Broad Infinity and Generation Principles

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Abstract

We introduce Broad Infinity, a new set-theoretic axiom scheme based on the slogan "Every time we construct a new element, we gain a new arity." It says that three-dimensional trees whose growth is controlled by a specified class function form a set. Such trees are called "broad numbers".

Assuming AC (the axiom of choice) or at least the weak version known as WISC (Weakly Initial Set of Covers), we show that Broad Infinity is equivalent to Mahlo's principle, which says that the class of all regular limit ordinals is stationary. Assuming AC or WISC, Broad Infinity also yields a convenient principle for generating a subset of a class using a "rubric" (family of rules); this directly gives the existence of Grothendieck universes, without requiring a detour via ordinals.

In the absence of choice, Broad Infinity implies that the derivations of elements from a rubric form a set; this yields the existence of Tarski-style universes.

Additionally, we reveal a pattern of resemblance between "Wide" principles, that are provable in ZFC, and "Broad" principles, that go beyond ZFC.

Note: this paper uses a base theory that is weaker than ZF but includes classical first-order logic and Replacement.

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Part I
Introduction

1 Broad Infinity vs Mahlo's Principle

1.1 Broad Infinity in a Nutshell

This paper is about a new axiom scheme of set theory, which is easy to state.

First, some preliminaries. For the sake of this introduction, assume either ZF or a variant that allows urelements. We write $\mathfrak{T}$ for the universal class and Set for the class of all sets; they are the same in ZF. The axiom of choice (AC) is not assumed.

We must say how to encode ordered pairs and the like.

Definition 1.1. Let C be a class.

(a) An *ordered pair encoding* on C is a binary operation $\langle -, - \rangle : C^2 \to C$ such that, for all $x, y, x', y' \in C$, if $\langle x, y \rangle = \langle x', y' \rangle$, then $x = x'$ and $y = y'$.

(b) A *unary Dedekind encoding* on C consists of an element $\mathsf{Nothing} \in C$ and a unary operation $\mathsf{Just} : C \to C$, such that

- for all $x \in C$, we have $\mathsf{Just}(x) \neq \mathsf{Nothing}$
- for all $x, x' \in C$, if $\mathsf{Just}(x) = \mathsf{Just}(x')$, then $x = x'$.

(c) A *binary Dedekind encoding* on C consists of an element $\mathsf{Begin} \in C$ and a binary operation $\mathsf{Make} : C^2 \to C$, such that

- for all $x, y \in C$, we have $\mathsf{Make}(x, y) \neq \mathsf{Begin}$
- for all $x, y, x', y' \in C$, if $\mathsf{Make}(x, y) = \mathsf{Make}(x', y')$, then $x = x'$ and $y = y'$.

The following encodings are fixed throughout the paper.

Definition 1.2.

(a) We give an ordered pair encoding on $\mathfrak{T}$ as follows:

$$\langle x, y \rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$$

(b) We give a unary Dedekind encoding on $\mathfrak{T}$ as follows:

$$\begin{aligned} \mathsf{Nothing} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Just}(x) &\stackrel{\text{def}}{=} \{x\} \end{aligned}$$

(c) We give a binary Dedekind encoding on $\mathfrak{T}$ as follows:

$$\begin{aligned} \mathsf{Begin} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Make}(x, y) &\stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\} \end{aligned}$$

For a class C and set K, we write C^K for the class of all functions from K to C.

The axiom of *Infinity* is included in ZF. As formulated by Zermelo [Zer08], it says that there is a set X with the following properties:

- $\mathsf{Nothing} \in X$.
- For any $x \in X$, we have $\mathsf{Just}(x) \in X$.

The new axiom scheme of *Simple Broad Infinity* is similar. It says that, for any function $F : \mathfrak{T} \to \mathsf{Set}$, there is a set X with the following properties:

- $\mathsf{Begin} \in X$.
- For any $x \in X$ and $y \in X^{Fx}$, we have $\mathsf{Make}(x, y) \in X$.

Here is a slogan: "Every time we construct a new element, we gain a new arity."

ZF extended with this scheme is called *Broad ZF*. The following sections will motivate this extension in light of a previously studied principle.

1.2 Regular Limits and Stationary Classes

We begin with some useful notions concerning ordinals. We write Ord for the class of all ordinals and Lim for the class of all *limit ordinals*—ordinals that are neither 0 nor a successor. An *initial* ordinal is one that is not the range of a function from a smaller ordinal; examples are the finite ordinals, ω and ω_1. (In ZFC, an initial ordinal is also called a "cardinal".)

A limit ordinal κ is *regular* when, for all $\alpha < \kappa$, the supremum function $\mathsf{Ord}^\alpha \to \mathsf{Ord}$ restricts to a function $\kappa^\alpha \to \kappa$. (See Section 13.3 for an alternative definition.) It follows that κ is initial, so ω is the only regular limit ordinal that is countable. We write Reg for the class of all regular limit ordinals.

For a function $F : \mathsf{Ord} \to \mathsf{Ord}$, we say that an limit ordinal λ is *F-closed* when F restricts to a function $\lambda \to \lambda$. Here are some examples:

- Let S be the successor function. Every limit ordinal is S-closed.
- For an ordinal α, let Const_α be the constant function $\gamma \mapsto \alpha$. A limit ordinal is Const_α-closed iff it is $> \alpha$.
- For functions $F, G : \mathsf{Ord} \to \mathsf{Ord}$, let $F \vee G$ be the pointwise maximum $\gamma \mapsto F(\gamma) \vee G(\gamma)$. A limit ordinal is $(F \vee G)$-closed iff it is both F-closed and G-closed.

A class of limit ordinals D is *stationary* when, for every function $F : \mathsf{Ord} \to \mathsf{Ord}$, there is an an F-closed member of D. (See Sections 13.1 and 14 for alternative definitions.) It follows that D is unbounded and that, for every function $F : \mathsf{Ord} \to \mathsf{Ord}$, there are stationarily many F-closed members of D.

1.3 Two Principles from the Literature

Next we look at two principles that use the above notions.

- *Mahlo's principle*, also known as "Ord is Mahlo", says that Reg is stationary [Ham03, Jor70, Lév60, May00, Wan77]. To illustrate its power, note that ZFC + Mahlo's principle proves that there are stationarily many inaccessible cardinals.[1] That is because, in ZFC, an inaccessible cardinal is precisely an uncountable F-closed regular limit ordinal, where F sends α to 2^α if α is a cardinal and to 0 otherwise.
- *Blass's axiom* [Bla83] says merely that Reg is unbounded. It follows from AC, but is it provable in ZF alone? To answer this question, Gitik [Git80] showed that, if ZFC + "Arbitrarily large strongly compact cardinals exist" is consistent, then so is ZF + "Every limit ordinal is the supremum of a strictly increasing ω-sequence". This means that ZF cannot even prove the existence of an uncountable regular limit ordinal, let alone prove Blass's axiom.

1.4 Limitations of Mahlo's Principle

Appealing though Mahlo's principle may be, I consider it deficient as an axiom scheme, in two respects.

Firstly, it does not meet the ZF standard of simplicity. Each ZF axiom, other than Extensionality and Foundation, expresses the idea that some easily grasped things form a set: the natural numbers (Infinity), the subsets of a set (Powerset), the elements of a set that satisfy a property (Separation), the images of a set's elements under a function (Replacement), and so forth. This is what makes these axioms so compelling. But Mahlo's principle does not do this.

The second problem is that Mahlo's principle, or indeed any addition to ZF that implies the existence of an uncountable regular limit ordinal, seems to be *entangled with choice* in light of Gitik's result. Admittedly this view is contentious, as some people would try to justify Mahlo's principle via the following

[1] In the absence of AC, there is no accepted notion of inaccessible. See [BDL07] for a comparative analysis.

choiceless argument: "For any $F : \mathsf{Ord} \to \mathsf{Ord}$, the property of being an F-closed regular limit can be reflected down from Absolute Infinity to an ordinal." But such thinking is avoided in this paper.

1.5 Motivating Broad Infinity

In light of the preceding discussion, my primary goal was to obtain an axiom scheme that

1. is equivalent to Mahlo's principle, assuming AC
2. asserts that some easily grasped things form a set
3. does not imply (given only ZF) that an uncountable regular limit ordinal exists.

To this end, I propose Simple Broad Infinity. Does it meet the requirements?

1. Assuming AC, we shall prove that Simple Broad Infinity is equivalent to Mahlo's Principle. So this requirement is met.
2. Simple Broad Infinity asserts, for each function $F : \mathfrak{T} \to \mathsf{Set}$, that the class of all simple F-broad numbers (explained in Section 7.2 below) is a set. Arguably this is "easily grasped", but the question is subjective and must be left to the reader's judgement.
3. In Broad ZF, I see no way to obtain the existence of an uncountable regular limit ordinal. However, an analogue of Gitik's result is currently lacking.

2 Goals and Structure of the Paper

2.1 Plausible vs Useful

Simple Broad Infinity has been designed to be as *plausible* as possible. In other words, I aimed to minimize the mental effort needed to believe it. This is surely a desirable feature for an axiom scheme. Furthermore, disentanglement from choice helps to achieve it because, even for a person who finds AC intuitively convincing (as I do), it is easier to accept one intuition at a time.

My second goal was different: to find an equivalent scheme that is as *useful* as possible. In other words, I wanted to minimize the effort needed to apply it. In particular, it should *obviously* imply the existence of Grothendieck universes, without requiring a detour via notions of ordinal or cardinal.

To this end, I propose a scheme called *Broad Set Generation*. For people who accept AC, this meets the stated goal. For those who do not, I offer instead a principle called *Broad Derivation Set*. The latter yields the existence of "Tarski-style" universes that are sometimes used in the literature [ML84].

2.2 Urelements and Non-Well-Founded Membership

In ZF, everything is a set and the membership relation is well-founded. But our results also hold in variants of ZF that allow urelements and/or non-well-founded membership [Wik23, Yao23, Wik24, Acz88]. Making this clear is the third goal.

2.3 Weak Choice Principle

Although—as stated above—some of our results depend on AC, the full strength of this axiom is not needed. More precisely, a weak form of choice known as WISC (Weakly Initial Set of Covers) suffices for our results. Explaining this fact is the fourth goal.

Caveat: we shall see different versions of WISC, and care must be taken to use an appropriate one. In ZF, they are all equivalent.

Without assuming the Axiom of Choice

Powerset + Infinity Wide Infinity * Wide Derivation Set † Injective Wide Set Generation †	$\longrightarrow$	Broad Infinity * Broad Derivation Set Injective Broad Set Generation
$\downarrow$		$\downarrow$
Powerset + Wide Supgeneration * Powerset + Blass's Axiom Wide Set Generation †	$\longrightarrow$	Powerset + Broad Supgeneration * Powerset + Mahlo's Principle Broad Set Generation

Assuming the Axiom of Choice or at least WISC

Powerset + Infinity Wide Infinity * Wide Derivation Set † Powerset + Wide Supgeneration * Powerset + Blass's Axiom Wide Set Generation †	$\longrightarrow$	Broad Infinity * Broad Derivation Set Powerset + Broad Supgeneration * Powerset + Mahlo's Principle Broad Set Generation

* The Simple and Full versions are equivalent.
† The Wide and Quasiwide versions are equivalent.

Figure 1: Diagram of theories, each extending the base theory

2.4 Wide vs Broad

We give the name "Broad" to the principles studied in this paper that go beyond ZFC. It turns out that each of them has a ZFC-provable counterpart that we call "Wide". For example, Mahlo's principle is Broad, and its Wide counterpart is Blass's axiom.

The fifth goal is to convey this pattern of resemblance, which is depicted in Figure 1, a summary of the results in the paper (using a base theory weaker than ZF). The rows within each block are equivalent, and each arrow represents inclusion of theories—i.e., reverse implication. The Wide principles appear on the left and the corresponding Broad principles on the right.

2.5 Summary of Goals

To summarize the previous sections, our goals are as follows.

1. To give a simple and plausible axiom scheme, disentangled from choice, that is equivalent over ZFC to Mahlo's principle. **Solution** Simple Broad Infinity.

2. To give an equivalent principle that is convenient for applications. **Solution** Broad Set Generation for those who accept AC, and Broad Derivation Set for those who do not.

3. To show that our results hold even when urelements and non-well-founded membership are allowed.

4. To show that, for the results that rely on AC, a weak choice principle suffices.

5. To convey the resemblance between Wide principles (which are provable in ZFC) and Broad principles (which are not, provided ZF is consistent).

Throughout the paper, we use a base theory that includes classical first-order logic, and do not consider the issue of logical complexity. Other versions of set theory are left to future work.

2.6 Related Work

Many formulations of Mahlo's principle have been studied [Jor70, May00, Lév60, Mon62, Dow11], and variations have been given for type theory [Rat00, Set00] and Explicit Mathematics [KS10]. Other principles have been considered that are equiconsistent with Mahlo's principle [Ham03, Mat77].

Another related topic—which inspired the Broad Derivation Set principle—is the treatment of "induction recursion" in type theory [DS06, GH16]. It is used in the proof assistant Agda, allows the formation of Tarski-style universes (as in Section 8.4), and was modelled in [DS06] using a Mahlo cardinal.

2.7 Structure of Paper

Before treating the wide and broad principles, the paper presents various foundational concepts in Part II, beginning in Section 3 with an introduction to sets and classes. Section 4 treats well-foundedness, and gives a way to generate subclasses and partial functions; this is is used throughout the paper, and especially to formulate the Derivation Set principles. Next, Section 5 is devoted to ordinals, and explains how to use an inductive chain to obtain a least prefixpoint. Lastly, Section 6 is devoted to category theory, notably the concept of an initial algebra.

Part III is devoted to those wide and broad principles that are concerned with sets and rubrics (not ordinals), beginning in Section 7 with the wide and broad infinity principles. This is followed in Section 8 by the useful principles of Set Generation and Derivation Set, with restricted versions of these principles considered in Section 9. In Section 10, we see how to deduce Set Generation principles from Derivation Set principles, by either imposing an injectivity condition or assuming AC or WISC.

Part IV presents the wide and broad principles for ordinals, beginning in Section 11 with "supgeneration" principles that connect the world of sets to that of ordinals. Section 12 explains the concept of Lindenbaum numbers, which is known from the literature on choiceless mathematics. This allows us in Section 13 to develop Mahlo's principle and establish all its relationships. Lastly, Section 14 presents the traditional use of Mahlo's principle (in a class setting) to prove the existence of various kinds of ordinal.

Part V wraps up the paper by summarizing the contributions and suggesting further work.

Some readers may just want to see the ZFC proof that Simple Broad Infinity is equivalent to Mahlo's principle. This is divided into several steps:

- Simple Broad Infinity is equivalent to Full Broad Infinity—Theorem 7.5(a).
- Full Broad Infinity implies Broad Derivation Set—Proposition 8.19(b).
- Broad Derivation Set implies Broad Set Generation—Proposition 10.9(c). Only this step uses AC.
- Broad Set Generation implies Full Broad Infinity—Proposition 9.2(b).
- Broad Set Generation is equivalent to Broad Supgeneration—Theorem 11.4(b).
- Broad Supgeneration is equivalent to Mahlo's principle—Theorem 13.11(b).
- Various definitions of stationarity, each giving a different formulation of Mahlo's principle, are equivalent—Proposition 14.9.

Part II
Foundations

3 Basic Theory of Sets

3.1 Our Base Theory

For this paper, I have chosen a base theory that differs from ZF in several ways:

- It allows urelements and non-well-founded membership.
- It excludes Powerset and Infinity, so that we can examine how these axioms relate to other principles.
- It allows undefined unary predicate symbols, also known as class variables.

For a given set Pred of predicate symbols, the syntax is as follows:

$$\begin{array}{rcl} \phi, \psi & ::= & P(x) \mid \mathsf{IsSet}(x) \mid x \in y \mid x = y \mid \mathsf{True} \mid \mathsf{False} \\ & & \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \Rightarrow \psi \mid \forall x.\, \phi \mid \exists x.\, \phi \end{array}$$

with P ranging over Pred. The formula $\mathsf{IsSet}(x)$ asserts that x is a set.

We define the *base theory* over Pred to be the classical first-order theory with equality, axiomatized as follows.

- Axiom of *Extensionality*: Any two sets with the same elements are equal.
- Axiom of *Inhabitation*: Anything that has an element is a set.
- Axiom scheme of *Replacement*: For any set A and binary predicate F such that each $x \in A$ has a unique F-image, there is a set $\{F(x) \mid x \in A\}$ of all F-images of elements of A.
- Axiom of *Twoity*: There are sets $0, 1, 2$ such that $0 = \{\}$ and $1 = \{0\}$ and $2 = \{0, 1\}$.
- Axiom of *Union Set*: For any set of sets $\mathcal{A}$, there s is a set $\bigcup \mathcal{A}$ of all elements of elements of $\mathcal{A}$.

Henceforth we assume the base theory. Pairing and Separation follow via

$$\begin{array}{rcl} \{x_0, x_1\} & \stackrel{\text{def}}{=} & \{x_i \mid i \in \{0, 1\}\} \\ \{x \in A \mid P(x)\} & \stackrel{\text{def}}{=} & \displaystyle\bigcup_{x \in A} \begin{cases} \{x\} & \text{if } P(x) \\ \emptyset & \text{otherwise} \end{cases} \end{array}$$

Related work For set theory without Powerset, see [GHJ16]. For set theory without Infinity, see [KW07].

3.2 Classes, Functions and Partial Functions

Since classes are so important in our story, much of this paper is devoted to studying them. As usual in set theory, a class is represented as a predicate formula with parameters. We write $\mathfrak{T}$ for the class of all things, Set for the class of all sets, and Ur for that of all urelements (things that are not sets). A class C is *inhabited* when it has an element—i.e., is not empty. For $x \in C$, the phrase "x is contained in C" mean $x \in C$.

Given classes A and B, we write

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \{\langle x, y\rangle \mid x \in A, y \in B\} \\ A + B &\stackrel{\text{def}}{=} \{\mathsf{inl}\ x \mid x \in A\} \cup \{\mathsf{inr}\ y \mid y \in B\} \end{aligned}$$

where $\langle x, y\rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$ and $\mathsf{inl}\ x \stackrel{\text{def}}{=} \langle 0, x\rangle$ and $\mathsf{inr}\ y \stackrel{\text{def}}{=} \langle 1, y\rangle$. The notation $A \subseteq B$ means that A is *included* in (i.e., a subclass of) B. The notation $F : A \to B$ means that F is a function sending each $x \in A$ to an element of B.

For a function F on a class A, the restriction of F to a subclass C of A is written $F\restriction_C$, or simply as F when C is clear from the context.

For a set K, a K*-tuple* is a function on K. It is written as $[x_k]_{k \in K}$ and envisaged as a column with K entries. It is *within* a class C when, for all $k \in K$, we have $x_k \in C$.

A *family* consists of a set I and a function x on I. More generally, a *class-family* consists of a class I and function x on I. It may be written as (I, x) or as $(x_i)_{i \in I}$. It is *injective* when the function x is injective, and its *range* is the range of x. It is *within* a class C when, for all $i \in I$, we have $x_i \in C$.

Given class-families $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$, we say that the former is *included* in the latter when $I \subseteq J$ and $x = y\restriction_I$. A *map* $(x_i)_{i \in I} \to (y_j)_{j \in J}$ is a function $f : I \to J$ such that, for all $i \in I$, we have $x_i = y_{f(i)}$. It is an *isomorphism* when f is bijective; note that isomorphic class-families have the same range.

The following will be useful.

Proposition 3.1. *Let C be a class. Any sets A, B and surjection $f : A \to B$ yield an injective function $C^f : C^B \to C^A$ sending $[x_b]_{b \in B}$ to $[x_{f(a)}]_{a \in A}$. Its range is the class of all A-tuples y such that, for all $a, a' \in A$ with the same f-image, we have $y_a = y_{a'}$.*

Proof. Straightforward. □

For a class C, we write $\mathcal{P}C$ for the class of all subsets of C, and $\mathcal{P}_{\mathsf{inh}}C$ for the class of all inhabited subsets of C. We write $\mathsf{Fam}(C)$ for the class of all families within C, and $\mathsf{InjFam}(C)$ for the class of all injective families within C. For any set I, we write C^I for the class of all I-tuples within C.

Given a classes A and B, a *partial function* $G : A \rightharpoonup B$ consists of a subclass $\mathsf{Dom}(G)$ of A and a function $\overline{G} : \mathsf{Dom}(G) \to B$. Put differently, it is a class-family (M, F) such that for all $x \in M$ we have $x \in A$ and $F(x) \in B$.

We also speak about "collections", although such talk is informal.

We write Class for the collection of all classes. Given a class C, we write $\mathsf{Sub}(C)$ for the collection of all subclasses of C. We write $\mathsf{ClassFam}(C)$ for the collection of all class-families within C, and $\mathsf{InjClassFam}(C)$ for the collection of all injective class-families within C.

Given a class A, we may speak of a function $B : A \to \mathsf{Class}$, also called an *A-tuple of classes* and written $[B_a]_{a \in A}$. It is represented as a binary predicate formula $\phi(x, y)$ with parameters, so that, for $x \in A$, we have $B_x = \{y \mid \phi(x, y)\}$. The pair (A, B), also written $(B_x)_{x \in A}$, is called a *class-family of classes*, or a *family of classes* if A is a set.

Given a class-family of classes $(B_x)_{x \in A}$, we form the class

$$\sum_{x \in A} B_x \stackrel{\text{def}}{=} \{\langle x, y\rangle \mid x \in A, y \in B_x\}$$

The notation $F \in \prod_{x \in A} B_x$ means that F is a function on A that sends each $x \in A$ to an element of B_x. Likewise, a *partial function* $G \in \prod_{x \in A} B_x$ consists of a subclass $\mathsf{Dom}(G)$ of A and function $\overline{G} \in \prod_{x \in \mathsf{Dom}(G)} B_x$. Put differently, it is a class-family (M, F) such that for all $x \in M$ we have $x \in A$ and $F(x) \in B_x$. It is *small* when it is a family—i.e., the domain is a set.

Two other kinds of function occur in the paper.

- Given a class A, we speak of a function $F : A \to \mathcal{B}$, where $\mathcal{B}$ is a collection, or—more generally—of a function $F \in \prod_{x \in A} \mathcal{B}_x$, where $\mathcal{B}$ is an A-tuple of collections. In each instance, it is obvious how F can be represented. For example, we can represent $F : A \to \mathsf{Class}^2$ as a pair of functions $A \to \mathsf{Class}$.
- Given collections $\mathcal{A}$ and $\mathcal{B}$, we speak of a function $\mathcal{A} \to \mathcal{B}$. For example, $\mathcal{P}$ is an endofunction on Class.

3.3 Class Reasoning: Arbitrariness and Predicativity

When speaking about classes, we must observe two disciplines.

Firstly, in order to assert that all classes have a given property, it is insufficient to prove this merely for classes that are definable from first-order parameters. Instead we must prove that an *arbitrary* class has the property. That is because our base theory's syntax includes class variables.

Secondly, in order to assert the existence of a class with a given property, we must prove this *predicatively*—i.e., without quantification over class variables. That is because our base theory's syntax does not provide such quantification.

Both requirements are illustrated in Section 3.7.

3.4 Powerset, Choice and Collection

We now present some additional principles, beginning with Powerset.

Proposition 3.2. *The following are equivalent.*

- Powerset*: For any set A, the class $\mathcal{P}A$ is a set.*
- Exponentiation*: For any sets A and B, the class B^A is a set.*
- *For any family of sets $(B_i)_{i \in I}$, the class $\prod_{i \in I} B_i$ is a set.*

Proof. Via the following constructions.

$$\begin{aligned}
\mathcal{P}A &\stackrel{\text{def}}{=} \{\{x \in A \mid f(x) = 1\} \mid f \in \{0,1\}^A\} \\
B^A &\stackrel{\text{def}}{=} \prod_{x \in A} B \\
\prod_{i \in I} B_i &\stackrel{\text{def}}{=} \{f \in \mathcal{P} \sum_{i \in I} B_i \mid \forall i \in I.\, \exists! b \in B_i.\, \langle i, b \rangle \in f\}
\end{aligned}$$

□

We continue with the following principles.

- The axiom of *Choice* (AC): For any family of inhabited sets $(A_i)_{i \in I}$, the class $\prod_{i \in I} A_i$ is inhabited.
- The axiom scheme of *Collective Choice*: For any family of inhabited classes $(A_i)_{i \in I}$, the class $\prod_{i \in I} A_i$ is inhabited.
- The axiom scheme of *Collection*: For any family of inhabited classes $(A_i)_{i \in I}$, the class $\prod_{i \in I} \mathcal{P}_{\mathsf{inh}} A_i$ is inhabited.

Proposition 3.3.

(a) *Collective Choice is equivalent to Collection + AC.*

(b) **In ZF** *Collection holds.*

Proof.

(a) For (⇒), it is obvious that AC holds, and Collection is proved as follows. Given a set I and I-tuple of inhabited classes A, we obtain $x \in \prod_{i \in I} A_i$ by Collective Choice, and then $i \mapsto \{x_i\}$ inhabits $\prod_{i \in I} \mathcal{P}_{\mathsf{inh}} A_i$. For (⇐), given a set I and I-tuple of inhabited classes A, we obtain $B \in \prod_{i \in I} \mathcal{P}_{\mathsf{inh}} A_i$ by Collection, and then—by AC—the subclass $\prod_{i \in I} B_i$ of $\prod_{i \in I} A_i$ is inhabited.

(b) We write $(V_\alpha)_{\alpha \in \mathsf{Ord}}$ for the cumulative hierarchy in the usual way. Given a set I and I-tuple of inhabited classes A, we proceed as follows. For each $i \in I$, define $t(i)$ to be the least ordinal α such that $A_i \cap V_\alpha$ is inhabited. Then $i \mapsto A_i \cap V_{t(i)}$ inhabits $\prod_{i \in I} \mathcal{P}_{\mathsf{inh}} A_i$. □

3.5 Ordered Collections

Order plays a large role in our story, so we present some useful notions.

Given an ordered collection $\mathcal{A}$, a subcollection $\mathcal{B}$ is *lower* when, for any $x \in \mathcal{B}$ and $y \leqslant x$, we have $y \in \mathcal{B}$.

Given ordered collections $\mathcal{A}$ and $\mathcal{B}$, a function $h : \mathcal{A} \to \mathcal{B}$ is *monotone* when $x \leqslant y$ implies $h(x) \leqslant h(y)$.

In this paper, a *standard ordered collection* is a collection $\mathcal{E}$ equipped with a class-family of classes $(B_x)_{x \in A}$, and a bijection $\mathcal{E} \cong \prod^{\rightharpoonup}_{x \in A} B_x$. This structure induces an order on $\mathcal{E}$, so we write $\bigvee$ for supremum and $\perp$ for the least element. It also induces a notion of smallness: we write $\mathcal{E}_{\mathsf{small}}$ for the class of all small elements. For $x \in \mathcal{E}$, we write $\downarrow x$ for the class of all $y \in \mathcal{E}_{\mathsf{small}}$ such that $y \leqslant x$.

For example, let C be a class.

- $\mathsf{Sub}(C)$ is a standard ordered collection, since subclasses of C correspond to partial functions $C \rightharpoonup 1$ via the bijection $X \mapsto (X, x \mapsto *)$. A small element is a subset of C.
- $\mathsf{ClassFam}(C)$ is a standard ordered collection, since class-families within C are the same thing as partial functions $\mathfrak{T} \rightharpoonup C$. A small element is a family within C.

The following is adapted from [Acz88, Tak69].

Definition 3.4. Let $\mathcal{D}$ and $\mathcal{E}$ be standard ordered collections.

(a) For any monotone function $f : \mathcal{D}_{\mathsf{small}} \to \mathcal{E}$, the monotone extension $\hat{f} : \mathcal{D} \to \mathcal{E}$ is defined as $x \mapsto \bigvee_{y \in \downarrow x} f(y)$.[2]

(b) A function $h : \mathcal{D} \to \mathcal{E}$ that arises from its restriction to $\mathcal{D}_{\mathsf{small}}$ in this way is said to be *set-continuous*. Explicitly, this means that h sends each $x \in \mathcal{D}$ to $\bigvee_{y \in \downarrow x} h(y)$.

For example, $\mathcal{P}$ is a set-continuous endofunction on Class. All functions between standard ordered collections considered in this paper are set-continuous (and hence monotone).

Note: since a set-continuous function $\mathcal{D} \to \mathcal{E}$ can be represented by its restriction to $\mathcal{D}_{\mathsf{small}}$, any quantifier ranging over set-continuous functions can be regarded as ranging over classes.

[2] In the language of category theory, $\hat{f}$ is the left Kan extension of f along the inclusion $\mathcal{D}_{\mathsf{small}} \subseteq \mathcal{D}$. That is, the least monotone function $g : \mathcal{D} \to \mathcal{E}$ such that $f \leqslant g \restriction_{\mathcal{D}_{\mathsf{small}}}$.

3.6 Fixpoints

We present some notions of fixpoints and order, as they are repeatedly used in the paper.

Let $\mathcal{E}$ be a collection, and h an endofunction on $\mathcal{E}$. Then an element $x \in \mathcal{E}$ is *h-fixed* or an *h-fixpoint* iff $h(x) = x$. The prefix h can be omitted when clear from the context.

Now let $\mathcal{E}$ be an ordered collection, and F a monotone endofunction on $\mathcal{E}$. An element $x \in \mathcal{E}$ is *h-prefixed* or an *h-prefixpoint* when $h(x) \leqslant x$, and *h-postfixed* or an *h-postfixpoint* when $x \leqslant h(x)$.[3] So x is fixed iff it is both prefixed and postfixed. Note that an infimum of prefixpoints is prefixed, and a supremum of postifxpoints is postfixed.

We say that h is *inflationary* when every $x \in \mathcal{E}$ is a postfixpoint, and *deflationary* when every $x \in \mathcal{E}$ is a prefixpoint.

The least prefixpoint of h, if it exists, is written μh. It is necessarily fixed; this fact is called *inductive inversion*. Dually the greatest postfixpoint of h, if it exists, is written νh and is necessarily fixed.

A prefixpoint x is *minimal* when the only prefixpoint y such that $y \leqslant x$ is x itself. A least prefixpoint is minimal, and conversely if $\mathcal{E}$ has binary meets, which is always the case for a standard ordered collection.

3.7 Natural Numbers

Bearing in mind that we do not assume Infinity, we must carefully define the class $\mathbb{N}$ of all natural numbers. Specifically, we shall construct the *Zermelo natural numbers*:

$$\mathbb{N} \quad = \quad \{\mathsf{Nothing}, \mathsf{Just}(\mathsf{Nothing}), \mathsf{Just}(\mathsf{Just}(\mathsf{Nothing})), \ldots\}$$

where $\mathsf{Nothing} \stackrel{\text{def}}{=} \emptyset$ and $\mathsf{Just}(x) \stackrel{\text{def}}{=} \{x\}$. Firstly, we define the monotone endofunction Maybe on Class that sends X to $\{\mathsf{Nothing}\} \cup \{\mathsf{Just}(x) \mid x \in X\}$.[4] So a class X is Maybe-prefixed iff it contains $\mathsf{Nothing}$ and, for any $x \in X$, contains $\mathsf{Just}(x)$. We want $\mathbb{N}$ to be the least Maybe-prefixed class—"least" means that $\mathbb{N}$ is included in an arbitrary Maybe-prefixed class.

We cannot simply define $\mathbb{N}$ to be the intersection of all Maybe-prefixed classes, as that would be impredicative. Instead we proceed as follows.

Proposition 3.5. *The class* $\mathbb{N} \stackrel{\text{def}}{=} \mu\mathsf{Maybe}$ *exists.*

Proof. First note that a class X is Maybe-postfixed iff every $x \in X$ is either $\mathsf{Nothing}$ or $\mathsf{Just}(x)$ for some $x \in X$. When this is so, say that a subclass U of X is *inductive* when it contains $\mathsf{Nothing}$ if X does, and, for all $x \in U$, contains $\mathsf{Just}(x)$ if X does.

For an inhabited class I, we note that Maybe preserves I-indexed intersections, and so any I-indexed intersection of Maybe-postfixpoints is Maybe-postfixed. Therefore, any thing x that is contained in a Maybe-postfixed set is contained in a least such set—viz., the intersection of all Maybe-postfixed sets that contain x. We call this set $\downarrow x$.

Define $\mathbb{N}$ to be the class of all x such that $\downarrow x$ exists and its only inductive subset is $\downarrow x$ itself. This class has the required properties. $\square$

We often write $0 \stackrel{\text{def}}{=} \mathsf{Nothing}$ and, for $n \in \mathbb{N}$, write $n + 1 \stackrel{\text{def}}{=} \mathsf{Just}(n)$. The standard properties of $\mathbb{N}$ hold, including the following.

Proposition 3.6 (Recursion over $\mathbb{N}$)**.** *For any sequence of classes $(B_n)_{n \in \mathbb{N}}$ and any $p \in B_0$ and $L \in \prod_{n \in \mathbb{N}}(B_n \to B_{n+1})$, there is a unique sequence $b \in \prod_{n \in \mathbb{N}} B_n$ such that $b_0 = p$ and, for all $n \in \mathbb{N}$, we have $b_{n+1} = L_n(b_n)$.*

The axiom of *Infinity* says that a Maybe-prefixed set exists; this is equivalent to $\mathbb{N}$ being a set.

[3] This terminology follows [SP82]. Some authors use the opposite terminology, following [MS78].

[4] This terminology comes from functional programming [Ha06].

4 Well-Foundedness and Scaffolds

4.1 Set-Based Relations

We shall now consider relations on a class, leading up to the key notion of well-foundedness.

Definition 4.1. Let $(C, <)$ be a class equipped with a relation.

(a) Let $x \in C$. An element $y \in C$ is

- a *child* of x when $y < x$.
- a *descendant* of x, written $y <^* x$, when there is a sequence

$$y = z_0 < \cdots < z_n = x$$

- a *strict descendant* of x, written $y <^+ x$, when there is such a sequence with $n > 0$.

We write $J_<(x)$ for the class of all children of x, and $J^*_<(x)$ for the class of all descendants, and $J^+_<(x)$ for the class of all strict descendants. The subscript $<$ may be omitted when clear from the context.

(b) A subclass X of C is *hereditary* when every child of an element of X is in X.

Thus, for $x \in C$, we see that $J^*_<(x)$ is the least hereditary subclass of C that contains x.

Example 4.2. Consider the membership relation on $\mathfrak{T}$. A class X is *membership-hereditary* or *transitive* when every element of an element of X is in X. For a thing x, we write $\mathcal{E}(x)$ for its *element set*, which is x or $\emptyset$ according as x is a set or an urelement. We write $\mathcal{E}^*(x)$ for the class of all membership-descendants of x, and $\mathcal{E}^+(x)$ for the class of all strict ones. Thus $\mathcal{E}^*(x)$ is the least transitive class containing x.

Definition 4.3. A relation $<$ on a class C is

- *set-based* when $J(x)$ is a set for all $x \in C$.
- *iteratively set-based* when each $x \in C$ is contained in a hereditary subset of C; this is equivalent to $J^*(x)$ being a set.

Thus $<$ is iteratively set-based iff $<^*$ is set-based.

Proposition 4.4. *Infinity is equivalent to the statement: "Every set-based relation on a class is iteratively set-based."*

Proof. For ($\Rightarrow$), let $<$ be a set-based relation on a class C, and $x \in C$. By induction on $n \in \mathbb{N}$, the class $J^n(x)$ of descendants of x at depth n is a set, since

$$\begin{aligned} J^0(x) &= \{x\} \\ J^{n+1}(x) &= \bigcup_{y \in J^n(x)} J(y) \end{aligned}$$

So the class $J^*(x) = \bigcup_{n \in \mathbb{N}} J^n(x)$ is a set.

For ($\Leftarrow$), the relation on $\mathfrak{T}$ given by $\{\langle x, y \rangle \mid y = \mathsf{Just}(x)\}$ is set-based, and the descendant class of $\mathsf{Nothing}$ is $\mathbb{N}$. So if set-based implies iteratively set-based, then $\mathbb{N}$ is a set. $\square$

Example 4.5. The membership relation on $\mathfrak{T}$ is set-based, and is iteratively set-based iff the *Transitive Containment* axiom holds: Every thing is contained in a transitive set. Thus, by Proposition 4.4, this axiom follows from Infinity.

4.2 Well-Founded Set-Based Relations

The notion of well-foundedness relies on the following concepts.

Definition 4.6. Let $(C, <)$ be a class equipped with a relation. Let X be a subclass.

(a) X is *inductive* when every element of C whose children are all in X is in X.

(b) An element of X is *minimal* when it has no child in X.

Now we are ready to formulate well-foundedness.

Proposition 4.7. *Let $(C, <)$ be a class equipped with a set-based relation. The following are equivalent:*

(a) The only inductive subclass of C is C itself.

(b) Every inhabited subclass of C has a minimal element.

(c) The relation $<$ is iteratively set-based, and every inhabited subset of C has a minimal element.

Proof. Conditions (a) and (b) are equivalent because a subclass of C is inductive iff its complement has no minimal element.

To prove (a) implies (c), we need only show that $<$ is iteratively set-based: for all $x \in C$, the class $J^*(x)$ is a set, by induction on x.

To show (c) implies (b), any subclass Y of C inhabited by x gives a subset $Y \cap J^*(x)$ of C inhabited by x. The latter has a minimal element, which is also a minimal element of Y. □

When the above conditions hold, we say that $<$ is a *well-founded set-based relation*.

Example 4.8. Consider again the membership relation on $\mathfrak{T}$. A class X is *membership-inductive* when every thing whose elements are all in X is in X. An element $x \in X$ is *membership-minimal* when it has no element in common with X. We can say that membership is well-founded in each of the following ways.

- The axiom scheme of *Membership Induction*: The only membership-inductive class is $\mathfrak{T}$.
- The axiom scheme of *Class Regularity*: Every inhabited class has a membership-minimal element.
- Transitive Containment + the axiom of *Regularity*: Every inhabited set has a membership-minimal element.

Here are some basic properties of well-founded relations.

Proposition 4.9. *Let $(A, <)$ and $(B, <')$ be classes equipped with a set-based relation, and $f : A \to B$ a function such that $x < y$ implies $f(x) <' f(y)$. If $<'$ is well-founded, then $<$ is too.*

Proof. Let X be an inductive subclass of A. We prove by induction on $y \in B$ that $f^{-1}(y) \subseteq X$. □

Proposition 4.10. *Let $(C, <)$ be a class equipped with a set-based relation.*

(a) The relation $<^+$ is well-founded iff $<$ is.

(b) If $<$ is well-founded, then there is no infinite sequence $\cdots < x_1 < x_0$.

Proof.

(a) The direction ($\Rightarrow$) is by Proposition 4.9. For ($\Leftarrow$), let X be a $<^+$-inductive subclass of C. For all $x \in C$, we prove $J^*(x) \subseteq C$ by induction on x.

(b) Fix such a sequence. We prove that every $x \in C$ fails to appear in it, by induction on x. ☐

Functions and partial functions can be defined by well-founded recursion:

Proposition 4.11. *Let $(A, <)$ be a class equipped with a well-founded set-based relation, and B an A-tuple of classes.*

(a) *For any $L \in \prod_{x \in A}((\prod_{y \in J(x)} B_y) \to B_x)$, there is a unique function $F \in \prod_{x \in A} B_x$ sending $x \in A$ to $L_x(F \upharpoonright_{J(x))})$. In other words, the endofunction Φ_L on $\prod_{x \in A} B_x$ sending F to $x \mapsto L_x(F \upharpoonright_{J(x)})$ has a unique fixpoint.*

(b) *For any $L \in \prod_{x \in A}((\prod_{y \in J(x)} B_y) \rightharpoonup B_x)$, let Ψ_L be the monotone endofunction on $\prod_{x \in A} B_x$ sending (M, F) to (N, G), where N is the class of all $x \in A$ such that $J(x) \subseteq M$ and $F \upharpoonright_{J(x)} \in \mathsf{Dom}(L_x)$, and G sends such an x to $\overline{L_x}(F \upharpoonright_{J(x)})$. Then Ψ_L has a least prefixpoint that is also a greatest postfixpoint and therefore a unique fixpoint.*

Proof. We first prove part (b). For a hereditary subclass M of A, an *attempt* on M is a function $F \in \prod_{x \in M} B_x$ such that, for all $x \in M$, we have $F \upharpoonright_{J(x)} \in \mathsf{Dom}(L_x)$ and $F(x) = \overline{L_x}(F_x \upharpoonright_{J(x)})$. Thus a Ψ_L-postfixpoint (M, F) consists of a hereditary subclass M of A, and an attempt F on M. Induction shows that

- any attempt on M and attempt on M' agree on $M \cap M'$
- any postfixpoint is included in any prefixpoint.

Let P be the class of all x such that there is a (necessarily unique) attempt on $J^*(x)$. Let H send each $x \in P$ to its image under the attempt on $J^*(x)$. Then (P, H) is a fixpoint, since any attempt g on $J^+(x)$ such that $g \upharpoonright_{J(x)} \in \mathsf{Dom}(L_x)$ extends to an attempt $g \cup \{\langle x, \overline{L_x}(g \upharpoonright_{J(x)})\rangle\}$ on $J^*(x)$. So part (b) is proved.

If L_x is total for all $x \in A$, then any Ψ_L-prefixpoint is total, and Φ_L is the restriction of Ψ_L to total functions, so part (a) follows. ☐

4.3 Generating a Subclass

Suppose we have a class C. We sometimes want to show that a given endofunction on $\mathsf{Sub}(C)$ has a least prefixpoint. The following makes this possible.

Definition 4.12.

(a) A *scaffold* on C consists of

- a subclass D
- a relation $<$ from C to D.

We call $x \in D$ a *parent* and $y < x$ a *child* of x. The scaffold is *set-based* when, for all $x \in D$, the class $J_<(x) \stackrel{\text{def}}{=} \{y \in C \mid y < x\}$ is a set.

(b) A scaffold $(D, <)$ on C gives rise to a set-continuous endofunction $\Gamma_{(D,<)}$ on $\mathsf{Sub}(C)$, sending X to the class of all $x \in D$ whose children are all in X.

Thus a subclass X of C is

- $\Gamma_{(D,<)}$-prefixed iff every parent whose children are all in X is in X
- $\Gamma_{(D,<)}$-postfixed iff it is a hereditary subclass of D.

Proposition 4.13. *Let $(D,<)$ be a scaffold on C.*

(a) (Generated subclass.) *If the scaffold is set-based, then $\Gamma_{(D,<)}$ has a least prefixpoint, which is also the greatest postfixpoint on which $<$ is well-founded.*

(b) (Cogenerated subclass.) *The endofunction $\Gamma_{(D,<)}$ has a greatest postfixpoint.*

Proof.

(a) First note that $<$ is a set-based relation on C, since for $x \in C \setminus D$, the class $J(x)$ is $\emptyset$.

Take the class of all $x \in C$ such that $J^*(x)$ is a subset of D whose only inductive subset is itself. This is clearly the least $\Gamma_{(D,<)}$-prefixed subclass. By inductive inversion, it is $\Gamma_{(D,<)}$-postfixed. The rest is straightforward, using the fact that any set-based well-founded relation is iteratively set-based.

(b) Take the class of all $x \in C$ such that $J^*(x) \subseteq D$. ☐

Note All scaffolds in this paper are on $\mathfrak{T}$, except in the proof of Theorem 7.3, where we use a scaffold on a set.

Example 4.14. The endofunction Maybe arises from the following scaffold on $\mathfrak{T}$: a parent is either $\mathsf{Nothing}$, which has no children, or $\mathsf{Just}(x)$, whose sole child is x. So Proposition 3.5 is an instance of Proposition 4.13(a).

Example 4.15. The endofunction $\Gamma_{(\mathfrak{T},\in)}$ sends a class X to $\mathsf{Ur} \cup \mathcal{P}X$. We define

$$V_{\mathsf{impure}} \stackrel{\text{def}}{=} \mu\Gamma_{(\mathfrak{T},\in)}$$

which is the least $\mathcal{P}$-prefixed class that includes Ur. Proposition 4.13(a) tells us that V_{impure} is the least membership-inductive class, and the greatest transitive class on which membership is well-founded. A member of this class is called a *vonniad*—the name alludes to "von Neumann iteration".

Example 4.16. A thing is *pure* when its membership-descendants are all sets. The class of all pure things is $\nu\mathcal{P}$, which is an instance of Proposition 4.13(b) since $\mathcal{P} = \Gamma_{(\mathsf{Set},\in)}$. Likewise, the class of all pure vonniads is given by

$$V_{\mathsf{pure}} \stackrel{\text{def}}{=} \mu\mathcal{P}$$

which is an instance of Proposition 4.13(a).

4.4 Generating a Partial Function

Now suppose we have a class-family of classes $(B_x)_{x\in a}$. We sometime want to show that a given endofunction on $\prod_{x\in A} B_x$ has a least prefixpoint. The following makes this possible.

Definition 4.17. Let $(D,<)$ be a set-based scaffold on A.

(a) A *functionalization* of $(D,<)$ on B is an $L \in \prod_{x\in D}((\prod_{y\in J(x)} B_y) \rightharpoonup B_x)$.

(b) Let L be such a functionalization. The set-continuous endofunction $\Delta^{L}_{(D,<)}$ on $\prod_{x\in A} B_x$ sends (M,F) to (N,G), where

- N is the class of $x \in D$ such that $J(x) \subseteq M$ and $F\restriction_{J(x)} \in \mathsf{Dom}(L_x)$
- G sends each such x to $\overline{L_x}(F\restriction_{J(x)})$.

Proposition 4.18 (Generated partial function)**.** *Let $(D,<)$ be a set-based scaffold on A with functionalization L on B. Then $\Delta^L_{(D,<)}$ has a least prefixpoint, which is also the greatest postfixpoint (M,F) such that $<$ is well-founded on M.*

Proof. Let E be the subclass of A generated by $(D,<)$. Since E is $\Gamma_{(D,<)}$-prefixed, $\Delta^L_{(D,<)}$ restricts to an endofunction on $\vec{\prod}_{x\in E} B_x$. By Proposition 4.11(b), the latter has a least prefixpoint (M,F) that is also a greatest postfixpoint, since $<$ is well-founded on E. Because (M,F) is a minimal prefixpoint in $\vec{\prod}_{x\in E} B_x$, which is a lower subcollection of $\vec{\prod}_{x\in A} B_x$, it is a minimal and therefore least prefixpoint in $\vec{\prod}_{x\in A} B_x$.

For any postfixpoint (N,G), the class N is $\Gamma_{(D,<)}$-postfixed. So if $<$ is well-founded on N, then $N \subseteq E$, giving $(N,G) \in \vec{\prod}_{x\in E} B_x$ and so $(N,G) \leqslant (M,F)$. □

4.5 Introspection

This section is not used in the sequel.

If membership is well-founded, then $\mathbb{N}$ is the *unique* Maybe-fixpoint, i.e., the unique class X such that $x \in X$ iff either $x = \mathsf{Nothing}$ or $x = \mathsf{Just}(y)$ for some $y \in X$. Various other classes defined in the paper, such as $\mathsf{Wide}(S)$ and $\mathsf{Broad}(G)$ and $\mathsf{Deriv}_{\mathcal{R}}$ and Ord, have a similar property. That is because they are "introspectively generated", in a sense that I now explain.

Definition 4.19. Let C be a class.

(a) A relation $<$ on C is *introspective* when $<$ is included in $\in^+$. In other words: when, for all $x \in C$, we have $J(x) \subseteq \mathcal{E}^+(x)$.

(b) Likewise, a scaffold $(D,<)$ on C is *introspective* when $<$ is included in $\in^+$.

Proposition 4.20.

(a) *Transitive Containment is equivalent to the statement: "Every introspective relation on a class is iteratively set-based."*

(b) *Membership Induction is equivalent to the statement: "Every introspective relation on a class is well-founded."*

Proof.

(a) Transitive Containment is equivalent to membership being iteratively set-based, which is equivalent to $\in^+$ being iteratively set-based, which is equivalent to every introspective relation being iteratively set-based.

(b) Similar. □

Now we come to the key result of the section:

Proposition 4.21. *Each of the following is equivalent to Membership Induction.*

(a) *For any class C, and any set-based introspective scaffold $(D,<)$ on C, the endofunction $\Gamma_{(D,<)}$ on $\mathsf{Sub}(C)$ has a unique fixpoint.*

(b) *For any class A and A-tuple of classes B, and any set-based introspective scaffold $(D,<)$ on A with functionalization L on B, the endofunction $\Delta^L_{(D,<)}$ on $\vec{\prod}_{x\in A} B_x$ has a unique fixpoint.*

Proof.

(a) Membership Induction implies this by Proposition 4.20(b) and Proposition 4.13(a), since a least prefixpoint that is also a greatest postfixpoint is a unique fixpoint. For the converse, since $\mathfrak{T}$ is a fixpoint of $\Gamma_{(\mathfrak{T},\in)}$, it must be the least prefixpoint and so $\in$ is well-founded over it.

(b) Membership Induction implies this by Proposition 4.20(b) and Proposition 4.18. For the converse, the scaffold $(\mathfrak{T}, \in)$ on $\mathfrak{T}$ has a functionalization L on $(1)_{x\in\mathfrak{T}}$ that at x takes $[*]_{y\in\mathcal{E}(x)}$ to $*$. Since the partial function $(\mathfrak{T}, x \mapsto *)$ is a fixpoint of $\Delta^L_{(\mathfrak{T},\in)}$, it must be the least prefixpoint and so $<$ is well-founded over $\mathfrak{T}$. $\square$

Example 4.22. Let us apply Proposition 4.21(a) to Example 4.14. We see that, if Membership Induction holds, then $\mathbb{N}$ is the unique Maybe-fixpoint.

5 Using Ordinals

5.1 Set-Based Well-Orderings

Our next task is to give the basic principles of ordinals, which requires us to first develop the notion of well-ordering. As in Section 4.2, we treat not only relations on a set, but also on a class.

Let C be a class with a relation $<$. Recall the notation $J(x) \stackrel{\text{def}}{=} \{y \in C \mid y < x\}$. We write $\sqsubseteq$ for the *extensional preorder*, given by $x \sqsubseteq y \stackrel{\text{def}}{\iff} J(x) \subseteq J(y)$. This is an order iff J is injective, and we then say that $<$ is *extensional*.

A *strict order* on a class C is an irreflexive transitive relation $<$. We write $\leqslant$ for the corresponding order, given by $x \leqslant y \stackrel{\text{def}}{\iff} x < y \vee x = y$. Thus a subclass is lower iff it is hereditary.

A *linear order* on a class C is a relation $<$ such that no two elements of C are mutually related and, for any $x, y \in C$, either $x = y$ or $x < y$ or $y < x$. It follows that $<$ is both extensional and a strict order, with $\sqsubseteq$ and $\leqslant$ coinciding.

Now let us formulate the notion of well-ordering.[5]

Proposition 5.1. *Let $(C, <)$ be a class equipped with a set-based relation. The following are equivalent:*

(a) $<$ is well-founded, extensional and transitive.

(b) $<$ is well-founded and, for any $x, y \in C$, either $x = y$ or $x < y$ or $y < x$.

(c) $<$ is a strict order, and any inhabited subclass has a least element.

(d) $<$ is a strict order, and any inhabited subset has a least element.

Moreover, when these conditions hold, the relations $\sqsubseteq$ and $\leqslant$ coincide.

Proof. Firstly, (b) implies (a) since a well-founded relation cannot mutually relate two elements, by Proposition 4.10(b).

For (a) $\Rightarrow$ (b) say that $x, y \in C$ are *comparable* when either $x = y$ or $x < y$ or $y < x$. We claim that, for all $a, b \in C$, if a is comparable with all $y \in J(b)$, and all $x \in J(a)$ with b, then a is with b. It follows by induction that any $a, b \in C$ are comparable.

To prove the claim, it suffices to show that $a \not< b$ and $b \not< a$ implies $a = b$. Any $x \in J(a)$ is comparable with b, and is therefore $< b$ as $b \leqslant x$ would imply $b < a$. Thus $J(a) \subseteq J(b)$, and likewise $J(b) \subseteq J(a)$. Extensionality gives $a = b$.

The rest follows from Proposition 4.7, noting that any set-based transitive relation is iteratively set-based. $\square$

[5] Cf. [Gra78, Page 93].

A relation satisfying the above conditions is called a *set-based well-ordering*. Henceforth, we often abbreviate $(C, <)$ as C. Now we consider how to compare set-based well-ordered classes.

Proposition 5.2. *Let A and B be set-based well-ordered classes. For a function $f : A \to B$, the following are equivalent.*

- *f is an isomorphism from A to a hereditary subclass of B.*
- *The square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J"'] & B \arrow[d, "J"] \\ \mathcal{P}A \arrow[r, "\mathcal{P}f"'] & \mathcal{P}B \end{tikzcd}$$

  commutes.[6] *Explicitly: for all $x \in A$, we have $J(f(x)) = \{f(y) \mid y \in J(x)\}$.*

Proof. ($\Rightarrow$) is straightforward. For ($\Leftarrow$), we show that $f(x) = f(x')$ implies $x = x'$ by induction on $x \in A$ as follows: $f(x) = f(x')$ implies that for all $y \in J(x)$ there is $y' \in J(x')$ such that $f(y) = f(y')$ and hence $y = y'$ so $J(x) \subseteq J(x')$, and the reverse inclusion likewise, so extensionality of A gives $x = x'$. □

A function satisfying the above conditions is called an *embedding*.

Proposition 5.3.

(a) For set-based well-ordered classes A, B, C, the composite of embeddings $A \to B$ and $B \to C$ is an embedding $A \to C$.

(b) For set-based well-ordered classes A and B, there is at most one embedding $A \to B$.

Proof.

(a) Follows from the fact that, for any hereditary subclass X of B, an embedding $B \to C$ restricts to an embedding $X \to C$.

(b) For embeddings $f, g : A \to B$, we show $f(x) = g(x)$ by induction on $x \in A$. □

Next we consider isomorphisms:

Proposition 5.4. *Let A and B be set-based well-ordered classes.*

(a) For a function $f : A \to B$, the following are equivalent:

- *f is an isomorphism.*
- *f is a surjective embedding.*
- *f is an embedding, and there is an embedding $B \to A$.*

(b) There is at most one isomorphism $A \cong B$.

Proof.

(a) Clearly, the first two conditions are equivalent and imply the third. Lastly, if f and $g : D \to C$ are embeddings, then Proposition 5.3(a) gives endo-embeddings $g \circ f$ on C and $f \circ g$ on D. By Proposition 5.3(b), both are identity maps.

[6] In the language of category theory, this says that f is a $\mathcal{P}$-coalgebra map.

(b) Special case of Proposition 5.3(b). ☐

Proposition 5.5. *Let C be a set-based well-ordered class.*

(a) *The only hereditary subclasses of C are $J(x)$, for $x \in C$, and C itself.*

(b) *These are pairwise non-isomorphic. In other words:*

- *For any $x, y \in C$, if $J(x) \cong J(y)$, then $x = y$.*
- *For any $x \in C$, we do not have $J(x) \cong C$.*

Proof.

(a) Let X be a hereditary subclass. If it has a strict upper bound, it has a strict supremum x and is $J(x)$. Otherwise it is C.

(b) For $x \in C$ and a hereditary subclass Y of C, any isomorphism $\theta : J(x) \cong Y$ is an embedding $J(x) \to C$. By Proposition 5.3(b), this must be the inclusion, so $J(x) = Y$. Since $x \notin Y$, we cannot have $Y = C$, nor $X = J(y)$ for $y > x$. Lastly, for $y < x$, we canot have $Y = J(y)$ since $y \in Y$. ☐

Proposition 5.6.

(a) *For set-based well-ordered classes C and D, either C embeds into D or vice versa.*

(b) *Let B be a subclass of a set-based well-ordered class $(C, <)$. Then $<$ is a set-based well-ordering on B, and there is a deflationary embedding $B \to C$.*

Proof.

(a) Define $R(C, D)$ to be the relation from C to D that relates $x \in C$ to $y \in D$ when $J(x) \cong J(y)$. It is an isomorphism from a hereditary subclass X of C to a hereditary subclass Y of D. If $X = J(x)$ and $Y = J(y)$, then $(x, y) \in R(C, D)$, contradiction. Therefore either $X = C$ or $Y = D$, which gives the two cases.

(b) On the subclass B, the relation $<$ is set-based and well-founded by Proposition 4.9, and also linear, hence a well-ordering. Obtain $R(B, C)$ as before. Induction shows that, for each $x \in B$, there is $y \leqslant x$ such that $(x, y) \in R(B, C)$. Thus R is total and deflationary. ☐

Proposition 5.7. *Let $(C, <)$ be a set-based well-ordered class. The following are equivalent.*

(a) *C is a proper class.*

(b) *Every subset of C has a strict supremum.*

(c) *Every hereditary subset of C has a strict supremum.*

(d) *Every set-based well-ordered class embeds into C.*

(e) *Every well-ordered set is isomorphic to $J(x)$ for some $x \in C$.*

Proof. Clearly (b) implies (c). For the converse, given a subset A of C, its hereditary closure $\{x \in C \mid \exists y \in A.\, x \leqslant y\}$ has the same strict upper bounds.

(c) implies (a), because C itself does not have a strict upper bound, so is not a set.

(a) implies (d) because, for any set-based well-ordered class B that does not embed into C, Proposition 5.6(a) gives an embedding $f : C \to B$. The range of f is a hereditary subclass of B that is not a set, and therefore is B by Proposition 5.5(a). So f is an isomorphism $C \cong B$, and its inverse is an embedding $B \to C$, contradiction.

Clearly (d) implies (e).

Lastly, (e) implies (c) because, for any hereditary subset A of C, there is an isomorphism $f : A \cong J(x)$ for some $x \in A$, which is the identity by Proposition 5.3(b). So x is a strict supremum of A. $\square$

We say that $(C, <)$ is *complete* when it has the above properties.

Proposition 5.8. *Any two complete set-based well-ordered classes are uniquely isomorphic.*

Proof. By Propositions 5.7(d) and 5.4. $\square$

5.2 Ordinals

Our next task is to define Ord. To do this, we write TrSet for the class of all transitive sets. Thus the function $\Gamma_{(\mathsf{TrSet},\in)}$ sends a class X to the class of all its transitive subsets. By Proposition 4.13(a), we can define

$$\mathsf{Ord} \quad \stackrel{\text{def}}{=} \quad \mu\Gamma_{(\mathsf{TrSet},\in)}$$

Alternatively, we can say that an *ordinal* is a transitive set of transitive pure vonniads. Here is yet another characterization of Ord:

Proposition 5.9. Ord *is the unique class of sets X such that $(X, \in)$ is a complete set-based well-ordered class.*

Proof. To prove uniqueness, let X, Y be two such classes. Proposition 5.8 gives an isomorphism $f : (X, \in) \cong (Y, \in)$. For all $x \in X$, we have $f(x) = x$, by induction on x: since $f(x)$ and x are sets that (by the inductive hypothesis) have the same elements, they are equal. So $X = Y$.

Now we show Ord has the required property. By inductive inversion, every ordinal is a transitive set of ordinals, so membership is extensional and transitive on Ord. Furthermore, membership is (set-based and) well-founded on Ord, so it is a well-ordering. Lastly, $(\mathsf{Ord}, \in)$ is complete since any transitive set of ordinals is an ordinal and its own supremum. $\square$

It is convenient to refer to a special value ∞ that is deemed greater than every ordinal. (Some authors call it "Absolute Infinity".) To ensure that it is not an ordinal, we define $\infty \stackrel{\text{def}}{=} \{\{0\}\}$. We obtain the ordered class $\mathsf{Ord}_\infty \stackrel{\text{def}}{=} \mathsf{Ord} \cup \{\infty\}$, whose elements are called *extended ordinals*. For any extended ordinals $\alpha \leqslant \beta$, we define intervals

$$(\alpha \mathinner{..} \beta) \quad \stackrel{\text{def}}{=} \quad \{\gamma \in \mathsf{Ord} \mid \alpha < \gamma < \beta\}$$
$$[\alpha \mathinner{..} \beta) \quad \stackrel{\text{def}}{=} \quad \{\gamma \in \mathsf{Ord} \mid \alpha \leqslant \gamma < \beta\}$$

We define $0 \stackrel{\text{def}}{=} \emptyset$ and, for any ordinal α, define $\mathsf{S}\alpha \stackrel{\text{def}}{=} \alpha \cup \{\alpha\}$. For each class I, we have the supremum function $\bigvee_I : \mathsf{Ord}_\infty^I \to \mathsf{Ord}_\infty$ and the strict supremum function $\mathsf{ssup}_I : \mathsf{Ord}^I \to \mathsf{Ord}_\infty$. They are connected via the equation $\mathsf{ssup}_{i \in I} \alpha_i = \bigvee_{i \in I} \mathsf{S}\alpha_i$.

By Proposition 5.5(a), we have a bijection from Ord_∞ to the collection of all lower classes of ordinals, sending α to $[0 \mathinner{..} \alpha)$. The inverse sends a lower class to its strict supremum.

By completeness, any set-based well-ordered class C is uniquely isomorphic to a lower class of ordinals, whose strict supremum is called the *order-type* of C.

More generally, let $(C, <)$ be a class equipped with a well-founded set-based relation. We recursively define the rank function $\rho : C \to \mathsf{Ord}$, sending x to $\mathsf{ssup}_{y \in J(x)} \rho(y)$, which by induction is also the strict supremum of the lower set $\{\rho(y) \mid y \in J^+(x)\}$. The range of ρ is a lower class, and its strict supremum called the *height* of $(C, <)$.

As an example, for $m, n \in \mathbb{N}$, define $m \prec n \overset{\text{def}}{\iff} m + 1 = n$. Since $\prec$ is a set-based well-founded relation on $\mathbb{N}$, we obtain the injection $\iota : \mathbb{N} \to \mathsf{Ord}$ sending $\mathsf{Nothing} \mapsto 0$ and $\mathsf{Just}(n) \mapsto \mathsf{S}(\iota n)$. The height of $(\mathbb{N}, \prec)$, denoted ω, is also the order-type of $(\mathbb{N}, <)$. It is a limit ordinal if Infinity holds, and ∞ otherwise.

5.3 Inductive Chains for Set-Continuous Functions

We recall a widely used notion in set theory:

Definition 5.10. An *ascending chain* within an ordered collection $\mathcal{A}$ is a monotone function $\mathsf{Ord} \to \mathcal{A}$. That is, a sequence $(x_\alpha)_{\alpha \in \mathsf{Ord}}$ such that $\alpha \leqslant \beta$ implies $x_\alpha \leqslant x_\beta$.

Proposition 5.11. *For standard ordered collections $\mathcal{D}$ and $\mathcal{E}$, a set-continuous function $h : \mathcal{D} \to \mathcal{E}$ preserves the supremum of every ascending chain $(x_\alpha)_{\alpha \in \mathsf{Ord}}$.*

Proof. Define $p \overset{\text{def}}{=} \bigvee_{\alpha \in \mathsf{Ord}} x_\alpha$.

First we show that $\downarrow p = \bigcup_{\alpha \in \mathsf{Ord}} \downarrow x_\alpha$. We just prove $\leqslant$, as $\geqslant$ is obvious. Suppose $\mathcal{E} = \prod^{\rightarrow}_{x \in A} B_x$ and $p = (N, F)$. We thus have $N = \bigcup_{\alpha \in \mathsf{Ord}} M_\alpha$, where for each $\alpha \in \mathsf{Ord}$ we have $x_\alpha = (M_\alpha, F \upharpoonright_{M_\alpha})$. Given an element $y \in \downarrow p$, we have $y = (K, N \upharpoonright_K)$, for some subset K of N. For each $k \in N$, let $\overline{k}$ be the least ordinal β such that $k \in M_\beta$. Put $\alpha \overset{\text{def}}{=} \bigvee_{k \in K} \overline{k}$, and we see that $K \subseteq M_\alpha$. Hence $y \in \downarrow x_\alpha$ as required.

It follows that

$$\begin{aligned} h(p) &= \bigvee_{y \in \downarrow p} h(y) \\ &= \bigvee_{\alpha \in \mathsf{Ord}} \bigvee_{y \in \downarrow x_\alpha} h(y) \\ &= \bigvee_{\alpha \in \mathsf{Ord}} x_\alpha \end{aligned}$$

□

The notion of ascending chain is used as follows.

Definition 5.12. Let h be a set-continuous endofunction on a standard ordered collection $\mathcal{E}$. An *inductive chain* for h is an ascending chain $(x_\alpha)_{\alpha \in \mathsf{Ord}}$ within $\mathcal{E}$ such that

$$\begin{aligned} x_0 &= \bot \\ x_{\mathsf{S}\alpha} &= h(x_\alpha) && \text{for any ordinal } \alpha \\ x_\alpha &= \textstyle\bigvee_{\beta < \alpha} x_\beta && \text{for any limit ordinal } \alpha. \end{aligned}$$

Equivalently: such that $X_\alpha = \bigvee_{\beta < \alpha} h(\mu^\beta h)$, for every ordinal α.

Clearly, there is at most one inductive chain. It it exists, it is written $(\mu^\alpha h)_{\alpha \in \mathsf{Ord}}$ and its supremum $\mu^\infty h$. Existence is unclear in general, since we cannot recursively define a sequence of classes, but we shall see various cases where it does exist:

- if h has a small prefixpoint (Proposition 5.14)

- if h preserves smallness (Proposition 5.15)
- if Collection holds (Proposition 5.16)
- if h arises from a scaffold (Proposition 5.17), or scaffold with functionalization (Proposition 5.18).

The key property of an inductive chain is that it yields a least prefixpoint:

Proposition 5.13. *Let h be a set-continuous endofunction on a standard ordered collection $\mathcal{E}$. If h has an inductive chain, then $\mu^\infty h$ is a least h-prefixpoint.*

Proof. Applying Proposition 5.11 to the inductive chain tells us that $\mu^\infty h$ is h-fixed. To show leastness, let z be a prefixpoint. Induction on $\alpha \in \mathsf{Ord}$ gives $\mu^\alpha h \leqslant z$. So $\mu^\infty h \leqslant z$. □

For an extended ordinal α, an inductive chain *stabilizes* at α when $\mu^\alpha h$ is h-prefixed, or equivalently when $\mu^\alpha h = \mu^\infty h$. Here is an application of this notion:

Proposition 5.14. *For a set-continuous endofunction h on a standard ordered collection $\mathcal{E}$, the following are equivalent.*

- *h has a small prefixpoint.*
- *h has a small least prefixpoint.*
- *h has an inductive chain within $\mathcal{E}_{\mathsf{small}}$ that stabilizes at an ordinal.*

Proof. If h has an inductive chain within $\mathcal{E}_{\mathsf{small}}$ that stabilizes at an ordinal α, then $\mu^\alpha h$ is a small least prefixpoint. Conversely, suppose h has a small prefixpoint x. Then we can define the inductive chain by well-founded recursion as a function $\mathsf{Ord} \to \downarrow x$. For stabilization, we prove a general fact: for any any ascending chain $(x_\alpha)_{\alpha \in \mathsf{Ord}}$ in $\mathcal{E}$, if $\bigvee_{\alpha \in \mathsf{Ord}} p \in \mathcal{E}_{\mathsf{small}}$, then there is $\beta \in \mathsf{Ord}$ such that $(xf\alpha)_{\alpha \geqslant \beta}$ is constant. Suppose $\mathcal{E} = \prod_{x \in A} B_x$, and $\bigvee_{\alpha \in \mathsf{Ord}} x_\alpha = (K, F)$. By hypothesis, K is a set. For each ordinal α, we have $x_\alpha = (M_\alpha, F \upharpoonright_{M_\alpha})$ for a subset M_α of N. For each $k \in K$, let $\overline{k}$ be the least ordinal α such that $k \in M_\alpha$. Then $\beta \stackrel{\text{def}}{=} \bigvee_{k \in K} \overline{k}$ has the required property. □

Here is the most important case (for our purposes) where inductive chains exist:

Proposition 5.15. *Let $\mathcal{E}$ be a standard ordered collection, and h a set-continuous endofunction on $\mathcal{E}$ that preserves smallness.*

(a) *h has an inductive chain within $\mathcal{E}_{\mathsf{small}}$, and a least prefixpoint.*

(b) *The following are equivalent:*

  - *h has a small prefixpoint.*
  - *μh is small.*
  - *The inductive chain stabilizes at an ordinal.*

Proof.

(a) We define the inductive chain by well-founded recursion as a function $\mathsf{Ord} \to \mathcal{E}_{\mathsf{small}}$.

(b) By Proposition 5.14. □

As an example of Proposition 5.15(a), if Powerset is assumed, then $\mathcal{P}$ has an inductive chain consisting of sets. This is the well-known cumulative hierarchy.

We see next that Collection guarantees the existence of inductive chains. This is adapted from [Tak69, Theorem 1] and [Acz88, Theorem 6.4] and [AR01, Theorem 5.1].

Proposition 5.16. (Assuming Collection.) *Let h be a set-continuous endofunction on a standard ordered collection $\mathcal{E}$. Then h has an inductive chain and a least prefixpoint.*

Proof. Via the bijection $\mathcal{E} \cong \vec{\prod}_{x\in A} B_x$ we identify elements of $\mathcal{E}$ with subclasses W of $\sum_{x\in A} B_x$ that are *single-valued*, meaning that, for all $x \in A$, there is at most one $y \in B_x$ such that $\langle x, y\rangle \in W$.

A subclass M of $\mathsf{Ord} \times \sum_{x\in A} B_x$ represents a sequence $(M_\alpha)_{\alpha\in\mathsf{Ord}}$ of subclasses of $\sum_{x\in A} B_x$ via $M_\alpha \stackrel{\text{def}}{=} \{x \in C \mid \langle \alpha, x\rangle \in M\}$. We say that M is *good* when $(M_\alpha)_{\alpha\in\mathsf{Ord}}$ is ascending and, for any ordinal α and $\langle x, y\rangle \in M_\alpha$, there is $\beta < \alpha$ and a single-valued subset W of M_β such that $\langle x, y\rangle \in h(W)$.

For a good class M, we show by induction on $\alpha \in \mathsf{Ord}$ that the class M_α is single-valued, as follows. For $x \in A$ and $y, y' \in B_x$, suppose $\langle x, y\rangle$ and $\langle x, y'\rangle$ are in M_α. We obtain $\beta < \alpha$ and a single-valued subset W of M_β such that $\langle x, y\rangle \in h(W)$, and likewise $\beta' < \alpha$ and a single-valued subset W' of $M_{\beta'}$ such that $\langle x, y'\rangle \in h(W')$. Without loss of generality we have $\beta \leqslant \beta'$, so W' is a subset of $M_{\beta'}$, and hence $W \cup W'$ is too. By the inductive hypothesis at β', the class M_β is single-valued, so the set $W \cup W'$ is too. Next, $\langle x, y\rangle \in h(W)$ gives $\langle x, y\rangle \in h(W \cup W')$, and likewise $\langle x, y'\rangle \in h(W \cup W')$. So $y = y'$.

Clearly, any union of good classes is good; in particular, the union R of all good sets. We show that, for any ordinal α, we have $R_\alpha = \bigvee_{\beta<\alpha} h(R_\beta)$. so that $(R_\alpha)_{\alpha\in\mathsf{Ord}}$ is an inductive chain. We just prove $\geqslant$, as $\leqslant$ is obvious. Given $\beta < \alpha$ and $\langle x, y\rangle \in h(R_\beta)$, there is a subset K of R_β such that $\langle x, y\rangle \in h(K)$. For each $k \in K$, we have $k \in R_\beta$ since $K \subseteq R_\beta$, so the class T_k of all good sets X such that $k \in X_\beta$ is inhabited. By Collection, there is $S \in \prod_{k\in K} \mathcal{P}_{\mathsf{inh}} T_k$. For each $k \in K$, the set $L_k \stackrel{\text{def}}{=} \bigcup S_k$ is a member of T_k. The set $N \stackrel{\text{def}}{=} \bigcup_{k\in K} L_k$ is good; and $K \subseteq N_\beta$ since, for all $k \in K$, we have $k \in (L_k)_\beta \subseteq N_\beta$. Putting $N' \stackrel{\text{def}}{=} \{\langle \alpha, \langle x, y\rangle\rangle\} \cup N$, we have $N'_\beta = N_\beta$, so $\langle x, y\rangle \in h(K) \subseteq h(N_\beta) = h(N'_\beta)$, so N' is good. Thus $\langle x, y\rangle \in N'_\alpha \subseteq R_\alpha$ as required.

The supremum $\bigvee_{\alpha\in\mathsf{Ord}}$ is a least prefixpoint by Proposition 5.13. ☐

Proposition 4.13(b) and Proposition 5.16 suggest the following question: for a class C, does every set-continuous endofunction on $\mathsf{Sub}(C)$ have a greatest postfixpoint? The question is addressed in [Acz88, Theorem 6.5] and [Acz08].

5.4 Inductive Chains for Scaffold Functions

We see next that every function arising from a set-based scaffold has an inductive chain.

Proposition 5.17. *Let $(D, <)$ be a set-based scaffold on a class C, generating $M \in \mathsf{Sub}(C)$.*

(a) For each extended ordinal α, we define $M_\alpha \stackrel{\text{def}}{=} \{x \in M \mid \rho(x) < \alpha\}$. Then $\Gamma_{(D,<)}$ has inductive chain $(M_\alpha)_{\alpha\in\mathsf{Ord}}$ with supremum M_∞.

(b) The height of $(M, <)$ is the least extended ordinal at which the inductive chain stabilizes.

(c) If $\Gamma_{(D,<)}$ preserves smallness, then M is a set iff the height of $(M, <)$ is an ordinal.

Proof.

(a) The requirements for zero, limit ordinals and ∞ are obvious. The requirement for $M_{\mathsf{S}\alpha}$ holds because it says that an element of M with rank $\leqslant \alpha$ is the same thing as an element of D whose children are all in M and have rank $< \alpha$.

(b) Since the height is the strict supremum of the ranks in M.

(c) By part (b) and Proposition 5.15. ☐

As an example of Proposition 5.17(a)–(b), we see that $\mathcal{P}$ has an inductive chain of classes, which does not stabilize at any ordinal, since every ordinal is its own rank. Its supremum (union) is V_{pure}. Another example: for any broad arity F, we see that $\mathsf{Maybe}^{\circ}_{F}$ has an inductive chain, whose supremum is $\mathsf{SimpleBroad}(F)$.

Here is the analogue of Proposition 5.17 for partial functions:

Proposition 5.18. *Let $(D,<)$ be a set-based scaffold on a class A with functionalization L on an A-tuple of classes B, generating $(M,F) \in \prod_{x\in A} B_x$.*

(a) For each extended ordinal α, we define $M_\alpha \stackrel{\text{def}}{=} \{x \in M \mid \rho(x) < \alpha\}$ and $p_\alpha \stackrel{\text{def}}{=} (M_\alpha, F\restriction_{M_\alpha})$. Then $(p_\alpha)_{\alpha\in Ord}$ is an inductive chain for $\Delta^L_{(D,<)}$ with supremum p_∞.

(b) The height of $(M,<)$ is the least extended ordinal at which the inductive chain stabilizes.

(c) If $\Delta^L_{(D,<)}$ preserves smallness, then M is a set iff the height of $(M,<)$ is an ordinal.

Proof. Similar to the proof of Proposition 5.17 □

6 Category Theory

6.1 Categories and Functors

To help organize parts of our story, we use some category theory:

- The fact that initial algebras of isomorphic functors are isomorphic.
- The notion of an "algebraically least" prefixpoint.
- The fact that class summation preserves connected limits.

For the sake of this paper, a *category* $\mathcal{C}$ consists of a collection $\mathsf{ob}\ \mathcal{C}$ and an indexed family of collections $(\mathcal{C}(x,y))_{x,y\in \mathsf{ob}\ \mathcal{C}}$, together with composition and identity morphisms satisfying the usual three laws. For example:

- $\mathbf{Class}$ is the category of all classes and functions.
- For a class E, we write $\mathbf{ClassFam}(E)$ for the category of all class-families within E and maps between them.

We also consider the following functors.

- $\mathcal{P}$ is an endofunctor on $\mathbf{Class}$, sending $f : A \to B$ to the function

$$\begin{array}{rcl} \mathcal{P}A & \to & \mathcal{P}B \\ U & \mapsto & \{f(x) \mid x \in U\} \end{array}$$

- Maybe is an endofunctor on $\mathbf{Class}$.
- Any class I gives a functor $\sum : \mathbf{Class}^I \to \mathbf{Class}$.
- Any set K gives a functor $\prod : \mathbf{Class}^K \to \mathbf{Class}$.

As stated in Section 3.2, all talk about collections—and therefore all talk about categories—is informal.

6.2 Initial Algebras

Since initial algebras frequently appear in our story, we present the key notions.

Definition 6.1. Let F be an endofunctor on a category $\mathcal{C}$.

(a) An *F-algebra* (x, θ) consists of a $\mathcal{C}$-object x (the *carrier*) and morphism $\theta : Fx \to x$ (the *structure*).

(b) Given F-algebras (x, θ) and (y, ϕ), a *map* $(x, \theta) \to (y, \phi)$ is a $\mathcal{C}$-morphism $f : x \to y$ such that the square

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\theta"'] & Fy \arrow[d, "\phi"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

commutes.

An F-algebra (x, θ) is *initial* when, for every F-algebra (y, ϕ), there is a unique map $(x, \theta) \to (y, \phi)$. It follows that θ is an isomorphism; this fact is called *Lambek's lemma*, and inductive inversion is a special case. Here are some examples of initial algebras.

- The endofunctor $\mathcal{P}$ on **Class** has initial algebra $(V_{\mathsf{pure}}, \mathsf{id}_{V_{\mathsf{pure}}})$.
- The endofunctor Maybe on **Class** has initial algebra $(\mathbb{N}, \mathsf{id}_{\mathbb{N}})$.

These statements hold because V_{pure} and $\mathbb{N}$ are each equipped with a well-founded relation by Proposition 4.13(a). So we can use Proposition 4.11(a) to construct a unique algebra map to any algebra.

We next see that initial algebras of isomorphic endofunctors are isomorphic. Recall that, for categories $\mathcal{C}, \mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural isomorphism* $\alpha : F \cong G$ associates to each object $x \in \mathcal{C}$ a $\mathcal{D}$-isomorphism $\alpha_x : Fx \cong Gx$ in such a way that, for each $\mathcal{C}$-morphism $f : x \to y$, the square

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"'] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\alpha_y"] & Gy \end{tikzcd}$$

commutes.

Proposition 6.2. *On a category $\mathcal{C}$, let F and G be endofunctors with initial algebras (x, θ) and (y, ϕ) respectively. Then any natural isomorphism $\alpha : F \cong G$ induces an isomorphism $x \cong y$.*

Proof. Let f be the unique F-algebra map $(x, \theta) \to (y, \phi \circ \alpha_y)$ and g the unique G-algebra map $(y, \phi) \to (x, \theta \circ \alpha_x^{-1})$. The diagram

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[dddd, "\theta"'] & Fy \arrow[r, "Fg"] \arrow[d, "\alpha_y"'] & Fx \arrow[d, "\alpha_x"'] \arrow[dd, bend left, "\mathsf{id}_{Fx}"] \\ & Gy \arrow[r, "Gg"] \arrow[ddd, "\phi"'] & Gx \arrow[d, "\alpha_x^{-1}"'] \\ & & Fx \arrow[dd, "\theta"] \\ & & \\ x \arrow[r, "f"'] & y \arrow[r, "g"'] & x \end{tikzcd}$$

commutes, so $g \circ f$ is an F-algebra endomap on (x, θ). Since id_x is too, they are equal. Likewise $f \circ g = \mathsf{id}_y$. □

6.3 Algebraically Least Prefixpoints

Observe that our examples V_{pure} and $\mathbb{N}$ serve as both least prefixpoints and initial algebras. We shall now formulate this situation in general.

Definition 6.3. An *ordered category* consists of

- a category $\mathcal{C}$
- an order $\leqslant$ on the collection $\mathsf{ob}\,\mathcal{C}$
- for each pair of objects $x \leqslant y$, an *inclusion* morphism $i_{x,y} : x \to y$

For each object x we must have $i_{x,x} = \mathsf{id}_x$, and for any objects $x \leqslant y \leqslant z$ the triangle
$$\begin{tikzcd} x \arrow[r, "i_{x,y}"] \arrow[dr, "i_{x,z}"'] & y \arrow[d, "i_{y,z}"] \\ & z \end{tikzcd}$$
must commute.

Our main examples of ordered categories are $\mathbf{Class}$ and $\mathbf{ClassFam}(E)$ for any class E.

Definition 6.4. Let $\mathcal{C}$ and $\mathcal{D}$ be ordered categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is *monotone* when, for any $\mathcal{C}$-objects $x \leqslant y$, we have $Fx \leqslant Fy$ and $Fi_{x,y} = i_{Fx,Fy}$.

Now let F be a monotone endofunctor on an ordered category $\mathcal{C}$. Any F-prefixpoint x gives an F-algebra $R(x) \stackrel{\text{def}}{=} (x, i_{Fx,x})$. Furthermore, for F-prefixpoints $x \leqslant y$, we have an algebra map $i_{x,y} : R(x) \to R(y)$.

Definition 6.5. A least F-prefixpoint a is *algebraically least* when the F-algebra $R(a)$ is initial.

For example:

- $\mathcal{P}$ is a monotone endofunctor on $\mathbf{Class}$, and V_{pure} is its algebraically least prefixpoint.
- Maybe is a monotone endofunctor on $\mathbf{Class}$, and $\mathbb{N}$ is its algebraically least prefixpoint.

Here is a final observation (not used in the sequel). As stated above, our main examples are where $\mathcal{C}$ is either $\mathbf{Class}$ or $\mathbf{ClassFam}(E)$ for a class E. In these cases, the requirement in Definition 6.5 for a to be least is redundant. For suppose that a is an F-prefixpoint such that $R(a)$ is initial. To show leastness of a, it suffices to show minimality, since the ordered collection $(\mathsf{ob}\,\mathcal{C}, \leqslant)$ has binary meets. So suppose x is an F-prefixpoint such that $x \leqslant a$. Since $R(a)$ is initial, the algebra morphism $i_{x,a} : R(x) \to R(a)$ has a section, so it is surjective, giving $x = a$.

6.4 Connected Limits

Before we can discuss limits, we have to formulate the notion of a diagram.

A *quiver* (also called a directed multigraph) consists of a set of nodes and a set of edges, with each edge having a source node and a target node. For nodes m, n we write $m \leftrightarrow n$ when either an edge $m \to n$ or an edge $m \to n$ exists. A quiver $\mathbb{M}$ is *connected* when there is a node a such that every node m satisfies $a \leftrightarrow^* m$.

Let $\mathbb{M}$ be a quiver. For a category $\mathcal{C}$, an $\mathbb{M}$-*indexed diagram* D in $\mathcal{C}$ consists of

- an object D_m for each node m
- a morphism $D_f : D_m \to D_n$ for each edge $f : m \to n$.

In the case that $\mathcal{C} = \mathbf{Class}$, we write $\lim_{m \in \mathbb{M}} D_m$ for the *limit* of D, i.e., the class of all $x \in \prod_{m \in \mathbb{M}} D_m$ satisfying $D_f(x_m) = x_n$ for every edge $f : m \to n$. Now we come to the promised result:

Proposition 6.6. *For any class I, the functor $\sum : \mathbf{Class}^I \to \mathbf{Class}$ preserves connected limits. More precisely, for a connected quiver $\mathbb{M}$ and $\mathbb{M}$-indexed diagram D in Class^I, the map*

$$\alpha : \sum_{i \in I} \lim_{m \in \mathbb{M}} D_{m,i} \to \lim_{m \in \mathbb{M}} \sum_{i \in I} D_{m,i}$$

sending $\langle i, [x_m]_{m \in \mathbb{M}} \rangle$ to $[\langle i, x_m \rangle]_{m \in \mathbb{M}}$ is bijective.[7]

Proof. Let $y \in \lim_{m \in \mathbb{M}} \sum_{i \in I} D_{m,i}$. For any edge $f : m \to n$, we see that y_m and y_n have the same left component, since

$$(\sum_{i \in I} D_{f,i})(y_m) \quad = \quad y_n$$

Hence, for any nodes m and n such that $m \leftrightarrow^* n$, we see that y_m and y_n have the same left component. By hypothesis, there is a node a such, for every node m, we have $a \leftrightarrow^* m$. Write i for the left component of y_a. Then for each node m, we can express y_m as $\langle i, x_m \rangle$. We obtain $x \in \lim_{m \in \mathbb{M}} D_{m,i}$, since for any edge $f : m \to n$, we have

$$\begin{array}{rcl} (\sum_{i \in I} D_{f,i})(y_m) & = & y_n \\ \text{i.e.,} \quad \langle i, D_{f,i}(x_m) \rangle & = & \langle i, x_n \rangle \\ \text{so} \quad D_{f,i}(x_m) & = & x_n \end{array}$$

So $\langle i, [x_m]_{n \in \mathbb{M}} \rangle$, which is the unique α-preimage of y, is in $\sum_{i \in I} \lim_{m \in \mathbb{M}} D_{m,i}$. $\square$

Part III

Wide and Broad principles for sets

7 Infinity Principles

7.1 Wide Infinity

Let us review what we have seen previously. The class $\mathbb{N}$ of all natural numbers is the algebraically least Maybe-prefixpoint. The axiom of Infinity says that a Maybe-prefixed set exists, which is equivalent to $\mathbb{N}$ being a set.

Now we continue. A set K (which in this context may be called an *arity*) gives rise to a monotone endofunctor Maybe°_K on $\mathbf{Class}$, sending X to $\mathsf{Maybe}\, X^K$. Thus a class X is Maybe°_K-prefixed iff it contains $\mathsf{Nothing}$ and, for any K-tuple y within X, contains $\mathsf{Just}(y)$.

The algebraically least such class is called the *class of all simple K-wide numbers*, denoted by $\mathsf{SimpleWide}(K)$. To show it exists, note that Maybe arises from the following set-based scaffold on $\mathfrak{T}$: a parent is either $\mathsf{Nothing}$, which has no children, or $\mathsf{Just}(y)$, for any K-tuple y, in which case the set of children is $\{y_k \mid k \in K\}$. So Proposition 4.13(a) gives a least Maybe°_K-prefixed class, and Proposition 4.11(a) gives the initial algebra property.

[7] This result is an instance of [nLa24, Theorem 4.2] via the equivalence

$$\mathbf{Class}^I \simeq \mathbf{ClassFam}(I) = \mathbf{Class}/I$$

that sends B to the class-family $(i)_{\langle i,x \rangle \in \sum_{i \in I} B_i}$.

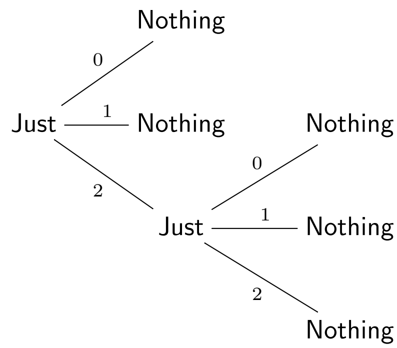

Figure 2: Visualization of a simple wide number

Example 7.1. Define K to be the arity $\{0,1,2\}$. The following are simple K-wide numbers:

- $\mathsf{Nothing}$
- $\mathsf{Just}\begin{bmatrix}\mathsf{Nothing}\\ \mathsf{Nothing}\\ \mathsf{Nothing}\end{bmatrix}$
- $\mathsf{Just}\begin{bmatrix}\mathsf{Nothing}\\ \mathsf{Nothing}\\ \mathsf{Just}\begin{bmatrix}\mathsf{Nothing}\\ \mathsf{Nothing}\\ \mathsf{Nothing}\end{bmatrix}\end{bmatrix}$

We can visualise a wide number as a two-dimensional well-founded tree. For example, the last number in Example 7.1 is visualized in Figure 2, using the vertical dimension for $\begin{bmatrix}\vdots\end{bmatrix}$ and the horizontal dimension for internal structure, with the root appearing at the left and the Nothing-marked leaves at the right. The axiom of *Simple Wide Infinity* says, for every arity K, that a $\mathsf{Maybe}^{\circ}_{K}$-prefixed set exists; this is equivalent to $\mathsf{SimpleWide}(K)$ being a set.

There is an alternative version, formulated as follows.

A *signature* is a family of sets $S = (K_i)_{i\in I}$, where we call $i \in I$ a *symbol* and the set K_i its *arity*. It gives rise to a monotone endofunctor H_S on **Class**, sending X to $\sum_{i\in I} X^{K_i}$. Thus a class X is H_S-prefixed iff, for any $i \in I$ and K_i-tuple y within X, it contains $\langle i, y\rangle$. As before, there is an algebraically least such class, which we call the *class of all S-wide numbers*, denoted by $\mathsf{Wide}(S)$.

The axiom of *Full Wide Infinity* says, for every signature S, that an H_S-prefixed set exists; this is equivalent to $\mathsf{Wide}(S)$ being a set. Previously this axiom has appeared in [vdB11, page 15] under the name "Smallness of W-types", alluding to the notion of W-type in type theory [ML84, MP00, AAG05].

Before giving the main result of the section, we give a result that allows us to work with injections rather than inclusions:

Proposition 7.2.

(a) Infinity is equivalent to Dedekind Infinity*: There is a set X and injection* $\mathsf{Maybe}\,X \to X$.[8]

(b) Let K be a set. Then $\mathsf{SimpleWide}(K)$ *is a set iff there is a set X and injection* $\mathsf{Maybe}^{\circ}_{K} X \to X$.

[8] For a class X, an injection $\mathsf{Maybe}\,X \to X$ corresponds to a unary Dedekind encoding on X. When such an injection exists, X is said to be *Dedekind-infinite.*.

(c) *Let S be a signature. Then* $\mathsf{Wide}(S)$ *is a set iff there is a set X and injection $H_S X \to X$.*

Proof. We just prove part (c), as the rest is similar. Any H_S-prefixed class X, such as μH_S, is equpped with an injection $i_{H_S X,X} : H_S X \to X$. Conversely, given a class X and injection $\theta : H_S X \to X$, the unique algebra morphism $f : R(\mu H_S) \to (X, \theta)$ is injective. (For this says that any $x \in \mu H_S$ is the only f-preimage of its f-image, which is proved by induction on x.) So if X is a set, then μH_S is a set. □

Theorem 7.3. *Simple Wide Infinity, Full Wide Infinity and Exponentiation + Infinity are all equivalent.*

Proof. To show that Full Wide Infinity implies Simple Wide Infinity: given a set K, define S to be the signature $(L_i)_{i\in\{0,1\}}$ where $L_0 \stackrel{\text{def}}{=} \emptyset$ and $L_1 \stackrel{\text{def}}{=} K$. We have a natural isomorphism $\mathsf{Maybe}^{\circ}_K \cong H_S$, so Proposition 6.2 tells us that $\mathsf{SimpleWide}(K)$ is a set iff $\mathsf{Wide}(S)$ is a set.

To show that Simple Wide Infinity implies Infinity, we have a natural isomorphism $\mathsf{Maybe} \cong \mathsf{Maybe}_1$, so Proposition 6.2 tells us that $\mathbb{N}$ is a set iff $\mathsf{SimpleWide}(1)$ is a set.

To show that Full Wide Infinity implies Exponentiation, let A and B be sets. Define the signature S with nullary symbols $(\mathsf{Leaf}_b)_{b\in B}$ and an A-ary symbol Node. Consider the injection $B^A \to \mathsf{Wide}(S)$ sending a function f to the S-wide number

$$\langle \mathsf{Node}, [\langle \mathsf{Leaf}_{f(a)}, [\,]\rangle]_{a\in A}\rangle$$

Since $\mathsf{Wide}(S)$ is a set, B^A is a set.

To show that Exponentiation+Infinity implies Full Wide Infinity, let S be a signature. By Exponentiation, H_S restricts to an endofunctor on $\mathbf{Set}$. Following [Bar93], we form the ω^{op}-chain

$$1 \xleftarrow{\langle\rangle} H_S 1 \xleftarrow{H_S\langle\rangle} H_S^2 1 \xleftarrow{H_S^2\langle\rangle} \cdots$$

which may be called the "coinductive chain". (Intuitively $H_S^n 1$ is the set of S-trees with stumps at level n.) Let M be the limit and $\theta : H_S M \to M$ the canonical map. (Intuitively M is the set of non-well-founded S-trees.) We note, by Proposition 6.6, that H_S preserves connected limits. So θ is bijective, and therefore $\mathsf{Wide}(S)$ is a set by Proposition 7.2(c).

To show that Simple Wide Infinity implies Full Wide Infinity, let $S = (K_i)_{i\in I}$ be a signature. Define the set $\overline{S} \stackrel{\text{def}}{=} I + \sum_{i\in I} K_i$, so $\mathsf{SimpleWide}(\overline{S})$ is a set. We obtain an injection $H_S\mathsf{SimpleWide}(\overline{S}) \to \mathsf{SimpleWide}(\overline{S})$ sending $\langle i, [a_k]_{k\in K_i}\rangle$ to $\mathsf{Just}([b_p]_{p\in\overline{S}})$, where

$$\begin{array}{rcll} b_{\mathsf{inl}\ i} & \stackrel{\text{def}}{=} & \mathsf{Just}([\mathsf{Nothing}]_{p\in\overline{S}}) & \\ b_{\mathsf{inl}\ j} & \stackrel{\text{def}}{=} & \mathsf{Nothing} & (\text{for } j \in I, j \neq i) \\ b_{\mathsf{inr}\ \langle i,k\rangle} & \stackrel{\text{def}}{=} & a_k & (\text{for } k \in K_i) \\ b_{\mathsf{inr}\ \langle j,k\rangle} & \stackrel{\text{def}}{=} & \mathsf{Nothing} & (\text{for } j \in I, j \neq i, k \in K_j) \end{array}$$

So $\mathsf{Wide}(S)$ is a set by Proposition 7.2(c). □

Related work The idea that every signature has an initial algebra (meaning: algebra carried by a set) has often appeared in the literature. Apparently, the earliest ZFC proof was given by Słomiński [Sł58], and the earliest ZF proof by Kerkhoff [Ker65]. Furthermore, as explained in [PS78, Bla83, MP00], the result holds in any topos with a natural number object.

7.2 Broad Infinity

Now we come to the main principle of the paper, which was briefly described in Section 1.1. We shall use the following notation. For any class A and A-tuple of classes B, we write

$$\sum_{x \in A}^{\mathsf{Maybe}} B(x) \quad \stackrel{\text{def}}{=} \quad \{\mathsf{Begin}\} \cup \{\mathsf{Make}(x, y) \mid x \in A, y \in B(x)\}$$

Thus we have a bijection $\sum_{x \in A}^{\mathsf{Maybe}} B(x) \cong \mathsf{Maybe} \sum_{x \in A} B(x)$, sending Begin to Nothing and $\mathsf{Make}(x, y)$ to $\mathsf{Just}\langle x, y\rangle$.

A *broad arity* is a function $F : \mathfrak{T} \to \mathbf{Set}$. It gives rise to a monotone endofunction $\mathsf{Maybe}^{\circ}_{F}$ on Class, sending X to $\sum_{x \in X}^{\mathsf{Maybe}} X^{Fx}$. Thus a class X is $\mathsf{Maybe}^{\circ}_{F}$-prefixed when it contains Begin and, for any $x \in X$ and Fx-tuple y within X, contains $\mathsf{Make}(x, y)$.

The least such class is called the *class of all simple F-broad numbers* and written $\mathsf{SimpleBroad}(F)$. To show it exists, note that $\mathsf{Maybe}^{\circ}_{F}$ arises from the following set-based scaffold on $\mathfrak{T}$: a parent is either Begin, which has no children, or $\mathsf{Make}(x, y)$, for a thing x and Fx-tuple y, in which case the set of children is $\{x\} \cup \{y_k \mid k \in K\}$. So we apply Proposition 4.13(a).

Although F is defined over $\mathfrak{T}$, only its restriction to $\mathsf{SimpleBroad}(F)$ matters. More precisely, for broad arities F and F' with the same restriction to $\mathsf{SimpleBroad}(F) \cap \mathsf{SimpleBroad}(F')$, we have $\mathsf{SimpleBroad}(F) = \mathsf{SimpleBroad}(F')$. Proof: the class $\mathsf{SimpleBroad}(F) \cap \mathsf{SimpleBroad}(F')$ is both $\mathsf{Maybe}^{\circ}_{F}$-prefixed and $\mathsf{Maybe}^{\circ}_{F'}$-prefixed.

Let us see some examples of simple broad numbers.

Example 7.4. Define F to be the broad arity that sends $\mathsf{Make}(\mathsf{Begin}, [\,])$ to $\{0, 1\}$, and everything else to $\emptyset$. The following are simple F-broad numbers:

- Begin
- $\mathsf{Make}(\mathsf{Begin}, [\,])$
- $\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, [\,]), \begin{bmatrix} \mathsf{Begin} \\ \mathsf{Make}(\mathsf{Begin}, [\,]) \end{bmatrix})$
- $\mathsf{Make}(\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, [\,]), \begin{bmatrix} \mathsf{Begin} \\ \mathsf{Make}(\mathsf{Begin}, [\,]) \end{bmatrix}), [\,])$

We can visualise a broad number as a well-founded three-dimensional tree, using the vertical dimension for $\begin{bmatrix} \vdots \end{bmatrix}$, the horizontal dimension for $\mathsf{Make}(-, -)$ and the depth dimension for internal structure. The root appears at the front, and the Begin-marked leaves at the rear.

The axiom scheme of *Simple Broad Infinity* says that, for every broad arity F, a $\mathsf{Maybe}^{\circ}_{F}$-prefixed set exists; this is equivalent to $\mathsf{SimpleBroad}(F)$ being a set.

There is an alternative version, formulated as follows. Note that $\mathsf{Fam}(\mathsf{Set})$ is the class of all signatures. Any function $G : \mathfrak{T} \to \mathsf{Fam}(\mathsf{Set})$, called a *broad signature*, gives rise to a monotone endofunction Maybe_{G} on Class, sending X to $\sum_{x \in X}^{\mathsf{Maybe}} H_{Gx} X$. Thus a class X is Maybe_{G}-prefixed iff it contains Begin and, for any $x \in X$ with $Gx = (K_i)_{i \in I}$ and any $i \in I$ and K_i-tuple y within X, contains $\mathsf{Make}(x, \langle i, y\rangle)$.

As before, there is a least such class, which we call the *class of all G-broad numbers*, denoted by $\mathsf{Broad}(G)$. For functions $G, G' : \mathfrak{T} \to \mathsf{Fam}(\mathsf{Set})$ with the same restriction to $\mathsf{Broad}(G) \cap \mathsf{Broad}(G')$, we have $\mathsf{Broad}(G) = \mathsf{Broad}(G')$.

The axiom scheme of *Full Broad Infinity* says that, for every broad signature G, a Maybe_{G}-prefixed set exists; this is equivalent to $\mathsf{Broad}(G)$ being a set.

Theorem 7.5.

(a) Simple Broad Infinity and Full Broad Infinity are equivalent.

(b) Full Broad Infinity implies Full Wide Infinity.

Proof.

(a) To show ($\Leftarrow$), let F be a broad arity. We recursively define the injection f on $\mathsf{SimpleBroad}(F)$ that sends

- Begin to Begin.
- $\mathsf{Make}(w, [a_k]_{k \in Fw})$ to $\mathsf{Make}(f(w), \langle 0, [f(a_k)]_{k \in Fw} \rangle)$.

Define the broad signature G sending

- $f(x)$, for $x \in \mathsf{SimpleBroad}(F)$, to $(Fx)_{i \in \{0\}}$
- everything else to the empty signature.

Observe that f sends each $w \in \mathsf{SimpleBroad}(F)$ to a G-broad number, by induction on w. Since $\mathsf{Broad}(G)$ is a set, $\mathsf{SimpleBroad}(F)$ is a set.

To show ($\Rightarrow$), for a signature $S = (K_i)_{i \in I}$, we write $\overline{S} \stackrel{\text{def}}{=} I + \sum_{i \in I} K_i$. Given a broad signature G, we recursively define an injection g on $\mathsf{Broad}(G)$ whose range does not contain Begin, as follows. It sends

- Begin to $\mathsf{Make}(\mathsf{Begin}, [\,])$
- $\mathsf{Make}(w, \langle i, [a_k]_{k \in K_i} \rangle)$, where $Gw = (K_i)_{i \in I}$, to $\mathsf{Make}(g(w), [b_p]_{p \in \overline{Gw}})$, using the definitions

$$
\begin{array}{rcll}
b_{\mathsf{inl}\ i} & \stackrel{\text{def}}{=} & \mathsf{Make}(\mathsf{Begin}, [\,]) & \\
b_{\mathsf{inl}\ j} & \stackrel{\text{def}}{=} & \mathsf{Begin} & (\text{for } j \in I, j \neq i) \\
b_{\mathsf{inr}\ \langle i,k \rangle} & \stackrel{\text{def}}{=} & g(a_k) & (\text{for } k \in K_i) \\
b_{\mathsf{inr}\ \langle j,k \rangle} & \stackrel{\text{def}}{=} & \mathsf{Begin} & (\text{for } j \in I, j \neq i, k \in K_j).
\end{array}
$$

Let F be the broad arity that sends

- $\mathsf{Just}(g(w))$, for $w \in \mathsf{Broad}(G)$, to $\overline{Gw}$
- everything else, including Begin, to $\emptyset$.

For any $w \in \mathsf{Broad}(G)$, we have $g(w) \in \mathsf{SimpleBroad}(F)$, by induction on w. Since $\mathsf{SimpleBroad}(F)$ is a set, $\mathsf{Broad}(G)$ is a set.

(b) Given a signature S, let G be the broad signature sending everything to S. Recursively define the injection $g : \mathsf{Wide}(S) \to \mathsf{Broad}(G)$ sending $\langle i, [a_k]_{k \in K_i} \rangle$ to $\mathsf{Make}(\mathsf{Begin}, \langle i, [g(a_k)]_{k \in Fa} \rangle)$. Since $\mathsf{Broad}(G)$ is a set, $\mathsf{Wide}(S)$ is a set. $\square$

8 Introducing Rubrics

8.1 Generating a subset

Having completed our presentation of the "plausible" principles, we now move on to the "useful" ones. First we consider how to generate a subset of a class using a suitable collection of rules, called a *rubric*.

Definition 8.1. Let C be a class.

(a) A *wide rule on C* consists of a set K (the *arity*) and a function $R : C^K \to \mathsf{Fam}(C)$. It is written $\langle K, R \rangle$, and the collection of all such is denoted $\mathsf{WideRule}(C)$.

(b) A *wide rubric on* C is a family of wide rules—i.e., a set I and function $r : I \to \mathsf{WideRule}(C)$. It is written $(r_i)_{i \in I}$, and the collection of all such is denoted $\mathsf{WideRub}(C)$.

(c) A *broad rule on* C consists of a set L (the *arity*) and a function $S : C^L \to \mathsf{WideRub}(C)$. It is written $\langle L, S \rangle$, and the collection of all such is denoted $\mathsf{BroadRule}(C)$.

(d) A *broad rubric on* C is a family of broad rules—i.e., a set J and function $s : J \to \mathsf{BroadRule}(C)$. It is written $(s_j)_{j \in J}$, and the collection of all such is denoted $\mathsf{BroadRub}(C)$.

Example 8.2. Here is a wide rubric on $\mathbb{N}$, consisting of two wide rules.

- Rule 0 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1 is nullary and sends $[\,] \mapsto (2p)_{p \geqslant 50}$.

Informally, these rules prescribe when an element of $\mathbb{N}$ is acceptable. Rule 0 says that, if m_0 and m_1 are acceptable, then $m_0 + m_1 + p$ is acceptable for all $p \geqslant 2m_0$. Rule 1 says that $2p$ is acceptable for all $p \geqslant 50$. So 100, 102 and 402 are acceptable, and by induction every acceptable number is $\geqslant 100$.

Example 8.3. Here is a broad rubric on $\mathbb{N}$, consisting of two broad rules. Broad rule 0 is nullary and sends $[\,]$ to the wide rubric described in Example 8.2. Broad rule 1 is unary. It sends $[7]$ to the the following wide rubric, consisting of one wide rule.

- Rule 0 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_0 + m_1 + 500p)_{p \geqslant 9}$.

It sends $[100]$ to the following wide rubric, consisting of three wide rules.

- Rule 0 is ternary and sends $\begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} \mapsto (m_0 + m_1 m_2 + p)_{p \geqslant 17}$.
- Rule 1 is nullary and sends $[\,] \mapsto (p)_{p \geqslant 1000}$.
- Rule 2 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_1 + p)_{p \geqslant 4}$.

And it sends $[n]$, for $n \in \mathbb{N} \setminus \{7, 100\}$, to the empty wide rubric.

Informally, these rules prescribe when an element of $\mathbb{N}$ is acceptable. For example, if 100 is acceptable and m_0, m_1, m_2 are too, then so is $m_0 + m_1 m_2 + p$ for all $p \geqslant 17$. So 100, 102, 402 and 107 are acceptable, and by induction every acceptable number is $\geqslant 100$.

To make the notion of "acceptable element" precise, we proceed as follows.

Definition 8.4. Let C be a class. A subclass X is

- $\langle K, R \rangle$*-closed*, for a wide rule $\langle K, R \rangle$ on C, when the family $R(x)$ is within X for all $x \in X^K$.
- $\mathcal{R}$*-complete*, for a wide rubric $\mathcal{R} = (r_i)_{i \in I}$ on C, when X is r_i-closed for all $i \in I$.
- $\langle L, S \rangle$*-closed*, for a broad rule $\langle L, S \rangle$ on C, when X is $S(y)$-complete for all $y \in X^L$.
- $\mathcal{S}$*-complete* for a broad rubric $\mathcal{S} = (s_j)_{j \in J}$ on C, when X is s_j-closed for all $j \in J$.

Our next task is to give an alternative formulation of completeness, using a notion of "plate". First we give preliminary notions that do not depend on a rubric.

Definition 8.5.

(a) A *wide preplate* is a triple $w = \langle i, [x_k]_{k\in K}, p\rangle$, where K can be any set. The *component set* of w is

$$\chi(w) \quad \stackrel{\text{def}}{=} \quad \{x_k \mid k\in K\}$$

(b) A *broad preplate* is a 5-tuple $w = \langle j, [y_l]_{l\in L}, i, [x_k]_{k\in K}, p\rangle$, where L and K can be any sets. The *component set* of w is

$$\chi(w) \quad \stackrel{\text{def}}{=} \quad \{y_l \mid l\in L\} \cup \{x_k \mid k\in K\}$$

(c) A preplate w is *within* a class X when $\chi(w) \subseteq X$.

Now we turn to rubrics.

Definition 8.6. Let C be a class.

(a) Let $\mathcal{R}$ be a wide rubric on C. An $\mathcal{R}$*-plate* is a wide preplate

$$w \quad = \quad \langle i, [x_k]_{k\in K}, p\rangle$$

within C such that

- writing $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in I}$, we have $i\in I$ and $K = K_i$
- writing $R_j(x) = (b_p)_{p\in P}$, we have $p\in P$.

The *result* of w is b_p.

(b) Let $\mathcal{S}$ be a broad rubric on C. An $\mathcal{S}$*-plate* is a broad preplate

$$w \quad = \quad \langle j, [y_l]_{l\in L}, i, [x_k]_{k\in K}, p\rangle$$

within C such that

- writing $\mathcal{S} = (\langle L_j, S_j\rangle)_{j\in J}$, we have $j\in J$ and $L = L_j$
- writing $S_j(y) = (\langle K_i, R_i\rangle)_{i\in I}$, we have $i\in I$ and $K = K_i$
- writing $R_i(x) = (b_p)_{p\in P}$, we have $p\in P$.

The *result* of w is b_p.

Definition 8.7. Let $\mathcal{R}$ be a rubric on a class C. We write $\Gamma_{\mathcal{R}}$ for the set-continuous endofunction on $\mathsf{Sub}(C)$ sending X to the class of all results of $\mathcal{R}$-plates within X.

Clearly a subclass of C is $\mathcal{R}$-complete iff $\Gamma_{\mathcal{R}}$-prefixed.

The least $\mathcal{R}$-complete subclass of C, if it exists, is said to be $\mathcal{R}$*-generated*. We shall see below (Proposition 10.3) that this class exists if Powerset or Collection holds. However, I do not know whether our base theory alone guarantees its existence. In any case, a rubric's purpose is to generate a *set*, not merely a class. So we formulate the following principles.

- The *Wide Set Generation* scheme says that any wide rubric on a class generates a subset.
- The *Broad Set Generation* scheme says that any broad rubric on a class generates a subset.

By Proposition 5.14, a rubric $\mathcal{R}$ on a class C generates a subset iff C has an $\mathcal{R}$-complete subset.

8.2 Application: Grothendieck Universes

For this section, **assume Powerset + Infinity**.

As promised in Section 2.1, we see the utility of Broad Set Generation: it gives Grothendieck universes without any detour via notions of cardinal or ordinal.

Definition 8.8. A *Grothendieck universe* is a transitive set $\mathfrak{U}$ with the following properties:

- For every set $A \in \mathfrak{U}$, we have $\mathcal{P}A \in \mathfrak{U}$.
- $\mathbb{N} \in \mathfrak{U}$.
- For every set of sets $\mathcal{A} \in \mathfrak{U}$, we have $\bigcup \mathcal{A} \in \mathfrak{U}$.
- For every set $K \in \mathfrak{U}$ and K-tuple $[a_k]_{k\in K}$ within $\mathfrak{U}$, we have $\{a_k \mid k \in K\} \in \mathfrak{U}$.

Proposition 8.9. *Broad Set Generation implies the "Axiom of Universes": For every set X, there is a least Grothendieck universe $\mathfrak{U}$ with $X \subseteq \mathfrak{U}$.*

Proof. We define a broad rubric $\mathcal{B}$ on $\mathfrak{T}$, consisting of two rules. Broad rule 0 is nullary and sends $[\,]$ to the following wide rubric indexed by $X + 4$.

- To achieve $X \subseteq \mathfrak{U}$, rule $\mathsf{inl}\ x$ (for $x \in X$) is nullary, and sends $[\,]$ to (x).
- To achieve transitivity, rule $\mathsf{inr}\ 0$ is unary, sending $[A]$ to $(b)_{b\in A}$ if A is a set, and the empty family otherwise.
- Rule $\mathsf{inr}\ 1$ is nullary, and sends $[\,]$ to $(\mathbb{N})$.
- Rule $\mathsf{inr}\ 2$ is unary, sending $[\mathcal{A}]$ to $(\bigcup \mathcal{A})$ if $\mathcal{A}$ is a set of sets, and the empty signature otherwise.
- Rule $\mathsf{inr}\ 3$ has arity 1, sending $[A]$ to $(\mathcal{P}A)$ if A is a set, and the empty signature otherwise.

Broad rule 1 is unary. For any set B, it sends $[B]$ to the rubric consisting of one B-ary rule that sends $[a_k]_{k\in B}$ to $(\{a_k \mid k \in B\})$. If b is not a set, then Broad Rule 1 sends $[b]$ to the empty rubric. This completes the definition of $\mathcal{B}$. The set that it generates is the desired Grothendieck universe. $\square$

8.3 Derivations

Intuitively, when we have a rubric on a class C, each acceptable element $x \in C$ has one or more "derivations" that explain why it is acceptable.

Example 8.10. For the wide rubric in Example 8.2:

- $\langle 1, [\,], 50\rangle$ derives 100.
- $\langle 1, [\,], 51\rangle$ derives 102.
- $\langle 0, \begin{bmatrix}\langle 1, [\,], 50\rangle \\ \langle 1, [\,], 50\rangle\end{bmatrix}, 202\rangle$ and $\langle 0, \begin{bmatrix}\langle 1, [\,], 50\rangle \\ \langle 1, [\,], 51\rangle\end{bmatrix}, 200\rangle$ derive 402.

Note that each derivation is a wide preplate whose components are derivations.

Example 8.11. For the broad rubric in Example 8.3:

- $\langle 0, [\,], 1, [\,], 50\rangle$ derives 100.
- $\langle 0, [\,], 1, [\,], 51\rangle$ derives 102.

- $\langle 0, [\langle 0, [\,], 1, [\,], 50\rangle], 2, \begin{bmatrix} \langle 0, [\,], 1, [\,], 50\rangle \\ \langle 0, [\,], 1, [\,], 51\rangle \end{bmatrix}, 5\rangle$ derives 107.

Note that each derivation is a broad preplate whose components are derivations.

Given a rubric $\mathcal{R}$ on a class C, we would like to define the class $\mathsf{Deriv}_{\mathcal{R}}$ of all $\mathcal{R}$-*derivations*. Each $\mathcal{R}$-derivation x will have an *overall result* $\mathsf{O}_{\mathcal{R}}(x) \in C$. We shall call $(\mathsf{Deriv}_{\mathcal{R}}, \mathsf{O}_{\mathcal{R}})$ the $\mathcal{R}$-*derivational* class-family within C—it is the algebraically least prefixpoint of an endofunctor $\Delta_{\mathcal{R}}$ on $\mathsf{ClassFam}(C)$ that we shall define. To do this, we adapt the notion of $\mathcal{R}$-plate (Definition 8.6). First we give preliminary notation that does not depend on a rubric.

Definition 8.12.

(a) For a wide preplate $w = \langle i, [x_k]_{k\in K}, p\rangle$, and a function h on $\chi(w)$, we write

$$\widehat{h}(w) \quad \stackrel{\text{def}}{=} \quad \langle i, [h(x_k)]_{k\in K}, p\rangle$$

(b) For a broad preplate $w = \langle j, [y_l]_{l\in L}, i, [x_k]_{k\in K}, p\rangle$, and function h on $\chi(w)$, we write

$$\widehat{h}(w) \quad \stackrel{\text{def}}{=} \quad \langle j, [h(y_l)]_{l\in L}, i, [h(x_k)]_{k\in K}, p\rangle$$

Now we turn to rubrics.

Definition 8.13. Let (M, F) be a class-family within a class C.

(a) Let $\mathcal{R}$ be a wide rubric on C. An $(\mathcal{R}, M, F)$-*plate* w is a wide preplate within M whose $\widehat{F}$-image is an $\mathcal{R}$-plate. The *result* of w is the result of $\widehat{F}(w)$.

(b) Let $\mathcal{S}$ be a broad rubric on C. An $(\mathcal{S}, M, F)$-*plate* w is a broad preplate within M whose $\widehat{F}$-image is an $\mathcal{S}$-plate. The *result* of w is the result of $\widehat{F}(w)$.

Definition 8.14. Let $\mathcal{R}$ be a rubric on a class C. We define the set-continuous endofunctor $\Delta_{\mathcal{R}}$ on $\mathbf{ClassFam}(C)$ sending

- a class-family (M, F) to (N, G), where N is the class of all $(\mathcal{R}, M, F)$-plates and G sends each such plate to its result
- a map $h : (M, F) \to (M', F')$ to the map $w \mapsto \widehat{h}(w)$.

Proposition 8.15. *Let $\mathcal{R}$ be a rubric on a class C. Then $\Delta_{\mathcal{R}}$ has an inductive chain and an algebraically least prefixpoint.*

Proof. We show $\Delta_{\mathcal{R}}$ has an inductive chain and a least prefixpoint by Proposition 5.18(a). Noting that $\mathsf{ClassFam}(C) = \prod_{x\in\mathfrak{T}} C$, we express $\Delta_{\mathcal{R}}$ as $\Delta^{L}_{(D,<)}$, for a set-based scaffold $(D, <)$ on $\mathfrak{T}$ with functionalization L on $x \mapsto C$.

We give the wide case, as the broad case is similar. Firstly, we have a set-based scaffold $(D, <)$ on $\mathfrak{T}$, where D is the class of all wide preplates and $<$ is componenthood.

Now let $\mathcal{R}$ be a wide rubric on C. Then $(D, <)$ has the following functionalization L to $(C)_{x\in\mathfrak{T}}$. For a wide preplate u, we define $\mathsf{Dom}(L_u)$ to be the class of all functions $f : J(u) \to C$ such that $\widehat{f}(u)$ is an $\mathcal{R}$-plate, and $\overline{L_u}$ sends such a function f to the result of $\widehat{f}(u)$.

It is then evident that $\Delta_{\mathcal{R}} = \Delta^{L}_{(D,<)}$, so we obtain a least $\Delta_{\mathcal{R}}$-prefixpoint, and Proposition 4.11(a) gives the initial algebra property. $\square$

Thus we define the $\mathcal{R}$-derivational class-family $(\mathsf{Deriv}_{\mathcal{R}}, \mathsf{O}_{\mathcal{R}}) \stackrel{\text{def}}{=} \mu\Delta_{\mathcal{R}}$. We want to know whether this is a family—i.e., whether $\mathsf{Deriv}_{\mathcal{R}}$ is a set. So we formulate the following principles.

- *Wide Derivation Set*: Any wide rubric on a class has a derivation set.
- *Broad Derivation Set*: Any broad rubric on a class has a derivation set.

8.4 Application: Tarski-style Universes

For this section, **assume Powerset + Infinity**.

In the type theory literature [ML84], a "Tarski-style universe" is a family of types that is closed under various constructions, such as $\sum$. Furthermore, the existence of such universes follows from various "induction-recursion" principles [DS06, GH16].

In a similar way, we show that Broad Derivation Set yields the existence of arbitrarily large Tarski-style universes.

Definition 8.16. A *Tarski-style universe* consists of the following data.

- A family of sets $(D_m)_{m \in M}$. Elements of M are called *codes*.
- For each code m, a code $\mathrm{pow}(m)$ and bijection $D_{\mathrm{pow}(m)} \cong \mathcal{P} D_m$.
- A code zero and bijection $D_{\mathrm{zero}} \cong \emptyset$.
- A code nat and bijection $D_{\mathrm{nat}} \cong \mathbb{N}$.
- For each code m and tuple of codes $[g_k]_{k \in D_m}$, a code $\mathrm{sigma}(m, g)$ and bijection $D_{\mathrm{sigma}(m,g)} \cong \sum_{k \in D_m} D_{g(m)}$.

Definition 8.17. For a family of sets $(B_a)_{a \in A}$, a *Tarski-style universe extension* consists of the following data:

- A Tarski-style universe $(D_m)_{m \in M}$.
- For each $a \in A$, a code $j(a)$ and bijection $D_{j(a)} \cong B_a$.

Proposition 8.18. *Broad Derivation Set implies that every family of sets $(B_a)_{a \in A}$ has a Tarski-style universe extension.*

Proof. We shall construct the extension so that all the required bijections are identities. To begin, define a broad rubric $\mathcal{B}$ on Set, consisting of two broad rules. Broad rule 0 is nullary and sends $[\,]$ to the following rubric indexed by $A + \{0, 1, 2\}$.

- Rule $\mathsf{inl}\ a$ (for $a \in A$) has arity 0 and sends $[\,]$ to (B_a).
- Rule $\mathsf{inr}\ 0$ has arity 1 and sends X to $(\mathcal{P}X)$.
- Rule $\mathsf{inr}\ 1$ has arity 0 and sends $[\,]$ to $(\emptyset)$.
- Rule $\mathsf{inr}\ 2$ has arity 0 and sends $[\,]$ to $(\mathbb{N})$.

Broad rule 1 is unary, and sends $[X]$ to the following rubric indexed by $\{0\}$.

- Rule 0 has arity X and sends $[Y_x]_{x \in X}$ to $(\sum_{x \in X} Y_x)$.

The $\mathcal{B}$-derivational family is the desired universe extension, where we define

$$
\begin{aligned}
j(a) &\stackrel{\text{def}}{=} \langle 0, [\,], \mathsf{inl}\ a, [\,], * \rangle \\
\mathrm{pow}(m) &\stackrel{\text{def}}{=} \langle 0, [\,], \mathsf{inr}\ 0, [m], * \rangle \\
\mathrm{zero} &\stackrel{\text{def}}{=} \langle 0, [\,], \mathsf{inr}\ 1, [\,], * \rangle \\
\mathrm{nat} &\stackrel{\text{def}}{=} \langle 0, [\,], \mathsf{inr}\ 2, [\,], * \rangle \\
\mathrm{sigma}(m, g) &\stackrel{\text{def}}{=} \langle 1, [m], 0, g, * \rangle
\end{aligned}
$$

$\square$

8.5 Proving the Derivation Set Principles

The following is the central result of the paper (at least for people who do not accept AC), since it says that *plausible principles entail useful ones*.

Proposition 8.19.

(a) Full Wide Infinity implies Wide Derivation Set.

(b) Full Broad Infinity implies Broad Derivation Set.

Proof.

(a) Let $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in I}$ be a wide rubric on a class C. Define the signature S to be $(K_i)_{i\in I}$. We recursively associate to each $t \in \mathsf{Wide}(S)$ a family (M_t, F_t) within C as follows. For $t = \langle i, [t_k]_{k\in K_i}\rangle$, an element of M_t is a triple $\langle i, [m_k]_{k\in K_i}, p\rangle$ where $i \in I$ and $m \in \prod_{k\in K_i} M_{t_k}$ with $R_i[F_{t_k}(m_k)]_{k\in K_i} = (b_p)_{p\in P}$ and $p \in P$, and F_t sends this element to b_p. For any $t, t' \in \mathsf{Wide}(S)$, if $M_t \cap M'_t$ is inhabited, then $t = t'$, by induction on t.

We define the family (M, F) within C to be the union of all these. Thus we define $M \stackrel{\text{def}}{=} \bigcup_{t\in\mathsf{Wide}(S)} M_t$, and F sends $m \in M_t$ to $F_t(m)$. It is then evident that (M, F) is the derivational family of $\mathcal{R}$.

(b) Let $\mathcal{S} = (\langle L_j, S_j\rangle)_{j\in J}$ be a broad rubric on a class C. We recursively define the function θ on $\mathsf{Deriv}_{\mathcal{S}}$ that sends $\langle j, [y_l]_{l\in L_j}, i, [x_k]_{k\in K_i}, p\rangle$ to

$$\mathsf{Make}(\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, \langle j, [\theta y_i]_{i\in L_j}\rangle), \langle i, [\theta x_k]_{k\in K_i}\rangle), \langle p, [\,]\rangle)$$

By induction, θ is injective. To show that $\mathsf{Deriv}_{\mathcal{S}}$ is a set, it suffices to give a broad signature G such that the range of θ is included in $\mathsf{Broad}(G)$, which by Full Broad Infinity is a set. Define $G : \mathfrak{T} \to \mathsf{Set}$ to send

- Begin to $(L_j)_{j\in J}$
- $\mathsf{Make}(\mathsf{Begin}, \langle j, [\theta y_l]_{l\in L_j}\rangle)$ obtained from
  - an index $j \in J$ and $y \in \mathsf{Deriv}_{\mathcal{S}}^{L_j}$, giving
  $$S_j[\mathsf{O}_{\mathcal{S}}(y_l)]_{l\in L_j} \quad = \quad (\langle K_i, R_i\rangle)_{i\in I}$$

  to $(K_i)_{i\in I}$
- $\mathsf{Make}(\mathsf{Make}(\mathsf{Begin}, \langle j, [\theta y_i]_{i\in L_j}\rangle), \langle i, [\theta x_k]_{k\in K_i}\rangle)$ obtained from
  - an index $j \in J$ and $y \in \mathsf{Deriv}_{\mathcal{S}}^{L_j}$, giving
  $$S_j[\mathsf{O}_{\mathcal{S}}(y_l)]_{l\in L_j} \quad = \quad (\langle K_i, R_i\rangle)_{i\in I}$$
  - and an index $i \in I$ and $x \in \mathsf{Deriv}_{\mathcal{S}}^{K_i}$, giving
  $$R_i[\mathsf{O}_{\mathcal{S}}(x_k)]_{k\in K_i} \quad = \quad (b_p)_{p\in P}$$

  to $(\emptyset)_{p\in P}$
- everything else to the empty signature.

By induction, for every $\mathcal{S}$-derivation x, we see that θx is a G-broad number, as required. □

9 Special Kinds of Rubric

9.1 Injective Rubrics

Definition 9.1. Let C be a class. A rubric $\mathcal{R}$ on C is *injective* when any two $\mathcal{R}$-plates with the same result are equal.

We now give the following principles.

- *Injective Wide Set Generation*: Any injective wide rubric on a class generates a subset.
- *Injective Broad Set Generation*: Any injective broad rubric on a class generates a subset.

Proposition 9.2.

(a) Injective Wide Set Generation implies Full Wide Infinity.

(b) Injective Broad Set Generation implies Full Broad Infinity.

Proof.

(a) Let $S = (K_i)_{i \in I}$ be a signature. A class is H_S-prefixed iff it is $\hat{S}$-complete, writing $\hat{S}$ for the following injective wide rubric on $\mathfrak{T}$: it consists of I rules, and rule $i \in I$ has arity K_i and sends a K_i-tuple x to the singleton $(\langle i, x \rangle)$. So $\mathsf{Wide}(S)$ is generated by $\hat{S}$, and is therefore a set by Injective Wide Set Generation.

(b) Let G be a broad signature. A class is Maybe_G-prefixed iff it is $\hat{G}$-complete, writing $\hat{G}$ for the injective broad rubric on $\mathfrak{T}$ consisting of two rules:

- Rule 0 is nullary and sends $[\,]$ to the wide rubric consisting of a nullary broad rule that returns the singleton $(\mathsf{Nothing})$.
- Rule 1 is unary and sends $[w]$, where $Gw = (K_i)_{i \in I}$ to the wide rubric consisting of I rules, where rule I has arity K_i and sends a K_i-tuple x to the singleton $(\mathsf{Make}(w, \langle i, x \rangle))$.

So $\mathsf{Broad}(G)$ is generated by $\hat{G}$, and is therefore a set by Injective Broad Set Generation. □

9.2 Comparing Rubrics

Suppose we have two rubrics $\mathcal{R}$ and $\mathcal{S}$ on the same class, and want to show they are "essentially the same". Giving a natural isomorphism $\Delta_{\mathcal{R}} \cong \Delta_{\mathcal{S}}$ suffices for our purposes:

Proposition 9.3. *Let C be a class. Let $\mathcal{R}$ and $\mathcal{S}$ be rubrics on C, and $\alpha : \Delta_{\mathcal{R}} \cong \Delta_{\mathcal{S}}$ a natural isomorphism.*

(a) $\mathcal{R}$ has a derivation set iff $\mathcal{S}$ does.

(b) The square

$$\begin{tikzcd} \mathsf{ClassFam}(C) \arrow[r, "\Delta_{\mathcal{S}}"] \arrow[d, "\Delta_{\mathcal{R}}"'] & \mathsf{ClassFam}(C) \arrow[d, "\mathsf{Range}"] \\ \mathsf{ClassFam}(C) \arrow[r, "\mathsf{Range}"'] & \mathsf{Sub}(C) \end{tikzcd}$$

commutes.

(c) The functions $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathcal{S}}$ are equal.

(d) $\mathcal{R}$ is injective iff $\mathcal{S}$ is.

Proof.

(a) By Proposition 6.2.

(b) Obvious.

(c) The range of $\Delta_{\mathcal{R}}(X, i_{X,C})$ is $\Gamma_{\mathcal{R}} X$, and the range of $\Delta_{\mathcal{S}}(X, i_{X,C})$ is $\Gamma_{\mathcal{S}} X$. These ranges are the same, by part (b).

(d) Since $\mathcal{R}$-plates are the same thing as $(\mathcal{R}, C, \mathsf{id}_C)$-plates. ☐

9.3 Quasiwide Rubrics

For any rubric $\mathcal{R}$, we shall now consider the class $\mathsf{Arit}(\mathcal{R})$ of all arities that appear inside it, defined explicitly as follows.

Definition 9.4. Let C be a class.

(a) For a wide rubric $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ on C, we obtain the set

$$\mathsf{Arit}(\mathcal{R}) \quad \stackrel{\text{def}}{=} \quad \{K_i \mid i \in I\}$$

(b) For a broad rule $\langle L, S \rangle$ on C, we obtain the class

$$\mathsf{Arit}(S) \quad \stackrel{\text{def}}{=} \quad \bigcup_{x \in C^L} \mathsf{Arit}(S(x))$$

(c) For a broad rubric $\mathcal{S} = (\langle L_j, S_j \rangle)_{j \in J}$ on C, we obtain the class

$$\mathsf{Arit}(\mathcal{S}) \quad \stackrel{\text{def}}{=} \quad \bigcup_{j \in J} (\{L_j\} \cup \mathsf{Arit}(S_j))$$

The fact that $\mathsf{Arit}(\mathcal{R})$ is a set for a wide rubric $\mathcal{R}$ motivates the following.

Definition 9.5. Let C be a class. A *quasiwide rubric* on C is a broad rubric $\mathcal{S}$ such that $\mathsf{Arit}(\mathcal{S})$ is a set.

We see that wide and quasiwide are essentially the same for our purposes:

Proposition 9.6. *Let C be a class.*

(a) *For any wide rubric $\mathcal{R}$ on C, there is a quasiwide rubric $\mathcal{S}$ on C and natural isomorphism $\Delta_{\mathcal{R}} \cong \Delta_{\mathcal{S}}$.*

(b) *Conversely, for any quasiwide rubric $\mathcal{S}$ on C, there is a wide rubric $\mathcal{R}$ on C and natural isomorphism $\Delta_{\mathcal{S}} \cong \Delta_{\mathcal{R}}$.*

Proof.

(a) Let $\mathcal{S}$ be given by a single nullary broad rule that sends $[\,]$ to $\mathcal{R}$. For a class-family (M, F) within C, the $(\mathcal{R}, M, F)$-plate $\langle i, [x_k]_{k \in K}, p \rangle$ corresponds to the $(\mathcal{S}, M, F)$-plate $\langle *, [\,], i, [x_k]_{k \in K}, p \rangle$.

(b) We introduce notation, for sets A and B. Given an A-tuple u and B-tuple v, we write $\mathsf{copair}(u, v)$ for the $A + B$ tuple whose rth component is u_a or v_b according as r is $\mathsf{inl}\ a$ or $\mathsf{inr}\ b$.

Given a quasiwide rubric $\mathcal{S}$ on C, we form a wide rubric indexed by the set $\mathsf{Arit}(\mathcal{S})^2$ as follows:

$$\mathcal{R} \quad \stackrel{\text{def}}{=} \quad (\langle L + K, R_{L,K} \rangle)_{L, K \in \mathsf{Arit}(\mathcal{S})}$$

where $R_{L,K}$ sends $\mathsf{copair}(y, x)$, for $y \in C^L$ and $x \in C^K$, to the family $(c_n)_{n \in N}$ defined as follows. An element of N is a triple $\langle j, i, p \rangle$ such that

- writing $\mathcal{S} = (\langle L_j, S_j \rangle)_{j \in J}$, we have $j \in J$ and $L_j = L$
- writing $S_j(y) = (\langle K_i, R_i \rangle)_{i \in I}$, we have $i \in I$ and $K_i = K$
- writing $R_i(x) = (z_p)_{p \in P}$, we have $p \in P$

with $c_{\langle j,i,p \rangle} \stackrel{\text{def}}{=} z_p$. For a class-family (M, F) within C, an $(\mathcal{S}, M, F)$-plate

$$\langle j, [y_l]_{l \in L}, i, [x_k]_{k \in K}, p \rangle$$

corresponds to the $(\mathcal{R}, M, F)$-plate

$$\langle \langle L, K \rangle, \mathsf{copair}(y, x), \langle j, i, p \rangle \rangle$$

□

Corollary 9.7.

(a) *Wide Set Generation is equivalent to* Quasiwide Set Generation*: Any quasiwide rubric on a class generates a subset.*

(b) *Wide Derivation Set is equivalent to* Quasiwide Derivation Set*: Any quasiwide rubric on a class has a derivation set.*

(c) *Injective Wide Set Generation is equivalent to* Injective Quasiwide Set Generation*: Any injective quasiwide rubric on a class generates a subset.*

9.4 Application: Rubrics on a Set

We now consider the special case of a rubric on a set. Clearly such a rubric generates a subset, but does it have a derivation set? The following answer is adapted from [HMG+13].

Proposition 9.8.

(a) *Exponentiation is equivalent to the assertion "Any broad rubric on a set is quasiwide."*

(b) *Full Wide Infinity is equivalent to the assertion "Any rubric on a set has a derivation set."*

Proof.

(a) Since ($\Rightarrow$) is evident, we just prove ($\Leftarrow$). For a function f on a set, we write $\mathsf{graph}(f)$ for f regarded as a set of ordered pairs.

Given sets A and B, define the broad rubric $\mathcal{S}$ on B consisting of one A-ary rule, sending a tuple x to the wide rubric consisting of one $\mathsf{graph}(x)$-ary rule, sending each tuple to the empty family. If $\mathcal{S}$ is quasiwide, then B^A is a set.

(b) For ($\Rightarrow$), Full Wide Infinity gives Wide Derivation Set by Proposition 8.19(a) and hence Quasiwide Derivation Set by Corollary 9.7(b). It also gives Exponentiation, so any broad rubric on a set is quasiwide by part (a), and therefore has a derivation set.

For ($\Leftarrow$), given a signature $S = (K_i)_{i \in I}$, form a wide rubric on 1 via

$$\mathcal{R} \quad \stackrel{\text{def}}{=} \quad (\langle K_i, [*]_{k \in K} \mapsto (*) \rangle)_{i \in I}$$

Recursively define the bijection $\theta : \mathsf{Wide}(S) \cong \mathsf{Deriv}_{\mathcal{R}}$ sending $\langle i, [x_k]_{k \in K_i} \rangle$ to $\langle i, [\theta(x_k)]_{k \in K}, * \rangle$. Thus $\mathsf{Wide}(S)$ is a set iff $\mathsf{Deriv}_{\mathcal{R}}$ is. □

10 Properties of Rubric Functions

10.1 Preservation of Smallness and Injectivity

Given a rubric $\mathcal{R}$ on a class C, does the endofunction $\Gamma_{\mathcal{R}}$ on $\mathsf{Sub}(C)$ restrict to one on $\mathcal{P}C$? Likewise, does the endofunction $\Delta_{\mathcal{R}}$ on $\mathsf{ClassFam}(C)$ restrict to one on $\mathsf{Fam}(C)$ or $\mathsf{InjClassFam}(C)$ or $\mathsf{InjFam}(C)$? The following results answer these questions.

Proposition 10.1. *Let $\mathcal{R}$ be a rubric on a class C. Then $\mathcal{R}$ is injective iff the endofunction $\Delta_{\mathcal{R}}$ restricts to one on $\mathsf{InjClassFam}(C)$.*

Proof. For ($\Rightarrow$), let (M, F) be an injective class-family, and let x and x' be $(\mathcal{R}, M, F)$-plates with the same result c. Then the $\mathcal{R}$-plates $\widehat{F}(x)$ and $\widehat{F}(x')$ have result c, so, by injectivity of $\mathcal{R}$, they are equal. Injectivity of F implies that $\widehat{F}$ is injective, so $x = x'$.

For ($\Leftarrow$), observe that an $\mathcal{R}$-plate is the same thing as an $(\mathcal{R}, M, \mathsf{id}_C)$-plate, with the same result. Thus the injectivity of the class-family $\Delta_{\mathcal{R}}(C, \mathsf{id}_C)$ means that $\mathcal{R}$ is injective. □

Proposition 10.2. *Each of the following is equivalent to Exponentiation.*

(a) For any rubric $\mathcal{R}$ on a class C, the endofunction $\Gamma_{\mathcal{R}}$ restricts to one on $\mathcal{P}C$.

(b) For any rubric $\mathcal{R}$ on a class C, the endofunction $\Delta_{\mathcal{R}}$ restricts to one on $\mathsf{Fam}(C)$.

(c) For any injective rubric $\mathcal{R}$ on a class C, the endofunction $\Gamma_{\mathcal{R}}$ restricts to one on $\mathcal{P}C$.

(d) For any injective rubric $\mathcal{R}$ on a class C, the endofunction $\Delta_{\mathcal{R}}$ restricts to one on $\mathsf{InjFam}(C)$.

Proof. Assume Exponentiation. The $\mathcal{R}$-plates within a set form a set, giving (a). For any family (M, F) within C, the $(\mathcal{R}, M, F)$-plates form a set, giving (b).

Clearly (a) implies (c), and (b) implies (d) by Proposition 10.1($\Rightarrow$).

For any set K, define the injective family $\overline{K} \stackrel{\text{def}}{=} (K, \mathsf{id}_K)$, and the injective wide rubric $\dot{K}$ on $\mathfrak{T}$ consisting of a single K-ary rule that sends $x \in \mathfrak{T}^K$ to (x). To deduce Exponentiation from (c) or (d), let A and B be sets. Then we have

$$\begin{array}{rcl} \Gamma_{\dot{A}}(B) & = & B^A \\ \Delta_{\dot{A}}(\overline{B}) & = & (x)_{\langle *, x, * \rangle \in 1 \times B^A \times 1} \\ & \cong & \overline{B^A} \quad \text{via } \langle *, x, * \rangle \mapsto x. \end{array}$$

So if either $\Gamma_{\dot{A}}(B)$ is a set or $\Delta_{\dot{A}}(\overline{B})$ is small, then B^A is a set. □

Here is an application:

Proposition 10.3. *Let $\mathcal{R}$ be a rubric on a class C. Suppose that either C has an $\mathcal{R}$-complete subset, or Powerset or Collection holds. Then $\Gamma_{\mathcal{R}}$ has an inductive chain and a least prefixpoint.*

Proof. If C has an $\mathcal{R}$-complete subset, apply Proposition 5.14. If Powerset holds, apply Proposition 5.15(a) using Proposition 10.2(a). If Collection holds, apply 5.16. □

10.2 Proving the Injective Set Generation Principles

In order to prove the set generation principles for a rubric $\mathcal{R}$, we establish a relationship between the functions $\Gamma_{\mathcal{R}}$ and $\Delta_{\mathcal{R}}$.

Proposition 10.4. *Let $\mathcal{R}$ be a rubric on a class C. The square*

$$\begin{tikzcd}
\mathsf{InjClassFam}(C) \arrow[r, "\Delta_{\mathcal{R}}"] \arrow[d, "\mathsf{Range}"'] & \mathsf{ClassFam}(C) \arrow[d, "\mathsf{Range}"] \\
\mathsf{Sub}(C) \arrow[r, "\Gamma_{\mathcal{R}}"'] & \mathsf{Sub}(C)
\end{tikzcd}$$

commutes.

Proof. Let (M, F) be an injective class-family within C. Then $\widehat{F}$ is a result-preserving bijection from the class of all $(\mathcal{R}, M, F)$-plates to that of all $\mathcal{R}$-plates within $\mathsf{Range}(M, F)$. So an element $c \in C$ is the result of an $(\mathcal{R}, M, F)$-plate iff it is the result of an $\mathcal{R}$-plate within $\mathsf{Range}(M, F)$. □

Proposition 10.5. *Let $\mathcal{R}$ be an injective rubric on a class C.*

(a) $\Gamma_{\mathcal{R}}$ has an inductive chain and least prefixpoint.

(b) For every extended ordinal α, the class-family $\mu^{\alpha}\Delta_{\mathcal{R}}$ is injective, and its range is $\mu^{\alpha}\Gamma_{\mathcal{R}}$.

(c) The class-family $\mu\Delta_{\mathcal{R}}$ is injective, and its range is $\mu\Gamma_{\mathcal{R}}$.

Proof. Firstly, for every extended ordinal α, injectivity of $\mu^{\alpha}\Delta_{\mathcal{R}}$ is proved by induction on α, using Proposition 10.1($\Rightarrow$) for the successor case.

All that remains is to show that $(\mathsf{Range}(\mu^{\alpha}\Delta_{\mathcal{R}}))_{\alpha \in \mathsf{Ord}}$ is an inductive chain for $\Gamma_{\mathcal{R}}$, with supremum $\mathsf{Range}(\mu^{\infty}\Delta_{\mathcal{R}})$, as part (c) follows by Proposition 5.13.

The zero and extended limit requirements are obvious. For the successor requirement, $\mathsf{Range}(\mu^{\mathsf{S}\alpha}\Delta_{\mathcal{R}})$ is $\mathsf{Range}(\Gamma_{\mathcal{R}}\mu^{\alpha}\Delta_{\mathcal{R}})$, which is $\Gamma_{\mathcal{R}}\mathsf{Range}(\mu^{\alpha}\Delta_{\mathcal{R}})$ by Proposition 10.4. □

Corollary 10.6. *Let $\mathcal{R}$ be an injective rubric on a class C. If $\mathsf{Deriv}_{\mathcal{R}}$ is a set, then $\mathcal{R}$ generates a subset of C.*

We arrive at our main result:

Theorem 10.7.

(a) Wide Infinity, Wide Derivation Set and Injective Wide Set Generation are equivalent.

(b) Full Broad Infinity, Broad Derivation Set and Injective Broad Set Generation are equivalent.

Proof. Follows from Propositions 8.19 and 9.2 and Corollary 10.6. □

10.3 Proving the Set Generation Principles, Assuming AC

We again present the relationship between $\Gamma_{\mathcal{R}}$ and $\Delta_{\mathcal{R}}$, this time assuming AC and ignoring injectivity.

Proposition 10.8. (Assuming AC.) *Let $\mathcal{R}$ be a rubric on a class C.*

(a) *The square*

$$\begin{tikzcd} \mathsf{Fam}(C) \arrow[r, "\Delta_{\mathcal{R}}"] \arrow[d, "\mathsf{Range}"'] & \mathsf{ClassFam}(C) \arrow[d, "\mathsf{Range}"] \\ \mathcal{P}C \arrow[r, "\Gamma_{\mathcal{R}}"'] & \mathsf{Sub}(C) \end{tikzcd}$$

commutes.

(b) *If Collection holds, then the square*

$$\begin{tikzcd} \mathsf{ClassFam}(C) \arrow[r, "\Delta_{\mathcal{R}}"] \arrow[d, "\mathsf{Range}"'] & \mathsf{ClassFam}(C) \arrow[d, "\mathsf{Range}"] \\ \mathsf{Sub}(C) \arrow[r, "\Gamma_{\mathcal{R}}"'] & \mathsf{Sub}(C) \end{tikzcd}$$

commutes.

Proof.

(a) Let (M, F) be a family within C. Then $\widehat{F}$ is a result-preserving surjection from the class of all $(\mathcal{R}, M, F)$-plates to that of all $\mathcal{R}$-plates within $\mathsf{Range}(M, F)$. To see surjectivity in the wide case (the broad case is similar), let $w = \langle i, [a_k]_{k \in K_i}, p \rangle$ be an $\mathcal{R}$-plate within $\mathsf{Range}(M, F)$. By AC and since M is a set, we can choose for each a_k an F-preimage $x_k \in M$, and then $\langle i, [x_k]_{k \in K_i}, p \rangle$ is a $\widehat{F}$-preimage of w.

(b) Since Collection + AC gives Collective Choice—Proposition 3.3(a⇐)—we use the same argument as in part (a), except that (M, F) is now a class-family. □

Proposition 10.9. (Assuming AC.) *Let $\mathcal{R}$ be a rubric on a class C. Suppose that either* $\mathsf{Deriv}_{\mathcal{R}}$ *is a set, or Powerset or Collection holds.*

(a) *$\Gamma_{\mathcal{R}}$ has an inductive chain and a least prefixpoint.*

(b) *For each extended ordinal α, the range of $\mu^{\alpha}\Delta_{\mathcal{R}}$ is $\mu^{\alpha}\Gamma_{\mathcal{R}}$.*

(c) *The range of $\mu\Delta_{\mathcal{R}}$ is $\mu\Gamma_{\mathcal{R}}$.*

Proof. If $\mathsf{Deriv}_{\mathcal{R}}$ is a set, then so is $\mu^{\alpha}\Delta_{\mathcal{R}}$, for every extended ordinal α. If Powerset holds, then $\mu^{\alpha}\Delta_{\mathcal{R}}$ is a set for every ordinal α, by induction on α, using Proposition 10.2(b) for the successor case.

All that remains is to show that $(\mathsf{Range}(\mu^{\alpha}\Delta_{\mathcal{R}}))_{\alpha \in \mathsf{Ord}}$ is an inductive chain for $\Gamma_{\mathcal{R}}$ with supremum $\mathsf{Range}(\mu^{\infty}\Delta_{\mathcal{R}})$, as part (c) follows by Proposition 5.13.

The zero and extended limit requirements are obvious. For the successor requirement, $\mathsf{Range}(\mu^{\mathsf{S}\alpha}\Delta_{\mathcal{R}})$ is $\mathsf{Range}(\Gamma_{\mathcal{R}}\mu^{\alpha}\Delta_{\mathcal{R}})$, which is $\Gamma_{\mathcal{R}}\mathsf{Range}(\mu^{\alpha}\Delta_{\mathcal{R}})$ by Proposition 10.8(a) in the case that $\mathsf{Deriv}_{\mathcal{R}}$ is a set or Powerset holds, and by Proposition 10.8(b) in the case that Collection holds. □

Corollary 10.10. (Assuming AC.) *Let $\mathcal{R}$ be a rubric on a class C. If* $\mathsf{Deriv}_{\mathcal{R}}$ *is a set, then $\mathcal{R}$ generates a subset of C.*

We arrive at our main result for those who accept AC.

Theorem 10.11. (Assuming AC.)

(a) *Wide Infinity, Wide Derivation Set and Wide Set Generation are equivalent.*

(b) *Broad Infinity, Broad Derivation Set and Broad Set Generation are equivalent.*

Proof. Follows from Propositions 8.19 and 9.2 and Corollary 10.10. □

10.4 WISC Principles

Our goal is to improve Theorem 10.11, by replacing AC with a weak form of choice called WISC, originally studied in [Str05, vdBM14]. We shall formulate three versions of WISC, using the following notions.

Definition 10.12. Let K be a set.

(a) A K*-cover* is a K-tuple of inhabited sets. More generally, a K*-class-cover* is a K-tuple of inhabited classes.

(b) For any K-class-cover A, the surjection $\pi : \sum A \to K$ sends $\langle k, a \rangle$ to k.

(c) Given K-class-covers A and B, a *map* $f : A \to B$ is a K-tuple of functions $[f_k : A_k \to B_k]_{k \in K}$.

(d) A *WISC* for K is a set $\mathcal{A}$ of K-covers that is weakly initial—i.e., for any K-cover B, there is $A \in \mathcal{A}$ and a map $f : A \to B$.

Although the definition of WISC does not mention class-covers, only covers, we note the following result.

Proposition 10.13. (Assuming Collection.) *Let K be a set, and $\mathcal{A}$ a WISC for K. Then, for any K-class-cover B, there is $A \in \mathcal{A}$ and a map $f : A \to B$.*

Proof. Collection yields an element $X \in \prod_{k \in K} \mathcal{P}_{\mathsf{inh}} B_k$. Since X is a K-cover, there is $A \in \mathcal{A}$ and a map $f : A \to X$, so we have $f : A \to B$. □

We continue without assuming Collection.

Definition 10.14. Let $\mathcal{K}$ be a class of sets. A *WISC function* on $\mathcal{K}$ sends each $K \in \mathcal{K}$ to a WISC for K.

Now consider the following principles.

- *Simple WISC*: Every set has a WISC.
- *Local WISC*: Every set of sets has a WISC function.
- *Global WISC*: The class Set has a WISC function.

These principles are related as follows.

Proposition 10.15.

(a) *AC implies Global WISC, which implies Local WISC, which is equivalent to Simple WISC.*

(b) **In ZF** *the three WISC principles are equivalent.*

Proof.

(a) To show that AC implies Global WISC: note that AC is equivalent to $K \mapsto \{[1]_{k \in K}\}$ being a global WISC function.

To show that Global WISC implies Local WISC: given a set of sets $\mathcal{K}$, restrict the global WISC function to it.

To show that Local WISC implies Simple WISC: for any set K, obtain a WISC function for the singleton $\{K\}$ and apply it to K.

To show that Simple WISC implies Local WISC: let $\mathcal{K}$ be a set of sets, and write $L \stackrel{\text{def}}{=} \sum_{K \in \mathcal{K}} K$. For each $K \in \mathcal{K}$, define $\acute{K}$ to be the functor from the category of L-covers to that of K-covers, sending $[A_l]_{l \in L}$ to $[A_{\langle K,k\rangle}]_{k \in K}$ and likewise for maps. Given an WISC $\mathcal{A}$ for L, define f to be the function sending $K \in \mathcal{K}$ to the set of K-covers $\{\acute{K}(A) \mid A \in \mathcal{K}\}$, which is weakly initial by the following argument. Any K-cover B is equal to $\acute{K}(C)$, where C is the L-cover whose $\langle M, k\rangle$-component is B_k if $M = K$ and 1 otherwise. Weak initiality of $\mathcal{A}$ gives an L-cover $A \in \mathcal{A}$ and map $g : A \to C$, so we obtain $\acute{K}(A) \in f(K)$ and $\acute{K}(g) : \acute{K}(A) \to \acute{K}(C) = B$. Thus f is a WISC function on $\mathcal{K}$.

(b) We write $(V_\alpha)_{\alpha \in \mathsf{Ord}}$ for the cumulative hierarchy in the usual way. Suppose Simple WISC holds. For each set K, define $t(K)$ to be the least ordinal α such that the set $(\mathcal{P}_{\mathsf{inh}} V_\alpha)^K$ of all "α-bounded" K-covers is weakly initial. Then $K \mapsto (\mathcal{P}_{\mathsf{inh}} V_{t(K)})^K$ is a global WISC function. □

It has been shown that the theory ZF + WISC is strictly between ZF and ZFC, provided ZF is consistent [Kar14, Rob15]. For applications of WISC, see [vdBM14, FPS22, PS21].

10.5 Proving the Set Generation Principles, Assuming WISC

Our task is to weaken the AC assumption of Theorem 10.11. Specifically, we shall prove that Local WISC suffices for part (a), and Global WISC for part (b).

First we shall provide some constructions. For a rubric $\mathcal{R}$ on a class C, and any WISC function f on $\mathsf{Arit}(\mathcal{R})$, we shall construct an "extended" rubric $\mathcal{R}_f$ on C. This is done as follows, using Proposition 3.1.

Definition 10.16. Let C be a class.

(a) Let $\langle K, R\rangle$ be a wide rule on C. For any K-cover A, define the wide rule $\langle K, R\rangle^A$ consisting of the arity $\sum A$ and function $C^{\sum A} \to \mathsf{Fam}(C)$ sending $C^\pi(x)$ to $R(x)$ and everything else to the empty family.

(b) Let $\mathcal{R} = (\langle K_i, R_i\rangle)_{i \in I}$ be a wide rubric on C. For any WISC function f on $\mathsf{Arit}(\mathcal{R})$, define the wide rubric

$$\mathcal{R}_f \quad \stackrel{\text{def}}{=} \quad (\langle K_i, R_i\rangle^A)_{i \in I, A \in f(K_i)}$$

(c) Let $\langle L, S\rangle$ be a broad rule on C. For any L-cover B and WISC function f on $\mathsf{Arit}(S)$, define the broad rule $\langle L, S\rangle^B_f$ consisting of the arity $\sum B$ and function $C^{\sum B} \to \mathsf{WideRub}(C)$ sending $C^\pi(x)$ to $S(x)_f$ and everything else to the empty rubric.

(d) Let $\mathcal{S} = (\langle L_j, S_j\rangle)_{j \in J}$ be a broad rubric on C. For any WISC function f on $\mathsf{Arit}(\mathcal{S})$, define the broad rubric

$$\mathcal{S}_f \quad \stackrel{\text{def}}{=} \quad (\langle L_j, S_j\rangle^B_f)_{j \in J, B \in f(L_j)}$$

Now we adapt Proposition 10.8 as follows.

Proposition 10.17. *Let $\mathcal{R}$ be a rubric on a class C, and f a WISC function on* $\mathsf{Arit}(\mathcal{R})$.

(a) The square

$$\begin{array}{ccc} \mathsf{Fam}(C) & \xrightarrow{\Delta_{\mathcal{R}_f}} & \mathsf{ClassFam}(C) \\ \Big\downarrow{\scriptstyle \mathsf{Range}} & & \Big\downarrow{\scriptstyle \mathsf{Range}} \\ \mathcal{P}C & \xrightarrow[\Gamma_{\mathcal{R}}]{} & \mathsf{Sub}(C) \end{array}$$

commutes.

(b) If Collection holds, then the square

$$\begin{tikzcd}
\mathsf{ClassFam}(C) \arrow[r, "\Delta_{\mathcal{R}_f}"] \arrow[d, "\mathsf{Range}"'] & \mathsf{ClassFam}(C) \arrow[d, "\mathsf{Range}"] \\
\mathsf{Sub}(C) \arrow[r, "\Gamma_{\mathcal{R}}"'] & \mathsf{Sub}(C)
\end{tikzcd}$$

commutes.

Proof.

(a) We just prove the wide case, as the broad case is similar. Let (M, F) be a family within C. For any $(\mathcal{R}_f, M, F)$-plate

$$m \quad = \quad \langle\langle i, A\rangle, [x_{k,a}]_{k\in K_i, a\in A_k}, p\rangle$$

the tuple $[F(x_{k,a})]_{k\in K_i, a\in A_k}$ is (uniquely) expressible as $C^\pi(u)$, and we obtain an $\mathcal{R}$-plate $\widetilde{F}(m) \stackrel{\text{def}}{=} \langle i, u, p\rangle$.

We see next that $\widetilde{F}$ is a result-preserving surjection from the class of all $(\mathcal{R}_f, M, F)$-plates to that of all $\mathcal{R}$-plates within $\mathsf{Range}(M, F)$. To prove surjectivity, let $n = \langle i, y, p\rangle$ be an $\mathcal{R}$-plate within $\mathsf{Range}(M, F)$. We obtain a K_i-cover $[F^{-1}(y_k)]_{k\in K_i}$, so there is $A \in f(K_i)$ and a K-cover map $g : A \to [F^{-1}(y_k)]_{k\in K_i}$. We then obtain an $(\mathcal{R}_f, M, F)$-plate

$$m \quad = \quad \langle\langle i, A\rangle, [g_k(a)]_{k\in K_i, a\in A_k}, p\rangle$$

Since $F(g_k(a)) = y_k$ for all $k \in K_i$ and $a \in A_k$, we have $\widetilde{F}(m) = n$.

So an element $c \in C$ is the result of an $(\mathcal{R}_f, M, F)$-plate iff it is the result of an $\mathcal{R}$-plates within $\mathsf{Range}(M, F)$.

(b) ISimilar to part (a), except that we use Proposition 10.13 and speak of classes rather than sets. □

Proposition 10.18. *Let $\mathcal{R}$ be a rubric on a class C, and f a WISC function on* $\mathsf{Arit}(\mathcal{R})$. *Suppose that either* $\mathsf{Deriv}_{\mathcal{R}_f}$ *is a set, or Powerset or Collection holds.*

(a) $\Gamma_{\mathcal{R}}$ has an inductive chain and a least prefixpoint.

(b) For each extended ordinal α, the range of $\mu^\alpha\Delta_{\mathcal{R}_f}$ is $\mu^\alpha\Gamma_{\mathcal{R}}$.

(c) The range of $\mu\Delta_{\mathcal{R}_f}$ is $\mu\Gamma_{\mathcal{R}}$.

Proof. Similar to the proof of Proposition 10.9, using Proposition 10.17 rather than Proposition 10.8. □

Corollary 10.19. *Let $\mathcal{R}$ be a rubric on a class C, and f a WISC function on* $\mathsf{Arit}(\mathcal{R})$. *If* $\mathsf{Deriv}_{\mathcal{R}_f}$ *is a set, then $\mathcal{R}$ generates a subset of C.*

The key question is whether $\mathsf{Arit}(\mathcal{R})$ has a WISC function, which we answer as follows.

Proposition 10.20.

(a) Local WISC is equivalent to the assertion "For any wide rubric $\mathcal{R}$ on a class C, the set $\mathsf{Arit}(\mathcal{R})$ *has a WISC function."*

(b) Global WISC is equivalent to the assertion: "For any broad rubric $\mathcal{R}$ on a class C, the class $\mathsf{Arit}(\mathcal{R})$ *has a WISC function."*

Proof. Since ($\Rightarrow$) is obvious, we just prove ($\Leftarrow$).

(a) Given a set of sets $\mathcal{K}$, define $\mathcal{R}$ to be the following wide rubric on $\emptyset$: it is indexed by $\mathcal{K}$, and rule K has arity K. Then $\mathsf{Arit}(\mathcal{R}) = \mathcal{K}$.

(b) Define $\mathcal{S}$ to be the broad rubric on Set consisting of a single unary rule, sending $[X]$ to the wide rubric consisting of a single X-ary rule, sending every tuple to empty family. Then $\mathsf{Arit}(\mathcal{S}) = \mathsf{Set}$. $\square$

We obtain our main result:

Theorem 10.21.

(a) (Assuming Local WISC.) *Wide Infinity, Wide Derivation Set and Wide Set Generation are equivalent.*

(b) (Assuming Global WISC.) *Broad Infinity, Broad Derivation Set and Broad Set Generation are equivalent.*

Proof. From Propositions 8.19 and 9.2, and Corollary 10.19 using Proposition 10.20. $\square$

Part IV
Wide and Broad principles for ordinals

11 From Rubrics to Supgeneration

Our next task will be to adapt Wide and Broad Set Generation into similar principles for ordinals. An extended ordinal that is neither 0 nor a successor is called an *extended limit*. Recall that Definition 8.4 gave us the notion of a class being closed or complete. Here are analogous properties for extended ordinals:

Definition 11.1. An extended limit λ is

- *K-supclosed*, for a set K, when $\bigvee_K$ (or equivalently ssup_K) restricts to a function $[0\,..\,\lambda)^K \to [0\,..\,\lambda)$.
- *$\mathcal{K}$-supcomplete*, for a class of sets $\mathcal{K}$, when it is K-supclosed for all $K \in \mathcal{K}$.
- *F-supclosed*, for a function $F : \mathsf{Ord} \to \mathsf{Set}$, when it is $F\alpha$-supclosed for all $\alpha < \lambda$.
- *H-supcomplete*, for a function $H : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$, when it is $H\alpha$-supcomplete for all $\alpha < \lambda$.

Here are some examples.

1. For a class of sets $\mathcal{K}$, let $\mathsf{Const}_{\mathcal{K}}$ be the constant function $\gamma \mapsto \mathcal{K}$. An extended limit is $\mathsf{Const}_{\mathcal{K}}$-supcomplete iff it is $\mathcal{K}$-supcomplete.
2. For functions $H, H' : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$, let $H \vee H'$ be the pointwise union $\gamma \mapsto H(\gamma) \cup H'(\gamma)$. An extended limit is $(H \vee H')$-supcomplete iff it is both H-supcomplete and H'-supcomplete.

Below (Proposition 13.6) we shall characterize supclosedness and supcompleteness in an explicit way.

Definition 11.2.

(a) For a set K, the *simply K-supgenerated* extended limit is the least K-supclosed one.

(b) For a class of sets $\mathcal{K}$, the *$\mathcal{K}$-supgenerated* extended limit is the least $\mathcal{K}$-supcomplete one.

(c) For a function $F : \mathsf{Ord} \to \mathsf{Set}$, the *simply F-supgenerated* extended limit is the least F-supclosed one.

(d) For a function $H : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$, the *$H$-supgenerated* extended limit is the least H-supcomplete one.

This leads to the following principles.

- *Simple Wide Supgeneration*: Any set simply supgenerates a limit ordinal.
- *Full Wide Supgeneration*: Any set of sets supgenerates a limit ordinal.
- *Simple Broad Supgeneration*: Any function $\mathsf{Ord} \to \mathsf{Set}$ simply supgenerates a limit ordinal.
- *Full Broad Supgeneration*: Any function $\mathsf{Ord} \to \mathcal{P}\mathsf{Set}$ supgenerates a limit ordinal.

Theorem 11.3.

(a) The two forms of Wide Supgeneration are equivalent.

(b) The two forms of Broad Supgeneration are equivalent.

(c) Full Broad Supgeneration implies Full Wide Supgeneration.

Proof.

(a) Full $\Rightarrow$ Simple is obvious. For the converse, we first note that, for a set of sets $\mathcal{K}$, any extended limit λ that is $\sum_{K \in \mathcal{K}} K$-supclosed is $\mathcal{K}$-supcomplete. That is because, for $K \in \mathcal{K}$ and $p \in [0 \,..\, \lambda)^K$, we have

$$\begin{aligned} \bigvee_{k \in K} p_k &= \bigvee_{\langle L, K \rangle \in \sum_{K \in \mathcal{K}} K} \begin{cases} p_k & (L = K) \\ 0 & \text{otherwise} \end{cases} \\ &< \lambda \end{aligned}$$

(b) Full $\Rightarrow$ Simple is obvious. For the converse: for a function $H : \mathsf{Ord} \to \mathcal{P}\mathsf{Set}$, any extended limit that is $(\beta \mapsto \sum_{K \in H\beta} K)$-supclosed is H-supcomplete, as before.

(c) Given a set of sets $\mathcal{K}$, the $\mathsf{Const}_{\mathcal{K}}$-supgenerated extended limit is $\mathcal{K}$-supgenerated. $\square$

Now we give the relationship between supgeneration and set generation.

Theorem 11.4.

(a) Wide Set Generation is equivalent to Powerset + Full Wide Supgeneration.

(b) Broad Set Generation is equivalent to Powerset + Full Broad Supgeneration.

Proof.

(a) For ($\Rightarrow$), we have Powerset by Theorem 7.3 and Proposition 9.2(a). To show Full Wide Supgeneration, let $\mathcal{K}$ be a set of sets. A $\mathcal{K}$-supcomplete ordinal is an $\mathcal{R}$-inductive subset of Ord, where the wide rubric $\mathcal{R}$ on Ord consists of the following.

- The unary rule sending $[\alpha]$ to $(\beta)_{\beta<\alpha}$. (A set of ordinals is closed under this rule iff it is transitive, i.e., an ordinal.)
- For each $K \in \mathcal{K}$, a K-ary rule sending $[\alpha_k]_{k\in K}$ to $(\bigvee_{k\in K} \alpha_k)$.

For ($\Leftarrow$), let $\mathcal{R} = (\langle K_i, R_i\rangle)_{i\in I}$ be a wide rubric on C. We show that the inductive chain of $\Gamma_{\mathcal{R}}$ (which preserves smallness by Exponentiation and Proposition 10.2(a)) stabilizes at an ordinal α. Define the set of sets

$$\mathcal{K} \quad \stackrel{\text{def}}{=} \quad \{K_i \mid i \in I\}$$

Let α be a $\mathcal{K}$-supcomplete limit ordinal. For any $x \in \mu^{\alpha}\Gamma_{\mathcal{R}}$, put $\overline{x}$ for the unique $\beta < \alpha$ such that $x \in \mu^{\mathsf{S}\beta}\Gamma_{\mathcal{R}} \setminus \mu^{\beta}\Gamma_{\mathcal{R}}$. Given an $\mathcal{R}$-plate

$$w \quad = \quad \langle i, x, p\rangle$$

within $\mu^{\alpha}\Gamma_{\mathcal{R}}$, put

$$\beta \quad \stackrel{\text{def}}{=} \quad \mathsf{ssup}_{l\in L_j}\overline{x_l}$$

Since α is K_i-supclosed, and $\overline{y_k} < \alpha$ for all $k \in K_i$, we obtain $\beta < \alpha$. Since w is an $\mathcal{R}$-plate within $\mu^{\beta}\Gamma_{\mathcal{R}}$, its result is in $\mu^{\mathsf{S}\beta}\Gamma_{\mathcal{R}}$, which is included in $\mu^{\alpha}\Gamma_{\mathcal{R}}$ since $\mathsf{S}\beta < \alpha$.

(b) For ($\Rightarrow$), to show Full Broad Supgeneration, let $H : \mathsf{Ord} \to \mathcal{P}\mathsf{Set}$ be a function. An H-supcomplete ordinal is an $\mathcal{R}$-inductive subset of Ord, where the broad rubric $\mathcal{R}$ on Ord consists of the following.

- The nullary rule returning the wide rubric consisting of just the unary rule sending $[\alpha]$ to $(\beta)_{\beta<\alpha}$. (A set of ordinals is closed under this rule iff it is transitive, i.e., an ordinal.)
- The unary rule sending $[\alpha]$ to the wide rubric consisting of, for each $K \in H\alpha$, the K-ary rule sending $[\alpha_k]_{k\in K}$ to $(\bigvee_{k\in K} \alpha_k)$.

For ($\Leftarrow$), let $\mathcal{S} = (\langle L_j, S_j\rangle)_{j\in J}$ be a broad rubric on C. We show that the inductive chain of $\Gamma_{\mathcal{S}}$ (which preserves smallness by Exponentiation and Proposition 10.2(a)) stabilizes at an ordinal α. Define the set

$$\mathcal{L} \quad \stackrel{\text{def}}{=} \quad \{L_j \mid j \in J\}$$

and the function $H : \mathsf{Ord} \to \mathcal{P}\mathsf{Set}$ sending β to

$$\bigcup_{j\in J} \quad \bigcup_{\substack{y\in(\mu^{\beta}\Gamma_{\mathcal{S}})^{L_j} \\ S_j(y)=(\langle K_i,R_i\rangle)_{i\in I}}} \quad \{K_i \mid i \in I\}$$

Let α be a limit ordinal that is $(\mathsf{Const}_{\mathcal{L}} \vee H)$-supcomplete—i.e., both $\mathcal{L}$-supcomplete and H-supcomplete. For any $x \in \mu^{\alpha}\Gamma_{\mathcal{S}}$, put $\overline{x}$ for the unique $\beta < \alpha$ such that $x \in \mu^{\mathsf{S}\beta}\Gamma_{\mathcal{S}} \setminus \mu^{\beta}\Gamma_{\mathcal{S}}$. Given an $\mathcal{S}$-plate

$$w \quad = \quad \langle j, y, i, x, p\rangle$$

within $\mu^{\alpha}\Gamma_{\mathcal{S}}$, put

$$\gamma \quad \stackrel{\text{def}}{=} \quad \mathsf{ssup}_{l\in L_j}\overline{y_l}$$
$$\beta \quad \stackrel{\text{def}}{=} \quad \mathsf{ssup}_{k\in K_i}\overline{x_k}$$

Since α is L_j-supclosed, and $\overline{y_l} < \alpha$ for all $l \in L_j$, we obtain $\gamma < \alpha$. We have $y \in (\mu^{\gamma}\Gamma_{\mathcal{S}})^{L_j}$ and $S_j(y) = (\langle K_i, R_i\rangle)_{i\in I}$, so $K_i \in H\gamma$, so α is K_i-supclosed. Since $\overline{x_k} < \alpha$ for all $k \in K$, we obtain $\beta < \alpha$. Since w is an $\mathcal{S}$-plate within $\mu^{\gamma\vee\beta}\Gamma_{\mathcal{S}}$, its result is in $\mu^{\mathsf{S}(\gamma\vee\beta)}\Gamma_{\mathcal{S}}$, which is included in $\mu^{\alpha}\Gamma_{\mathcal{S}}$ since $\mathsf{S}(\gamma \vee \beta) < \alpha$. $\square$

12 Lindenbaum Numbers

We interrupt our journey towards Mahlo's principle to give some useful constructions that relate sets to ordinals. First we give some notation:

Definition 12.1. Let A and B be sets.

(a) We write $A \preccurlyeq B$ when there is an injection $A \to B$.

(b) We write $A \preccurlyeq^* B$ when there is a partial surjection $B \to A$. Equivalently: when either $A = \emptyset$ or there is a surjection $B \to A$.

Thus $A \preccurlyeq B$ implies $A \preccurlyeq^* B$, and conversely if AC holds or B is well-orderable.

Definition 12.2. Let K be a set.

(a) Define R to be the class of all pairs $(X, <)$, consisting of a subset X of K, and a well-order $<$ on X. Then we obtain a lower class

$$\{\text{order-type}(X, <) \mid (X, <) \in R\}$$

whose strict supremum is denoted $\aleph(K)$.

(b) Let S be the class of all triples $(X, \sim, <)$, consisting of a subset X of K, and an equivalence relation $\sim$ on X, and a well-order $<$ on the set $X/\sim$ of all equivalence classes. Then we obtain a lower class

$$\{\text{order-type}(X/\sim, <) \mid (X, \sim, <) \in S\}$$

whose strict supremum is denoted $\aleph^*(K)$.

The extended ordinals $\aleph(K)$ and $\aleph^*(K)$ are called the *Hartogs number* and the *Lindenbaum number* of K, respectively.[9] Note that $\aleph(K) \leqslant \aleph^*(K)$, with equality if AC holds or K is well-orderable. For any ordinal α, we have $\alpha \preccurlyeq K$ iff $\alpha < \aleph(K)$, and $\alpha \preccurlyeq^* K$ iff $\alpha < \aleph^*(K)$. Thus we have $\aleph(K) \not\preccurlyeq K$ and $\aleph^*(K) \not\preccurlyeq^* K$.

These constructions are often applied to an ordinal β, giving $\beta < \aleph(\beta) \leqslant \aleph^*(\beta)$.

For the sake of Section 13.4 below, we introduce some axioms about Lindenbaum numbers. The first is *Full Lindenbaum*: For any set K, the extended ordinal $\aleph^*(K)$ is an ordinal.

Proposition 12.3.

(a) Powerset implies Full Lindenbaum.

(b) Wide Supgeneration implies Full Lindenbaum.

Proof.

(a) By Powerset, the classes R and S in Definition 12.2 are sets.

(b) For a set K, let λ be a K-closed limit ordinal. If $\lambda < \aleph^*(K)$, then there is a surjection $f : K \to \lambda$, so $\lambda = \sup_{k \in K} f(k) < \lambda$, contradiction. Thus $\aleph^*(K) \leqslant \lambda$, so $\aleph^*(K)$ is an ordinal. $\square$

We divide Full Lindenbaum into two parts:

- *Ordinal Lindenbaum*: For any ordinal α, the extended ordinal $\aleph^*(\alpha)$ is an ordinal.

[9] See [KRS24] for analysis of the range of possibilities.

- *Relative Lindenbaum*: For any set K, there is an ordinal α such that $\aleph^*(K) \subseteq \aleph^*(\alpha)$.

Proposition 12.4. *Full Lindenbaum is equivalent to Ordinal Lindenbaum + Relative Lindenbaum.*

Proof. ($\Leftarrow$) is obvious, and clearly Full Lindenbaum implies Ordinal Lindenbaum. To show that it implies Relative Lindenbaum, put $\alpha \stackrel{\text{def}}{=} \aleph^*(K)$ so that $\aleph^*(K) < \aleph^*(\alpha)$. $\square$

Lastly we consider the *Well-orderability* axiom: Every set is well-orderable. This principle has the following properties:

Proposition 12.5.

(a) AC + Powerset implies Well-orderability.

(b) Well-orderability implies AC + Relative Lindenbaum.

Proof. We prove only Well-orderability $\Rightarrow$ Relative Lindenbaum, as the rest is standard. Given a set K, define α to be the least order-type of a well-ordering of K. Since $\alpha \cong K$, we have $\aleph^*(K) = \aleph^*(\alpha)$. $\square$

13 From Supgeneration to Mahlo's Principle

13.1 Unbounded and Stationary Classes

Now at last, it is time to treat Mahlo's principle; but we approach it more slowly than in Section 1. To begin, we revisit the notions of unbounded and stationary class from Section 1.2.

Definition 13.1. Let A be a set-based well-ordered class.

(a) A subclass B is *cofinal* when, for all $x \in A$, there is $y \in B$ such that $y \geqslant x$. Equivalently: when it has no strict upper bound.

(b) A subclass B is *strictly cofinal* when, for all $x \in A$, there is $y \in B$ such that $y > x$. Equivalently: when it has no upper bound.

If A has no greatest element (e.g., when $A = \mathsf{Ord}$), then "cofinal" and "strictly cofinal" are equivalent, and the word "unbounded" is also used.

We turn next to ordinal functions.

Definition 13.2. An extended limit λ is

- G*-based*, for a function $G : \mathsf{Ord} \to \mathsf{Ord}_\infty$, when, for all $\alpha < \lambda$, we have $G(\alpha) \leqslant \lambda$.
- F*-closed*, for a function $F : \mathsf{Ord} \to \mathsf{Ord}$, when, for all $\alpha < \lambda$, we have $F(\alpha) < \lambda$. In short: when F restricts to an endofunction on λ.

Proposition 13.3. *For a class of limit ordinals D, the following are equivalent.*

(a) For every function $G : \mathsf{Ord} \to \mathsf{Ord}$, there is a G-based limit ordinal in D.

(b) For every function $F : \mathsf{Ord} \to \mathsf{Ord}$, there is a F-closed limit ordinal in D.

Proof. Since F-closed is the same as $\mathsf{S}F$-based and implies F-based. $\square$

A class of limit ordinals with these properties is said to be *stationary*. It is then unbounded, and, for any function $F : \mathsf{Ord} \to \mathsf{Ord}$, contains stationarily many F-closed elements. Here is an application:

Proposition 13.4. *Each of the following is equivalent to Infinity.*

(a) Lim *is unbounded.*

(b) Lim *is stationary.*

Proof. If Infinity does not hold, then Lim is empty.

Assume Infinity. To show Lim is stationary, let $F : \mathsf{Ord} \Rightarrow \mathsf{Ord}$. Define $G : \mathsf{Ord} \Rightarrow \mathsf{Ord}$ sending α to $\mathsf{ssup}_{\beta<\alpha}(\mathsf{S}\beta \vee F\beta)$. We show that $\lambda \stackrel{\text{def}}{=} \bigvee_{n\in\mathbb{N}} G^n(1)$ is an F-closed limit ordinal. Firstly, $0 < 1 = G^0(1) \leqslant \lambda$. If $\beta < \lambda$ then there is $n \in \mathbb{N}$ such that $\beta < G^n(1)$, so $\mathsf{S}\beta < G^{n+1}(1) \leqslant \lambda$, and likewise $F\beta < \lambda$. ☐

13.2 Cofinality

The treatment of cofinality relies on the following result:

Proposition 13.5. *For any extended ordinal* α *and function* $f : [0\,..\,\alpha) \to \mathsf{Ord}$*, the range of* f *has a cofinal subclass of order-type* $\leqslant \alpha$*.*

Proof. Let K be the class of all $i < \alpha$ such that $f(i)$ is a strict upper bound of $\{f(j) \mid j < i\}$. We prove by induction on $i < \alpha$ that there is $k \leqslant i$ such that $k \in K$ and $f(k) \geqslant f(i)$, as follows. If $i \in K$, put $k \stackrel{\text{def}}{=} i$, and if not, then there is $j < i$ such that $f(i) \leqslant f(j)$, and we apply the inductive hypothesis to it.

Thus the range of $f\restriction_K$ is cofinal within that of f, and (since $f\restriction_K$ is strictly monotone) has the same order-type as K, which is $\leqslant \alpha$ by Proposition 5.6(b). ☐

Let λ be an extended limit. The cofinal and strictly cofinal subclasses of $[0\,..\,\lambda)$ are the same (as stated above), and the order-type of each is an extended limit. The least such order-type is called the *cofinality* of λ, and written $\mathsf{cf}(\lambda)$. Clearly it satisfies $\mathsf{cf}(\lambda) \leqslant \lambda$ and $\mathsf{cf}(\mathsf{cf}(\lambda)) = \mathsf{cf}(\lambda)$.

Now we use cofinality to characterize supclosedness and supcompleteness.

Proposition 13.6. *Let* λ *be an extended limit. It is*

(a) K*-supclosed, for a set* K*, iff* $\aleph^*(K) \leqslant \mathsf{cf}(\lambda)$*.*

(b) $\mathcal{K}$*-supcomplete, for a class of sets* $\mathcal{K}$*, iff* $\bigvee_{K\in\mathcal{K}} \aleph^*(K) \leqslant \mathsf{cf}(\lambda)$*.*

(c) F*-supclosed, for a function* $F : \mathsf{Ord} \to \mathsf{Set}$*, iff* $\mathsf{cf}(\lambda)$ *is* $(\alpha \mapsto \aleph^*(F(\alpha)))$*-based.*

(d) H*-supcomplete, for a function* $H : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$*, iff* $\mathsf{cf}(\lambda)$ *is* $(\alpha \mapsto \bigvee_{K\in H(\alpha)} \aleph^*(K))$*-based.*

Proof. We prove part (a), from which the other parts follow.

For ($\Rightarrow$), take a cofinal subclass B of $[0\,..\,\lambda)$ with order-type $\mathsf{cf}(\lambda)$. If $\mathsf{cf}(\lambda) < \aleph^*(K)$, then we have a surjection $K \to [0\,..\,\mathsf{cf}(\lambda))$, and hence $K \to B$, so λ is B-supclosed, so $\bigvee_{\beta<B} \beta < \lambda$, a contradiction.

For ($\Leftarrow$), the range of any $f : K \to [0\,..\,\lambda)$ has order-type in $\aleph^*(K)$ and hence in $\mathsf{cf}(\lambda)$, so its supremum is $< \lambda$. ☐

Where the sets in questions are ordinals, we give a simpler characterization:

Proposition 13.7. *Let* λ *be an extended limit. It is*

(a) α*-supclosed, for an ordinal* α*, iff* $\alpha < \mathsf{cf}(\lambda)$*.*

(b) ρ*-supcomplete, for an extended ordinal* ρ*, iff* $\rho \leqslant \mathsf{cf}(\lambda)$*.*

(c) F*-supclosed, for a function* $F : \mathsf{Ord} \to \mathsf{Ord}$*, iff* $\mathsf{cf}(\lambda)$ *is* F*-closed.*

(d) G*-supcomplete, for a function* $G : \mathsf{Ord} \to \mathsf{Ord}_\infty$*, iff* $\mathsf{cf}(\lambda)$ *is* G*-based.*

Proof. Part (a) is by Proposition 13.5, and the rest follows. ☐

13.3 Regularity

We revisit the notion of regularity from Section 1.2. First we note that Proposition 13.7(b) gives the following:

Corollary 13.8. *Let λ be an extended limit. It is λ-supcomplete iff $\mathsf{cf}(\lambda) = \lambda$.*

We say that λ is *regular* when it satisfies these conditions. Thus the cofinality of any extended limit is regular. Here are more ways of obtaining examples:

Proposition 13.9. *All of the following are regular.*

- *The extended limit that is simply supgenerated by a set K.*
- *The extended limit that is supgenerated by a class of sets $\mathcal{K}$.*
- *The extended limit that is simply supgenerated by a function $F : \mathsf{Ord} \to \mathsf{Set}$.*
- *The extended limit that is supgenerated a function $H : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$.*

Proof. We first prove that, for a class of sets $\mathcal{K}$, a minimal $\mathcal{K}$-supcomplete extended limit λ is regular. Fix $\alpha < \lambda$ and a tuple $[x_i]_{i<\alpha}$ within λ. We shall show that $\bigvee_{i<\alpha} x_i < \lambda$. For $\beta < \lambda$, put $\overline{\beta} \stackrel{\text{def}}{=} \bigvee_{i<\alpha \wedge \beta} x_i$. The class $P \stackrel{\text{def}}{=} \{\beta < \lambda \mid \overline{\beta} < \lambda\}$ is a $\mathcal{K}$-supcomplete limit $\leqslant \lambda$ by the following reasoning.

- For $\gamma \leqslant \beta \in P$ we have $\gamma \in P$, since $\overline{\gamma} \leqslant \overline{\beta}$. So P is an ordinal $\leqslant \lambda$.
- We have $0 \in P$ since $\overline{0} = 0$, and for any $\beta \in P$ we have $\mathsf{S}\beta \in P$, since $\overline{\mathsf{S}\beta}$ is $\overline{\beta} \wedge x_\beta$ if $\beta < \alpha$ and $\overline{\beta}$ otherwise So P is a limit.
- For any $K \in \mathcal{K}$ and tuple $[\beta_k]_{k\in K}$ within P, we have $\bigvee_{k\in K} \beta_k \in P$ since $\overline{\bigvee_{k\in K} \beta_k} = \bigvee_{k\in K} \overline{\beta_k}$. So P is $\mathcal{K}$-supcomplete.

Minimality of λ gives $P = \lambda$, so $\alpha \in P$, meaning that $\bigvee_{i<\alpha} x_i = \overline{\alpha} < \lambda$ as required.

Lastly, for $H : \mathsf{Ord} \to \mathsf{Sub}(\mathsf{Set})$, any minimal H-supcomplete limit λ is also a minimal $(\bigcup_{\beta<\lambda} H\beta)$-supcomplete limit, and therefore regular. $\square$

13.4 Blass's Axiom and Mahlo's Principle

Now we revisit the principles from Section 1.3: *Blass's axiom* says that Reg is unbounded, and *Mahlo's principle* that Reg is stationary. We begin with basic consequences.

Proposition 13.10.

(a) Blass's axiom implies Ordinal Lindenbaum + Infinity.

(b) Mahlo's principle implies Blass's axiom.

Proof.

(a) Given an ordinal α, take a regular limit ordinal $\lambda > \alpha$. It is is α-supclosed by Proposition 13.7(a). We show $\aleph^*(\alpha) \leqslant \lambda$. For any $\beta < \aleph^*(\alpha)$, we have a partial surjection $\alpha \to \beta$, so λ is β-supclosed, so $\beta < \lambda$ by Proposition 13.7(a). Infinity holds since $\mathsf{Reg} \subseteq \mathsf{Lim}$.

(b) Since stationary implies unbounded. $\square$

We arrive at the main result of the section:

Theorem 13.11.

(a) Wide Supgeneration is equivalent to Blass's axiom + Relative Lindenbaum.

(b) Broad Supgeneration is equivalent to Mahlo's principle + Relative Lindenbaum.

Proof. We prove part (b), as part (a) is similar.

For ($\Rightarrow$), Broad Supgeneration implies Relative Lindenbaum by Proposition 12.3(b). To show that it implies Mahlo's principle, let $F : \mathsf{Ord} \Rightarrow \mathsf{Ord}$. The simply F supgenerated limit ordinal is F-supclosed, so by Proposition 13.7(c) it is F-closed. It is regular by Proposition 13.9.

For ($\Leftarrow$), let $F : \mathsf{Ord} \to \mathsf{Set}$. Define $G : \mathsf{Ord} \to \mathsf{Ord}$ sending α to the least ordinal β such that $\aleph^*(F(\alpha)) \leqslant \aleph^*(\beta)$. Then there is a regular limit ordinal that is G-closed. By Proposition 13.7(c) it is G-supclosed, so by Proposition 13.6(c) it is F-supclosed. □

Corollary 13.12. (Assuming Powerset or Well-orderability.)

(a) Wide Supgeneration is equivalent to Blass's axiom.

(b) Broad Supgeneration is equivalent to Mahlo's principle.

Proof. Immediate from Theorem 13.11, using Proposition 12.3(a) and 12.5(b). □

14 The Power of Stationarity

14.1 Club Classes and Continuous Functions

Our final task is to develop the traditional theory of stationarity, in which "iterated inaccessibles" of various kinds are obtained from Mahlo's principle. Throughout Section 14, *class* will always mean a class of ordinals, and *function* an endofunction on Ord. We use the following constructions:

Definition 14.1.

(a) For any monotone function H, we write $\mathsf{Pref}(H)$ for the class of all its prefixpoints.

(b) For any family of classes $(C_i)_{i \in I}$, the *intersection* is given by

$$\bigcap_{i \in I} C_i \quad \stackrel{\text{def}}{=} \quad \{\alpha \in \mathsf{Ord} \mid \forall i \in I.\, \alpha \in C_i\}$$

(c) For any family of monotone functions $(H_i)_{i \in I}$, the *supremum* is given by

$$\bigvee_{i \in I} H_i \quad : \quad \alpha \quad \mapsto \quad \bigvee_{i \in I} H_i(\alpha)$$

$$\text{so that} \quad \mathsf{Pref}(\bigvee_{i \in I} H_i) \quad = \quad \bigcap_{i \in I} \mathsf{Pref}(H_i)$$

(d) For any sequence of classes $(C_\alpha)_{\alpha \in \mathsf{Ord}}$, the *diagonal intersection* is given by

$$\Delta_{\alpha \in \mathsf{Ord}} C_\alpha \quad \stackrel{\text{def}}{=} \quad \{\alpha \in \mathsf{Ord} \mid \forall \beta < \alpha.\, \alpha \in C_\beta\}$$

(e) For any sequence of monotone functions $(H_\alpha)_{\alpha \in \mathsf{Ord}}$, the *diagonal supremum* is given by

$$\nabla_{\alpha \in \mathsf{Ord}} H_\alpha \quad : \quad \alpha \quad \mapsto \quad \bigvee_{\beta < \alpha} H(\beta)$$

$$\text{so that} \quad \mathsf{Pref}(\nabla_{\alpha \in \mathsf{Ord}} H_\alpha) \quad = \quad \Delta_{\alpha \in \mathsf{Ord}} \mathsf{Pref}(H_\alpha)$$

Definition 14.2. A function H is *continuous* when it is monotone and sends every limit ordinal λ to $\bigvee_{\alpha<\lambda} H(\alpha)$.

Here are some ways to obtain continuous functions:

Proposition 14.3.

(a) For every ordinal α, the function Const_α is continuous.

(b) For any family of continuous functions $(H_i)_{i\in I}$, the supremum $\bigvee_{i\in I} H_i$ is continuous.

(c) For any sequence of continuous functions $(H_\alpha)_{\alpha\in\mathsf{Ord}}$, the diagonal supremum $\nabla_{\alpha\in\mathsf{Ord}} H_\alpha$ is continuous.

Proof.

(a) Obvious.

(b) Straightforward.

(c) Monotonicity is obvious. For continuity, let λ be a limit ordinal. Then

$$\begin{aligned}
(\nabla_{\alpha\in\mathsf{Ord}} H_\alpha)(\lambda) &= \bigvee_{\alpha<\lambda} H_\alpha(\lambda) \\
&= \bigvee_{\alpha<\lambda} \bigvee_{\beta<\lambda} H_\alpha(\beta) \\
&\leqslant \bigvee_{\alpha<\lambda} \bigvee_{\beta<\lambda} H_\alpha(\beta \vee \mathsf{S}\alpha) \\
&\leqslant \bigvee_{\gamma<\lambda} \bigvee_{\alpha<\gamma} H_\alpha(\gamma) \\
&= \bigvee_{\gamma<\lambda} (\nabla_{\alpha\in\mathsf{Ord}} H_\alpha)(\gamma)
\end{aligned}$$

□

Definition 14.4. Let C be a class.

(a) A *limit point* of C is a limit ordinal λ such that $\lambda \cap C$ is unbounded in λ. The class of all such is written $\mathsf{LimPt}(C)$.

(b) A class C is *closed* when it is LimPt-prefixed, i.e., contains every limit ordinal λ such that $\lambda \cap C$ is unbounded in λ.

Here are some ways to obtain closed classes:

Proposition 14.5.

(a) For any continuous function H, the class $\mathsf{Pref}(H)$ is closed.

(b) For any class C, the class $\mathsf{LimPt}(C)$ is closed.

(c) For a family of closed classes $(C_i)_{i\in I}$, the intersection $\bigcap_{i\in I} C_i$ is closed.

(d) For a sequence of closed classes $(C_\alpha)_{\alpha\in\mathsf{Ord}}$, the diagonal intersection $\Delta_{\alpha\in\mathsf{Ord}} C_\alpha$ is closed.

Proof.

(a) We must show that a limit point λ of $\mathsf{Pref}(H)$ is in $\mathsf{Pref}(H)$. For any $\gamma < \lambda$, there is $\beta \in [\gamma \mathinner{..} \lambda) \cap \mathsf{Pref}(H)$, giving $H(\gamma) \leqslant H(\beta) \leqslant \beta < \lambda$. So we have $H(\lambda) = \bigvee_{\gamma<\lambda} H(\gamma) \leqslant \lambda$ as required.

(b) We show that a limit point λ of $\mathsf{LimPt}(C)$ is a limit point of C. For any $\alpha < \lambda$, there is $\beta \in (\alpha \mathinner{..} \lambda) \cap \mathsf{LimPt}(C)$. Since $\beta \in \mathsf{LimPt}(C)$, there is $\gamma \in [\alpha \mathinner{..} \beta) \cap C$. Thus we have $\gamma \in [\alpha \mathinner{..} \lambda) \cap C$ as required.

(c) Since an infimum of prefixpoints is a prefixpoint.

(d) It suffices to show that $\mathsf{LimPt}(\Delta_{\alpha\in\mathsf{Ord}} C_\alpha) \subseteq \Delta_{\alpha\in\mathsf{Ord}} \mathsf{LimPt}(C_\alpha)$. This means that any limit point λ of $\Delta_{\alpha\in\mathsf{Ord}} C_\alpha$ is, for all $\alpha < \lambda$, a limit point of C_α. For any $\beta \in (\alpha \mathinner{..} \lambda)$, there is $\gamma \in [\beta \mathinner{..} \lambda) \cap \Delta_{\alpha\in\mathsf{Ord}} C_\alpha$. Since $\alpha < \beta \leqslant \gamma$, we have $\gamma \in C_\alpha$ as required. $\square$

Next we consider unbounded classes.

Definition 14.6.

(a) A *closure operator* (on Ord) is a function that is inflationary and idempotent.

(b) For an unbounded class C, we write H_C for the unique closure operator whose range is C. Explicitly, it sends α to the least element of C that is $\geqslant \alpha$.

Thus we have a bijection between the collection of all unbounded classes and that of all closure operators. Moreover it is *dual*—i.e., for unbounded classes C and D, we have $H_C \leqslant H_D$ iff $D \subseteq C$.

Proposition 14.7. *For any unbounded class C, the following are equivalent:*

- *C is closed.*
- *H_C is continuous.*

Proof. Firstly, H_C is continuous at every $\lambda \not\in \mathsf{LimPt}(C)$, since there is $\alpha < \lambda$ such that $[\alpha \mathinner{..} \lambda)$ has no element in C, so H_C sends every ordinal in this interval to $H_C(\lambda)$. As for $\lambda \in \mathsf{LimPt}(C)$, we have $\bigvee_{\gamma<\lambda} H_C(\gamma) = \lambda$, so H_C is continuous at λ iff $\lambda \in C$. $\square$

Thus we have a dual bijection between the collection of all closed unbounded classes, known as *club classes*, and that of all continuous closure operators. Here are some ways to obtain club classes:

Proposition 14.8. *Each of the following is equivalent to Infinity.*

(a) *For any continuous function H, the class $\mathsf{Pref}(H)$ is club.*

(b) *For any unbounded class C, the class $\mathsf{LimPt}(C)$ is club.*

(c) *For a family of club classes $(C_i)_{i\in I}$, the intersection $\bigcap_{i\in I} C_i$ is club.*

(d) *For a sequence of club classes $(C_\alpha)_{\alpha\in\mathsf{Ord}}$, the diagonal intersection $\Delta_{\alpha\in\mathsf{Ord}} C_\alpha$ is club.*

Proof. Firstly, if Infinity does not hold, then a club class is just an unbounded class of natural numbers, and the statements are all false:

(a) $\mathsf{Pref}(\mathsf{S})$ is empty.

(b) $\mathsf{LimPt}(\mathsf{Ord})$ is empty.

(c) Let C be the class of all even numbers and D that of all odd numbers. Each is club, but $C \cap D$ is empty.

(d) For each ordinal n, let C_n be the class of ordinals $\geqslant n+2$. Each of these is club, but $\Delta_{n \in \mathsf{Ord}} C_n = \{0\}$.

Now assume Infinity. Because of Proposition 14.5, we need only prove unboundedness.

(a) Let α be an ordinal. Form a strictly increasing sequence of ordinals $(x_n)_{n\in\mathbb{N}}$ via $x_0 \stackrel{\text{def}}{=} \alpha$ and $x_{n+1} \stackrel{\text{def}}{=} \mathsf{S}(x_n) \vee H(x_n)$. Its supremum λ is a limit ordinal and satisfies

$$\begin{aligned} H(\lambda) &= \bigvee_{\gamma<\lambda} H(\beta) \\ &= \bigvee_{n\in\mathbb{N}} H(x_n) \\ &\leqslant \bigvee_{n\in\mathbb{N}} x_{n+1} \text{ (since } H(x_n) \leqslant x_{n+1}) \\ &= \lambda \end{aligned}$$

so λ is an H-prefixpoint $\geqslant \alpha$.

(b) Let α be an ordinal. Form a strictly increasing sequence of ordinals $(x_n)_{n\in\mathbb{N}}$ via $x_0 \stackrel{\text{def}}{=} \alpha$ and $x_{n+1} \stackrel{\text{def}}{=}$ the least ordinal in C that is greater than x_n. Then $\bigvee_{n\in\mathbb{N}} x_n$ is in $\mathsf{LimPt}(C)$ and is $> \alpha$.

(c) Since

$$\begin{aligned} \bigcap_{i\in I} C_i &= \bigcap_{i\in I} \mathsf{Pref}(H_{C_i}) \\ &= \mathsf{Pref}(\bigvee_{i\in I} H_{C_i}) \end{aligned}$$

which is club by part (a), using Proposition 14.3(b) .

(d) Similar. ☐

Proposition 14.9. *For a class of limit ordinals D, the following are equivalent.*

(a) D is stationary.

(b) Every continuous function has a prefixpoint in D.

(c) Every club class has an element in D.

Proof. To show that (a) implies (b), let H be a continuous function. Then any H-based limit ordinal is H-prefixed. To show the converse, let G be a function. Then the function $H \stackrel{\text{def}}{=} \nabla_{\alpha\in\mathsf{Ord}} \mathsf{const}_{G(\alpha)}$ is continuous by Proposition 14.3, and a limit ordinal is G-based iff it is H-prefixed.

To show that (b) implies (c): for any club class C, we have $C = \mathsf{Pref}(H_C)$ and H_C is continuous. To show the converse, we note that the club class Ord has an element in $D \subseteq \mathsf{Lim}$. So Infinity holds and we can apply Proposition 14.8(a). ☐

Corollary 14.10. *Let D be a stationary class of limit ordinals. The intersection of D with any club class is stationary.*

14.2 Application: Iterated Inaccessibility

In order to formulate iterated inaccessibility, we use the following result.

Proposition 14.11. *Let D be a class of limit ordinals.*

(a) There is a sequence of classes $(X_\alpha)_{\alpha \in \mathsf{Ord}}$ and class X_∞ uniquely specified by

$$\begin{aligned} X_\alpha &= D \cap \bigcap_{\beta<\alpha} \mathsf{LimPt}(X_\beta) \\ X_\infty &= D \cap \Delta_{\beta \in \mathsf{Ord}} \mathsf{LimPt}(X_\beta) \end{aligned}$$

(b) If D is stationary, then so are all these classes.

Proof.

(a) We cannot define a sequence of classes recursively, so we proceed as follows. For any ordinal ρ, we recursively define a sequence $(X^\rho_\alpha)_{\beta \in \mathsf{Ord}}$ of subsets of ρ via

$$X^\rho_\alpha = \rho \cap D \cap \bigcap_{\beta<\alpha} \mathsf{LimPt}(X^\rho_\beta)$$

These sequences are compatible in the sense that, for $\rho \leqslant \sigma$, we have $X^\rho_\alpha = \rho \cap X^\sigma_\alpha$. We define

$$X_\alpha \stackrel{\text{def}}{=} \{\rho \in \mathsf{Ord} \mid \rho \in X^{\mathsf{S}\rho}_\alpha\}$$

and obtain the required properties.

(b) Firstly, since D is stationary, Infinity holds and we can use Proposition 14.8.

We cannot simply prove X_α stationary by induction on α, as stationarity involves second-order quantification. Instead, we prove unboundedness of X_α by induction on α, as follows. For all $\beta < \alpha$, the class X_β is unbounded, so $\mathsf{LimPt}(X_\beta)$ is club. So $\bigcap_{\beta<\alpha} X_\beta$ is club, making X_α stationary and hence unbounded. This completes the induction.

Next, for any ordinal α, we see (again) that $\bigcap_{\beta<\alpha} \mathsf{LimPt}(X_\beta)$ is club, making X_α stationary. Likewise $\nabla_{\beta \in \mathsf{Ord}} \mathsf{LimPt}(X_\beta)$ is club, making X_∞ stationary. ☐

For an application **assuming Powerset** $+$ **Infinity** $+$ **AC**, let D be the class of all inaccessible cardinals. Then X_α is the class of all α-*inaccessible* cardinals, and X_∞ the class of all *hyper-inaccessible* cardinals. Proposition 14.11(b) tells us that these classes are stationary if Broad Infinity holds.

This construction can be further iterated, giving hyper-hyper-inaccessible cardinals and more. In Carmody's work [Car17], this is achieved by generalizing the subscripts used in Proposition 14.11 to a system of "meta-ordinals".

Related work A standard treatment of stationarity is given in [Jec03], not for classes but for subsets of a given limit ordinal. So the predicativity issue does not arise, and additional results are obtained, such as Fodor's pressing-down lemma and Solovay's partitioning theorem. See [GHK21] for an analysis of whether Fodor's lemma applies to classes.

Part V
Wrapping up

15 Conclusions

15.1 Summary of Achievements

We have now established all the relationships in Figure 1. The main technical achievement was proving the equivalence (assuming Powerset + AC) of Simple Broad Infinity and Mahlo's principle. The centre-piece is the implication Full Broad Infinity $\Rightarrow$ Broad Derivation Set, which relies on Proposition 4.18 to generate the $\mathcal{R}$-derivational class-family.

On the philosophical side, I claim that the notion of F-broad number (for a broad arity F) is easily grasped, making Simple Broad Infinity a plausible axiom scheme. This is for the reader to judge.

On the practical side, we have seen several equivalent principles that are convenient for applications. Specifically:

- Broad Derivation Set yields the existence of Tarski-style universes.
- Broad Set Generation yields the existence of Grothendieck universes.
- Mahlo's principle in the form "Every club class contains a regular limit ordinal" yields the existence of α-inaccessibles and hyper-inaccessibles.

As promised in Section 2.5, we have developed our results in a setting that allows urelements and non-well-founded membership, proved the sufficiency of (a version of) WISC for our main AC-reliant results, and seen the pattern of resemblance between Wide and Broad principles throughout the paper.

15.2 Further Work

Beyond the above contributions, more work remains to be done. Firstly, there are unanswered questions, particularly about the power of Broad ZF.

1. By analogy with Gitik's work [Git80], can it be shown, under some consistency hypothesis for large cardinals, that Broad ZF does not prove the existence of an uncountable regular limit ordinal?
2. Does Broad ZF + Blass's axiom prove Mahlo's principle?
3. Jech [Jec82] showed in ZF that the class of all hereditarily countable sets is a set, and his result has been extended to other cardinalities [Die92, Hol14]. Can a stronger version be proved in Broad ZF? For example, given a broad arity $F : \mathfrak{T} \to \mathsf{Set}$, let $H(F)$ denote the least class X that contains Begin and, for any $x \in X$ and $y \in X^{F(x)}$, contains $\mathsf{Make}(x, \mathsf{Range}(y))$. This exists by Proposition 4.13(a). Does Broad ZF prove, for every broad arity F, that $H(F)$ is a set?

Everything in this paper has been done in a base theory that—like ZF—uses classical first-order logic and ignores logical complexity. But some other versions of set theory use intuitionistic logic and/or restrict the use of logically complex sentences [Cro20, Mat01]. The task of adapting our results to such theories (as far as possible) is left to future work.

Lastly, the link between type-theoretic work on induction-recursion [DS06, GH16, GMNFS17] and the principles in this paper remains to be developed.

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Errata

In this version, the following corrections have been made.

- In Section 1.3, an inaccessible is an uncountable F-closed *regular* limit ordinal.
- In the proof of Proposition 4.21(b), the extra parenthesis in $\Delta^{L}_{(\mathfrak{T},\in))}$ has been removed.
- In the proof of Proposition 8.19(b), the formula $\mathsf{Deriv}_{\mathcal{R}}$ has been replaced by $\mathsf{Deriv}_{\mathcal{S}}$. Also, in the third subitem, the index j has been replaced by i, and the set L_j by K_i.
- In the proof of Proposition 11.4(a)($\Rightarrow$) and 11.4(b)($\Rightarrow$), in the parenthetical remark of the first item, "set" has been replaced by "set of ordinals" for clarity.