EEP/IAS 118 - Introductory Applied Econometrics, Section 5

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Agenda

- 1 Review: population parameters and sample estimators
- 2 Hypothesis testing overview
- 3 Hypothesis testing: sample mean
- 4 Hypothesis testing: binary variables
- 5 Hypothesis testing: difference in means
- 6 Hypothesis testing: regression coefficients

Sample Estimators

Since we don't observe the population, we estimate parameters of interest in a sample. We have formulas that give us the best estimate for a given parameter in the population. For example, under SLR/MLR 1-4, the OLS estimators $\hat{\beta}$ are unbiased estimates of the true population β values.

Remember that estimators themselves are **random variables** because they depend on the random sample we happen to draw from the population, hence they have a certain probability distribution, with a certain mean and a certain variance. We usually can't calculate the population variance of an estimator, so use another estimator for this population variance to generate what we call the standard error.

Summary: X as continuous variable

	Symbol	Formula
Population parameters	μ	$\sum_{j=1}^k x_j f(x_j)$
	σ_X^2	$E[(X - E(X))^2]$
	σ_{X}	$\sqrt{E[(X-E(X))^2]}$
Sample estimators	\bar{X}	$\frac{1}{n}\sum_{i}X_{i}$
	s_X^2	$\frac{1}{n-1}\sum_{i}(X_{i}-\bar{X})^{2}$
	s_X	$\sqrt{\frac{1}{n-1}\sum_i(X_i-\bar{X})^2}$
Estimator parameters	$E(\bar{X})$	μ
	$Var(\bar{X})$	$\frac{\sigma_X^2}{n}$
	$Sd(\bar{X})$	$\frac{\sigma_X}{\sqrt{n}}$
SE of estimator	$Se(\bar{X})$	$\frac{s_X}{\sqrt{n}}$

Summary: X as binary variable

	Symbol	Formula
Population parameters	μ	р
	σ_X^2	p(1-p)
	σ_X	$\sqrt{p(1-p)}$
Sample estimators	X	p̂
	s_X^2	$\hat{p}(1-\hat{p})$
	s_X	$\sqrt{\hat{p}(1-\hat{p})}$
Estimator parameters	$E(\bar{X})$	p
	$Var(\bar{X})$	p(1-p)/n
	$Sd(\bar{X})$	$\sqrt{p(1-p)/n}$
SE of estimator	$Se(\bar{X})$	$\frac{s_X}{\sqrt{n}}$

Summary: Regression Coefficients - Simple Linear Regression

	Symbol	Formula
Population estimators	β_0	
	β_1	
Sample estimators	\hat{eta}_0	$ar{y} - \hat{eta}_1 ar{x}$
	\hat{eta}_1	$\frac{\bar{y} - \hat{\beta}_1 \bar{x}}{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}$ $\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2}$
Estimator parameters*	$E(\hat{eta}_0)$	eta_0
	$E(\hat{\beta}_1)$	β_1
	$Var(\hat{eta}_1)$	$\frac{\sigma_u^2}{SST_x}$
	$Sd(\hat{\beta}_1)$	$\frac{\sigma_u}{\sqrt{SST_x}}$
SE of estimator	$Se(\hat{eta}_1)$	$\frac{\hat{\sigma}_u}{\sqrt{SST_x}}$

^{*}We don't show $Var(\hat{\beta}_0)$, $Sd(\hat{\beta}_0)$, or $Se(\hat{\beta}_0)$ because we rarely care

Summary: Regression Coefficients - Multiple Linear Regression

- Don't worry about specific formula for $\hat{\beta}_j$; tedious with large number of x_i variables
- Important for inference: revised formula for $Var(\hat{\beta}_j) = \frac{\sigma_u^2}{SST_j(1-R_j^2)}$ (note mistake in L10 slides formula)
- Recall that when we don't know σ_u^2 we use the estimate $\hat{\sigma}_u^2 = \frac{SSR}{n-k-1}$ where k is the number of regressors in X

Hypothesis Testing

Last time we discussed how to construct/ interpret confidence intervals. Sometimes we may want to go beyond just generating a range of probable values for our population parameter:

- Often we want to test a hypothesis that our parameter is a specific value
- For example: "vocational training schools are effective at increasing employment rates"
- We will use our sample of data to test whether some hypothesis we have about the true population is likely or not.

Hypothesis Testing Process

To actually test a hypothesis, we need to follow five steps:

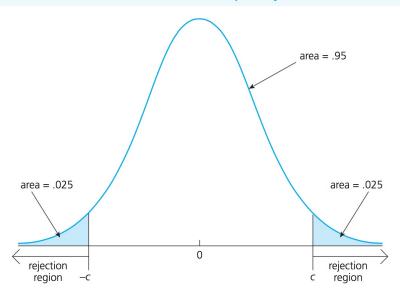
- 1 State the **null** (H_0) and **alternative** (H_1) hypotheses
 - Most common is testing that the population parameter is zero $H_0: \mu=0$ against the alternative $H_1: \mu \neq 0$
 - The form of the alternative will determine whether you perform a one-sided or two-sided test
- 2 Choose and calculate a test statistic with a known distribution. The test statistic is just a function of our random sample (a transformation of the estimator)
 - The test statistic provides a measure of how far our sample estimator (for example, \bar{X}) is from our hypothesized population value (in this case, μ) relative to the standard error of the estimator $(se(\bar{X}))$

Hypothesis Testing Process

- 3 Choose significance level (α) and find critical value (c)
 - Sig. level is typically 10%, 5%, or 1%. This value is the probability that we reject the null when the null is true
 - Critical value is found on the appropriate table (t, z, or F) for the chosen significance level and given degrees of freedom
 - Be careful, the critical value will change depending on whether the test is one-sided or two-sided!
- 4 Define our rejection rule. Reject H_0 if and only if:
 - Two-sided test: $|t| > |c_{\alpha/2}|$
 - Positive One-sided: $t > c_{\alpha}$
 - Negative One-sided: $t < c_{\alpha}$ (where c is negative)
- 5 Interpret. We either:
 - a. Reject the null
 - b. Fail to reject the null

We **NEVER** accept the null

Rejection region for a 5% significance level test against the two-sided alternative H,: $\mu \neq \mu_{\rm o}$.



The t-statistic

What do we use for our test statistic? If we do not know the variance of our estimator, we use a t-statistic. For a parameter θ :

$$t = \frac{\hat{\theta} - \theta}{se(\theta)}$$

Why is this useful and intuitive?

- The numerator is the deviation between the estimator and our hypothesized population parameter
- We "normalize" this difference based on the standard error of our estimator, so we can think about how "surprising" the estimator would be if the null were true
- The t-stat has a known (t) distribution so we can easily compare it to critical values

Let's say we want to test hypothesis that the true mean of UCB student's GPA is 3.1

- 1 Define hypotheses:
 - Since we don't have a good reason to think the average GPA should be higher or lower than 3.1, our hypotheses are:

$$H_0: \mu = 3.1$$

$$H_1: \mu \neq 3.1$$

 Note that the ≠ in the alternative hypothesis means we are running a two-sided test

- 2 Compute the test statistic. Lets assume that we draw a sample of n=101 students, and we calculate the sample estimators:
 - $\bar{X} = 2.984$
 - $s_x = 0.3723$

The we calculate:

$$t = \frac{\bar{X} - \mu}{\frac{s_x}{\sqrt{n}}} = \frac{2.984 - 3.1}{\frac{0.3723}{\sqrt{101}}} = -3.13$$

Note that we inserted the value of our null hypothesis for μ as "true" population parameter

- 3 Choose a significance level and find critical value.
 - Let's choose $\alpha = 0.05$
 - Look at t-table for critical value with chosen significance level for a two-tailed test and n-1=100 degrees of freedom
 - Make sure I remember whether this is a two sided or one-sided test

TABLE B: #-DISTRIBUTION CRITICAL VALUES

	. Tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0003
1	1.000	1,376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	.765	.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	.741	.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	.727	.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.86
6	.718	.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.95
7	.711	.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.40
8	.706	.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5:04
9	.703	.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.78
10	.700	.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.58
11	.697	.876	1.088	1.363	1.796	2,201	2.328	2.718	3.106	3.497	4.025	4.43
12	.695	.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.31
13	.694	.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.22
14	.692	.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	.691	.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3,733	4.073
16	.690	.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252-	3.686	4.01:
17	.689	.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	.688	.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	.688	.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	.687	.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	.686	.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	.686	.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	.685	.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	685	.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467.	3.745
25	.684	.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	.684	.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	.684	.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	.683	.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	.683	.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	.683	.854	1.055	1.310	1.697	2.042	2.147	2:457	2.750	3.030	3.385	3.646
40	.681	.851	1.050	1.303	1.684	2.021	2,123	2.423	2.704	2.971	3.307	3.551
50	.679	.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	.679	.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	.678	.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	.677	.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	.675	.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
00	.674	.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3 291

TABLE B: 1-DISTRIBUTION CRITICAL VALUES

	- N	Tail probability p										
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
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00	674	841	1.036	1.282	1.645	1.960	2 054	2 326	2 576	2.807	3.001	3 201

- 3 Choose a significance level and find critical value.
 - Let's choose $\alpha = 0.05$
 - Look at t-table for critical value with chosen significance level for a two-tailed test and n-1=100 degrees of freedom
 - Make sure I remember whether this is a two sided or one-sided test

$$c = 1.984$$

4 Do we reject the null? We only reject if:

And we have

$$|-3.13| > |1.984|$$

So we reject the null!

- 5 How do we interpret this:
 - We reject the null hypothesis. There is statistical evidence at the 5% level that the average student GPA is different from 3.1
 - What if we had failed to reject the null? We would say:
 "There is no statistical evidence at the 5% level that average GPA is different from 3.1"

Two-tailed vs. One-tailed

What if we had instead posed the hypotheses

$$H_0: \mu = 3.1$$

$$H_1: \mu < 3.1$$

This is a **one-tailed** test. We are only concerned with determining if the GPA is *lower* than 3.1

Let's go back to the t-table from before

• What is the new critical value for our test?

Two-tailed vs. One-tailed

What if we had instead posed the hypotheses

$$H_0: \mu = 3.1$$

$$H_1: \mu < 3.1$$

This is a **one-tailed** test. We are only concerned with determining if the GPA is *lower* than 3.1

Let's go back to the t-table from before

- What is the new critical value for our test?
- c = 1.660

Helpful rule:

- If two-sided, look for column with value $\alpha/2$
- If one-sided, look for column with value α

Hypothesis Testing Binary Variables

Let's say we wanted to test a hypothesis on a **binary** variable (e.g. a yes / no vote). Conceptually the process is the same as before, but we have two important (related) differences:

1 When we do hypothesis testing, we **assume** the null is true say that p=.65 for example. Now recall the formula for the variance of our estimator \bar{X} when X is binary:

$$Var(\bar{X}) = \frac{p(1-p)}{n}$$

Notice how this variance **only** depends on the population parameter p.

Because we are assuming that p is equal to our null hypothesis, this implies that we **DON'T** have to estimate the variance!

Hypothesis Testing Binary Variables

The second difference follows from above:

2 Because we don't estimate the variance, we never lose the normal distribution for the test statistic! Therefore:

$$z = \frac{\hat{p} - p}{\sigma_x / \sqrt{n}} \sim N(0, 1)$$

This implies that we need to look at a z table to find the critical values when dealing with a binary random variable

Hypothesis Testing Binary Variable Example

We want to test that in a yes / no vote 65% of people voted "yes" or if the true proportion is lower than 65%. We sample 200 people and we find that 115 voted yes.

1

$$H_0: p = 0.65$$

$$H_1: p < 0.65$$

- 2 Calculate our test statistic:
 - $\hat{p} = \frac{115}{200} = 0.575$
 - $\sigma_x^2/n = \frac{p(1-p)}{n} = \frac{.65(1-.65)}{200} = 0.0011375$
 - $z = \frac{\hat{p} p}{\sigma_x / \sqrt{n}} = \frac{.575 .65}{0.033727} = -2.224$

Hypothesis Testing Binary Variable Example

3 Let's choose a 1% significance level. Using the z-table, for 1% significance that

$$c = -2.32$$

- 4 Reject if z < c (since it is a negative one-sided test)
 - We have -2.224 > -2.32, therefore, here we fail to reject
- 5 We fail to reject the null hypothesis (at the 1% significance level) that the proportion of people who voted yes is 65%

Hypothesis Testing Difference in Means

Let's say we wanted to test if two separate groups have different means of some outcome. For example, is income different between rural and urban areas? How does this change our process?

1 In the numerator of the test statistic we put the difference between our sample means for each group:

$$\hat{D} = \overline{income_U} - \overline{income_R}$$

Our null hypothesis will then be about \hat{D}

2 In the denominator, we now need to divide by the standard error of $\overline{income_U} - \overline{income_R}$. This is slightly more complicated because both of these values are random variables

Hypothesis Testing Difference in Means

Since Var(a - b) = Var(a) + Var(b), the formula for this standard error is

$$se(\overline{income_U} - \overline{income_R}) = \sqrt{\frac{s_U^2}{n_U} + \frac{s_R^2}{n_R}}$$

In our data we calculate

- **Urban**: $\overline{income_U} = 7061.63$, $s_u = 5198$, and $n_u = 1112$
- Rural: $\overline{income_R} = 3661.3$, $s_r = 8974$, and $n_r = 1112$

Therefore, the standard error is:

$$se(\hat{D}) = \sqrt{\frac{5198^2}{1112} + \frac{8974^2}{1112}} = 310$$

and

$$\hat{D} = 7061.63 - 3661.3 = 3400.33$$

Hypothesis Testing Difference in Means

Finally we can calculate the test statistic:

$$t = \frac{\hat{D}}{se(\hat{D})} = \frac{3400.33}{310} = 10.97 \sim t_{n_u + n_r - 2 = 1112 + 1112 - 2}$$

- **NOTE:** The degrees of freedom for our t-stat (distributed t) is equal to $n_u + n_r 2$: df = n k where k is the number of estimators you have constructed
- The rest of hypothesis testing is the same

Hypothesis Testing β

How does all this apply to Econometrics?

- We want to be able to run a hypothesis test for β .
- In general the testing process for β is the same. However in order to get a known distribution for $\hat{\beta}$ we have to add one more regression assumption
 - MLR6: The population error u is normally distributed with mean 0 and variance σ^2 : $u \sim N(0, \sigma^2)$
- If we assume MLR1 5 and MLR6, we know that

$$\hat{\beta}_j \sim Normal[\beta_j, Var(\hat{\beta}_j)]$$

and therefore

$$\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim N(0,1), \quad \text{and} \quad \frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t(0,1)_{n-k-1}$$

Hypothesis Testing β

Using this very useful fact:

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t(0,1)_{n-k-1}$$

We have df=n-k-1 where k is the number of β parameters you've estimated and the 1 accounts for the intercept. We can test the hypothesis that our true population parameter β_j is equal to any value using our estimator $\hat{\beta}_j$.

A quick example. We have this population model:

$$log(rent) = \beta_0 + \beta_1 log(pop) + \beta_2 log(avginc) + \beta_3 pctstu + u$$

Which relates rental prices to population, average income, and the percent of the population that are students.

Hypothesis Testing β

$$log(rent) = \beta_0 + \beta_1 log(pop) + \beta_2 log(avginc) + \beta_3 pctstu + u$$

We want to test if the percent of students in population has no effect on the rental prices using a sample of 2,000

- 1 Hypotheses: $H_0: \beta_3 = 0, H_1: \beta_3 \neq 0.$
- 2 To get calculate the test statistic, we need to find the find $\hat{\beta}$ as well as $se(\hat{\beta})$. Both of these are in the R output:

```
Call:
lm(formula = lrent ~ lpop + lavainc + pctstu, data = rent[rent$year ==
   90, 1)
Residuals:
    Min
              10 Median
-0.22706 -0.09469 -0.02827 0.03806 0.48271
Coefficients:
           Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.042780 0.843875 0.051
                                          0.960
lpop
                               1.696
           0.065868
                      0.038826
                                          0.095 .
lavainc
                               6.272 4.29e-08 ***
           0.507015
                      0.080836
                                3.232
                                         0.002 **
pctstu
           0.005630 0.001742
Signif. codes: 0 '*** 0.001 '** 0.01 '* 0.05 '.' 0.1 ' ' 1
```

Therefore the t-stat is

$$t = \frac{0.005630 - 0}{0.001742} = 3.23$$

- 3 Significance level and critical value: 5%, and then c=1.96
- 4 We reject the null because |3.23| > |1.96|
- 5 At the 5% significance level, we can reject the null that the size of the student body relative to the population has no effect on expected monthly rents holding population and average income in the city constant.

We can now assess the statistical significance of our β estimates when we interpret them, along with the size and sign. We think of **statistical significance** (can we reject a null hypothesis that $\beta=0$?) as distinct from **economic significance** (is the magnitude of the association economically meaningful?).