Hida Theory tutorials - NCMW 2023

December 2023

1 Problem set 2: p-ordinarity, Eisenstein series and supersingular elliptic curves

1. Let $M \cong \mathbb{Z}_p^d$ denote a free $\mathbb{Z}_p$-module. Let $U : M \to M$ denote a $\mathbb{Z}_p$-linear endomorphism. We obtain an induced linear transformation on the $\overline{\mathbb{Q}}_p$-vector space $M \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$.

$$\widetilde{U}_M : M \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \to M \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p.$$

   (a) Prove that the eigenvalues of the linear transformation $\widetilde{U}_M$ lie in $\overline{\mathbb{Z}}_p$.

   (b) Let $S_{unit,M}$ denote the set of eigenvalues of $\widetilde{U}_M$ whose p-adic valuations equal zero. Let $S_{non-unit,M}$ denote the remaining eigenvalues of $\widetilde{U}_M$.

   (c) Prove that
   $$e := \lim_n U^{n!}$$
   defines an idempotent operator on M (that is as endomorphisms over M, we have $e^2 = e$).

   (d) Show that $e \cdot M$ is a free $\mathbb{Z}_p$-module.

   (e)
   - If x is an element of $\overline{\mathbb{Q}}_p$ with positive p-adic valuation, show that $\lim_n x^{n!} = 0$.
   - If x is a root of unity in $\overline{\mathbb{Q}}_p$, show that $\lim_n x^{n!} = 1$.
   - If x is a 1-unit in $\overline{\mathbb{Q}}_p$, show that $\lim_n x^{n!} = 1$.

   (f) Prove that
   $$S_{unit,e\cdot M} = S_{unit,M}, \qquad S_{non-unit,eM} = \emptyset.$$

2. Prove that, for each even integer $2k > 4$, the normalized Eisenstein series

$$E_{2k} :- \frac{(2k-1)!}{2 \times (2\pi i)^{2k}} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0\}} \frac{1}{(m+nz)^{2k}}$$

is a modular form in $M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and its q-expansion equals

$$\frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left(\sum_{\substack{d \geq 1 \\ d|n}} d^{k-1} \right) q^n.$$

3. (a) Prove that q-expansion at ∞ of the normalized $E^*_{p-1} := \dfrac{E_{p-1}}{\frac{\zeta(2-p)}{2}}$ belongs to $\mathbb{Z}_p[[q]]$.

   (b) Prove that all the coefficients of q^n in the q-expansion of E^*_{p-1} belong to $p\mathbb{Z}_p$, for each $n \geq 1$.
   (Hint: von Staudt–Clausen theorem)

4. Let E denote an elliptic curve over a field k with characteristic $p > 0$. Prove taht the following statements are equivalent:

   (a) $E(k^{\mathrm{sep}})[p] = \{0\}$.

   (b) $E(k^{\mathrm{sep}})[p] \neq \mathbb{Z}/p\mathbb{Z}$.

   (c) The map induced by Verschiebung on global sections of the sheaf of differentials

   $$V : H^0(E, \Omega_E) \to H^0(E, \Omega_E)$$

   is the zero map.