Hida Theory tutorial set - NCMW 2023

December 2023

1 Problem set 1: Review of complex theory

References

- Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.
- Toshitsune Miyake. *Modular forms*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, English edition, 2006.
- Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994.

Problems

1. For every integer N, prove that the natural map
$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$
is surjective.

2. (a) Prove that the only torsion elements in $\mathrm{SL}_2(\mathbb{Z})$ have order 1, 2, 3 or 6.

   (b) Prove that the torsion elements in $\mathrm{SL}_2(\mathbb{Z})$ are conjugate to
$$\pm I_2, \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

   (c) If $N \geq 4$, prove that $\Gamma_1(N)$ has no torsion elements.

   (d) *If $N \geq 4$, prove that $\Gamma_1(N)$ is a free group.*

3. (a) Prove that we have an isomorphism
$$\frac{\mathbb{C}}{\Lambda_1} \cong \frac{\mathbb{C}}{\Lambda_2}$$
of complex elliptic curves if and only if there exists a non-zero complex number λ such that

i. $\lambda\Lambda_1 = \Lambda_2$

ii. $f(z + \Lambda_1) = \lambda z + \Lambda_2$.

(b) Prove that the torsion subgroup of a complex elliptic curve is isomorphic to $\left(\frac{\mathbb{Q}}{\mathbb{Z}}\right)^2$.

(c) Prove that there is a natural bijection of sets:

$$Y_1(N) \to \{(E, P) : \ E \text{ is a complex elliptic curve, } P \text{ is a } \Gamma_1(N)\text{-level structure on } E\} / \sim,$$
$$\tau \mapsto \left(\frac{\mathbb{C}}{Z \cdot 1 \oplus \mathbb{Z} \cdot \tau}, \frac{1}{N}\right).$$

4. Let Γ denote a congruence subgroup. Prove the q-expansion principle over the complex numbers. That is, prove that the following natural map, given by taking the q-expansion at ∞, is an injective ring homomorphism:

$$\bigoplus_{k=2}^{\infty} M_k(\Gamma) \hookrightarrow \mathbb{C}[\![q]\!]. \tag{1}$$

5. Prove that the ring of modular forms with level $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to the power series $\mathbb{C}[X, Y]$. In fact, prove that

$$\bigoplus_{k=2}^{\infty} M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C}) \cong \mathbb{C}[E_4, E_6]. \tag{2}$$

Here, E_4 and E_6 are the (unique, up to scalars) Eisenstein series with level $\mathrm{SL}_2(\mathbb{Z})$ and with weights 4 and 6 respectively.

6. Let $k \geq 2$ denote a positive integer. Let $\mathrm{T}_{\mathbb{C}}^{\mathrm{good}}$ denote the $\mathbb{C}$-algebra generated by the Hecke operators T_l, for all primes l not dividing N, acting on $S_k(\Gamma_1(N), \mathbb{C})$.

   (a) Show that $\mathrm{T}_{\mathbb{C}}^{\mathrm{good}}$ is a commutative subring of the $\mathbb{C}$-endomorphism ring $\mathrm{End}(S_k(\Gamma_1(N), \mathbb{C})$.

   (b) Show that, for all natural numbers $n \geq 2$ not dividing the level N, the Hecke operators T_n have an adjoint for their action on the (Petersson) inner product space $S_k(\Gamma_1(N), \mathbb{C})$.

   (c) Prove that the ring $\mathrm{T}_{\mathbb{C}}^{\mathrm{good}}$ is reduced.

7. Let $k \geq 2$ denote a positive integer. Prove that there exists modular forms $f_1, \cdots f_n$ in $S_k(\Gamma_1(N), \mathbb{Z})$ such that the set $\{f_1, \cdots f_n\}$ forms a $\mathbb{C}$-basis for $S_k(\Gamma_1(N), \mathbb{C})$.