On the Hilbert polynomials and Hilbert series of homogeneous projective varieties

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Among all complex projective varieties $X \hookrightarrow \mathbb{P}(V)$, the equivarient embeddings of homogeneous varieties—those admitting a transitive action of a semi-simple comples algebraic group G—are the easiest to study. These include projective spaces, Grassmannians, non-singular quadrics, Segre varieties, and Venonese varieties. In Joe Harris beautiful book "*Algebraic geometry–a first course*" [H], he computes the dimension $d = \dim(X)$ and degree $\deg(X)$ of $X \hookrightarrow \mathbb{P}(V)$ for many homogeneous varieties, in a geometric fashion.

In this expository paper we redo these calculations algebraically, using the representation theory of G to determine the Hilbert polynomial $h(t)$ of the coordinate ring of $X \hookrightarrow \mathbb{P}(V)$ since

$$h(t) = \deg(X) \cdot \frac{t^d}{d!} + (\text{lower order terms})$$

with $d = \dim(X)$, this gives formula for the two invariants. As a byproduct, we find that $h(t)$ is the product of linear factors over $\mathbb{Q}$.

We now state the results precisely. Fix a maximal torus T contained in a Borel subgroup B of G. The projective varieties X which admit a transitive action of G correspond to the 2^n subgroups P of G which contain B (where $n = \dim(T)$). These varieties depend only on G up to isogeny, so there is no loss of generality in assumlng that G is simply-connected, and we will henceforth do so. The equivariant projective embeddings π_λ of $X = G/P$ into $\mathbb{P}(V)$ then correspond bijectively to the dominant weights λ for T which lie in a certain face of the closed Weyl chamber corresponding to B.

The Hilbert polynomial $h_\lambda(t)$ of the coordinate algebra of $\pi_\lambda : X \hookrightarrow P(V)$ factors as the product

$$h_\lambda(t) = \prod_\alpha (1 + c_\lambda(\alpha)t).$$

This product is taken over the set of positive roots α of G which satisfy $\langle \lambda, \alpha^\vee \rangle \neq 0$; the number d of such roots is equal to the dimension of X. In the product, $c_\lambda(\alpha)$ is the positive rational number

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where ρ is half the sum of the positive roots. Hence

$$\deg(X) = d! \prod_\alpha \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where the product is taken over the same subset of positive roots. This simple formula for the degree was obtained by Borel and Hirzebruch [B-H, Theorem 24.10], using characteristic classes for the compact form of G.

Using the same methods we also calculate the Hilbert series of the image of the equivariant embedding corresponding to λ.

After sketching the proof of these results, which follows from the Borel-Weil theorem and Weyl's dimension formula, we illustrate it by calculating the degrees and Hilbert series of several equivariant embeddings.

1. Equivariant embeddings

Let G be a semi-simple, simply-connected, complex algebraic group. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup. the choice of B determines a set of positive roots for G—those characters of T which occur in $\mathrm{Lie}(B)/\mathrm{Lie}(T)$—as well as a Weyl chamber of dominant weights in the character group of T. We say a weight λ is dominant if the integer $\langle \lambda, \alpha^\vee \rangle$ is $\geqslant 0$ for all positive roots α. Here $\alpha^\vee$ is the corresponding co-root, denoted H_α in [S]. If ρ is half the sum of the positive roots, then ρ is a dominant weight in the interior of the Weyl chamber: $\langle \rho, \alpha^\vee \rangle$ is strictly positive for all positive roots α.

Associated to every dominant weight λ for T there is an irreducible representation $V = V_\lambda$ of G over $\mathbb{C}$ with highest weight λ for B. Let V^* be the dual representation, and let $\langle f \rangle$ be the unique line in $V^* = \mathrm{Hom}(V, \mathbb{C})$ fixed by B; the character of T on this line is $i(\lambda)$, where i is the opposition involution of G. Let $P \supset B$ be the parabolic subgroup of G which stabilizes the line $\langle f \rangle$ in V^*, or equivalently which stabilizes the hyperplane H annihilated by f in V.

Let $\mathbb{P}(V)$ denote the projective space of *all* hyperplanes in V. This has coordinate ring

$$A(\mathbb{P}(V)) = \mathrm{Sym}^\bullet(V) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n(V)$$

Associated to λ, we have the equivariant embedding

$$\pi_\lambda : X = G/P \hookrightarrow \mathbb{P}(V)$$

defined by mapping the coset gP to the hyperplane $g(H)$. The image of π_λ is the unique closed orbit of G on $\mathbb{P}(V)$, and is a homogeneous, nonsingular projective variety [F-H,].

2. The Hilbert polynomial

We fix an equivariant embedding

$$\pi_\lambda : X \hookrightarrow \mathbb{P}(V).$$

The line bundle $\mathcal{L} = \pi^*\mathcal{O}(1)$ on X is equivariant and has sections

$$H^0(X, \mathcal{L}) = V = V_\lambda$$

Then $\mathcal{L}^n = \pi^*\mathcal{O}(n)$ is also equivariant, and by the theorem of Borel and Weil (cf. [F-H, 393])

$$H^0(X, \mathcal{L}^n) = V_{n\lambda}$$

for all $n \geqslant 0$.

Since the restriction homomorphism $(n \geqslant 0)$

$$\begin{array}{ccc} H^0(\mathbb{P}(V), \mathcal{O}(n)) & \longrightarrow & H^0(X, \mathcal{L}^n) \\ \| & & \| \\ \mathrm{Sym}^n(V) & \longrightarrow & V_{n\lambda} \end{array}$$

is G-equivariant and non-zero, and $V_{n\lambda}$ is irreducible, it must be surjective for all $n \geqslant 0$. Hence the embedding of X is projectively normal, and the coordinate ring of X is given by

$$A(X) = \bigoplus_{n \geqslant 0} V_{n\lambda}.$$

In particular, the Hilbert polynomial $h_\lambda(t)$ of $\pi_\lambda : X \hookrightarrow \mathbb{P}(V)$ satisfies

$$h_\lambda(n) = \dim V_{n\lambda}$$

for $n \gg 0$.

But the Weyl dimension formula states that

$$\dim V_{n\lambda} = \prod_{\alpha>0} \frac{\langle n\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where the product is taken over all positive roots α. Hence

$$\dim V_{n\lambda} = \prod_{\alpha>0} (1 + n \cdot c_\lambda(\alpha))$$

with

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}.$$

Therefore the polynomial

$$h_\lambda(t) = \prod_{\alpha>0} (1 + t \cdot c_\lambda(\alpha))$$

satisfies $h_\lambda(n) = \dim A_n(X)$ for all $n \geqslant 0$. This completes the determination of the Hilbert polynomial of $X \hookrightarrow \mathbb{P}(V)$ using representation theory.

3. Hilbert series

One can also calculate the Hilbert series of the image of the projective embedding of G/P corresponding to λ. The observations in section 2 imply that the series is given by (cf. [A-M, pg 116-118])

$$H(q) = \sum_{n \geq 0} (\dim V_{n\lambda})\, q^n.$$

This series must represent a rational function of the form

$$\frac{g(q)}{(1-q)^{d+1}}$$

with $g(q)$ a polynomial with integer coefficients and $d = \dim G/P$. We note that $g(1)$ is the degree of the embedding. The Weyl dimension formula implies that we can write

$$H(q) = \sum_{n \geq 0} \left(\prod_{\langle \lambda, \check{\alpha} \rangle > 0} \frac{\langle n\lambda + \rho, \check{\alpha} \rangle}{\langle \rho, \check{\alpha} \rangle} \right) q^n = \sum_{n \geq 0} \left(\prod_{\langle \lambda, \check{\alpha} \rangle > 0} (nc_\lambda(\alpha) + 1) \right) q^n.$$

Let $\beta_1, ..., \beta_d$ be an enumeration of the set of roots α such that $\langle \lambda, \check{\alpha} \rangle > 0$. Let e_j be the jth elementary symmetric function in d variables then we have (after a bit of manipulation)

$$H(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), ..., c_\lambda(\beta_d)) \sum_{n \geq 0} n^j q^n.$$

Thus to complete the determination of $g(q)$ we must calculate $f_j(q) = \sum_{n \geq 0} n^j q^n$. This rational function has a long history but for the sake of completeness we will give the simplest (from our perspective) route to it. We note that

$$q \frac{d}{dq} f_j(q) = f_{j+1}(q).$$

Thus since $f_0(q) = \frac{1}{(1-q)}$ we must have

$$f_j(q) = \left(q\frac{d}{dq}\right)^j \frac{1}{1-q} = \frac{\phi_j(q)}{(1-q)^{j+1}}$$

and since $q\frac{d}{dq}$ preserves degree $\phi_j(q)$ is a polynomial of degree j. We write

$$\phi_j(q) = \sum a_{j,i}q^i.$$

We note that $a_{j,0} = 0$. If we arrange the $a_{j,i}$ with $i = 1, ..., j$ in a triangle with jth row $a_{j,1}, ..., a_{j,j}$ we have

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

This triangle is called Euler's triangle and it has been studied intensively. We note one property. Consider the diagonals (the second diagonal is 1,4,11,26,...) then the element in the ith diagonal and the nth row is the number of permutations with exactly i descents.

The upshot is that

$$H(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), ..., c_\lambda(\beta_d))\frac{\phi_j(q)}{(1-q)^{j+1}}.$$

This implies that

$$g(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), ..., c_\lambda(\beta_d))\phi_j(q)(1-q)^{d-j}.$$

In particular, since $g(1) = \deg \pi_\lambda$, we have the formula

$$\deg \pi_\lambda = e_d(c_\lambda(\beta_1), c_\lambda(\beta_2), ..., c_\lambda(\beta_d))\phi_d(1).$$

This agrees with the formula for the degree in the introduction as $\phi_d(1) = d!$.

There is another more suggestive way of writing the above formula for $H(q)$. We note that if we consider the case of the standard Segre embedding of $P^1 \times \cdots \times P^1$ (j copies) into $\mathbb{P}(\otimes^j \mathbb{C}^2)$ then the Hilbert series is

$$\sum_{n \geq 0} (n+1)^j q^n = \frac{\frac{\phi_j(q)}{q}}{(1-q)^{j+1}}.$$

So the degree of this embedding is $d!$. This also says that the formula above for $H(q)$ expresses the Hilbert series of $\pi_\lambda(G/P)$ in terms of the Hilbert series of $\times^j \mathbb{P}^1$ for $j = 1, ..., d$. The simplest example is the case of G/B with $\lambda = \rho$. Then the formula becomes

$$H_{G/B}(q) = H_{\times^d \mathbb{P}^1}(q)$$

with d equal to the number of positive roots.

We can summarize with the following result.

Theorem. *The Hilbert series of the embedding π_λ of G/P is*

$$\prod_{\langle \lambda, \check{a} \rangle > 0} \left(\frac{\langle \lambda, \check{a} \rangle}{\langle \rho, \check{a} \rangle} q \frac{d}{dq} + 1\right) \frac{1}{1-q}.$$

4. Veronese varieties

For the first examples, we observe that the variety X remains unchanged as we scale λ by an integer $m \geqslant 1$. If $\dim(X) = d$, then

$$\deg(\pi_{m\lambda}) = m^d \cdot \deg(\pi_\lambda)$$

as every factor $c(\alpha)$ in the product for the degree is scaled by m.

We apply this to $G = SL(V)$ and $V = V_\lambda$ the standard representation. Then $X = \mathbb{P}(V) = \mathbb{P}^n$, where $\dim(V) = n + 1$, and $\deg(\pi_\lambda) = 1$. Hence the Veronese embedding

$$\pi_{m\lambda} : \mathbb{P}^n \longrightarrow \mathbb{P}(\mathrm{Sym}^m V) = \mathbb{P}^{\binom{m+n}{n}-1}$$

has degree $= m^n$.

For $n = 1$, this is the rational normal curve, of degree m in $\mathbb{P}^m$. For $n = 2$ and $m = 2$ this gives the degree $(= 4)$ of the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

5. The flag variety

Another simple case is the embedding of the full flag variety $X = G/B$ using the representation V_ρ. (The dominant weight ρ is the simplest weight in the interior of the Weyl chamber; the stabilizer of its highest weight vector $\langle v_\rho \rangle$ is equal to B.)

In this case, $\dim(V_\rho) = 2^d$ by the Weyl dimension formula, where $d = \dim(X)$ is the number of positive roots. Moreover, for every positive root α we have

$$c_\rho(\alpha) = \frac{\langle \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = 1.$$

Hence $h_\rho(t) = (t+1)^d$ and

$$\pi_\rho : X = G/B \hookrightarrow \mathbb{P}^{2^d - 1}$$

has degree $= d!$. Compare this to the linear system $|2\Theta|$ on a principally polarized abelian variety A of dimension d, which maps $A \to \mathbb{P}^{2^d-1}$ with degree $2^d \cdot d!$.

6. Segre varieties

We next consider the representation of $G = SL(W) \times SL(U)$ on $V = \mathrm{Hom}(W, U) = V_\lambda$. The closed orbit X of G on $\mathbb{P}(V)$ consists of the linear maps of rank 1; this gives the Segre embedding

$$\pi_\lambda = \pi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{mn+m+n}$$

where $n + 1 = \dim(W)$ and $m + 1 = \dim(U)$.

Let $\{e_1, \ldots, e_{n+1}\}$ be the weights for $SL(W)$ on W and $\{f_1, \ldots, f_{m+1}\}$ be the weights for $SL(U)$ on U. The highest weight of $V_\lambda = W^* \otimes U = \bigwedge^n W \otimes U$ is

$$\lambda = (e_1 + e_2 + \cdots + e_n) + f_1.$$

There are $(n + m) = d$ positive roots α with $c_\lambda(\alpha) \neq 0$:

$$\alpha = e_i - e_{n+1} \qquad i = 1, 2, \ldots, n$$

$$\alpha = f_1 - f_j \qquad j = 2, 3, \ldots, m + 1.$$

Since $\rho = ne_1 + (n - 1)e_2 + \cdots + e_n + mf_1 + (m - 1)f_2 + \cdots + f_m$ we find

$$\begin{aligned} c_\lambda(\alpha) &= \frac{1}{(n + 1 - i)} \qquad &&\text{in the first case} \\ &= \frac{1}{(j - 1)} &&\text{in the second case.} \end{aligned}$$

Hence

$$\begin{aligned} \deg(\pi_{n,m}) &= d! \prod c_\lambda(\alpha) \\ &= (m + n)! \cdot \frac{1}{n!} \cdot \frac{1}{m!} \\ &= \binom{m + n}{n}. \end{aligned}$$

For example, the degree of the Segre 3-fold

$$\pi_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

is equal to $\binom{3}{1} = 3$.

Using the same data we can compute the Hilbert series of $\mathbb{P}^n \times \mathbb{P}^m$ for $n \leq m$ yielding

$$\frac{\sum_{1\leq j\leq n} \binom{n}{j}\binom{m}{j} q^j}{(1-q)^{n+m+1}}.$$

7. Grassmannians

We now consider the Plucker embedding of the Grassmannian $G(k,n)$ of $(n-k)$ – planes (i.e., subspaces of codimension k) in $\mathbb{C}^n$. In this case $G = SL_n$ and $V = V_\lambda = \bigwedge^k \mathbb{C}^n$.

The highest weight λ of V is

$$\lambda = e_1 + e_2 + \cdots + e_k$$

and there are $d = k(n-k)$ positive roots α with $c_\lambda(\alpha) = \langle \lambda, \alpha^\vee \rangle / \langle \rho, \alpha^\vee \rangle$ non-zero. We recall that

$$\rho = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}.$$

The relevant roots are those of the form $\alpha = e_i - e_j$ with $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$. All of these roots have $\langle \lambda, \alpha^\vee \rangle = 1$, and we find that $c_\lambda(\alpha) = 1/(j-i)$. Hence

$$\begin{aligned} \deg(G(k,n)) &= d! \prod_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant n}} \frac{1}{(j-i)} \\ &= (k(n-k))! \prod_{1 \leqslant i \leqslant k} \frac{(k-i)!}{(n-i)!}. \end{aligned}$$

For example, the degree of $X = Gr(2, n+2)$ in $\mathbb{P}(\bigwedge^2 \mathbb{C}^{n+2}) = \mathbb{P}^{(n^2+3n)/2}$ is equal to

$$(2n)! \frac{1}{(n+1)!} \frac{1}{n!} = \frac{1}{n+1} \binom{2n}{n},$$

the Catalan number c_n.

The corresponding Hilbert series for X is

$$\frac{\sum_{1 \le j \le n} \frac{1}{n} \binom{n}{j} \binom{n}{j-1} q^{j-1}}{(1-q)^{2n+1}}.$$

The polynomial in the numerator has coefficients the Narayana numbers. If these numbers are laid out in a triangle they yield the so called Catalan triangle.

A similar case is the Lagrangian Grassmannian X of maximal isotropic subspaces (of dimension n) in a symplectic space of dimension $2n$. Here $G = Sp_{2n}$ and $V = V_\lambda = \bigwedge^n \mathbb{C}^{2n} - \bigwedge^{n-2} \mathbb{C}^{2n}$ has dimension $\frac{1}{(n+2)(n+1)}(4n+2)\binom{2n}{n}$. The highest weight is $\lambda = e_1 + e_2 + \cdots + e_n$ and there are $d = n(n+1)/2$ positive roots with $c_\lambda(\alpha)$ non-zero. These roots have the form $\alpha = e_i + e_j$ with $1 \leqslant i \leqslant j \leqslant n$. We have $c_\lambda(\alpha) = 2/(2n+2-i-j)$, so

$$deg(X) = 2^d d! \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{(2n+2-i-j)}.$$

8. An exceptional homogeneous variety

We now consider an exceptional variety $X \hookrightarrow \mathbb{P}^{26}$ of dimension $d = 16$. Here $G = E_6$ and $V = V_\lambda$ is a minuscule representation of dimension 27. In the notation of [B] the positive roots α with $\langle \lambda, \alpha^\vee \rangle \neq 0$ have the form

$$\alpha = \frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i \right)$$

with $\sum_{i=1}^{5} \nu(i)$ even. They all satisfy $\langle \lambda, \alpha^\vee \rangle = 1$ so it suffices to compute their inner products with

$$\rho = 4(e_8 - e_7 - e_6 + e_5) + 3e_4 + 2e_3 + e_2$$

We find

$$\langle \rho, \alpha^\vee \rangle = 6 + 2(-1)^{\nu(5)} + \frac{3}{2} \cdot (-1)^{\nu(4)} + (-1)^{\nu(3)} + \frac{1}{2}(-1)^{\nu(2)}$$

Here is a table

$\nu(5)$	$\nu(4)$	$\nu(3)$	$\nu(2)$	$\langle \rho, \alpha^\vee \rangle$
0	0	0	0	11
0	0	0	1	10
0	0	1	0	9
0	0	1	1	8
0	1	0	0	8
0	1	0	1	7
0	1	1	0	6
0	1	1	1	5
1	0	0	0	7
1	0	0	1	6
1	0	1	0	5
1	0	1	1	4
1	1	0	0	4
1	1	0	1	3
1	1	1	0	2
1	1	1	1	1

Hence we find

$$\begin{aligned} \deg(X) &= 16!/11!(8.7.6.5.4) \\ &= 16.15.14.13.12./8.7.6.5.4 \\ &= 78 \end{aligned}$$

Is there any reason that this degree is equal to the dimension of the algebraic group E_6 which acts on X?

We note that if we use the formula for the Hilbert series of this embedding and the above table we find the formula

$$\frac{1 + 10q + 28q^2 + 28q^3 + 10q^4 + q^5}{(1-q)^{17}.}$$

Similarly, for the minuscule representation V_λ of dimension 56 for the exceptional group E_7, we find that X has dimension $d = 27$ and degree $= 13110 = 2.3.5.19.23$ in $\mathbb{P}^{55}$. The Hilbert series of this embedding is given by the formula

$$\frac{(1 + 28q + 273q^2 + 1248q^3 + 3003q^4 + 4004q^5 + 3003q^6 + 1248q^7 + 273q^8 + 28q^9 + q^{10})}{(1-q)^{28}.}$$

9. The ideal of X

Since the coordinate algebra of $X \hookrightarrow \mathbb{P}(V_\lambda)$ is equal to

$$A(X) = \bigoplus_{n \geqslant 0} V_{n\lambda}$$

it follows that the quadrics in the kernel of the map

$$\mathrm{Sym}^2(V_\lambda) \to V_{2\lambda}$$

lie in the ideal $I(X)$. In fact, Kostant proved that these quadrics generate $I(X)$ (cf. [P, pg 368], [W]).

This being said, one can ask for a full resolution of the ideal $I(X)$. Such a resolution is known in a number of simple cases, such as for the rational normal curve $X = \mathbb{P}^1 \hookrightarrow \mathbb{P}(\mathrm{Sym}^n\mathbb{C}^2) = \mathbb{P}^n$. For example, the ideal $I(X)$ of the twisted cubic $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is generated by the 3 independent quadrics Q_i in the kernel of the map [H, pg. 9]:

$$\mathrm{Sym}^2(\mathrm{Sym}^3\mathbb{C}^2) \to \mathrm{Sym}^6\mathbb{C}^2,$$

which is isomorphic to the representation $\mathrm{Sym}^2\mathbb{C}^2$. The only syzygies between these quadrics are two independent linear relations of the form $\sum_{i=1}^3 L_i Q_i = 0$. Hence we obtain a complete resolution:

$$0 \to S(-3) \otimes \mathrm{Sym}^1\mathbb{C}^2 \to S(-2) \otimes \mathrm{Sym}^2\mathbb{C}^2 \to S \to A \to 0$$

as representations of $G = SL(\mathbb{C}^2)$, with $S = \mathrm{Sym}^\bullet(\mathrm{Sym}^3\mathbb{C}^2)$. In most of the other cases where a complete resolution of $I(X)$ is known, G has an open orbit on V_λ.

10. Bibliography

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