Variable

- Evaluation $\mathscr{E}[\![x]\!]\eta = \eta[\![x]\!]$
- Well Typed Definition

(i) $\bar{p} \mid x_\tau$ is wt iff it is standard, and *either*

(a) λx_τ or *fix* x_τ is active in $\bar{p}$, *or*

(b) *let* x_σ is active in $\bar{p}$, and τ is a generic instance of σ.

- Proof

(i) x_τ . Then *either* λx_τ or *fix* x_τ is active in $\bar{p}$, and $\eta[\![x]\!] : \tau$, so $\mathscr{E}[\![x]\!]\eta : \tau$, *or let* x_σ is active in $\bar{p}$, and $\eta[\![x]\!] : \sigma$; but then $\tau \leqslant \sigma$, so $\mathscr{E}[\![x]\!]\eta = \eta[\![x]\!] : \tau$, by Proposition 1.

Application

- Evaluation

$$\mathscr{E}[\![(e_1 e_2)]\!]\eta = v_1 \mathsf{E} F \underset{\text{wrong}}{\to} (v_2 \mathsf{E} W \to \text{wrong}, (v_1 \mid F) v_2),$$

$$\text{where } v_i \text{ is } \mathscr{E}[\![e_i]\!]\eta \qquad (i = 1, 2).$$

- Well Typed Definition

(ii) $\bar{p} \mid (\bar{e}_\rho \bar{e}'_\sigma)_\tau$ is wt iff $\bar{p} \mid \bar{e}$ and $\bar{p} \mid \bar{e}'$ are both wt, and $\rho = \sigma \to \tau$.

- Proof

(ii) $(\bar{e}_{\sigma\to\tau}\bar{e}_\sigma)_\tau$. Then $\bar{p} \mid \bar{e}_{\sigma\to\tau}$ is wt, so $\mathscr{E}[\![e]\!]\eta : \sigma \to \tau$, and similarly $\mathscr{E}[\![e']\!]\eta : \sigma$. Then from the semantic equation (remembering that wrong has no type) and by Proposition 2 we get $\mathscr{E}[\![d]\!]\eta : \tau$.

Conditional

- Evaluation $\mathscr{E}[\![\mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3]\!]\eta = v_1 \mathsf{E}\ B_0 \to (v_1 \mid B_0 \to v_2\ , v_3)$, wrong
  where v_i is $\mathscr{E}[\![e_i]\!]\eta \qquad (i = 1, 2, 3)$

- Well Typed Definition

(iii) $\bar{p} \mid (\mathit{if}\ \bar{e}_\rho\ \mathit{then}\ \bar{e}'_\sigma\ \mathit{else}\ \bar{e}''_\sigma)_{\tau'}$ is wt iff $\bar{p} \mid \bar{e}$, $\bar{p} \mid \bar{e}'$ and $\bar{p} \mid \bar{e}''$ are all wt, $\rho = \iota_0$, and $\sigma = \tau = \tau'$.

- Proof

(iii) (if $\bar{e}_{\iota_0}$ *then* $\bar{e}'_\sigma$ *else* $\bar{e}''_\sigma$). Straightforward; the only extra detail needed here is that $\perp_V$ has every type.

Lambda Abstraction

$$\mathscr{E}[\![\lambda x \cdot e]\!]\eta = (\lambda v \cdot \mathscr{E}[\![e]\!]\,\eta\{v/x\}) \text{ in } V$$

(iv) $\quad \bar{p} \mid (\lambda x_\rho \cdot \bar{e}_\sigma)_\tau$ is wt iff $\bar{p} \cdot \lambda x_\rho \mid \bar{e}$ is wt and $\tau = \rho \to \sigma$.

(iv) $\quad (\lambda x_\rho \cdot \bar{e}_\sigma)_{\rho \to \sigma}$. Then $\bar{p} \cdot \lambda x_\rho \mid \bar{e}_\sigma$ is wt. Now we require $(\lambda v \cdot \mathscr{E}[\![e]\!]\,\eta\{v/x\})$ in $V : \rho \to \sigma$. Denote this function by f in V. The inverse of Proposition 2 does not hold, that is, to show f in $V : \rho \to \sigma$ it is not sufficient (though it is necessary) that whenever $v : \rho$, $fv : \sigma$. What is required is that for every $\eta \to \nu \leqslant \rho \to \sigma$, f in $V : \mu \to \nu$.

Suppose then that $\mu \to \nu \leqslant \rho \to \sigma$. Then there is a substitution S, involving only the type variables in ρ and σ, such that $\mu \to \nu = S(\rho \to \sigma)$. Then, since none of these type variables is generic in $\bar{p} \cdot \lambda x_\rho \mid \bar{e}_\sigma$, it follows that $S(\bar{p}) \cdot \lambda x_\mu \mid S(\bar{e})_\nu$ is wt by Proposition 4. Moreover η respects $S(\bar{p})$ (since by Proposition 1 whenever $\eta[\![x]\!] : \sigma'$ and $\tau' \leqslant \sigma'$, $\eta[\![x]\!] : \tau'$) so for any $v : \mu$ we also have $\eta\{v/x\}$ respects $S(\bar{p}) \cdot \lambda x_\mu$.

It then follows by induction that $\mathscr{E}[\![e]\!]\,\eta\{v/x\} : \nu$, so we have shown that $v : \mu$ implies $fv : \nu$, and this yields f in $V : \mu \to \nu$ as required.

Fix

$$\mathscr{E}[\![\mathit{fix}\ x \cdot e]\!]\eta = Y(\lambda v \cdot \mathscr{E}[\![e]\!]\,\eta\{v/x\})$$

(v) $\bar{p} \mid (\mathit{fix}\ x_\rho \cdot \bar{e}_\sigma)_\tau$ is wt iff $\bar{p} \cdot \mathit{fix}\ x_\rho \mid \bar{e}$ is wt and $\rho = \sigma = \tau$.

(v) $(\mathit{fix}\ x_\rho \cdot e_\rho)_\rho$. Then $\bar{p} \cdot \mathit{fix}\ x_\rho \mid \bar{e}_\rho$ is wt. Now we require that $v : \rho$, where

$$v = Y(\lambda v' \cdot \mathscr{E}[\![e]\!]\,\eta\{v'/x\}).$$

Now $v = \bigsqcup_i v_i$, where $v_0 = \bot_V$, $v_{i+1} = \mathscr{E}[\![e]\!]\,\eta\{v_i/x\}$, and by the directed completeness of types we only have to show $v_i : \rho$ for each i.

Clearly $v_0 : \rho$. Assume $v_i : \rho$. Since η respect $\bar{p}$, we have that $\eta\{v_i/x\}$ respects $\bar{p} \cdot \mathit{fix}\ x_\rho$, so by the main induction hypothesis $v_{i+1} : \rho$ also, and we are done.

Let x=e in e'

$$\mathscr{E}[\![let\ x = e_1\ in\ e_2]\!]\eta = v_1\ \mathsf{E}\ W \rightarrow \text{wrong},\ \mathscr{E}[\![e_2]\!]\ \eta\{v_1/x\}$$
$$\text{where } v_1 = \mathscr{E}[\![e_1]\!]\eta.$$

(vi) $\bar{p} \mid (let\ x_\rho = \bar{e}_\rho\ in\ \bar{e}'_\sigma)_\tau$ is wt iff $\bar{p} \mid \bar{e}$ and $\bar{p} \cdot let\ x_\rho \mid \bar{e}'$ are both wt, and $\sigma = \tau$.

(vi) $(let\ x = \bar{e}_\rho\ in\ \bar{e}'_\sigma)_\sigma$. Then $\bar{p} \mid \bar{e}_\rho$ is wt, so we immediately have $v : \rho$, where $v = \mathscr{E}[\![e]\!]\eta$. We require $\mathscr{E}[\![e']\!]\ \eta\{v/x\} : \sigma$.

Now $\bar{p} \cdot let\ x_\rho \mid \bar{e}'_\sigma$ is also wt, and because $v : \rho$ we have that $\eta\{\sigma/x\}$ respects $\bar{p} \cdot let\ x_\rho$; the rest follows by the induction hypothesis. ▮