1 Introduction

1.1 Dedekind–Zeta function

Let K be a totally real number field. Let $\mathfrak{a}$ be an ideal inside the ring of integers $\mathrm{O_K}$. Consider the Dedekind-Zeta function:

$$\zeta_K(s) = \sum_{\mathfrak{b}} \frac{1}{\mathrm{Norm}(\mathfrak{b})^s},$$

along with the associated partial Dedekind–Zeta functions:

$$\zeta_K(\mathfrak{a}, s) = \sum_{\mathfrak{b}\sim\mathfrak{a}} \frac{1}{\mathrm{Norm}(\mathfrak{b})^s},$$

The first sum is taken with respect to all the integral ideals $\mathfrak{b}$ in $\mathrm{O_K}$. The second sum is taken with respect to all the integral ideals $\mathfrak{b}$ that are in the same class as $\mathfrak{a}$ in the narrow ray class group of K.

1.2 Motivating questions

Here's an overarching and sequential list of motivating questions:

- (**analytic continuation**) Do the zeta functions $\zeta_K(s)$ and $\zeta_K(\mathfrak{a}, s)$ have an analytic (or meromorphic) continuation to the entire complex plane?
- (**rationality**) Having established analytic continuation, can we conclude that the zeta values *at non-positive integers* are rational?
- (**integrality**) Having established rationality, for what primes p are these zeta-values p-integral?
- (**Kummer congruences**) Having established p-integrality, how do we construct *p-adic L-functions* associated to these zeta values?

1.3 Rationality

The main motivating question for our study group for this semester will be the question of *rationality* of these zeta values. This is known as the *Siegel–Klingen* rationality theorem. There are three celebrated proofs of this result, attributed to:

1. Siegel–Klingen (*constant terms of Eisenstein series*),
2. Shintani, and
3. Sczech (*cohomological*).

Our goal will be to read the methods of **Shintani** and **Sczech**. As a reminder to our eventual goal of studying p-adic L-functions, it is worth noting that in this set up, the constructions of the p-adic L-functions by

1. Deligne–Ribet [6],
2. Cassou–Noguès [1, 2], and
3. Charollois–Dasgupta [3].

rely on the methods of Siegel–Klingen, Shintani and Sczech respectively.

$$\text{Siegel–Klingen} \rightarrowtail \text{Deligne–Ribet}$$

$$\text{Shintani} \rightarrowtail \text{Cassou–Noguès}$$

$$\text{Sczech} \rightarrowtail \text{Charollois–Dasguptas}$$

2 Plan for the study group: Shintani's method

References for Shintani's method

1. Shintani's article [11]
2. Hida's book [8]

The study group will meet on **Tuesdays, 10:30 am**.

Date	Speaker	Section	Su
18 Jan	Bharath	Introduction	Di
25 Jan	Bharath	Theorem 1 in [11] $\implies$ Siegel–Klingen rationality	Pa
1 Feb	Shaunak	Proposition 1 and Corollary to Proposition 1 in [11]	Pa
8 Feb	Radhika	Lemma 2 and Corollary to Lemma 2 in [11]	Pa
15 Feb	Mihir	Lemma 3 and Proposition 4 in [11]	Pa
22 Feb	Mahesh	Proposition 4 and Theorem 1 in [11]	Pa

3 Plan for the study group: Sczech's method

References for Sczech's method

1. Sczech's Inventiones article [10].

2. Sczech's Commentarii article [9].

In our talks, we will have to take the Q-limit formula (Theorem 2 in [10], Theorem 1 in [9]) for granted.

Date	Speaker	Section
Mar 1	Bharath	Introduction + Eisenstein co cycle
Mar 8	Bharath	Limit formula + Trignometri and Bernoulli cocycles
Mar 15	Radhika	L-functions in real quadrati fields
Mar 22	Mahesh	Properties of the rational co cycle and the Eisenstein cocy cle
Mar 29	Mahesh	Properties of the rational co cycle and the Eisenstein cocy cle
Apr 5	Shaunak	Special values of L-series i terms of Eisenstein cocycle
Apr 12	Shaunak	Proof of Lemma 6
Apr 19	Mihir	A finite expression for th Eisenstein cocycle and ratic nality

More references:

1. An article by Gunnells and Sczech [7].
2. An article by Charollois and Sczech[4].
3. An article by Chinta, Gunnells and Sczech [5].
4. An article by Charollois–Dasgupta [3].

References

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[4] P. Charollois and R. Sczech. Elliptic functions according to Eisenstein and Kronecker: an update. *Eur. Math. Soc. Newsl.*, (101):8–14, 2016.

[5] G. Chinta, P. E. Gunnells, and R. Sczech. Computing special values of partial zeta functions. In *Algorithmic number theory (Leiden, 2000)*, volume 1838 of *Lecture Notes in Comput. Sci.*, pages 247–256. Springer, Berlin, 2000.

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[9] R. Sczech. Eisenstein cocycles for $\mathrm{GL}_2\mathbf{Q}$ and values of L-functions in real quadratic fields. *Comment. Math. Helv.*, 67(3):363–382, 1992.

[10] R. Sczech. Eisenstein group cocycles for $\mathrm{G}L_n$ and values of L-functions. *Invent. Math.*, 113(3):581–616, 1993.

[11] T. Shintani. On evaluation of zeta functions of totally real algebraic number fields at non-positive integers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 23(2):393–417, 1976.