Quantum cohomology of partial flag varieties

Wintersemester 2020/2021

The goals of this seminar are to:

- introduce quantum cohomology, as a deformation of the usual cohomology of a variety using Gromov–Witten invariants;
- describe methods to work with the quantum cohomology of partial flag varieties;
- give structural results of the quantum cohomology of partial flag varieties, and understand their link to the structure of the derived category;

with a view towards current research.

It is a natural follow-up of the previous graduate seminar "Exceptional collections for finite-dimensional algebras and partial flag varieties", involving more advanced topics. But for the majority of the seminar this is not a prerequisite at all, and the results we will use from the previous graduate seminar can easily be black-boxed.

Any questions or comments can be emailed to `pbelmans@math.uni-bonn.de`

1 First part: quantum cohomology

We will base ourselves on the excellent lecture notes [7] (occasionally known as FP-Notes), and treat §1–§7 essentially as black boxes from algebraic geometry. Additional references are [5, §7, §8] and the very accessible [9] and [14].

Talk 1: Motivation The goal of the first talk is to *set the stage*: introduce the main player of the seminar (i.e. quantum cohomology), and explain how it relates via Dubrovin's conjecture to the structure of the derived category. Therefore this talk will also introduce some concepts which will only reappear in the second half of the seminar.

- Dubrovin's conjecture, and motivation from mirror symmetry
- overview of exceptional collections for G/P
- overview of (generic) semisimplicity of quantum cohomology for G/P

Talk 2: Geometric preliminaries and presentation of the black boxes The road to quantum cohomology involves lots of complicated algebraic geometry, which we would like to (mostly) ignore. But for general geometric culture and to understand the later constructions we want to understand what we are putting away in black boxes and how to use these black boxes.

- briefly introduce stacks: use the j-line parametrising elliptic curves as an example
- discuss weighted projective lines as an example
- $\mathcal{M}_g$, $\mathcal{M}_{g,n}$, their compactifications and their coarse moduli spaces; with an emphasis on $g = 0$ (and therefore $n \geq 3$)
- discuss the road to $\overline{\mathrm{M}_{0,n}}(X, \beta)$

Talk 3: Gromov–Witten invariants This is based on [7, §0 and §7], and introduces Gromov–Witten invariants using algebro-geometric techniques.

- define Gromov–Witten invariants, for G/P, following [7, §7]
- explain (non)enumerativeness of the invariants
- discuss the number of rational curves of degree d through $3d - 1$ points on $\mathbb{P}^2$, following [9] and [14]

Talk 4: Big quantum cohomology The geometry used to define Gromov–Witten invariants allows us to define a (formal) deformation of the ordinary cohomology $\mathrm{H}^\bullet(X, \mathbb{C})$; where associativity of the algebra reflects these geometric properties. This gives an algebro-geometric definition of quantum cohomology (an approach using symplectic geometry also exists).

- define big quantum cohomology, following [7, §8], see also [5, §8.2]
- explain how associativity of quantum cohomology is encoding the number of rational curves of degree d through $3d - 1$ points on $\mathbb{P}^2$
- (time permitting) discuss the case of $\mathrm{Gr}(2, 4)$ using [6, §3.4]

Talk 5: Small quantum cohomology Many interesting properties of quantum cohomology can already be seen on the level of small quantum cohomology, which uses less parameters in the deformation of ordinary cohomology (namely only H^2).

- define small quantum cohomology, following [7, §10]
- discuss basic examples of small and big quantum cohomology

2 Second part: quantum cohomology of partial flag varieties

For the second part of the seminar we have the following goals:

- understand various tools for the quantum cohomology of G/P, known as *quantum Schubert calculus*: quantum Giambelli, quantum Pieri, quantum Chevalley, ...
- obtain presentations of quantum cohomology of G/P in certain cases, and study semisimplicity

What we must get from this discussion is an understanding of semisimplicity results, as these are essential for understanding Dubrovin's conjecture later on.

We could spend a whole semester on discussing results of this nature, and we might add talks to this part of the seminar as we deem fit. But for now we just want to get a glimpse of the techniques and results which appear.

Talk 6: Quantum cohomology of Grassmannians The goal of this talk is to obtain a presentation of the (small) quantum cohomology of a Grassmannian, following [15, §1–§3], which is an exposition of [1]. If time permits one can talk about types B, C and D (using [15, §5] and [1]), without giving proofs.

Talk 7: Quantum cohomology for (co)minuscule varieties In this talk one gives an overview of the series of papers [2, 3, 4]. These extend the results of the previous talk to exceptional types, and one can obtain semisimplicity of the small quantum cohomology in these cases in a *completely uniform matter*.

3 Third part: Fusion rings

Before tackling aspects of Dubrovin's conjecture, we talk about how the quantum cohomology of Grassmannians is an object which relates to many other objects, using [10].

Talks 8 and 9: Fusion rings, Verlinde algebras and quantum cohomology The goal is to give an overview of [10].

4 Fourth part: Dubrovin's conjecture

We now start relating the structure of quantum cohomology to the structure of the derived category, as predicted by Dubrovin's conjecture.

Talk 10: The isotropic Grassmannian $\mathrm{IGr}(2,6)$ In this talk we discuss the necessity for *big* quantum cohomology in the formulation of Dubrovin's conjecture, by studying [8].

- briefly describe the geometry of isotropic Grassmannians, and $\mathrm{IGr}(2,6)$ in particular

- discuss the exceptional collection for $\mathbf{D}^{\mathrm{b}}(\mathrm{IGr}(2,6))$ from [11]
- discuss the quantum cohomology of $\mathrm{IGr}(2,6)$
- discuss how to deform the small quantum cohomology to a semisimple algebra

Talk 11: Lefschetz collections and the structure of quantum cohomology In this talk we discuss how additional structure present in the quantum cohomology of G/P should reflect additional properties that exceptional collections for $\mathbf{D}^{\mathrm{b}}(G/P)$ might exhibit; this enhancement of Dubrovin's conjecture is introduced by Kuznetsov–Smirnov [12].

- introduce Lefschetz structures
- briefly explain motivation from homological projective duality
- state the Kuznetsov–Smirnov conjecture
- discuss the known cases (see also [13])

Talk 12: Recent progress on semisimplicity and exceptional collections Work-in-progress of Maxim with Nicolas and Anton (or some subset thereof).

References

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