

1 Introduction

1.1 Dedekind–Zeta function

Let K be a totally real number field. Let \mathfrak{a} be an ideal inside the ring of integers \mathcal{O}_K . Consider the Dedekind-Zeta function:

$$\zeta_K(s) = \sum_{\mathfrak{b}} \frac{1}{\text{Norm}(\mathfrak{b})^s},$$

along with the associated partial Dedekind–Zeta functions:

$$\zeta_K(\mathfrak{a}, s) = \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{1}{\text{Norm}(\mathfrak{b})^s},$$

The first sum is taken with respect to all the integral ideals \mathfrak{b} in \mathcal{O}_K . The second sum is taken with respect to all the integral ideals \mathfrak{b} that are in the same class as \mathfrak{a} in the narrow ray class group of K .

1.2 Motivating questions

Here’s an overarching and sequential list of motivating questions:

- **(analytic continuation)** Do the zeta functions $\zeta_K(s)$ and $\zeta_K(\mathfrak{a}, s)$ have an analytic (or meromorphic) continuation to the entire complex plane?
- **(rationality)** Having established analytic continuation, can we conclude that the zeta values *at non-positive integers* are rational?
- **(integrality)** Having established rationality, for what primes p are these zeta-values p -integral?
- **(Kummer congruences)** Having established p -integrality, how do we construct p -adic L -functions associated to these zeta values?

1.3 Rationality

The main motivating question for our study group for this semester will be the question of *rationality* of these zeta values. This is known as the *Siegel–Klingen* rationality theorem. There are three celebrated proofs of this result, attributed to:

1. Siegel–Klingen (*constant terms of Eisenstein series*),
2. Shintani, and
3. Sczech (*cohomological*).

Our goal will be to read the methods of **Shintani** and **Sczech**. As a reminder to our eventual goal of studying p -adic L -functions, it is worth noting that in this set up, the constructions of the p -adic L -functions by

1. Deligne–Ribet [4],
2. Cassou–Noguès [1, 2], and
3. Charolmois–Dasgupta [3].

rely on the methods of Siegel–Klingen, Shintani and Sczech respectively.

Siegel–Klingen \rightsquigarrow Deligne–Ribet

Shintani \rightsquigarrow Cassou–Noguès

Sczech \rightsquigarrow Charolmois–Dasguptas

References for Shintani’s method

1. Shintani’s article [8]
2. Hida’s book [5]

References for Sczech’s method

1. Sczech’s inventiones article [6].
2. Sczech’s Kyushu article [7].

2 Plan for the study group

The study group will meet on **Tuesdays, 10:30 am.**

| Date | Speaker | Section | |
|--------|---------|--|----|
| 18 Jan | Bharath | Introduction | Su |
| 25 Jan | Bharath | Theorem 1 in [8] \implies Siegel–Klingen rationality | Di |
| 1 Feb | Shaunak | Proposition 1 and Corollary to Proposition 1 in [8] | Pa |
| 8 Feb | Radhika | Lemma 2 and Corollary to Lemma 2 in [8] | Pa |
| 15 Feb | Mihir | Lemma 3 and Proposition 4 in [8] | Pa |
| 22 Feb | Mahesh | Proposition 4 and Theorem 1 in [8] | Pa |

References

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- [2] P. Cassou-Noguès. Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p -adiques. *Invent. Math.*, 51(1):29–59, 1979.
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- [4] P. Deligne and K. A. Ribet. Values of abelian L -functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.
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- [6] R. Sczech. Eisenstein group cocycles for GL_n and values of L -functions. *Invent. Math.*, 113(3):581–616, 1993.
- [7] R. Sczech. Eisenstein cocycles for arithmetic groups and values of zeta functions. Number 958, pages 46–48. 1996.
- [8] T. Shintani. On evaluation of zeta functions of totally real algebraic number fields at non-positive integers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 23(2):393–417, 1976.