Spacetime coercive wave equation tests

Notes for the wave equation coercive formulation numerical tests

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Introduction

Local operators

To construct the operators of the bilinear form, introduce the local operators for the space and time component, S is for space, T is for time. Apices will be used to denote the order of derivation for the trial and test functions: for example, S^{00} is the **local mass matrix** for the space component. In general S^{ij} is the local matrix associated to the local operator

$$\int_{\hat{R}} \partial_i u \cdot \partial_j v dx,$$

letting u and v vary among all the space local basis functions, where $\hat{R}$ is the reference space element. Similarly T^{ij} is associated to the local operator

$$\int_0^1 \partial_i u \cdot \partial_j v dt,$$

letting u and v vary among all the time local basis functions.

$H^1(Q)$ terms

Scalar terms of the $H^1(Q)$ space are easy to compute.

$$\int_Q u_t v_t = S^{00} \otimes T^{11}$$
$$\int_Q \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = S^{11} \otimes T^{00}$$

Least square term

Least square term comes from the integral $\int_Q WuWv$. Expanding, we get

$$\int_Q \left[u_{tt} v_{tt} - c^2 u_{tt} \Delta v - c^2 v_{tt} \Delta v + c^4 \Delta u \Delta v \right],$$

Each terms is

$$\int_Q u_{tt} v_{tt} = S^{00} \otimes T^{22}$$
$$\int_Q u_{tt} \Delta v = S^{20} \otimes T^{02}$$
$$\int_Q v_{tt} \Delta u = S^{02} \otimes T^{20}$$
$$\int_Q \Delta u \Delta v = S^{22} \otimes T^{00}$$

$\int_Q ZuWv$ term

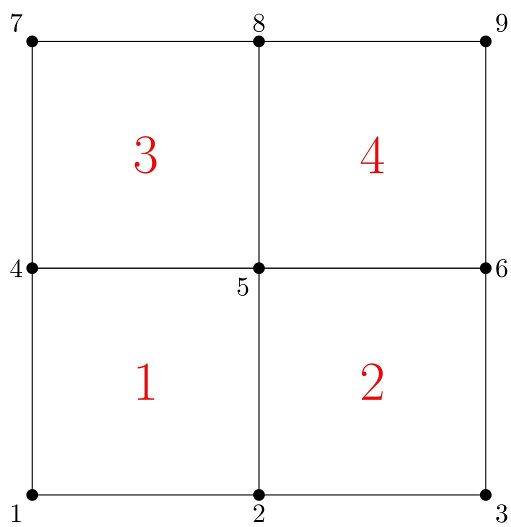
Expanding all terms we get

$$\int_Q ZuWv = ...$$

$$\int_Q tu_t v_{tt} = S^{00} \otimes T_t^{12}$$
$$\int_Q tu_t \Delta v = S^{02} \otimes T_t^{10}$$
$$\int_Q u_t v_{tt} = S^{00} \otimes T^{12}$$
$$\int_Q u_t \Delta v = S^{02} \otimes T^{10}$$
$$\int_Q \boldsymbol{x} \cdot \boldsymbol{\nabla} u v_{tt} = S_x^{10} \otimes T^{02}$$
$$\int_Q \boldsymbol{x} \cdot \boldsymbol{\nabla} u \Delta v = S_x^{12} \otimes T^{00}$$

Appendix A: FEM matrix assembly

Consider $\Omega = [0, 1]^2$ as the domain, and the following subdivision:



Assume we have a local matrix called A, which is easy to compute. Now construct the *connectivity matrix* which has in each column the list of vertices of the element.

$$T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 2 & 3 & 5 & 6 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \end{pmatrix}$$

Now, assembly the global matrix by

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for e = 1:n_elems
    el_nodes = T[:,e]
    K[el_nodes,el_nodes] = K[el_nodes,el_nodes] + A
end
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How do we treat more general elements? …

Suppose we have a finite element that has k local degree of freedom, then wo assemble the stiffness matrix we would generally

Appendix B: Load vector assembly