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Eisenstein group cocycles for GL_n and values of L-functions

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1.1 Let F be a totally real number field of degree n over $\mathbb{Q}$ and b, f two relatively prime ideals in the ring of integers $\mathfrak{O}_F$. The partial zeta function associated with the ray class b mod f is defined by

$$\zeta(b, f; s) = \sum_{a \equiv b(f)} N(a)^{-s}, \quad \mathrm{Re}(s) > 1$$

where a runs over all integral ideals in $\mathfrak{O}_F$ such that the fractional ideal ab^{-1} is a principal ideal generated by a totally positive number in the coset $1 + fb^{-1}$. This function has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$. It was originally conjectured by Hecke, but first proved by Siegel and Klingen, that its special values at non-positive integral s are rational numbers: $\zeta(b, f; 1-s) \in \mathbb{Q}$, $s = 1, 2, 3, \ldots$. This result was supplemented by Shintani who established a finite formula for these rational numbers in terms of Bernoulli numbers, generalizing the classical formula of Euler: $\zeta(1-s) = -B_s/s$, $s = 2, 3, 4 \ldots$. In this paper, we show that the special values $\zeta(b, f; 1-s)$ admit a cohomological interpretation in terms of the Eilenberg-MacLane group cohomology of the unimodular group $\Gamma = \mathrm{GL}_n\mathbb{Z}$. We construct a group cocycle Ψ on Γ (called the Eisenstein cocycle) which represents a nontrivial cohomology class in $H^{n-1}(\Gamma, M)$ with values in a function space M. Restricting Ψ to U, the group of totally positive units in $1 + f$ (embedded in Γ via a regular representation), and evaluating the elements of M on U-invariant points, gives rise to cohomology classes in $H^{n-1}(U, \mathbb{C})$ with trivial coefficients. Among these classes, there is a sequence of rational cohomology classes $\eta(b, f; s) \in H^{n-1}(U, \mathbb{Q})$, $s = 1, 2, \ldots$, which give the numbers $\zeta(b, f; 1-s)$ by evaluation on a fundamental cycle in $H_{n-1}(U, \mathbb{Z})$. The Eisenstein cocycle Ψ is universal in the sense that it parameterizes all those special values of Hecke L-functions in every totally real number field of degree n which are known to be either an algebraic number or an algebraic number times a power of π (cf. Theorem 5).

1.2 To define Ψ, we start with the rational function

$$f(\sigma)(x) = \frac{\det(\sigma_1, \sigma_2, \ldots, \sigma_n)}{\langle x, \sigma_1\rangle \langle x, \sigma_2\rangle \cdots \langle x, \sigma_n\rangle}$$

of a row vector $x \in \mathbb{R}^n$ and n nonzero column vectors $\sigma_j \in \mathbb{R}^n$ (where $\langle x, y \rangle = \sum x_j y_j$). This function is well defined outside the hyperplanes $\langle x, \sigma_j \rangle = 0$. More generally, for every homogenous polynomial $P(X_1, \ldots, X_n)$, we apply the differential operator $P(-\partial_{x_1}, \ldots, -\partial_{x_n})$ to $f(\sigma)(x)$, and define

$$f(\sigma)(P, x) = P(-\partial_{x_1}, \ldots, -\partial_{x_n}) f(\sigma)(x),$$

where ∂_{x_j} denotes the partial derivative with respect to the variable x_j. Let $G = \mathrm{GL}_n$ and $G(\mathbb{R})$ be the group of real points of G. We consider n-tuples $\mathfrak{A} = (A_1, \ldots, A_n)$ of matrices A_i in $G(\mathbb{R})$, and write A_{ij} for the jth column of A_i. Then for every non-zero $x \in \mathbb{R}^n$ and every i, there is at least one column A_{ij} such that $\langle x, A_{ij} \rangle \neq 0$. For given $x \neq 0$, we denote by A_{ij_i} the first column in A_i with this property, and define

$$\psi(\mathfrak{A})(P, x) = f(A_{1j_1}, \ldots, A_{nj_n})(P, x).$$

For $x = 0$, it is convenient to set $\psi(\mathfrak{A})(P, 0) = 0$. Then ψ is well defined for all $x \in \mathbb{R}^n$. As a function of $\mathfrak{A}$, the map ψ is a group cocycle on $G(\mathbb{R})$ (all cycles and cocycles are assumed to be homogenous unless stated otherwise) with values in the $G(\mathbb{R})$-module of all complex valued functions $f(P, x)$ with $x \in \mathbb{R}^n$ and P a homogenous polynomial of n variables (cf. Theorem 3). We call ψ the rational cocycle, and construct the Eisenstein cocycle Ψ by averaging the values of ψ (with respect to the variable x) over a coset $\mathbb{Z}^n + u$ of the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. This process requires considerable attention since the series $\sum \psi(x)$, $x \in \mathbb{Z}^n + u$, converges only conditionally. We define its value by the following "Q-limit",

$$\sum_{x \in \mathbb{Z}^n + u} \psi(x) \Big|_Q \underset{\mathrm{def}}{=} \lim_{t \to \infty} \Bigg(\sum_{\substack{x \in \mathbb{Z}^n + u \\ |Q(x)| < t}} \psi(x) \Bigg), \tag{1}$$

where Q is a finite product of (say $m \geqq 1$) linear forms $L_k \in \mathbb{R}[X_1, \ldots, X_n]$,

$$Q(X) = \prod_{k=1}^{m} L_k(X), \quad L_k(X) = \sum_{j=1}^{n} l_{kj} X_j$$

such that for every fixed k, the coefficients l_{kj} $(j = 1, 2, \ldots, n)$ are rationally independent real numbers. For such a Q and a *rational* n-tuple $\mathfrak{A}$ of matrices A_i in $G(\mathbb{Q})$, we define the Eisenstein cocycle Ψ by

$$\Psi(\mathfrak{A})(P, Q, u, v) = \sum_{x \in \mathbb{Z}^n + u} \mathbf{e}((u - x)v)\, \psi(\mathfrak{A})(P, x) \Big|_Q, \tag{2}$$

where u is a row vector in $\mathbb{R}^n$, v a column vector in $\mathbb{R}^n/\mathbb{Z}^n$, and $\mathbf{e}(z)$ is the additive character $\exp(2\pi i z)$. Then as a function on $G(\mathbb{Q})^n$, Ψ satisfies the identity

$$\sum_{i=0}^{n} (-1)^i\, \Psi(A_0, \ldots, \hat{A}_i, \ldots, A_n) = 0, \quad A_i \in G(\mathbb{Q}),$$

while the action of $G(\mathbb{Q})$ on the values $\Psi(\mathfrak{A})$ is given by (18). Let M be the set of all complex valued functions $f(P, Q, u, v)$ with P, Q, u, v as above. M is a left Γ-module under the action

$$Af(P, Q, u, v) = \det(A) f(A^t P, A^{-1} Q, uA, A^{-1} v), \quad A \in \Gamma,$$

where the implied Γ-action on homogenous polynomials is given by $AP(X) = P(XA)$. With respect to this action, the map $\Psi\colon \Gamma^n \to M$ has the property

$$\Psi(A\mathfrak{A}) = A\Psi(\mathfrak{A}) \quad \text{for } A \in \Gamma,\ \mathfrak{A} \in \Gamma^n$$

which implies that Ψ is a homogenous cocycle on Γ representing a cohomology class in $H^{n-1}(\Gamma, M)$. Note that Ψ extends to the group ring $\mathbb{Z}[\Gamma^n]$ by linearity. In order to state our main result, we make the following observation. If Γ_0 is any subgroup of $\mathrm{SL}_n\mathbb{Z}$, and $\mathfrak{Z} \in \mathbb{Z}[\Gamma_0^n]$ a representative of a $\mathbb{Z}$-valued chain

$$[\mathfrak{Z}] \in C_{n-1}(\Gamma_0, \mathbb{Z}) = \mathbb{Z}[\Gamma_0^n] \otimes_{\Gamma_0} \mathbb{Z},$$

then the number $\Psi(\mathfrak{Z})(P, Q, u, v)$ depends only on the chain $[\mathfrak{Z}]$ provided the point (P, Q, u, v) is invariant under the induced action of Γ_0. Returning to the partial zeta function $\zeta(b, f; s)$, we associate to the ray class b mod f a cycle $[\mathfrak{Z}]$ and a U-invariant point $(P, Q, 0, v)$ by choosing a $\mathbb{Z}$-basis W for $fb^{-1} = \sum \mathbb{Z} W_j$ together with the dual basis W^* determined by $\mathrm{tr}(W_i^* W_j) = \delta_{ij}$ $(i, j = 1, \ldots, n)$. Then

$$P(X) = N(b)N(\textstyle\sum X_i W_i), \quad Q(X) = N(\textstyle\sum X_i W_i^*), \quad v_j = \mathrm{tr}(W_j^*), \quad j = 1, \ldots, n\,.$$

Let $\varrho\colon U \to \mathrm{SL}_n\mathbb{Z}$ be the regular representation of U with respect to the basis W^*, and let $A_i = \varrho(\varepsilon_i)$ for a fixed set of generators $\varepsilon_1, \ldots, \varepsilon_r$ $(r = n-1)$ of U. The cycle $[\mathfrak{Z}] \in Z_{n-1}[\varrho(U), \mathbb{Z}]$ is then represented by

$$\mathfrak{Z} = \rho \sum_{\pi} \mathrm{sign}(\pi)\, [A_{\pi(1)} | \cdots | A_{\pi(r)}], \quad \rho = \pm 1\,,$$

where $[A_1 | \cdots | A_r] = (1, A_1, A_1 A_2, \ldots, A_1 \ldots A_r) \in \Gamma^n$, and π runs over all permutations of $\{1, \ldots, r\}$. To fix the orientation ρ, we identify $\alpha \in F$ with the row vector $(\alpha^{(1)}, \ldots, \alpha^{(n)}) \in \mathbb{R}^n$ using the n different embeddings of F in $\mathbb{R}$. This allows us to view W as an element in $\mathrm{GL}_n\mathbb{R}$, and to define ρ by

$$\rho = (-1)^r \mathrm{sign}(\det W)\mathrm{sign}(R), \quad R = \det(\log|\varepsilon_i^{(j)}|), \quad 1 \leqq i, j \leqq r\,.$$

With this definition of the orientation ρ, the homology class of $[\mathfrak{Z}]$ in $H_{n-1}(\varrho(U), \mathbb{Z})$ does not depend on the choice of the generators ε_i. Our main result can now be stated as follows.

Theorem 1 $\zeta(b, f; 1-s) = (2\pi i)^{-ns}\, \Psi(\mathfrak{Z})(P^{s-1}, Q, 0, v)$, $s = 1, 2, 3, \ldots$.

The right side does not depend on the choice of a $\mathbb{Z}$-basis for fb^{-1} – fact which is easily checked directly using the cocycle property of Ψ together with Lemma 4. It can therefore be viewed as the evaluation of a cohomology class $\eta(b, f, s) \in H^{n-1}(U, \mathbb{Q})$ on a fundamental class in $H_{n-1}(U, \mathbb{Z})$.

Since the special values do not vanish identically, Theorem 1 implies in particular:

Corollary. *The Eisenstein cocycle Ψ represents a nontrivial class in $H^{n-1}(\Gamma, M)$.*

1.3 The main difficulty in the construction of Ψ is to show that the Q-limit (1) does exist. Ultimately, this problem comes down to the study of the series

$$\mathscr{C}_e(u, v, Q) = \sum_{p \in \mathbb{Z}^n + u}{}' \left. \frac{\mathbf{e}((u-p)v)}{p^e} \right|_Q, \tag{3}$$

where e is an n-tuple of positive integers e_i and, for convenience of notation, p^e stands for $\Pi p_i^{e_i}$. A substantial part of our paper is devoted to the proof of a limit formula which evaluates this series in completely elementary terms. To state the limit formula, we consider first the case $n = 1$. Since the Q-limit is independent of Q for $n = 1$ (summation according to increasing values of $|p|$), we write $\mathscr{C}_e(u, v)$

instead $\mathscr{C}_e(u, v, Q)$ in that case. By results of Euler, Cauchy and Kronecker, we have for $u \in \mathbb{C}\backslash\mathbb{Z}$,

$$\mathscr{C}_1(u, v) = \begin{cases} 2\pi i \dfrac{e(u(v - [v]))}{e(u) - 1}, & v \notin \mathbb{Z} \\ 2\pi i \dfrac{1}{e(u) - 1} + \pi i, & v \in \mathbb{Z} \end{cases} . \tag{4}$$

The value of $\mathscr{C}_e(u, v)$ with $e > 1$ and $u \notin \mathbb{Z}$ can be obtained from this result by differentiation with respect to u, and is thus always an elementary function of u and v. For $u \in \mathbb{Z}$, one has

$$\mathscr{C}_e(u, v) = -\frac{(-2\pi i)^e}{e!} \mathbf{e}(uv) B_e(v - [v]), \quad u \in \mathbb{Z},\ v \in \mathbb{R}\backslash\mathbb{Z}, \tag{5}$$

with the Bernoulli polynomial $B_e(v)$, defined by the power series expansion

$$\frac{x \exp(xv)}{\exp(x) - 1} = \sum_{e=0}^{\infty} \mathrm{B}_e(v) \frac{x^e}{e!}, \quad (|x| < 2\pi).$$

Formula (5) remains valid for integral v provided the right side is replaced by zero if $e = 1$. It follows that $\mathscr{C}_e(u, v)$ is continuous in v except for $\mathscr{C}_1(u, v)$, which has a discontinuity at each $v \in \mathbb{Z}$. Let

$$\mathscr{C}_e(u, v^{\pm}) = \lim_{\varepsilon \to +0} \mathscr{C}_e(u, v \pm \varepsilon).$$

Then $\mathscr{C}_e(u, v^{\pm}) = \mathscr{C}_e(u, v)$ unless $e = 1$ and $v \in \mathbb{Z}$ where $\mathscr{C}_1(u, v^{\pm}) = \mathscr{C}_1(u, v) \mp \pi i$.

We are now ready to consider the general case $n \geqq 1$.

Theorem 2 *Let $Q(X)$ be a product of $m \geqq 1$ linear forms $L_k(X) = \sum l_{kj} X_j$ with non-zero real coefficients l_{kj}. Then for $u \in \mathbb{C}^n$* and *$v \in \mathbb{R}^n$,*

$$\mathscr{C}_e(u, v, Q) = \frac{1}{2}(h(u, v) + (-1)^{|e|} h(-u, -v)) \quad \textit{where } |e| = \sum e_j \quad \textit{and}$$

$$h(u, v) = \frac{1}{m} \sum_{k=1}^{m} \prod_{j=1}^{n} \mathscr{C}_{e_j}(u_j, v_j^{\operatorname{sign}(l_{kj})}).$$

Moreover, the sequence of partial sums (1) *defining this Q-limit converges uniformly for v in a compact subset of the open cube* $(0, 1)^n$ *and u in a compact subset of $\mathbb{C}^n$ on which $h(u, v)$ is bounded.*

Two special cases are worth mentioning. If there is no j with $e_j = 1$ and $v_j \in \mathbb{Z}$, the theorem says that $\mathscr{C}_e(u, v, Q) = \prod \mathscr{C}_{e_j}(u_j, v_j)$ is independent of Q. This is obvious if all e_j are greater than 1 because the series is then absolutely convergent. In the opposite extreme case where all $e_j = 1$ and $v_j \in \mathbb{Z}$,

$$h(u, v) = \frac{1}{m} \sum_{k=1}^{m} \prod_{j=1}^{n} (\pi \cot(\pi u_j) - \operatorname{sign}(l_{kj}) \pi i)$$

assuming no u_j is integral. If in addition all u_j are integral, then the theorem says

$$\sum_{p \in \mathbb{Z}^n}{}' \left.\frac{1}{p}\right|_Q = \operatorname{Re} \frac{(\pi i)^n}{m} \sum_{k=1}^{m} \operatorname{sign}\left(\prod_{j=1}^{n} l_{kj} \right),$$

which is equal to $\mathrm{Re}(\pi i)^n$ if all l_{kj} have the same sign. In particular,

$$\lim_{t\to\infty} \sum_{|p_1+\cdots+p_n|<t}' \frac{1}{p_1\cdots p_n} = \mathrm{Re}(\pi i)^n .$$

Theorem 2 does not only show that the Eisenstein cocycle Ψ is well defined, but also leads to a finite formula for Ψ in terms of the cyclotomic functions $\mathscr{C}_e$. For a precise statement of this complicated formula (generalized Dedekind sums), we refer to Theorem 6 in Sect. 4.1. It follows from this theorem that the values

$$(2\pi i)^{-n-\deg(P)}\Psi(\mathfrak{A})(P, Q, u, v)$$

are cyclotomic (resp. rational) numbers for $P \in \mathbb{Q}[X_1, \ldots, X_n]$, $v \in \mathbb{Q}^n$ and $u \in \mathbb{Q}^n$ (resp. $u = 0$) which obey a reciprocity law under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In the case $n = 2$, Theorem 7 implies in particular that Ψ is identical with the classical Dedekind-Rademacher cocycle for GL_2.

We now briefly review the content of the individual sections of our paper. Section 2.1 contains various elementary observations about the rational cocycle ψ. In Sect. 2.2, we define the Eisenstein cocycle Ψ in a slightly more general context, and study its behavior under the action of $G(\mathbb{Q})$. In Sect. 3.1, we show how a natural rearrangement of the series defining Ψ leads to special values of Hecke L-functions and partial zeta functions. Section 3.2 contains a key lemma in the proof of Theorem 1. In Sect. 4.1 we derive a finite formula for the value of Ψ on an arbitrary n-tuple of matrices in $G(\mathbb{Q})$. This result is supplemented in 4.2 by a somewhat simpler expression for the value of Ψ on very special n-tuples which we call "diagonal." The final Sect. 5 is devoted to the proof of the limit formula stated in Theorem 2. The proof is elementary, but lengthy. An overview is given in Sect. 5.1. Section 5 was written separately from the rest of the paper since we believe that its content is of independent interest.

Historical remark. In some sense, our paper can be viewed as a partial generalization of Eisenstein's paper "Genaue Untersuchung der unendlichen Doppelproducte, aus denen die elliptischen Functionen als Quotienten zusammengesetzt sind" [Eis, W] which is the historic origin of the notion of Eisenstein series. In this paper Eisenstein defines cyclotomic and elliptic functions by their partial fraction decompositions and shows how to derive non-trivial relations among these functions (addition theorems) starting with identities between rational functions. The connection with the present paper arises from the observation that these rational identities are all special cases of the cocycle property of the rational cocycle ψ. Indeed, the addition theorem for cyclotomic functions which Eisenstein derives by averaging the values of a rational function over a coset mod $\mathbb{Z}^2$, is a special case of the cocycle property of Ψ for GL_2, cf. [Sc1]. As Eisenstein indicates in his paper, he learned this approach to the theory of cyclotomic functions from his high school teacher Karl Heinrich Schellbach [Sch].

2.1 The rational cocycle

For n column vectors $\sigma_1, \ldots, \sigma_n \in \mathbb{R}^n$, and a row vector $x \in \mathbb{R}^n$, all different from zero, let

$$f(\sigma)(x) = f(\sigma_1, \sigma_2, \ldots, \sigma_n)(x) = \frac{\det(\sigma_1, \sigma_2, \ldots, \sigma_n)}{\langle x, \sigma_1\rangle \langle x, \sigma_2\rangle \cdots \langle x, \sigma_n\rangle} .$$

Here $\langle x, \sigma_j \rangle$ denotes the usual scalar product on $\mathbb{R}^n$. Note that, as a function of x, $f(\sigma)$ is well defined outside the hyperplanes $\langle x, \sigma_j \rangle = 0$, $j = 1, 2, \ldots, n$. Since $f(\sigma)$ does not change if σ_j is replaced by $\lambda\sigma_j$, $\lambda \in \mathbb{R}^*$, we may also view the σ_j as points on the projective space $\mathbb{P}^{n-1}(\mathbb{R})$. The n points σ_j decompose $\mathbb{P}^{n-1}(\mathbb{R})$ into 2^{n-1} simplices. If $\langle x, \sigma_j \rangle \neq 0$ for all j, then the hyperplane $\langle x, y \rangle = 0$ in $\mathbb{P}^{n-1}$ given by x meets all but one of these simplices. It was shown by Hurwitz [Hu] that $f(\sigma)(x)$ is the value of the integral

$$(n-1)! \int_S \Omega_x, \quad \Omega_x = \langle x, y \rangle^{-n} \sum_{i=1}^{n} (-1)^{i-1} y_i dy_1 \ldots d\hat{y}_i \ldots dy_n$$

over this unique simplex S in $\mathbb{P}^{n-1}$ defined by the pair (σ, x).

Lemma 1 $$\sum_{i=0}^{n} (-1)^i f(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_n) = 0 . \tag{6}$$

Proof. Consider the $(n+1) \times (n+1)$ determinant

$$\det \begin{pmatrix} \langle x, \sigma_0 \rangle & \langle x, \sigma_1 \rangle & \ldots & \langle x, \sigma_n \rangle \\ \sigma_0 & \sigma_1 & \ldots & \sigma_n \end{pmatrix} = 0 .$$

This determinant clearly vanishes because the first row is a linear combination of the other rows. Expanding it along the first row, and dividing by $\prod \langle x, \sigma_i \rangle$, gives the statement of the lemma.

A further obvious property of f is $f(A\sigma)(x) = \det(A) f(\sigma)(xA)$, i.e.,

$$f(A\sigma_1, A\sigma_2, \ldots, A\sigma_n)(x) = \det(A) f(\sigma_1, \sigma_2, \ldots, \sigma_n)(xA) \tag{7}$$

for every $A \in \mathrm{GL}_n\mathbb{R}$. If we restrict x to a subspace $V \subset \mathbb{R}^n$, then we have the relation

$$\det(\tau) f(\sigma)(x) = \det(\sigma) f(\tau)(x)$$

for every set of vectors τ_j such that $\tau_j - \sigma_j \in V^\perp$, the orthogonal complement of V in $\mathbb{R}^n$. Let $\mathscr{H} \subset \mathbb{R}[X_1, X_2, \ldots, X_n]$ be the subspace of homogenous polynomials $P(X) = P(X_1, \ldots, X_n)$. For $A \in \mathrm{GL}_n\mathbb{R}$ and $P \in \mathscr{H}$, let $AP \in \mathscr{H}$ be the polynomial defined by $AP(X) = P(XA)$. Then $(AB)P = A(BP)$, i.e. the map $P \mapsto AP$ defines a GL_n action on $\mathscr{H}$ from left. Consider the space N of all functions $\phi: \mathscr{H} \times \mathbb{R}^n \mapsto \mathbb{R}$ satisfying

$$\phi(\lambda^{\deg(P)} P, \lambda x) = \lambda^{-n} \phi(P, x), \quad \lambda \neq 0 .$$

This property implies that the projective group $PG = \mathrm{PGL}_n\mathbb{R}$ acts on N from left by

$$A\phi(P, x) = \det(A) \phi(A^t P, xA) \quad \text{for } \phi \in N, \ A \in G.$$

We extend $f(\sigma)$ to an element of N by the definition

$$f(\sigma)(P, x) = P(-\partial_{x_1}, \ldots, -\partial_{x_n}) f(\sigma)(x) \tag{8}$$

where ∂_{x_j} denotes the partial derivative with respect to the variable x_j. For fixed P, this is again a rational function of x which is well defined outside the hyperplanes $\langle x, \sigma_j \rangle = 0$, and clearly satisfies the identity (6). It is also clear that $f(\sigma)(P, x)$ is homogenous of degree 0 in every σ_j.

Lemma 2 $$Af(\sigma) = f(A\sigma).$$

Proof. By the chain rule,

$$(\partial_{x_1}, \ldots, \partial_{x_n}) f(xA) = ((\partial_{y_1}, \ldots, \partial_{y_n})^t A) f(y), \quad y = xA .$$

Therefore,

$$\begin{aligned} f(A\sigma)(P, x) &= P(-\partial_{x_1}, \ldots, -\partial_{x_n}) f(A\sigma)(x) \\ &= P(-\partial_{x_1}, \ldots, -\partial_{x_n})(\det(A) f(\sigma)(xA)) \\ &= \det(A) A^t P(-\partial_{y_1}, \ldots, -\partial_{y_n}) f(\sigma)(y) \\ &= \det(A) f(\sigma)(A^t P, xA) . \end{aligned}$$

As a corollary to Lemma 2, we observe that $f(\sigma)(P, x)$ admits the representation

$$f(\sigma)(P, x) = \det(\sigma) \sum_r P_r(\sigma) \prod_{j=1}^{n} \frac{r_j!}{\langle x, \sigma_j \rangle^{1+r_j}}, \tag{9}$$

where r runs over all decompositions of $\deg(P) = r_1 + \cdots + r_n$ into non-negative parts $r_j \geqq 0$, and $P_r(\sigma)$ is a homogenous polynomial in the σ_{ij} defined by the expansion

$$P(y\sigma^t) = \sum_r P_r(\sigma) \prod_{j=1}^{n} y_j^{r_j} .$$

In the case of a divided power $P(x) = x_1^{(g_1)} \cdots x_n^{(g_n)}$ with $z^{(k)} = z^k/k!$, we have

$$P_r(\sigma) = \sum \prod_{i,\, j=1}^{n} \sigma_{ij}^{(r_{ij})} ,$$

where the last sum runs over all simultaneous decompositions $r_j = r_{1j} + \cdots + r_{nj}$, $r_{ij} \geqq 0$ $(j = 1, \ldots, n)$ which satisfy the condition $r_{i1} + \cdots + r_{in} = g_i$.

Our next goal is to interpret Lemma 1 as a group cocycle property. To this end let $A_1, \ldots, A_n$ be n matrices in $GL_n\mathbb{R}$ representing elements in PG. We denote the columns of A_k by $A_{kj}, j = 1, \ldots, n$. Then for every k and every $x \in \mathbb{R}^n$, $x \neq 0$, there is a smallest index $j = j_k(x)$ such that $\langle x, A_{kj} \rangle \neq 0$. With this definition of $j_k = j_k(x)$, we let for $x \neq 0$,

$$\psi(A_1, \ldots, A_n)(P, x) = f(A_{1j_1}, A_{2j_2}, \ldots, A_{nj_n})(P, x) , \tag{10}$$

and set $\psi(A_1, \ldots, A_n)(P, x = 0$ for $x = 0$. (Note that the distinction between $x = 0$ and $x \neq 0$ is natural as these are the two orbits of $GL_n\mathbb{R}$ acting on $\mathbb{R}^n$.) By construction, $\psi(A_1, \ldots, A_n)$ is an element in the G-module N. Since this definition is crucial for the rest of the paper, we want to restate it in a second way. Let $d = (d_1, \ldots, d_n)$ be an n-tuple of integers, $1 \leqq d_j \leqq n$. Consider the space $A(d) \subseteq \mathbb{R}^n$ generated by all columns A_{ij} with $j < d_i$, and let $A(d)^\perp$ be the orthogonal complement of $A(d)$ in $\mathbb{R}^n$. Furthermore let

$$X(d) = A(d)^\perp \setminus \bigcup_{i=1}^{n} A_{id_i}^\perp .$$

Note that $X(d)$, if nonempty, is a subspace of $\mathbb{R}^n$ from which a finite number of codimension 1 subspaces is removed. Let

$$D = D(A_1, \ldots, A_n) = \{d \mid X(d) \neq \emptyset\} .$$

By construction of $X(d)$, we have associated with $(A_1, \ldots, A_n)$ a finite decomposition $\mathbb{R}^n \setminus \{0\}$,

$$\mathbb{R}^n \setminus \{0\} = \dot{\bigcup_{d \in D}} X(d) \qquad \text{(disjoint union).} \tag{11}$$

In terms of this decomposition, the definition of ψ is given by

$$\psi(A_1, \ldots, A_n)(P, x) = f(A_{1d_1}, A_{2d_2}, \ldots, A_{nd_n})(P, x) \quad \text{for } x \in X(d) \,. \tag{10a}$$

The cardinality of D, i.e. the number of nonempty sets $X(d)$ depends in general on the matrices A_i, but it is at most δ_n where

$$\delta_n = \binom{2n-1}{n-1} \sim \frac{4^n}{\sqrt{4\pi n}} \,.$$

The cardinality of D reaches its maximum in the generic case, that is when

$$\dim A(d) = \sum_i (d_i - 1)$$

holds for all nonempty $X(d)$. In this case every $d \in D$ satisfies the condition $\sum d_i < 2n$. Counting the number of all d satisfying the last inequality gives δ_n. This proves the estimate $\# D \leqq \delta_n$.

Theorem 3 *The map $\psi : PG^n \mapsto N$ represents a nontrivial cohomology class in $H^{n-1}(PG, N)$.*

Proof. By construction, ψ is a homogenous $(n-1)$ cocycle on G; that is, ψ has the properties

$$\psi(A\mathfrak{A}) = A\psi(\mathfrak{A}), \quad \mathfrak{A} = (A_1, \ldots, A_n) \in G^n \,, \tag{12}$$

$$\sum_{i=0}^{n} (-1)^i \psi(A_0, \ldots, \hat{A}_i, \ldots, A_n) = 0 \,. \tag{13}$$

In other words, ψ represents a cohomology class in $H^{n-1}(G, N)$. Since ψ is invariant under the center of G, it also represents a class in $H^{n-1}(PG, N)$. Let T be a subgroup of $\mathrm{SL}_n(\mathbb{R})$, and (P, x) a point in $\mathscr{H} \times \mathbb{R}^n$ such that the map $\psi(P, x)$ is invariant under the action of T on (P, x). Then the cocycle $\eta : T^n \to \mathbb{R}$ given by $\eta(\mathfrak{A}) = \psi(\mathfrak{A})(P, x)$ represents a class in $H^{n-1}(T, \mathbb{R})$ with trivial coefficients. Therefore, in order to show that ψ is not a coboundary, it suffices to construct a cycle in $Z_{n-1}(T, \mathbb{Z})$ on which η does not vanish. To this end let $T \simeq \mathbb{Z}^{n-1}$ be the free abelian subgroup generated by the unipotent matrices $A_i \in G$ $(i = 2, \ldots, n)$ whose entries are zero except for the diagonal elements and the $(i, 1)$ entry which are equal to 1. This subgroup fixes $P = 1$ and the unit vector $x = e_1$. It follows from Lemma 5 that $Z \in \mathbb{Z}[T^n]$,

$$Z = \sum_\pi \operatorname{sign}(\pi) \, [A_{\pi(2)} | \cdots | A_{\pi(n)}]$$

where $[A_2 | \cdots | A_n]$ denotes the n-tuple $(1, A_2, A_2 A_3, \ldots, A_2 \ldots A_n)$ and π runs over all permutations of $\{2, \ldots, n\}$, does represent a cycle $[Z] \in Z_{n-1}(T, \mathbb{Z})$. A calculation yields $\eta(Z) = (n-1)!$ which proves that the homology class of $[Z]$ and the cohomology class of η (and therefore of ψ) are both nontrivial.

For the applications in Sect. 3, we need a lemma about averaging ψ along a subgroup of G. The lemma is valid under the following more general conditions.

Let G be a group, $\chi: G \to \mathbb{C}^*$ a character, $H \subseteq G$ a subgroup with $\chi(H) = 1$, $\mathscr{S}$ a right G-set, and $M = \{\phi: \mathscr{S} \to \mathbb{C}\}$. Then M is a left G module: $(g\phi)(s) = \chi(g)\phi(sg)$. Let ψ be any $n-1$ cocycle on G with values in M. For $\mathfrak{A} \in G^n$, $\mathfrak{U} \in C(H)$ (set of conjugates of H in G) and $s \in \mathscr{S}/\mathfrak{U}$ represented by $r \in s$, we define (assuming absolute convergence if $\mathfrak{U}$ is infinite),

$$\psi_0(\mathfrak{A})(\mathfrak{U}, s) = \sum_{A \in \mathfrak{U}} \psi(A\mathfrak{A})(r) .$$

Lemma 3 *ψ_0 is a well-defined $n-1$ cocycle on G with values in $M_0 = \{\phi_0: \mathscr{S}_0 \to \mathbb{C}\}$, where*

$$\mathscr{S}_0 = \{(\mathfrak{U}, s): \mathfrak{U} \in C(H),\, s \in \mathscr{S}/\mathfrak{U}\} ,$$

and the action of G on $\mathscr{S}_0$ is given by $(\mathfrak{U}, s)g = (g^{-1}\mathfrak{U}g, sg)$. In particular, the restriction of ψ_0 to H defines an $n-1$ cocycle on H with values in the trivial H module $\{\phi: \mathscr{S}/H \to \mathbb{C}\}$.

We skip the very simple proof.

Lemma 4 *For any homogenous cocycle ψ on G and $B \in G$, let ψ_B be the cocycle defined by $\psi_B(\mathfrak{A}) = \psi(\mathfrak{A}B)$. Then the cohomology class of ψ_B is independent of the choice of B.*

As Jacob Sturm pointed out to me, this fact is a special case of Proposition 3 in [AW, p. 99].

The definition of the cocycle ψ as given in this section is a direct generalization of the inhomogenous 1-cocycle ϕ which was introduced in my earlier paper [Sc1] in the case $n = 2$. The connection is given by

$$\phi(A) = \psi(T, AT), \quad T = (\tau_1, -\tau_2, \tau_2, \tau_1) \in \mathrm{GL}_2\mathbb{R} \text{ fixed.}$$

It was shown in that paper that averaging the values $\phi(A)(x)$ over the lattice $\mathbb{Z}^2 + u$ with respect to x led to the classical Dedekind-Rademacher sums and special values of Hecke L-functions in real quadratic fields. It is therefore reasonable to expect that averaging the values of ψ over a lattice will lead to an object with similarly interesting arithmetic properties.

2.2 The Eisenstein cocycle

Let V be a subspace of row vectors in $\mathbb{Q}^n$, and $G_V \subseteq \mathrm{GL}_n\mathbb{Q}$ the subgroup of all matrices $A \in \mathrm{GL}_n\mathbb{Q}$ with the property $VA \subseteq V$. If $V = \mathbb{Q}^n$ or $\{0\}$, then clearly $G_V = \mathrm{GL}_n\mathbb{Q}$, but if V is a proper non-zero subspace of $\mathbb{Q}^n$, then G_V is the maximal parabolic subgroup of $\mathrm{GL}_n\mathbb{Q}$ associated to V. In this section, we fix a non-zero subspace $V \subseteq \mathbb{Q}^n$, and denote by $L = L(V) = \mathbb{Z}^n \cap V$ the lattice of integral points in V. Our goal is to study the series $\sum \psi(\mathfrak{A})(P, x)$ as x runs over the coset $L + u$ with some fixed vector $u \in X = V \otimes \mathbb{R}$ and an n-tuple $\mathfrak{A} \in G_V^n$. In general, this series converges only conditionally, so the order of terms does matter. To define its sum, we choose a decomposable form $Q(x) = \prod L_k(x)$ such that the coefficients l_{kj} of every linear form

$$L_k(x) = \sum_{j=1}^{n} l_{kj}x_j, \quad 1 \leqq k \leqq m ,$$

are rationally independent real numbers. This property of Q is clearly preserved under the action of $\mathrm{GL}_n\mathbb{Q}$, i.e. for $A \in \mathrm{GL}_n\mathbb{Q}$, the coefficients of AQ are again rationally independent real numbers. Let $\mathscr{Q} \subset \mathscr{H}$ be the set of such forms. For $Q \in \mathscr{Q}$ and a vector $v^t \in X$, we define $\Psi = \Psi_V$ by

$$\Psi(\mathfrak{A})(P, Q, u, v) = \sum_{x \in L+u} \mathbf{e}((u-x)v)\psi(\mathfrak{A})(P, x)\Big|_Q \tag{14}$$

where the Q-limiting process $|_Q$ is defined in Eq. (1) of the introduction. The existence of this limit is not obvious, but it will follow from the calculations in Sect. 4.1, where we establish a finite expression for it using Theorem 2. However, it is clear from the definition (1) that this limit does not change if Q is replaced by a non-zero multiple of any positive integral power of Q. Note that $\Psi = \psi$ if $V = \{0\}$ which explains why $V \neq 0$ is the only case of interest. Summing the relation (13) term by term for fixed (P, Q, u, v), we get for $A_0, \ldots, A_n \in G_V$,

$$\sum_{i=0}^{n} (-1)^i \Psi(A_0, \ldots, \hat{A}_i, \ldots, A_n) = 0 . \tag{15}$$

The action of $A \in G_V$ on ψ does not always carry over to Ψ. Calculating formally, we get

$$\begin{aligned}\Psi(A\mathfrak{A})(P, Q, u, v) &= \det(A) \sum_{x \in L+u} \mathbf{e}((u-x)v)\psi(\mathfrak{A})(A^tP, xA) \\ &= \det(A) \sum_{p \in LA} h(p), \quad p = (x-u)A , \\ h(p) &= \mathbf{e}(-pA^{-1}v)\psi(\mathfrak{A})(A^tP, p + uA) .\end{aligned} \tag{16}$$

At this point it is convenient to assume that A is integral. This is no loss of generality as we can always replace A by λA with some rational number $\lambda \neq 0$. Then LA is a sublattice of L with index $|L:LA|$. In order to make use of the character relations for the finite group L/LA, we introduce the dual of L in V,

$$L^* = \{r \in V: \langle r, p\rangle \in \mathbb{Z} \text{ for all } p \in L\} .$$

Then

$$\begin{aligned}|L:LA| \sum_{p \in LA} h(p) &= \sum_{q \in L} \sum_{r \in L^*/AL^*} \mathbf{e}(-qA^{-1}r)h(q) \\ &= \sum_{r \in L^*/AL^*} \sum_{q \in L} \mathbf{e}(-qA^{-1}(r+v))\psi(\mathfrak{A})(A^tP, q + uA) \\ &= \sum_{r \in L^*/AL^*} \Psi(\mathfrak{A})(A^tP, A^{-1}Q, uA, A^{-1}(r+v)) .\end{aligned} \tag{17}$$

To justify this formal manipulation, we only have to observe that every x in (16) can be written as $(q + uA)A^{-1}$ for some q in (17), thus the condition $|Q(x)| < t$ can be rewritten as $|(A^{-1}Q)(q + uA)| < t$. It follows that for integral $A \in G_V$, we have

$$\Psi(A\mathfrak{A})(P, Q, u, v) = \frac{\det A}{|L:LA|} \sum_{r \in L^*/AL^*} \Psi(\mathfrak{A})(A^tP, A^{-1}Q, uA, A^{-1}(r+v)) . \tag{18}$$

Let M be the space of complex valued functions ϕ on $\mathscr{H} \times \mathscr{Q} \times X \times X/L^*$ satisfying the distribution law

$$\phi(P, Q, u, v) = \frac{\lambda^n}{|\lambda|^{\dim X}} \sum_{\substack{w \in X/L^* \\ \lambda w = v}} \phi(\lambda^{\deg P} P, \lambda^{-\deg Q} Q, \lambda u, w)$$

for every non-zero integer λ. This property allows us to define an action of the projective group PG_V on M by

$$[A]\phi(P, Q, u, v) = \frac{\det A}{|L:LA|} \sum_{r \in L^*/AL^*} \phi(A^t P, A^{-1}Q, uA, A^{-1}(r + v)),$$

where A is an integral representative of an element $[A]$ in PG_V. If $A \in \Gamma_V$, $\Gamma_V = \mathrm{Aut}(L) \subset G_V$, then

$$[A]\phi(P, Q, u, v) = \det(A)\phi(A^t P, A^{-1}Q, uA, A^{-1}v).$$

In terms of this action, the relations (18) and (15) mean that the map $\Psi\colon PG_V^n \to M$ is a homogenous cocycle for the projective group PG_V. We call Ψ the *Eisenstein cocycle* associated to the parabolic group G_V. Note that the definition given in the introduction corresponds to the case $V = \mathbb{Q}^n$ and the restriction of Ψ to $\Gamma_V = GL_n\mathbb{Z}$. This case deserves special attention because of the connection to special values of L-functions. However, in the systematic study of the cohomology of GL_n, it is of interest to consider all Eisenstein cocycles.

Theorem 4 *The Eisenstein cocycle Ψ represents a nontrivial class in $H^{n-1}(PG_V, M)$.*

As in the proof of Theorem 3, it suffices to construct a cycle for a subgroup of G_V on which a suitable restriction of Ψ (representing a cohomology class with trivial coefficients) does not vanish. In the most interesting case $V = \mathbb{Q}^n$, such a cycle will be constructed in the next chapter out of a unit group in a totally real number field F of degree n over $\mathbb{Q}$. The nontriviality of the class represented by Ψ follows then from the non-vanishing of special values of Hecke L-functions in F (Theorem 5). In the opposite extreme case $\dim(V) = 1$, the relevant subgroup is not a group of units, but the unipotent subgroup T introduced in the proof of Theorem 3 (assuming V is spanned by the unit vector e_1). It is easy to see that for the cycle $[Z]$ and the cocycle η constructed there, we have

$$\Psi(Z)(1, Q, u_1 e_1, v) = \eta(Z)\mathscr{C}_n(u_1, v_1)$$

with the cyclotomic function $\mathscr{C}_n$ defined in (3). As an element of M, the left side is invariant under the action of T. Therefore, for a fixed choice of (u_1, v_1), the right side is the value of a cohomology class in $H^{n-1}(T, \mathbb{R})$ on the cycle $[Z]$. Since $\eta(Z) \neq 0$, this cohomology class is nontrivial iff $\mathscr{C}_n(u_1, v_1)$ does not vanish. The nontriviality of the class $[\Psi]$ represented by Ψ is therefore equivalent to the fact that the cyclotomic functions do not vanish identically. Since half the special values of Dirichlet L-functions (in $\mathbb{Q}$) are linear combinations of the division values of cyclotomic functions, the nontriviality of $[\Psi]$ follows again from the the non-vanishing of special values of L-functions. A similar argument also works in the case $1 < \dim(V) < n$ except that the unipotent subgroup T has to be replaced now by a semidirect product of a unit group in a totally real number field of degree $= \dim(V)$ with a unipotent group of rank $n - \dim(V)$. In effect, this reduces the proof to the case $\dim(V) = n$ which will be treated in Sect. 3.

The definition of the Eisenstein cocycle Ψ can be generalized slightly. Let $L \subseteq V \subseteq \mathbb{Q}^n$ be any lattice (not necessarily in $\mathbb{Z}^n$), and $\mathfrak{A}$ any n-tuple of matrices in $GL_n\mathbb{Q}$. Define $\Psi_L(\mathfrak{A})(P, Q, u, v)$ by the right side of (14). Then we have the obvious relation

$$\Psi_L(A\mathfrak{A})(P, Q, u, v) = \det(A)\Psi_{LA}(\mathfrak{A})(A^t P, A^{-1}Q, uA, A^{-1}v). \tag{19}$$

If L and LA are commensurable, then we can assume $LA \subseteq L$, and the right side of (19) is equal to the right side of (18). The cocycle property (15) remains true for Ψ_L with arbitrary $A_0, \ldots, A_n \in \mathrm{GL}_n\mathbb{Q}$. It's natural to ask whether the definition of Ψ_L makes also sense for a lattice L in $\mathbb{R}^n$ which is not similar to a lattice in $\mathbb{Q}^n$. No simple answer to this question is known as the convergence of the series defining Ψ_L poses a very delicate problem for a lattice $L \subset \mathbb{R}^n$ which does not project into the projective space $\mathbb{P}^{n-1}(\mathbb{Q}) \subset \mathbb{P}^{n-1}(\mathbb{R})$.

3.1 Special values of L-series in terms of the Eisenstein cocycle

Let $F/\mathbb{Q}$ be a totally real number field of degree n over $\mathbb{Q}$. We identify $\alpha \in F$ with the row vector $(\alpha^{(1)}, \ldots, \alpha^{(n)}) \in \mathbb{R}^n$ where the $\alpha^{(j)}$ are the n different embeddings of α into the field of real numbers. In that way F becomes a commutative $\mathbb{Q}$-algebra in $\mathbb{R}^n$ with componentwise multiplication. Let $W = (W_1, \ldots, W_n)$ be a $\mathbb{Q}$-basis for F. We consider W as a matrix in $\mathrm{GL}_n\mathbb{R}$ whose rows are the basis vectors W_j. It is convenient to assume $\det(W) > 0$. The row vectors of the transpose inverse matrix $W^{-t} = (W_1^*, \ldots, W_n^*)$ form the dual basis W^*. Associated to these bases are the norm forms

$$Q(x) = N_{F/\mathbb{Q}}(\xi), \quad \xi = \sum x_i W_i\,,$$
$$P(x) = N_{F/\mathbb{Q}}(\xi^*), \quad \xi^* = \sum x_i W_i^*\,.$$

Moreover, the basis W induces a representation $\varrho = \varrho_W : F^* \to \mathrm{GL}_n\mathbb{Q}$ of the multiplicative group of F given by

$$\varrho(x) = W\delta(\alpha)\,W^{-1}\,,$$

where $\delta(\alpha)$ denotes the diagonal matrix with entries $\delta_{ii} = \alpha^{(i)}$. A calculation yields

$$(\varrho(\alpha)Q)(x) = N_{F/\mathbb{Q}}(\alpha)Q(x), \quad (\varrho(\alpha)^t P)(x) = N_{F/\mathbb{Q}}(\alpha)P(x)\,. \tag{20}$$

Let $\Lambda = \mathbb{Z}W_1 + \cdots + \mathbb{Z}W_n$ be the lattice generated by the W_j. It's known that Λ is a fractional ideal with respect to the multiplier ring $\mathfrak{O}_\Lambda = \{\mu \in F : \mu\Lambda \subseteq \Lambda\}$ of Λ. The units in $\mathfrak{O}_\Lambda$ are exactly the $\varepsilon \in \mathfrak{O}_\Lambda$ with $\varrho(\varepsilon) \in \mathrm{GL}_n\mathbb{Z}$. In other words,

$$\mathfrak{O}_\Lambda^* = \varrho^{-1}(\varrho(F^*) \cap \mathrm{GL}_n\mathbb{Z})\,.$$

According to the Dirichlet unit theorem, the rank of $\mathfrak{O}_\Lambda^*$ equals $r = n - 1$. Let $\varepsilon_1, \ldots, \varepsilon_r$ be any totally positive units which generate a subgroup U of finite index in $\mathfrak{O}_\Lambda^*$. This is exactly the case if $\varepsilon_j \in \mathfrak{O}_\Lambda^*$, $\varepsilon_j \gg 0$ for all j, and the regulator

$$R = R(\varepsilon_1, \ldots, \varepsilon_r) = \det(\log|\varepsilon_i^{(j)}|), \quad 1 \leq i, j \leq r\,, \tag{21}$$

is different from zero. Let $\mathfrak{U} = \varrho(U)$ be the subgroup of $\mathrm{SL}_n\mathbb{Z}$ generated by the $A_j = \varrho(\varepsilon_j)$. We consider the chain $[\mathfrak{Z}] \in C_r(\mathfrak{U}, \mathbb{Z})$ represented by

$$\mathfrak{Z} = \rho \sum_\pi \mathrm{sign}(\pi)\,[A_{\pi(1)}|\cdots|A_{\pi(r)}], \; \rho = (-1)^r \mathrm{sign}(R),$$

where $[A_1|\cdots|A_r] = (1, A_1, A_1A_2, \ldots, A_1 \ldots A_r)$, and π runs over all permutations of $\{1, \ldots, r\}$.

Lemma 5 $[\mathfrak{Z}]$ *is a cycle whose homology class in* $H_r(\mathfrak{U}, \mathbb{Z})$ *is independent of the choice of the* A_j.

Proof. By definition of the boundary operator ∂, we have $\partial[\mathfrak{Z}] = \sum(-1)^i[\mathfrak{Z}_i]$, where for $0 < i < r$,

$$\mathfrak{Z}_i = \rho \sum_\pi \operatorname{sign}(\pi)\,[A_{\pi(1)}|\cdots|A_{\pi(i-1)}|A_{\pi(i)}A_{\pi(i+1)}|A_{\pi(i+2)}|\cdots|A_{\pi(r)}]\,.$$

Let τ_i be the transposition which interchanges i and $i+1$. Then the two terms in $\mathfrak{Z}_i$ corresponding to π and $\pi\tau_i$ cancel each other. This shows $\mathfrak{Z}_i = 0$ for $0 < i < r$. For the other two terms, we can write

$$\mathfrak{Z}_0 + (-1)^r\mathfrak{Z}_r = \rho\left\{\sum_\pi \operatorname{sign}(\pi)A_{\pi(1)}[A_{\pi(2)}|\cdots|A_{\pi(r)}]\right.$$

$$\left. + (-1)^r \sum_\sigma \operatorname{sign}(\sigma)\,[A_{\sigma(1)}|\cdots|A_{\sigma(r-1)}]\right\}.$$

Let τ be the cycle $(1, 2, 3, \ldots, r)$. Letting $\sigma = \pi\tau$ in the second sum, we see again that the above expression represents the zero element in $C_{r-1}(\mathfrak{U}, \mathbb{Z})$. Now let $\eta_1, \ldots, \eta_r$ be another set of generators for U. The basis change $\varepsilon \to \eta$ corresponds to an element $h \in \mathrm{GL}_r\mathbb{Z}$ with the property $R(\eta) = \det(h)R(\varepsilon)$. To prove that the homology class $cl(\mathfrak{Z})$ of $[\mathfrak{Z}]$ is independent of the choice of the ε_j in U means to show that $cl(\mathfrak{Z})$ is invariant under the induced action of $\mathrm{GL}_r\mathbb{Z}$ on $[\mathfrak{Z}]$. It is enough to verify this property for a set of generators for $\mathrm{GL}_r\mathbb{Z}$. Using a particular set of 3 generators determined by Hua and Reiner [HR], the proof reduces to a lengthy calculation which we prefer not to reproduce here.

Consider $\lambda \in F$ such that for all $\varepsilon \in U$,

$$\varepsilon(\Lambda + \lambda) = \Lambda + \lambda\,, \tag{22}$$

that is, the $\varepsilon \in U$ act as multiplicative automorphisms on the coset $\Lambda + \lambda$. For given $\varepsilon = \varepsilon_1, \ldots, \varepsilon_r$, there are only finitely many $\lambda + \Lambda$ with this property. Conversely, for arbitrary $\lambda \in F$, it follows from Dirichlet's unit theorem that there are always independent units $\varepsilon_1, \ldots, \varepsilon_r$ in $\mathfrak{O}_\Lambda^*$ such that (22) holds. In the following, we assume that $U = U(\Lambda + \lambda)$ is the group of all totally positive units ε satisfying (22). Let $u \in \mathbb{Q}^n$ be a row vector such that $\lambda = \sum u_i W_i$. Then (22) means that

$$(\mathbb{Z}^n + u)A = \mathbb{Z}^n + u \quad \text{for all } A \in \mathfrak{U}\,. \tag{23}$$

We recall the definition of the Eisenstein cocycle Ψ associated to $V = \mathbb{Q}^n$,

$$\Psi(\mathfrak{A})(P, Q, u, v) = \sum_{x \in \mathbb{Z}^n + u} \mathbf{e}((u - x)v)\psi(\mathfrak{A})(P, x)\bigg|_Q\,,$$

and observe that because of (20), (23) and Lemma 2, we can rearrange the series over x according to the orbits of $\mathfrak{U}$ in $\mathbb{Z}^n + u$. Doing so, we want to assume that the additive character $\phi(x) = \mathbf{e}((u - x)v)$ is a character of the orbit $x\mathfrak{U}$, that is, $\phi(x) = \phi(xA)$ for $A \in \mathfrak{U}$. This condition is satisfied if the column vector $v \in \mathbb{Q}^n$ is chosen in such a way that

$$Av \equiv v \bmod \mathbb{Z}^n \quad \text{and} \quad uAv \equiv uv \mod \mathbb{Z} \tag{24}$$

holds for all $A \in \mathfrak{U}$. Thus, for P, Q, u, v satisfying (20), (23) and (24), we get the identity

$$\Psi(\mathfrak{A})(P, Q, u, v) = \sum_{x \in \mathbb{Z}^n + u/\mathfrak{U}} \mathbf{e}((u - x)v) \sum_{A \in \mathfrak{U}} \psi(A\mathfrak{A})(P, x)\bigg|_Q\,.$$

Since Q is invariant under the action of $\mathfrak{U}$, this ordering of the series is compatible with the Q-limiting process. By definition of ψ, the inner series is zero if $x = 0$. But the main point is that for a special choice of $\mathfrak{A}$ this series can be evaluated in simple terms for every $x \neq 0$. For $\mathfrak{A} = \mathfrak{Z}Y$, a translation of $\mathfrak{Z}$ by a fixed element $Y \in \mathrm{GL}_n\mathbb{Q}$ from the right, we have

Lemma 6 $$\sum_{A \in \mathfrak{U}} \psi(A\mathfrak{Z}Y)(P^{s-1}, x)\Big|_Q = \det(W)\frac{((s-1)!)^n}{Q(x)^s}, \quad s = 1, 2, 3, \ldots .$$

This identity allows us to express values of certain Hecke L-functions in F by the Eisenstein cocycle Ψ. Denote by χ_0 and χ_1 the two signature characters $\chi_0(\mu) = 1$, $\chi_1 = \operatorname{sign} N(\mu)$. Associated to these characters are the two Hecke L-functions $L_0(s)$, $L_1(s)$, given for $\mathrm{Re}(s) > 1$ by

$$L_j(s) = \sum_{\xi \in \Lambda + \lambda/U}{}' \chi_j(\xi)\frac{\mathrm{e}(\mathrm{tr}(\lambda - \xi)v)}{|N(\xi)|^s}$$

with $v = \sum v_i W_i^*$. For positive integral $s \equiv j(2)$, we can write

$$L_j(s) = \sum_{x \in \mathbb{Z}^n + u/\mathfrak{U}}{}' \frac{\mathrm{e}((u - x)v)}{Q(x)^s}\Big|_Q .$$

This is clear for $s > 1$ since this series converges then absolutely. If $s = 1$, the series $L_1(1)$ converges conditionally, and according to a well known lemma about convergence of Dirichlet series, the analytic continuation of $L_1(s)$ to $s = 1$ is given by the limiting process $|_Q$.

Theorem 5a. *For $s = 1, 2, 3, \ldots$ with $s \equiv j(2)$, we have*

$$((s-1)!)^n \det(W) L_j(s) = \Psi(\mathfrak{Z}Y)(P^{s-1}, Q, u, v) .$$

The L-function $L_j(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ if $j = 0$ and $v \in \Lambda^*$, the dual lattice of Λ given by $\Lambda^* = \sum \mathbb{Z} W_i^*$. Let

$$L_j^*(s) = \sum_{\xi \in \Lambda^* + v/U}{}' \chi_j(\mu)\frac{\mathrm{e}(\mathrm{tr}(v - \xi)\lambda)}{|N(\xi)|^s}, \quad \mathrm{Re}(s) > 1 .$$

According to a well known result of Hecke [Si1], L_j and L_j^* are related by

$$\theta G(s) L_j(s) = G(1-s) L_j^*(1-s) ,$$

$$\theta = i^{nj}\det W, \quad G(s) = \pi^{-ns/2}\Gamma\left(\frac{j+s}{2}\right)^n .$$

Using this relation, we can calculate the analytic continuation of $L_j(s)$ for $\mathrm{Re}(s) < 0$ in terms of $L_j^*(s)$ for $\mathrm{Re}(s) > 1$. In particular, for $s = 1, 2, 3, \ldots$ with $j \equiv s(2)$, we get

$$\begin{aligned} L_j(1-s) &= \det W^* \left(\frac{2\Gamma(s)}{(2\pi i)^s}\right)^n L_j^*(s) \\ &= \det W^* \left(\frac{2\Gamma(s)}{(2\pi i)^s}\right)^n \sum_{x \in \mathbb{Z}^n + v^t/\mathfrak{U}^t}{}' \frac{\mathrm{e}((v^t - x)u^t)}{P(x)^s}\Big|_P \end{aligned}$$

where $\mathfrak{U}^t = \{A^t : A \in E\}$. Let $\mathfrak{Z}^t$ be the element arising from $\mathfrak{Z}$ by replacing $A_1, \ldots, A_r$ by $A_1^t, \ldots, A_r^t$. Replacing $(\mathfrak{Z}, \mathfrak{U}, W, P, Q, u, v)$ by $(\mathfrak{Z}^t, \mathfrak{U}^t, W^*, Q, P, v^t, u^t)$, and applying Lemma 6 again, we get

Theorem 5b. *For $s = 1, 2, 3, \ldots$ with $s \equiv j(2)$, we have*

$$L_j(1 - s) = 2^n (2\pi i)^{-ns} \Psi(\mathfrak{Z}^t Y)(Q^{s-1}, P, v^t, u^t).$$

This theorem can be given a simpler shape by introducing the partial zeta function

$$\zeta(\Lambda, \lambda, v; s) = \sum_{\substack{\xi \in \Lambda + \lambda/U \\ \xi \gg 0}} \frac{\mathrm{e}(\mathrm{tr}(\lambda - \xi)v)}{N(\xi)^s}, \quad \mathrm{Re}(s) > 1.$$

It was observed by Serre and Siegel [Si1], that Hecke's functional equation implies for $s = 1, 2, 3, \ldots$,

$$\zeta(\Lambda, \lambda, v; 1 - s) = 2^{-n} L_j(1 - s) \quad \text{where } j \equiv s(2).$$

Corollary. $\zeta(\Lambda, \lambda, v; 1 - s) = (2\pi i)^{-ns} \Psi(\mathfrak{Z}^t Y)(Q^{s-1}, P, v^t, u^t), \quad s = 1, 2, 3, \ldots.$

In particular, for $\Lambda = fb^{-1}$, $\lambda = 1$ and $v = 0$, this equation reduces to the statement of Theorem 1 in the introduction. Contrary to Theorem 5, no reference is being made here to the parity j. The simplicity of this equation reflects the fact that no simple result can be expected for the values $\zeta(\Lambda, \lambda, v; s)$ at $s = 1, 2, 3, 4, \ldots$ if $n > 1$. Finally, it should be noticed that the special values covered by Theorem 5a resp. 5b are exactly the critical points of $L_j(s)$ in the sense of Deligne. These are the only special values of Hecke L-functions in totally real number fields which are known to be either an algebraic number or an algebraic number times a power of π.

3.2 Proof of Lemma 6

It is more convenient to work with the units ε_j instead of the matrices A_j. Let $\xi = xW$, and

$$\mathfrak{A}(\eta, \pi) = \delta(\eta)\,[\delta(\varepsilon_{\pi(1)})| \cdots |\delta(\varepsilon_{\pi(r)})]$$

for $\eta \in U$. Using the action of W on ψ, we can then rewrite Lemma 6 as

$$\rho \sum_{\eta \in U} \sum_{\pi} \mathrm{sign}(\pi) \psi(\mathfrak{A}(\eta, \pi) W^{-1} Y)(W^t P^{s-1}, \xi) = \frac{((s-1)!)^n}{N(\xi)^s}. \tag{25}$$

The series on the left converges absolutely. This fact is implicitly contained in the definition of the Eisenstein cocycle, and will be proved in Sect. 5.12 by an elementary estimate. Lemma 3 and Lemma 4 imply therefore that the left side is independent of the choice of $Y \in \mathrm{GL}_n\mathbb{Q}$. This simple observation is of great importance in the following proof of (25).

Let $D \subseteq \mathrm{SL}_n\mathbb{R}$ be the subgroup of diagonal matrices with positive entries on the diagonal, and $\mathbf{l}: D \to \mathbb{R}^n$ the logarithm map given by $\mathbf{l}(x) = y$, $y_j = \log x_{jj}$. The image of D under this map is the hypersurface $H \subset \mathbb{R}^n$ defined by $\sum y_j = 0$. The element $\mathfrak{A}(\eta, \pi)$ defines in H an oriented simplex $S(\eta, \pi)$ whose k-th vertex is the image of the k-th component of $\mathfrak{A}(\eta, \pi)$ under the map $\mathbf{l}$. If $v_1, \ldots, v_n$ are the vertices of any oriented $(n-1)$-simplex in H, we define its orientation by the sign of

$$\det(e, v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1}), \quad e = (1, 1, \ldots, 1)^t. \tag{26}$$

It is easy to see that with respect to this definition, the orientation of $S(\eta, \pi)$ equals $\rho\,\mathrm{sign}(\pi)$. Let

$$H_j = \{y \in H : y_i < 0 < y_j \text{ for all } i \neq j\}, \quad j = 1, \ldots, n.$$

We choose n units $\theta_j \in U$ such that $\mathrm{l}(\delta(\theta_j)) \in H_j$ for every j, and claim that the simplex with vertices $\mathrm{l}(\delta(\theta_j))$, $j = 1, 2, \ldots, n$, is positively oriented. This assertion is not obvious. As a matter of fact, it is a key step in the classical proof of the Dirichlet unit theorem [La, p. 107] to show that any $n-1$ of the vertices $\mathrm{l}(\delta(\theta_j))$ are linearly independent. By continuity, it follows that the sign of the determinant (26) is independent of the particular choice of the θ_j as long as the condition $\mathrm{l}(\delta(\theta_j)) \in H_j$ is maintained. To calculate this sign, we pass for every j to a limit $v_j = \lim \mathrm{l}(\delta(\theta_j)) \in H$ such that

$$v_{n1} < 0, \quad v_{jj} > 0, \quad v_{ji} = 0 \quad \text{for } i \neq j, n .$$

A routine calculation shows then that for such v_j the determinant (26) is indeed positive. Our next goal is to calculate the limit

$$\lim_{a \to +\infty} \psi(\mathfrak{G} W^{-1} Y)(W^t P^{s-1}, \xi), \text{ where} \tag{27}$$

$$\mathfrak{G} = \mathfrak{G}(a) = (\delta(\theta_1^a), \ldots, \delta(\theta_n^a)) .$$

We begin with the observation that the rows of W^{-1} are conjugated over $\mathbb{Q}$. The same is true for $W^{-1}Y$ (since $Y \in \mathrm{GL}_n\mathbb{Q}$), and therefore all entries of $W^{-1}Y$ are nonzero real numbers. It follows that $\delta(\theta_j^a) W^{-1} Y$ modulo projective equivalence converges for $a \to +\infty$ to a matrix whose nonzero entries are exactly the entries in the j-th row. According to the definition of P,

$$W^t P^{s-1}(\xi_1, \ldots, \xi_n) = (\xi_1 \cdots \xi_n)^{s-1} ,$$

and therefore by (8) and (10), the limit (27) is equal to

$$(-1)^{n(s-1)} (\partial_{\xi_1} \cdots \partial_{\xi_n})^{s-1} f(1_n)(\xi) = \frac{((s-1)!)^n}{N(\xi)^s} .$$

It remains to show that the left side of (25) is equal to the limit (27). To this end we consider the simplex $S = S(a)$ in H corresponding to $\mathfrak{G}(a)$ under the map l for a positive integer a. This simplex meets only finitely many of the simplices $S(\eta, \pi)$, and thus determines the element

$$\mathfrak{T} = \mathfrak{T}(a) = \sum_{\substack{\eta, \pi \\ S(\eta,\pi) \cap S \neq \emptyset}} \rho \operatorname{sign}(\pi) \mathfrak{A}(\eta, \pi) .$$

We want to estimate the difference

$$\psi(\mathfrak{T} W^{-1} Y) - \psi(\mathfrak{G} W^{-1} Y)$$

evaluated on $(W^t P^{s-1}, \xi)$ for large integral values of a. Since every individual term in $\mathfrak{T}$ has the same (positive) orientation as $\mathfrak{G}$, and since ψ is a cocycle, the value $\psi(\mathfrak{T} W^{-1} Y)$ does not change if $\mathfrak{T}$ is replaced by a homologous element of the form $\mathfrak{G} + \sum_i \mathfrak{B}_i$. Then

$$\psi(\mathfrak{T} W^{-1} Y) - \psi(\mathfrak{G} W^{-1} Y) = \sum_i \psi(\mathfrak{B}_i W^{-1} Y) . \tag{28}$$

Let B_i be the $(n-1)$ simplex in H corresponding to the n-tuple $\mathfrak{B}_i$, and let T be the chain in H corresponding to $\mathfrak{T}$ under the map l. We can assume that every B_i is the join of a subsimplex of S with a subsimplex in the boundary of T such that $B_i \cap S \subseteq \partial S$. [One way to construct such a chain $\sum B_i$ is to start with a vertex v of

S and join v with every $(n-2)$ simplex in the boundary of T which is visible from v. After repeating this process with every vertex of S, continue by joining the 1-subsimplices of S which are not covered yet with the visible $(n-3)$ subsimplices in the boundary of T. Repeat this process will all higher dimensional subsimplices of S until $T-S$ is completely covered]. It is clear that the number of simplices B_i in the chain $\sum B_i$ grows polynomially with a. To complete the proof of (25), it suffices therefore to prove that every term $\psi(\mathfrak{B}_i W^{-1} Y)$ in (28) tends to zero exponentially fast for $a \to \infty$. At this point we take advantage of the fact that the left side of (25) is independent of Y. Let $\tau \in F$ be such that τ^t is the first column of $W^{-1}Y$. We choose $Y \in GL_n\mathbb{Q}$ such that $\xi\tau$ is totally positive. Writing $\mathfrak{B}_i = (\delta(\eta_1), \ldots, \delta(\eta_n))$ with units $\eta_j \in U$, we have then

$$\psi(\mathfrak{B}_i W^{-1} Y)(W^t P^{s-1}, \xi) = \pm(\partial_{\xi_1} \cdots \partial_{\xi_n})^{s-1}\left(\frac{\det((\eta_1\tau)^t, \ldots, (\eta_1\tau)^t)}{\operatorname{tr}(\xi\eta_1\tau)\ldots\operatorname{tr}(\xi\eta_n\tau)}\right)$$

since all $\xi\eta_j\tau$ $(j = 1, \ldots, n)$ are totally positive. The assumption on the position of B_i relative to S implies the existence of an index $k = k(i)$ such that for every j, $\eta_j^{(k)} = O(t^a)$ for $a \to \infty$, where

$$t = \max_{j \neq l} \theta_j^{(l)} < 1 .$$

Expanding the determinant along the k-th row, we conclude that

$$\det((\eta_1\tau)^t, \ldots, (\eta_1\tau)^t) = O\left(t^a \prod_j \max_l \eta_j^{(l)}\right), \; a \to \infty .$$

Combining this with (9) gives the desired estimate

$$\psi(\mathfrak{B}_i W^{-1} Y)(W^t P^{s-1}, \xi) = O(t^a), \; a \to \infty .$$

This finishes the proof of Lemma 6. An alternative proof can be given using the result of Hurwitz stated in Sect. 2.1. Lemma 6 reduces then to an obvious statement about a fundamental domain $\mathscr{F}$ for U: the projective volumes of all translates of $\mathscr{F}$ by U must add up to the projective volume of the space on which U acts. Viewed in that way, it is clear that Lemma 6 is only a variation of a basic identity Hecke has used extensively in order to establish the functional equation of L-functions associated with number fields. The important difference is that all terms in Lemma 6 are rational expressions, while the corresponding terms in Hecke's work are non-elementary integrals. The existence of such a "rational" version of Hecke's identity was first discovered by Siegel [Si2] who studied the case $n = 2$ only. The present paper as well as an earlier conjecture of mine, recently proved by Colmez [Co], were inspired by Siegel's work. The paper of Colmez is also based on Lemma 6, but the proof given there (and its simplification by Weselmann [We]) does not use the cocycle property of ψ and is, for that reason, rather complicated.

4.1 A finite expression for the Eisenstein cocycle

We return to the notation of Sect. 2: Let V be any non-zero subspace of $\mathbb{Q}^n$ and $\mathfrak{A} = (A_1, \ldots, A_n)$ with integral matrices A_i in G_V, the maximal parabolic subgroup associated to V. The n-tuple $\mathfrak{A}$ defines a stratification (11) of $\mathbb{R}^n$ which is indexed by D. For $d \in D$, we write $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = A_{id_i}$. By definition of Ψ, we have the decomposition

$$\Psi_V(\mathfrak{A})(P, Q, u, v) = \sum_{d \in D} F(d)\,,$$

$$F(d) = \sum_{m \in X(d)} \mathbf{e}((u-m)v)\, f(\sigma)(P, m)\Big|_Q\,.$$

In the last equation, m denotes a general element of $L + u$, so $m \in X(d)$ has to be understood as $m \in (L + u) \cap X(d)$. According to (9), we have $F(d) = 0$ unless $\det(\sigma) \neq 0$ where

$$F(d) = \det(\sigma) \sum_{|r| = \deg(P)} P_r(\sigma) G_r(\sigma)\,,$$

$$G_r(\sigma) = \sum_{m \in X(d)} \mathbf{e}((u-m)v) \prod_{j=1}^{n} \frac{r_j!}{\langle m, \sigma_j \rangle^{1+r_j}}\Bigg|_Q\,. \tag{29}$$

In order to establish a finite formula for $\Psi(\mathfrak{A})$, it suffices therefore to evaluate $G_r(\sigma)$ in finite terms for all $d \in D$ with $\det(\sigma) \neq 0$. We can assume that the summation set $(L + u) \cap X(d)$ is not empty for otherwise $G_r(\sigma) = 0$. Then

$$(L + u) \cap A(d)^{\perp} = M + w, \quad M = L \cap A(d)^{\perp} \tag{30}$$

with some fixed vector $w \in \mathbb{R}^n$ satisfying $u - w \in L$. As a preparation for the general case, we consider first the special case $M = L = \mathbb{Z}^n$. (This is the only case to consider if $V = \mathbb{Q}^n$ and the components u_j $(j = 1, \ldots, n)$ of $u \in \mathbb{R}^n$ are rationally independent). Then $L\sigma$ is a sublattice of finite index in L generated by the rows of σ. As in (17), we introduce the character group of $L/L\sigma$ which we identify with $\sigma^{-1}(L^*/\sigma L^*)$, where L^* is the lattice of integral column vectors (while L is the lattice of integral row vectors). We write

$$G_r(\sigma) = \sum_{p \in (L+u)\sigma}{}' h(p)\Big|_R\,, \quad p = m\sigma, \quad R = \sigma^{-1}Q\,,$$

$$h(p) = \mathbf{e}((u\sigma - p)\sigma^{-1}v)\frac{r!}{p^e}, \quad r! = \prod (r_j!), \quad e_j = r_j + 1\,,$$

and apply the character relations:

$$|L/L\sigma| \sum_{p \in L\sigma + u\sigma}{}' h(p)\Big|_R = \sum_{p \in L + u\sigma}{}' h(p) \sum_{\mu \in L^*/\sigma L^*} \mathbf{e}((u\sigma - p)\sigma^{-1}\mu)\Big|_R$$

$$= \sum_{\mu \in L^*/\sigma L^*} \sum_{p \in L + u\sigma}{}' \mathbf{e}((u\sigma - p)\sigma^{-1}(\mu + v))\frac{r!}{p^e}\Bigg|_R\,. \tag{31}$$

Applying Theorem 2 here to the inner series leads to a representation of $|L/L\sigma| G_r(\sigma)$ as a finite sum which generalizes the classical Dedekind sum,

$$\mathrm{Ded}_r(\sigma, u, v, Q) \underset{\mathrm{def}}{=} r! \sum_{\mu \in L^*/\sigma L^*} \mathscr{C}_e(u\sigma, \sigma^{-1}(\mu + v), \sigma^{-1}Q)\,. \tag{32}$$

Corollary. *Let $L = \mathbb{Z}^n$. If the components $u_j (j = 1, \ldots, n)$ of $u \in \mathbb{R}^n$ are rationally independent, then*

$$\Psi(\mathfrak{A})(P, Q, u, v) = \operatorname{sign}(\det\sigma) \sum_{|r| = g} P_r(\sigma)\,\mathrm{Ded}_r(\sigma, u, v, Q)$$

with the integral matrix $\sigma = (A_{11}, A_{21}, \ldots, A_{n1})$.

The above calculation relies essentially on the assumption that the rank of M equals n. To deal with the case $\operatorname{rank}(M) < n$ we need a suitable generalization of the character sum over μ in (31). To this end let $\lambda_1, \ldots, \lambda_l$ be a minimal set of integral column vectors such that

$$M + w = \{m \in \mathbb{Z}^n + u \mid \langle m, \lambda_j \rangle = 0 \quad \text{for } j = 1, \ldots, l\}\,. \tag{33}$$

Note that the λ_j form an integral basis for the subspace $V^\perp + A(d) \subseteq \mathbb{Q}^n$. In particular, if $V = \mathbb{Q}^n$, the λ_j can be chosen among the A_{ij} with $j < d(i)$. Denote by λ the matrix with column's λ_j, and let x be a real variable in the unit cube $C = [0,1]^l$. Finally, let

$$\chi(m) = \int_C \mathbf{e}(m\lambda x)\,dx = \int_0^1 \cdots \int_0^1 \mathbf{e}\left(\sum_{j=1}^{l} \langle m, \lambda_j \rangle x_j\right) dx_1 \ldots dx_l\,.$$

Then for $m \in \mathbb{Z}^n + u$, we have $\chi(m) = 0$ unless $m \in M + w$ where $\chi(m) = 1$. Therefore,

$$\begin{aligned}
G_r(\sigma) &= \sum_{m \in M + w}{}' \left. h(m\sigma) \right|_Q \\
&= \sum_{m \in \mathbb{Z}^n + w}{}' \left. \chi(m)\, h(m\sigma) \right|_Q \\
&= \sum_{p \in (\mathbb{Z}^n + w)\sigma}{}' \left. \chi(p\sigma^{-1})\, h(p) \right|_R \\
&= |\det(\sigma)|^{-1} \sum_{\mu \in \mathbb{Z}^n/\sigma\mathbb{Z}^n} H(\mu),
\end{aligned}$$

$$H(\mu) = \sum_{p \in \mathbb{Z}^n + w\sigma}{}' \left. \mathbf{e}((w\sigma - p)\sigma^{-1}\mu)\,\chi(p\sigma^{-1})\,h(p) \right|_R\,.$$

Taking into account $w\lambda = 0$, we observe that

$$\begin{aligned}
&\mathbf{e}((w\sigma - p)\sigma^{-1}\mu)\,\mathbf{e}(p\sigma^{-1}\lambda x)\,\mathbf{e}((u\sigma - p)\sigma^{-1}v) \\
&\quad = \mathbf{e}((u - w)v)\,\mathbf{e}((w\sigma - p)\sigma^{-1}(\mu + v - \lambda x)),
\end{aligned}$$

and therefore,

$$\begin{aligned}
H(\mu) &= \mathbf{e}((u - w)v)\, r! \sum_{p \in \mathbb{Z}^n + w\sigma}{}' \int_C \left. \frac{\mathbf{e}((w\sigma - p)\sigma^{-1}(\mu + v - \lambda x))}{p^e}\,dx \right|_R \\
&= \mathbf{e}((u - w)v)\, r! \int_C \mathscr{C}_e(w\sigma, \sigma^{-1}(\mu + v - \lambda x), \sigma^{-1}Q)\,dx\,.
\end{aligned}$$

The interchange of summation and integration is justified by the dominated convergence theorem since the sequence of partial sums defining the Q-limit $\mathscr{C}_e(u, v, Q)$ is bounded by an integrable function of v. This will be proved in Sect. 5 (Lemma 10 and 16). In order to emphasize the analogy to the previous case, we introduce the function

$$\mathscr{C}_e(\lambda, u, v, Q) = \int_C \mathscr{C}_e(u, v - \lambda x, Q)\,dx \tag{34}$$

which is well-defined for arbitrary $n \times l$ matrices λ. For integral λ, the above calculation shows that

$$\mathscr{C}_e(\lambda, u, v, Q) = \sum_{\substack{p \in \mathbb{Z}^n + u \\ (p-u)\lambda = 0}}{}' \left. \frac{\mathbf{e}((u-p)v)}{p^e} \right|_Q$$

depends only on the columnspace of λ but not on λ itself. Our result can now be written in the form

$$|\det(\sigma)| G_r(\sigma) = \mathbf{e}((u-w)v) \operatorname{Ded}_r(M, \sigma, w, v, Q),$$

$$\operatorname{Ded}_r(M, \sigma, w, v, Q) \underset{\text{def}}{=} r! \sum_{\mu \in \mathbb{Z}^n / \sigma \mathbb{Z}^n} \mathscr{C}_e(\sigma^{-1}\lambda, w\sigma, \sigma^{-1}(\mu + v), \sigma^{-1} Q).$$

Note that this definition contains (32) as the special case $M = \mathbb{Z}^n$ (corresponding to $\lambda = 0 \in \mathbb{Z}^n$). Although this sum is independent of λ, its individual terms do depend in general on the choice of λ.

Theorem 6. *For a given n-tuple $\mathfrak{A}$ of integral matrices $A_i \in G_V$, let D_0 be the set of all $d \in D$ such that $M + w = L + u \cap A(d)^{\perp} \neq \emptyset$ and $\det(\sigma) \neq 0$ for $\sigma = (A_{1d_1}, \dots, A_{nd_n})$. Then*

$$\Psi_V(\mathfrak{A})(P, Q, u, v) = \sum_{d \in D_0} \operatorname{sign}(\det \sigma) \mathbf{e}((u-w)v) \sum_{|r| = \deg P} P_r(\sigma) \operatorname{Ded}_r(M, \sigma, w, v, Q)\,.$$

According to the definition of the coefficient $P_r(\sigma)$, the inner sum over r is the value of the differential operator $P(\partial_y \sigma^t)$ at $y = 0$ applied to the generating function

$$\operatorname{Ded}(M, \sigma, w, v, Q)(y) = \sum_r \operatorname{Ded}_r(M, \sigma, w, v, Q) \frac{y^r}{r!}$$

of the variable $y = (y_1, \dots, y_n)$, where r runs over all n-tuples of non-negative integers.

Corollary.

$$\Psi_V(\mathfrak{A})(P, Q, u, v) = \sum_{d \in D_0} \operatorname{sign}(\det \sigma) \, \mathbf{e}((u-w)v) \, P(\partial_y \sigma^t) \operatorname{Ded}(M, \sigma, w, v, Q)(y) \Big|_{y=0}\,.$$

To get a rationality statement about the values of the Eisenstein cocycle Ψ, we need a corresponding statement about the values of the cyclotomic functions $\mathscr{C}_e$.

Lemma 7 *If $u, v \in \mathbb{Q}^n$ and $u\lambda = 0$, then $(2\pi^i)^{-|e|} \mathscr{C}_e(\lambda, u, v, Q) \in \mathbb{Q}(\mathbf{e}(u_j), \mathbf{e}(u_j v_j)$, $j = 1, \dots, n)$.*

Corollary. $(2\pi i)^{-n-\deg(P)} \Psi(\mathfrak{A})(P, Q, 0, v) \in \mathbb{Q}$ *for $v \in \mathbb{Q}^n$ and $P \in \mathbb{Q}[x_1, \dots, x_n]$.*

For the proof of Lemma 7, we observe that according to Theorem 2, it suffices to consider the values

$$\int_C h_e(u, v - \lambda x, Q)\, dx, \quad h_e(u, v) = \prod_{j=1}^n \mathscr{C}_{e_j}(u_j, v_j^+)\,.$$

Using (4), one calculates easily the generating function

$$\sum_e h_e(u, v)\, y^e = (2\pi i)^n \prod_{j=1}^n \frac{y_j e((u_j - y_j)\{v_j\})}{e(u_j - y_j) - 1}\,,$$

where $\{t\} = t - [t]$ denotes the fractional part of t. Therefore, we have to show that the coefficient of $(2\pi i)^{-|e|} y^e$ in the Taylor expansion (at the origin) of

$$\int_C \prod_{j=1}^{n} \mathbf{e}\left((u_j - y_j)\left\{v_j - \sum_i \lambda_{ji} x_i\right\}\right) dx$$

belongs to the field specified in Lemma 7. The integrand here has discontinuities on the intersection of C with a finite set of hyperplanes given by the rational numbers v_j, λ_{ji}. It follows that C can be subdivided into a finite set of simplices S with vertices in $\mathbb{Q}^n$ such that the above integrand is smooth inside every S. Lemma 7 follows now from the fact that the coefficient of $(2\pi i)^{-|e|} y^e$ in the Taylor expansion of the integral of $\mathbf{e}(y\lambda x)\,dx$ over S is a rational number.

Although it is clear that the values $\mathscr{C}_e(\lambda, u, v, Q)$ can be calculated in finitely many steps as shown above, the question remains open whether there is another evaluation of the integral (34) which avoids the cumbersome subdivision of C into simplices. In the next section, we discuss a special case where this integral can be avoided altogether.

4.2 A special case: diagonal n-tuples

Let $V_d = V \cap A(d)^\perp$ be the subspace generated by the lattice M defined in (30). We consider the following hypothesis.

There are non-zero rational numbers α_j such that the set

$$\{\alpha_j \sigma_j \bmod V_d^\perp : j = 1, \ldots, n\} \text{ is a } \mathbb{Q}\text{-basis for } \mathbb{Q}^n / V_d^\perp. \tag{35}$$

If this condition is satisfied, then we can assume in addition that $\alpha_j = 1$ and $\sigma_j \in \mathbb{Z}^n$ for all j since $F(d)$ does not change if σ_j is replaced by $\alpha_j \sigma_j$. The hypothesis says then that for a generic point $x \in V_d$, the number of different scalar products among the $\langle x, \sigma_j \rangle$, $j = 1, \ldots, n$, is equal to the dimension of V_d. In other words, the subspace $V_d \sigma$ is a diagonal embedding of $\mathbb{Q}^l$ in $\mathbb{Q}^n$, $l = \dim(V_1)$. This means that there is a partition of $J = \{1, \ldots, n\}$ into subsets $J_i, 1 \leq i \leq l$, such that the components of $p \in V_d \sigma$ determined by the subset J_i are all equal. Motivated by this fact, we call a tuple $\mathfrak{A} = [A_1, \ldots, A_n]$ of integral matrices *V-diagonal* iff $\sigma = (A_{1d_1}, \ldots, A_{nd_n})$ satisfies condition (35) for all $d \in D_0$. Despite the complicated description, diagonal n-tuples exist for every n. An example is an n-tuple where all A_i are permutation matrices. More generally, $\mathfrak{A}$ is $\mathbb{Q}^n$-diagonal if the chambers corresponding to A_i belong to *one* apartment in the Tits building of $GL_n\mathbb{Q}$. If $n = 2$, then every n-tuple is automatically V-diagonal. In the case $n = 3$, it can be shown that every n-tuple is homologous to a linear combination of diagonal tuples, but it's not clear whether this remains true for $n > 3$. Diagonal $\mathfrak{A}$ are characterized by the fact that the series $G_r(\sigma)$ contributing to $\Psi(\mathfrak{A})$ are all of the same type as in the special case $M = L = \mathbb{Z}^n$ which we discussed in the beginning of the previous section. To see that, consider the lattice $M_d = \mathbb{Z}^n \cap V_d \sigma$. Then $M\sigma$ is a sublattice of finite index in M_d, and we can repeat the calculation (31) as follows:

$$|M_d/M\sigma| \sum_{p \in M\sigma + w\sigma}' h(p)\Big|_R = \sum_{p \in M_d + w\sigma}' h(p) \sum_{\mu \in M^*/\sigma M_d^*} \mathbf{e}((w\sigma - p)\sigma^{-1}\mu)\Big|_R$$

$$= \mathbf{e}((u - w)v) \sum_{\mu \in M^*/\sigma M_d^*} \sum_{p \in M_d + w\sigma}' \mathbf{e}((w\sigma - p)\sigma^{-1}(\mu + v)) \frac{r!}{p^e}\Big|_R,$$

where M^* resp. M_d^* is the dual of M resp. M_d in the corresponding space with respect to the scalar product $\langle\,,\rangle$. In order to apply Theorem 2 to the inner series over p, let $\iota: \mathbb{Q}^l \to \mathbb{Q}^n$ be the diagonal embedding of $\mathbb{Q}^l$ in $\mathbb{Q}^n$ associated with $V_d\sigma = \iota(\mathbb{Q}^l)$ and determined by a partition $\{J_i\}$ of J. Write

$$e_i = \sum_{j\in J_i} 1 + r_j, \quad \mu_i^* = \sum_{j\in J_i} (\sigma^{-1}(\mu + v))_j \text{ for } 1 \leqq i \leqq l.$$

Then μ^* has the property $\iota(q)\sigma^{-1}(\mu + v) = q\mu^*$ for $q \in \mathbb{Q}^l$. Since $w \in V_d$, the vector $w^* = \iota^{-1}(w\sigma)$ is well-defined. Using these quantities, we can evaluate

$$\begin{aligned}\sum_{p\in M_d + w\sigma} \mathbf{e}((w\sigma - p)\sigma^{-1}(\mu + v)) \frac{r!}{p^e}\bigg|_R &= \sum_{q\in\mathbb{Z}^l + w^*} \mathbf{e}((w^* - q)\mu^*)) \frac{r!}{q^e}\bigg|_{R^*} \\ &= r!\,\mathscr{C}_e(w^*, \mu^*, R^*), \quad R^* = R\circ\iota,\end{aligned}$$

by Theorem 2. We summarize the result in the following theorem.

Theorem 7 *Assume that $\mathfrak{A}$ is V-diagonal. Then*

$$\Psi_V(\mathfrak{A})(P, Q, u, v) = \sum_{d\in D_0} \frac{\mathbf{e}((u - w)v)}{|M_d/M\sigma|} \sum_{|r| = \deg P} r!\,P_r(\sigma) \sum_{\mu\in M^*/\sigma M_d^*} \mathscr{C}_e(w^*, \mu^*, R^*).$$

As a corollary to this theorem, we show that the Eisenstein cocycle Ψ is a generalization of the classical Dedekind-Rademacher cocycle in the case $n = 2$. Consider for $A \in \mathrm{GL}_2\mathbb{Q}$,

$$\Phi(A)(Q, v) = -\frac{1}{2\pi^2}\Psi(1_2, A)(1, Q, 0, v).$$

Then Φ is an inhomogenous 1-cocycle $\Phi(AB) = \Phi(A) + A\Phi(B)$. Since $\Phi(A) = \Phi(\lambda A)$ for $\lambda \in \mathbb{Q}^*$, we can assume that the coefficients a, c in $A = \left(\begin{smallmatrix} a & d \\ c & d\end{smallmatrix}\right)$ are relatively prime integers. If $c \neq 0$, we can assume in addition $c > 0$. Writing $\mathscr{B}_e(u) = B_e(\{u\})$ for the periodic Bernoulli functions, it follows from Theorem 7 that for $c > 0$,

$$\begin{aligned}\Phi\begin{pmatrix} a & b \\ c & d\end{pmatrix}(Q, v) &= \frac{a}{c}\mathscr{B}_2(v_2) + \frac{d}{c}\mathscr{B}_2(cv_1 - av_2) \\ &\quad + 2\sum_{\mu(c)} \mathscr{B}_1\left(v_1 - a\frac{\mu + v_2}{c}\right)\mathscr{B}_1\left(\frac{\mu + v_2}{c}\right) \quad \text{if } v \notin \mathbb{Z}^2.\end{aligned}$$

This formula remains also valid in the case $v \in \mathbb{Z}^2$ provided a correction term depending on Q is added on the right side. For instance, in the case $Q(x, y) = \alpha x - \beta y$ with rationally independent real numbers α, β, the correction terms equals $-\frac{1}{2}\operatorname{sign}(\beta(\alpha c + \beta a))$. For the above formula and related material, see [Ra].

In conclusion, it must be said that the evaluation of the Eisenstein cocycle Ψ in terms of generalized Dedekind sums as given by Theorem 6 and 7, although theoretically satisfying, is of limited value for practical calculations because of the potentially very large number of terms in these sums. This phenomenon is a reflection of the fact that we did not use the cocycle property in deriving these expressions. To calculate $\Psi(\mathfrak{A})$ more effectively, one should take advantage of the cocycle property of Ψ and replace $\mathfrak{A}$ by a homologous chain of "smaller" n-tuples.

In the case $n = 2$, this procedure is known as the "continued fraction algorithm" and leads to a very efficient calculation of $\Psi(\mathfrak{A})$. No such algorithm seems to be known for $n > 2$, but one can hope that a suitable refinement of the algorithm of Ash and Rudolph [AR] might lead to a fast calculation of $\Psi(\mathfrak{A})$ in the general case.

5 Proof of Theorem 2

5.1 Overview

Let $w = (u - \bar{u})/2 \in i\mathbb{R}^n$ be the imaginary part of $u \in \mathbb{C}^n$. In the following we use the letter p exclusively to denote an element of the translated lattice $\mathbb{Z}^n + u$, thus we can abbreviate the expression $p \in (\mathbb{Z}^n + u) \cap (X + w)$, where $X \subseteq \mathbb{R}^n$ is any set, by writing simply $p \in X + w$. Let

$$S(X) = \sum_{p \in X + w}' \frac{\mathbf{e}((u-p)v)}{p^e},$$

$$R(X) = \sum_{p \in B(X) + w}' |p_1 \dots p_n|^{-\rho}, \quad 1 - 1/n < \rho < 1,$$

where

$$B(X) = \left\{ x + y : x \in \partial \bar{X}, y \in \mathbb{R}^n, \max_j |y_j| \leqq 1 \right\}$$

is a set of points in $\mathbb{R}^n$ which are "close" to the boundary of $\bar{X}$. In what follows we will always assume that X is open in $\mathbb{R}^n$, and that $R(tX)$ and $S(tX)$ are both absolutely convergent for all positive $t > 0$. Let $\sigma_j : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection map along the j-th coordinate hyperplane ($j = 1, 2, \dots, n$), i.e., $\sigma_j(x)_i = x_i$ for $i \neq j$, and $\sigma_j(x)_j = -x_j$. The basic idea behind our proof of Theorem 2 is very simple, and relies on the following three lemmata.

Lemma 8 *Assume* $0 \in X$, *and let* $\sigma_j(X) = X$ *for all* $j = 1, 2, \dots, n$. *Then*

$$S(tX) = \prod_{j=1}^{n} \mathscr{C}_{e_j}(u_j, v_j) + \mathrm{O}(R(tX)) \quad \textit{for } t \to \infty. \tag{36}$$

Proof. Let $n = 1$. It suffices to consider the case $e = 1$. The estimate is then obvious for $v \in \mathbb{Z}$, while for $v \notin \mathbb{Z}$, it follows from

$$\sum_{|p| < t}' \frac{\mathbf{e}((u-p)v)}{p} = \mathscr{C}_1(u, v) + \mathrm{O}(\min\{1, |vt|^{-\rho}\}) \tag{37}$$

which is valid for $t \to \infty$ and every $|v| < 1/2, \rho \leqq 1$, with the O-constant independent of u. The last estimate is sharp for $\rho = 1$ and shows the non-uniform convergence of this series near $v = 0$. It suffices to prove it in the case $u = 0$, that is,

$$-\frac{\pi}{2} + \pi v + \sum_{k=1}^{t} \frac{\sin(2\pi k v)}{k} = \mathrm{O}(\min\{1, |vt|^{-1}\}), \quad 0 < v < \frac{1}{2}.$$

Let t be a positive integer. Taking the derivative with respect to v, and calculating the sum of the resulting series of cosines, we can rewrite the left side as

$$-\frac{\pi}{2}+\int_0^{\pi v}\frac{\sin(2t+1)x}{\sin x}\,dx = -\frac{\pi}{2}+I_1+I_2,$$

$$\begin{aligned}-\frac{\pi}{2}+I_1 &= -\frac{\pi}{2}+\int_0^{\pi v}\frac{\sin(2t+1)x}{x}\,dx\\ &= -\int_{\pi v}^{\infty}\frac{\sin(2t+1)x}{x}\,dx = \mathrm{O}(\min\{1,|vt|^{-1}\}),\end{aligned}$$

while the second integral,

$$I_2=\int_0^{\pi v}\sin((2t+1)x)\,\frac{x-\sin x}{x\sin x}\,dx\,,$$

is easily seen, by partial integration, to be $\mathrm{O}(t^{-1})$ uniformly in v. This proves (37) and therefore (36) in the case $n=1$. Lemma 8 follows now by induction for $n>1$. In particular, if $0<|v_j|<1/2$ for all j, then

$$S(tX)=\prod_{j=1}^{n}\mathscr{C}_{e_j}(u_j,v_j)+\mathrm{O}(|\textstyle\prod v_j|^{-\rho}R(tX))\quad\text{for } t\to\infty\,. \tag{38}$$

Lemma 9 *Assume that the integral*

$$I(X)=\int_X\prod_{i=1}^{n}\frac{dx_i}{x_i}$$

converges absolutely, and that $R(tX)=\mathrm{o}(1)$ *for* $t\to\infty$. *Then*

$$\lim_{t\to\infty}S(tX)=\begin{cases}I(X) \text{ if all } e_j=1 & \text{and } v_j\in\mathbb{Z}\\ 0 & \text{otherwise}\,.\end{cases}$$

Moreover, the sequence of partial sums $S(tX)$, as t *runs, is uniformly bounded in* v.

Proof. Without loss of generality let $X\subseteq\mathbb{R}^n_+$. We assume $u\in\mathbb{R}^n$ (i.e. $w=0$) for a moment, and consider first the case $e_j=1$ and $v_j\in\mathbb{Z}$ for all j. Then for $p\in tX\backslash B(tX)$, we have

$$\prod_{i=1}^{n}\int_{p_i}^{p_i+1}\frac{dx_i}{x_i}<\prod_{i=1}^{n}\frac{1}{p_i}<\prod_{i=1}^{n}\int_{p_i-1}^{p_i}\frac{dx_i}{x_i}\,.$$

Summing up these inequalities for all $p\in tX\backslash B(tX)$, and observing $I(X)=I(tX)$, it follows that

$$\lim_{t\to\infty}S(tX)=I(X). \tag{39}$$

Now assume that $e_k>1$ for some k, and let $X_\varepsilon=\{x\in X:x_k>\varepsilon\}$ for $\varepsilon>0$. Then from (39) we get the estimate $S(tX_\varepsilon)=\mathrm{O}(1/\varepsilon t)$ for $t\to\infty$, which implies

$$\lim_{t\to\infty}S(tX)=0\quad\text{if some } e_k>1\,. \tag{40}$$

We drop the assumption $u \in \mathbb{R}^n$, and consider the general case $u \in \mathbb{C}^n$ now. Then clearly (40) remains valid. In the previous case (all $e_j = 1$ and $v_j \in \mathbb{Z}$), we use the estimate

$$\frac{1}{p_k} = \frac{1}{\mathrm{Re}(p_k)} + \mathrm{O}(|p_k|^{-2}) \quad \text{for } |p_k| \to \infty$$

to conclude that (39) remains valid for all $u \in \mathbb{C}^n$. Finally, if all $r_j = 1$ but $v_k \notin \mathbb{Z}$ for some k, then partial summation applied to

$$\sum_{a < \mathrm{Re}(p_k) < b} \frac{\mathbf{e}((u_k - p_k)v_k)}{p_k},$$

shows that the absolute value of this sum can be estimated by $\mathrm{O}(1/a)$ for $a \to +\infty$. It follows that $S(tX) = \mathrm{O}(R(tX))$ for $t \to \infty$, and therefore

$$\lim_{t \to \infty} S(tX) = 0 \quad \text{if some } v_k \notin \mathbb{Z}\,.$$

Finally, the uniform boundedness of the partial sums $S(tX)$ in v follows from (39). This finishes the proof of Lemma 9. In order to state Lemma 10, let $K \subseteq I = \{1, 2, \ldots, n\}$ be a fixed subset, and $H \subseteq \mathbb{R}^n$ the hyperplane

$$H = H_K = \{x \in \mathbb{R}^n : x_k = 0 \text{ for } k \in K\}$$

together with the projection map $\pi : \mathbb{R}^n \to H$ and the associated differential form $\Omega = \Omega_k$ given by

$$\pi(x)_i = \begin{cases} 0, & i \in K \\ x_i, & i \notin K \end{cases}, \qquad \Omega = \prod_{i \notin K} \frac{dx_i}{x_i}\,.$$

Lemma 10 *Assume that X has the following properties with respect to a fixed subset $K \subseteq I$.*

1) $\sigma_k(X) = X$ *for all* $k \in K$.

2) $\int_{X \cap H} \Omega$ *converges absolutely.*

3) $R(tX) = \mathrm{o}(1)$ *for* $t \to \infty$.

Then $\lim S(tX) = 0$ *unless* $e_i = 1$ *and* $v_i \in \mathbb{Z}$ *for all* $i \in I \setminus K$ *where*

$$\lim_{t \to \infty} S(tX) = \left(\prod_{k \in K} \mathscr{C}_{e_k}(u_k, v_k)\right) \int_{X \cap H} \Omega$$

Moreover, the partial sums $S(tX)$ are continuous functions of v which are bounded by

$$M \left| \prod_{k \in K} v_k \right|^{-\rho}, \quad \text{if } 0 < |v_k| < 1/2 \text{ for all } k, \tag{41}$$

with some constant M independent of t.

Proof. Consider all p in $tX + w$ with $\pi(p - w) \notin tX \cap H$. The estimate

$$\sum_{a < |p_k| < b} \mathbf{e}((u_k - p_k)v_k)\, p_k^{-e_k} = \mathrm{O}\left(\frac{1}{a}\right),$$

valid for $a > 0$ and uniform in b, shows that the contribution of these p to $S(tX)$ is at most $O(R(tX))$ for $t \to \infty$. On the other hand, the contribution of those $p \in tX + w$ with $\pi(p - w) \in tX \cap H$ can be calculated with the help of Lemma 8 and 9, and leads to the stated result. The estimate (41) follows from (38) and Lemma 9.

Now let $X = \{x \in \mathbb{R}^n : |Q(x)| < 1\}$. In order to prove Theorem 2, we have to determine the limit of $S(tX)$ for $t \to \infty$. (This is clear for $w = 0$; if $w \neq 0$, it follows from a precise estimate of $R(tX)$ in Sect. 5.6.) To this end we shall construct a decomposition

$$X = \bigcup_{K \subseteq I} X_K \quad \text{(disjoint union)} \tag{42}$$

such that Lemma 10 can be applied to every X_K. Let

$$J = \{j \in I : e_j > 1 \text{ or } v_j \notin \mathbb{Z}\}.$$

According to Lemma 10, $\lim S(tX_K) = 0$ if J is not a subset of K. Therefore,

$$\lim_{t \to \infty} S(tX) = \sum_{J \subseteq K \subseteq I} \lim_{t \to \infty} S(tX_K) = \sum_{J \subseteq K \subseteq I} C_K I_K ,$$

where

$$C_K = \prod_{k \in K} \mathscr{C}_{e_k}(u_k, v_k), \quad I_K = \int_{X_K \cap H_K} \Omega_K .$$

Lemma 11 I_K converges absolutely for every $K \subseteq I$, and has the value

$$I_K = \frac{1}{2m} (1 + (-1)^{n - |K|}) \sum_{k=1}^{m} \prod_{j \notin K} \pi i \operatorname{sign}(l_{kj}) .$$

The proof of this lemma is not easy, and will be given in Sect. 5.3 and 5.4. Using

$$\mathscr{C}_e(x, y) = (-1)^e \mathscr{C}_e(-x, -y),$$

it follows from Lemma 11 that

$$\lim_{t \to \infty} S(tX) = \frac{1}{2m} C_J (g(u, v) + (-1)^{n - |J|} g(-u, -v)) ,$$

where

$$g(u, v) = \sum_{k=1}^{m} \prod_{j \notin J} (\mathscr{C}_1(u_j, v_j) - \pi i \operatorname{sign}(l_{kj}))$$

$$= \sum_{k=1}^{m} \prod_{j \notin J} \mathscr{C}_1(u_j, v_j^{\operatorname{sign}(l_{kj})}) .$$

Since $\mathscr{C}_{e_j}(u_j, v_j^{\pm}) = \mathscr{C}_{e_j}(u_j, v_j)$ for $j \in J$, we finally get the statement of Theorem 2,

$$\lim_{t \to \infty} S(tX) = \tfrac{1}{2} (h(u, v) + (-1)^{|e|} h(-u, -v))$$

with

$$h(u, v) = \frac{1}{m} \sum_{k=1}^{m} \prod_{j=1}^{n} \mathscr{C}_{e_j}(u_j, v_j^{\operatorname{sign}(l_{kj})}) .$$

In order to complete the proof of this limit formula, we have to define the decomposition (42), verify that Lemma 10 does apply to all X_K, and prove Lemma 11.

5.2 *The decomposition of X*

For a subset $J \subseteq I$, let

$$X(J) = \{x \in X : \sigma_j(x) \in X \text{ iff } j \in J\}$$

$$= X \cap \bigcap_{j \in J} \sigma_j(X) \bigcap_{i \notin J} C(\sigma_i(X))$$

where $C(\sigma_i(X))$ denotes the complement of $\sigma_i(X)$ in $\mathbb{R}^n$. Then X is the disjoint union of all the $X(J)$, but $X(J)$ may not be invariant under all reflections σ_j with $j \in J$. To get a decomposition of X with this property, we consider the sets

$$Y = X(J_1)(J_2) \cdots (J_l)$$

where $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_l$ is a monotonic sequence of subsets $J_k \subseteq I$ (only such sequences are of interest because otherwise $Y = \emptyset$). From the definition of $X(J)$, it follows that Y can be written as

$$Y = \bigcup_{(\mathfrak{K}, \mathfrak{L})} \left(\bigcap_{K \in \mathfrak{K}} \sigma_K(X) \bigcap_{K \in \mathfrak{L}} C(\sigma_L(X)) \right) \tag{43}$$

where every $\mathfrak{K}$ and $\mathfrak{L}$ is a certain set of subsets of I, and by definition, $\sigma_L = \prod_{l \in L} \sigma_l$ for every $L \subseteq I$. Since the intersections on the right lead only to a finite number of different subsets of X, it is clear that the number of different Y's is also finite. Therefore, for l large enough, Y has the property that $Y(J_l) = Y$ which implies $\sigma_j(Y) = Y$ for all $j \in J_l$. For a fixed subset $K \subseteq I$, let

$$X_K = \bigcup_{J_1 \supseteq \cdots \supseteq J_l = K} Y\,, \quad Y = X(J_1) \cdots (J_l)$$

where the union is taken over all sequences $J_1 \supseteq \cdots \supseteq J_l$ such that $K = J_l$ and $Y(J_l) = Y$. Then

$$X = \bigcup_{K \subseteq I} X_K \qquad \text{(disjoint union)}$$

is the desired decomposition (42) of X where every part X_K is invariant under all reflections σ_k with $k \in K$. For the proof of Theorem 2, it is crucial to show that the integral

$$I_K = \int_{X_K \cap H_K} \prod_{i \notin K} \frac{dx_i}{x_i}$$

converges absolutely for every $K \subseteq I$. Since $X_K \cap H_K$ is a lower dimensional analogue of $X_\emptyset = X_\emptyset \cap H_\emptyset$ for $K \neq \emptyset$, it suffices to consider the case $K = \emptyset$ only. The following lemma gives a simple property of $X_\emptyset$ for estimating $I_\emptyset$.

Lemma 12 *For each $i \in I$ and every $x \in X_\emptyset$, there is a subset $K = K(i, x) \subseteq I$ such that*

$$\sigma_K(x) \in X \quad \text{and} \quad \sigma_i \circ \sigma_K(x) \notin X\,. \tag{44}$$

Proof. Since $\sigma_i^2 = \mathrm{id}$, it suffices to show the existence of $L \subseteq I$ with the property

$$\sigma_i \circ \sigma_L(x) \in X \quad \text{and} \quad \sigma_L(x) \notin X\,. \tag{45}$$

Let $i \in I$ and $x \in X_\emptyset$ be given. By definition of $X_\emptyset$, there is $Y = X(J_1)\cdots(J_l)$ with $Y(\emptyset) = Y$ and $x \in Y$. It follows from (43) that there is a pair $(\mathfrak{K}, \mathfrak{L})$ with

$$x \in \bigcap_{K \in \mathfrak{K}} \sigma_K(X) \bigcap_{L \in \mathfrak{L}} \mathrm{C}(\sigma_L(X))\,. \tag{46}$$

But $x \in Y(\emptyset)$ implies that $\sigma_i(x) \notin Y$. Therefore

$$\sigma_i(x) \in \bigcup_{K \in \mathfrak{K}} \mathrm{C}(\sigma_K(X)) \bigcup_{L \in \mathfrak{L}} \sigma_L(X)\,. \tag{47}$$

From (46) and (47), it follows that (44) or (45) must be true for some K or $L \subseteq I$. This proves the lemma. Note that K may not be unique for given i and x. However, since $Q(x)$ is a continuous function of x, we can assume that $Q(\sigma_K(x))$ is a continuous function on $X_\emptyset$. We shall use this observation in the next section. A second property every X_K must satisfy, is condition 3) in Lemma 10. Since every point on the boundary of $\bar{X}_K$ satisfies the equation $|Q(\sigma x)| = 1$ for some reflection $\sigma = \sigma_L$, it suffices to check condition 3) for X itself. This will be done in Sect. 5.6.

5.3 The convergence of $I_\emptyset$

In this section, our goal is to prove the absolute convergence of $I_\emptyset$. In view of Lemma 12, it suffices to estimate the integral of $\prod dx_i/x_i$ $(i = 1, 2, \ldots, n)$ over the set $G \subset \mathbb{R}_+^n$ of all points $x \in \mathbb{R}_+^n$ which satisfy the inequalities

$$|Q(\sigma_K x)| < 1 \leqq |Q(\sigma_i \circ \sigma_K x)|, \qquad K = K(i, x)$$

for all $i = 1, 2, \ldots, n$ (where $Q(\sigma_K x)$ is a continuous function of x for every i). To this end we introduce new coordinates

$$u_n = 1/x_n, \quad u_j = x_j/x_n, \quad j = 1, 2, \ldots, n-1.$$

Then

$$\prod_{i=1}^n \frac{dx_i}{x_i} = \prod_{i=1}^n \frac{du_i}{u_i},$$

and the defining inequalities for G in terms of the u_j are

$$|Q(\sigma_K u)| < u_n^m \leqq |Q(\sigma_i \sigma_K u)|, \quad 1 \leqq i \leqq n\,,$$

with $u = (u_1, \ldots, u_{n-1}, 1)$ and all u_j positive. Integration over u_n gives

$$\int_G \prod_{i=1}^n \frac{dx_i}{x_i} \leqq I_0, \quad I_0 := \int_{\mathbb{R}_+^{n-1}} M(u) \prod_{i=1}^{n-1} \frac{du_i}{u_i},$$

$$M(u) = \min_{1 \leqq i \leqq n} \left| \log \left| \frac{Q(\sigma_i \sigma_K u)}{Q(\sigma_K u)} \right| \right|.$$

Let I_1 be the integral which arises from I_0 by restricting the domain of integration to $[1, \infty)^{n-1}$. We claim that, in order to prove the convergence of I_0, it suffices to prove the convergence of I_1. To verify that claim, consider for fixed $r \neq n$ the map

$$v_n = 1, \quad v_r = 1/u_r, \quad v_i = u_i/u_r, \quad i \neq n, r\,. \tag{48}$$

Under this transformation, the set

$$\{u: 0 < u_r \leqq 1,\, u_i \geqq a_i \text{ for } i \neq n, r\}, \quad a_i = 0 \text{ or } 1\,,$$

is mapped into $\{v: v_r \geqq 1, v_i \geqq a_i \text{ for } i \neq n, r\}$. Moreover, it's clear that $M(u) = M'(u)$ where $M'(u)$ arises from M by interchanging the coefficients l_{kn} and l_{kr} in Q (for every k), and by replacing the chosen sets K by another choice K'. Thus the shape of the differential form in I_0 does not change under the transformation (48) which implies that we may restrict our attention to I_1. Concerning the singularities of I_1, we notice that for every linear form $L_k(u)$, the function $|\log|L_k(u)||$ is integrable over any compact subset of $\mathbb{R}^{n-1}$. In other words, I_1 can only fail to converge because of the singularities near $u_i = \infty$. Motivated by this remark, we are led to consider the set U,

$$U = \{(u_1, \ldots, u_{n-1}, 1): 1 \leqq u_i \leqq u_1, i \geqq 2\}\,.$$

Let I_2 be the integral arising from I_1 by restricting the domain of integration to U. It suffices to prove the convergence of I_2 which we write as an iterated integral

$$I_2 = \int_1^\infty \frac{du_1}{u_1} \int_1^{u_1} \frac{du_2}{u_2} \cdots \int_1^{u_1} \frac{du_{n-1}}{u_{n-1}} M(u)\,.$$

We claim that the inner integral in I_2 is

$$\int_1^{u_1} \frac{du_2}{u_2} \cdots \int_1^{u_1} \frac{du_{n-1}}{u_{n-1}} M(u) = \mathrm{O}\left(\frac{\log^{n-2} u_1}{u_1}\right) \quad \text{for } u_1 \to \infty$$

which clearly implies the convergence of I_2. For this claim, it's enough to show that for any linear form $L(x) = \sum l_i x_i$ with nonzero coefficients $l_i \neq 0$,

$$\int_1^{u_1} \frac{du_2}{u_2} \cdots \int_1^{u_1} \frac{du_{n-1}}{u_{n-1}} \left|\log\left|\frac{L(\sigma_n u)}{L(u)}\right|\right| = \mathrm{O}\left(\frac{\log^{n-2} u_1}{u_1}\right) \quad \text{for } u_1 \to \infty\,. \tag{49}$$

This follows from the obvious estimate

$$M(u) \leqq \sum_{k=1}^m \left|\log\left|\frac{L_k(\sigma_n \sigma u)}{L_k(\sigma u)}\right|\right|$$

with σu the image of u under some reflection $\sigma = \sigma_K$. In the following proof of (49) we assume without loss of generality that $l_n = 1$. We begin by estimating the innermost integral.

$$\int_1^{u_1} \frac{du_{n-1}}{u_{n-1}} \left|\log\left|\frac{L(\sigma_n u)}{L(u)}\right|\right| < \frac{P(\log u_1, |\log|\lambda||)}{1 + |\lambda|}\,,$$

where $\lambda = L(u_1, \ldots, u_{n-2}, 0, 0)$, and $P(\eta, \xi)$ is a polynomial of degree 1 with positive coefficients.

Proof. The part of the last integral where $A := |\lambda + l_{n-1} u_{n-1}| \leqq 2$, can be estimated by

$$2\int_\alpha^\beta \frac{dy}{y} \log\left|\frac{|\lambda' + y| + 1}{|\lambda' + y| - 1}\right|, \quad \lambda' = \lambda\, \mathrm{sign}(l_{n-1}), \quad \alpha = \max\{|l_{n-1}|, |\lambda|\}, \quad \beta = |\lambda| + 2$$

$$\leqq \frac{2}{\alpha} \int_0^2 \log\left|\frac{y+1}{y-1}\right| dy \leqq \frac{P}{1 + |\lambda|}, \quad P = 4\max\{1, |l_{n-1}|^{-1}\},$$

but if $A > 2$, then

$$\left|\log\left|\frac{L(\sigma_n u)}{L(u)}\right|\right| = \log\left|\frac{A+1}{A-1}\right| < \frac{4}{1+A},$$

and we can apply the following lemma with $g = 0$ and $A = |\lambda + ly|$.

Lemma 13 *For fixed l and integral $g = 0, 1, 2, \ldots$, there is a polynomial $P(\eta, \xi)$ of degree $g + 1$ with positive coefficients (depending only upon g and l) such that for real λ,*

$$\int_1^t \frac{dy}{y} \frac{|\log|\lambda + ly||^g}{1 + |\lambda + ly|} < \frac{P(\log t, |\log|\lambda||)}{1 + |\lambda|}.$$

Applying this lemma repeatedly, we get for the integral in (49),

$$\int_1^{u_1} \frac{du_2}{u_2} \cdots < \int_1^{u_1} \frac{du_2}{u_2} \frac{P(\log u_1, |\log|l_1 u_1 + l_2 u_2||)}{1 + |l_1 u_1 + l_2 u_2|} = O\left(\frac{\log^{n-2} u_1}{u_1}\right) \quad \text{for } u_1 \to \infty .$$

Modulo Lemma 13, the convergence of $I_\emptyset$ is now completely proved. As a closing remark, we note that $I_\emptyset$ converges also under the weaker assumption that for every k, at least $(n-1)$ coefficients of L_k do not vanish. This can be proved along the same lines, but it requires a more elaborate estimate of I_2.

Proof of Lemma 13. It is easy to see that we may restrict our attention to the case $l = 1$. Let

$$E = \int_1^t \frac{dy}{y} \frac{\phi(\lambda + y)}{1 + |\lambda + y|}, \quad \phi(x) = |\log|x||^g.$$

We distinguish $\lambda \geqq 0$ from $\lambda < 0$. If $\lambda \geqq 0$, then

$$E \leqq \frac{\phi(\lambda + t)}{1 + \lambda} \int_1^t \frac{dy}{y} \leqq \frac{\phi(2\lambda) + \phi(2t)}{1 + \lambda} \log t .$$

In the case $\lambda < 0$, we break up E into three parts,

$$E = \int_1^t = \int_1^\alpha + \int_\alpha^\beta + \int_\beta^t = E_1 + E_2 + E_3 ,$$

$$\alpha = \max\{1, |\lambda| - 1\}, \ \beta = \min\{|\lambda| + 1, t\} ,$$

and consider every piece separately. Clearly, $E_1 = 0$ unless $\lambda < -2$ which implies $\phi(\lambda + 1) < \phi(\lambda)$. Therefore, if $E_1 \neq 0$, then

$$E_1 \leqq \phi(\lambda + 1) \int_1^\alpha \frac{dy}{y} \frac{1}{1 + |\lambda| - y} \leqq \frac{\phi(\lambda)}{1 + |\lambda|} \int_1^\alpha dy \left(\frac{1}{y} + \frac{1}{1 + |\lambda| - y}\right)$$

$$\leqq \frac{\phi(\lambda)}{1 + |\lambda|} (\log(|\lambda| - 1) + \log|\lambda| - \log 2) \leqq \frac{2|\log|\lambda||^{g+1}}{1 + |\lambda|}.$$

Next,

$$E_2 = \int_\alpha^\beta \frac{dy}{y} \frac{\phi(\lambda + y)}{1 + |\lambda + y|} \leqq \frac{1}{\alpha} \int_{|\lambda|-1}^{|\lambda|+1} \phi(\lambda + y)\, dy$$

$$= \frac{2}{\alpha} \int_0^1 \phi(y)\, dy = \frac{2g!}{\alpha} \leqq \frac{6g!}{1 + |\lambda|} .$$

Finally, $E_3 = 0$ unless $t > 1 + |\lambda|$, where

$$E_3 \leqq \frac{\phi(t)}{1+|\lambda|} \int_{1+|\lambda|}^{t} \frac{dy}{1+|\lambda+y|} < \frac{(\log t)^{g+1}}{1+|\lambda|},$$

which finishes the proof of Lemma 13.

5.4 *Evaluation of* $I_\emptyset$

Having established the absolute convergence of I_K, our next goal is to determine its value. As in the previous section, it suffices to consider $I_\emptyset$. If n is odd, then clearly $I_\emptyset = 0$ because $X_\emptyset = -X_\emptyset$. Thus we can assume that n is even. The easiest way to calculate $I_\emptyset$ is to consider it as the (Cauchy) principal value of the divergent integral

$$\int_X \Omega, \quad \Omega = \prod_{i=1}^{n} \frac{dx_i}{x_i}, \quad X = \{x \in \mathbb{R}^n : |Q(x)| < 1\}.$$

This can be done as follows. Let $R_i \subset \mathbb{R}^n$, $i = (i_1, \dots, i_n)$, $i_j = 1, 2, 3, \dots$, be a monotone sequence of sets (i.e. $R_k \subseteq R_l$ whenever $k_j \leqq l_j$ for all j) such that

1) $\sigma_j(R_i) = R_i$ for all $j = 1, 2, \dots, n$
2) $\bigcup_i R_i = (\mathbb{R} \backslash \{0\})^n$
3) $\int_{X \cap R_i} \Omega$ converges absolutely for all i.

Then, applying the decomposition (42) to $X \cap R_i$, it follows from 1) and 3) that

$$\int_{X \cap R_i} \Omega = \int_{X_\emptyset \cap R_i} \Omega.$$

Therefore, by the dominated convergence theorem,

$$I_\emptyset = \lim_{i \to \infty} \int_{X \cap R_i} \Omega = \lim_{i_1 \to \infty} \cdots \lim_{i_n \to \infty} \int_{X \cap R_i} \Omega.$$

In the following, we write for the limit on the right side,

$$PV \int_X \Omega := \lim_{i \to \infty} \int_{X \cap R_i} \Omega.$$

We will define the sequence R_i step by step as we proceed with the calculation of $I_\emptyset$. We start by requiring that

$$1/i_n < |x_n| < i_n \quad \text{for } x \in R_i.$$

This allows us to carry out the integration over x_n for $x \in X \cap R_i$ with fixed $y = (x_1, \dots, x_{n-1})$. Clearly, only those points contribute to this integral which satisfy simultaneously

$$|Q(y, x_n)| < 1 \leqq |Q(y, -x_n)|, \quad 1/i_n < |x_n| < i_n, \quad y \text{ fixed}.$$

These inequalities do not change if we replace x by $-x$, so we may assume $x_n > 0$. As in the previous section, we introduce the coordinates

$$u = y/x_n, \quad u_n = 1/x_n.$$

In terms of these coordinates, the inequalities above are

$$|Q(u,1)| < u_n^m \leqq |Q(u,-1)|, \quad 1/i_n < u_n < i_n .$$

Moreover, these coordinates allow us to specify one further inequality defining R_i,

$$1/i_{n-1} < |u_{n-1}| < i_{n-1} \quad \text{for all } x \in R_i .$$

Let

$$\Omega_1 = P(u) \prod_{i=1}^{n-1} \frac{du_i}{u_i}, \quad P(u) = \log\left|\frac{Q(u,-1)}{Q(u,1)}\right| .$$

Integrating over u_n and passing to the limit $i_n \to \infty$, we get

$$\lim_{i_n \to \infty} \int_{X \cap R_i} \Omega = \frac{2}{m} \int_{P(u)>0} \Omega_1 ,$$

where in the last integral some additional restrictions on u have to be specified yet. Notice that Ω_1 remains unchanged under $u \to -u$, but $P(u) > 0$ iff $P(-u) < 0$, therefore

$$\lim_{i_n \to \infty} \int_{X \cap R_i} \Omega = \frac{1}{m} PV \int_{\mathbb{R}^{n-1}} \Omega_1 = \frac{1}{m} \sum_{k=1}^{m} PV \int_{\mathbb{R}^{n-1}} \left(\prod_{i=1}^{n-1} \frac{du_i}{u_i}\right) \log\left|\frac{L_k(u,-1)}{L_k(u,1)}\right| . \quad (50)$$

Here every integral converges absolutely if u is restricted to a compact set in $\mathbb{R}^{n-1}$ outside the hyperplanes $u_j = 0$. Let

$$W = \{w \in \mathbb{R} : 1/i_{n-1} < |w| < i_{n-1}\} .$$

In the case $n = 2$, we have

$$\lim_{i_1 \to \infty} \int_W \frac{du_1}{u_1} \log\left|\frac{L_k(u_1,-1)}{L_k(u_1,1)}\right| = \int_{-\infty}^{\infty} \frac{du_1}{u_1} \log\left|\frac{l_{k1}u_1 - l_{k2}}{l_{k1}u_1 + l_{k2}}\right| = -\pi^2 \operatorname{sign}(l_{k1} l_{k2}) . \quad (51)$$

The last equation follows from

$$\int_0^1 \frac{dw}{w} \log\left(\frac{1+w}{1-w}\right) = \frac{3}{2}\zeta(2)$$

using Euler's result on $\zeta(2)$. Note that the integral (51) converges absolutely. Now let $n \geqq 4$. In order to simplify notation, we write L instead of L_k, and let

$$u = (v, w), \quad v = (u_1, \dots, u_{n-2}), \quad w = u_{n-1} .$$

For fixed v, consider the integral

$$T = \int_W \frac{dw}{w} \log\left|\frac{L(v,w,-1)}{L(v,w,1)}\right| .$$

The substitution $w \to -w$ gives

$$T = \int_W \frac{dw}{w} \log\left|\frac{L(-v,w,-1)}{L(-v,w,1)}\right| .$$

Adding these two integrals, and passing to the limit $i_{n-1} \to \infty$, we get

$$2T = \int_W \frac{dw}{w} \left(\log\left|\frac{L(v,w,-1)}{L(-v,w,1)}\right| + \log\left|\frac{L(-v,w,-1)}{L(v,w,1)}\right|\right),$$

$$\lim_{i_{n-1} \to \infty} 2T = -\pi^2 \operatorname{sign}(l_{n-1}) \{\operatorname{sign} L(-v,0,1) + \operatorname{sign} L(v,0,1)\} . \quad (52)$$

Let $\tilde{L}(v) = L(v,0,0)/l_n$. Then we can rewrite (52) as

$$- \pi^2 \operatorname{sign}(l_{n-1} l_n)\{\operatorname{sign}(1 - \tilde{L}(v)) + \operatorname{sign}(1 + \tilde{L}(v))\} .$$

It follows that

$$\lim_{i_{n-1} \to \infty} T = \begin{cases} -\pi^2 \operatorname{sign}(l_{n-1} l_n), & |\tilde{L}(v)| < 1 \\ 0, & |\tilde{L}(v)| > 1 . \end{cases}$$

Applying this to (50), we get

$$PV \int_{\mathbb{R}^{n-1}} \left(\prod_{i=1}^{n-1} \frac{du_i}{u_i}\right) \log\left|\frac{L(u, -1)}{L(u,1)}\right| = (\pi i)^2 \operatorname{sign}(l_{n-1} l_n)\, PV \int_V \prod_{i=1}^{n-2} \frac{dv_i}{v_i} ,$$

$$V = \{v \in \mathbb{R}^{n-2} : |\tilde{L}(v)| < 1\} .$$

But the last integral is of the same shape as the integral we started with, so we can apply induction to conclude that

$$I_\emptyset = PV \int_X \Omega = \frac{(\pi i)^n}{m} \sum_{k=1}^m \operatorname{sign}\left(\prod_{j=1}^n l_{kj}\right) .$$

This finishes the proof of Lemma 11.

5.5 *A remark concerning Lemma 11*

It was essential for the evaluation of the integral $I_\emptyset$ that the coefficients l_{kj} were real. However, the integral $I_\emptyset$ makes also sense for complex coefficients l_{kj}, and it might be an interesting problem to evaluate $I_\emptyset$ in this more general case. Very likely, the value of $I_\emptyset$ can be expressed by polylogarithms, functions which were studied recently in algebraic K-theory in connection with the regulator map and values of zeta functions. This guess is supported by the following result which settles the case $n = 2$.

Lemma 14 *Let $Q(x) = \prod_{k=1}^m (l_{k1} x_1 + l_{k2} x_2)$ with nonzero complex numbers l_{kj}, and let*

$$X_\emptyset = \{x \in \mathbb{R}^2 : |Q(x_1, x_2)| < 1 \leqq |Q(x_1, -x_2)|\} .$$

Then

$$\int_{X_\emptyset} \frac{dx_1\, dx_2}{x_1 x_2} = -\pi^2 + \frac{2\pi}{m} \sum_{k=1}^m |\arg(l_{k1}/l_{k2})|$$

where arg *denotes the principal branch of the argument function.*

Note that $|\arg \tau| = |\arg \bar{\tau}|$ is continuous on $\mathbb{C}\backslash\{0\}$. For the proof, only (51) has to be replaced by

$$\int_{-\infty}^{\infty} \frac{du_1}{u_1} \log\left|\frac{l_{k1} u_1 - l_{k2}}{l_{k1} u_1 + l_{k2}}\right| = 2\pi |\arg(l_{k1}/l_{k2})| - \pi^2 .$$

Let $\tau = l_{k2}/l_{k1}$, $\operatorname{Im}(\tau) > 0$. Then the last equation follows from

$$\int_{-\infty}^{\infty} \frac{dy}{y} \left(\log\left(\frac{y - \tau}{y + \tau}\right) - \log\left(\frac{y - i}{y + i}\right)\right) = 2\pi i \log\left(\frac{i}{\tau}\right)$$

which can be verified by taking the derivative with respect to τ.

5.6 Contribution from the boundary

In order to complete the proof of Theorem 2, we have to prove yet that $R(tX) = o(1)$ for $t \to \infty$ (cf. Lemma 10). In Lemma 16 we show that in fact $R(tX) = O(t^{n(1-\rho)-1})$ for $t \to \infty$. The exponent of t is here negative since $1 - 1/n < \rho < 1$ by assumption. Finally, we prove in Lemma 17 that

$$\sum_{p \in tX + w} |p_1 \cdots p_n|^{-\rho} = O(t^{n(1-\rho)}) \quad \text{for } t \to \infty ,$$

which shows that $S(tX)$ is well defined for all $t > 0$. In the following, we consider more generally

$$Q(p) = \prod_{k=1}^{m} L_k(p), \quad L_k(p) = \alpha_k + \sum_{j=1}^{n} l_{kj} p_j$$

with complex numbers α_k, and let

$$B(t) = \{x + y : x, y \in \mathbb{R}^n, \quad |Q(x)| = t^m, \quad \max |y_j| \leqq 1\}$$

be the set of points near the boundary of tX.

Lemma 15 *Let $\varepsilon > 0$ be fixed, and $\alpha = \min|\alpha_k|$. Then for $t + \alpha \to \infty$,*

$$\sum_1 := \sum_{\substack{p \in B(t) + w \\ |p_j| > \varepsilon t \text{ for all } j}} |p_1 \cdots p_n|^{-\rho} = O((t + \alpha)^{n(1-\rho)-1})$$

where the O-constant depends only on ε and the l_{kj}.

Proof. We proceed by induction. For $n = 1$, $B(t) + w \cap \mathbb{Z} + u$ is a finite set whose cardinality is bounded by $6m$. This implies $\sum_1 = O(t^{-\rho})$ for $t \to \infty$. On the other hand, from $|Q(x)| = t^m$, it follows that $|l_k x + \alpha_k| \leqq t$ holds for at least one k. Therefore, for $\alpha > t$, we can also write $\sum_1 = O(\alpha^{-\rho})$. This proves the lemma for $n = 1$. For $n > 1$, we write

$$Q(p) = \prod_{k=1}^{m} \left(\beta_k + \sum_{j=1}^{n-1} l_{kj} p_j \right), \quad \beta_k = \alpha_k + l_{kn} p_n .$$

Assuming the lemma is true for $n - 1$, we conclude that

$$\sum_1 = O\left(\sum_{q > \varepsilon t} q^{-\rho} (t + \beta)^{(n-1)(1-\rho)-1} \right), \quad \beta = \min|\beta_k|, \quad q = |p_n| .$$

Here the partial sum with $q < t + \alpha$ contributes

$$O\left((t + \alpha)^{(n-1)(1-\rho)-1} \sum_{q < t + \alpha} q^{-\rho} \right) = O((t + \alpha)^{n(1-\rho)-1}) ,$$

whereas the rest can be estimated by

$$O\left(\sum_{q > t + \alpha} q^{-\rho} q^{(n-1)(1-\rho)-1} \right) = O((t + \alpha)^{n(1-\rho)-1}) .$$

This proves Lemma 15. We need it to prove

Lemma 16 *For given $l_{kj} \neq 0$ there is an $\varepsilon > 0$ such that for all α_k with $|\alpha_k| < \varepsilon t$, we have*

$$R(t) = \sum_{p \in B(t)+w}' |p_1 \cdots p_n|^{-\rho} = O(t^{n(1-\rho)-1}) \quad \text{for } t \to \infty$$

with an O-constant depending only on u and the l_{kj}.

Proof: If $n = 1$, it follows from $|Q(x)| = t^m$ that

$$\frac{t}{\lambda} \leqq \left|\frac{\alpha_k}{l_k} + x\right|, \quad \lambda = \left|\prod_{k=1}^{m} l_{k1}\right|^{1/m},$$

holds at least for one k. Therefore, $|\alpha_k/l_k| < t/(2\lambda)$ implies $|x| > t/(2\lambda)$ which gives $R(t) = O(t^{-\rho})$ for $t \to \infty$. Now let $n > 1$ and i be fixed $(1 \leqq i \leqq n)$. By induction hypothesis, there is $\varepsilon_i > 0$ such that for all $|\alpha_k| < \varepsilon_i t/2$, the partial sum over those $p \in B(t) + w$ with $|l_{ki} p_i| < \varepsilon_i t/2$ for all k, can be estimated by

$$O\left(t^{(n-1)(1-\rho)-1} \sum_{\max_k |l_{ki} p_i| < \varepsilon_i t/2} |p_i|^{-\rho}\right) = O(t^{n(1-\rho)-1}).$$

Taking $\varepsilon = \min_i \max_k |\varepsilon_i/(2l_{ki})|$, the assertion follows now from Lemma 15.

Lemma 17 $\displaystyle\sum_{p \in tX+w}' |p_1 \cdots p_n|^{-\rho} = O(t^{n(1-\rho)})$ *for $t \to \infty$.*

Proof. $Q(p)| < t^m$ implies $|L_k(p)| < t$ for some k, so we can assume $m = 1$. The estimate is then obvious for $n = 1$. If $n > 1$, then the induction hypothesis implies that this estimate holds for the partial sum over those $p \in tX + w$ with at least one $|p_j| \leqq t$. For the rest we write

$$\sum_{\substack{p \in tX+w \\ |p_j| > t \text{ for all } j}} = \sum_{1 \leqq k \leqq t} \left(\sum_{\substack{k-1 \leqq |Q(p)| < k \\ |p_j| > t \text{ for all } j}} \right)$$

and observe that Q is a linear form. Comparing this expression with the definition of $B(t)$, we find that Lemma 15 does apply to the inner sum and gives the desired estimate

$$O\left(\sum_{1 \leqq k \leqq t} k^{n(1-\rho)-1}\right) = O(t^{n(1-\rho)}).$$

Theorem 2 is now completely proved.

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