Hilbert schemes of points: symmetries and deformations

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make notes and follow along: `https://pbelmans.ncag.info/slides.pdf`

Introducing Hilbert schemes of points

Definition

- A a finitely generated (commutative) $\mathbb{C}$-algebra, e.g. $\mathbb{C}[x, y]$
- $X = \operatorname{Spec} A$ the affine variety with coordinate ring A

the Hilbert scheme of n points on X parametrises codimension n ideals of A, i.e.

$$X^{[n]} = \{ I \triangleleft A \mid \dim_{\mathbb{C}} A/I = n \}$$

Grothendieck ('60) this is a quasiprojective algebraic variety so we can do geometry with the set of all such ideals

we have $X^{[1]} \cong X$ so $n \geq 2$ throughout

Interpretation

let $p_1, \ldots, p_n$ be n distinct closed points, we define

$$I = \{f \in A \mid f(p_1) = \ldots = f(p_n) = 0\} \triangleleft A$$

we have that $I \in X^{[n]}$: $X^{[n]}$ at least parametrises n distinct points on X

however, there is more: e.g. for $A = \mathbb{C}[x, y]$ and $n = 2$ we take $I_\alpha = (x^2 - \alpha, y) \triangleleft \mathbb{C}[x, y]$ for $\alpha \in \mathbb{C}$

- $\alpha \neq 0$ $\operatorname{Spec} A/I_\alpha$ is two distinct points $(\alpha, 0)$ and $(-\alpha, 0)$
- $\alpha = 0$ $\operatorname{Spec} A/I_\alpha \cong \operatorname{Spec} \mathbb{C}[x]/(x^2)$ is the origin, together with tangent direction

Motivation

- Hilbert–Chow morphism sends I to support of $\operatorname{Spec} A/I$, counted with multiplicities

$$X^{[n]} \to \operatorname{Sym}^n X = \overbrace{X \times \ldots \times X}^{n \text{ times}} / \mathfrak{S}_n$$

$\dim X = 1$	Hilbert–Chow morphism is isomorphism
$\dim X = 2$	assume X smooth, then Hilbert–Chow morphism is resolution of singularities

- representation theory of $\mathfrak{S}_n$ versus the geometry of $X^{[n]}$: Haiman's work on combinatorics, Göttsche's generating series for invariants, Nakajima's Heisenberg algebra action on cohomology, ...

Hilbert schemes of points on surfaces

don't need X to be affine: from now on let S a smooth and projective surface, "classified" by Enriques–Kodaira

Fogarty ('66) $S^{[n]}$ is again smooth projective, of dimension $2n$

for $\dim X \geq 3$ this fails badly, unless $n = 2, 3$

very interesting for

- representation theory
- birational geometry
- first examples of hyperkähler varieties

Today:

1. symmetries of $S^{[n]}$
2. deformations of $S^{[n]}$

Three running examples of surfaces

the following three examples will **illustrate everything**:

1. the projective plane $\mathbb{P}^2$
2. a quartic surface in $\mathbb{P}^3$, e.g. $x^4 + y^4 + z^4 + w^4 = 0$
3. a quintic surface in $\mathbb{P}^3$, e.g. $x^5 + y^5 + z^5 + w^5 = 0$

have very different behavior:

1. the easiest del Pezzo surface
2. the first example of a K3 surface
3. one of many surfaces of general type

Symmetries

Motivation

We want to understand the automorphism groups $\mathrm{Aut}(S)$ and $\mathrm{Aut}(S^{[n]})$:

- symmetries often allow us to simplify a problem
- they are an interesting invariant of a variety: distinguish varieties
- they induce useful actions on *other* invariants, such as cohomology

every automorphism of S induces an automorphism of $S^{[n]}$

$$\mathrm{Aut}(S) \subseteq \mathrm{Aut}(S^{[n]})$$

Question Is this an equality?

Linearisation of a problem

understand an object by understanding it locally

calculus	function	derivative	linear function
algebraic geometry	variety	tangent space	vector space

1. dimension of tangent space is dimension of variety (if smooth)
2. $\mathrm{Aut}(X)$ is also smooth
3. tangent space of $\mathrm{Aut}(X)$ (at identity) is isomorphic to $\mathrm{H}^0(X, \mathrm{T}_X)$, a finite-dimensional vector space

we get an approximation of the size of $\mathrm{Aut}(S)$ and $\mathrm{Aut}(S^{[n]})$

Comparing the sizes

Boissière ('12) $\dim_{\mathbb{C}} \mathrm{H}^0(S, \mathrm{T}_S) = \dim_{\mathbb{C}} \mathrm{H}^0(S^{[n]}, \mathrm{T}_{S^{[n]}})$, so they have the same size

three running examples

	S	$\dim_{\mathbb{C}} \mathrm{H}^0(S, \mathrm{T}_S)$	$\dim_{\mathbb{C}} \mathrm{H}^0(S^{[n]}, \mathrm{T}_{S^{[n]}})$
1.	$\mathbb{P}^2$	8	8
2.	quartic	0	0
3.	quintic	0	0

- $\mathbb{P}^2$ has many symmetries: $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$
- automorphisms of quartics (and other K3 surfaces) turn out very interesting: possibly infinite, but discrete in nature
- surfaces of general type have finite automorphism groups

Beauville's example

Let $S \subseteq \mathbb{P}^3$ be a quartic K3 surface, not containing any lines, and consider $S^{[2]}$:

1. $I \in S^{[2]}$ describes 2 points on S (or a point and a tangent direction)
2. therefore I spans a line $\mathbb{P}^1$ in $\mathbb{P}^3$
3. the intersection $S \cap \mathbb{P}^1$ consists of 4 points (generically)
4. define an automorphism by sending $I \in S^{[2]}$ to $I^\perp \in S^{[2]}$ describing the complement of the 2 points

Beauville ('85) this extends to an honest automorphism of $S^{[2]}$, which moreover does not come from S

Comparing the automorphism groups

So at least for some K3 surfaces, we have

$$\mathrm{Aut}(S) \subsetneq \mathrm{Aut}(S^{[n]})$$

- $S^{[n]}$ is a hyperkähler variety
- automorphisms of hyperkähler varieties via Torelli theorem and lattice theory

In stark contrast to this:

Theorem (B–Oberdieck–Rennemo, '19)

If ω_S or $\omega_S^\vee$ is big and nef, and if $n = 2$ assume moreover that $S \neq C_1 \times C_2$, then

$$\mathrm{Aut}(S) = \mathrm{Aut}(S^{[n]})$$

Comments

- this applies in particular to cases 1 and 3, for all $n \geq 2$
- for $n = 2$ and $S = C_1 \times C_2$ there is a $\mathbb{Z}/2\mathbb{Z}$ contribution: away from diagonal we take

$$(x_1, y_1) + (x_2, y_2) \mapsto (x_1, y_2) + (x_2, y_1),$$

  extends to $(C_1 \times C_2)^{[2]}$, essentially because $\mathfrak{S}_2$ is the only abelian symmetric group
- this does not cover all possible surfaces, but covers a large part the classification

Deformations

Motivation

Given a variety X, can we find other varieties which are similar to X? Understand classification problems!

deformation theory gives us the tools for this: deformation functor Def_X encodes the deformation theory, and linearisation of the problem tells us to look at

$$\mathrm{H}^1(X, \mathrm{T}_X)$$

again a finite-dimensional vector space

we have $\mathrm{Def}_S \subseteq \mathrm{Def}_{S^{[n]}}$, so

$$\mathrm{H}^1(S, \mathrm{T}_S) \subseteq \mathrm{H}^1(S^{[n]}, \mathrm{T}_{S^{[n]}})$$

Question Are these equalities?

Comparison

comparing $\mathrm{H}^1(X, \mathrm{T}_X)$ in the **three running examples**

	S	$\dim_{\mathbb{C}} \mathrm{H}^1(S, \mathrm{T}_S)$	$\dim_{\mathbb{C}} \mathrm{H}^1(S^{[n]}, \mathrm{T}_{S^{[n]}})$
1.	$\mathbb{P}^2$	0 (rigid)	10 (not rigid)
2.	quartic	20	21
3.	quintic	35	35

1. big jump: from no deformations to plenty of deformations
2. small jump
3. no jump

Results

Fantechi ('95) if S is of general type, then $\mathrm{Def}_S \cong \mathrm{Def}_{S^{[n]}}$
so no jump in case 3 holds generally

Hitchin ('12) if $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, then we have short exact sequence

$$0 \to \mathrm{H}^1(S, \mathrm{T}_S) \to \mathrm{H}^1(S^{[n]}, \mathrm{T}_{S^{[n]}}) \to \mathrm{H}^0(S, \omega_S^\vee) \to 0$$

this explains the jumps:

1. $\mathrm{H}^0(S, \omega_S^\vee) \cong \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \cong \mathbb{C}[x, y, z]_3 \cong \mathbb{C}^{\oplus 10}$
2. $\mathrm{H}^0(S, \omega_S^\vee) \cong \mathrm{H}^0(S, \mathcal{O}_S) \cong \mathbb{C}$

1. deformations have description in terms of noncommutative $\mathbb{P}^2$'s
2. deformations have description in terms of moduli spaces of sheaves on K3 surfaces

Alternative point of view

These proofs depend heavily on the geometry of the Hilbert–Chow morphism, and don't generalise to other moduli spaces of sheaves.

Alternative approach take appearance of noncommutative surfaces seriously:

- use deformation theory of categories, not just varieties, encoded in Hochschild cohomology $\mathrm{HH}^\bullet(X)$
- this is an invariant of the derived category $\mathbf{D}^{\mathrm{b}}(X)$

linearisation of deformation functor is

$$\mathrm{HH}^2(X) \overset{\mathrm{HKR}}{\cong} \underbrace{\mathrm{H}^0\left(X, \bigwedge^2 \mathrm{T}_X\right)}_{\text{noncommutative}} \oplus \underbrace{\mathrm{H}^1(X, \mathrm{T}_X)}_{\text{geometric}} \oplus \underbrace{\mathrm{H}^2(X, \mathcal{O}_X)}_{\text{gerby}}$$

Deformation theory via fully faithful functors

Keller ('05) Hochschild cohomology is not functorial, but if $F\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ is fully faithful, then we can get restriction morphism

$$\mathrm{HH}^\bullet(Y) \to \mathrm{HH}^\bullet(X)$$

Krug–Sosna ('15) If $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$, then universal ideal sheaf $\mathcal{I} \in \operatorname{coh} S \times S^{[n]}$ gives

$$\Phi_{\mathcal{I}}\colon \mathbf{D}^{\mathrm{b}}(S) \to \mathbf{D}^{\mathrm{b}}(S^{[n]}) \text{ fully faithful}$$

Theorem (B–Fu–Raedschelders, '19)
If $\mathrm{H}^1(S, \mathcal{O}_S) = \mathrm{H}^2(S, \mathcal{O}_S) = 0$, then

$$\mathrm{H}^1(S^{[n]}, \mathrm{T}_{S^{[n]}}) \overset{!}{\cong} \mathrm{HH}^2(S) \overset{\mathrm{HKR}}{\cong} \mathrm{H}^1(S, \mathrm{T}_S) \oplus \mathrm{H}^0(S, \omega_S^\vee)$$

Comments

the proof of the fully faithfulness still uses Hilbert–Chow, by virtue of the Bridgeland–King–Reid–Haiman equivalence

$$\mathbf{D}^{\mathrm{b}}(S^{[n]}) \cong \mathbf{D}^{\mathrm{b}}([S^n/\mathfrak{S}_n])$$

the proof of the isomorphism is independent of Hilbert–Chow: only ingredients are

- fully faithfulness of Fourier–Mukai for universal sheaf
- analysis of local-to-global relative Ext spectral sequence
- understanding of pushforward of universal sheaves

hence it could be possible to

- generalise this to other moduli spaces on surfaces
- generalise this to higher dimensions

In higher dimensions?

- if $\dim X \geq 3$ then $X^{[n]}$ is singular, unless $n = 2, 3$
- e.g. $X = \mathbb{P}^d$ for $d \geq 3$ also has noncommutative deformations

Question What can we say about the deformation theory (and automorphisms) of e.g. $\mathbb{P}^{d,[2]}$?

Results in higher dimensions

Theorem (B–Fu–Raedschelders, '19)
If $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X$, then

$$\Phi_{\mathcal{I}} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(X^{[2]}) \text{ fully faithful}$$

Theorem (B–Fu–Raedschelders, '19)
If $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim X$, then

$$\mathrm{H}^i(X^{[2]}, \mathrm{T}_{X^{[2]}}) \cong \mathrm{H}^i(X, \mathrm{T}_X)$$

so no new deformations!

Theorem (B–Oberdieck–Rennemo, '19)
At least for $\mathbb{P}^d$: $\mathrm{Aut}(\mathbb{P}^d) \cong \mathrm{Aut}(\mathbb{P}^{d,[2]})$