Some parametricity isomorphisms

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1 Polymorphic λ-calculus (System F)

Types

$$
\begin{aligned}
A \quad ::= \quad &\textstyle\sum_{i \in I} A_i \ \mid\ \prod_{i \in I} A_i \ \mid\ A \to A \ \mid\ \mathtt{X} \\
&\mid\ \textstyle\sum \mathtt{X}.A[\mathtt{X}, \mathtt{X}] \ \mid\ \prod \mathtt{X}.A[\mathtt{X}, \mathtt{X}] \qquad \text{where } A[-, +] \\
&\mid\ \mu \mathtt{X}.A[\mathtt{X}] \ \mid\ \nu \mathtt{X}.A[\mathtt{X}] \qquad\qquad \text{where } A[+]
\end{aligned}
$$

Parametricity isomorphisms			for all
$\prod \mathtt{X}.A$	$\cong$	A	A
$\prod \mathtt{X}.(B[\mathtt{X}, \mathtt{X}] \to \mathtt{X}) \to A[\mathtt{X}, \mathtt{X}]$	$\cong$	$\prod \mathtt{X}.A[\mathtt{X}, \mu \mathtt{Y}.(\mathtt{X} + B[\mathtt{X}, \mathtt{Y}])]$	$A[-, +],\ B[-, +]$
$\prod \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}, \mathtt{X}]) \to A[\mathtt{X}, \mathtt{X}]$	$\cong$	$\prod \mathtt{X}.A[\nu \mathtt{Y}.(\mathtt{X} \times B[\mathtt{X}, \mathtt{Y}]), \mathtt{X}]$	$A[-, +],\ B[-, +]$
$\prod \mathtt{X}.A[\mathtt{X}, C \times \mathtt{X}]$	$\cong$	$\prod \mathtt{X}.A[C \to \mathtt{X}, \mathtt{X}]$	$A[-, +],\ C$
$\prod \mathtt{X}.A[\mathtt{X}, \mathtt{X} \to B]$	$\cong$	$\prod \mathtt{X}.A[\mathtt{X} \to B, \mathtt{X}]$	$A[-, -],\ B$
We deduce			
$\prod \mathtt{X}.(B[\mathtt{X}, \mathtt{X}] \to \mathtt{X}) \to A[\mathtt{X}]$	$\cong$	$A[1]$	$A[-],\ B[-, +]$
$\prod \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}, \mathtt{X}]) \to A[\mathtt{X}]$	$\cong$	$A[0]$	$A[+], B[-, +]$
$\prod \mathtt{X}.(B[\mathtt{X}] \to \mathtt{X}) \to A[\mathtt{X}]$	$\cong$	$A[\mu \mathtt{Y}.B[\mathtt{Y}]]$	$A[+],\ B[+]$
$\prod \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}]) \to A[\mathtt{X}]$	$\cong$	$A[\nu \mathtt{Y}.B[\mathtt{Y}]]$	$A[-],\ B[+]$
We also deduce isomorphisms dual to the above ones:			
$\sum \mathtt{X}.A$	$\cong$	A	A
$\sum \mathtt{X}.(B[\mathtt{X}, \mathtt{X}] \to \mathtt{X}) \times A[\mathtt{X}, \mathtt{X}]$	$\cong$	$\sum \mathtt{X}.A[\mu \mathtt{Y}.(\mathtt{X} + B[\mathtt{X}, \mathtt{Y}]), \mathtt{X}]$	$A[-, +],\ B[-, +]$
$\sum \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}, \mathtt{X}]) \times A[\mathtt{X}, \mathtt{X}]$	$\cong$	$\sum \mathtt{X}.A[\mathtt{X}, \nu \mathtt{Y}.(\mathtt{X} \times B[\mathtt{X}, \mathtt{Y}])]$	$A[-, +],\ B[-, +]$
$\sum \mathtt{X}.A[\mathtt{X}, C \to \mathtt{X}]$	$\cong$	$\sum \mathtt{X}.A[C \times \mathtt{X}, \mathtt{X}]$	$A[-, +],\ C$
$\sum \mathtt{X}.A[\mathtt{X} \to B, \mathtt{X}]$	$\cong$	$\sum \mathtt{X}.A[\mathtt{X}, \mathtt{X} \to B]$	$A[+, +],\ B$
$\sum \mathtt{X}.(B[\mathtt{X}, \mathtt{X}] \to \mathtt{X}) \times A[\mathtt{X}]$	$\cong$	$A[1]$	$A[+],\ B[-, +]$
$\sum \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}, \mathtt{X}]) \times A[\mathtt{X}]$	$\cong$	$A[0]$	$A[-], B[-, +]$
$\sum \mathtt{X}.(B[\mathtt{X}] \to \mathtt{X}) \times A[\mathtt{X}]$	$\cong$	$A[\mu \mathtt{Y}.B[\mathtt{Y}]]$	$A[-],\ B[+]$
$\sum \mathtt{X}.(\mathtt{X} \to B[\mathtt{X}]) \times A[\mathtt{X}]$	$\cong$	$A[\nu \mathtt{Y}.B[\mathtt{Y}]]$	$A[+],\ B[+]$

2 Polymorphic call-by-push-value

Types

$$
\begin{aligned}
A \quad ::= \quad & U\underline{B} \ \mid \ \textstyle\sum_{i\in I}A_i \ \mid \ 1 \ \mid \ A\times A \ \mid \ \mathtt{X} \\
& \mid \ \textstyle\sum\mathtt{X}.A[\mathtt{X},\mathtt{X}] && \text{where } A[-,+] \\
& \mid \ \textstyle\sum\underline{\mathtt{X}}.A[\underline{\mathtt{X}},\underline{\mathtt{X}}] && \text{where } A[\underline{-},\underline{+}] \\
& \mid \ \nu\mathtt{X}.A[U\underline{B}[\mathtt{X}]] && \text{where } A[+],\ \underline{B}[+] \\
\underline{B} \quad ::= \quad & FA \ \mid \ \textstyle\prod_{i\in I}\underline{B}_i \ \mid \ A\to\underline{B} \ \mid \ \underline{\mathtt{X}} \\
& \mid \ \textstyle\prod\mathtt{X}.\underline{B}[\mathtt{X},\mathtt{X}] && \text{where } \underline{B}[-,+] \\
& \mid \ \textstyle\prod\underline{\mathtt{X}}.\underline{B}[\underline{\mathtt{X}},\underline{\mathtt{X}}] && \text{where } \underline{B}[\underline{-},\underline{+}] \\
& \mid \ \mu\underline{\mathtt{X}}.\underline{B}[FA[\underline{\mathtt{X}}]] && \text{where } A[\underline{+}],\ \underline{B}[\underline{+}]
\end{aligned}
$$

Parametricity isomorphisms			for all
$\textstyle\sum\mathtt{X}.A$	$\cong$	A	A
$\textstyle\prod\mathtt{X}.\underline{A}$	$\cong$	$\underline{A}$	$\underline{A}$
$\textstyle\sum\mathtt{X}.\mathtt{X}^n\times A[\mathtt{X},\mathtt{X}^m]$	$\cong$	$\textstyle\sum\mathtt{X}.A[m\times\mathtt{X}+n,\mathtt{X}]$	$A[-,+],\ m,\ n$
$\textstyle\prod\mathtt{X}.\mathtt{X}^n\to\underline{A}[\mathtt{X}^m,\mathtt{X}]$	$\cong$	$\textstyle\prod\mathtt{X}.\underline{A}[\mathtt{X},m\times\mathtt{X}+n]$	$\underline{A}[-,+],\ m,\ n$
$\textstyle\sum\mathtt{X}.U(\mathtt{X}\to\underline{B}[\mathtt{X},\mathtt{X}])\times A[\mathtt{X},\mathtt{X}]$	$\cong$	$\textstyle\sum\mathtt{X}.A[\mathtt{X},\nu\mathtt{Y}.(\mathtt{X}\times U\underline{B}[\mathtt{X},\mathtt{Y}])]$	$A[-,+],\ \underline{B}[-,+]$
$\textstyle\prod\mathtt{X}.U(\mathtt{X}\to\underline{B}[\mathtt{X},\mathtt{X}])\to\underline{A}[\mathtt{X},\mathtt{X}]$	$\cong$	$\textstyle\prod\mathtt{X}.\underline{A}[\nu\mathtt{Y}.(\mathtt{X}\times U\underline{B}[\mathtt{X},\mathtt{Y}]),\mathtt{X}]$	$\underline{A}[-,+],\ \underline{B}[-,+]$
$\textstyle\sum\mathtt{X}.A[U(X\to\underline{B}),X]$	$\cong$	$\textstyle\sum\mathtt{X}.A[X,U(X\to\underline{B})]$	$A[+,+],\ \underline{B}$
$\textstyle\prod\mathtt{X}.\underline{A}[X,U(X\to\underline{B})]$	$\cong$	$\textstyle\prod\mathtt{X}.\underline{A}[U(X\to\underline{B}),X]$	$\underline{A}[-,-],\ \underline{B}$
$\textstyle\sum\underline{\mathtt{X}}.A[\underline{\mathtt{X}}]$	$\cong$	$A[1_\Pi]$	$\underline{A}[\underline{+}]$
$\textstyle\prod\underline{\mathtt{X}}.\underline{A}[\underline{\mathtt{X}}]$	$\cong$	$\underline{A}[1_\Pi]$	$\underline{A}[\underline{-}]$
$\textstyle\sum\underline{\mathtt{X}}.U(B[U(C\to\underline{\mathtt{X}}),\underline{\mathtt{X}}]\to\underline{\mathtt{X}})\times A[\underline{\mathtt{X}},U(C\to\underline{\mathtt{X}})]$			
$\cong \quad \textstyle\sum\mathtt{X}.A[\mu\underline{\mathtt{Y}}.F(C\times\mathtt{X}+B[\mathtt{X},\underline{\mathtt{Y}}]),\mathtt{X}]$			$A[\underline{-},+],\ B[-,\underline{+}],\ C$
$\textstyle\prod\underline{\mathtt{X}}.U(B[U(C\to\underline{\mathtt{X}}),\underline{\mathtt{X}}]\to\underline{\mathtt{X}})\to\underline{A}[U(C\to\underline{\mathtt{X}}),\underline{\mathtt{X}}]$			
$\cong \quad \textstyle\prod\mathtt{X}.\underline{A}[\mathtt{X},\mu\underline{\mathtt{Y}}.F(C\times\mathtt{X}+B[\mathtt{X},\underline{\mathtt{Y}}])]$			$\underline{A}[-,\underline{+}],\ B[-,\underline{+}],\ C$
We deduce			
$\textstyle\sum\mathtt{X}.\mathtt{X}^n\times A[\mathtt{X}]$	$\cong$	$A[n]$	$A[-],\ n$
$\textstyle\prod\mathtt{X}.\mathtt{X}^n\to\underline{A}[\mathtt{X}]$	$\cong$	$\underline{A}[n]$	$\underline{A}[+],\ n$
$\textstyle\sum\mathtt{X}.U(\mathtt{X}\to\underline{B}[\mathtt{X},\mathtt{X}])\times A[\mathtt{X}]$	$\cong$	$A[0]$	$A[-],\ \underline{B}[-,+]$
$\textstyle\prod\mathtt{X}.U(\mathtt{X}\to\underline{B}[\mathtt{X},\mathtt{X}])\to\underline{A}[\mathtt{X}]$	$\cong$	$\underline{A}[0]$	$\underline{A}[+],\ \underline{B}[-,+]$
$\textstyle\sum\underline{\mathtt{X}}.U(B[\underline{\mathtt{X}}]\to\underline{\mathtt{X}})\times A[\underline{\mathtt{X}}]$	$\cong$	$A[\mu\underline{\mathtt{Y}}.FB[\underline{\mathtt{Y}}]]$	$A[\underline{-}],\ B[\underline{+}]$
$\textstyle\prod\underline{\mathtt{X}}.U(B[\underline{\mathtt{X}}]\to\underline{\mathtt{X}})\to\underline{A}[\underline{\mathtt{X}}]$	$\cong$	$\underline{A}[\mu\underline{\mathtt{Y}}.FB[\underline{\mathtt{Y}}]]$	$\underline{A}[\underline{+}],\ B[\underline{+}]$
$\textstyle\sum\mathtt{X}.(U(\mathtt{X}\to\underline{B}[\mathtt{X}])\times A[\mathtt{X}])$	$\cong$	$A[\nu\mathtt{Y}.U\underline{B}[\mathtt{Y}]]$	$A[+],\ \underline{B}[+]$
$\textstyle\prod\mathtt{X}.U(\mathtt{X}\to\underline{B}[\mathtt{X}])\to\underline{A}[\mathtt{X}]$	$\cong$	$\underline{A}[\nu\mathtt{Y}.U\underline{B}[\mathtt{Y}]]$	$\underline{A}[-],\ \underline{B}[+]$

3 Polymorphic calculus of no return

Types

$$A \quad ::= \quad \neg A \mid \sum_{i \in I} A_i \mid 1 \mid A \times A \mid \mathtt{X}$$
$$\mid \sum \mathtt{X}.A[\mathtt{X}, \mathtt{X}] \qquad \text{where } A[-, +]$$
$$\mid \nu \mathtt{X}.A[\neg A'[\mathtt{X}]] \qquad \text{where } A[+], A'[-]$$

Parametricity isomorphisms			for all
$\sum \mathtt{X}.A$	$\cong$	A	A
$\sum \mathtt{X}.\mathtt{X}^n \times A[\mathtt{X}, \mathtt{X}^m]$	$\cong$	$\sum \mathtt{X}.A[m \times \mathtt{X} + n, \mathtt{X}]$	$A[-, +]$, m, n
$\sum \mathtt{X}.\neg(\mathtt{X} \times B[\mathtt{X}, \mathtt{X}]) \times A[\mathtt{X}, \mathtt{X}]$	$\cong$	$\sum \mathtt{X}.A[\mathtt{X}, \nu \mathtt{Y}.(\mathtt{X} \times \neg B[\mathtt{Y}, \mathtt{X}])]$	$A[-, +]$, $B[-, +]$
$\sum \mathtt{X}.A[\neg(X \times B), X]$	$\cong$	$\sum \mathtt{X}.A[X, \neg(X \times B)]$	$A[+, +]$, B

We deduce

$\sum \mathtt{X}.\mathtt{X}^n \times A[\mathtt{X}]$	$\cong$	$A[n]$	$A[-]$, n
$\sum \mathtt{X}.\neg(\mathtt{X} \times B[\mathtt{X}, \mathtt{X}]) \times A[\mathtt{X}]$	$\cong$	$A[0]$	$A[-]$, $B[-, +]$
$\sum \mathtt{X}.\neg(\mathtt{X} \times B[\mathtt{X}]) \times A[\mathtt{X}]$	$\cong$	$A[\nu \mathtt{Y}.\neg B[\mathtt{Y}]]$	$A[+]$, $B[-]$

4 Transforms

The following transforms convert each listed parametricity isomorphism into an instance of another one:

- the trivialization transform from call-by-push-value to λ-calculus

- the state passing transform from call-by-push-value to call-by-push-value, using a value type S

- the exception transform from call-by-push-value to call-by-push-value, using a value type E, and more generally the I/O transform from call-by-push-value to call-by-push-value using $P[\underline{+}]$

- the CPS transform from call-by-push-value to calculus of no return

- the result transform from calculus of no return to call-by-push-value, using a computation type $\underline{R}$.