INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

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INTERNATIONAL TRANSACTIONS IN OPERATIONAL RESEARCH

Intl. Trans. in Op. Res. 00 (2019) 1–30
DOI: 10.1111/itor.12629

Deviation measure in second-order stochastic dominance with an application to enhanced indexing

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Received 28 January 2018; received in revised form 1 January 2019; accepted 2 January 2019

Abstract

Deviation measures form a separate class of functionals applied to the difference of a random variable to its mean value. In this paper, we aim to introduce a deviation measure in the second-order stochastic dominance (SSD) criterion to select an optimal portfolio having a higher utility of deviation from its mean value than that of the benchmark portfolio. A new strategy, called deviation SSD (DSSD), is proposed in portfolio optimization. Performance of the proposed model in an application to enhanced indexing is evaluated and compared to other two standard portfolio optimization models that employ SSD criterion on lower partial moments of order one (LSSD) and tail risk measure (TSSD). We use historical data of 16 global indices to assess the performance of the proposed model. In addition to this, we solved the models on simulated data, capturing the joint dependence between indices using historical data of two data sets, each data set comprising six indices. The simulated data are generated by first fitting the autoregressive moving average–Glosten–Jagannathan–Runkle–generalized autoregressive conditional heteroscedastic model on historical data to determine marginal distributions, and thereafter capture the dependence structure using the best fitted regular vine copula. The portfolios from DSSD model achieve higher excess mean return than TSSD model and lower variance, downside deviation, and conditional value-at-risk than LSSD model. Also, portfolios from DSSD model demonstrate higher values of information ratio, stable tail-adjusted return ratio and Sortino ratio than the other two models. Lastly, DSSD model is observed to produce well-diversified portfolios compared to LSSD model.

Keywords: second-order stochastic dominance; deviation measures; lower partial moment; tail risk measure; enhanced indexing; simulated data

1. Introduction

Utility theory is long-recognized to rank alternatives under uncertainty (Neumann and Morgenstern, 1947). Although utility theory imparts powerful ordering between two distributions, yet its

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practical applications are hampered by insufficient knowledge of the exact utility function. An excellent alternative to it is the concept of stochastic dominance (SD), which for different orders relate to the different utility functions (Levy, 1992). The SD ranking of first (FSD) order (Quirk and Saposnik, 1962), second order (SSD; Hadar and Russell, 1969), and third order (TSD; Whitmore, 1970) reflects the behavior of rational investors, rational risk-averse investors, and rational risk-averse and ruin-averse investors, respectively. Comprehensive details can be found in Levy (2016).

The underpinning of SD lies in the theory of majorization (Olkin and Marshall, 2016), which involves ranking of real-valued random vectors. One can refer to Hadar and Russell (1969) and Hanoch and Levy (1969) for early developments on SD. Roman and Mitra (2009) discussed how the SSD selection criterion in portfolio optimization improves the traditional mean-risk models by incorporating more information from return distribution. Fidan Keçeci et al. (2016) reported a favorable outcome for SSD ranking in portfolio selection. Lozano and Gutiérrez (2008) applied SSD ranking in combination with data envelopment analysis to assess the performance of mutual funds. Since mean-risk models are computationally easier to solve than the optimization models involving SSD criterion, Ogryczak and Ruszczyński (1999), Ogryczak and Ruszczyński (2001), and Gotoh and Konno (2000) investigated whether a portfolio from a mean-risk model is SSD or TSD efficient.

Apart from its theoretical benefits, SSD gained much attention due to its efficient implementation in portfolio selection (Fábián et al., 2011; Bruni et al., 2012; Roman et al., 2013; Sharma et al., 2017; Sharma and Mehra, 2017). Recently, Liang et al. (2018) and Singh and Dharmaraja (2017) explored the applications of SD criteria in disappointment theory and option pricing, respectively. Dentcheva and Ruszczyński (2003) used lower partial moment (LPM) of order one to develop the SSD ranking concerning the benchmark portfolio. We call this strategy as LSSD. Fábián et al. (2011) proposed the SSD model for portfolio selection using tail risk measure at different confidence levels. We call this strategy as TSSD. The optimization models incorporating these strategies result in equivalent linear programs for finite realizations of alternatives, which have been efficiently solved using the cutting plane algorithms (Rudolf and Ruszczyński, 2008; Fábián et al., 2011).

Typically, the market index is used as a benchmark portfolio. Index tracking (Beasley et al., 2003) and enhanced indexing (Canakgoz and Beasley, 2009) are two well-studied applications of portfolio investment based on movements of the market index. While index tracking aims to mimic the index, enhanced indexing intents to outperform it in terms of returns. A portfolio that dominates the market index in SSD rule is known to possess higher expected returns (Ogryczak and Ruszczyński, 2002) than the market, therefore, making SSD criterion a natural choice for use in enhanced indexing. Roman et al. (2013) presented two variants of enhanced indexing models using tail risk measures (unscaled tails) and conditional value-at-risk (CVaR; scaled tails). Sharma et al. (2017) relaxed the SSD criterion relative to the market index in enhanced indexing model to earn a higher mean return. Bruni et al. (2017) proposed cumulative zero-order ϵ-SD in enhanced indexing. Post and Kopa (2017) used TSD in enhanced indexing. Valle et al. (2017) proposed a method of reshaping the benchmark distribution in SSD optimization models and used long–short strategies to obtain portfolio with enhanced return distribution. Applications of SSD are not restricted to portfolio optimization. Cheong et al. (2007) and Nie et al. (2012) applied SSD ranking in electric energy problems and optimal path problems, respectively. More recent and relevant literature on applications of SSD includes Sriboonchita et al. (2017).

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Several authors designed and developed optimization problems and statistical inference tests for testing the null hypothesis of nondominance of a portfolio over the set of all admissible portfolios in the sense of SSD rule (see, for instance, Bawa et al., 1985; Kaur et al., 1994; Davidson and Duclos, 2000; Post, 2003; Kuosmanen, 2004; Davidson, 2009; Scaillet and Topaloglou, 2010; Linton et al., 2014; Kopa and Post, 2015). Hodder et al. (2014) used the nondominance test of Davidson (2009) to test whether the SSD-based portfolio dominates the benchmark portfolio in out-of-sample periods.

In extension to the existing SSD ranking rule, the present study proposes a portfolio optimization model to select a portfolio whose return deviation from its mean value dominates the return deviation of the benchmark portfolio from its mean value under the SSD framework. The proposed formulation includes the deviation measure in SSD criterion to construct an optimal portfolio. The deviation measures form a separate class of functionals satisfying a specific set of axioms and having one to one correspondence with the class of expectation-bounded risk measures (Rockafellar et al., 2002). A distinguishing feature of our approach is to replace the random return R_x of the portfolio x and return I of the benchmark portfolio by their return deviations from the respective mean values thereby using $R_x - E(R_x)$ and $I - E(I)$ in the SSD criterion. Equivalently, the proposed strategy aims to achieve $U(R - E(R_x)) \geq U(I - E(I))$, for all nondecreasing concave utility functions U. The resulting ranking rule is called deviation SSD (DSSD). Similar to LSSD and TSSD strategies, optimization model based on the proposed DSSD rule is transformed into an equivalent linear program (LP) that can be solved in a smaller runtime using the cutting plane algorithm.

We phrase our empirical analysis of models based on LSSD, DSSD, and TSSD strategies with an application to enhanced indexing by taking the market index as a benchmark portfolio. We apply a rolling window scheme on historical data of weekly closing prices of stocks from 16 global indices. We also analyze the performance of the three strategies on two simulated data sets comprising six assets each, where simulations are conducted using GARCH vine copula approach in which the marginal distribution of each asset is captured via GARCH model, and their joint distribution is modeled using regular vine (R-Vine) copulas.[1]

Performance of the proposed strategy is discussed and compared to the LSSD and TSSD strategies. We observe that the obtained portfolios from the proposed model yield higher excess mean return (EMR) over an index without incurring more downside risk concerning downside deviation and CVaR in out-of-sample periods. While LSSD suffers from highest risk, TSSD produces least EMR over all indices. Acknowledging the limitations of LSSD and TSSD strategies, DSSD attempts to create a balanced risk-return profile by achieving higher Sortino ratio, stable tail-adjusted return (STARR) ratio, and information ratio for a significantly larger number of indices than the former two. It also has been observed that the optimization model based on DSSD produces diversified portfolios on those data sets in which LSSD allocates 70% or more weight to a single stock.

The rest of this paper is organized as follows. Section 2 explains the three portfolio optimization models based on LSSD, TSSD, and DSSD strategies. Section 3 reports a comparative evaluation of portfolios obtained from the three models using historical data. Section 4 explains the procedure of applying a time series model along with the copula theory to simulate instances of return series. The section also presents experimental analysis performed with the simulated data sets. Section 5 concludes the paper.

[1]The R-Vine copula (Bedford and Cooke, 2001) model allows dynamics and dependence structure to be captured in a multivariate return distribution. For more details on R-Vine copula, see Chollete et al. (2009) and Dissmann et al. (2013).

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2. Notations and models

A portfolio is an $n \times 1$ decision vector $x = (x_1, \dots, x_n)^t$, where x_i denotes the proportion of investment in ith stock, $i = 1, \dots, n$. Let $R = (r_1, \dots, r_n)$ be a random vector of returns, where r_i denotes random return from x_i. The return of the portfolio x is given by $R_x = \sum_{i=1}^{n} r_i x_i$, assuming $E(|R_x|) < \infty$. We present details of the SSD criterion for ranking two portfolios.

2.1. Second-order stochastic dominance

For a given distribution function F_{R_x} of portfolio return R_x, consider the following recursive relation:

$$F_{R_x}^{(k)}(\eta) = \int_{-\infty}^{\eta} F_{R_x}^{(k-1)}(r)dr, \quad k \geq 2,$$

$$\text{and} \quad F_{R_x}^{(1)}(\eta) = F_{R_x}(\eta), \quad \text{where } \eta \in \mathbb{R}.$$

A return R_x of portfolio $x \in \mathbb{R}^n$ is preferred to return R_y of portfolio $y \in \mathbb{R}^n$ weakly in the kth order SD, or we say R_x is stochastically larger than R_y in a weak sense, denoted by $R_x \succeq_{(k)} R_y$, if and only if

$$F_{R_x}^{(k)}(\eta) \leq F_{R_y}^{(k)}(\eta), \quad \forall \, \eta \in \mathbb{R}. \tag{1}$$

The strict dominance relations $\succ_{(k)}$ is defined as follows:

$$R_x \succ_{(k)} R_y \Leftrightarrow R_x \succeq R_y \text{ and } R_y \not\succeq R_x, \tag{2}$$

that is, $R_x \succ_{(k)} R_y$ if and only if $F_{R_x}^{(k)}(\eta) \leq F_{R_y}^{(k)}(\eta), \forall \eta \in \mathbb{R}$, with at least one strict inequality.

For $k = 2$, in Equations (1) and (2), $R_x \succ_{(SSD)} R_y$ if and only if $F_{R_x}^{(2)}(\eta) \leq F_{R_y}^{(2)}(\eta)$, $\forall \, \eta \in \mathbb{R}$, with at least one strict inequality.

We recall the following equivalent characterizations of SSD relation from the literature.

(ES1) Hadar and Russell (1969): $R_x \succ_{(SSD)} R_y$ if and only if $E(U(R_x)) \geq E(U(R_x))$, for every nondecreasing and concave utility function $U \in U_2 = \{U : U'(x) \geq 0, \; U''(x) \leq 0\}$, with at least one strict inequality.

(ES2) Hadar and Russell (1969): Let R_x and R_y are discrete random variables with T equally probable outcomes $R_x = \{R_j(x), \; j = 1, \dots, T\}$ and $R_y = \{R_j(y), \; j = 1, \dots, T\}$. Arrange the outcomes of R_x and R_y in the ascending order as $R_{(1)}(x) \leq \cdots \leq R_{(T)}(x)$ and $R_{(1)}(y) \leq \cdots \leq R_{(T)}(y)$. Then, $R_x \succ_{(SSD)} R_y$ if and only if $\sum_{i=1}^{j} R_{(j)}(x) \geq \sum_{i=1}^{j} R_{(j)}(y), \; \forall \, j = 1, \dots, T$, with at least one strict inequality.

(ES3) Ogryczak and Ruszczyński (1999): $R_x \succ_{(SSD)} R_y$ if and only if $E(\eta - R_x)^+ \leq E(\eta - R_y)^+, \; \forall \, \eta \in \mathbb{R}$, with at least one strict inequality. Here, $E(\eta - R_x)^+ = E(\max\{\eta - R_x, 0\})$, is the LPM of order one for target point $\eta \in \mathbb{R}$, or simply an expected downside risk.

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(ES4) Ogryczak and Ruszczyński (2002), Roman et al. (2013): The second quantile function $F_{R_x}^{(-2)} : \mathbb{R} \to \bar{\mathbb{R}}$ is defined by

$$F_{R_x}^{(-2)}(p) = \begin{cases} \int_0^p F_{R_x}^{(-1)}(q)dq, & p \in (0, 1] \\ 0, & p = 0 \\ +\infty, & p \notin [0, 1], \end{cases}$$

where $F_{R_x}^{(-1)}(p) = \inf\{\eta : F_{R_x}(\eta) \geq p\},\ 0 < p < 1$.

Let $p \in (0, 1)$. If there exists $\eta \in \mathbb{R}$ such that $P(R_x \leq \eta) = p$, then it is shown that $F_{R_x}^{(-2)}(p) = pE(R_x|R_x \leq \eta) = \text{Tail}_p(R_x)$. Hence, $R_x \succ_{(SSD)} R_y$ if and only if $\text{Tail}_p(R_x) \geq \text{Tail}_p(R_y),\ \forall\, p \in (0, 1]$, with at least one strict inequality.

(ES5) Ogryczak and Ruszczyński (2002): Since $F_{R_x}^{(-2)}(p)/p = E(R_x|R_x \leq \eta) = CVaR_p(R_x)$. Therefore, $R_x \succ_{(SSD)} R_y$ if and only if $F_{R_x}^{(-2)}(p)/p \geq F_{R_y}^{(-2)}(p)/p,\ \forall\, p \in (0, 1]$, or $CVaR_p(R_x) \geq CVaR_p(R_y),\ \forall\, p \in (0, 1]$, with at least one strict inequality.

2.2. Optimization model incorporating SSD by lower partial moments of order one

We present the optimization models proposed by Dentcheva and Ruszczyński (2003) and Fábián et al. (2011) to select a portfolio to outperform the market index in SSD ordering. Using the SSD equivalent condition (ES3), the optimization model by Dentcheva and Ruszczyński (2003) is as follows:

$$\begin{aligned} (P1) \quad & \max \quad E(R_x) \\ & \text{subject to} \\ & E((\eta - R_x)^+) \leq E((\eta - I)^+), \quad \forall\, \eta \in \mathbb{R}, \qquad (3) \\ & x \in X, \end{aligned}$$

where I denotes the random variable for market index and $X = \{(x_1, \ldots, x_n)^t : \sum_{i=1}^n x_i = 1,\ x_i \geq 0,\ i = 1, \ldots, n\}$, is the set of all admissible portfolios.

It is reasonable to observe prices of any asset at discrete time points called scenarios. We take T scenarios of each stock return over the investment horizon Γ with probability vector $p = ((p_1, \ldots, p_T)^t;\ p^t e = 1,\ p_j \geq 0,\ \forall\, j = 1, \ldots, T)$. If we denote the jth outcome of ith stock by r_{ij}, with probability p_j, the mean return from the ith asset is given by $\mu_i = \sum_{j=1}^T r_{ij} p_j,\ i = 1, \ldots, n$. The jth outcome of a portfolio return R_x is then calculated by $R_j(x) = \sum_{i=1}^n r_{ij} x_i$, with probability $p_j,\ j = 1, \ldots, T$. Thus, the portfolio return R_x is finitely distributed over $\{R_1(x), \ldots, R_T(x)\}$, with probabilities $p_j,\ j = 1, \ldots, T$, and $E(R_x) = \sum_{i=1}^n \mu_i x_i$.

Problem $(P1)$ is a semi-infinite programming problem which for a finitely T known scenarios of I is translated into the following relaxed problem $(P2)$:

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$$
\begin{aligned}
(P2)\quad & \max \quad E(R_x) \\
& \text{subject to} \\
& E((I_k - R_x)^+) \leq E((I_k - I)^+), \quad k = 1, \ldots, T, \qquad (4) \\
& x \in X,
\end{aligned}
$$

where I_k is the kth realization of I, $k = 1, \ldots, T$. Using T^2 number of auxiliary variables $s_{jk} = (I_k - \sum_{i=1}^{n} r_{ij}x_i)^+$, $j,\ k = 1, \ldots, T$, problem $(P2)$ is simplified to the following LP:

$$
\begin{aligned}
(P3)\quad & \max \quad \sum_{i=1}^{n} \mu_i x_i \\
& \text{subject to} \\
& \sum_{i=1}^{n} r_{ij}x_i + s_{jk}, \geq I_k, \quad j,\ k = 1, \ldots, T, \\
& \sum_{j=1}^{T} p_j s_{jk} \leq v_k, \quad k = 1, \ldots, T, \\
& s_{jk} \geq 0, \quad j,\ k = 1, \ldots, T, \\
& x \in X,
\end{aligned}
$$

where $v_k = E(I_k - I)^+$, $k = 1, \ldots, T$. If $(x^* = (x_i^*, i = 1, \ldots, n,),\ s^* = (s_{jk}^*),\ j,\ k = 1, \ldots, T)$ is an optimal solution of $(P3)$, then x^* is an optimal solution of $(P1)$, and it is said to dominate the market index I in LSSD sense.

2.3. Optimization model incorporating SSD by tail risk measures

Using an equivalent SSD condition (ES4), optimization model proposed by Fábián et al. (2011) is originally the following multiobjective program with T objective functions:

$$\max_{x \in X} \{\mathrm{Tail}_{1/T}(R_x), \mathrm{Tail}_{2/T}(R_x), \ldots, \mathrm{Tail}_{T/T}(R_x)\}.$$

The above multiobjective problem can be translated into a single objective problem by maximizing the worst outcome of $\mathrm{Tail}_{j/T}(R_x)$ relative to $\mathrm{Tail}_{j/T}(I)$, as follows:

$$
\begin{aligned}
(P4)\quad & \max \quad \nu \\
& \text{subject to} \\
& \nu \leq \mathrm{Tail}_{j/T}(R_x) - \mathrm{Tail}_{j/T}(I), \quad j = 1, \ldots, T, \\
& x \in X.
\end{aligned}
$$

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Using $\mathrm{Tail}_{j/T}(R_x) = -\frac{j}{T} CVaR_{(1-(j/T))}(-R_x)$, along with the LP model for CVaR optimization (Rockafellar and Uryasev, 2000; Fábián et al., 2011) obtained the following LP equivalent to $(P4)$:

$$
\begin{aligned}
(P5) \quad & \max \quad \nu \\
& \text{subject to} \\
& \nu \leq z_k - I_k, \quad k = 1, \ldots, T, \\
& z_k = \frac{1}{T}\left(kt_k - \sum_{j=1}^{T} d_{kj}\right), \quad k = 1, \ldots, T, \\
& t_k - \sum_{i=1}^{n} x_i r_{ij} \leq d_{kj}, \ d_{kj} \geq 0, \quad j, k = 1, \ldots, T, \\
& x \in X.
\end{aligned}
$$

If $x^* = (x_i^*, i = 1, \ldots, n,)$ is an optimal solution of $(P5)$, then it is said to dominate the market index I in TSSD sense. Note that model $(P5)$ always generates a feasible solution irrespective of the market index chosen by an investor.

2.4. *The proposed strategy: deviation SSD rule and its optimization model*

The SSD criterion induces a ranking among portfolios on the basis of worst realizations of returns while ignoring how they perform against their mean returns. To incorporate the latter, we propose to employ a deviation measure in SSD ranking. The deviation measures form a class of functionals satisfying all four properties of coherent risk measure established for the deviation type of risk measures. We recall the definition of deviation measure from Rockafellar et al. (2002) as follows:

Definition 1. *A measure $D : L \to [0, \infty]$ defined on a subspace L of all random variable X, is called a deviation risk measure if it satisfies the following four properties:*

(d1) $D(X + C) = D(X), \ \forall \ X$ *and constant* C.
(d2) $D(0) = 0,$ *and* $D(\lambda X) = \lambda D(X), \ \forall \ X$ *and* $\lambda > 0$.
(d3) $D(X + Y) \leq D(X) + D(Y), \ \forall \ X$ *and* Y.
(d4) $D(X) > 0, \ \forall \ X$ *and* $D(X) = 0$ *whenever* X *is constant.*

The standard deviation $\sigma(X) = \sqrt{E(X - E(X))^2}$ and mean-absolute deviation $E|X - E(X)|$ are examples of deviation risk measures. Deviation measure can be obtained by replacing X with $X - E(X)$ in the expectation-bounded risk measures $R(X)$, that is, $D(X) = R(X - E(X))$ (Rockafellar et al., 2002). The deviation associated with $VaR_\delta(X)$ (the value-at-risk at δ confidence level)[2] is denoted by $VaR_\delta^\triangle(X)$, and defined by $VaR_\delta(X - E(X))$. The $VaR_\delta^\triangle(X)$ fails to satisfy property

[2] $VaR_\delta(X)$ of the random variable X is the maximum possible loss at δ confidence level that can occur in X over a holding period

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(d3), and hence it is not a deviation measure, while the deviation of $CVaR_\delta(X)$, that is, $CVaR_\delta^\triangle(X) = CVaR_\delta(X - E(X))$, is found to be a deviation measure.

Standard deviation and mean-absolute deviation are symmetric deviation risk measures, whereas $CVaR_\delta^\triangle(X)$ is asymmetric deviation risk measure. For more insight and properties of deviation measures, one can refer to Rockafellar et al. (2002, 2006). More recently, Goel et al. (2017) applied the STARR ratio model with deviation mixed CVaR to an application of enhanced indexing and reported favorable results. The orientation of applying deviation risk measure should be clear in the context. If the concern is on the extent by which the random variable X drops below $E(X)$, then one needs to work with $X - E(X)$ rather than X.

In this paper, we attempt to analyze the SSD criterion in deviation framework. Also, to maintain consistency in notation, we use "△" to point out the deviation counterpart of SSD, whenever required.

We say that the portfolio return R_x is preferred to the portfolio return R_y weakly in kth order SD under deviation framework, denoted by $R_x \succeq_{(\triangle k)} R_y$, if and only if

$$F_{R_x}^{(\triangle k)}(\eta) \leq F_{R_y}^{(\triangle k)}(\eta), \quad \forall\, \eta \in \mathbb{R}, \tag{5}$$

where

$$F_{R_x}^{(\triangle k)}(\eta) = \int_{-\infty}^{\eta} F_{R_x}^{(\triangle(k-1))}(r)dr, \quad k \geq 2,$$

and $F_{R_x}^{(\triangle 1)}(\eta) = F_{R_x - E(R_x)}(\eta) = P(R_x - E(R_x) \leq \eta) = F_{R_x}(\eta + E(R_x)),\ \eta \in \mathbb{R}.$

For $k = 2$, we have $R_x \succ_{(DSSD)} R_y$ if and only if $F_{R_x}^{(\triangle 2)}(\eta) \leq F_{R_y}^{(\triangle 2)}(\eta),\ \forall\, \eta \in \mathbb{R}$, with at least one strict inequality.

We have the following equivalent characterizations of DSSD ordering similar to the ones we have for SSD ordering.

(DES1): $R_x \succ_{(DSSD)} R_y$ if and only if $E(U(R_x - E(R_x))) \geq E(U(R_y - E(R_y)))$, for every non-decreasing and concave utility function $U \in U_2 = \{U : U^{'}(x) \geq 0,\ U^{''}(x) \leq 0\}$, with at least one strict inequality.

(DES2): Let R_x and R_y be discrete random variables with T equally probable outcomes $R_x = \{R_j(x),\ j = 1, \ldots, T\}$ and $R_y = \{R_j(y),\ j = 1, \ldots, T\}$. Arrange the outcomes according to their corresponding deviation series $R_x - E(R_x)$ and $R_y - E(R_y)$ in an ascending order as $(R(x) - E(R_x))_{(1)} \leq \cdots \leq (R(x) - E(R_x))_{(T)}$ and $(R(y) - E(y))_{(1)} \leq \cdots \leq (R(y) - E(y))_{(T)}$. Then, $R_x \succ_{(DSSD)} R_y$ if and only if $\sum_{i=1}^{j}(R(x) - E(R_x))_{(j)} \geq \sum_{i=1}^{j}(R(y) - E(R_y))_{(j)},\ \forall\, j = 1, \ldots, T$, with at least one strict inequality.

(DES3): $R_x \succ_{(DSSD)} R_y$ if and only if $E(\eta - (R_x - E(R_x)))^+ \leq E(\eta - (R_y - E(R_y)))^+,\ \forall\, \eta \in \mathbb{R}$, with at least one strict inequality.

The deviation counterpart of $E(\eta - R_x)^+$, that is, $E^\triangle(\eta - R_x)^+ = E(\eta - (R_x - E(R_x)))^+$, fails to satisfy properties (d2) and (d4), and hence it is not a deviation measure. However, it can be a good choice in risk management for reason that it represents the under performance of the random variable $R_x - E(R_x)$ below the threshold η. In

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application to enhanced indexing, $E(\eta - (R_x - E(R_x)))^+$ should be less than or equal to $E(\eta - I - E(I)))^+$, for all $\eta \in \mathbb{R}$.

(DES4): The second quantile function $F_{R_x}^{(-2\triangle)} : \mathbb{R} \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, is defined by

$$F_{R_x}^{(-2)}(p) = \begin{cases} \int_0^p F_{R_x}^{(-1\triangle)}(q)dq, & p \in (0, 1] \\ 0, & p = 0 \\ +\infty, & p \notin [0, 1]. \end{cases}$$

where $F_{R_x}^{(-1\triangle)}(p) = \inf\{\eta : F_{R_x}^{\triangle 1}(\eta) \geq p\} = \inf\{\eta + E(R_x) : F_{R_x}(\eta + E(R_x)) \geq p\}$, $0 < p < 1$. Let $p \in (0, 1)$. If there exists $\eta \in \mathbb{R}$ such that $P(R_x - E(R_x) \leq \eta) = p$, then it can be shown that $F_{R_x}^{(-2\triangle)}(p) = pE(R_x - E(R_x)|R_x - E(R_x) \leq \eta) = \mathrm{Tail}_{\mathrm{p}}^{\triangle}(\mathrm{R_x})$. Hence, $R_x \succ_{(DSSD)} R_y$ if and only if $\mathrm{Tail}^{\triangle}{}_p(R_x) \geq \mathrm{Tail}^{\triangle}{}_p(R_y)$, $\forall\, p \in (0, 1]$, with at least one strict inequality.

(DES5): As $F_{R_x}^{(-2\triangle)}(p)/p = E(R_x - E(R_x)|R_x - E(R_x) \leq \eta) = CVaR_p(R_x - E(R_x)) = CVaR_p^{\triangle}(R_x)$, so, $R_x \succ_{(DSSD)} R_y$ if and only if $CVaR_p^{\triangle}(R_x) \geq CVaR_p^{\triangle}(R_y)$, $\forall\, p \in (0, 1]$, with at least one strict inequality.

The proofs of above equivalent characterizations of DSSD follows the similar lines as proofs of (ES1)–(ES5) in the literature with X replaced by $X - E(X)$.

Remark 1. If either $E(R_x) = E(R_y) = 0$, or the average returns of all the assets comprising portfolios x and y are the same, then the ranking between R_x and R_y under SSD and DSSD is similar.

We propose the following optimization model to formulate a portfolio to dominate the market index in the DSSD ordering:

$$\begin{aligned} (P6) \quad & \max \quad E(R_x) \\ & \text{subject to} \\ & E((\bar{I}_k - (R_x - E(R_x)))^+) \leq E((\bar{I}_k - (I - E(I)))^+), \quad k = 1, \ldots, T, \qquad (6) \\ & x \in X, \end{aligned}$$

where $\bar{I}_k$ is the kth realization of $I - E(I)$.

Similar to obtaining model $(P3)$ from $(P2)$, we can derive the following model from $(P6)$:

$$\begin{aligned} (P7) \quad & \max \quad \sum_{i=1}^{n} \mu_i x_i \\ & \text{subject to} \\ & \sum_{i=1}^{n} (r_{ij} - \mu_i)x_i + \bar{s}_{jk} \geq \bar{I}_k, \quad j, k = 1, \ldots, T, \end{aligned}$$

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$$\sum_{j=1}^{T} p_j \bar{s}_{jk} \leq \bar{v}_k, \quad k = 1, \ldots, T,$$

$$\bar{s}_{jk} \geq 0, \quad j, k = 1, \ldots, T,$$

$$x \in X,$$

where $\bar{v}_k = E((\bar{I}_k - (I - E(I)))^+), k = 1, \ldots, T, \bar{s}_{jk} = (\bar{I}_k - \sum_{i=1}^{n} (r_{ij} - \mu_i)x_i)^+, j, k = 1, \ldots, T.$

If $(x^* = (x_i^*, i = 1, \ldots, n,),\ \bar{s}^* = (\bar{s}_{jk}^*),\ j, k = 1, \ldots, T)$ is an optimal solution of $(P7)$, then x^* is an optimal solution of $(P6)$. Note that the number of variables and constraints in $(P7)$ are same as in $(P3)$.

In the presence of a large number of auxiliary variables in $(P3)$, $(P5)$, and $(P7)$, which grow quadratically with the number of scenarios T and linearly with the number of stocks n, these models take much time to solve on traditional LP solvers especially for 500 or more scenarios (Rudolf and Ruszczyński, 2008; Roman et al., 2013). Therefore, as suggested in studies in the past, we solve $(P5)$ applying the cutting plane method described by Fábián et al. (2011), and $(P3)$ and $(P7)$ applying the cutting plane method described by Rudolf and Ruszczyński (2008).

In the following two sections, we examine the performance of $(P3)$, $(P5)$, and $(P7)$ in an application to enhanced indexing. For convenience to recall the models, portfolios generated by $(P3)$, $(P5)$, and $(P7)$ are named LSSDP, TSSDP, and DSSDP, respectively.

Our data can be put in two groups: historical data containing weekly closing prices of stocks from 16 global indices, and simulated data generated after fitting the regular vine copulas in the daily closing prices of two data sets each comprising six indices.

To solve the three models empirically, we consider T equiprobable scenarios for stock returns (and hence for the portfolio return R_x) over the investment horizon Γ with probability vector $p = (p_j = 1/T, \forall j = 1, \ldots, T)$. Some research in the literature suggest to obtain the probability vector p by minimizing its distance with the reference probability vector. For instance, Post et al. (2018) lately minimized the Kullback Leibler divergence of p to the sample probabilities subject to the set of moment conditions with its application to SD decision criterion. However, we stick to use the uniform probability vector to solve the three models in our present study due to its wide acceptability in the empirical studies of portfolio optimization.

We use R software with CPLEX solver interface on Windows 64 bits Intel(R) Core(TM) i7-6700 CPU @3.40 GHz processor for solving all optimization problems.

3. Empirical analysis on historical data

3.1. Historical data

The following three data sets are used in empirical analysis:

- The first data set is available on Beasley OR library (Canakgoz and Beasley, 2009). It comprises 291 weekly closing prices of the following eight indices for the period March 1992 to September 1997.

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1. Hang Seng (HSI, Hong Kong), indtrack1.txt, 31 assets;
2. DAX 100 (Germany), indtrack2.txt, 85 assets;
3. FTSE 100 (UK), indtrack3.txt, 89 assets;
4. S&P 100 (USA), indtrack4.txt, 98 assets;
5. Nikkei 225 (Japan), indtrack5.txt, 225 assets;
6. S&P 500 (USA), indtrack6.txt, 457 assets;
7. Russell 2000 (USA), indtrack7.txt, 1318 assets;
8. Russell 3000 (USA), indtrack8.txt, 2151 assets.

- The second data set contains weekly closing prices of five indices. It is listed in the supplementary material by Bruni et al. (2016).

1. Dow Jones (Dow Jones Industrial Average, USA), 28 assets, 1363 observations, February 1990–April 2016;
2. FTSE 100 (Financial Times Stock Exchange, UK), 83 assets, 717 observations, July 2002–April 2016;
3. FF 49 (Fama and French 49 Industry portfolios, USA), 49 portfolios considered as assets (using the subsample where returns of 49 industries are available, July 1969–July 2015;
4. S&P 500 (Standard and Poor's, USA), 442 assets, 595 observations November 2004–April 2016;
5. NASDAQ 100 (National Association of Securities Dealers Auto-mated Quotation, USA), 82 assets, 596 observations, November 2004–April 2016.

- The third data set is extracted from Thomson Reuters data stream on EIKON software. It consists of weekly closing prices of following three indices where period of each data are taken according to the availability of data for maximum number of stocks in that period.

1. S&P BSE Sensex: It consists of 30 largest and most liquid stocks listed in the Bombay Stock Exchange (BSE), India. The BSE is a long-established stock exchange in Asia. The data are taken for the period January 2011 to April 2017.
2. CNX Nifty: It is the key index of 50 major stocks listed in the National Stock Exchange, India. The data are for 48 assets from March 2009 to April 2017.
3. MSCI world index: It is a broad global equity benchmark that contains large and mid-cap of 1646 equities from 23 developed markets across the globe. The data are taken for the period May 2007to April 2017 for 1444 equities.

3.2. Performance measures

The performance of the three models is assessed on several performance measures described briefly as follows (for details, see Bacon, 2011):

Excess mean return: It is an average of the difference between portfolio return and market index return. It is used to describe reward over index, that is,

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$$EMR = \frac{1}{T}\sum_{j=1}^{T}(R_j(x) - I_j),$$

where $R_j(x)$ and I_j are the jth realizations of portfolio and market index, respectively. Higher values of EMR are preferable.

Downside deviation (DD): It is defined by

$$\frac{1}{\sqrt{T}}\sqrt{\sum_{j=1}^{T}\min[R_j(x) - I_j, 0]^2},$$

to depict the under achievement of the portfolio from the market index. The lower values are preferable.

Sortino ratio: It measures return per unit risk where return and risk are taken as EMR and DD, respectively. It is calculated as follows:

$$\text{Sortino ratio} = \begin{cases} \dfrac{EMR}{DD}, & EMR > 0 \\ 0, & EMR \leq 0. \end{cases}$$

Higher values of Sortino ratio are desirable.

Conditional value-at-risk ($CVaR_\delta(I - R_x)$): It is the CVaR value of return series of $I - R_x$ depicting worst under achievement of portfolio return R_x from the market index I at δ level of confidence, $\delta \in (0, 1)$. The CVaR is one of the most famous downside risk measures to capture the risk of extreme losses. Its lower values, for $\delta \to 1$, indicate safety against worst outcomes.

The STARR ratio (STARR_δ): It is the ratio of EMR to $CVaR_\delta(I - R_x)$:

$$\text{STARR}_\delta = \begin{cases} \dfrac{EMR}{CVaR_\delta(I - R_x)}, & EMR > 0 \\ 0, & EMR \leq 0. \end{cases}$$

The higher values of STARR_δ $(\delta \to 1)$ are desirable.

We report values of $CVaR_\delta(I - R_x)$ and $STARR_\delta$ for $\delta = 0.97$ and 0.95 in the empirical analysis.

Sharpe ratio: Sharpe ratio is the average return (earned over the risk-free rate) per unit of volatility as measured by the standard deviation, and is defined as:

$$\text{Sharpe ratio} = \begin{cases} \dfrac{E(R_x) - R_f}{\sigma(R_x)}, & E(R_x) > R_f \\ 0, & E(R_x) \geq R_f, \end{cases}$$

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where R_f and $\sigma(\cdot)$ denote the risk-free rate and standard deviation, respectively. The higher values of Sharpe ratio are preferable.

Information ratio (IR): It is the ratio of EMR to the standard deviation of excess returns from the market index, and is given by:

$$IR = \begin{cases} \dfrac{EMR}{\sigma(R_x - I)}, & EMR > 0 \\ 0, & EMR \leq 0. \end{cases}$$

The higher values of IR are preferable.

Turnover ratio (TR): It is the average absolute value of trades among n stocks, defined by:

$$TR = \frac{1}{N} \sum_{r=1}^{N} \sum_{i=1}^{n} |w_i^r - w_i^{r-1}|,$$

where N is the total number of rebalancing windows, w_i^r and w_i^{r-1} are weights of ith asset in rth and $(r-1)$th windows, respectively. The smaller values of TR are desirable for it implies lower transaction costs.

SSD rule: We also check whether the portfolios from three models dominate the corresponding market indices in SSD sense or not. We follow the method described by Vinod (2008) (Chapter 4) to check SSD dominance in the out-of-sample period.

A portfolio R_x dominates a portfolio R_y in the SSD sense if

$$\epsilon_{R_x,R_y} := \int_{\psi^*}^{\psi} F_{R_xR_y}(\Psi)d\Psi \leq 0, \ \forall \ \psi \in [\psi^*, \psi^U],$$

where $F_{R_xR_y} = F_{R_x} - F_{R_y}$, is the difference of two distributions. The integral is calculated using modified trapezoidal rule to accommodate unequal interval widths.

3.3. Methodology

We follow a rolling window scheme of 12 weeks on historical data sets. The in-sample periods (training) and out-of-sample periods (testing) comprise of, respectively, 52 and 12 weeks. By sliding in-sample period by 12 weeks in each window, we get a total of 19 windows for data set 1 (all indices have the same period of study), 109 for Dow Jones, 55 for FTSE 100, 189 for FF 49, 45 for S&P 500, 45 for NASDAQ 100, 23 for S&P BSE Sensex, 31 for CNX Nifty, and 32 for MSCI world index.

Weekly returns are calculated using $r_{ij} = \log \frac{P_{ij}}{P_{ij-1}}, \ i = 1, \ldots, n, \ j = 1, \ldots, T$, where P_{ij} and P_{ij-1} are the closing prices of ith stock on jth and $(j-1)$th weeks, respectively.

We often faced diversification issue in portfolios from ($P3$), especially for indices having less than 50 stocks. Diversification is the practice of spreading investments to avoid exposure to unsystematic risk in a handful of stocks.

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Table 1
Total number of rolling windows in the in-sample periods in which allocation to a single stock in LSSDP, DSSDP, and TSSDP falls in the intervals $I_1 = [70\%,\ 80\%)$, $I_2 = [80\%,\ 90\%)$, $I_3 = [90\%,\ 99\%)$, $I_4 = [99\%,\ 100\%]$

		I_1	I_2	I_3	I_4	Max allocation (%)
Track1	LSSDP	1	2	0	0	84.65
(19)	DSSDP	0	0	0	0	53.88
	TSSDP	0	0	0	0	62.21
Dow Jones	LSSDP	7	5	1	2	100
(109)	DSSDP	1	1	1	1	100
	TSSDP	1	1	0	0	80.44
FF 49	LSSDP	20	15	9	33	100
(189)	DSSDP	7	6	4	4	100
	TSSDP	22	8	6	7	100
BSE Sensex	LSSDP	1	2	1	2	100
(23)	DSSDP	1	1	1	0	95.19
	TSSDP	0	0	0	0	42.61
CNX Nifty	LSSDP	1	0	0	2	100
(31)	DSSDP	0	0	0	0	60.66
	TSSDP	0	0	0	0	52.45

The analysis is reported on five indices, Track 1 (Hang Seng), Dow Jones, FF 49, BSE Sensex, and CNX Nifty comprising 50 or fewer stocks. The total number of rolling windows is quoted in parentheses in the first column alongside each index.

To highlight the concern on diversification by portfolios from three models, Table 1 reports the total number of windows when an allocated weight to a single stock in an optimal portfolio falls in the intervals [70%, 80%), [80%, 90%), [90%, 99%), and [99%, 100%]. The analysis is reported on five indices—Track 1 (Hang Seng), Dow Jones, FF 49, BSE Sensex, and CNX Nifty—comprising 50 or fewer stocks.

Table 1 indicates that LSSDP from ($P3$) frequently allocate more than 70% weight to a single asset than the other two models. A serious problem of diversification is faced in FF 49 index where, out of a total 189 in-sample windows, LSSDP allocates almost 100% weight to a single stock on 33 occasions, and 70% or more weight to a single stock on 77 occasions. In contrast, DSSDP and TSSDP perform better on diversification front.

Table 2 summarizes the out-of-sample performance statistics of LSSDP and DSSDP.

We apply one tailed t-test (described in Appendix B) on null hypothesis $H_0 : \mu_s \leq 0$ against the alternative hypothesis $H_a : \mu_s > 0$ to test if the out-of-sample mean return from s (LSSDP, DSSDP, or TSSDP) is statistically significant.

We also examine if the out-of-sample variance from DSSDP (strategy s_1) is significantly smaller than that from LSSDP (strategy s_2), using one-tailed F-test (reported in Appendix B) with null hypothesis $H_0 : \sigma^2_{s_1} \geq \sigma^2_{s_2}$ against the alternative hypothesis $H_a : \sigma^2_{s_1} < \sigma^2_{s_2}$.

Except for index Track 5, LSSDP and DSSDP produce significantly positive mean returns. Furthermore, DSSDP yields significantly lower variance compared to LSSDP on all data sets indicating lower volatility in the proposed DSSD model.

Table 3 presents the out-of-sample values of performance measures, viz. EMR, DD, $CVaR_\delta(I - R_x)$, Sortino ratio, STARR ratio (STARRδ), TR, Sharpe ratio, and IR, for the three portfolios. We

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Table 2
Performance statistics of the out-of-sample returns from DSSDP and LSSDP on the historical data sets of 16 global indices, where the values are quoted by mean = mean $\times 10^{-3}$, variance = variance $\times 10^{-3}$, and median = median $\times 10^{-3}$

Index	Model	Mean	Variance	Skewness	Kurtosis	Median	Min	Max
Track 1	DSSDP	7.154***	1.052***	−0.0573	0.515	9.44	−0.091	0.11
	LSSDP	10.205***	1.626	0.008	−0.1524	10.224	−0.111	0.116
Track 2	DSSDP	7.704***	0.533***	0.330414	0.794861	6.714	−0.06	0.088
	LSSDP	10.31***	1.157	0.69	1.733	7.878	−0.077	0.155
Track 3	DSSDP	4.139***	0.371***	0.09	0.268	4.99	−0.047	0.07
	LSSDP	4.041***	0.524	0.28	−0.031	4.31	−0.06	0.08
Track 4	DSSDP	4.428***	0.441***	0.125	0.202	3.8	−0.054	0.07
	LSSDP	4.084**	0.822	−0.0335	0.7315	3.24	−0.113	0.081
Track 5	DSSDP	0.55	0.794***	0.13	0.703	−0.12	−0.092	0.088
	LSSDP	−0.63	1.138	0.4834	1.095	−0.27	−0.082	0.118
Track 6	DSSDP	6.766***	1.61***	−0.037	0.8	7.931	−0.133	0.14
	LSSDP	11.462***	3.511	0.0916	0.75	7.241	−0.177	0.2055
Track 7	DSSDP	9.111***	3.09***	−0.2	1.6	11.43	−0.192	0.246
	LSSDP	7.882*	8.046	0.363	2.74	6.9	−0.288	0.42
Track 8	DSSDP	8.664***	2.756***	−0.104	1.102	13.025	−0.146	0.231
	LSSDP	7.769*	7.973	0.443	3.99	11.701	−0.319	0.463
Dow Jones	DSSDP	2.783***	0.592***	−0.287	2.238	3.538	−0.125	0.114
	LSSDP	3.766***	0.964	−0.36	2.55	3.764	−0.18	0.13
FTSE 100	DSSDP	4.444***	0.733***	−0.262	3.12	5.528	−0.154	0.113
	LSSDP	5.436***	1.364	0.125	3.39	5.82	−0.166	0.206
FF 49	DSSDP	5.482***	0.625***	−0.326	11.2	6.867	−0.2	0.26
	LSSDP	6.015***	0.919	−0.083	5.7013	6.417	−0.165	0.216
SP 500	DSSDP	2.834**	1.035***	−0.089	6.44	4.209	−0.164	0.198
	LSSDP	3.187*	2.13	−0.19	5.403	4.113	−0.246	0.286
NADAQ 100	DSSDP	4.075***	1.296***	−0.19	7.48	5.214	−0.222	0.234
	LSSDP	5.911***	2.295	−0.346	4.526	5.78	−0.294	0.231
BSE Sensex	DSSDP	1.264**	0.105***	−0.187	0.21	1.4	−0.034	0.027
	LSSDP	1.205**	0.145	−0.3	0.53	1.511	−0.043	0.028
CNX Nifty	DSSDP	1.831***	0.114***	−0.29	0.57	1.77	−0.04	0.029
	LSSDP	2.141***	0.187	−0.018	0.502	2.025	−0.05	0.044
MSCI	DSSDP	1.045**	0.161***	−0.598	3.11	1.515	−0.067	0.048
	LSSDP	1.421*	0.424	−0.212	2.482	2.443	−0.1	0.086

The ***, **, and *, are used to mark the significant values in the statistical tests for mean and variance at the significance levels $0 < \alpha \leq 0.01$, $0.01 < \alpha \leq 0.05$, and $0.05 < \alpha \leq 0.1$, respectively.

also calculate the values of $\epsilon_{s,I}$ for each portfolio from the three models with respect to the respective market index I.

To examine whether the out-of-sample EMR by s (LSSDP, DSSDP, and TSSDP) is significantly positive, we apply a one-tailed t-test (see Appendix B) on null hypothesis $H_0: \mu_s - \mu_I = 0$ against

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Table 3
Out-of-sample performance of optimal portfolios DSSDP, LSSDP, and TSSDP. The best values amongst the three models are highlighted in bold

Index	Model	EMR	Sortino	DD	$CVaR_{0.95}$	$CVaR_{0.97}$	$STARR_{0.95}$	$STARR_{0.97}$	TR	Sharpe	IR	$\epsilon_{s,I}$
Track 1	DSSDP	0.00278	0.203396	**0.0137**	**0.0023**	**0.0029**	1.2324	0.9465	**0.696**	0.22059	0.12634	**−0.01890**
	LSSDP	**0.0058****	**0.3266**	0.01787	0.0046	0.00607	**1.2726**	**0.962**	0.8142	**0.25309**	**0.18404**	**−0.01451**
	TSSDP	−0.0013	0	0.01835	0.0031	0.0045	0	0	0.716	0.10497	0	0.00075
Track 2	DSSDP	0.00416**	0.3533	0.0118	0.0022	0.003	1.89032	1.4098	**0.894**	0.33378	0.19723	**−0.00826**
	LSSDP	**0.0068*****	**0.414**	0.01634	0.0043	0.00544	1.5664	1.2437	1.0138	0.30306	**0.21490**	**−0.00184**
	TSSDP	0.0035**	0.344	**0.0102**	**0.0013**	**0.00154**	**2.784**	**2.2602**	0.959	**0.37589**	0.20253	**−0.01208**
Track 3	DSSDP	**0.0018**	**0.2213**	0.00792	0.0007	0.00085	**2.5173**	**2.0738**	**1.05634**	0.21495	**0.13753**	**−0.00136**
	LSSDP	0.0017	0.147	0.0113	0.0012	0.00137	1.37431	1.2087	1.2071	0.17645	0.09229	0.00627
	TSSDP	0.0013	0.1772	**0.0074**	**0.0005**	**0.0007**	2.4321	1.995	1.1122	**0.21923**	0.11765	**−0.00477**
Track 4	DSSDP	**0.0009**	**0.0964**	0.00901	0.0008	0.001	**1.0783**	**0.8942**	**1.123**	**0.21089**	**0.06418**	0.00503
	LSSDP	0.0005	0.033	0.016	0.0025	0.0034	0.2081	0.153	1.225	0.14245	0.02251	0.06543
	TSSDP	−0.0006	0	**0.0075**	**0.0005**	**0.0006**	0	0	1.1963	0.18457	0	0.00288
Track 5	DSSDP	**−0.0005**	0	0.0142	0.002	0.0024	0	0	1.2737	**0.01953**	0	0.01212
	LSSDP	−0.0017	0	0.0179	0.0031	0.0041	0	0	1.275	0	0	0.03320
	TSSDP	−0.00083	0	**0.0104**	**0.0009**	**0.0011**	0	0	**1.1561**	0.01113	0	**−0.01056**
Track 6	DSSDP	0.00554**	0.31812	0.0174	0.00379	0.00516	**1.4618**	**1.072**	1.2956	0.16864	0.20262	**−0.01122**
	LSSDP	**0.0102*****	**0.376**	0.0272	0.01116	0.0145	0.9168	0.7058	**1.2643**	**0.19343**	**0.21326**	0.02475
	TSSDP	0.0022	0.1595	**0.0138**	**0.0021**	**0.0029**	1.03072	0.7655	1.3012	0.12658	0.10412	**−0.00817**
Track 7	DSSDP	**0.0081****	0.375	0.0217	0.0062	0.0081	1.3162	1.0053	1.2891	0.16391	0.22032	**−0.04937**
	LSSDP	0.0069	0.1504	0.046	0.0267	0.0366	0.2594	0.189	**1.281**	0.08787	0.09451	0.32514
	TSSDP	0.007**	**0.4992**	**0.0141**	**0.0031**	**0.0042**	**2.287**	**1.689**	1.3135	**0.21078**	**0.27445**	**−0.04285**
Track 8	DSSDP	**0.0075****	**0.3206**	0.0233	0.0069	0.0092	1.0846	0.8157	**1.329**	0.16504	**0.19269**	**−0.01749**
	LSSDP	0.0066	0.1338	0.0492	0.0306	0.0429	0.215	0.1535	1.3458	0.08701	0.08461	0.45262
	TSSDP	0.005**	0.3063	**0.0164**	**0.0037**	**0.0048**	**1.3502**	**1.0408**	1.4017	**0.16950**	0.17357	**−0.01829**
Dow Jones	DSSDP	0.00116	0.0967	0.01196	0.00143	0.002	**0.80654**	**0.5783**	0.9899	0.11440	0.06835	**−0.00952**
	LSSDP	**0.0021****	**0.1285**	0.016645	0.00302	0.00406	0.7089	0.528	1.0678	**0.12131**	**0.08824**	0.21366
	TSSDP	0.00004	0.0034	**0.0107**	**0.0011**	**0.0015**	0.0325	0.024	**0.906**	0.07402	0.00245	0.01187
FTSE 100	DSSDP	0.0035***	**0.2878**	0.0123	0.00216	0.00287	1.633	1.2308	1.0668	**0.16415**	**0.17679**	**−0.12891**
	LSSDP	**0.0045*****	0.2395	0.0189	0.0052	0.0071	0.8698	0.6335	1.147	0.14717	0.15096	**−0.05613**
	TSSDP	0.0025**	0.254	**0.0097**	**0.0013**	**0.0017**	**1.9434**	**1.456**	**1.0056**	0.15707	0.15690	**−0.13756**
FF 49	DSSDP	0.0011*	**0.1198**	0.0094	0.0011	0.0016	**1.0191**	**0.7188**	1.0277	0.21929	**0.08066**	**−0.40376**
	LSSDP	**0.0017****	0.119	0.014	0.00245	0.0035	0.6804	0.4727	1.088	0.19843	0.07659	0.22729
	TSSDP	0.0002	0.0207	**0.0093**	**0.001**	**0.0014**	0.197	0.1413	**0.9207**	**0.21933**	0.01446	**−0.53819**
S&P 500	DSSDP	0.00154	**0.0998**	0.0154	0.0029	0.00422	**0.526**	0.3641	1.369	0.08810	**0.06689**	**−0.01060**
	LSSDP	**0.002**	0.075	0.0252	0.00714	0.0095	0.265	0.1985	1.402	0.06906	0.05199	0.37994
	TSSDP	0.0008	0.0667	**0.012**	**0.0016**	**0.0021**	0.4984	**0.383**	**1.2774**	**0.09076**	0.04593	**−0.04800**
NASDAQ 100	DSSDP	0.0018	0.1136	0.01582	0.0027	0.0039	**0.6675**	**0.4583**	**1.1387**	0.11319	0.08072	**−0.03758**
	LSSDP	**0.0036***	**0.1503**	0.02417	0.0074	0.0098	0.4937	0.3712	1.2444	**0.12340**	**0.09915**	0.41777
	TSSDP	0.0005	0.0352	**0.0141**	**0.0018**	**0.0024**	0.2752	0.208	1.1535	0.10204	0.02589	**−0.03265**
BSE Sensex	DSSDP	**0.0003**	**0.0449**	0.00658	0.00038	0.00047	**0.7848**	**0.62283**	**0.9418**	**0.12351**	**0.03182**	0.00067
	LSSDP	0.00024	0.02964	0.00799	0.00057	0.00071	0.41607	0.3342	1.0221	0.10011	0.02087	0.00313
	TSSDP	−0.0002	0	**0.0048**	**0.0002**	**0.0002**	0	0	0.978	0.09402	0	**−0.00036**
CNX Nifty	DSSDP	0.0012*	**0.2214**	0.0052	0.0003	0.00038	**3.8212**	**3.0391**	1.03772	**0.17181**	**0.13794**	**−0.00302**
	LSSDP	**0.0015****	0.20005	0.00731	0.00061	0.0008	2.4041	1.9417	1.1608	0.15639	0.12561	0.00165
	TSSDP	0.00064	0.144	**0.0044**	**0.0002**	**0.0003**	3.1148	2.4486	**0.9556**	0.14491	0.09793	**−0.00425**
MSCI	DSSDP	**0.0008**	**0.1048**	0.0072	0.00054	0.0007	1.3969	1.1146	**1.322**	0.08244	**0.07151**	0.00636
	LSSDP	0.0011	0.0981	0.0116	0.0017	0.0023	0.6683	0.5037	1.386	0.06902	0.06485	0.05573
	TSSDP	0.0005	0.0903	**0.0052**	**0.0003**	**0.0004**	**1.6158**	**1.2767**	1.3225	**0.10095**	0.06084	**−0.02688**

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the alternative hypothesis $H_a : \mu_s - \mu_I > 0$. The significance levels are those considered in Table 2, and the significant values are expressed using "star."

Further, we also apply the SSD test proposed in Scaillet and Topaloglou (2010) to test the statistical significance of SSD dominance between two strategies. The null hypothesis H_0 for the test is set to a strategy s that dominates the corresponding market index I while the alternative hypothesis is that s does not dominate I in SSD sense. We set the levels of confidence as 0.95, 0.97, and 0.99 to reject the alternative hypothesis.

We observe the following points from Table 3.

(1) *EMR:* DSSDP and LSSDP achieve positive EMR on all indices except for Track 5 while TSSDP has negative EMR for four indices. All three portfolios yield negative EMR for Track 5 (Nikkei 225) however, DSSDP results in least negative among them. Also, DSSDP have highest EMR for seven indices and never achieves the lowest value of EMR among three portfolios.

(2) *Risk measures:* In terms of risks, TSSDP outperforms the other two portfolios on DD (except for Track 1) and $CVaR_\delta(I - R_x)$ (except for Track 1 and Track 4), whereas portfolios LSSDP suffer the most on these three risk parameters for all 16 indices (except on DD for Track 1).

(3) *Performance ratios:*
   - (3a) DSSDP exhibits highest Sortino and IR ratios on a total of nine indices, while LSSDP and TSSDP for five indices and one index, respectively.
   - (3b) DSSDP reports highest STARR ratio for nine indices (at $\delta = 0.95$ and 0.97), while LSSDP perform worst by getting maximum value only for Track 1.
   - (3c) DSSDP, LSSDP, and TSSDP record the best values of TR on a total of eight, two, and six indices, respectively, and the best Sharpe ratio on five, four, and seven number of indices, respectively.

(4) The values of $\epsilon_{s,I}$ are negative for a total of 3, 12, and 13 indices when $s =$ LSSDP, DSSDP, and TSSDP, respectively. This implies that LSSDP fails to dominate the respective market index for a larger number of data sets under consideration.

   Further, the alternative hypothesis of SSD test (Scaillet and Topaloglou, 2010) on DSSDP and TSSDP at confidence levels 0.99, 0.97, 0.95 is rejected on all 16 data sets. On the other hand, we can reject the alternative hypothesis of SSD on LSSDP for a total number of 8, 9, and 11 indices at 0.95, 0.97 and 0.99 confidence levels, respectively. We also apply the same SSD test with an alternative hypothesis that DSSDP does not dominate LSSDP. This hypothesis stands rejected at 0.99, 0.97, and 0.95 confidence levels on all 16 data sets.

To summarize, the proposed DSSD strategy demonstrates better performance when compared to those achieved by the other two strategies. Although portfolios TSSDP exhibit better risk hedging capacity on risk of extreme losses yet it performs poorly on EMR. On the other hand, LSSDP offers an advantage on EMR but suffers worst risk of extreme losses (measured by CVaR) and uncertainty (measured by DD). In contrast, the proposed model appears balanced out on EMR and risk. It never suffers the worst result on any performance measure while yielding best EMR on 7 indices of 16 considered.

To confirm and strengthen our findings on three models, we perform empirical experiments by simulating data using two historical data sets. The results are detailed in the following section.

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4. Empirical analysis using simulations

The process of data simulation is decomposed into two steps: first we identify the marginal distributions of returns of each stock using historical data, and second apply the regular vine copulas to find the joint distribution among stock returns.

We use ARMA-GJR-GARCH (AutoRegressive Moving Average–Glosten–Jagannathan–Runkle–Generalized Autoregressive Conditional Heteroscedastic) model for modeling the marginal distributions of each stock. The reason for choosing ARMA-GJR-GARCH model comes from the stylized features present in the return series. The presence of such features is confirmed by applying several tests in the analysis presented below.

4.1. ARMA-GJR-GARCH model for estimating marginal distributions

The GARCH model (Engle and Bollerslev, 1986), being symmetric, is not applicable to capture leverage effects in financial data series. The leverage effect is described as a tendency of volatility to increase faster following a large price drop than a price rise of the same magnitude (Mohammadi and Su, 2010). Advanced GARCH models such as EGARCH, GJR-GARCH, TGARCH, FIGARCH, IGARCH (Tsay, 2005), to name a few, have been developed in the literature to model various stylized features in financial data series.

We use ARMA(p, q)-GJR-GARCH(1,1) model to find marginal distribution of each stock considered in this section. The (1,1) ordering in GARCH is known to explain volatility clustering adequately in stock returns (see Brooks, 2014, p. 430).

Using the Kolmogorov–Smirnov (KS) test or the Lagrange multiplier (LM) test, we confirm the goodness of fit of ARMA(p, q)-GJR-GARCH(1,1) model in estimating the marginal distribution of return series.

The ARMA(p, q)-GJR-GARCH(1,1) model is expressed as follows:

$$r_j = \mu + \sum_{\wp=1}^{p} \phi_{\wp} r_{j-\wp} + \sum_{\ell=1}^{q} \theta_{\ell} \epsilon_{j-\ell} + \epsilon_j, \tag{7}$$

$$\epsilon_j = \sigma_j z_j, \tag{8}$$

$$\sigma_j^2 = \omega + \beta_1 \sigma_{j-1}^2 + \alpha_1 \epsilon_{j-1}^2 + \gamma_1 I_{j-1} \epsilon_{j-1}^2, \tag{9}$$

where $\omega > 0,\ \alpha_1,\ \beta_1 \geq 0,$ and $\alpha_1 + \gamma_1 \geq 0$. The indicator function $I_{j-1} = 1$ if $\epsilon_{j-1} < 0$ and $I_{j-1} = 0$ if $\epsilon_{j-1} \geq 0$.

In Equation (7), the dependent variable return r_j is explained by a constant term μ, p lags of its own past movements $r_{j-1}, \ldots, r_{j-p}$, q lags of the residual terms $\epsilon_{j-1}, \ldots, \epsilon_{j-q}$, and the residual term ϵ_j. In Equation (8), the residuals are defined as the product of conditional volatility σ_j and standardized residual z_j, which is a sequence of independently and identically distributed random variables each with zero mean and unit variance.

Equation (9) defines the evolution of conditional volatility in which symmetric effect is captured by the parameter α_1, while the parameter β_1 measures conditional variance. If β_1 takes a large value

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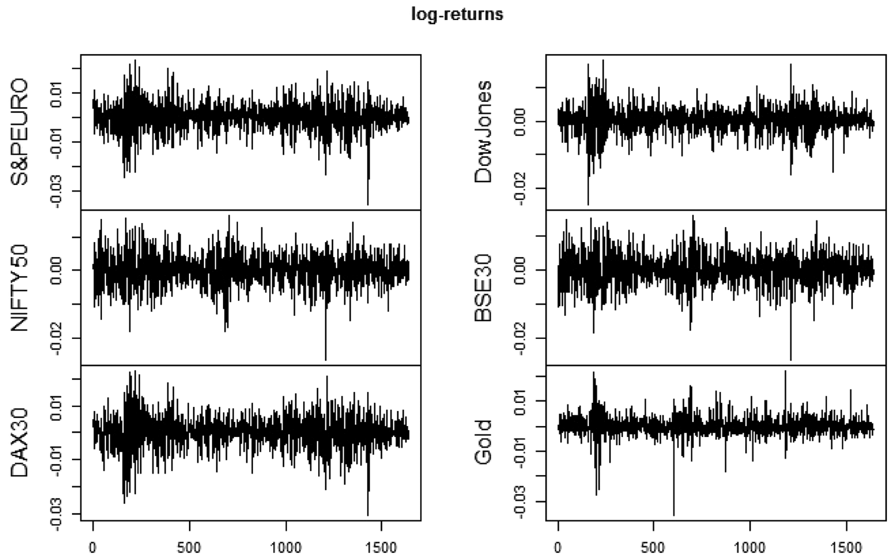

Fig. 1. In-sample period daily return of each index in data set A.

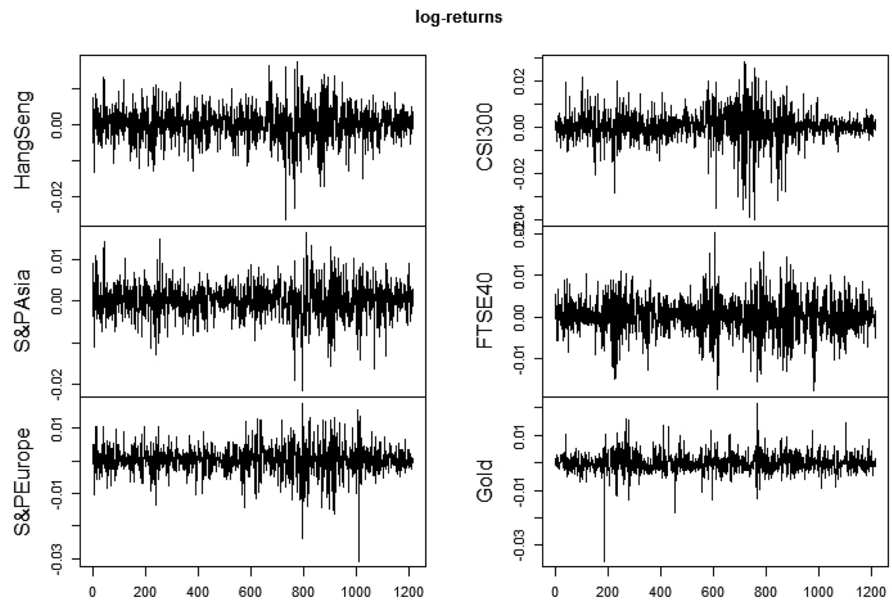

Fig. 2. In-sample period daily return of indices in data set B.

then it consumes a longer time for the conditional variance to die out under a market crisis and vice-versa. The parameter γ_1 represents the magnitude of the leverage effect. If $\gamma_1 < 0$ then negative news generate more pronounced effect than positive news while the opposite happens if $\gamma_1 > 0$ (Dennis et al., 2006). The sign of indicator term I_{j-1} reflects the influence of positive and negative news on the volatility of returns of a stock.

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4.2. Data and empirical findings

We consider historical data of two data sets A and B. data set A comprises the daily closing price of six indices—S&P EURO, CNX Nifty, DAX 30, Dow Jones, BSE Sensex, and Gold ETF. The data set B comprises the daily closing price of another six indices—Hang Seng, S&P Asia, S&P Europe, CSI 300, FTSE 40, and Gold ETF. The historical data are acquired from Thomson Reuters data stream on EIKON software for period January 2011 to April 2017 for data set A and July 2012–June 2017 for data set B.

The in-sample period is considered from January 1, 2010 to December 31, 2015 (1305 observations) and the out-of-sample validation period is taken from 1 January 2016 to 13 April 2017 (335 observations) for data set A. The in-sample period is July 2012 to July 2016 (985 observations) and the rest 231 observations as out-of-sample period for data set B. Figures 1 and 2 depict daily returns from indices in data set A and data set B, respectively, in their in-sample periods. Further, the benchmark index is considered to be the naive $1/n$ portfolio.

The real in-sample period data are used to learn the dependence structure in the returns of indices. A step-wise procedure for simulating data using the in-sample historical data are presented in Algorithm 1.

Algorithm 1. A step-wise procedure for modeling marginals and joint dependence via copula

1: **procedure** METHODOLOGY
2: Transform the prices of the stocks into log returns using

$$r_{ij} = \log \frac{P_{ij}}{P_{ij-1}}; i = 1, \ldots, n; \quad j = 1, \ldots, T,$$

where P_{ij} and P_{ij-1} are the prices of ith stock on jth and $(j-1)$th day, respectively, and T is the total number of time points.
3: Use ARMA-GJR-GARCH process and estimate the marginals F_i, $i = 1, \ldots, n$, for each return series. Filter the return series by fitting an ARMA(p, q)-GJR-GARCH(1,1) model, to get the residuals[3]. Standardize the residuals to obtain the standardized residuals.
4: Using the marginal distribution F_i of each return series, transform the standardized residuals to the uniform random variable U_i, $i = 1, \ldots, n$, in the interval [0,1] so as to fit the copula. Use the KS test to ensure that the distribution transformed residuals is uniform $U[0, 1]$.
5: Using the regular vine copula model (see Appendix A for detail), we find the best fit copula for the transformed marginals from Step 3 and estimate the copula parameters.
6: The estimated copula is then used to simulate N random vectors from the estimated joint probability distribution. We form a matrix $U = [U_1, \ldots, U_n]_{N \times n}$, where each $U_i \in \mathbb{R}^N$ is a vector of simulated marginal and n is the number of indices in the data set.
7: Transform the simulated random samples to the original scales of the log returns using the inverse quantile function of the marginals, that is, $F_i^{-1}(U_i)$.
8: Reintroduce the autocorrelations and heteroscedasticity observed in the original return series using the mean and variance equations of the fitted ARMA(p, q)-GJR-GARCH(1,1) model to get a matrix $[R_1 \quad R_2 \ \cdots \ R_n]_{N \times n}$; $R_i \in \mathbb{R}^N$ of the simulated returns for each associated marginal distribution.
9: Use these simulated returns data of n stocks to solve three optimization models.

[3]For each return series, the most appropriate values of p and q in the model are identified on the basis of AIC values.

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Table 4
Summary statistics for indices in data set A in the in-sample period along with results of hypothesis tests

	S&P EURO	CNX Nifty	DAX 30	Dow Jones	BSE Sensex	Gold ETF
Mean	0.8	0.9	1.5	1.4	0.8	0.9
Min	−0.0240	−0.0265	−0.0260	−0.0248	−0.0266	−0.0357
Max	0.0231	0.0162	0.0226	0.0180	0.0161	0.0218
SD	0.0055	0.0045	0.0057	0.0039	0.0044	0.00397
Skewness	−0.1811	−0.1525	−0.2198	−0.4337	−0.1639	−0.4786
Kurtosis	5.1563	4.861	5.1886	7.224	4.8623	13.883
ADF test	1	1	1	1	1	1
JB test	1	1	1	1	1	1
ARCH test	1	1	1	1	1	1

Also, read values of mean = mean $\times 10^{-4}$. The value 1 indicates rejection of the null hypothesis for a given statistical test, otherwise 0 is used.

We provide the procedural details of Algorithm 1 on data set A. Similar procedure is followed for data set B.

In Table 4, along with the descriptive statistics for each index in data set A, we also display results of three statistical tests, Augmented Dickey-Fuller (ADF), Jarque-Bera (JB), and ARCH-LM test. The negative values of skewness and high values of kurtosis implying nonnormal fat-tailed left-skewed return distribution of each index which is further confirmed by JB test. The ADF test results suggest rejecting the null hypothesis of a unit root in all six returns series at 3% critical level. The ARCH-LM test indicates the presence of conditional heteroscedasticity, thus it is appropriate to use GARCH model to get the marginal distributions of return series.

We use GJR-GARCH(1,1) to model conditional volatility and $\mathrm{ARMA}(p, q)$ to account for serial correlation present in each return series. We estimate the parameters p and q in $\mathrm{ARMA}(p, q)$ modeling by minimizing Bayes information criterion (BIC). We report the results for the ordered pair $(p, q) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ in Table 5. The $(p, q) = (1, 1)$ is observed to be the best pair.

The other parameters for the best fitted model (Equations (7) and (8)) are listed in Table 6 for ARMA(1,1)-GJR-GARCH(1,1). The asymmetric parameter ψ is observed to be less than one for five indices indicating left skewness in their returns (i.e., a small negative price change is encountered more often than a large positive price change), whereas it is more than one for Gold ETF depicting safety in Gold investment. Also, β_1 is greater than 0.8 for each data indicating greater time dependency in their conditional volatility. The shape parameter values indicate a low degree of freedom (less than 15) therefore, implying appropriateness of skewed Student's t-distribution in the residual term.

Furthermore, we apply the Ljung-Box test to check serial dependence in residuals followed by KS test for $U[0, 1]$ transformed variables. Since none of the marginals reject the null hypothesis of the two tests at 1% level, we can safely conclude goodness of ARMA(1,1)-GJR-GARCH(1,1) model in estimating the marginals accurately.

Figure 3 provides a scatter plot of residuals of returns in indices suggesting correlation among them.

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Table 5
The BIC values for different ARMA models on in-sample returns data of six indices from the real data set A. The best values of BIC are highlighted in bold

(p, q)	S&P EURO	CNX Nifty	DAX 30	Dow Jones	BSE Sensex	Gold ETF
(1,1)	**−7.77693**	**−8.05546**	**−7.73033**	**−8.58077**	**−8.08871**	**−8.52211**
(1,2)	−7.77155	−8.05015	−7.72466	−8.57584	−8.08333	−8.51992
(2,1)	−7.77152	−8.05020	−7.72466	−8.57605	−8.08337	−8.51999
(2,2)	−7.77104	−8.04533	−7.72210	−8.57971	−8.07846	−8.51694

Table 6
Estimated values of the parameters for best fit ARMA(1,1)-GJR-GARCH(1,1) model in Equations (7) and (8), where read the values of $\mu = \mu \times 10^{-4}$, $\omega = \omega \times 10^{-4}$ and $\alpha_1 = \alpha_1 \times 10^{-4}$

	S&P EURO	NIFTY 50	DAX 30	Dow Jones	BSE Sensex 30	Gold ETF
μ	0.99	0.38	1.5	0.894	0.364	0.397
ϕ_1	0.6633	0.1318	−0.4561	−0.0035	0.1925	0.8596
θ_1	−0.6925	−0.0522	0.4661	−0.0422	−0.1218	−0.8362
ω	0.5	0.5	0.5	0.6	0.5	0.5
α_1	0.5	12.4	0.3	0.0	19.34	974.1
β_1	0.9042	0.9226	0.9021	0.8162	0.9224	0.8969
γ_1	0.1598	0.1143	0.1769	0.3034	0.1145	−0.0614
ψ	0.9096	0.9801	0.9098647	0.8547	0.9835	1.0782
Shape	7.3481	6.5532	6.5008	8.6766	6.6912	4.4321

Table 7
The Pearson correlation matrix of six residuals series obtained from ARMA(1,1)-GJR-GARCH(1,1) model applied on real data of the in-sample period of each index comprising data set A

	S&P EURO	CNX Nifty	DAX 30	Dow Jones	BSE Sensex	Gold ETF
S&P EURO	1	0.3358	0.9429	0.6101	0.3461	0.0131
CNX Nifty		1	0.3254	0.2232	0.9921	0.0052
DAX 30			1	0.5929	0.3358	0.0126
Dow Jones				1	0.2277	−0.0001
BSE Sensex					1	0.0053
Gold ETF						1

Table 7 presents the Pearson's correlation matrix for residuals confirming correlated indices in real data set A. Observe that the correlation between Gold ETF and other indices is comparatively low pointing safety component in Gold investment. The same is observable in Fig. 3.

Using the uniform marginals, we estimate the best fit regular vine (R-Vine) copula and the parameters. In order to build on the R-Vine tree, we use the fact that copulas specified in the first tree have the strongest dependence and greatest influence (Dissmann, 2010) on dependency. We use Kendall's tau to determine a tree that maximizes the cumulative pairwise dependency. After finalizing the R-Vine structure, we choose the best copulas by applying the minimum BIC criterion that rewards the goodness of fit of a model and penalizes the increase in the number of parameters.

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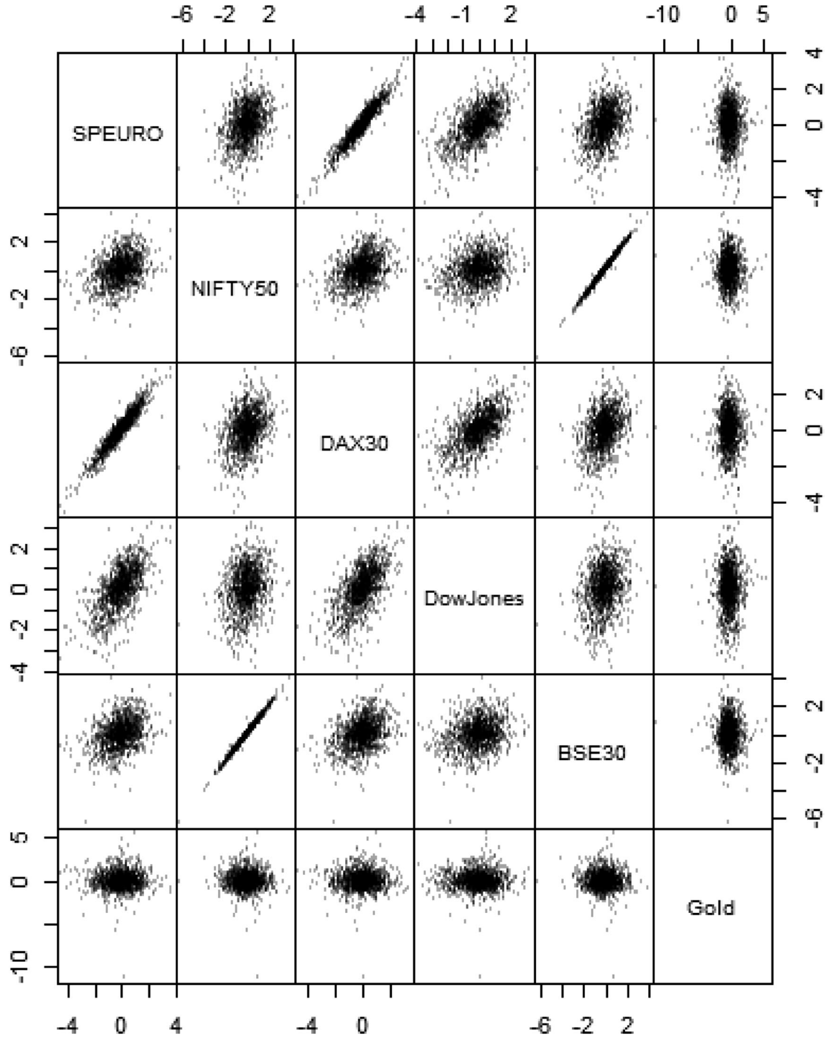

Fig. 3. The scatter plot of residuals series of indices comprising data set A.

We refer to Appendix A for a brief explanation of the copula theory. To cover a bandwidth of possible dependence structures, we consider all available copulas in VineCopula library of R software.

Subsequently, using the copula structure and parameters obtained, we generate 100 paths for returns series of six indices, each path having 500 scenarios. In other words, we simulated 100 matrices each of size 500×6 to serve as input return data matrices corresponding to data set A in models ($P3$), ($P5$), and ($P7$). The benchmark index (naive $1/n$ portfolio) returns are obtained using the simulated returns. We obtain optimal portfolios, 100 for each model, which are then used to compute out-of-sample portfolios returns using the real data corresponding to the out-of-sample period. The results are averaged across hundred portfolios to obtain the out-of-sample average return from each model.

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Table 8
Out-of-sample performance analysis of three models where the models are solved using simulated data for two data sets A and B. The best values amongst the three models are highlighted in bold

	Data Set A			Data Set B		
	DSSDP	LSSDP	TSSDP	DSSDP	LSSDP	TSSDP
Mean	**0.1136**	0.1135	*0.103*	**0.224**	0.219	*0.203*
SD	0.236	*0.242*	**0.222**	0.152	*0.159*	**0.146**
Skewness	−0.534	−0.562	−0.43	−0.574	−0.757	−0.474
Kurtosis	2.675	2.857	2.087	1.862	3.175	1.336
Min	−1.19	*−1.26*	**−1.04**	−0.72	*−0.85*	**−0.65**
Max	**0.76**	0.75	*0.71*	0.45	**0.49**	*0.44*
EMR	**−0.13**	−0.132	*−0.24*	**0.069**	0.015	*−0.142*
DD	0.59	**0.58**	*0.74*	0.35	*0.47*	**0.28**
$CVaR_{95}$	**0.181**	0.182	*0.224*	0.102	*0.14*	**0.083**
$CVaR_{97}$	0.2041	**0.2040**	*0.251*	0.116	*0.158*	**0.092**
Sharpe ratio	**0.048**	0.047	0.046	**0.148**	0.138	0.139
Information ratio	0	0	0	0.014	**0.017**	0
Sortino ratio	0	0	0	**1.96**	0.32	0
$STARR_{0.95}$	0	0	0	**0.68**	0.101	0
$STARR_{0.97}$	0	0	0	**0.6**	0.095	0
$\epsilon_{s,I}$	**−0.00039**	**−0.00035**	**−0.00043**	0.000038	.000077	.000033

Read values of mean = mean × 10^{-3}; SD = SD × 10^{-2}; min = min × 10^{-2}; max = max × 10^{-2}; EMR = EMR × 10^{-4}; Sortino ratio = Sortino ratio × 10^{-2}; DD = DD × 10^{-3}; $CVaR_{0.95} = CVaR_{0.95} \times 10^{-2}$; $CVaR_{0.97} = CVaR_{0.97} \times 10^{-2}$; $STARR_{0.95} = STARR_{0.95} \times 10^{-2}$; $STARR_{0.97} = STARR_{0.97} \times 10^{-2}$.

Table 8 summarizes the performance comparison of portfolios from three models in the out-of-sample period. The TSSDP from ($P5$) offers lower risk especially for data set B. The DSSDP outperforms in return-risk ratios (STARR ratio, Sharpe ratio, and Sortino ratio) endorsing its superiority over LSSDP and TSSDP in constructing a balanced trade-off in risk-return profile for investors. Furthermore, the SSD test by Scaillet and Topaloglou (2010) produces favorable results for all three portfolios with respect to the market indices at 0.95, 0.97, and 0.99 confidence levels.

Moreover, when we applied the same test with the alternative hypothesis that DSSDP does not dominate LSSDP, we reject the alternative hypothesis at 0.95, 0.97, and 0.99 confidence levels on both the data sets.

5. Conclusions

Several equivalent conditions for SSD criterion exist in the literature leading to different optimization models for constructing optimal portfolios that can dominate the benchmark portfolio. The LSSD and TSSD strategies are being two such examples. However, an investor may rather be interested in knowing the utility of improvement in a portfolio return from its mean value and not merely the return outcomes of the portfolio. To cater to this need, we proposed a DSSD ranking by replacing portfolio return R_x and benchmark portfolio I by their respective deviation measures $R_x - E(R_x)$ and $I - E(I)$ in constraints of model corresponding to the LSSD ranking. The

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performance of portfolios following the DSSD strategy is accessed on historical data of 16 global indices as well as on two simulated data sets; each having six indices. The data sets are simulated after fitting the ARMA-GJR-GARCH and R-Vine copula models on the two real data sets. Portfolios from the proposed DSSD strategy are shown to outperform those from LSSD regarding risks and TSSD regarding EMRs over the benchmark index. Overall, the DSSD strategy to construct the portfolio is found to be more suitable in application to enhanced indexing for a risk-averse investor by achieving best values of risk-reward ratios for a substantial number of data sets considered in this study. Besides, portfolios from DSSD strategy are more diversified in cases when portfolios from the LSSD strategy allocate 70% or more weight to a single stock.

Acknowledgments

The authors are thankful to the esteemed reviewers for their insightful and constructive commendations that led to an improved version of the current article. The authors acknowledge the positive support from the editor-in-chief and associate editor. The first author is also thankful to the Council of Scientific and Industrial Research (CSIR), India, for the financial grant (no. 09/086(1215)/2015-EMR-I).

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Appendix A

Copula theory

In October 2010, the Basel Committee on Banking Supervision published a document emphasizing use of the copula approach in risk management and computing minimum economic capital in the working of banks. Since copula models work directly with the percentile measures (conceptually visualized by uniform marginals of copula), they are well suited for aggregating financial risks. However, the real challenge of using copula in practice lies in identifying the most appropriate copula to model the joint distribution of a multivariate data set. While there is a wide variety of copula families for modeling the bivariate case, the choices in higher dimensions are very limited.

Joe (1996) and Nelsen (2006) defined copula as a function, which joins or couples the finite number of marginal distributions into a multivariate distribution. Copulas are multivariate distribution functions whose one-dimensional marginals are uniform. In terms of probability theory, an n-dimensional copula is defined as follows:

$$C(u_1, \ldots, u_n) = P(U_1 \leq u_1, \ldots, U_n \leq u_n),$$

where $U_1, \ldots, U_n$ are $U[0, 1]$ distributed random variables and $u_i \in [0, 1]$, $i = 1, \ldots, n$.

Theorem A.1. ***Sklar's thorem*** *(Sklar, 1959): Let $F_1, \ldots, F_n$ be the marginals of an n-dimensional distribution function F. Then there exists an n-dimensional copula C such that*

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \quad \boldsymbol{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n. \tag{A1}$$

Conversely, if C is an n-dimensional copula and $F_1, \ldots, F_n$ are one-dimensional distribution functions, then the function F defined in (A1) *is an n-dimensional distribution function with marginals $F_1, \ldots, F_n$. More precisely, we have*

$$C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where $F_i^{-1}(\cdot)$ denotes the quasi inverse of the marginal F_i.

Regular vine copula

While the class of bivariate copula families is fairly rich, scarcity of the same is felt when it comes to higher dimensions. Although multivariate Student's t-copula and multivariate Gaussian copula have been available, they are not exhaustive enough to accurately model the variety of dependence structures that can exist in financial data. Moreover, these two are symmetric and fail to capture the asymmetric tail dependence structure.

Joe et al. (2010) proposed a flexible graphical model to obtain multivariate copula using a cascade of bivariate copulas. Such copulas are called vines and the process of obtaining them is called pair-copula construction (PCC). The PCC allows enormous flexibility in modeling dependence structure including tail dependence and asymmetries to build more parsimonious models (Aas et al., 2009).

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To look at the construction of vine copulas, first we derive a relation between probability density function (pdf) and copulas using the Sklar's theorem.

Definition A.2. *Let f be the multivariate pdf associated with the distribution function F, and $f_1, \dots, f_n$ be the univariate pdfs associated with the marginals $F_1, \dots, F_n$. The copula density function $c(u_1, \dots, u_n)$ associated with the copula $C(u_1, \dots, u_n)$ is defined as*

$$c(u_1, \dots, u_n) = \frac{\partial^n C(u_1, \dots, u_n)}{\partial u_1 \dots \partial u_n} = \frac{f(x_1, \dots, x_n)}{\prod_{i=1}^n f_i(x_i)}.$$

To obtain the density f of the n-dimensional distribution F, we use the following relationship:

$$f(x_1, \dots, x_n) = c(F_1(x_1), \dots, F_n(x_n)) \prod_{i=1}^{n} f_i(x_i). \tag{A2}$$

We also know that the joint density function is described by

$$f(x_1, \dots, x_n) = \prod_{i=2}^{n} f(x_i \mid x_1, \dots, x_{i-1}) \times f_1(x_1), \tag{A3}$$

where $f(\cdot \mid \cdot)$ represents the conditional density function and $f_1(x_1)$ denotes the pdf of random variable X_1. In general, $f_i(x_i)$ denotes the pdf of X_i. Using (A2), we can represent $f(x_i \mid x_j)$ as follows:

$$f(x_i \mid x_j) = c_{ij}(F_i(x_i), F_j(x_j)) \times f_i(x_i),$$

where c_{ij} denotes the bivariate copula density and $F(\cdot \mid \cdot)$ denotes the conditional distribution function. Let $c_{i,j|i_1,\dots,i_m}(F(x_i \mid x_{i_1}, \dots, x_{i_m}), F(x_j \mid x_{i_1}, \dots, x_{i_m}))$ be denoted by $c_{i,j|i_1,\dots,i_m}$ for distinct indices $i,\ j,\ i_1, \dots, i_m$, satisfying $i < j$ and $i_1 < \dots < i_m$. Hence, we can write $f(x_i \mid x_1, \dots, x_{i-1})$ using recursion as follows:

$$\begin{aligned} f(x_i \mid x_1, \dots, x_{i-1}) &= c_{1,i|2,\dots,i-1} \times f(x_i \mid x_2, \dots, x_{i-1}) \\ &= \prod_{j=1}^{i-2} c_{j,i|j+1,\dots,i-1} \times c_{i-1,i} \times f_i(x_i). \end{aligned} \tag{A4}$$

Using (A4) in (A3), we have

$$f(x_1, \dots, x_n) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{i,i+j|i+1,\dots,i+j-1} \times \prod_{s=1}^{n} f_s(x_s). \tag{A5}$$

Since the decomposition of joint density function in terms of conditional density function (A3) is not unique, there exist many such iterative PCCs in (A5). Bedford and Cooke (2001, 2002) introduced the graphical model called regular vine (R-Vine) to organize the possible decompositions of joint density. It is noteworthy that the applications of copulas are not limited to the continuous

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distributions, see, for example, Channouf and L'Ecuyer (2012) in which normal copula is used to model discrete multivariate distribution.

Appendix B: Statistical test

We briefly explain the statistical tests used for mean and variance in this paper.

1. The t-test is used to test if the mean μ of the population is ($\geq$ or $\leq$) to some constant μ_0 when variance of the population is unknown. To test the following hypothesis

   $$H_0 : \mu \leq \mu_0 \quad \text{against} \quad H_a : \mu > \mu_0,$$

   the test statistics is given by $t = \frac{\bar{x}-\mu_0}{\frac{S}{\sqrt{N}}}$, where S and $\bar{x}$ denote the sample variance and mean, respectively, and N is the sample size. The critical region for the test is $t > t_{\alpha,N-1}$, where α is the significance level, and $t_{\alpha,N-1}$ denotes the critical value of the t-distribution with $N-1$ degrees of freedom. The p-value can be obtained as $P(t_{N-1} > t)$ and we reject the null hypothesis if p-value is less than α.
2. The F-test is used to test if the variances of two populations are equal. In order to test the following hypothesis

   $$H_0 : \sigma_1^2 \geq \sigma_2^2 \quad \text{against} \quad H_a : \sigma_1^2 < \sigma_2^2.$$

   The test statistics is given by $F = \frac{S_1^2}{S_2^2}$, where S_1^2 and S_2^2 denote the sample variances of two populations, respectively. The critical region for the test is $F < F_{1-\alpha,N_1-1,N_2-1}$, where α is the significance level, N_1 and N_2 are the sample sizes of two populations, $F_{1-\alpha,N_1-1,N_2-1}$ denotes the critical value of F-distribution with $N_1 - 1$ and $N_2 - 1$ degrees of freedom. The p-value can be obtained as $P(F_{N_1-1,N_2-1} < F)$ and we reject the null hypothesis if p-value is less than α.
3. The SSD test is used to test if one portfolio dominates other in the SSD sense. We opted for block bootstrap method, with nonoverlapping blocks, to simulate the p-values for this test. For a detail description of the test, one can refer to Scaillet and Topaloglou (2010, pp. 170–172).

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