has been to encourage school leaders and teachers alike to focus on fractions in class and during professional learning opportunities, not only in mathematics but across the curriculum. We hope we have succeeded in spurring you on to put the focus on fractions!

Just as whole-school, cross-curriculum approaches to literacy learning led to significant gains in student achievement over the past decade, they will lead to improvements in mathematics learning over the next one. And as fractions understanding improves, so too will achievement in other disciplines.

Some Other Examples of Applying Fractions across the Curriculum

Fractions in Art scaling, colour-blocking, quilting, perspective, geometric and symmetrical designs and portioned space

Fractions in Science logarithmic scales, exponential growth and decay, measurement (physics – calculating slope, speed, torque)

Fractions in Geography scaling graphs and cross-sections, changes in measure (population, GDP)

Fractions in History historical timelines

Fractions in Health food sharing, cooking, serving portions, nutrition, medical doses, heart beats per minutes, steps per day

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ISSN: 1913 8482 (Print)
ISSN: 1913 8490 (Online)