The Effects of Spline Interpolation on Power Spectral Density

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Abstract—This paper discusses the power spectral effects of spline interpolators. A general technique is given for finding the steady-state spectral effects of splines of all orders, when applied following uniform sampling of the input function. The following observations are made: 1) the even order splines that were examined (second and fourth order) possessed divergent steady-state frequency transfer functions, 2) the degree of preservation of the power spectral density of the input process increased with the order of the (odd order) spline used for interpolation, and 3) the reconstruction of a stationary random process over a finite record length will, on the average, have less power than indicated by the steady-state transfer function.

The Effects of Spline Interpolation on Power Spectral Density

There are many techniques for approximately reconstructing a signal from samples of the signal. One family of such techniques is that of the spline interpolators [1]-[4]. An (n)th order spline will generate, from a sequence of samples, a function characterized by a distinct nth order polynomial over each sample interval, which passes through every sample point and for which the $(n-1)$st derivative is continuous everywhere. Make the following definitions (where k is an index over the samples):

$\{\cdots, t_k, \cdots\}$: set of sample times associated with signal $x(t)$.

$d_k = t_{k+1} - t_k$: kth sample time increment.

$\{\cdots, x_k, \cdots\}$: set of samples of signal $x(t)$ at sample times $\{\cdots, t_k, \cdots\}$.

$p(t)$: spline interpolation function of $x(t)$ over the entire record length.

$\{\cdots, y_k, \cdots\}$: values of $p^{(n-1)}(t)$, the $(n-1)$st derivative of $p(t)$, at $(t = t_k)$. These values are initially undetermined.

Because the interpolation function is to be described as an nth order polynomial over each sample interval, and the $(n-1)$st derivative of $p(t)$ is constrained to be continuous, we may write $p^{(n-1)}(t)$ as a linear interpolation in time of the values of $p^{(n-1)}(t)$ at the sample times (the set $\{y_k\}$, as yet undetermined). That is

$$p^{(n-1)}(t) = \frac{(t-t_k)(y_{k+1}-y_k)}{d_k} + y_k \quad \text{over } [t_k, t_{k+1}]. \tag{1}$$

Integrating (1) $(n-1)$ times with respect to (t), from (t_k) to (t), we obtain

$$p^{(n-2)}(t) = -\frac{y_k(t_{k+1}-t)^2}{2d_k} + \frac{y_{k+1}(t-t_k)^2}{2d_k} + c_{1,k} \quad \text{over } [t_k, t_{k+1}]$$

$$p^{(n-3)}(t) = \frac{y_k(t_{k+1}-t)^3}{6d_k} + \frac{y_{k+1}(t-t_k)^3}{6d_k} + c_{1,k}t + c_{2,k} \quad \text{over } [t_k, t_{k+1}]$$

$$\vdots$$

$$p(t) = \frac{(-1)^{n-1} y_k(t_{k+1}-t)^n}{(n!)d_k} + \frac{y_{k+1}(t-t_k)^n}{(n!)d_k} + \frac{c_{1,k}t^{(n-2)}}{(n-2)!} + \frac{c_{2,k}t^{(n-3)}}{(n-3)!} + \cdots + c_{(n-1),k} \quad \text{over } [t_k, t_{k+1}]$$

where $c_{(\cdot),k}$ are, as yet, unknown constants of integration.

We thus have $(n-1)$ integration constants for each interval over which the spline is defined. Consider the case in which the spline is generated over (r) intervals. There are $(r-1)$ interior sample points and two end sample points. From the preceding development we see that there are $(n-1)(r)$ constants of integration to be determined to define the spline over (r) intervals. The following linear algebraic equations may be solved to obtain the constants in terms of the sets $\{x_k\}$ and $\{y_k\}$. First, since the spline is constrained to pass through all sample points

$$\begin{aligned} p(t_i)|_{[t_i, t_{i+1}]} &= x_i \qquad i = k, \cdots, r-1+k \\ p(t_i)|_{[t_{i-1}, t_i]} &= x_i \qquad i = k+1, \cdots, r+k. \end{aligned} \tag{2}$$

Where the notation $p(t_i)|_{[t_i, t_{i+1}]}$ means the value of $p(t)$ as expressed over the interval $[t_i, t_{i+1}]$, evaluated at interval boundary point (t_i). Next, since $(n-1)$ derivatives of the spline are continuous, we have

$$p^{(j)}(t_i)|_{[t_i, t_{i+1}]} = p^{(j)}(t_i)|_{[t_{i-1}, t_i]} \qquad \begin{aligned} i &= 1+k, \cdots, r-1+k \\ j &= 1, \cdots, n-2. \end{aligned} \tag{3}$$

(There are no constants of integration in the $(n-1)$st derivative.) This produces a total number of equations given by

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$$2r + (r-1)(n-2) = nr + 2 - n\,.$$

In order to have a unique solution for $(n-1)(r)$ constants

$$nr + 2 - n \geqslant nr - r$$
$$r \geqslant n - 2\,.$$

Equations (2) and (3) must be solved over $(n-2)$ intervals. Because there are $(n-1)$ constants for each interval, the equations will involve $(n-1)(n-2)$ constants (30 for the seventh-order spline). Substitution of the constants into (3) with $(i = r + k)$ and $(j = n - 2)$ yields (using the fact that $c_{1,r} = c_{1,i}|_{i=r}$) a difference equation for $\{y_k\}$ in terms of $\{x_k\}$. The difference equation depends on the particular spline. Carrying out the preceding equations for the cases of the linear interpolator (first-order spline), quadratic spline, cubic spline,[1] fourth-order spline, and fifth-order spline yields (for uniform sampling, where $d_k = \widetilde{T}$ for all k)

1) *Linear interpolator (first-order spline):* $y_k = x_k$.

2) *Quadratic spline:* $y_k + y_{k-1} = (2x_k - 2x_{k-1})\dfrac{1}{\widetilde{T}}$.

3) *Cubic spline:* $\frac{1}{6}(y_{k+1} + 4y_k + y_{k-1}) = (x_{k+1} - 2x_k + x_{k-1})\dfrac{1}{\widetilde{T}^2}$.

4) *Fourth-order spline:* $\frac{1}{24}(y_{k+1} + 11y_k + 11y_{k-1} + y_{k-2}) = (x_{k+1} - 3x_k + 3x_{k-1} - x_{k-2})\dfrac{1}{\widetilde{T}^3}$.

5) *Fifth-order spline:* $\frac{1}{120}(y_{k+1} + 26y_k + 66y_{k-1} + 26y_{k-2} + y_{k-3}) = (x_{k+1} - 4x_k + 6x_{k-1} - 4x_{k-2} + x_{k-3})\dfrac{1}{\widetilde{T}^4}$.

The spectral effects of the spline interpolators for nonuniform sampling are not discussed here. The preceding difference equations (of order greater than 2) are not implemented recursively (on-line) using initial conditions. They are instead implemented in a noncausal manner using both initial and final boundary conditions. Reasons for this will be given later.

The power spectral effects of the spline functions are now examined (for a different approach in this direction, see [8]). First, define the z-transform of the sequence $\{y(k)\}$.

$$Y(z) = \sum_{k=-\infty}^{\infty} y(k) z^{-k} \tag{4}$$

where z is a complex parameter. Substituting (4) into the difference equations for the splines, we obtain

1) *Linear interpolator:*

$$H_1(z) = Y(z)/X(z) = 1. \tag{5}$$

2) *Quadratic spline:*

$$H_2(z) = Y(z)/X(z) = \left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\left(\frac{2}{\widetilde{T}}\right). \tag{6}$$

3) *Cubic spline:*

$$H_3(z) = Y(z)/X(z) = \left(\frac{z - 2 + z^{-1}}{z + 4 + z^{-1}}\right)\left(\frac{6}{\widetilde{T}^2}\right). \tag{7}$$

4) *Fourth-order spline:*

$$H_4(z) = Y(z)/X(z) = \left(\frac{z - 3 + 3z^{-1} - z^{-2}}{z + 11 + 11z^{-1} + z^{-2}}\right)\left(\frac{24}{\widetilde{T}^3}\right). \tag{8}$$

5) *Fifth-order spline:*

$$H_5(z) = Y(z)/X(z) = \left(\frac{z - 4 + 6z^{-1} - 4z^{-2} + z^{-3}}{z + 26 + 66z^{-1} + 26z^{-2} + z^{-3}}\right)\left(\frac{120}{\widetilde{T}^4}\right). \tag{9}$$

We shall now discuss the method used to generate the spline reconstruction of a signal. Function $x(t)$ is first sampled uniformly at times $\{t_k\}$. The samples of $x(t)$ are then processed according to the appropriate difference equation. The resultant output is samples of the $(n-1)$st derivative of the interpolation function. These samples are linearly interpolated to construct the continuous $(n-1)$st derivative of the interpolation function. The $(n-1)$st derivative is finally integrated $(n-1)$ times to yield the spline interpolation function, $p(t)$. Our aim is to find the power spectral density of $p(t)$ in terms of the power spectral density of $x(t)$. Fig. 1 illustrates the procedure for obtaining $p(t)$ from $x(t)$, the input function. The Fourier transform of a discrete sequence $\{x(k)\}$ is defined as[2]

$$X_d(e^{j\omega_d}) = \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega_d k} = X(z)\big|_{z=e^{j\omega_d}} \tag{10}$$

where ω_d is digital frequency, and $X(z)$ is the z-transform defined in (4).

Our first aim is to find the Fourier transform of a sequence of samples as a function of the Fourier transform of the continuous function $x(t)$, being sampled. Let $x_s(t)$ represent the sampled version of $x(t)$:

$$x_s(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\widetilde{T})x(t)\,.$$

From (10), we see that

$$X_d(e^{j\omega_d}) = \sum_{k=-\infty}^{\infty} x(k\widetilde{T})\, e^{-j\omega_d k}\,.$$

Inspection of the preceding two equations shows that

[1] For the cubic spline, (2) and (3) are solved, and the difference equation obtained, by Carnahan *et al.* [1].

[2] Note that the Fourier transform of a discrete sequence is periodic in digital frequency ω_d, with period 2π.

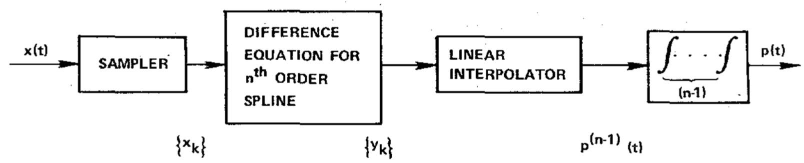

Fig. 1. ***n*th-order spline interpolation of $x(t)$.**

$$X_d(e^{j\omega_d}) = \int_{-\infty}^{\infty} x_s(t)\, e^{\frac{-j\omega_d t}{\widetilde{T}}}\, dt = F\,[x_s(t)] \Big|_{f=\frac{\omega_d}{2\pi\widetilde{T}}}$$

where $F\,[\cdot]$ is the Fourier transform operator.

It is well known that

$$F\,[x_s(t)] = \frac{1}{\widetilde{T}} \sum_{k=-\infty}^{\infty} X_c(f - k/\widetilde{T})$$

where $X_c(f)$ represents the Fourier transform of continuous function $x(t)$. Thus

$$X_d(e^{j\omega_d}) = \frac{1}{\widetilde{T}} \sum_{k=-\infty}^{\infty} X_c\left(\frac{[\omega_d/2\pi] - k}{\widetilde{T}}\right). \quad (11)$$

Equations (5)–(9) may be converted to Fourier transform expressions by substituting $(e^{j\omega_d})$ for (z). We then have for $H_n(e^{j\omega_d})$ $(n = 1\text{–}5)$:

1) *Linear interpolator:*

$$H_1(e^{j\omega_d}) = \frac{Y_d(e^{j\omega_d})}{X_d(e^{j\omega_d})} = 1\,. \quad (12)$$

2) *Quadratic spline:*

$$H_2(e^{j\omega_d}) = \frac{Y_d(e^{j\omega_d})}{X_d(e^{j\omega_d})} = \frac{\sin\left(\frac{\omega d}{2}\right)}{\cos\left(\frac{\omega d}{2}\right)} \left(\frac{2j}{\widetilde{T}}\right). \quad (13)$$

3) *Cubic spline:*

$$H_3(e^{j\omega_d}) = \frac{Y_d(e^{j\omega_d})}{X_d(e^{j\omega_d})} = \frac{(\cos\omega_d - 1)}{(2 + \cos\omega_d)} \left(\frac{6}{\widetilde{T}^2}\right). \quad (14)$$

4) *Fourth-order spline:*

$$H_4(e^{j\omega_d}) = \frac{Y_d(e^{j\omega_d})}{X_d(e^{j\omega_d})} = \left(\frac{e^{j\omega_d} - 3 + 3e^{-j\omega_d} - e^{-j2\omega_d}}{e^{j\omega_d} + 11 + 11e^{-j\omega_d} + e^{-2j\omega_d}}\right) \left(\frac{24}{\widetilde{T}^3}\right). \quad (15)$$

5) *Fifth-order spline:*

$$H_5(e^{j\omega_d}) = \frac{Y_d(e^{j\omega_d})}{X_d(e^{j\omega_d})} = \left(\frac{6 - 8\cos\omega_d + 2\cos 2\omega_d}{66 + 52\cos\omega_d + 2\cos 2\omega_d}\right) \left(\frac{120}{\widetilde{T}^4}\right). \quad (16)$$

Equations (12)–(16), in conjunction with (11), permit $Y_d(e^{j\omega_d})$ to be found as a function of $X_c(f)$.

Having found the Fourier transform of the sequence $\{y(n)\}$, we now find the Fourier transform of the linear interpolation of samples $\{y(n)\}$, which is the $(n-1)$st derivative of the spline function[3]

$$p^{(n-1)}(t) = \sum_{k=-\infty}^{\infty} \left[\frac{(y_{k+1} - y_k)(t - t_k)}{\widetilde{T}} + y_k\right] \cdot \left[u_{-1}(t - t_k) - u_1(t - t_{k+1})\right]$$

$$F(p^{(n-1)}(t)) = \sum_{k=-\infty}^{\infty} \int_{t_k}^{t_{k+1}} \left[\frac{(y_{k+1} - y_k)(t - t_k)}{\widetilde{T}} + y_k\right] e^{-j2\pi ft}\, dt.$$

The preceding equation is obtained by interchanging[4] the order of integration and summation. After integration by parts and manipulation

$$F(p^{(n-1)}(t)) = \widetilde{T} \left(\frac{\sin \pi f\widetilde{T}}{\pi f\widetilde{T}}\right)^2 Y_d(e^{j2\pi f\widetilde{T}}). \quad (17)$$

Now due to the fact that

$$F(p(t)) = \left(\frac{1}{j2\pi f}\right)^{n-1} F\left(p^{(n-1)}(t)\right)$$

we have

$$F(p(t)) = \left(\frac{1}{j2\pi f}\right)^{n-1} \widetilde{T} \left(\frac{\sin \pi f\widetilde{T}}{\pi f\widetilde{T}}\right)^2 \cdot Y_d(e^{j2\pi f\widetilde{T}}). \quad (18)$$

Using (11)–(16) we have

$$F(p(t)) = \left(\frac{1}{j2\pi f}\right)^{n-1} \left(\frac{\sin \pi f\widetilde{T}}{\pi f\widetilde{T}}\right)^2 \cdot H_n(e^{j\omega_d}) \Big|_{\omega_d = 2\pi f\widetilde{T}} \sum_{k=-\infty}^{\infty} X_c\left(f - \frac{k}{\widetilde{T}}\right) \quad (19)$$

where the Fourier transform relation of the difference equation of the spline is substituted for $H_n(e^{j\omega_d})$, and (n) is the order of the spline. When $x(t)$ is a random process, $p(t)$ will also be a random process. In this case, define

[3] The unit step function $u_{-1}(t)$, is defined as

$$u_{-1}(t) = \begin{Bmatrix} 0: & t < 0 \\ 1: & t \geqslant 0 \end{Bmatrix}.$$

[4] A discussion of the justifications for interchanging orders of summation and integration is given by Rektorys [6].

$$X_T(f) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi ft}\, dt \tag{20}$$

$$S_{xx}(f, T) = T\,|X_T(f)|^2. \tag{21}$$

It can be shown that [5]

$$\lim_{T \to \infty} E[S_{xx}(f, T)] = S_{xx}(f) \tag{22}$$

where $E(\cdot)$ is the ensemble expectation operator and $S_{xx}(f)$ is the power spectral density of random process $x(t)$. Using (22) and (19), we find that

$$S_{pp}(f) = \left|\frac{1}{j2\pi f}\right|^{2(n-1)} \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^4 \cdot |H_n(e^{j\omega_d})|^2 \left|\sum_{k=-\infty}^{\infty} S_{xx}\left(f - \frac{k}{\tilde{T}}\right)\right|_{\omega_d = 2\pi f\tilde{T}}. \tag{23}$$

(The crossterms in the summation cancel.)

The power spectral density of $p(t)$ is given by first repeating the power spectral density of $x(t)$ and then multiplying by a function of frequency. In the case where the repeated version of $S_{xx}(f)$ is flat for all frequencies ($x(t)$ being white noise band limited to $|f| \leqslant 1/2\tilde{T}$) the power spectral density of $p(t)$ is given by

$$S_{pp}(f) = \left|\frac{1}{j2\pi f}\right|^{2(n-1)} \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^4 \cdot |H_n(e^{j\omega_d})|^2 \,\Big|_{\omega_d = 2\pi f\tilde{T}}. \tag{24}$$

For the splines treated in (12) through (16), we have the following.

Linear interpolator:

$$S_{pp}(f) = \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^4. \tag{25}$$

Quadratic spline:

$$S_{pp}(f) = \frac{1}{\tilde{T}^2 \pi^2 f^2} \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^4 (\tan [\pi f\tilde{T}])^2. \tag{26}$$

Cubic spline:

$$S_{pp}(f) = \left(\frac{3}{2 + \cos 2\pi f\tilde{T}}\right)^2 \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^8. \tag{27}$$

Fourth-order spline:

$$S_{pp}(f) = \left[\frac{576}{2^6 \pi^6 f^6 \tilde{T}^6}\right] \cdot \left[\frac{20 - 30 \cos 2\pi f\tilde{T} + 12 \cos 4\pi f\tilde{T} - 2 \cos 6\pi f\tilde{T}}{244 + 286 \cos 2\pi f\tilde{T} + 44 \cos 4\pi f\tilde{T} + 2 \cos 6\pi f\tilde{T}}\right] \cdot \left[\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right]^4. \tag{28}$$

Fifth-order spline:

$$S_{pp}(f) = \frac{(120)^2}{(2\pi f\tilde{T})^8} \left(\frac{6 - 8 \cos 2\pi f\tilde{T} + 2 \cos 4\pi f\tilde{T}}{66 + 52 \cos 2\pi f\tilde{T} + 2 \cos 4\pi f\tilde{T}}\right)^2 \cdot \left(\frac{\sin \pi f\tilde{T}}{\pi f\tilde{T}}\right)^4. \tag{29}$$

Note from (26) and (28) that the spectral densities of the outputs of the two even-order splines diverge at $(f = (2k + 1)/2\tilde{T})$ for $k = 0, \pm 1, \pm 2, \cdots$. Outputs of these methods are hence unstable. Graphs of (25), (27), and (29) appear in Figs. 2 and 3, plotted on linear–linear and log–log scales, respectively.

The output power spectral densities in Figs. 2 and 3 are the results of band-limited white noise being sampled uniformly and reconstructed using the various spline interpolators. The power spectral density of the output of the fifth-order spline more closely approaches the power spectral density of the input process than does that of the cubic spline. The cubic spline, in turn, is seen to more closely preserve the power spectral density of the input process than does the linear interpolator (first-order spline).

The cause of the divergence of power spectral density for the outputs of the even-order splines may be seen by examining (6) and (8). Because the Fourier transform relation [which to be stable must converge on the $j\omega$ axis) is found (using (10))] by substituting $e^{j\omega_d}$ for z, a necessary condition for stability is the absence of poles of the z-transform relation on the unit circle in the z-plane. The denominators of both (6) and (8) have zeros at $(z = -1)$ on the unit circle, *resulting in instability. The cubic spline and fifth-*order spline have poles at $z = (-0.268, -3.732)$ and $z = (-0.043, -0.431, -2.322, -23.204)$, respectively. In order to guarantee stability, regions of convergence must be chosen which include the unit circle; hence $(0.268 < |z| < 3.732)$ for the cubic spline and $(0.431 < |z| < 2.322)$ for the fifth-order spline. In order that a causal implementation (all boundary conditions for the difference equation set at the start of the (possibly infinite) observation interval) be stable, it would be necessary that the z-transform converge everywhere outside some circle, centered at the origin of the z-plane, of radius less than unity. In order that a "totally noncausal" implementation (all boundary conditions set at the end of the (possibly infinite in past extent) observation interval) be stable, it would be necessary that the z-transform converge everywhere inside some circle, centered at the origin of the z-plane, of radius greater than unity. Because the regions mentioned are in fact annular, satisfying neither of these two conditions, in practical implementations it is best to set boundary conditions at both ends of the sample interval (implying that a finite-length sample interval must be used), and to solve for the values $\{y_k\}$ using simultaneous linear

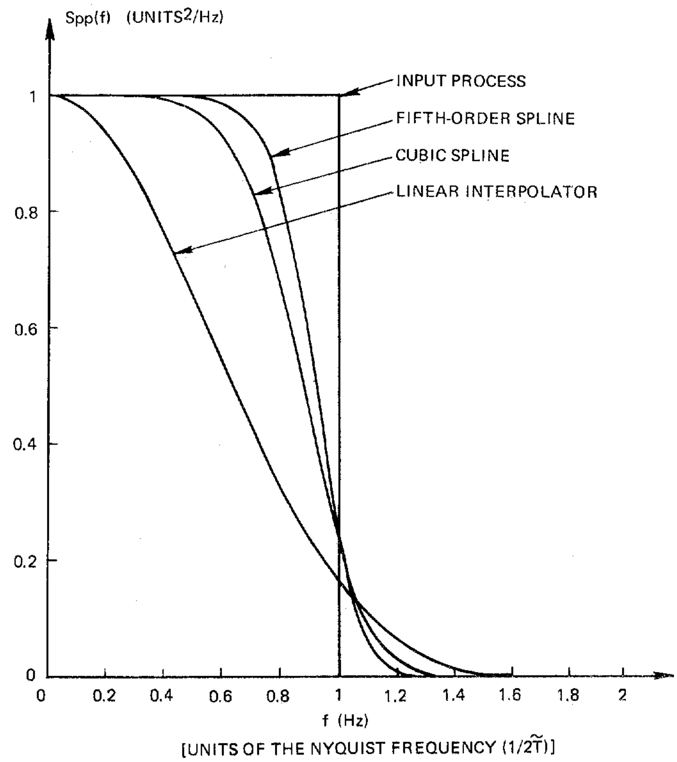

Fig. 2. Power spectral density of outputs of spline interpolators for uniformly spaced samples of band-limited white noise (linear scale).

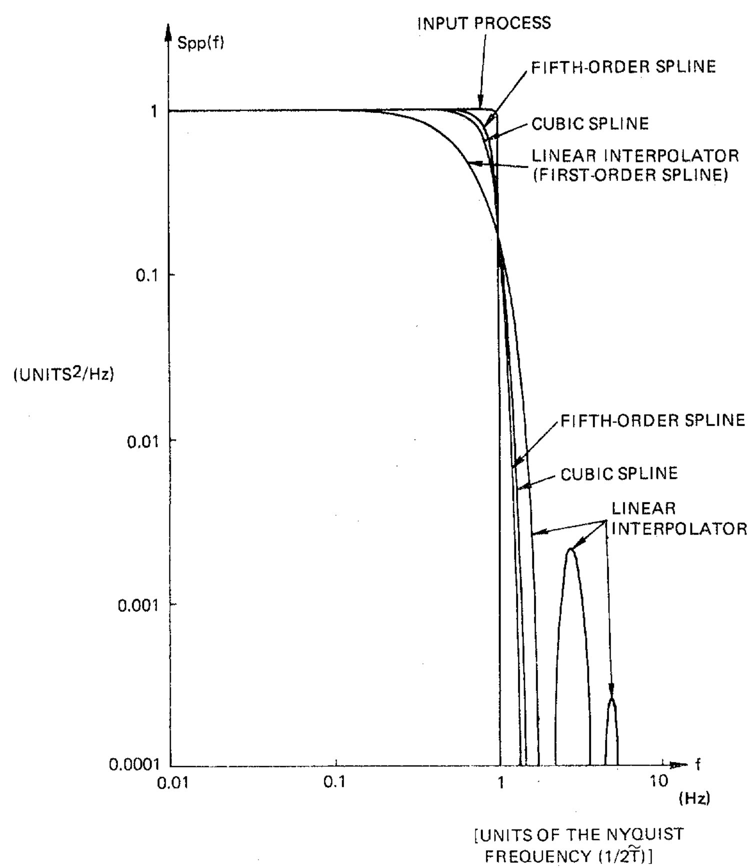

Fig. 3. Power spectral density of outputs of spline interpolators for uniformly spaced samples of band-limited white noise (log–log scale).

difference equations, one for each interval, in terms of $\{x_k\}$. It is then possible to generate $p(t)$ from $\{x_k\}$ and $\{y_k\}$ [2].

As demonstrated, the spline interpolation function, for splines of odd order greater than one, over each sample interval is dependent on all past and future samples of the input function. This dependence indicates (heuristically) that some amount of power, albeit decreasing with the distance of the samples from the given interval, is carried by the interpolator into the interval from every sample of the input function. Hence the spline reconstruction of a finite record length of a stationary random process can be expected, on the average, to have less power in each interval than indicated by the transfer function of the given spline, this effect being due to the fact that the frequency transfer functions given apply exactly only in steady state (or, equivalently, for infinite record lengths). This phenomenon can be observed in the author's previous work [7].

Summary

From the results obtained we conclude that close preservation of power spectral density can be realized through the use of high-order spline interpolators. In applications where power spectral preservation is important, high-order spline interpolators can be used in place of, for example, the conventional zero-order hold. One disadvantage of the spline interpolators is that they require "off-line" implementation, a second is that the reconstruction of a finite record length of a stationary random process will, on the average, have less power than indicated by the transfer function of the spline. Determination of the number of intervals over which a spline must be applied before its transfer function (as derived in this paper) is realized to within a given tolerance appears to be a worthwhile project for future work.

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Realization of Canonical Digital Networks

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Abstract—**Formal realization procedures for digital networks (simulation diagrams, state diagrams) are presented. The methods provide step-by-step realization of all network elements either by repeated divisions and order reductions, or a continued fraction expansion. A total of 14 basic canonical forms has been obtained. The determination of multiplier constants in all cases involves only simple divisions, and is particularly simple.**

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I. Introduction

Probably due to the fact that an arbitrary digital transfer function is always realizable in the well known direct form and canonic[1] form, little work has been done in the past on other realization techniques for digital networks [1]. Meanwhile, it became well known that the adverse effects of quantization and roundoff accumulation noises are deeply influenced by the particular configuration of the realized network. It is conceded now that the cascade and parallel forms are, in general, superior to the direct and canonic forms, and that each second-order filter in the former forms is better realized in the canonic form than the direct one [1]–[5] for standard fixed-wordlength digital computers.

Since most noise analyses and comparisons were done for these four forms (cascade, parallel, direct, and canonic), the questions remain whether there are

[1] In the literature of digital filter, the term canonic form generally refers to the realization variously known as the first Kalman form, controllable canonical form, or 1*D* form.