1 Introduction

1.1 Dedekind–Zeta function

Let K be a totally real number field. Let $\mathfrak{a}$ be an ideal inside the ring of integers $\mathrm{O_K}$. Consider the Dedekind-Zeta function:

$$\zeta_K(s) = \sum_{\mathfrak{b}} \frac{1}{\mathrm{Norm}(\mathfrak{b})^s},$$

along with the associated partial Dedekind–Zeta functions:

$$\zeta_K(\mathfrak{a}, s) = \sum_{\mathfrak{b} \sim \mathfrak{a}} \frac{1}{\mathrm{Norm}(\mathfrak{b})^s},$$

The first sum is taken with respect to all the integral ideals $\mathfrak{b}$ in $\mathrm{O_K}$. The second sum is taken with respect to all the integral ideals $\mathfrak{b}$ that are in the same class as $\mathfrak{a}$ in the narrow ray class group of K.

1.2 Motivating questions

Here's an overarching and sequential list of motivating questions:

- (**analytic continuation**) Do the zeta functions $\zeta_K(s)$ and $\zeta_K(\mathfrak{a}, s)$ have an analytic (or meromorphic) continuation to the entire complex plane?
- (**rationality**) Having established analytic continuation, can we conclude that the zeta values *at non-positive integers* are rational?
- (**integrality**) Having established rationality, for what primes p are these zeta-values p-integral?
- (**Kummer congruences**) Having established p-integrality, how do we construct *p-adic L-functions* associated to these zeta values?

1.3 Rationality

The main motivating question for our study group for this semester will be the question of *rationality* of these zeta values. This is known as the *Siegel–Klingen* rationality theorem. There are three celebrated proofs of this result, attributed to:

1. Siegel–Klingen (*constant terms of Eisenstein series*),
2. Shintani, and
3. Sczech (*cohomological*).

Our goal will be to read the methods of **Shintani** and **Sczech**. As a reminder to our eventual goal of studying p-adic L-functions, it is worth noting that in this set up, the constructions of the p-adic L-functions by

1. Deligne–Ribet [4],
2. Cassou–Noguès [1, 2], and
3. Charollois–Dasgupta [3].

rely on the methods of Siegel–Klingen, Shintani and Sczech respectively.

$$\text{Siegel–Klingen} \rightarrowtail \text{Deligne–Ribet}$$

$$\text{Shintani} \rightarrowtail \text{Cassou–Noguès}$$

$$\text{Sczech} \rightarrowtail \text{Charollois–Dasguptas}$$

References for Shintani's method

1. Shintani's article [8]
2. Hida's book [5]

References for Sczech's method

1. Sczech's inventiones article [6].
2. Sczech's Kyushu article [7].

2 Plan for the study group

The seminar will meet on **Tuesdays, 10:30 am**.

Date	Speaker	Section	Su
18 Jan	Bharath	Introduction	Di
25 Jan	Bharath	[8, Theorem 1] $\implies$ Siegel–Klingen rationality	Pa
31 Jan	Shaunak	[8, Proposition 1 and Corollary to Proposition 1]	Pa
8 Feb	Radhika	[8, Lemma 2 and Corollary to Lemma 2]	Pa
15 Feb	Mihir	[8, Lemma 3 and Proposition 4]	Pa
22 Feb	Mahesh	[8, Proposition 4 and Theorem 1]	Pa

References

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[4] P. Deligne and K. A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[5] H. Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[6] R. Sczech. Eisenstein group cocycles for GL_n and values of L-functions. *Invent. Math.*, 113(3):581–616, 1993.

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