On evaluation of zeta functions of totally real algebraic number fields at non-positive integers

Dedicated to Professor C. L. Siegel for his eightieth birthday

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Introduction

0-1. Let F be a totally real algebraic number field of degree n. For mutually prime integral ideals $\mathfrak{b}$ and $\mathfrak{f}$ of F, set $\zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\mathfrak{g}} N(\mathfrak{g})^{-s}$, where the summation is over all integral ideals $\mathfrak{g}$ of F which are in the same narrow ray class modulo $\mathfrak{f}$ as $\mathfrak{b}$. In his papers [8], [9] and [10], C. L. Siegel established an algorithm to compute the values of $\zeta(\mathfrak{b}, \mathfrak{f}, s)$ at non-positive integers (which turn out to be rational). In particular, for $\mathfrak{f} = \mathfrak{o}_F$ (the ring of integers of F), he obtained a striking explicit formula (see (22) of [9]). His method is based on the theory of elliptic modular forms. In this paper, we present a different method of evaluating the special values of these zeta functions.

For an $r \times n$ $(1 \leq r \leq n)$ matrix $A = (a_{jl})$ with non-zero entries and an r-tuple $x = (x_1, \cdots, x_r)$ of complex numbers, let $(m!)^{-n} B_m(A, x)^{(k)}$ $(m = 1, 2, \cdots;\ 1 \leq k \leq n)$ be the coefficient of $u^{n(m-1)}(t_1 \cdots t_{k-1} t_{k+1} \cdots t_n)^{m-1}$ in the Laurent expansion at the origin of the function

$$\prod_{j=1}^{r} \frac{\exp(u x_j L_j(t))}{\exp(u L_j(t)) - 1}\bigg|_{t_k=1},$$

where $u, t_1, \cdots, t_n$ are independent variables and L_j is a linear form in $t_1, \cdots, t_n$ given by

$$L_j(t) = a_{j1} t_1 + \cdots + a_{jn} t_n \qquad (1 \leq j \leq r).$$

Set $B_m(A, x) = \sum_{k=1}^{n} B_m(A, x)^{(k)}/n$. We note that for $r = n = 1$ and $A = a$, $B_m(a, x)$ is given by $a^{m-1} B_m(x)$ where $B_m(x)$ is the usual m-th Bernoulli polynomial.

For linearly independent vectors $v_1, v_2, \cdots, v_r$ in the n-dimensional real vector space $\boldsymbol{R}^n$, we denote by $C(v_1, \cdots, v_r)$ the *open simplicial cone* with generators $v_1, \cdots, v_r$. More precisely, $C(v_1, \cdots, v_r)$ is the subset of $\boldsymbol{R}^n$ consisting of all linear combinations of $v_1, \cdots, v_r$ with *positive* coefficients.

For an $x \in F$, we denote by $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ the n conjugates of x with respect to the rational number field. We embed F into $\boldsymbol{R}^n$ via the mapping:

$$x \longrightarrow (x^{(1)}, x^{(2)}, \cdots, x^{(n)}).$$

Then, via componentwise multiplication, $F-\{0\}$ acts on R^n as a group of linear transformations. Denote by $E(\mathfrak{f})_+$ the group of totally positive units of F which are congruent to 1 modulo $\mathfrak{f}$. It is shown that the fundamental domain of R_+^n with respect to the action of $E(\mathfrak{f})_+$ is a disjoint union of a *finite* number of open simplicial cones *with generators in* $\mathfrak{f}$. More precisely, we have, for a suitable choice of a finite number of open simplicial cones C_j $(j \in J)$,

$$R_+^n = \bigcup_{j \in J} \bigcup_{u \in E(\mathfrak{f})_+} uC_j \qquad \text{(disjoint union)}, \tag{0.1}$$

where J is a *finite* set of indices and $C_j = C_j(v_{j1}, \cdots, v_{jr(j)})$ is an open simplicial cone with generators $v_{j1}, \cdots, v_{jr(j)} \in \mathfrak{f}$. For each $j \in J$, and for each subset S of F, let $R(j, S)$ be the set of all $r(j)$-tuples $x = (x_1, \cdots, x_{r(j)})$ of *rational* numbers such that

$$\begin{cases} 0 < x_1, \cdots, x_{r(j)} \leq 1 \quad \text{and} \\ x_1 v_{j1} + x_2 v_{j2} + \cdots + x_{r(j)} v_{jr(j)} \in S. \end{cases} \tag{0.2}$$

It is easy to see that $R(j, S)$ is a *finite* set if S is a subset of a fractional ideal of F. Furthermore, denote by A_j the $r(j) \times n$ matrix given by

$$A_j = \begin{pmatrix} v_{j1}^{(1)} & v_{j1}^{(2)} & \cdots & v_{j1}^{(n)} \\ \vdots & \vdots & & \vdots \\ v_{jr(j)}^{(1)} & v_{jr(j)}^{(2)} & \cdots & v_{jr(j)}^{(n)} \end{pmatrix}.$$

Then we have:

THEOREM 1. *Notations being as above,*

$$\zeta(\mathfrak{b}, \mathfrak{f}, 1-m) = m^{-n} N(\mathfrak{b})^{m-1} \sum_{j \in J} \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} (-1)^{r(j)} B_m(A_j, x),$$

where $\mathfrak{b}^{-1}\mathfrak{f}+1$ *is the set consisting of all* $x \in F$ *such that* $x-1 \in \mathfrak{b}^{-1}\mathfrak{f}$.

0-2. Denote by $\mathfrak{o}$ the ring of integers in F and set $E_+ = E(\mathfrak{o})_+$. Choose and fix a system $\{C_j(v_{j1}, \cdots, v_{jr(j)});\ j \in J\}$ of a finite number of simplicial cones with generators in $\mathfrak{o}$ which satisfies (0.1) for $\mathfrak{f} = \mathfrak{o}$. Let K be a totally imaginary quadratic extension of F with the relative discriminant $\mathfrak{d}$. Let χ be the quadratic character of the group of narrow ideal classes with the conductor $\mathfrak{d}$ which corresponds to the extension K in class field theory. Let $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_h$ be a complete set of representatives of ideal classes of F.[0] Denote by R_F (resp. R_K) the regulator of F (resp. K) and denote by w the number of roots of unity in K. Then we have

[0] Ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_h$ are all assumed to be integral.

THEOREM 2. *Notations and assumptions being as above, the relative class number H/h of K with respect to F is given by the following formula:*

$$\frac{H}{h}=2^{n-1}\frac{wR_F}{R_K[E,E_+]}\sum_{m=1}^{h}\sum_{j\in J}\sum_{x\in R(j,(\mathfrak{a}_m\mathfrak{d})^{-1})}\chi\left(\left(\sum_{k=1}^{r(j)}x_k v_{jk}\right)\mathfrak{a}_m\mathfrak{d}\right)$$
$$\times\frac{(-1)^{r(j)}}{n}\sum_{l}\prod_{k=1}^{r(j)}\frac{B_{l_k}(x_k)}{l_k!}\,\mathrm{tr}\left(\prod_{k=1}^{r(j)}v_{jk}^{l_k-1}\right),$$

where the summation with respect to l is over all $r(j)$-tuples $l=(l_1,\cdots,l_{r(j)})$ of non-negative integers which satisfy

$$l_1+l_2+\cdots+l_{r(j)}=r(j),$$

$[E,E_+]$ is the group index of E_+ in the group E of all units of F.

We note that if $[E,E_+]=2^n$, then $2^{n-1}(R_F/R_K[E,E_+])=2^{-n}$. The above formula may be regarded as *an affirmative answer to the Hecke conjecture that the relative class number of K with respect to F admits an elementary arithmetic expression in terms of the relative discriminant $\mathfrak{d}$.*

0-3. This paper consists of two sections. The first section is divided into five subsections. In 1, we evaluate the Dirichlet series:

$$\sum_{z_1,\cdots,z_r=0}^{\infty}\prod_{j=1}^{n}L_j^*(z_1+x_1,\cdots,z_r+x_r)^{-s}, \tag{0.3}$$

where $L_1^*,\cdots,L_j^*$ are linear forms with positive coefficients and $x_1,\cdots,x_r$ are positive numbers, at non-positive integers. In 2 and 3, we prove that a finite system of simplicial cones with the property (0.1) is available for any totally real field F. This enables us to transform the zeta-function $\zeta(\mathfrak{b},\mathfrak{f},s)$ into finite linear combinations of Dirichlet series of type (0.3) (cf. Zagier [12]). In 4 and 5, Theorem 1 and Theorem 2 are proved. The second section consists of three subsections. In 1 we will show that, if F is real quadratic, Theorem 1 is equivalent to Satz 1 of Siegel [8]. An application of our result to continued fractions of quadratic irrationalities is given in 2. In 3, a numerical example is discussed.

Recently, in [3], Hida discussed the evaluation of zeta-functions of totally real algebraic number fields. His method is based on Siegel's formula (22) of [9] and his results are of quite different nature from those of ours.

Notation. As usual, we denote by $\boldsymbol{Z},\boldsymbol{Q},\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field respectively. The set of positive real numbers is denoted by $\boldsymbol{R}_+$. We denote by $\Gamma(s),\zeta(s)$ and by $B_m(x)$ the gamma function, the Riemann zeta function and the

m-th Bernoulli polynomial, respectively.

§1.

1. Let A be an $r\times n$ matrix $(r\leq n)$ with *positive* entries a_{jk} $(1\leq j\leq r, 1\leq k\leq n)$. Denote by L_j $(j=1,\cdots,r)$ (resp. L_k^* $(k=1,\cdots,n)$) a linear form in n (resp. r) variables given by $L_j(t_1,\cdots,t_n)=\sum_{k=1}^{n}a_{jk}t_k$ (resp. $L_k^*(z_1,\cdots,z_r)=\sum_{j=1}^{r}a_{jk}z_j$). For an r-tuple $x=(x_1,x_2,\cdots,x_r)$ of *positive* real numbers and an r-tuple $\chi=(\chi_1,\chi_2,\cdots,\chi_r)$ of non-zero complex numbers with *modulus not larger than* 1, we denote by $\zeta(s,A,x,\chi)$ a Dirichlet series in s given by the following formula:

$$\zeta(s,A,x,\chi)=\sum_{z_1,\cdots,z_r=0}^{\infty}\prod_{k=1}^{r}\chi_k^{z_k}\prod_{j=1}^{n}L_j^*(z+x)^{-s}$$
$$=\sum_{z_1,\cdots,z_r=0}^{\infty}\prod_{k=1}^{r}\chi_k^{z_k}\prod_{j=1}^{n}\left\{\sum_{l=1}^{r}a_{lj}(z_l+x_l)\right\}^{-s}. \tag{1.1}$$

PROPOSITION 1. *Notations being as above, the Dirichlet series* $\zeta(s,A,x,\chi)$ *given by* (1.1) *is absolutely convergent if* $\operatorname{Re}s>r/n$ *and has an analytic continuation to a meromorphic function in the whole complex plane. Moreover, if one puts* $1-x=(1-x_1,1-x_2,\cdots,1-x_r)$, *the value at* $s=1-m$ $(m=1,2,\cdots)$ *is evaluated as follows:*

$$\zeta(1-m,A,x,\chi)=(-1)^{n(m-1)}m^{-n}\sum_{k=1}^{n}B_m(A,1-x,\chi)^{(k)}/n,$$

where $(m!)^{-n}B_m(A,y,\chi)^{(k)}$ *is the coefficient of* $u^{(m-1)n}(t_1\cdots t_{k-1}t_{k+1}\cdots t_n)^{m-1}$ *in the Laurent expansion at the origin of the function*

$$\prod_{j=1}^{r}\frac{\exp(uy_jL_j(t))}{\exp(uL_j(t))-\chi_j}\Bigg|_{t_k=1}$$

in $u,t_1,\cdots,t_{k-1},t_{k+1},\cdots,t_n$.

PROOF. It is easy to see that the Dirichlet series is absolutely convergent for $\operatorname{Re}s>r/n$. If $\operatorname{Re}s>r/n$,

$$\Gamma(s)^n\prod_{k=1}^{n}L_k^*(z+x)^{-s}=\int_0^{\infty}\cdots\int_0^{\infty}\exp\left\{-\sum_{k=1}^{n}t_kL_k^*(z+x)\right\}(t_1\cdots t_n)^{s-1}dt_1\cdots dt_n.$$

Since $\sum_{k=1}^{n}t_kL_k^*(z+x)=\sum_{j=1}^{r}(z_j+x_j)L_j(t)$, we have, for $\operatorname{Re}s>r/n$,

$$\Gamma(s)^n\zeta(s,A,x,\chi)=\int_0^{\infty}dt_1\cdots\int_0^{\infty}dt_n\prod_{j=1}^{r}\frac{\exp((1-x_j)L_j(t))}{\exp(L_j(t))-\chi_j}(t_1\cdots t_n)^{s-1}.$$

Denote by D_k $(k=1,2,\cdots,n)$ the subset of $\boldsymbol{R}^n$ given as follows:

$$D_k=\{t\in \boldsymbol{R}^n;\ 0\leq t_l\leq t_k,\ l=1,\cdots,k-1,k+1,\cdots,n\}.$$

For simplicity, set

$$g(t)=g(t_1,\cdots,t_n)=\prod_{j=1}^{r}\frac{\exp\,((1-x_j)L_j(t))}{\exp\,(L_j(t))-\chi_j}.$$

It is obvious that

$$\begin{aligned}(1.2)\qquad \zeta(s,A,x,\chi)&=\Gamma(s)^{-n}\int_0^\infty\cdots\int_0^\infty g(t)(t_1\cdots t_n)^{s-1}dt_1\cdots dt_n\\ &=\Gamma(s)^{-n}\sum_{k=1}^{n}\int_{D_k}g(t)(t_1\cdots t_n)^{s-1}dt_1\cdots dt_n.\end{aligned}$$

In D_k, we make the following change of variables:

$$t=uy=u(y_1,y_2,\cdots,y_n),$$

where $0<u$, $0\leq y_l\leq 1$ for $l\neq k$ and $y_k=1$. Then we have

$$\begin{aligned}(1.3)\qquad &\Gamma(s)^{-n}\int_{D_k}g(t)(t_1\cdots t_n)^{s-1}dt_1\cdots dt_n\\ &=\Gamma(s)^{-n}\int_0^\infty du\int_0^1\cdots\int_0^1 g(uy)u^{ns-1}(\prod_{l\neq k}y_l)^{s-1}\prod_{l\neq k}dy_l.\end{aligned}$$

For a positive number $\varepsilon<1$, denote by $I_\varepsilon(1)$ (resp. $I_\varepsilon(+\infty)$) the integral path in $\boldsymbol{C}$ consisting of the interval $[1,\varepsilon]$ (resp. $[+\infty,\varepsilon]$), counterclockwise circle of radius ε around the origin and of the interval $[\varepsilon,1]$ (resp. $[\varepsilon,+\infty]$).

Since $L_1,L_2,\cdots,L_r$ are linear forms with *positive* coefficients, for sufficiently small ε, the right side of (1.3) is equal to

$$\frac{\Gamma(s)^{-n}}{(e^{2n\pi is}-1)(e^{2\pi is}-1)^{n-1}}\int_{I_\varepsilon(+\infty)}du\int_{I_\varepsilon(1)^{n-1}}g(uy)u^{ns-1}(\prod_{l\neq k}y_l)^{s-1}\prod_{l\neq k}dy_l.$$

It is easy to see that, as a function of s, the above integral is meromorphic in the whole complex plane. Moreover, since

$$\Gamma(s)^{-n}(e^{2n\pi is}-1)^{-1}(e^{2\pi is}-1)^{1-n}=(2\pi i)^{-n}\Gamma(1-s)^n(e^{2\pi is}-1)(e^{2n\pi is}-1)^{-1}e^{-n\pi is},$$

the value of the integral at $s=1-m$ is equal to $(-1)^{n(m-1)}\Gamma(m)^n/n$ times the coefficient of $u^{n(m-1)}(\prod_{l\neq k}y_l)^{m-1}$ in the Laurent expansion at the origin of the function

$$g(uy_1,uy_2,\cdots,uy_{k-1},u,uy_{k+1},\cdots,uy_n)=\prod_{j=1}^{r}\frac{\exp\,(u(1-x_j)L_j(y))}{\exp\,(uL_j(y))-\chi_j}\bigg|_{y_k=1},$$

which is holomorphic in $u,t_1,\cdots,t_{k-1},t_{k+1},\cdots,t_n$ in the direct product of n copies of the disk with radius ε punctured at the origin. Thus, the integral (1.3) is, for

$s=1-m$, equal to $(-1)^{n(m-1)}m^{-n}B_m(A, 1-x, \chi)^{(k)}/n$. Thus, it follows from (1.2) that

$$\zeta(1-m, A, x, \chi)=(-1)^{n(m-1)}m^{-n}n^{-1}\sum_{k=1}^{n}B_m(A, 1-x, \chi)^{(k)}.$$

For $\chi=(\overbrace{1, 1, \cdots, 1}^{r})$, we put

$$\tag{1.4} \begin{cases} \zeta(s, A, x)=\zeta(s, A, x, \chi), \\ B_m(A, x)^{(k)}=B_m(A, x, \chi)^{(k)} \quad \text{and} \\ B_m(A, x)=\sum_{k=1}^{n}B_m(A, x)^{(k)}/n. \end{cases}$$

It is easy to see that $B_m(A, 1-x)=(-1)^{n(m-1)+r}B_m(A, x)$.

COROLLARY TO PROPOSITION 1. *The value of the Dirichlet series*

$$\zeta(s, A, x)=\sum_{z_1,\cdots,z_r=0}^{\infty}\prod_{k=1}^{n}L_k^*(z+x)^{-s} \quad at \quad s=1-m \ \ (m=1, 2, \cdots)$$

is equal to $(-1)^r m^{-n}B_m(A, x)$ *and*

$$\frac{B_m(A, x)}{(m!)^n}=\sum_{p}\frac{B_{p_1}(x_1)\cdots B_{p_r}(x_r)}{p_1!p_2!\cdots p_r!}c(A, p)+\frac{1}{n}\sum_{S}\sum_{q}\left\{\prod_{j\in S}\frac{B_{q(j)}(x_j)}{q(j)!}\right\}\sum_{k=1}^{n}c(S, q, A)^{(k)}$$

($B_k(t)$ is the usual k-th Bernoulli polynomial), where the summation with respect to p is taken over all r-tuples of positive integers $p=(p_1, p_2, \cdots, p_r)$ which satisfy $p_1+\cdots+p_r=n(m-1)+r$, $C(A, p)$ is the coefficient of $(t_1\cdots t_n)^{m-1}$ in the polynomial $\prod_{j=1}^{r}L_j(t)^{p_j-1}$, the summation with respect to S is taken over all the proper and non-empty subsets of indices $\{1, 2, \cdots, n\}$, for each S, the summation with respect to q is over all the mappings from S to the set of positive integers which satisfy $\sum_{j\in S}q(j)=n(m-1)+r$, $c(S, q, A)^{(k)}$ is the coefficient of $(t_1\cdots t_{k-1}t_{k+1}\cdots t_n)^{m-1}$ in the Taylor expansion of the function $\prod_{j\in S}L_j(t)^{q(j)-1}/\prod_{j\notin S}L_j(t)\big|_{t_k=1}$ at the origin.

2. Let V be an n-dimensional real vector space. For $\boldsymbol{R}$-linearly independent vectors $v_1, \cdots, v_i$ of V, we denote by $C(v_1, \cdots, v_i)$ the i-dimensional *open simplicial cone* given as follows:

$$C(v_1, \cdots, v_i)=\{t_1v_1+\cdots+t_iv_i;\ t_1, \cdots, t_i\in\boldsymbol{R}_+\}.$$

We call $v_1, \cdots, v_i$ the *generators* of the open simplicial cone C. It is easy to see that generators of an open simplicial cone are unique up to mutual permutations and multiplications by positive scalars. A closed (resp. open) *half space* in V is a subset of V of the form $\{x\in V;\ L(x)\geq 0\}$ (resp. $\{x\in V;\ L(x)>0\}$), where L is a non-zero $\boldsymbol{R}$-linear form on V. A *polyhedral cone* is the intersection of a finite number of (open or closed) half spaces. It is said to be a closed polyhedral cone if it is given as an intersection of a finite number of closed half spaces. A closed polyhedral cone is said to be *proper* if it has a point other than the origin. Let k be a subfield of the real field and assume that a k-structure is assigned to V. Namely, an n-dimensional k-vector space V_k is embedded in V so that $V=V_k\underset{k}{\otimes}\boldsymbol{R}$. An $\boldsymbol{R}$-linear form on V is said to be *k-rational* if it is k-valued on V_k. An open simplicial cone C is said to be *k-rational* if, for a suitable choice of generators, all the generators of C are in V_k. The k-rationality of a polyhedral cone is defined in a similar manner.

LEMMA 2. *A proper k-rational closed polyhedral cone is a disjoint union of a finite number of k-rational open simplicial cones and the origin.*

PROOF. Let P be the proper closed k-rational polyhedral cone given as follows:

$$P=\{x\in V;\ L_i(x)\geq 0\ (i=1,2,\cdots,m)\},$$

where $L_1,\cdots,L_m$ are non-zero k-rational linear forms on V. If $n=1$ or 2, the lemma is obvious. Assume that the lemma has been proved when the dimension of V is smaller than n. If P has not an interior point, there is a linear form L among $L_1,L_2,\cdots,L_m$ such that P is in the hyperplane $\{x\in V;\ L(x)=0\}$. In this case, the lemma follows easily from the induction hypothesis. Now assume that P has an interior point u. Since V_k is dense in V, we may assume that u is in V_k. We have $L_1(u)>0, L_2(u)>0,\cdots,L_m(u)>0$. For each i $(1\leq i\leq m)$, set $\partial_iP=\{x\in P;\ L_i(x)=0\}$. If $\partial_iP\neq\{0\}$, by the induction hypothesis, $\partial_iD-\{0\}$ is a disjoint union of a finite number of k-rational open simplicial cones of dimensions smaller than n. It is easy to see that if a simplicial cone C in ∂_iP has a non-empty intersection with $\partial_j(P)$ $(j\neq i)$, then C is contained in $\partial_i(P)\cap\partial_j(P)$. Hence $\partial_1(P)\cup\partial_2(P)\cup\cdots\cup\partial_m(P)-\{0\}$ is a disjoint union of a finite number of k-rational simplicial cones of dimensions smaller than n. We have

$$\partial_1P\cup\partial_2P\cup\cdots\cup\partial_m(P)-\{0\}=\bigcup_{j\in J}C_j\qquad\text{(disjoint union)},$$

where $C_j=C(v_1,v_2,\cdots,v_{d_j})$ $(v_1,v_2,\cdots,v_{d_j}\in V_k)$ is a k-rational open simplicial cone of dimension $d_j<n$ and J is a finite set of indices. For each $C_j=C(v_1,\cdots,v_{d_j})$ $(j\in J)$, set $C_j(u)=C(v_1,\cdots,v_{d_j},u)$. Then $C_j(u)$ is a k-rational (d_j+1)-dimensional open simplicial cone. We claim that

$$P-\{0\}=\bigcup_{j\in J}C_j\cup\bigcup_{j\in J}C_j(u)\cup\boldsymbol{R}_+u\qquad\text{(disjoint union)}.$$

In fact, if $x\in P-\{0\}$ is on the boundary of P, it is in some of $\partial_1 P, \partial_2 P, \cdots, \partial_m P$. Hence $x\in \bigcup_{j\in J} C_j$. If x is in the interior of P, $L_1(x), L_2(x), \cdots, L_m(x)$ are all positive. If x is a scalar multiple of u, $x\in \boldsymbol{R}_+ u$. Assume that x is not a scalar multiple of u. Denote by s the minimum of $L_1(x)/L_1(u), \cdots, L_{m-1}(x)/L_{m-1}(u)$ and of $L_m(x)/L_m(u)$. Then s is positive and $x-su$ is on the boundary of P. Since it is not the origin, there exists a unique $j\in J$ such that $x-su\in C_j$. Hence, there exists a unique $j\in J$ such that $x\in C_j(u)$. Thus,

$$P-\{0\}=\bigcup_{j\in J} C_j \cup \bigcup_{j\in J} C_j(u)\cup \boldsymbol{R}_+ u \qquad \text{(disjoint union)}.$$

COROLLARY TO LEMMA 2. *A k-rational polyhedral cone is, if it is neither closed nor empty, a disjoint union of a finite number of k-rational simplicial cones.*

PROOF. Set

$$P=\{x\in V;\ L_i(x)\geq 0\ (1\leq i\leq m),\ M_j(x)>0\ (1\leq j\leq l)\},$$

where L_i and M_j are non-zero k-rational linear forms on V. Further, set

$$\bar{P}=\left\{x\in V;\ \begin{array}{ll} L_i(x)\geq 0 & (1\leq i\leq m) \\ M_j(x)\geq 0 & (1\leq j\leq l) \end{array}\right\}.$$

If P is not closed and non-empty, $\bar{P}$ is a proper closed polyhedral cone. Hence, $\bar{P}-\{0\}$ is a disjoint union of a finite number of k-rational open simplicial cones. For each j $(1\leq j\leq l)$, set $\partial_j\bar{P}=\{x\in\bar{P};\ M_j(x)=0\}$. If a simplicial cone, which is contained in $\bar{P}$, has a non-empty intersection with $\partial_j\bar{P}$, it is contained in $\partial_j\bar{P}$. Since $P=\bar{P}-\bigcup_{j=1}^{l}\partial_j\bar{P}$, P is also a disjoint union of a finite number of k-rational open simplicial cones.

3. In this paragraph, we set $V=\boldsymbol{R}^n$ and regard V as an $\boldsymbol{R}$-algebra (summation and multiplication are defined componentwise).

Let F be a *totally real algebraic number field* of degree n. For each $x\in F$, we denote by $x^{(1)}, x^{(2)}, \cdots, x^{(n)}$ the n distinct embeddings of F into $\boldsymbol{R}$. We identify F with a $\boldsymbol{Q}$-subalgebra of V via the mapping: $x\mapsto (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$. We fix the $\boldsymbol{Q}$-structure of V such that $V_{\boldsymbol{Q}}=F$. For $x\in V$, we put $\operatorname{tr} x=x_1+x_2+\cdots+x_n$. We set $V_+=\boldsymbol{R}_+^n\subset V$ and denote by E_+ the group of all totally positive units of F. Set

$$\bar{D}=\{x\in V;\ \operatorname{tr} x\leq \operatorname{tr} xu \quad \text{for any} \quad u\in E_+\}.$$

Further, let D be the interior of $\bar{D}$.

LEMMA 3.

(i) *The set $\bar{D}$ is a closed $\boldsymbol{Q}$-rational polyhedral cone in V and*

$$V_+ \cup \{0\} = \bigcup_{u \in E_+} u\bar{D}.$$

(ii) *There are only a finite number of* $u \in E_+$ *such that*

$$u\bar{D} \cap \bar{D} \neq \{0\}.$$

Moreover

$$uD \cap D = \varnothing \quad \textit{if} \quad u \neq 1.$$

PROOF. First we will show that $\bar{D} - \{0\} \subset V_+$. For each *proper* non-empty subset S of the set of n indices $\{1, 2, 3, \cdots, n\}$, there exists a totally positive unit $u(S)$ of F which satisfies $u(S)^{(i)} > 1$ for $i \in S$ and $u(S)^{(i)} < 1$ for $i \notin S$. For each S, we choose such a unit $u(S)$. For each j $(1 \leq j \leq n)$, take a totally positive unit u_j which satisfies $u_j^{(j)} > n$. Set

$$X = \left\{ x \in V; \quad \begin{matrix} \operatorname{tr} x \leq \operatorname{tr} u(S)x & \varnothing \neq {}^{\forall} S \subsetneqq \{1, 2, \cdots, n\} \\ \operatorname{tr} x \leq \operatorname{tr} u_j x & 1 \leq j \leq n \end{matrix} \right\}.$$

Take an $x \in X - \{0\}$ and set $S = \{i;\ x_i \leq 0\}$. Assume $S = \{1, 2, \cdots, n\}$ and let t be the maximum of $-x_1, -x_2, \cdots$, and of $-x_n$. Then t is positive and $t = -x_j$ for some j. Then $\operatorname{tr} xu_j \leq -tu_j^{(j)} < -nt$ while $\operatorname{tr} x \geq -nt$. This is impossible since $\operatorname{tr} x \leq \operatorname{tr} xu_j$. Next assume S to be a non-empty proper subset of $\{1, 2, \cdots, n\}$. Then

$$\operatorname{tr} u(S)x - \operatorname{tr} x = \sum_{i \in S} (u(S)^{(i)} - 1)x_i + \sum_{i \notin S} (u(S)^{(i)} - 1)x_i < 0.$$

This is impossible. Thus $X - \{0\} \subset V_+$. Since $\bar{D}$ is a subset of X, $\bar{D} \subset V_+ \cup \{0\}$. For each i $(1 \leq i \leq n)$, set $S_i = \{i\}$ and take a positive number t_i which satisfies

$$t_i \geq 1 + \{u(S_i)^{(i)} - 1\}\{1 - u(S_i)^{(j)}\}^{-1}$$

for any $j \neq i$. Take a $u \in E_+$. If $u^{(i)} > t_i$ for some i, then, for any $x \in X - \{0\}$ $\operatorname{tr} xu - x_i > (t_i - 1)x_i$. Since $(u(S_i)^{(i)} - 1)x_i \geq \sum_{j \neq i} (1 - u(S_i)^{(j)})x_j$,

$$(t_i - 1)x_i \geq \sum_{j \neq i} x_j. \tag{1.5}$$

Thus $\operatorname{tr} ux > \operatorname{tr} x$.

Let N be the set consisting of all totally positive units u which satisfy $u^{(i)} \leq t_i$ for $i = 1, 2, \cdots, n$. Set $M = N \cup \{u_1, u_2, \cdots, u_n\} \cup \{u(S);\ \varnothing \neq S \subsetneqq \{1, 2, \cdots, n\}\}$. Then M is a finite subset of E_+. We have proved that

$$\bar{D} = \{x \in V;\ \operatorname{tr} x \leq \operatorname{tr} ux, \quad {}^{\forall} u \in M\}$$

and

$$\begin{aligned} D &= \{x \in V;\ \operatorname{tr} x < \operatorname{tr} ux, \quad 1 \neq {}^{\forall} u \in M\} \\ &= \{x \in V;\ \operatorname{tr} x < \operatorname{tr} ux, \quad 1 \neq {}^{\forall} u \in E_+\}. \end{aligned} \tag{1.6}$$

Thus $\bar{D}$ is a closed polyhedral cone. It follows easily from (1.6), that $D\cap uD=\emptyset$ for $1\neq u\in E_+$. For each $x\in V_+$, there exists a $u\in E_+$ such that $\operatorname{tr} ux\leq \operatorname{tr} vx$ for any $v\in E_+$. Then $ux\in\bar{D}-\{0\}$. Thus $V_+\cup\{0\}=\bigcup_{u\in E_+} u\bar{D}$.

Next assume that $\bar{D}\cap v\bar{D}\neq\{0\}$ for some $1\neq v\in E_+$. There exists an $x\in\bar{D}-\{0\}$ such that $vx\in\bar{D}$. Since $x\in\bar{D}-\{0\}$, it follows from (1.5) that

$$x_it_i\geq \operatorname{tr} x=\operatorname{tr} vx>v^{(i)}x_i$$

for $i=1,2,\cdots,n$. Thus, $v\in N$. Hence $\bar{D}\cap v\bar{D}=\{0\}$ except for a finite number of $v\in E_+$.

PROPOSITION 4. *There are a finite number of Q-rational simplicial cones $\{C_j;\ j\in J\}$ such that*

$$V_+=\bigcup_{u\in E_+}\bigcup_{j\in J} uC_j \qquad (\textit{disjoint union}).$$

PROOF. Set $U=\{u\in E_+, \bar{D}\cap u\bar{D}\neq\{0\}\}$. By Lemma 3, U is a finite subset of E_+. For each $x\in\bar{D}-\{0\}$, set $E_x=\{u\in E_+;\ ux\in\bar{D}\}$[1]. Then E_x is a subset of U. If $x, vx\in\bar{D}-\{0\}$ for some $v\in E_+$, it is easy to see that $E_{vx}=v^{-1}E_x$. For any subset T of U, set $\bar{D}_T=\{x\in\bar{D};\ E_x=T\}$. It is easy to see that $u\bar{D}_T\cap\bar{D}\neq\emptyset$ for some $u\in E_+$ implies that $u\bar{D}_T=\bar{D}_{u^{-1}T}$. Furthermore, since E_+ is torsion free, $u\bar{D}_T\cap\bar{D}_T\neq\emptyset$ for some $u\in E_+$ implies $u=1$. Let W be the set of subsets T of U such that $\bar{D}_T\neq\emptyset$. Two elements T_1, T_2 of W are said to be *equivalent* if there exists a $u\in E_+$ such that $uT_1=T_2$. Let W' be a complete set of representatives of the equivalence classes in W. Then W' is a set consisting of a certain finite number of subsets of W. It follows now easily from Lemma 3 that

$$V_+=\bigcup_{u\in E_+}\bigcup_{T\in W'} u\bar{D}_T \qquad \text{(disjoint union)}.$$

It is easy to see that $\bar{D}_T$ is a finite disjoint union of Q-rational polyhedral cones. Hence, by Corollary to Lemma 2, $\bar{D}_T$ is a disjoint finite union of Q-rational open simplicial cones.

4. As in 3, let F be a totally real algebraic number field of degree n. We keep notations in 3. In particular, F is regarded as a Q-subalgebra of the R-algebra R^n via the embedding: $x\mapsto(x^{(1)}, x^{(2)}, \cdots, x^{(n)})$. Let $\mathfrak{f}$ be an integral ideal of F. Two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F are said to be in the same narrow ray class modulo $\mathfrak{f}$ if both are prime to $\mathfrak{f}$ and $\mathfrak{a}^{-1}\mathfrak{b}=(\mu)$ for a suitable totally positive element μ of F which is congruent to 1 modulo $\mathfrak{f}$. For an integral ideal $\mathfrak{b}$ of F which is prime

[1] $E_x=\{u\in E_+;\ \operatorname{tr} ux=\operatorname{tr} x\}$.

to $\mathfrak{f}$, set

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) = \sum_{\mathfrak{g}} N(\mathfrak{g})^{-s},$$

where the summation is over all integral ideals $\mathfrak{g}$ of F which are in the same narrow ray class as $\mathfrak{b}$ modulo $\mathfrak{f}$. Let $E(\mathfrak{f})_+$ be the group of totally positive units of F which are congruent to 1 modulo $\mathfrak{f}$. Then $E(\mathfrak{f})_+$ is a subgroup of finite index of the group E_+ of all totally positive units of F. Hence, by Proposition 4, there exist a finite number of open simplicial cones $C_j(v_{j1}, \cdots, v_{jr(j)})$ ($j \in J$: a finite set of indices) with generators $v_{j1}, v_{j2}, \cdots, v_{jr(j)} \in F \cap \boldsymbol{R}_+^n$ ($r(j) \leq n$) such that

$$\text{(1.7)} \qquad \boldsymbol{R}_+^n = \bigcup_{j \in J} \bigcup_{u \in E(\mathfrak{f})_+} uC_j(v_{j1}, \cdots v_{jr(j)}) \qquad \text{(disjoint union)}.$$

We may assume (multiplying by suitable positive integers if necessary) that, for each $j \in J$, all of $v_{j1}, \cdots, v_{jr(j)}$ are in $\mathfrak{f}$. For each $j \in J$, and for each subset S of F, denote by $R(j, S)$ the set of $r(j)$-tuples $x = (x_1, x_2, \cdots, x_{r(j)})$ of *rational numbers* which satisfy the following conditions (i) and (ii):

(i) $0 < x_k \leq 1 \qquad (k = 1, 2, \cdots, r(j))$,

(ii) $\sum_{k=1}^{r(j)} x_k v_{jk} \in S$.

We note that the set $R(j, S)$ is finite provided S is contained in a fractional ideal of F. Further denote by A_j ($j \in J$) an $r(j) \times n$ matrix whose (l, m) entry is $v_{jl}^{(m)}$. Then A_j is a matrix with positive entries.

THEOREM 1. *Notations being as above, $\zeta(\mathfrak{b}, \mathfrak{f}, 1-m)$ $(m = 1, 2, \cdots)$ is equal to*

$$N(\mathfrak{b})^{m-1} m^{-n} \sum_{j \in J} (-1)^{r(j)} \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} B_m(A_j, x),^{2)}$$

where $\mathfrak{b}^{-1}\mathfrak{f}+1$ is the set consisting of all elements μ of F which satisfy $\mu - 1 \in \mathfrak{b}^{-1}\mathfrak{f}$.

PROOF. It follows from the definition of $\zeta(\mathfrak{b}, \mathfrak{f}, s)$ that $\zeta(\mathfrak{b}, \mathfrak{f}, s) = N(\mathfrak{b})^{-s} \sum_{\mu}' N(\mu)^{-s}$, where the summation is over all totally positive numbers μ of F which satisfy $\mu - 1 \in \mathfrak{b}^{-1}\mathfrak{f}$ and are not associated with each other under the action of the group $E(\mathfrak{f})_+$. Hence, it follows from (1.7) that

$$\zeta(\mathfrak{b}, \mathfrak{f}, s) = N(\mathfrak{b})^{-s} \sum_{j \in J} \sum_{\mu \in C_j \cap (\mathfrak{b}^{-1}\mathfrak{f}+1)} N(\mu)^{-s}.$$

Since $C_j = C_j(v_{j1}, \cdots, v_{jr(j)})$ is the simplicial cone with generators $v_{j1}, \cdots, v_{jr(j)} \in \mathfrak{f}$, each $\mu \in C_j \cap (\mathfrak{b}^{-1}\mathfrak{f}+1)$ has a unique expression:

$$\mu = \sum_{k=1}^{r(j)} (x_k + z_k) v_{jk},$$

2) For the definition of $B_m(A, x)$, see (1.4) (see also the Introduction).

for a suitable $x=(x_1, \cdots, x_{r(j)}) \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)$ and a suitable $r(j)$-tuple $z=(z_1, \cdots, z_{r(j)})$ of non-negative integers. Thus,

$$\sum_{\mu \in C_j \cap (\mathfrak{b}^{-1}\mathfrak{f}+1)} N(\mu)^{-s} = \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} \zeta(s, A_j, x)$$

(for notations see (1.1) and (1.4)). Hence, it follows from Proposition 1 and its Corollary that

$$\sum_{\mu \in C_j \cap (\mathfrak{b}^{-1}\mathfrak{f}+1)} N(\mu)^{-s} = m^{-n}(-1)^{r(j)} \sum_{x \in R(j, \mathfrak{b}^{-1}\mathfrak{f}+1)} B_m(A^{(j)}, x).$$

COROLLARY TO THEOREM 1 (Siegel-Klingen). *The values of* $\zeta(\mathfrak{b}, \mathfrak{f}, s)$ *at* $s=0, -1, -2, \cdots$ *are all rational numbers.*

PROOF. Let $a_1, \cdots, a_r$ be non-zero numbers of F and let A be the $r \times n$ matrix whose (j, k)-entry is $a_j^{(k)}$. It is sufficient to show that $B_m(A, x)$ is rational for any r-tuple x of rational numbers. Let K be the Galois closure of F with respect to $\boldsymbol{Q}$ and let σ be an element of the Galois group of K with respect to $\boldsymbol{Q}$. Then σ induces a permutation of indices $\{1, 2, \cdots, n\}$ such that

$$\sigma \cdot a_j^{(k)} = a_j^{(\sigma(k))} \qquad (1 \le j \le r, k=1, \cdots, n).$$

Remember that $B_m(A, x)$ is given by $\sum_{k=1}^{n} B_m(A, x)^{(k)}/n$, and that $B_m(A, x)^{(k)}$ is the coefficient of $u^{n(m-1)+r}(t_1 \cdots t_{k-1}t_{k+1} \cdots t_n)^{m-1}$ in the Taylor expansion at the origin of the function

$$u^r \prod_{j=1}^{r} \frac{\exp(x_j u L_j(t))}{\exp(u L_j(t)) - 1}\bigg|_{t_k=1},$$

where $L_j(t) = a_j^{(1)}t_1 + a_j^{(2)}t_2 + \cdots + a_j^{(n)}t_n$. Thus, it is easy to see that $B_m(A, x)^{(k)}$ is in K and $\sigma B_m(A, x)^{(k)} = B_m(A, x)^{(\sigma(k))}$. Hence, $B_m(A, x)$ is left invariant under the action of the Galois group of K with respect to $\boldsymbol{Q}$ and is in $\boldsymbol{Q}$.

5. We keep notations in 4. Denote by $\mathfrak{o}(F)$ the ring of integers of F. Choose and fix a *finite* system $\{C_j(v_{j1}, \cdots, v_{jr(j)});\ j \in J\}$ of simplicial open cones *with generators in* $\mathfrak{o}(F)$ which satisfies

$$\text{(1.8)} \qquad \boldsymbol{R}_+^n = \bigcup_{j \in J} \bigcup_{u \in E_+} u C_j(v_{j1}, v_{j2}, \cdots, v_{jr(j)}) \quad (\text{disjoint union, } v_{j1}, \cdots, v_{jr(j)} \in \mathfrak{o}(F) \cap \boldsymbol{R}_+^n).$$

We note that the existence of such a system is guaranteed by Proposition 3. Let K be a *totally imaginary quadratic extension* of F. Denote by $\mathfrak{d}$ the relative discriminant of K with respect to F. Let χ be the quadratic character of the group of the narrow ideal classes of F with the conductor $\mathfrak{d}$ which is associated to the quadratic extension K of F in class field theory. Denote by $E(K)$ (resp. E) the group of all the units of K (resp. F).

The relative norm $N_{K/F}$ of K with respect to F gives a homomorphism of $E(K)$ into E_+, the group of all totally positive units of F. Denote by $N_{K/F}E(K)$ the image of $E(K)$ by this homomorphism. Let h and H be the class numbers of F and K respectively. Take a complete set of representatives $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_h$ of the ideal classes of F such that each $\mathfrak{a}_m$ is integral. The set $R(j, (\mathfrak{a}_m\mathfrak{d})^{-1})$ $(j \in J, 1 \leq m \leq h)$ is the set of all $r(j)$-tuples of positive *rational* numbers which satisfy (0.2) for $S = (\mathfrak{a}_m\mathfrak{d})^{-1}$. It is finite. Denote by $w(K)$ the cardinality of the set of roots of unity in K. The following result may be regarded as *an affirmative answer to the Hecke conjecture* that the relative class number of K with respect to F admits an elementary arithmetic expression in terms of the relative discriminant $\mathfrak{d}$ (see Hecke [2]).

THEOREM 2. *Notations being as above, the relative class number H/h of K with respect to F is given by the following formula:*

$$\frac{H}{h} = \frac{2^n w(K)}{[E, E_+]^2 [E_+, N_{K/E}E(K)]} \sum_{m=1}^{h} \sum_{j \in J} \sum_{x \in R(j, (\mathfrak{a}_m\mathfrak{d})^{-1})} \chi\left(\left(\sum_{k=1}^{r(j)} x_k v_{jk}\right)\mathfrak{a}_m\mathfrak{d}\right)$$
$$\times \frac{(-1)^{r(j)}}{n} \sum_{l} \prod_{k=1}^{r(j)} \frac{B_{l_k}(x_k)}{l_k!} \operatorname{tr}\left(\prod_{k=1}^{r(j)} v_{jk}^{l_k - 1}\right),$$

where the summation with respect to l is taken over all $r(j)$-tuples $l = (l_1, l_2, \cdots, l_{r(j)})$ of non-negative integers which satisfy

$$l_1 + l_2 + \cdots + l_{r(j)} = r(j).$$

PROOF. Let ζ_K be the Dedekind zeta function of K. It is known that

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{(2\pi)^n R_K}{w(K)\sqrt{D_K}} H,$$

where R_K and D_K are the regulator and the discriminant of K, respectively. On the other hand, $\zeta_K(s) = \zeta_F(s) L_F(s, \chi)$, where $\zeta_F(s)$ is the Dedekind zeta function of F and $L_F(s, \chi)$ is the L-function of F associated with the character χ. We know that $\lim_{s \to 1} (s-1)\zeta_F(s) = (2^{n-1} R_F / \sqrt{D_F}) h$, where R_F and D_F are the regulator and the discriminant of F, respectively. Hence, $H/h = \sqrt{D_K/D_F}(w(K)R_F/2R_K)\pi^{-n} L_F(1, \chi)$. On the other hand, the Dirichlet series $L_F(s, \chi)$ satisfies the following functional equation:

$$A^s \Gamma\left(\frac{s+1}{2}\right)^n L_F(s, \chi) = A^{1-s} \Gamma\left(\frac{2-s}{2}\right)^n L_F(1-s, \chi),$$

where $A=\sqrt{N(\mathfrak{d})D_F/\pi^n}$. Hence,

$$L_F(1,\chi)=\frac{\pi^n}{\sqrt{D_F N(\mathfrak{d})}}L_F(0,\chi)\,.$$

Since $\sqrt{D_K/D_F}=\sqrt{D_F N(\mathfrak{d})}$, we have

$$\frac{H}{h}=\frac{w(K)R_F}{2R_K}L_F(0,\chi)\,.$$

It is easy to see that

$$[E,E_+]L_F(s,\chi)=\sum_{m=1}^{h}N(\mathfrak{a}_m\mathfrak{d})^{-s}\sum_{\alpha\in(\mathfrak{a}_m\mathfrak{d})^{-1}/\sim}\chi((\alpha)\mathfrak{a}_m\mathfrak{d})\,|N(\alpha)|^{-s},$$

where the summation with respect to α is over all non-zero numbers in $(\mathfrak{a}_m\mathfrak{d})^{-1}$ *which are not associated with each other under the action of the group E_+.*

However it is also known (see p. 18～p. 19 of [10]) that

$$\sum_{\alpha\in(\mathfrak{a}_m\mathfrak{d})^{-1}/\sim}\chi((\alpha)\mathfrak{a}_m\mathfrak{d})\,|N(\alpha)|^{-s}|_{s=0}=2^n\sum_{0<\alpha\in(\mathfrak{a}_m\mathfrak{d})^{-1}/\sim}\chi((\alpha)\mathfrak{a}_m\mathfrak{d})\,|N(\alpha)|^{-s}|_{s=0}, \tag{1.9}$$

where the summation in the right side is over all totally positive numbers in $(\mathfrak{a}_m\mathfrak{d})^{-1}$ which are not associated with each other under the action of the group E_+. Since the conductor of χ is $\mathfrak{d}$, we see that the series in the right side of (1.9) is equal to

$$\sum_{j\in J}\sum_{x\in R(j,(\mathfrak{a}_m\mathfrak{d})^{-1})}\chi\left(\left(\sum_{k=1}^{r(j)}x_k v_{jk}\right)\mathfrak{a}_m\mathfrak{d}\right)\sum_{z_1,\cdots,z_{r(j)}=0}^{\infty}\left|N\left(\sum_{k=1}^{r(j)}(x_k+z_k)v_{jk}\right)\right|^{-s}.$$

It follows from Corollary to Proposition 1 that the Dirichlet series

$$\sum_{z_1,\cdots,z_{r(j)}=0}^{\infty}\left|N\left(\sum_{k=1}^{r(j)}(x_k+z_k)v_{jk}\right)\right|^{-s}$$

is, at $s=0$, equal to

$$\frac{(-1)^{r(j)}}{n}\sum_{\substack{l_1,\cdots,l_{r(j)}\geq 0\\ l_1+\cdots+l_{r(j)}=r(j)}}\prod_{k=1}^{r(j)}\frac{B_{l_k}(x_k)}{l_k!}\operatorname{tr}\prod_{k=1}^{r(j)}v_{jk}^{l_k-1}.$$

Since

$$2^n\frac{w}{2}\frac{R_F}{R_K}\frac{1}{[E,E_+]}=\frac{2^n\cdot w}{[E,E_+]^2[E_+,N_{K/F}E(K)]},$$

the theorem follows.

REMARK 1. If $[E,E_+]=2^n$, $[E_+,N_{K/F}E(K)]=1$.

REMARK 2. When $F=\boldsymbol{Q}$, our formula for H/h coincides with Dirichlet's class number formula for imaginary quadratic fields. For a real quadratic field F, the class number formula for totally imaginary quadratic extensions of F was given in Hecke [2], Meyer [6] and in Siegel [11]. Our result is consistent with theirs. For a real cubic field F, Reidemeister discussed the class number formula for totally imaginary extensions in [7].

6. (This subsection is added on March 29, 1976.)

We keep notations in 5.

Let χ be a character of the group of ideal classes of F modulo $\mathfrak{f}$ which is given, for a principal integral ideal (μ), by the following formula:

$$\chi((\mu))=\chi_0(\mu)\prod_{k=1}^{n}\left(\frac{\mu^{(k)}}{|\mu^{(k)}|}\right)^{a_k}, \tag{1.10}$$

where χ_0 is a character of the residue class group modulo $\mathfrak{f}$ and $a_k=0$ or 1 $(1\leq k\leq n)$. Denote by $L_F(s,\chi)$ the L-series of F associated with the character χ and set

$$\xi(s,\chi)=\sqrt{\frac{D_F N(\mathfrak{f})}{\pi^n}}^{\,s}\left\{\prod_{k=1}^{n}\Gamma\left(\frac{s+a_k}{2}\right)\right\}L_F(s,\chi)\,. \tag{1.11}$$

If χ is primitive, ξ is known to satisfy the following functional equation:

$$\xi(1-s,\chi)=w(\chi)\xi(s,\chi^{-1}), \qquad \text{(see (45) of Hecke [1])}$$

where $w(\chi)$ is a complex number of modulus 1 which depends only on χ. Denote by r the number of a_k's which are equal to 1. The functional equation reduces the evaluation problem of $L_F(1,\chi)$ to that of the $(n-r)$-th derivative of $L_F(s,\chi)$ at $s=0$. In particular, if $r=n$, namely if

$$\chi(\mu)=\chi_0(\mu)\operatorname{sgn}(N_Q^F(\mu)), \tag{1.12}$$

the value of $L_F(1,\chi)$ is evaluated by the method in the proof of Theorem 2. More precisely, choose a complete set of representatives $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_h$ of the ideal classes of F[3] and let $\{C_j;\, j\in J\}$ be a finite system of open simplicial cones with the property (1.8). Then we have

THEOREM 3. *Notations being as above, if χ is a primitive character of the ideal class group modulo $\mathfrak{f}$ which is of the form* (1.12), *then*

$$\frac{[E,E_+]}{(2\pi)^n}L_F(1,\chi)=w(\chi)\sqrt{D_F N(\mathfrak{f})}^{-1}\sum_{m=1}^{h}\sum_{j\in J}\sum_{x\in R(j,(\mathfrak{a}_m\mathfrak{f})^{-1})}\chi^{-1}\left(\mathfrak{a}_m\mathfrak{f}\left(\sum_{k=1}^{r(j)}x_k v_{jk}\right)\right)$$
$$\times\frac{(-1)^{r(j)}}{n}\sum_{l_1,\cdots,l_{r(j)}}\prod_{k=1}^{r(j)}\frac{B_{l_k}(x_k)}{l_k!}\operatorname{tr}\left(\prod_{k=1}^{r(j)}v_{jk}^{l_k-1}\right)$$

3) All ideals $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_h$ are assumed to be integral.

where the summation with respect to $l_1, \cdots, l_{r(j)}$ is over all $r(j)$-tuples of non-negative integers which satisfy $l_1+\cdots+l_{r(j)}=r(j)$.

REMARK. If $0\leq r<n$, namely if some of a_k's in (1.10) are actually equal to 0, via the method in the proof of Theorem 2, the evaluation problem of $L_F(1,\chi)$ is reduced to that of the $(n-r)$-th derivatives of Dirichlet series $\zeta(s, A, x)$ (see (1.4)) at $s=0$. On the other hand, by the method in the proof of Proposition 1, one gets an integral representation for the values of derivatives of $\zeta(s, A, x)$ at $s=0$. Thus, one obtains an integral representation for $L_F(1,\chi)$ which is, to the best of the author's knowledge, not discussed in the previous literature. In particular, if $r=n-1$, the obtained integral representation for $L_F(1,\chi)$ is, at least if F is real quadratic, further transformed into a formula which has a striking resemblance to the classical Kronecker limit formula. These subjects will be discussed in subsequent papers. Here, we only indicate a formula for the first derivative of $\zeta(s, A, x)$ at $s=0$.

PROPOSITION 1′. *Notations being as in 1 of §1, we have*

$$\frac{d}{ds}\zeta(s, A, x)\Big|_{s=0}=(n\gamma-(2n-1)\pi i)\zeta(0, A, x)+\sum_{k=1}^{n}\sum_{l=1}^{n} I_{kl}(A, x)\ ,$$

where γ is the Euler constant and $I_{kl}(A, x)$ is given by the following formula:

$$I_{kk}=\frac{1}{2\pi\sqrt{-1}}\int_{I_\varepsilon(+\infty)}\prod_{j=1}^{r}\frac{\exp(1-x_j)a_{jk}t}{\exp(a_{jk}t)-1}\frac{\log t}{t}dt\ ,$$

if $l\neq k$,

$$I_{kl}=\frac{1}{(2\pi\sqrt{-1})^2 n}\int_{I_\varepsilon(+\infty)}\frac{dt}{t}\int_{I_\varepsilon(1)}\prod_{j=1}^{r}\frac{\exp\{(1-x_j)t(a_{jl}u+a_{jk})\}}{\exp\{t(a_{jl}u+a_{jk})\}-1}\frac{\log u}{u}du\ .$$

We note that the integrals I_{kl} $(l\neq k)$ are evaluated in terms of elementary functions.

§2

1. Let $F=\boldsymbol{Q}(\sqrt{d})$ be the real quadratic field with discriminant d. Denote by $\mathfrak{o}$ (resp. $\mathfrak{o}_f$) the maximal order (resp. the order with the conductor f) of F. Let L be a lattice in F and assume that the order of L has the conductor f:

$$\{x\in F;\ xL\subset L\}=\mathfrak{o}_f\ .$$

Let $E(f)_+$ be the group of totally positive units of F which are in $\mathfrak{o}_f$. Take an $x\in F$ which satisfies the congruence $ux-x\in L$ for any $u\in E(f)_+$ and set

$$\zeta_+(L, x, s) = \sum_{\mu}{}' |N(\mu)|^{-s},$$

where the summation is over all totally positive numbers μ of F which satisfy the congruence $\mu - x \in L$ and are not associated with each other under the action of the group $E(f)_+$. The Dirichlet series $\zeta_+(L, x, s)$ is a generalization, in the case of the real quadratic field F, of the Dirichlet series $\zeta(\mathfrak{b}, \mathfrak{f}, s)$ introduced in the previous section.

For an $\omega \in F$, we denote by ω' the conjugate of ω with respect to $\boldsymbol{Q}$. Then the mapping: $\omega \mapsto (\omega, \omega')$ embeds F into $\boldsymbol{R}^2$. Let $\varepsilon > 1$ be the generator of an infinite cyclic group $E(f)_+$. *Assume that the smallest positive number in $L \cap \boldsymbol{Q}$ is* 1. Take an $\omega \in L$ such that $\{1, \omega\}$ is a $\boldsymbol{Z}$-base of L and set $\varepsilon = a - c\omega$ $(a, c \in \boldsymbol{Z})$ and $x = p + q\varepsilon$ $(p, q \in \boldsymbol{Q})$. Without loss of generality we may assume that $c > 0$. It is easy to see that

$$\boldsymbol{R}_+^2 = \bigcup_{\eta \in E(f)_+} \eta C_1 \cup \bigcup_{\eta \in E(f)_+} \eta C_2 \qquad \text{(disjoint union)},$$

where we put $C_1 = C(1, \varepsilon)$ and $C_2 = C(1)$.[4] Denote by R_1 the set of pairs $y = (y_1, y_2)$ of rational numbers which satisfy the inequalities $0 < y_1, y_2 \leq 1$ and the condition

$$y_1 + y_2\varepsilon - x \in L\,.$$

For each $z \in \boldsymbol{R}$ we denote by $\langle z \rangle$ the unique number in $(0, 1]$ such that $\langle z \rangle - z \in \boldsymbol{Z}$. Then it is easy to see that

$$R_1 = \bigcup_{z=0}^{c-1} \left(\left\langle p - \frac{az}{c} \right\rangle, \ \left\langle q + \frac{z}{c} \right\rangle \right).$$

Thus, setting $A_1 = \begin{pmatrix} 1 & 1 \\ \varepsilon & \varepsilon' \end{pmatrix}$, we have

$$\zeta_+(L, x, s) = \sum_{z=0}^{c-1} \zeta\left(s, A_1, \left(\left\langle p - \frac{az}{c} \right\rangle, \left\langle q + \frac{z}{c} \right\rangle \right)\right) + \begin{cases} 0 & \text{if } \ cq \notin \boldsymbol{Z} \\ \zeta(2s, 1, \langle p + qa \rangle) & \text{if } \ cq \in \boldsymbol{Z} \end{cases}$$

(for notations, see (1.1) and (1.4)).

Thus, by Proposition 1 and its corollary, we have

$$\zeta_+(L, x, 1-s) = \sum_{k=1}^{2s-1} \left\{ \frac{\Gamma(s)^2}{k!\,(2s-k)!} \sum_{t=0}^{k-1} \binom{k-1}{t} \binom{2s-k-1}{s-1-t} \varepsilon^{k-2t-1} \times \sum_{z=0}^{c-1} B_k\left(\left\langle p - \frac{az}{c} \right\rangle \right) B_{2s-k}\left(\left\langle q + \frac{z}{c} \right\rangle \right) \right\} \tag{2.1}$$

4) For notations, see 1 of §1 or the Introduction.

$$+\frac{\Gamma(s)^2}{2(2s)!}\sum_{z=0}^{c-1}\left\{B_{2s}\left(\left\langle p-\frac{az}{c}\right\rangle\right)+B_{2s}\left(\left\langle q+\frac{z}{c}\right\rangle\right)\right\}$$
$$\times\sum_{k=0}^{s-1}(-1)^k(\varepsilon^{2k+1}+\varepsilon'^{2k+1})\binom{2s-1}{s-1-k}$$
$$+\begin{cases}0 & \text{if}\quad qc\notin \boldsymbol{Z}\\ -\dfrac{B_{2s-1}(\langle p+qa\rangle)}{2s-1} & \text{if}\quad qc\in\boldsymbol{Z}\end{cases}$$

for $s=1,2,\cdots$.

We will show that the above formula (2.1) is, essentially, identical with Satz 1 of Siegel [8]. Since the lattice L (which is generated by 1 and ω) is invariant under the multiplication by ε, there exists an integral 2×2 unimodular matrix $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ which satisfies

$$(m-\omega n)\varepsilon=(am+bn)-\omega(cm+dn)$$

for any $m, n\in\boldsymbol{Z}$.

Set $\Delta=(\omega-\omega')^2>0$ and $x=v+\omega u$ $(u, v\in\boldsymbol{Q})$. We note that $\varepsilon=a-\omega c$ and $\varepsilon'=d+\omega c$. Since $\varepsilon x-x$ and $\varepsilon' x-x$ are both in L, we have

$$\begin{aligned}u&\equiv au+cv\\ v&\equiv bu+dv\end{aligned}\quad(\text{mod. } \boldsymbol{Z})\,. \tag{2.2}$$

Then $\zeta_+(L, x, 1-s)$ $(s=1,2,\cdots)$ is evaluated by the formula (2.1), where one should put

$$p=v+\frac{au}{c}\qquad\text{and}\qquad q=-\frac{u}{c}\,. \tag{2.3}$$

On the other hand, it follows from the functional equation of the Dirichlet series $\zeta_+(L, x, s)$ that, for $s=1,2,\cdots$,

$$\zeta_+(L, x, 1-s)=(2\pi)^{-2s}\Gamma(s)^2\Delta^{s-1/2}\sum{}'\frac{\exp -2\pi\sqrt{-1}\,(mu+nv)}{(m-n\omega)^s(m-n\omega')^s}\,, \tag{2.4}$$

where the summation is over all non-zero pairs of integers (m, n) which are not associated with each other under the action of the group generated by γ.

By Satz 1 of Siegel [8], the right side of (2.4) is equal to

$$\sum_{k=0}^{2s-1}\frac{(-1)^k c^{2s-k-1}}{k!\,(2s-k)}R_s^{(k)}\left(\frac{a}{c}\right)\sum_{l\,(\text{mod.}\,c)}P_k\left(a\frac{u+l}{c}+v\right)P_{2s-k}\left(\frac{u+l}{c}\right) \tag{2.5}$$
$$-\begin{cases}0 & \text{unless}\quad (u, v)\in\boldsymbol{Z}^2\quad\text{and}\quad s=1\,,\\ \dfrac{1}{4} & \text{if}\quad (u, v)\in\boldsymbol{Z}^2\quad\text{and}\quad s=1\,,\end{cases}$$

where $R_s(x)=\int_{-d/c}^{x}\{(x-\omega)(x-\omega')\}^{s-1}dx$ and $P_k(x)$ $(k=0,1,2,\cdots)$ is the periodic function with period 1 which coincides with $B_k(x)$ on the open interval $(0,1)$ and is equal to $\{B_k(0)+B_k(1)\}/2$ at the origin.

For $k\geq 1$,

$$\begin{aligned}R_s^{(k)}\left(\frac{a}{c}\right)&=\frac{d^{k-1}}{dx^{k-1}}\{(x-\omega)(x-\omega')\}^{s-1}\Big|_{x=a/c}\\&=c^{1+k-2s}\sum_{t=0}^{k-1}\binom{k-1}{t}\frac{(s-1)!^2}{(s-t-1)!\,(s+t-k)!}\,\varepsilon^{k-2t-1}\\&=\frac{\Gamma(s)^2c^{1+k-2s}}{(2s-k-1)!}\sum_{t=0}^{k-1}\binom{k-1}{t}\binom{2s-k-1}{s-1-t}\varepsilon^{k-2t-1}.\end{aligned}$$

Applying the integration by part, we have

$$\begin{aligned}R_s\left(\frac{a}{c}\right)&=\int_{-d/c}^{a/c}\{(x-\omega)(x-\omega')\}^{s-1}dx\\&=\frac{c^{1-2s}}{(2s-1)!}\Gamma(s)^2\sum_{t=0}^{s-1}(-1)^t(\varepsilon^{2t+1}+\varepsilon'^{2t+1})\binom{2s-1}{s-1-t}.\end{aligned}$$

We note that, if x is not an integer or if $k\geq 2$, then $P_k(x)=B_k(\langle x\rangle)$. However, $P_1(x)=B_1(\langle x\rangle)-\dfrac{1}{2}$ if x is an integer. Since

$$\sum_{l=0}^{c-1}P_k\left(x+\frac{l}{c}\right)=c^{1-k}P_k(cx)\qquad(k\geq 2),$$

we have, by (2.2) and (2.3),

$$\begin{aligned}\sum_{l=0}^{c-1}B_{2s}\left(\left\langle p-\frac{al}{c}\right\rangle\right)&=\sum_{l=0}^{c-1}B_{2s}\left(\left\langle q+\frac{l}{c}\right\rangle\right)\\&=\sum_{l=0}^{c-1}P_{2s}\left(\frac{u+l}{c}\right).\end{aligned}$$

Moreover, if u is not an integer (we note that $u\notin \boldsymbol{Z}$ implies $cp\notin \boldsymbol{Z}$),

$$\sum_{l=0}^{c-1}P_k\left(a\frac{u+l}{c}+v\right)P_{2s-k}\left(\frac{u+l}{c}\right)=(-1)^k\sum_{l=0}^{c-1}B_k\left(\left\langle p-\frac{al}{c}\right\rangle\right)B_{2s-k}\left(\left\langle q+\frac{l}{c}\right\rangle\right)\qquad(1\leq k\leq 2s-1).$$

Assume $u\in\boldsymbol{Z}$, then the above equality still holds for $2\leq k\leq 2s-2$. However, for $s\geq 2$,

$$\begin{aligned}&\sum_{l=0}^{c-1}P_{2s-1}\left(a\frac{u+l}{c}+v\right)P_1\left(\frac{u+l}{c}\right)\\&\quad=-\sum_{l=0}^{c-1}B_{2s-1}\left(\left\langle p-\frac{al}{c}\right\rangle\right)B_1\left(\left\langle q+\frac{l}{c}\right\rangle\right)+\frac{1}{2}B_{2s-1}(v)\end{aligned}$$

and

$$\sum_{l=0}^{c-1} P_1\left(a\frac{u+l}{c}+v\right)P_{2s-1}\left(\frac{u+l}{c}\right)$$
$$=-\sum_{l=0}^{c-1} B_1\left(\left\langle p-\frac{al}{c}\right\rangle\right)B_{2s-1}\left(\left\langle q+\frac{l}{c}\right\rangle\right)+\frac{1}{2}B_{2s-1}(v)$$

$\Big($here, we have made use of the congruences $\frac{l+u}{c}\equiv d\left(a\frac{u+l}{c}+v\right)-dv \pmod{\boldsymbol{Z}}$ and $dv\equiv v \pmod{\boldsymbol{Z}}\Big)$. If $s=1$ and $u\in\boldsymbol{Z}$,

$$\sum_{l=0}^{c-1} P_{2s-1}\left(a\frac{u+l}{c}+v\right)P_1\left(\frac{u+l}{c}\right)$$
$$=-\sum_{l=0}^{c-1} B_{2s-1}\left(\left\langle p-\frac{al}{c}\right\rangle\right)B_1\left(\left\langle q+\frac{l}{c}\right\rangle\right)+B_{2s-1}(v) \qquad \text{if} \quad v\notin\boldsymbol{Z}.$$

If $s=1$ and $(u,v)\in\boldsymbol{Z}^2$,

$$\sum_{l=0}^{c-1} P_{2s-1}\left(a\frac{u+l}{c}+v\right)P_1\left(\frac{u+l}{c}\right)$$
$$=-\sum_{l=0}^{c-1} B_{2s-1}\left(\left\langle p-\frac{al}{c}\right\rangle\right)B_1\left(\left\langle q+\frac{l}{c}\right\rangle\right)+\frac{1}{4}.$$

Moreover, for $k=1$ or $2s-1$,

$$\frac{\Gamma(s)^2}{k!\,(2s-k)!}\sum_{t=0}^{k-1}\binom{k-1}{t}\binom{2s-k-1}{s-1-t}\varepsilon^{k-2t-1}=\frac{1}{2s-1}.$$

Thus, we have seen that formulas (2.1) and (2.5) are identical.

2. As in 1, let F be the real quadratic field with discriminant d. An $\omega\in F$ is said to be *reduced* (in a modified sense) if $\omega>1>\omega'>0$. It is known that any reduced ω has a purely periodic expansion to (modified) continued fraction:

$$\omega=a_1-\cfrac{1}{a_2-\cfrac{\ddots}{\ddots-\cfrac{1}{a_n-\cfrac{1}{a_1-\ddots}}}} \tag{2.6}$$

$(a_1,\dots,a_n$ are integers $\geq 2)$.

When ω has a purely periodic expansion (2.6) into continued fraction, $(a_1, a_2,\dots,a_n)$ is said to be the primitive period of the continued fraction of ω. Two numbers ω_1

and ω_2 of F are said to be *equivalent* if there exists a $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL_2(\boldsymbol{Z})$ which satisfies $\omega_2 = \dfrac{\alpha\omega_1+\beta}{\gamma\omega_1+\delta}$. It is known that every element of F is equivalent to some reduced element of F. Two reduced numbers of F are equivalent if and only if primitive periods of their continued fractions are obtained from each other by cyclic permutations.

PROPOSITION 5. *Let ω be a reduced number of F and let $(a_1, a_2, \cdots, a_n)$ be the primitive period of its continued fraction. Denote by L the lattice in F generated by 1 and ω and denote by ε $(\varepsilon>1)$ the fundamental unit of L.*

(i) *If $N(\varepsilon)=-1$, $a_1+a_2+\cdots+a_n=3n$.*

(ii) *If $N(\varepsilon)=1$, let τ be a reduced number which is equivalent to $\dfrac{1}{\omega}$ and let $(b_1, \cdots, b_m)$ be the primitive period of the continued fraction of τ. Then*

$$a_1+\cdots+a_n+b_1+\cdots+b_m=3(n+m)\ .$$

PROOF. We prove only the second half of the proposition, since the proof of the first half is similar but simpler. Set $\zeta(L, s)=\sum' |N(x)|^{-s}$, where the summation with respect to x is over all non-zero numbers of L which are not associated with each other under the action of the unit group of L. Then $\zeta(L, s)$ satisfies the following functional equation:

$$\zeta(L, 1-s)=(2\pi)^{-2s}\Gamma(s)^2 4\cos^2\frac{s\pi}{2} D^{s-1/2}\zeta(L, s)\ ,$$

where $D=(\omega-\omega')^2$. Since $s=1$ is a simple pole of $\zeta(L, s)$, we have $\zeta(L, 0)=0$. There exists a $\boldsymbol{Z}$-base $\{\nu_1, \nu_2\}$ of L with the following properties (i) and (ii):

(i) $N(\nu_1)<0$,

(ii) the ratio $\lambda=\nu_2/\nu_1$ is reduced.

Then λ is equivalent to τ. We embed F into $\boldsymbol{R}^2$ via the mapping: $x\to(x, x')$. Assuming that the fundamental unit ε of L has norm 1, we have

$$\zeta(L, s)=\sum_{x\in L\cap C(1,\varepsilon^{-1})} N(x)^{-s}+\sum_{x\in L\cap C(\nu_1,\nu_1\varepsilon^{-1})} |N(x)|^{-s}+\zeta(2s)(1+|N(\nu_1)|^{-s})\ .$$

Denote by $(c_1, \cdots, c_m)$ the primitive period of the continued fraction of λ. Set $\omega_{-1}=\omega$, $\omega_0=1$ and $\omega_i=a_i\omega_{i-1}-\omega_{i-2}$ $(1\leq i\leq n)$. Further set $\lambda_{-1}=\lambda$, $\lambda_0=1$ and $\lambda_i=c_i\lambda_{i-1}-\lambda_{i-2}$ $(1\leq i\leq m)$ (cf. 6. of Zagier [12]). Then it is known that $\omega_n=\lambda_m=\varepsilon^{-1}$, $\omega_{n-1}=\omega\varepsilon^{-1}$, $\lambda_{m-1}=\lambda\varepsilon^{-1}$. Moreover,

$$C(1, \varepsilon^{-1})=\bigcup_{i=1}^{n} C(\omega_{i-1}, \omega_i)\cup\bigcup_{i=1}^{n-1} \boldsymbol{R}_+\omega_i\ ,$$

$$C(\nu_1, \nu_1\varepsilon^{-1}) = \bigcup_{i=1}^{m} C(\nu_1\lambda_{i-1}, \nu_1\lambda_i) \cup \bigcup_{i=1}^{m-1} R_+\nu_1\lambda_i \,.$$

It is easy to see that

$$L \cap C(\omega_{i-1}, \omega_i) = \{k\omega_{i-1} + l\omega_i;\ k, l = 1, 2, \cdots\} \qquad (1 \leq i \leq n)\,,$$

$$L \cap R_+\omega_i = \{k\omega_i;\ k = 1, 2, \cdots\} \qquad (1 \leq i \leq n-1)\,,$$

$$L \cap C(\nu_1\lambda_{i-1}, \nu_1\lambda_i) = \{k\nu_1\lambda_{i-1} + l\nu_1\lambda_i;\ k, l = 1, 2, \cdots\} \qquad (1 \leq i \leq m)\,,$$

$$L \cap R_+\nu_1\lambda_i = \{k\nu_1\lambda_i;\ k = 1, 2, \cdots\} \qquad (1 \leq i \leq m-1)\,.$$

Hence

$$\begin{aligned}\zeta(L, s) = {} & \sum_{i=1}^{n} \sum_{k,l=1}^{\infty} N(k\omega_{i-1} + l\omega_i)^{-s} \\ & + \sum_{j=1}^{m} \sum_{k,l=1}^{\infty} N(k\lambda_{i-1} + l\lambda_i)^{-s} |N(\nu_1)|^{-s} \\ & + \zeta(2s)\{\sum_{i=0}^{m-1} N(\omega_i)^{-s} + |N(\nu_1)|^{-s} \sum_{i=0}^{n-1} N(\lambda_i)^{-s}\}\,.\end{aligned}$$

Applying Proposition 1 and its corollary, we have

$$\sum_{i=1}^{n} \left\{B_1^2 + \frac{B_2}{4} \operatorname{tr}\left(\frac{\omega_{i-1}}{\omega_i} + \frac{\omega_i}{\omega_{i-1}}\right)\right\} + \sum_{i=1}^{m} \left\{B_1^2 + \frac{B_2}{4} \operatorname{tr}\left(\frac{\lambda_{i-1}}{\lambda_i} + \frac{\lambda_i}{\lambda_{i-1}}\right)\right\} - \frac{1}{2}(m+n) = 0\,.$$

Since

$$B_1^2 = \frac{1}{4}\,, \quad B_2 = \frac{1}{6}\,, \quad \frac{\omega_i}{\omega_{i-1}} = a_i - \frac{\omega_{i-2}}{\omega_{i-1}} \qquad (1 \leq i \leq n)$$

and

$$\frac{\lambda_i}{\lambda_{i-1}} = c_i - \frac{\lambda_{i-2}}{\lambda_{i-1}} \qquad (1 \leq i \leq m)\,,$$

we have

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{m} c_i = 3(m+n)\,.$$

As $(c_1, \cdots, c_m)$ is a suitable cyclic permutation of $(b_1, \cdots, b_m)$, we have

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i = 3(m+n)\,.$$

REMARK. M. Inoue communicated to the author that he obtained Proposition 5 in his study on analytic surfaces (see Proposition (5.3) of [5]). He also pointed out that the first part of the proposition has been obtained by Hirzebruch (see p. 222 of [4]). The author wishes to express his gratitude to Inoue.

3. Set $F=\boldsymbol{Q}(\omega)$, where $\omega=2\cos\frac{2\pi}{7}$. Then F is a totally real cubic field. A $\boldsymbol{Z}$-base of the lattice of integers $\mathfrak{o}_F$ of F is given by $\{1, \omega, \omega^2\}$. It is known that the class number of F is one and that -1, ω and $1+\omega$ generate the group E of units of F. Hence, $\varepsilon=2+\omega=(\omega+1)^2\omega^{-2}$ and $\eta=\omega^2$ generate the group E_+ of totally positive units of F. For $w=x+y\omega+z\omega^2\in F$ $(x, y, z\in \boldsymbol{Q})$ we put $w'=x+y\omega'+z\omega'^2$ and $w''=x+y\omega''+z(\omega'')^2$, where $\omega'=2\cos\frac{6\pi}{7}$ and $\omega''=2\cos\frac{4\pi}{7}$. We embed F into $\boldsymbol{R}^3$ via the mapping: $w\mapsto(w, w', w'')$. In the following we use notations in 4 of §1 without further comment.

Set $\bar{D}=\{x\in \boldsymbol{R}^3;\ \mathrm{tr}\, x\leq \mathrm{tr}\, xu \quad \forall u\in E_+\}$. It is not difficult to see that

$$\bar{D}=\{x\in \boldsymbol{R}^3,\ \mathrm{tr}\, x\leq \mathrm{tr}\, ux \quad \text{for} \quad u=\varepsilon, \eta, \varepsilon^{-1}, \eta^{-1}, \varepsilon\eta, \varepsilon^{-1}\eta^{-1}\}\,.$$

Thus, $\bar{D}$ is a closed polyhedral cone in $\boldsymbol{R}^3$ spanned by

$$1+\omega+\omega^2=(2-\omega)\varepsilon\eta,\ (2-\omega)^2\varepsilon\eta,\ 2-\omega,\ (2-\omega)^2\varepsilon,\ (2-\omega)\varepsilon \quad \text{and} \quad (2-\omega)^2\varepsilon^2\eta\,.$$

Thus $\bar{D}=(2-\omega)\bar{C}(1, \varepsilon\eta, 1+\omega+\omega^2)\cup(2-\omega)\eta^{-1}\bar{C}(\eta, 1+\omega+\omega^2, \varepsilon\eta)\cup(2-\omega)\varepsilon\bar{C}(1, 1+\omega+\omega^2, \eta)\cup(2-\omega)\bar{C}(1, \varepsilon, \varepsilon\eta)$. Since $1+\omega+\omega^2=(1+\eta+\varepsilon\eta)/2$, it is easy to see that $\boldsymbol{R}^3_+=\bigcup_{j=1}^{6}\bigcup_{u\in E_+} uC_j$, where $C_1=C(1, \varepsilon, \varepsilon\eta)$, $C_2=C(1, \eta, \eta\varepsilon)$, $C_3=C(1, \varepsilon)$, $C_4=C(1, \eta)$, $C_5=C(1, \varepsilon\eta)$ and $C_6=C(1)$.

Set $K=\boldsymbol{Q}(\zeta)$, where $\zeta=\exp 2\pi\sqrt{-1}/7$. Then K is a totally imaginary quadratic extension of F with the relative discriminant $(2-\omega)$. Moreover, the quadratic character χ of the group of narrow ideal classes of F with the conductor $(2-\omega)$ which corresponds to the quadratic extension K of F in class field theory is given by

$$\chi(x)=\mathrm{sgn}\, N(x)\left(\frac{N(x)}{7}\right) \qquad (x\in\mathfrak{o}(F))\,,$$

where $N(x)$ is the norm of x and $\left(\frac{\cdot}{7}\right)$ is the Legendre Symbol. Now we employ notations given by (0-2). It is easy to see that

$$R(1, (2-\omega)^{-1})=\left\{(1, 1, 1)\frac{k}{7};\quad 1\leq k\leq 7\right\},$$

$$R(2, (2-\omega)^{-1})=\left\{\left(\left[\frac{k}{14}\right], \left[\frac{11}{14}k\right], \left[\frac{9}{14}k\right]\right);\quad 1\leq k\leq 14\right\},$$

where $[x]$ is the Gauss symbol,

$$R(j, (2-\omega)^{-1})=\{(1, 1)\} \quad \text{for} \quad j=3, 4, 5,$$
$$R(6, (2-\omega)^{-1})=\{(1)\}\,.$$

It is easy to see that

$$\chi((1+\varepsilon+\varepsilon\eta)k(1-\omega)/7)=\left(\frac{k}{7}\right)$$

and

$$\chi\left(\left\{\left[\frac{k}{14}\right]+\left[\frac{11}{14}k\right]\eta+\left[\frac{9}{14}k\right]\varepsilon\eta\right\}(2-\omega)\right)=-\left(\frac{k}{7}\right).$$

The number of roots of unity in K is 14. Since the class number of F is 1 and $[E, E_+]=8$, the class number of K is given by the following (see Theorem 2 in the Introduction)

$$\frac{14}{8}\times\frac{(-1)}{3}\left\{\sum_{k=1}^{7}\left(\frac{k}{7}\right)\sum_{l}\frac{B_{l_1}\left(\frac{k}{7}\right)B_{l_2}\left(\frac{k}{7}\right)B_{l_3}\left(\frac{k}{7}\right)}{l_1!l_2!l_3!}\operatorname{tr}\left(\frac{\varepsilon^{l_2+l_3}\eta^{l_3}}{\varepsilon^2\eta}\right)\right.$$
$$\left.-\sum_{k=1}^{14}\left(\frac{k}{7}\right)\sum_{l}\frac{B_{l_1}\left(\left[\frac{k}{14}\right]\right)B_{l_2}\left(\left[\frac{11}{14}k\right]\right)B_{l_3}\left(\left[\frac{9}{14}k\right]\right)}{l_1!l_2!l_3!}\operatorname{tr}\left(\frac{\eta^{l_2+l_3}\varepsilon^{l_3}}{\eta^2\varepsilon}\right)\right\}$$

where the summation with respect to l is over all 3-tuples (l_1, l_2, l_3) of non-negative integers which satisfy the relation $l_1+l_2+l_3=3$. After some computations, we see that the above expression is equal to one.

References

[1] Hecke, E., Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen, zweite Mitteilung, Werke 249-289.

[2] Hecke, E., Bestimmung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern, Werke 290-312.

[3] Hida, H., On the values of Hecke's L-functions at non-positive integers (to appear).

[4] Hirzebruch, F., Hilbert modular surfaces, L'Enseignement Mathématique **19** (1973), 183-281.

[5] Inoue, M., New surfaces with no meromorphic functions II (to appear).

[6] Meyer, C., Die Berechnung der Klassenzahl abelscher Körper über quadratischen Zahlkörper, Akademie-Verlag, Berlin, 1957.

[7] Reidemeister, K., Über die Relativklassenzahl gewisser relativquadratischer Zahlkörper, Abh. Math. Sem. Hamb. Band **1** (1929), 27-48.

[8] Siegel, C. L., Bernoullische Polynome und quadratische Zahlkörper, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. 1968, 7-38.

[9] Siegel, C. L., Berechnung von Zetafunktionen an ganzzahligen Stellen, ibid. 1969, 87-102.

[10] Siegel, C. L., Über die Fourierschen Koeffizienten von Modulformen, ibid. 1970, 15-54.

[11] Siegel, C. L., Lectures on advanced analytic number theory, Tata Institute of Fundamental Research, 1961.

[12] Zagier, D., A Kronecker limit formula for real quadratic fields, Math. Ann. **213** (1975), 153–184.

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