# EVALUATION OF DEDEKIND SUMS, EISENSTEIN COCYCLES, AND SPECIAL VALUES OF $L$-FUNCTIONS

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**Abstract**
*We define higher-dimensional Dedekind sums that generalize the classical Dedekind-Rademacher sums as well as Zagier's sums, and we show how to compute them effectively using a generalization of the continued-fraction algorithm.*

*We present two applications. First, we show how to express special values of partial zeta functions associated to totally real number fields in terms of these sums via the Eisenstein cocycle introduced by R. Sczech. Hence we obtain a polynomial time algorithm for computing these special values. Second, we show how to use our techniques to compute certain special values of the Witten zeta function, and we compute some explicit examples.*

## 1. Introduction

*1.1*
Let $L$ be a lattice of rank $\ell \geq 1$, and let $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. For $r \geq \ell$, we denote by $\sigma = (\sigma_1, \ldots, \sigma_r) \in (L^*)^r$ a tuple of nonzero linear forms of maximal rank. For a tuple $e = (e_1, \ldots, e_r)$ of positive integers, let $|e| = \sum e_i$. Finally, let $v \in L^*_{\mathbb{R}} := L^* \otimes \mathbb{R}$. We associate to the data $(L, \sigma, e, v)$ the *Dedekind sum*

$$D = D(L, \sigma, e, v) := (2\pi \mathrm{i})^{-|e|} \sum_{x \in L}{}' \frac{\mathbf{e}(\langle x, v\rangle)}{\langle x, \sigma_1\rangle^{e_1} \cdots \langle x, \sigma_r\rangle^{e_r}}, \tag{1}$$

where $\langle\ ,\ \rangle \colon L \times L^* \to \mathbb{Z}$ is the pairing, $\mathbf{e}(t)$ is $\exp(2\pi \mathrm{i} t)$, and the prime next to the summation means that the terms for which the denominator vanishes are to be omitted. This series converges absolutely if all $e_j > 1$, but may only conditionally converge if $e_j = 1$ for some $j$. In this latter case, choose a finite product $Q$ of real-

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valued linear forms on $L_{\mathbb{R}}$ that does not vanish on $L_{\mathbb{Q}} \smallsetminus \{0\}$, and define

$$D(L, \sigma, e, v)\big|_Q := \lim_{t\to\infty}\Bigl((2\pi \mathrm{i})^{-|e|} \sideset{}{'}\sum_{\substack{x\in L\\ |Q(x)|<t}} \frac{\mathbf{e}(\langle x, v\rangle)}{\langle x, \sigma_1\rangle^{e_1}\cdots\langle x, \sigma_r\rangle^{e_r}}\Bigr). \tag{2}$$

This limit always exists (see [17, Theorem 2]), and the value depends on $Q$ in a rather simple way. Nevertheless, to keep notation to a minimum, we assume for now that the series (1) converges absolutely.

*1.2*

The arithmetic nature of $D$ is known: If $v \in L_{\mathbb{Q}}$, then $D(L, \sigma, e, v) \in \mathbb{Q}$ (see [17], [21]). Moreover, special cases of the sums $D$ are well known. For example, if $L = \mathbb{Z}$ and $\sigma$ is the identity, then

$$D(L, \sigma, e, v) = -\frac{1}{e!}\mathscr{B}_e(v), \tag{3}$$

where $\mathscr{B}_e(v)$ is the periodic function (with period lattice $\mathbb{Z}$) that coincides with the classical Bernoulli polynomial $B_e(v)$ on the interval $0 < v < 1$.

More generally, let $L = \mathbb{Z}^\ell$, and identify $L^*$ with $L$ via the canonical basis. Identify $\sigma$ with the $\ell \times r$ integral matrix with columns $\sigma_i$. Then if $r = \ell$, we show in Proposition 2.7 that

$$D(L, \sigma, e, v) = \frac{(-1)^\ell}{e!|\det\sigma|}\sum_{z\in L/\sigma L}\mathscr{B}_{e_1}(u_1)\cdots\mathscr{B}_{e_\ell}(u_\ell), \tag{4}$$

where

$$u = \sigma^{-1}(z+v) \in L_{\mathbb{R}} \qquad \text{and} \qquad e! := \prod_{j=1}^{\ell} e_j!. \tag{5}$$

The finite sum on the right of (4) is a classical Dedekind-Rademacher sum if $\ell = 2$ and $e = (1, 1)$; for $\ell > 2$ it generalizes the higher-dimensional Dedekind sums studied by D. Zagier [20] (see Example 3.6 for a precise comparison).

There is one other case when the series (1) can easily be converted to a finite sum. Call two linear forms $\sigma_1, \sigma_2$ *proportional* if $\sigma_1\big|_L = \alpha\sigma_2\big|_L$ for some nonzero $\alpha$. We call a Dedekind sum $D$ *diagonal* if among all the $\sigma_i$ there are exactly $\ell$ distinct linear forms, up to proportionality. Then it is clear that, modulo trivial modifications, (4) can be used to explicitly sum any diagonal $D$.

*1.3*

The Dedekind sum D arises naturally from the definition of the Eisenstein cocycle introduced in [17]. This is a degree $(n-1)$ group cocycle on $\mathrm{GL}(n, Z)$ which can be used to express special values of the partial zeta functions attached to totally real number fields as a finite linear combination of the sums $D(L, \sigma, e, v)$ (see §§6,7). Unfortunately, these Dedekind sums are almost never diagonal, so the finite sum expression (4) does not apply. Moreover, even if they are diagonal, the number of terms in (4) is usually prohibitively large. The question is therefore whether there is any efficient method for calculating the sums $D$. In this paper we answer this question affirmatively. Our main result is an effective algorithm for computing any given Dedekind sum $D(L, \sigma, e, v)$. To state the result precisely, we require one more notion. As in the second example in §1.2, identify $L$ with $\mathbb{Z}^\ell$ and $\sigma$ with an $\ell \times r$ integral matrix.

*Definition 1.4*
The *index* of $D$, denoted $\|D\|$, is defined to be the largest among the absolute values of the maximal minors of $\sigma$. We say $D$ is *unimodular* if $\|D\| = 1$.

Our main result can now be stated as follows.

THEOREM 1.5
*Every Dedekind sum $D(L, \sigma, e, v)$ can be expressed as a finite rational linear combination of unimodular diagonal sums. If $|e|$ and $\ell$ are fixed, then this expression can be computed in time polynomial in $\log \|D\|$. Moreover, the number of terms in this expression is bounded by a polynomial in $\log \|D\|$.*

*1.6*
The main tool in the proof of Theorem 1.5 is a certain reciprocity law satisfied by the Dedekind sums (Theorem 3.3). This law coincides with the classical Dedekind-Rademacher law if $\ell = r = 2$, $e = (1, 1)$, but for higher rank $L$ it is new (Examples 3.5 and 3.6). By combining this law with the modular symbol algorithm of A. Ash and L. Rudolph [1], we prove Theorem 1.5.

For applications we review the connection between the Eisenstein cocycle and our sums and show how to use our result to compute the special values of partial zeta functions. We also discuss the connection to *Witten's zeta function* and show how to compute its special values at even positive integers.

*1.7*
To the best of our knowledge, the story of Dedekind sums started with a paper of L. Kronecker [15] written in 1885, seven years before the widely quoted paper of R. Dedekind [7] on Riemann's work. In that paper, Kronecker obtains a Dedekind sum by calculating the logarithm of the quadratic residue symbol using the Gauss lemma.

Higher-dimensional Dedekind sums have been studied more recently by L. Carlitz [6], Zagier [20], F. Hirzebruch and Zagier [12], M. Beck and S. Robins [4], and S. Hu and D. Solomon [13]. The point of view of our paper (Dedekind sums as infinite series over a lattice) is also pursued in Zagier [21], M. Brion and M. Vergne [5], and A. Szenes [18], who expressed these sums as iterated residues.

## 2. The modular symbol algorithm and the Q-limit formula

*2.1*

Let $\sigma_1, \ldots, \sigma_n \in \mathbb{Z}^n$ be nonzero points, and let $\delta = |\det(\sigma_1, \ldots, \sigma_n)|$. Let $w \in \mathbb{Z}^n$ be another nonzero point, and define

$$\delta_i(w) := |\det(\sigma_1, \ldots, \hat{\sigma}_i, \ldots, \sigma_n, w)|, \quad i = 1, \ldots, n.$$

Here $\hat{\sigma}_i$ means to delete $\sigma_i$. The following basic result plays a key role.

PROPOSITION 2.2 (see [1], [2])
*If $\delta > 1$, then there exists $w \in \mathbb{Z}^n \smallsetminus \{0\}$ such that*

$$0 \le \delta_i(w) < \delta^{(n-1)/n}, \quad i = 1, \ldots, n, \tag{6}$$

*and at least one $\delta_i(w) \neq 0$. Moreover, for fixed $n$, the point $w$ can be constructed in polynomial time.*

*Proof (Sketch)*
Here we show only that $w$ exists. Let $P$ be the open parallelotope

$$P := \Big\{ \sum \lambda_i \sigma_i \;\Big|\; |\lambda_i| < \delta^{-1/n} \Big\}.$$

Then $P$ is an $n$-dimensional centrally symmetric convex body with volume $2^n$. By Minkowski's theorem (cf. [8, §IV.2.6]), $P \cap \mathbb{Z}^n$ contains a nonzero point. This is the desired point $w$. □

*Remark 2.3*
Ash and Rudolph [1] show that $w$ satisfies $0 \le \delta_i(w) < \delta$ and show how to construct $w$ using the Euclidean algorithm. The stronger estimate (6) and the statement about polynomial time are due to I. Barvinok [2, Lemma 5.2]. In practice, to efficiently construct $w$ such that $\delta_i(w)$ is small, one may use $LLL$-reduction (see [11, §3]).

*2.4*

Recall that the series $D(L, \sigma, e, v)$ may only converge conditionally if some $e_j = 1$ and that in this case we have defined the sum through the *Q-limit process*: We choose a set of $m \geq 1$ linear forms $Q_i \in L_{\mathbb{R}}^*$ that do not vanish on $L_{\mathbb{Q}} \setminus \{0\}$ and put

$$Q(y) := \prod_{i=1}^{m} \langle y, Q_i \rangle.$$

Then we order the terms of $D$ via formula (2). The purpose of this section is to explain how the value of $D$ depends on the choice of $Q$ for diagonal Dedekind sums. Hence we assume until further notice that $r = \ell$.

Let $L = \mathbb{Z}^\ell$ with its standard basis, and identify $L$ and $L^*$ via the standard dual basis. Then for each form $Q_i$ and any point $y \in \mathbb{R}^\ell$, we may write

$$\langle y, Q_i \rangle = \sum_{j=1}^{n} Q_{ij} y_j,$$

and we may identify $Q$ with an $(m \times \ell)$-matrix with real *rows* $Q_i$. Similarly, we may identify $\sigma$ with an $(\ell \times \ell)$-matrix with integral *columns* $\sigma_i$.

*Definition 2.5*
Let $e = (e_1, \ldots, e_\ell)$ be a vector of positive integers, and let $v \in \mathbb{R}^\ell$. We define a function $\mathbb{B}_e(v, Q)$ as follows. Let

$$J = \left\{ j \,\middle|\, e_j = 1 \text{ and } v_j \in \mathbb{Z} \right\}.$$

Then if $\#J$ is even, define

$$\mathbb{B}_e(v, Q) = \frac{1}{m} \sum_{i=1}^{m} \left( \prod_{j \in J} \frac{\operatorname{sgn} Q_{ij}}{2} \right) \prod_{j \notin J} \mathscr{B}_{e_j}(v_j); \tag{7}$$

otherwise, let $\mathbb{B}_e(v, Q) = 0$. In particular, if $J = \varnothing$, then

$$\mathbb{B}_e(v, Q) = \prod_{j=1}^{n} \mathscr{B}_{e_j}(v_j).$$

We first consider the case when $\sigma = \mathrm{Id}_\ell$, the $\ell \times \ell$ identity matrix.

THEOREM 2.6 ([17, Theorem 2])
*We have*

$$D(\mathbb{Z}^\ell, \mathrm{Id}_\ell, e, v)\big|_Q = \frac{(-1)^\ell}{e!} \mathbb{B}_e(v, Q). \tag{8}$$

Next, we consider a general diagonal Dedekind sum.

PROPOSITION 2.7
*Suppose that $r=\ell$. Then*

$$D(\mathbb{Z}^\ell,\sigma,e,v)\big|_Q=\frac{(-1)^\ell}{e!|\det\sigma|}\sum_{z\in\mathbb{Z}^\ell/\sigma\mathbb{Z}^\ell}\mathbb{B}_e\big(\sigma^{-1}(z+v),Q\sigma^{-t}\big). \tag{9}$$

*Proof*
By definition, we have

$$D(\mathbb{Z}^\ell,\sigma,e,v)\big|_Q=(2\pi\mathrm{i})^{-|e|}\sum_{x\in\mathbb{Z}^\ell}{}' \frac{\mathbf{e}(\langle x,v\rangle)}{\langle x,\sigma_1\rangle^{e_1}\cdots\langle x,\sigma_\ell\rangle^{e_\ell}}\bigg|_Q. \tag{10}$$

Letting $Q'=Q\sigma^{-t}$ and $v'=\sigma^{-1}v$, the right-hand side of (10) becomes

$$(2\pi\mathrm{i})^{-|e|}\sum_{y\in\sigma^t\mathbb{Z}^\ell}{}' \frac{\mathbf{e}(\langle y,v'\rangle)}{y_1^{e_1}\cdots y_\ell^{e_\ell}}\bigg|_{Q'}. \tag{11}$$

Inserting the character relations

$$\sum_{z\in\mathbb{Z}^\ell/\sigma\mathbb{Z}^\ell}\mathbf{e}(\langle y,\sigma^{-1}z\rangle)=\begin{cases}0, & y\in\mathbb{Z}^\ell\smallsetminus\sigma^t\mathbb{Z}^\ell,\\ \#(\mathbb{Z}^\ell/\sigma\mathbb{Z}^\ell), & y\in\sigma^t\mathbb{Z}^\ell,\end{cases} \tag{12}$$

we obtain

$$D(\mathbb{Z}^\ell,\sigma,e,v)\big|_Q=\frac{(2\pi\mathrm{i})^{-|e|}}{|\det\sigma|}\sum_{z\in\mathbb{Z}^\ell/\sigma\mathbb{Z}^\ell}\sum_{y\in\mathbb{Z}^\ell}{}' \frac{\mathbf{e}\big(\langle y,\sigma^{-1}(z+v)\rangle\big)}{y_1^{e_1}\cdots y_\ell^{e_\ell}}\bigg|_{Q'} \tag{13}$$

$$=\frac{1}{|\det\sigma|}\sum_{z\in\mathbb{Z}^\ell/\sigma\mathbb{Z}^\ell}D(\mathbb{Z}^\ell,\mathrm{Id}_\ell,e,u)\big|_{Q'}, \tag{14}$$

where $u=\sigma^{-1}(z+v)$. The proposition follows now from the $Q$-limit formula (8). □

## 3. The reciprocity law

*3.1*
To state and prove the reciprocity law the sums $D(L,\sigma,e,v)$ satisfy, it is convenient to introduce a different setup, in which we replace the abstract lattice $L$ with a lattice embedded in a larger Euclidean lattice.

*Definition 3.2*
Let $n\geq\ell$ be an integer, and suppose that $L$ is a rank $\ell$ saturated sublattice of $\mathbb{Z}^n$; that is, suppose that any $\mathbb{Z}$-basis of $L$ can be extended to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Let $\sigma$ be an

integral $(n \times n)$-matrix with primitive columns, whose columns $\sigma_i$ we identify with linear forms on $\mathbb{Z}^n$. Let $v \in \mathbb{R}^n$, which we identify with a real-valued linear form on $\mathbb{Z}^n$. Then we define

$$S(L, \sigma, v) := (2\pi \mathrm{i})^{-n} \sum_{x \in L}{}' \frac{\mathbf{e}(\langle x, v \rangle)}{\langle x, \sigma_1 \rangle \cdots \langle x, \sigma_n \rangle}$$

if $\det \sigma \neq 0$, and we put $S(L, \sigma, v) := 0$ otherwise. In the case of conditional convergence, we define $S(L, \sigma, v)\big|_Q$ as in (2). The sum $S(L, \sigma, v)$ is said to have *rank* $\ell$.

The sums $S$ are equivalent to the sums $D$ in the sense that any sum $D$ can be expressed as a sum $S$, and vice versa. For example, given a sum $D(L, \sigma, e, v)$, where $L$ has rank $\ell$, let $n = |e| = e_1 + \cdots + e_r$. Assume (without loss of generality) that none of the $\sigma_i$ are proportional on $L$. Then choose an isomorphism from $L$ to the span $Z^\ell$ of the first $\ell$ standard basis vectors of $\mathbb{Z}^n$. One can then choose a matrix $\sigma'$ with the first $e_1$ columns inducing the linear form $\sigma_1$ on $Z^\ell$, the next $e_2$ columns inducing $\sigma_2$, and so forth. Of course, one must guarantee that $\det \sigma' \neq 0$, but this is easy. By choosing $v'$ and $Q'$ appropriately, we can easily achieve $S(Z^\ell, \sigma', v')\big|_{Q'} = D(L, \sigma, e, v)\big|_Q$. Passing from the sums $S$ to the sums $D$ is similar.

We are now ready to state the reciprocity law. For any linear form $\varphi$, let $\varphi^\perp$ be its kernel.

THEOREM 3.3
*Let $\sigma_0, \ldots, \sigma_n \in \mathbb{Z}^n$ be nonzero points. For $j = 0, \ldots, n$, let $\sigma^j$ be the matrix with columns $\sigma_0, \ldots, \hat{\sigma}_j, \ldots, \sigma_n$, and let $\delta_j = \det \sigma^j$. Fix $L \subseteq \mathbb{Z}^n$. Then for any $v \in \mathbb{R}^n$, we have*

$$\sum_{j=0}^{n} (-1)^j \delta_j S(L, \sigma^j, v)\big|_Q = \sum_{j=0}^{n} (-1)^j \delta_j S(L \cap \sigma_j^\perp, \sigma^j, v)\big|_Q. \tag{15}$$

*Proof*
We have an identity for rational functions in $\mathbb{R}(x_1, \ldots, x_n)$,

$$\sum_{j=0}^{n} (-1)^j \frac{\delta_j \langle x, \sigma_j \rangle}{\langle x, \sigma_0 \rangle \cdots \langle x, \sigma_n \rangle} = 0, \tag{16}$$

valid for any $x \in \mathbb{R}^n$ satisfying $\langle x, \sigma_j \rangle \neq 0$ for $j = 0, \ldots, n$. To see this, consider the $((n+1) \times (n+1))$-matrix

$$\begin{pmatrix} \langle x, \sigma_0 \rangle & \ldots & \langle x, \sigma_n \rangle \\ \sigma_0 & \ldots & \sigma_n \end{pmatrix}. \tag{17}$$

This matrix is singular since the first row is a linear combination of the others. Expanding by minors along the top row and dividing by $\prod\langle x,\sigma_k\rangle$ yields (16).

To pass from (16) to (15), we need to incorporate the exponential character and sum over $L$ using $Q$. There is no obstruction to doing this, although we must omit terms where *any* linear form vanishes. We obtain

$$\sum_{j=0}^{n}(-1)^j \sum \mathbf{e}(\langle x,v\rangle)\frac{\delta_j}{\prod_{k\neq j}\langle x,\sigma_k\rangle}\bigg|_Q = 0, \tag{18}$$

where the inner sum is taken over all $x\in L$ with $\langle x,\sigma_k\rangle\neq 0$ for $k=0,\ldots,n$.

Up to powers of $2\pi\mathrm{i}$, the $j$th inner sum in (18) coincides with the Dedekind sum $S(L,\sigma^j,v)$, except for the terms with $\langle x,\sigma_j\rangle=0$ and $\langle x,\sigma_k\rangle\neq 0$ for $k\neq j$. In other words, to make the $j$th sum into a Dedekind sum, we must add to it

$$\sum_{x\in L\cap\sigma_j^{\perp}}{}' (-1)^j\mathbf{e}(\langle x,v\rangle)\frac{\delta_j}{\prod_{j\neq k}\langle x,\sigma_k\rangle}\bigg|_Q. \tag{19}$$

Simultaneously adding and subtracting (19) to and from (18) yields (15). □

*Remark 3.4*
The sum $S(L\cap\sigma_j^{\perp},\sigma^j,v)\big|_Q$ on the right-hand side of (15) vanishes trivially if $\sigma_j$ is proportional on $L$ to any $\sigma_k$ with $k$ different from $j$ since in this case the product of the linear forms vanishes on $L\cap\sigma_j^{\perp}$.

*Example 3.5*
We illustrate the reciprocity law by explaining the connection with the classical sums (see [16]). Without loss of generality, we can assume that $h$ and $k$ are positive relatively prime integers. Then the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) := \sum_{0<r<k}\mathscr{B}_1\Big(\frac{r}{k}\Big)\mathscr{B}_1\Big(\frac{rh}{k}\Big) \tag{20}$$

and satisfies

$$s(k,h)+s(h,k)=\frac{1}{12}\Big(\frac{k}{h}+\frac{1}{kh}+\frac{h}{k}\Big)-\frac{1}{4}. \tag{21}$$

Let $L=\mathbb{Z}^2$, thought of as a sublattice of itself, and let $v=(0,0)^t$. Let $\sigma_0,\sigma_1,\sigma_2$ be the columns of the matrix

$$\begin{pmatrix} 1 & h & 0\\ 0 & k & 1\end{pmatrix}.$$

Applying (15) to this data, we obtain six sums, three of rank 2 and three of rank 1. Letting $S_j^{\ell}$ be the rank $\ell$ sum obtained by deleting $\sigma_j$, we see that the reciprocity law takes the form

$$\delta_0S_0^2-\delta_1S_1^2+\delta_2S_2^2=\delta_0S_0^1-\delta_1S_1^1+\delta_2S_2^1. \tag{22}$$

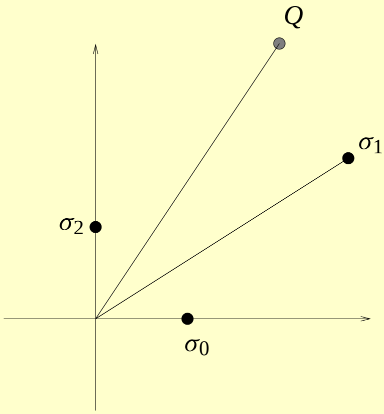


Figure 1

The three rank 1 sums converge absolutely and are easy to compute directly. For instance,

$$S_0^1 = -\frac{1}{(2\pi)^2 k} \sum_{x\in\mathbb{Z}}{}' \frac{1}{x^2} = -\frac{1}{2\pi^2 k}\,\zeta(2) = -\frac{1}{12k}.$$

In reciprocity law this value appears multiplied by the factor $\delta_0 = h$. The other two rank 1 sums may be similarly evaluated, and we obtain for the right-hand side of (22)

$$-\frac{1}{12}\Big(\frac{k}{h} + \frac{1}{kh} + \frac{h}{k}\Big). \tag{23}$$

Now consider the rank 2 sums. These sums converge only conditionally, and so we must choose some linear forms and apply the $Q$-limit formula. Applying Proposition 2.7, we see that these correction terms only arise from the zero coset in the finite sum (9). Denoting the correction term for $S_j^2$ by $C_j$ and applying (20), we find for the left-hand side of (22)

$$C_0 - C_1 + C_2 - s(h,k) - s(k,h). \tag{24}$$

Finally, we investigate the contribution $C = C_0 - C_1 + C_2$. Note that $C$ is independent of $Q$, so that it suffices to compute it for one linear form. Consider the three points $\sigma_j$, which we think of as points in $\mathbb{Z}^2 \subset \mathbb{R}^2$. Any linear form $Q$ can be identified with a point in $\mathbb{R}^2$. If we take the linear form shown in Figure 1, we obtain $C = -1/4$. (In fact, there are only three possible positions of $Q$, up to a global sign, and it is easy to check that all three give the same value of $C$, as well as of any finite product of linear forms.) Equation (21) then follows from (23) and (24).

*Example 3.6*

We explain the relationship to Zagier's higher-dimensional Dedekind sums [20]. (Similar sums were studied by Carlitz [6].) Let $k$ be a positive integer, and let $\{h_1, \ldots, h_n\}$ be a set of $n$ integers relatively prime to $k$, where $n \geq 2$ is even. Zagier introduced the sum

$$s(\{h_1, \ldots, h_n\}, k) := 2^n \sum_{r \in R} \mathscr{B}_1\Big(\frac{r_1}{k}\Big) \cdots \mathscr{B}_1\Big(\frac{r_n}{k}\Big),$$

where

$$R := \big\{ r \in \mathbb{Z}^n \;\big|\; 0 < r_1, \ldots, r_n < k,\ h_1 r_1 + \cdots + h_n r_n \equiv 0 \bmod k \big\}.$$

To express these sums in terms of our sums $D$, we choose integers $a_j \equiv h_1^{-1} h_j \bmod k$, $2 \leq j \leq n$, and let

$$\sigma = \begin{pmatrix} 1 & a_2 & \cdots & a_n \\ & k & & \\ & & \ddots & \\ & & & k \end{pmatrix}, \qquad \sigma^{-1} = \frac{1}{k} \begin{pmatrix} k & -a_2 & \cdots & -a_n \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then

$$\Lambda := \sigma^{-1}\mathbb{Z}^n = \frac{1}{k}\big\{ r \in \mathbb{Z}^n \mid h_1 r_1 + \cdots + h_n r_n \equiv 0 \bmod k \big\}.$$

Let $Q$ be a linear form with real coefficients $Q_j$ that are linearly independent over the field of rational numbers. We write $Q' = Q\sigma^t$, let $I = \{1, \ldots, n\}$, and put $\mathbf{1} = (1, \ldots, 1)$. According to Proposition 2.7, we have

$$D(\mathbb{Z}^n, \sigma, \mathbf{1}, 0)\big|_{Q'} = \frac{1}{k^{n-1}} \sum_{\substack{J \subset I \\ \#J \text{ even}}} \sum_{\substack{z \in \Lambda/\mathbb{Z}^n \\ z_j \in \mathbb{Z} \Leftrightarrow j \in J}} \Big( \prod_{j \in J} \frac{\operatorname{sgn} Q_j}{2} \Big) \prod_{j \notin J} \mathscr{B}_1(z_j).$$

For simplicity we write $h_J = \{h_j \mid j \in J\}$ for any $J \subset I$, and we define $s(\varnothing, k) = 1$. Then the above expression can be written as

$$2^n D(\mathbb{Z}^n, \sigma, \mathbf{1}, 0)\big|_{Q'} = \frac{1}{k^{n-1}} \sum_{\substack{J \subset I \\ \#J \text{ even}}} \Big( \prod_{j \in J} \operatorname{sgn} Q_j \Big) s(h_{I \setminus J}, k). \tag{25}$$

For the rest of this section, we assume that all $Q_j$ are positive. The left-hand side of (25) then depends only on $k$ and the set $h_I$, so it makes sense to write

$$D(h_I, k) := D(\mathbb{Z}^n, \sigma, \mathbf{1}, 0)\big|_{Q'}.$$

The relation (25) now translates into

$$(2k)^n D(h_I, k) = k \sum_{\substack{J \subset I \\ \#J \text{ even}}} s(h_J, k).$$

This last equation can be inverted using the Euler numbers $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$, defined by

$$\frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

If we define $D(\varnothing, k) = k$, we obtain the following relation.

THEOREM 3.7
*We have*

$$ks(h_I, k) = \sum_{\substack{j=0 \\ j \text{ even}}}^{n} (2k)^j E_{n-j} \sum_{\substack{J \subset I \\ \#J = j}} D(h_J, k).$$

Combining Theorem 3.7 with Theorem 1.5, we conclude that Zagier's sums can be calculated in polynomial time as well.* It is interesting to note that, except for the case $n = 2$, this conclusion does not seem to follow from the reciprocity law satisfied by these sums; according to [20, §3],

$$\sum_{j=0}^{n} s(\{h_0, \ldots, \hat{h}_j, \ldots, h_n\}, h_j) \tag{26}$$

is a rational function of the coefficients $h_j$, provided they are relatively prime in pairs. In comparison, the reciprocity law (Theorem 3.3) reduces a linear combination of sums of rank $n$ to a combination of sums of lower rank.

## 4. Algorithms

*4.1*
In this section we prove that any Dedekind sum is a $\mathbb{Q}$-linear combination of diagonal, unimodular sums. For this section we do not need to consider the $v$ parameter in the Dedekind sum since it is not affected by our manipulations. Hence for simplicity we abbreviate the sums by $D(L, \sigma, e)$ and $S(L, \sigma)$. Also, in the case of conditional convergence, we do not need to mention $Q$ explicitly since all operations are compatible with the $Q$-summation.

*This result had been obtained using different techniques by A. Barvinok and J. Pommersheim [3].

*4.2*
We begin by examining the correspondence between the sums $D$ and $S$ more closely to establish notation. Given a sum $D(L, \sigma, e)$, write $\ell$ for the rank of $L$ and $n$ for $|e|$. Choosing a basis, we identify $L$ with $\mathbb{Z}^\ell$ and $\sigma$ with an $\ell \times r$ integral matrix with columns $\sigma_1, \ldots, \sigma_r$. We assume without loss of generality that none of the $\sigma_i$ are proportional on $L$. Let $\pi : \mathbb{Z}^n \to \mathbb{Z}^\ell$ be the projection given by taking the first $\ell$ coordinates, and let $Z^\ell \subset \mathbb{Z}^n$ be the sublattice generated by the first $\ell$ standard basis vectors. Choose an $n \times n$ integral matrix $\tau$ with columns labelled

$$(\tau_1^1, \ldots, \tau_1^{e_1}, \tau_2^1, \ldots, \tau_2^{e_2}, \ldots, \tau_r^1, \ldots, \tau_r^{e_r}) \tag{27}$$

such that $\pi(\tau_i^j) = \sigma_i$. Then we have

$$S(Z^\ell, \tau) = D(\mathbb{Z}^\ell, \sigma, e).$$

We call the tuple $(e_1, \ldots, e_r)$ the *type* of $D$ (or $S$). Note that the Dedekind sum is diagonal if and only if $r = \ell$.

We come now to our first main result.

THEOREM 4.3
*Let $D = D(\mathbb{Z}^\ell, \sigma, e)$ be a Dedekind sum. Then we may write*

$$D = \sum_{\mu \in M} q_\mu D\big(\mathbb{Z}^\ell, \mu, e(\mu)\big) + \sum_{\nu \in N} q_\nu D\big(L_\nu, \nu, e(\nu)\big), \tag{28}$$

*where*
(1) *the sets $M$ and $N$ are finite,*
(2) $q_\mu, q_\nu \in \mathbb{Q}$,
(3) *each $D(\mathbb{Z}^\ell, \mu)$ is diagonal,*
(4) *each $D(L_\nu, \nu)$, $\nu \in N$, has rank $\ell - 1$, and*
(5) *each of the Dedekind sums on the right-hand side of (28) has index less than or equal to $\|D(\mathbb{Z}^\ell, \sigma, e)\|$.*

*Proof*
Permuting columns of $\sigma$ and $\tau$ if necessary, we may assume that $\sigma_1, \ldots, \sigma_\ell$ are the columns used to compute $\|D\|$. Since these columns must be linearly independent, there exist rational numbers $a_i$ such that

$$a_1\sigma_1 + \cdots + a_\ell\sigma_\ell = \sigma_r. \tag{29}$$

Choose points $\tau_i^j \in \mathbb{Z}^n$ as in (27) such that $\pi(\tau_i^j) = \sigma_i$ and such that $S(Z^\ell, \tau) = D(\mathbb{Z}^\ell, \sigma, e)$. We can modify the columns of $\tau$ if necessary without changing the value of $S$, so that the rational point

$$\omega := a_1\tau_1^1 + \cdots + a_\ell\tau_\ell^1$$

is *integral*.

The main step now is to apply the reciprocity law (15) using the list of $(n+1)$-vectors $\tau_1^1, \dots, \tau_r^{e_r}, \omega$, ordered as in (27). Removing from that list the column $\tau_i^j$, we denote the resulting matrix by $\tau_i^j(\omega)$. Then applying (15), we find that

$$\begin{aligned} S(Z^\ell, \tau) = S(Z^\ell \cap \omega^\perp, \tau) &+ \sum (\varepsilon_i^j \delta_i^j) S\big(Z^\ell, \tau_i^j(\omega)\big) \\ &- \sum (\varepsilon_i^j \delta_i^j) S\big(Z^\ell \cap (\tau_i^j)^\perp, \tau_i^j(\omega)\big). \end{aligned} \tag{30}$$

Here $\delta_i^j$ is the ratio $\det(\tau_i^j(\omega))/\det\tau$, and the sign $\varepsilon_i^j \in \{\pm 1\}$ is determined by the reciprocity law. Both summations are taken over pairs $(i, j)$ satisfying $1 \le i \le r$ and $1 \le j \le e_i$.

We claim that the sums in (30) actually have at most $\ell$ terms. This follows since the points

$$\tau_1^1, \dots, \tau_\ell^1, \omega$$

are dependent. Hence any sum such that $\tau_i^j(\omega)$ contains these columns is killed by $\delta_i^j$. Moreover, the sum $S(Z^\ell \cap \omega^\perp, \tau)$ is zero by definition since the last column of $\tau$ induces a linear form vanishing on $Z^\ell \cap \omega^\perp$. In view of this and the fact that $\delta_i^1 = a_i$, (30) becomes

$$S(Z^\ell, \tau) = \sum_{i=1}^{\ell} (\varepsilon_i^1 a_i) S\big(Z^\ell, \tau_i^1(\omega)\big) - \sum_{i=1}^{\ell} (\varepsilon_i^1 a_i) S\big(Z^\ell \cap (\tau_i^1)^\perp, \tau_i^1(\omega)\big). \tag{31}$$

Now consider the types of the rank $\ell$ Dedekind sums on the right-hand side of (31). If the type of $S$ is

$$(e_1, \dots, e_i, \dots, e_\ell, \dots, e_r),$$

then the type of $S(Z^\ell, \tau_i^1(\omega))$ is

$$(e_1, \dots, e_i - 1, \dots, e_\ell, \dots, e_r + 1).$$

Hence, by induction, we can write $S$ as a finite $\mathbb{Q}$-linear combination of diagonal rank $\ell$ Dedekind sums plus sums of lower rank. This and the equivalence between sums of type $S$ and $D$ proves items (1)–(4) of the statement.

To complete the proof, we must show that the indices of the Dedekind sums on the right-hand side of (31) are no larger than $\|S\|$. For the sums of rank $\ell$ this is clear, and for those of rank $\ell - 1$ it can be seen as follows. For a rank $\ell - 1$ sum to appear, it must be the case that some $e_i = 1$. Without loss of generality, assume $e_1 = 1$. The statement follows if we can show that

$$\|D(\mathbb{Z}^\ell, \sigma, e)\| \ge \|D(L', \sigma', e')\|,$$

where $L' = \mathbb{Z}^\ell \cap (\sigma_1)^\perp$, $\sigma'$ is the matrix with columns $(\sigma_2, \dots, \sigma_r)$, and $e' = (e_2, \dots, e_{r-1}, e_r + 1)$.

To see this, choose $\gamma \in \mathrm{GL}_\ell(\mathbb{Z})$ such that $\gamma\sigma_1$ is the first basis vector in $\mathbb{Z}^\ell$. Then the matrix $\gamma\sigma$ has the form

$$\gamma\sigma = \begin{pmatrix} 1 & | & | & \cdots & | \\ 0 & \gamma\sigma_2 & \gamma\sigma_3 & \cdots & \gamma\sigma_r \\ \vdots & | & | & \cdots & | \\ 0 & | & | & \cdots & | \end{pmatrix}. \tag{32}$$

Since multiplication by $\gamma$ preserves determinants, $\|D(L', \sigma', e')\|$ is the largest absolute value of the $(\ell-1) \times (\ell-1)$ determinants taken from the lower right $((\ell-1) \times (r-1))$-block of $\gamma\sigma$. Any such determinant equals the absolute value of a maximal minor of $\gamma\sigma$ containing $\gamma\sigma_1$ since this vector is the first basis vector of $\mathbb{Z}^\ell$. But our assumption that $\|D\| = |\det(\sigma_1, \dots, \sigma_\ell)| = |\det(\gamma\sigma_1, \dots, \gamma\sigma_\ell)|$ implies that the absolute values of these maximal minors are $\leq \|D(\mathbb{Z}^\ell, \sigma, e)\|$. This completes the proof. □

THEOREM 4.4

*With the notation as in Theorem 4.3, we can write*

$$D = \sum_{\mu \in M} q_\mu D\big(\mathbb{Z}^\ell, \mu, e(\mu)\big) + \sum_{\nu \in N} q_\nu D\big(L_\nu, \nu, e(\nu)\big), \tag{33}$$

*where the sums of rank $\ell$ on the right are diagonal and unimodular, and the rank $\ell - 1$ sums have index less than or equal to $\|D\|$.*

*Proof*

By Theorem 4.3, we may take $D$ to be diagonal. Choose a sum $S(Z^\ell, \tau)$ representing $D$ as in §4.2. Suppose that

$$\delta = |\det(\sigma_1, \dots, \sigma_\ell)| > 1.$$

Then by Proposition 2.2, there exists $w \in \mathbb{Z}^\ell$ such that

$$\delta_j = |\det(\sigma_1, \dots, \hat{\sigma}_j, \dots, \sigma_\ell, w)|$$

satisfies $0 \leq \delta_j < \delta^{(\ell-1)/\ell}$. As in the proof of Theorem 4.3, write $w = \sum_{i=1}^{\ell} a_i\sigma_i$, and set $\omega = \sum_{i=1}^{\ell} a_j \tau_j^1$.

Now apply Proposition 3.3 using $\omega$ and the columns of $\tau$ to write $S$ as a finite $\mathbb{Q}$-linear combination of new Dedekind sums. These sums are not diagonal, but we can apply the proof of Theorem 4.3 with $\omega$ playing the role of $\sigma_r$. The resulting sums include $\omega$ among their linear forms and are diagonal. All the rank $\ell$ sums have index less than $\|S\|$, and the sums of lower rank satisfy the conditions in the statement of Theorem 4.3. By induction on the index, this completes the proof. □

COROLLARY 4.5
*Any Dedekind sum can be written as a finite $\mathbb{Q}$-linear combination of diagonal, unimodular sums.*

*Proof*
First, it is easy to see that any rank 1 Dedekind sum is automatically unimodular and diagonal. The result then follows by applying Theorems 4.3 and 4.4 and descending induction on the rank. □

## 5. Complexity

*5.1*
In this section we discuss the computational complexity of Corollary 4.5. In particular, we show that if $|e|$ and $\ell$ are fixed, then we can form a finite $\mathbb{Q}$-linear combination of diagonal unimodular Dedekind sums

$$D(\mathbb{Z}^\ell, \sigma, e, v) = \sum_{\mu \in M} q_\mu D\big(L_\mu, \mu, e(\mu), v(\mu)\big),$$

where $\#M$ is bounded by a polynomial in $\log \|D\|$. As a corollary, we obtain the fact that this expression can be computed in polynomial time.

To do this, we must make a more detailed analysis of the proofs in §4. As before, put $n = |e|$, and drop $v$ and $Q$ from the notation. We begin by analyzing diagonality.

LEMMA 5.2
*Let $D = D(\mathbb{Z}^\ell, \sigma, e)$ be a Dedekind sum, and write*

$$D = \sum_{\mu \in M} q_\mu D\big(\mathbb{Z}^\ell, \mu, e(\mu)\big) + \sum_{\nu \in N} q_\nu D\big(L_\nu, \nu, e(\nu)\big) \tag{34}$$

*as in Theorem 4.3, so that the rank $\ell$ sums in (34) are diagonal. If $\ell > 1$, there exists a constant $A_{n,\ell}$ such that*

$$\#M \leq A_{n,\ell} \qquad \textit{and} \qquad \#N \leq A_{n,\ell}.$$

*Proof*
Let $(e_1, \ldots, e_r)$ be the type of $D$, where $\ell \leq r \leq n$. By the proof of Theorem 4.3, we know how to pass from a sum of type

$$(e_1, \ldots, e_i, \ldots, e_\ell, \ldots, e_r) \tag{35}$$

to a linear combination of sums of types

$$(e_1, \ldots, e_i - 1, \ldots, e_\ell, \ldots, e_r + 1), \quad i = 1, \ldots, \ell. \tag{36}$$

By iterating this, we pass from $D$ to a linear combination of sums with types

$$(e'_1, \ldots, e'_{i-1}, 0, e'_{i+1}, \ldots, e'_\ell, \ldots, e'_r), \quad i = 1, \ldots, \ell. \tag{37}$$

We bound the number of rank $\ell$ and rank $\ell - 1$ sums produced in passing from (35) to (37) by a constant $A^{(r)}_{n,\ell}$. We can then take

$$A_{n,\ell} = \prod_{r=\ell+1}^{n} A^{(r)}_{n,\ell}.$$

To describe what happens in going from (35) to (37), we use a geometric construction. Let $X = X(e_1, \ldots, e_\ell)$ be the set

$$X = \left\{ (x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell \;\middle|\; 0 \le x_i \le e_i,\ i = 1, \ldots, \ell \right\}.$$

The points in $X$ correspond to types of intermediate sums in the passage from (35) to (37). In particular, passing from (35) to (36) can be encoded by moving from $(x_1, \ldots, x_\ell)$ to $(x_1, \ldots, x_i - 1, \ldots, x_\ell)$ in $X$. Moreover, the sums of the form (37) correspond to the subset of points $X_0 \subset X$ with exactly one coordinate zero. (See Figure 2.)

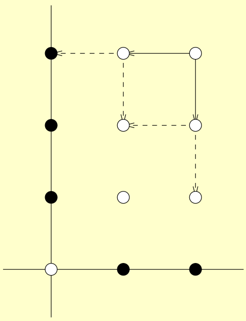

Figure 2. The set $X(2, 3)$. The arrows represent applications of Theorem 3.3. The black dots are the set $X_0$.

Now the constant $A^{(r)}_{n,\ell}$ is given by $\operatorname{Max} \# X_0$ as $X(e_1, \ldots, e_\ell)$ ranges over all possibilities for fixed $n$, $\ell$, and $r$. If the $e_i$ become continuous parameters, then a simple computation shows that the maximum occurs when $e_1 = \cdots = e_\ell = (n - r + \ell)/\ell$. Hence we have

$$A^{(r)}_{n,\ell} \le \ell \left( \frac{n - r + \ell}{\ell} \right)^{\ell - 1}. \tag{38}$$

To finish the proof, we must consider the rank $\ell - 1$ sums. According to the proof of Theorem 3.3, a rank $\ell - 1$ sum is produced as a correction term to account for a linear form that disappears from the denominator of a Dedekind sum when applying (15). In terms of the set $X$, such a sum appears when passing from a point in $X \setminus X_0$ to a point in $X_0$. Hence for fixed $n$, $\ell$, and $r$ the number of these sums is also bounded by $\operatorname{Max} \# X_0$, which implies that $\# N \leq A_{n,\ell}$. This completes the proof. □

PROPOSITION 5.3
*If $n$ and $\ell$ are fixed, then (34) can be constructed in constant time, independent of $\|D\|$.*

*Proof*
This follows easily from the proof of Lemma 5.2. Forming the expression (34) depends only on the type of $D$ and makes no reference to $\|D\|$. In particular, the number of steps needed can be bounded for fixed $n$ and $\ell$. □

*5.4*
Now we investigate the size of the output in Theorem 4.4. The main step of the proof of Theorem 4.4 shows how, given $D$, one may write

$$D = \sum_{\mu \in M} q_\mu D\big(\mathbb{Z}^\ell, \mu, e(\mu)\big) + \sum_{\nu \in N} q_\nu D\big(L_\nu, \nu, e(\nu)\big), \tag{39}$$

where the sums of rank $\ell$ on the right are diagonal and satisfy $\|D(\mathbb{Z}^\ell, \mu, e(\mu))\| < \|D\|^{(n-1)/n}$.

LEMMA 5.5
*Let $C_{n,\ell} = \# M$, the number of rank $\ell$ sums on the right-hand side of (39). Then $C_{n,\ell} \leq A^{(\ell)}_{n,\ell}$. Furthermore, the number $\# N$ of rank $\ell - 1$ sums on the right-hand side of (39) is bounded by $A^{(\ell)}_{n,\ell} + 1$.*

*Proof*
The proof is similar to the that of Lemma 5.2, although with the following twist. The first step is to pass from a diagonal sum of type $(e_1, \ldots, e_\ell)$ to a collection of sums of types

$$(e_1, \ldots, e_i - 1, \ldots, e_\ell, 1), \quad i = 1, \ldots, \ell.$$

The resulting sums can be diagonalized, and we can bound the number of rank $\ell$ sums produced by $\# X_0(e_1, \ldots, e_\ell)$, which is in turn bounded by

$$\ell \left(\frac{n}{\ell}\right)^{\ell - 1} = A^{(\ell)}_{n,\ell}.$$

For the rank $\ell-1$ sums, we argue similarly, except that the first step also produces a rank $\ell-1$ sum. □

*5.6*
Now consider the expression (39). To complete the proof of Theorem 4.4, we repeat the process that produced (39) until all rank $\ell$ sums are diagonal and unimodular, and we obtain the expression (33). Using Lemma 5.5 and estimate (6), we can bound the number of rank $\ell$ sums produced.

PROPOSITION 5.7 (cf. [2, Theorem 5.4])
*Write $D$ as a sum of diagonal, unimodular rank $\ell$ sums and lower rank sums as in (33). Then the number of rank $\ell$ sums on the right-hand side of (33) is bounded by*

$$C'(\log\|D\|)^{\alpha\log C_\ell}, \tag{40}$$

*where $\alpha=\log(n/(n-1))^{-1}$, and $C'$ is a constant depending on $n$ and $\ell$ but not $\|D\|$.*

*Proof*
In (39) we have

$$0\le\left\|D\big(\mathbb{Z}^\ell,\mu,e(\mu)\big)\right\|<\left\|D\right\|^{(n-1)/n}.$$

Thus after $t$ iterations we have

$$0\le\left\|D\big(\mathbb{Z}^\ell,\mu,e(\mu)\big)\right\|<\left\|D\right\|^{((n-1)/n)^t}.$$

Since the index of a Dedekind sum is always an integer, the condition for termination is that for some $\varepsilon>0$ we have

$$\|D\|^{((n-1)/n)^t}\le 2-\varepsilon \qquad \text{or} \qquad t\ge\frac{\log\log\|D\|-\log\log(2-\varepsilon)}{\log n-\log(n-1)}. \tag{41}$$

On the other hand, by Lemma 5.5, we know that $t$ iterations produce no more than $C_\ell^t$ sums of rank $\ell$. So fix $\varepsilon>0$, set $\alpha=(\log n-\log(n-1))^{-1}$, and let

$$C'=C_{n,\ell}^{1-\alpha\log\log(2-\varepsilon)}.$$

Then

$$C_{n,\ell}^t\le C'\big(\log\|D\|\big)^{\alpha\log C_\ell},$$

which completes the proof. □

*5.8*
We are now ready to discuss the complexity of our algorithms.

THEOREM 5.9

*Let $D = D(\mathbb{Z}^\ell, \sigma, e)$ be a Dedekind sum. Using Corollary 4.5, write $D$ as a $\mathbb{Q}$-linear combination of diagonal unimodular sums. Then there exists a polynomial $P_{n,\ell}$ such that the number of terms in the expression is bounded by $P_{n,\ell}(\log \|D\|)$. Moreover, we have*

$$\deg P_{n,\ell} \leq \alpha \log\Big(\frac{\ell!\, n^{\ell(\ell-1)/2}}{2^1 3^2 \cdots \ell^{(\ell-1)}}\Big),$$

*where $\alpha = \log(n/(n-1))^{-1}$.*

*Proof*

Fix $n$. We proceed by induction on $\ell$.

First, if $\ell = 1$, then it is easy to see that the sum $D(\mathbb{Z}^1, \sigma, e)$ is already diagonal and unimodular. Hence we may take $P_{n,1} = 1$.

Next, assume that the statement is true for sums of rank $\ell - 1$, and let $P_{n,\ell-1}$ be the corresponding polynomial. We first claim that without loss of generality, we need only consider the case when $D$ is diagonal. Indeed, apply Theorem 4.3, and write

$$D = \sum_{\mu \in M} q_\mu D\big(\mathbb{Z}^\ell, \mu, e(\mu)\big) + \sum_{\nu \in N} q_\nu D\big(L_\nu, \nu, e(\nu)\big), \tag{42}$$

where the rank $\ell$ Dedekind sums are diagonal and all the Dedekind sums have index less than or equal to $\|D\|$. By Lemma 5.2, the sets $M$ and $N$ have a bounded number of elements independent of $\|D\|$. Hence we may bound the output for diagonal $D$ and then multiply this by a constant to obtain the final bound, which does not change the degree of the final bound.

Now we apply Theorem 4.4 and count the number of Dedekind sums produced. By Proposition 5.7, we know that the total number of rank $\ell$ sums is bounded by

$$C'(\log \|D\|)^{\alpha \log C_\ell}. \tag{43}$$

Furthermore, each sum of lower rank produced in the proof of Theorem 4.4 can be written as a sum of less than or equal to $P_{n,\ell-1}(\|D\|)$ Dedekind sums by induction. To find the total output, we must count these lower-rank sums.

We can do this as follows. Let $Q = Q_{n,\ell} := (1 + C_{n,\ell})P_{n,\ell-1}$, where $C_{n,\ell}$ is the constant in Lemma 5.5. Represent the process of reducing the diagonal sum $D$ to unimodularity by the following diagram:

$$\begin{tikzcd}
\text{rank } \ell: & 1 \arrow[r] \arrow[dr] & C_{n,\ell} \arrow[r] \arrow[dr] & C_{n,\ell}^2 \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & C_{n,\ell}^t \\
\text{rank } \ell - 1: & & Q & QC_{n,\ell} & \cdots & QC_{n,\ell}^{t-1}
\end{tikzcd} \tag{44}$$

The top row represents the bound on the number of rank $\ell$ sums at each step of the algorithm, and the bottom row is the number of rank $\ell - 1$ sums produced by each step.

According to (44), we have produced

$$Q \sum_{i=0}^{t-1} C_{n,\ell}^{i} = Q\, \frac{C_{n,\ell}^{t} - 1}{C_{n,\ell} - 1} \leq Q\, \frac{C'(\log \|D\|)^{\alpha \log C_{n,\ell}} - 1}{C_{n,\ell} - 1}$$

sums of ranks less than or equal to $\ell - 1$. Adding this estimate to (43), we find that the total number of Dedekind sums produced is a polynomial of degree

$$\deg Q + \alpha \log C_{n,\ell} = \deg P_{n,\ell-1} + \alpha \log C_{n,\ell}.$$

To complete the proof, we compute the degree of $P_{n,\ell}$ by induction. Indeed, using the estimate

$$C_{n,\ell} \leq A_{n,\ell}^{(\ell)} = \ell \left(\frac{n}{\ell}\right)^{\ell-1}$$

and the fact that $\deg P_{n,1} = 0$, an easy computation shows that

$$\deg P_{n,\ell} \leq \alpha \log \left(\frac{\ell!\, n^{\ell(\ell-1)/2}}{2^1 3^2 \cdots \ell^{(\ell-1)}}\right),$$

as required. □

*Example 5.10*
Table 1 is a table of the bound of $\deg P_{n,\ell}$ for small values of $n$ and $\ell$.

Table 1

| $n, \ell$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | | | | | | |
| 3 | 2 | 5 | | | | | |
| 4 | 4 | 10 | 15 | | | | |
| 5 | 7 | 16 | 25 | 33 | | | |
| 6 | 9 | 23 | 37 | 50 | 60 | | |
| 7 | 12 | 30 | 50 | 69 | 86 | 99 | |
| 8 | 15 | 38 | 64 | 90 | 114 | 135 | 150 |

COROLLARY 5.11
*Keeping the same notation as in Theorem 5.9, for fixed $n$ and $\ell$ we may express $D$ as a finite $\mathbb{Q}$-linear combination of diagonal unimodular sums in time polynomial in* $\log \|D\|$.

*Proof*

First, the vector $w$ constructed in Proposition 2.2 can be found in polynomial time in the size of the coefficients of $D$. In fact, investigation of [2, Lemma 5.2] shows that the rational numbers $a_i$ in (29) in the proof of Theorem 4.3 can also be constructed in polynomial time. This implies that $w$ and the $a_i$ can be found in time polynomial in $\log \|D\|$. The proof of Theorem 5.9 then shows that the final expression can be computed in polynomial time. □

## 6. The Eisenstein cocycle

*6.1*

In this section we briefly review the construction of the Eisenstein cocycle introduced in [17]. In particular, we show that this is a finite object that can be calculated effectively using Corollary 4.5. Roughly speaking, the Eisenstein cocycle represents a generalization of the classical Bernoulli polynomial within the arithmetic of the unimodular group $\Gamma = \mathrm{GL}_n(\mathbb{Z})$.

*6.2*

Let $\mathscr{A} = (A_1, \ldots, A_n)$ be an $n$-tuple of matrices $A_i \in \mathrm{GL}_n(\mathbb{R})$. For an $n$-tuple $d = (d_1, \ldots, d_n)$ of integers $1 \le d_i \le n$, let $\mathscr{A}(d) \subseteq \mathbb{R}^n$ be the subspace generated by all columns $A_{ij}$ such that $j < d_i$. (Here $A_{ij}$ denotes the $j$th column of $A_i$.) Writing $\mathscr{A}(d)^\perp$ for the orthogonal complement of $\mathscr{A}(d)$ in $\mathbb{R}^n$, we let

$$X(d) = \mathscr{A}(d)^\perp \smallsetminus \bigcup_{i=1}^{n} \sigma_i^\perp, \quad \text{where } \sigma_i = A_{i d_i}. \tag{45}$$

The $n$-tuple $\mathscr{A}$ then determines the stratification

$$\mathbb{R}^n \smallsetminus \{0\} = \bigsqcup_{d \in D} X(d) \tag{46}$$

indexed by the finite set

$$D = D(\mathscr{A}) = \big\{ d \bigm| X(d) \ne \varnothing \big\}.$$

Associated to this decomposition is the collection of rational functions $\psi(\mathscr{A})$ on $\mathbb{R}^n \smallsetminus \{0\}$ defined by

$$\psi(\mathscr{A})(x) = \frac{\det(\sigma_1, \ldots, \sigma_n)}{\langle x, \sigma_1 \rangle \cdots \langle x, \sigma_n \rangle} \quad \text{for } x \in X(d) \text{ and } \sigma_i = A_{i d_i}.$$

*6.3*

More generally, if $P(X_1, \ldots, X_n)$ is any homogeneous polynomial, we form the differential operator $P(-\partial_{x_1}, \ldots, -\partial_{x_n})$ in the partial derivatives $\partial_{x_i} := \partial/\partial x_i$ and define

$$\psi(\mathscr{A})(P, x) = P(-\partial_{x_1}, \ldots, -\partial_{x_n})\psi(\mathscr{A})(x).$$

The last expression can be written more explicitly as

$$\psi(\mathscr{A})(P, x) = \det(\sigma) \sum_r P_r(\sigma) \prod_{j=1}^{n} \frac{1}{\langle x, \sigma_j \rangle^{1+r_j}}, \tag{47}$$

where $r$ runs over all decompositions of $\deg(P) = r_1 + \cdots + r_n$ into nonnegative parts $r_j \geq 0$, and $P_r(\sigma)$ is the homogeneous polynomial in the $\sigma_{ij}$ defined by the expansion

$$P(X\sigma^t) = \sum_r P_r(\sigma) \prod_{j=1}^{n} \frac{X_j^{r_j}}{r_j!}.$$

In the excluded case $x = 0$, it is convenient to set $\psi(\mathscr{A})(P, 0) = 0$.

The definition of the Eisenstein cocycle $\Psi$ is now easy to state:

$$\Psi(\mathscr{A})(P, Q, v) := (2\pi \mathrm{i})^{-n-\deg P} \sum_{x \in \mathbb{Z}^n} \mathbf{e}(\langle x, v \rangle)\psi(\mathscr{A})(P, x)\Big|_Q. \tag{48}$$

The series on the right-hand side converges, provided that all components $A_i$ of $\mathscr{A}$ are in $\mathrm{GL}_n(\mathbb{Q})$. However, since the convergence is only conditional, we are forced to introduce the additional parameter $Q$ specifying the limiting process.

*6.4*

Let $M$ be the set of all complex-valued functions $f(P, Q, v)$ with $P$, $Q$, $v$ as above ($v \in \mathbb{R}^n$). Then $M$ is a left $\Gamma$-module under the action

$$Af(P, Q, v) = \det(A) f(A^t P, A^{-1}Q, A^{-1}v), \quad A \in \Gamma,$$

where the implied $\Gamma$-action on homogeneous polynomials is given by $(AP)(X) = P(XA)$. With respect to this action, the map $\Psi\colon \Gamma^n \to M$ has the property

$$\Psi(A\mathscr{A}) = A\Psi(\mathscr{A}), \quad A \in \Gamma,\ \mathscr{A} \in \Gamma^n, \tag{49}$$

$$\sum_{i=0}^{n} \Psi(A_0, \ldots, \hat{A}_i, \ldots, A_n) = 0, \quad A_i \in \Gamma. \tag{50}$$

In other words, $\Psi$ is a homogeneous cocycle on $\Gamma$. It is known that $\Psi$ represents a nontrivial cohomology class in $H^{n-1}(\Gamma; M)$ (see [17, Theorem 4]).

*6.5*

To express $\Psi$ in terms of Dedekind sums, we need to incorporate a parameter $e$ into the sum $S$:

$$S(L, \sigma, e, v) := (2\pi \mathrm{i})^{-|e|} \sum_{x \in L}{}' \frac{\mathbf{e}(\langle x, v\rangle)}{\langle x, \sigma_1\rangle^{e_1} \cdots \langle x, \sigma_n\rangle^{e_n}}.$$

Combining (45)–(48), we see that $\Psi$ is a finite linear combination of Dedekind sums

$$\Psi(\mathscr{A})(P, Q, v) = \sum_{d \in D} \sum_{r} \det(\sigma) P_r(\sigma) S(L, \sigma, e, v)\big|_Q.$$

Here $\sigma$ is the matrix with columns $A_{id_i}$ for $i = 1, \ldots, n$, $L$ is the lattice $\mathscr{A}(d)^\perp \cap \mathbb{Z}^n$, and $e_j = 1 + r_j$. The case $P = 1$ is of special interest:

$$\Psi(\mathscr{A})(1, Q, v) = \sum_{d \in D} \det(\sigma) S(L, \sigma, v)\big|_Q.$$

This case yields the classical Dedekind-Rademacher sums if $n = 2$, and, more importantly, it corresponds to special values of partial zeta functions at $s = 0$.

## 7. Values of partial zeta functions

*7.1*

Let $F$ be a totally real number field of degree $n$ over $\mathbb{Q}$, and let $\mathfrak{f}$, $\mathfrak{b}$ be two relatively prime ideals in the ring of integers $\mathscr{O}_F$. The partial zeta function to the ray class $\mathfrak{b}$ mod $\mathfrak{f}$ is defined by

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) := \sum_{\mathfrak{a} \equiv \mathfrak{b} \bmod \mathfrak{f}} N(\mathfrak{a})^{-s}, \quad \Re(s) > 1,$$

where $\mathfrak{a}$ runs over all all integral ideals in $\mathscr{O}_F$ such that the fractional ideal $\mathfrak{a}\mathfrak{b}^{-1}$ is a principal ideal generated by a totally positive number in the coset $1 + \mathfrak{f}\mathfrak{b}^{-1}$. According to the work of H. Klingen and K. Siegel, the special values $\zeta(\mathfrak{b}, \mathfrak{f}, 1 - s)$, where $s = 1, 2, 3, \ldots,$ are well-defined rational numbers. In this section we give a formula for calculating these numbers in terms of the Eisenstein cocycle $\Psi$.

*7.2*

The formula depends on the choice of a $\mathbb{Z}$-basis $W$ for the fractional ideal $\mathfrak{f}\mathfrak{b}^{-1} = \sum \mathbb{Z} W_j$, together with the dual basis $W^*$ determined by $\mathrm{Tr}(W_i^* W_j) = \delta_{ij}$. Here we identify $\alpha \in F$ with the row vector $(\alpha^{(1)}, \ldots, \alpha^{(n)}) \in \mathbb{R}^n$, where the $\alpha^{(j)}$ are the $n$ different embeddings of $\alpha$ into the field of real numbers. Then $W$ can be identified with a matrix in $\mathrm{GL}_n(\mathbb{R})$ whose $j$th row is the basis vector $W_j$. Let

$$P(X) = N(\mathfrak{b}) \prod_i \sum_j X_j W_j^{(i)},$$
$$Q(X) = \prod_i \sum_j X_j (W_j^*)^{(i)},$$

and let $v \in \mathbb{Q}^n$ be defined by $v_j = \mathrm{Tr}(W_j^*)$.

The formula also depends on the choice of generators $\varepsilon_1, \ldots, \varepsilon_\nu$, where $\nu = n-1$, for the group $U \subset \mathscr{O}_F^\times$ of totally positive units. Using the regular representation $\rho\colon U \to \Gamma$ defined via $\rho(\varepsilon) = W\delta(\varepsilon)W^{-1}$, where $\delta(\varepsilon)$ is the matrix $\mathrm{diag}(\varepsilon^{(1)}, \ldots, \varepsilon^{(n)})$, we identify the units $\varepsilon_j$ with elements $A_j = \rho(\varepsilon_j)^t \in \Gamma$. (Note that $\rho$ is the *row* regular representation.)

*7.3*
Using the bar notation

$$[A_1|\cdots|A_\nu] := (1, A_1, A_1A_2, \ldots, A_1\cdots A_\nu) \in \Gamma^n,$$

we have the following proposition expressing the zeta values in terms of the Eisenstein cocycle.

PROPOSITION 7.4
*Let $U_\mathfrak{f}$ be the subgroup $U \cap (1+\mathfrak{f})$, and let $\pi$ run through all permutations of $\{1, \ldots, \nu\}$. Then for $s = 1, 2, 3, \ldots,$*

$$\zeta(\mathfrak{b}, \mathfrak{f}, 1-s) = \eta \sum_{\varepsilon \in U/U_\mathfrak{f}} \sum_\pi \mathrm{sgn}(\pi) \Psi\big([A_{\pi(1)}|\cdots|A_{\pi(\nu)}]\big)\big(P^{s-1}, Q, \rho(\varepsilon)^t v\big).$$

*Here the sign $\eta = \pm 1$ is determined by*

$$\eta = (-1)^\nu \,\mathrm{sgn}(\det W)\,\mathrm{sgn}(R),$$

*where $R = \det(\log \varepsilon_i^{(j)})$, $1 \le i, j \le \nu$.*

*Proof*
This follows from [17, Corollary, page 595] by writing the fundamental cycle of $U_\mathfrak{f}$ in terms of the $A_j$. □

*Example 7.5*
We work out the above formula in the case of a real quadratic field $F$. Let $\varepsilon > 1$ be the fundamental unit of $U$, the group of totally positive units in $\mathscr{O}_F$, and let $\mathbb{Z}w_1 + \mathbb{Z}w_2 =$

$\mathfrak{f}\mathfrak{b}^{-1}$ be a $\mathbb{Z}$-basis of $\mathfrak{f}\mathfrak{b}^{-1}$. Such a basis determines a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and a vector $v \in \mathbb{Q}^2$ via

$$\begin{pmatrix} \varepsilon w_1 \\ \varepsilon w_2 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad v_1 w_1 + v_2 w_2 = 1.$$

In addition, we get the normforms

$$P(X) = N(\mathfrak{b}) N(X_1 w_1 + X_2 w_2), \qquad Q(X) = N(X_1 w_2 - X_2 w_1).$$

Let $p$ be the smallest positive integer such that $(A^p - 1)v \in \mathbb{Z}^2$. Then

$$\zeta_F(\mathfrak{b}, \mathfrak{f}, 1-s) = \eta \sum_{k \bmod \mathrm{p}} \Psi(1, A)(P^{s-1}, Q, A^k v), \tag{51}$$

where $\eta = \mathrm{sgn}(w_2 w_1^{(1)} - w_1 w_2^{(1)})$, and if $s = 1$,

$$\Psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(1, Q, v) = \frac{a}{2c}\mathscr{B}_2(v_2) + \frac{d}{2c}\mathscr{B}_2(cv_1 - av_2) \tag{52}$$

$$- \sum_{j \bmod c} \mathscr{B}_1\Big(\frac{j + v_2}{|c|}\Big)\mathscr{B}_1\Big(a\frac{j + v_2}{c} - v_1\Big) \tag{53}$$

with an additional correction term $-\mathrm{sgn}(c)/4$ on the right if $v \in \mathbb{Z}^2$. The finite sum (53) is the classical Dedekind-Rademacher sum $S\left(\mathbb{Z}^2, \left(\begin{smallmatrix} 1 & a \\ 0 & c \end{smallmatrix}\right), v\right)\big|_Q$. Note that the number of terms in that sum equals $|c| = |(\varepsilon - \varepsilon')/(w - w')|$, where $w = w_2/w_1$, and the prime is Galois conjugation. Depending on $\varepsilon$, this number can be very large. To get a more efficient formula for calculating $\Psi$, we apply the Euclidean algorithm to the first column of $A$ and obtain a product decomposition

$$A = B_1 \cdots B_t, \quad t \geq 1, \qquad B_j = \begin{pmatrix} b_j & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\Psi(1, A) = \sum_{\ell=0}^{t-1} (B_1 \cdots B_\ell)\Psi(1, B_{\ell+1}). \tag{54}$$

Here $t$ is roughly $\log|c|$. Thus the number of terms is effectively reduced from $|c|$ to $\log|c|$ since

$$\Psi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}\right)(P^{s-1}, Q, v)$$
$$= \sum_r \left[ bP_r \begin{pmatrix} 0 & b \\ 1 & 1 \end{pmatrix} \frac{\mathscr{B}_{2s}(v_2)}{(2s)!} + P_r \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \frac{\mathscr{B}_{1+r_1}(v_1 - bv_2)}{(1+r_1)!}\frac{\mathscr{B}_{1+r_2}(v_2)}{(1+r_2)!}\right], \tag{55}$$

where the rational numbers $P_r\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ are the coefficients of the polynomial

$$P(\alpha X_1 + \beta X_2, \gamma X_1 + \delta X_2)^{s-1} = \sum_r P_r \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \frac{X_1^{r_1}}{r_1!} \frac{X_2^{r_2}}{r_2!}.$$

In the exceptional case $s = 1$, $\mathfrak{f} = (1)$, the correction term

$$-\frac{1}{8}\left\{ \operatorname{sgn}(w + b) + \operatorname{sgn}(w' + b)\right\}$$

must be added to the right side of (55).

As a numerical example, we choose $F = \mathbb{Q}(\sqrt{5})$, $\varepsilon = (3 + \sqrt{5})/2$, $\mathfrak{f} = \mathfrak{b} = (1)$, $w_1 = -\varepsilon$, $w_2 = 1$. Then $\eta = +1$, $A = \left(\begin{smallmatrix}3 & -1\\ 1 & 0\end{smallmatrix}\right)$, $P_{11}\left(\begin{smallmatrix}1 & 3\\ 0 & 1\end{smallmatrix}\right) = 3$, while $P_{20}\left(\begin{smallmatrix}0 & 3\\ 1 & 1\end{smallmatrix}\right) = 2$, $P_{11}\left(\begin{smallmatrix}0 & 3\\ 1 & 1\end{smallmatrix}\right) = -7$, $P_{02}\left(\begin{smallmatrix}0 & 3\\ 1 & 1\end{smallmatrix}\right) = 2$. Hence, according to (51) and (55), we get for the value of the Dedekind zeta function of $F$ at $s = -1$,

$$\zeta_F(-1) = \zeta\big((1), (1), -1\big) = 3(2 - 7 + 2)\Big(-\frac{1}{720}\Big) + 3\Big(\frac{1}{12}\Big)^2 = \frac{1}{30}.$$

*Example 7.6*

As a second example, we consider the cubic field $\mathbb{Q}(\theta)$ of discriminant 148 given by $\theta^3 - \theta^2 - 3\theta + 1 = 0$. According to [14], the group of totally positive units $U$ is generated by $\varepsilon_1 = -3\theta^2 + 2\theta + 10$ and $\varepsilon_2 = 5\theta^2 + 6\theta - 2$.

Let $\mathfrak{f} = (2)$ and $\mathfrak{b} = (1)$. Then $\mathfrak{f}\mathfrak{b}^{-1} = \mathbb{Z}w_1 + \mathbb{Z}w_2 + \mathbb{Z}w_3$, where $w_1 = 2$, $w_2 = 2\theta$, and $w_3 = 2\theta^2$. With respect to this basis, we find that

$$A_1 = \begin{pmatrix} 10 & 3 & 1 \\ 2 & 1 & 0 \\ -3 & -1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -2 & -5 & -11 \\ 6 & 13 & 28 \\ 5 & 11 & 24 \end{pmatrix},$$

$$A_1 A_2 = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 3 & 6 \\ 0 & 2 & 5 \end{pmatrix}.$$

Then $v = (1/2, 0, 0)^t$ and $\eta = 1$. Let $V \subset \mathbb{Q}^3$ be a complete set of representatives for the orbit of $v + \mathbb{Z}^3$ under the action of $U$ (via $A_1$ and $A_2$) on $\mathbb{Q}^3/\mathbb{Z}^3$. Note that $V$ is a finite set. Thus

$$\zeta(\mathfrak{b}, \mathfrak{f}, 0) = \sum_{v \in V} \big(\Psi(1, A_1, A_1 A_2) - \Psi(1, A_2, A_1 A_2)\big)(1, Q, v). \tag{56}$$

Each term on the right-hand side of (56) breaks up into 10 Dedekind sums: one of rank 3, three of rank 2, and six of rank 1. Note that the rank 3 and rank 1 sums are diagonal, whereas the rank 2 sums are not. After making all sums diagonal, we find that each $\Psi$ in (56) has 30 terms. Applying the summation formula for diagonal sums

Table 2. $\zeta((1), (N), 0)$ for the cubic field of discriminant 148

| $N$ | $N \cdot \zeta$ | $N$ | $N \cdot \zeta$ | $N$ | $N \cdot \zeta$ | $N$ | $N \cdot \zeta$ | $N$ | $N \cdot \zeta$ |
|---|---|---|---|---|---|---|---|---|---|
| | | 11 | −18 | 21 | −78 | 31 | 74 | 41 | 382 |
| 2 | 0 | 12 | 5 | 22 | 68 | 32 | 15 | 42 | −228 |
| 3 | 2 | 13 | −22 | 23 | 12 | 33 | 62 | 43 | −366 |
| 4 | 1 | 14 | −20 | 24 | 23 | 34 | 50 | 44 | 1 |
| 5 | −4 | 15 | 42 | 25 | 106 | 35 | −54 | 45 | 254 |
| 6 | 4 | 16 | 7 | 26 | −24 | 36 | −43 | 46 | 6 |
| 7 | 2 | 17 | 100 | 27 | −190 | 37 | 20 | 47 | −570 |
| 8 | 3 | 18 | −32 | 28 | 25 | 38 | 22 | 48 | −13 |
| 9 | −10 | 19 | 82 | 29 | 242 | 39 | 156 | 49 | −222 |
| 10 | −2 | 20 | 4 | 30 | 6 | 40 | 2 | 50 | 178 |

(Proposition 2.7), we see that to evaluate $\Psi$ on any element of $V$, we must sum 76 terms.

Since $\varepsilon_2^2 \equiv \varepsilon_1\varepsilon_2 \equiv 1 \bmod \mathfrak{f}$, we can take $V = \{v, A_2v\}$. This yields 152 terms altogether, all of which sum to zero, and thus $\zeta((1), (2), 0) = 0$. This agrees with [14] and also with the observation that, since $-1$ preserves the congruence class of $1 + \mathfrak{f}$ and the norm of $-1$ is $-1$, the special value at $s = 0$ must vanish.

Now let $\mathfrak{f} = (3)$. Then we may take $A_1, A_2$ as above, and $v = (1/3, 0, 0)^t$ and $\eta = 1$. Since $\varepsilon_2^{13} \equiv \varepsilon_1\varepsilon_2^5 \equiv 1 \bmod \mathfrak{f}$, we must sum $13 \cdot 76 = 988$ terms, and we find that $\zeta((1), (3), 0) = 2/3$, again in agreement with [14].

Note that to compute $\zeta((1), (N), 0)$ for various $N \in \mathbb{Z}$, we must only compute $A_1, A_2$, and thus $\Psi(1, A_1, A_1A_2) - \Psi(1, A_2, A_1A_2)$ once.* After this it is routine to compute special values at $s = 0$, and the complexity in (56) comes from $\#V$, which can be large, even for small values of $N$. The values of $\zeta((1), (N), 0)$ for several rational integers $N$ are given in Table 2. The values for $N = 2, 3, 5, 7$ are also in [14]. Partial zeta values for all totally real cubic fields of discriminant less than 3000 and for conductors $\mathfrak{f}$ of norm less than 750 can be obtained by request from P. Gunnells.

## 8. Witten's zeta function

*8.1*

To give another illustration, we recall the definition of Witten's zeta function [19] and show how our algorithms can be used to compute special values of this function

*This is a special feature since $\mathfrak{f}$ is a rational principal ideal and $\mathfrak{b} = (1)$. In general, the cocycle must be computed for each ideal class modulo $\mathfrak{f}$.

at even integers. For unexplained notions from representation theory, the reader may consult [9] and [10].

Let $\mathfrak{g}$ be a simple complex lie algebra, and let $R$ be the associated root system with root lattice $\Lambda_R$. Let $R^+$ (resp., $R^-$) be a subset of positive roots (resp., negative roots), and let $\Delta \subset R^+$ be the set of simple roots. Let $\Omega$ be the set of fundamental weights, and let $\Lambda_W$ be the weight lattice.

The lattices $\Lambda_R \subset \Lambda_W$ sit in an $\ell$-dimensional real vector space $E$ endowed with an inner product $( \ , \ )$. Let $W$ be the Weyl group of $R$. This is a finite group that acts on $E$ via reflections and preserves the inner product and the lattices $\Lambda_R$ and $\Lambda_W$. There is a decomposition of $E$ into a finite union of rational polyhedral cones, and $W$ acts by permuting these cones. Let $C^+$ be the closed top-dimensional cone generated by $\Omega$.

Let $\Pi$ denote the set of isomorphism classes of complex irreducible representations of $\mathfrak{g}$. It is known that elements of $\Pi$ are in bijection with the set $\Lambda_W \cap C^+$, the dominant weights. Given $\lambda$ from this latter set, we denote the corresponding representation by $\pi_\lambda$. Then the definition of the zeta function associated to $\mathfrak{g}$ is

$$\zeta_{\mathfrak{g}}(s) := \sum_{\lambda \in \Lambda_W \cap C^+} (\dim \pi_\lambda)^{-s}. \tag{57}$$

*8.2*

Let $m > 1$ be an integer. The special value $\zeta_{\mathfrak{g}}(2m)$ can be computed using a Dedekind sum as follows.

Let $\rho$ be one-half the sum of the positive roots. An application of the Weyl character formula (see [9, Corollary 24.6]) shows that for any dominant weight $\lambda$, we have

$$\dim \pi_\lambda = \prod_{\alpha \in R^+} \frac{(\rho + \lambda, \alpha)}{(\rho, \alpha)}. \tag{58}$$

It is known that any dominant weight $\lambda$ can be written as a nonnegative integral linear combination of the fundamental weights. Using this in (58), a computation shows that (57) becomes

$$\zeta_{\mathfrak{g}}(2m) = M^{2m} \sum_{x \in (\mathbb{Z}^{>0})^\ell} \frac{1}{\prod_{i=1}^r \langle a_i, x \rangle^{2m}}. \tag{59}$$

Here $M$ is the integer $\prod_{\alpha \in R^+} (\rho, \alpha)$, $r$ is the number of positive roots, and the $a_i \in (\mathbb{Z}^{>0})^\ell$ are the coefficients of the positive roots in terms of $\Delta$. The pairing $\langle \ , \ \rangle$ is the usual scalar product on $\mathbb{R}^\ell$.

*8.3*

We obtain a Dedekind sum by extending the sum (59) to the whole lattice. Let $Z^\ell \subset \mathbb{R}^r$ be the span of the first $\ell$ basis vectors, and let $\sigma = \sigma(\mathfrak{g})$ be an $r \times r$ integral matrix such that $\langle \sigma_i, x\rangle = \langle a_i, x\rangle$ for $x \in Z^\ell$, and such that $\det \sigma = 1$. Let $e = (2k, \ldots, 2k) \in \mathbb{R}^r$.

PROPOSITION 8.4 (cf. [21, page 507])
*We have*

$$(2\pi)^{-|e|}\zeta_{\mathfrak{g}}(2k) = \frac{M^{2k}}{\#W} S\big(Z^\ell, \sigma(\mathfrak{g}), e, 0\big).$$

Hence these special values can be computed in polynomial time using our techniques. We conclude with two examples: $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$.* We recommend verification of these formulas to the interested reader for a pleasant combinatorial exercise.

PROPOSITION 8.5
*Let $\zeta(s)$ be the Riemann zeta function. Then we have the following:*

$$\frac{6}{2^{2m}}\zeta_{\mathfrak{sl}_3}(2m) = 8 \sum_{\substack{0\le i\le 2m\\ i\equiv 0 \bmod 2}} \binom{4m-i-1}{2m-1}\zeta(i)\zeta(6m-i), \tag{60}$$

$$\frac{24}{12^{2m}}\zeta_{\mathfrak{sl}_4}(2m) = 16 \sum_{0\le i\le 2m} \binom{4m-i-1}{2m-1}(A+B+C+D), \tag{61}$$

*where*

$$A = \sum_{\substack{0\le j\le 2m\\ 0\le t\le 4m+i-j\\ j,t\equiv 0 \bmod 2}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-t-1}{2m-1}\zeta(j)\zeta(t)\zeta(12m-j-t), \tag{62}$$

$$B = \sum_{\substack{0\le j\le 2m\\ 0\le u\le 2m\\ j,u\equiv 0 \bmod 2}} \binom{2m+i-j-1}{i-1}\binom{6m+i-j-u-1}{4m+i-j-1}\zeta(j)\zeta(u)\zeta(12m-j-u), \tag{63}$$

*For $\mathfrak{g} = \mathfrak{sl}_n$, the corresponding Dedekind sum is unimodular but not diagonal. In fact, the number of vectors in $\sigma$ is the largest possible size of a configuration with this property.

$$C = \sum_{\substack{0\le k\le i\\ 0\le v\le 4m+i-k\\ k,v\equiv 0 \bmod 2}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-v-1}{2m-1}\zeta(k)\zeta(v)\zeta(12m-k-v), \tag{64}$$

$$D = \sum_{\substack{0\le k\le i\\ 0\le w\le 2m\\ k,w\equiv 0 \bmod 2}} \binom{2m+i-k-1}{i-k}\binom{6m+i-k-w-1}{4m+i-k-1}\zeta(k)\zeta(w)\zeta(12m-k-w). \tag{65}$$

*Remark 8.6*
Formula (60) was independently discovered by D. Zagier, S. Garoufalidis, and L. Weinstein (see [21, page 506]).

*Example 8.7*
Tables 3 and 4 show some special values of $\zeta_{\mathfrak{sl}_3}$ and $\zeta_{\mathfrak{sl}_4}$.

Table 3

| $2m$ | $(6m+1)!\cdot 6\cdot\zeta_{\mathfrak{sl}_3}(2m)/(2^{2m}\cdot(2\pi)^{6m})$ |
|---|---|
| 2 | $1/(2\cdot 3)$ |
| 4 | $19/(2\cdot 3\cdot 5)$ |
| 6 | $1031/(3\cdot 7)$ |
| 8 | $(11\cdot 43\cdot 751)/(2\cdot 7)$ |
| 10 | $(5\cdot 13\cdot 27739097)/(3\cdot 11)$ |
| 12 | $(17\cdot 29835840687589)/(3\cdot 5\cdot 7\cdot 13)$ |
| 14 | $(2\cdot 17\cdot 19\cdot 89\cdot 127\cdot 6353243297)/7$ |
| 16 | $(19\cdot 23\cdot 31\cdot 22113713266984288663)/(2\cdot 5^2\cdot 13\cdot 17)$ |

Table 4

| $2m$ | $(12m+1)! \cdot (6m+1) \cdot (4m+1) \cdot 24 \cdot \zeta_{\mathfrak{sl}_4}(2m)/(12^{2m} \cdot (2\pi)^{12m})$ |
|---|---|
| 2 | $23/2$ |
| 4 | $(3 \cdot 7 \cdot 14081)/2$ |
| 6 | $(757409 \cdot 23283173)/(5 \cdot 7)$ |
| 8 | $(3 \cdot 11 \cdot 1021 \cdot 5529809 \cdot 754075957)/2$ |
| 10 | $(13 \cdot 116763209 \cdot 1872391681 \cdot 3187203549787)/(5 \cdot 11)$ |
| 12 | $(17 \cdot 1798397149 \cdot 5509496891 \cdot 61272058469885714847 43)/(3 \cdot 7 \cdot 13)$ |

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