Oles Embeddings
(work in progress)

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Background

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It's connected to numerous structures in the semantics of effects:

- Power and Plotkin's lookup-update algebras
- Power and Plotkin's and Melliès' redundancy theorems for lookup equations
- Power and Shkaravska's account of arrays as comodels
- Hyland, Plotkin and Power's combination of a functor and a monad
- My account and Mossakowski and Schröder's account of monads supporting exception handling
- Hermida and Tennent's account of monoidal indeterminates
- Johnson et al's account of lenses.

Three levels of generality

1. Oles embedding in a category
2. Oles embedding across an action
3. Base for a monad

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Most of the talk will be about (1).

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections.

Complementors

The complementor of an injection $f : A \rightarrowtail B$
is the function $f^{\mathsf{c}} : B \to B + A$ sending

- $f(a) \mapsto \mathsf{inr}\ a$
- $b \mapsto \mathsf{inl}\ b$ if $b \notin \mathrm{range}(f)$.

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We then have the equations:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\mathsf{inr}"'] & B \arrow[d, "f^{\mathsf{c}}"] \\ & B + A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{\mathsf{c}}"] \arrow[dr, "\mathsf{id}"'] & B + A \arrow[d, "{[\mathsf{id}, f]}"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{\mathsf{c}}"] \arrow[d, "f^{\mathsf{c}}"'] & B + A \arrow[d, "f^{\mathsf{c}} + A"] \\ B + A \arrow[r, "\mathsf{inl} + A"'] & (B + A) + A \end{tikzcd}$$

Basic definition

Let $\mathcal{C}$ be a category with binary coproducts and initial object.
We form a category $\mathrm{Oles}(\mathcal{C})$ with the same objects as $\mathcal{C}$.

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An Oles embedding $f : A \rightarrowtail B$ consists of

- an map $f^{\mathsf{i}} : A \to B$ (the injection)
- a map $f^{\mathsf{c}} : B \to B + A$ (the complementor)

satisfying the equations:

$$\begin{tikzcd} A \arrow[r, "f^{\mathsf{i}}"] \arrow[dr, "\mathsf{inr}"'] & B \arrow[d, "f^{\mathsf{c}}"'] \\ & B + A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{\mathsf{c}}"] \arrow[dr, "\mathsf{id}"'] & B + A \arrow[d, "{[\mathsf{id}, f^{\mathsf{i}}]}"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{\mathsf{c}}"] \arrow[d, "f^{\mathsf{c}}"'] & B + A \arrow[d, "f^{\mathsf{c}} + A"] \\ B + A \arrow[r, "\mathsf{inl} + A"'] & (B + A) + A \end{tikzcd}$$

In **Set**, f^{i} is an injection and f^{c} is its complementor.

Alternative: don't mention the injector

An Oles embedding $A \rightarrowtail B$ can be described as
a map $g \ : \ B \to B + A$ such that

$$B \xrightarrow{g} B + A \underset{\mathsf{inl}+A}{\overset{g+A}{\rightrightarrows}} (B + A) + A$$

is an equalizer.

Making a category

The identity on A has injector id_A and complementor

$$\mathsf{inr} \; : \; A \to A + A$$

The composite of $f \; : \; A \rightarrowtail B$ and $g \; : \; B \rightarrowtail C$ has injector

$$A \xrightarrow{f^{\mathsf{i}}} B \xrightarrow{g^{\mathsf{i}}} C$$

and complementor

$$C \xrightarrow{g^{\mathsf{c}}} C + B \xrightarrow{C+f^{\mathsf{c}}} C + (B + A) \xrightarrow{[\mathsf{inl}, g^{\mathsf{i}}+A]} C + A$$

Basic properties

- $(\mathrm{Oles}(\mathcal{C}), 0, +)$ is a symmetric monoidal category.
- Its groupoid of isomorphisms is the same as that of $\mathcal{C}$.
- 0 is a strict initial object. (Strict means: any morphism to it is an isomorphism.)

Oles embeddings in extensive categories

In an extensive category:

- The forgetful functor $-^{\mathsf{i}} : \mathrm{Oles}\mathcal{C} \to \mathcal{C}$ is faithful, so complementors are redundant. (Not true in $\mathbf{Set}^{\mathsf{op}}$)
- Injectors are monic. (Not true in $\mathbf{Set}^{\mathsf{op}}$)

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In an extensive category with 1, an Oles embedding $A \rightarrowtail B$ can be described as

- a map $\chi_f : B \to 1 + 1$ (the characteristic map)
- and an isomorphism $A \cong \chi_f^* \top$.

From coproduct embeddings to Oles embeddings

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These give a symmetrical monoidal bicategory with the same objects as $\mathcal{C}$.

A 2-cell from (X, α) to (Y, β) is $h : X \to Y$ such that

$$\begin{tikzcd} X + A \arrow[r, "\alpha"] \arrow[d, "h+A"'] & B \\ Y + A \arrow[ur, "\beta"'] & \end{tikzcd}$$

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Every coproduct embedding gives rise to an Oles embedding,
so there's a symmetric monoidal functor
from the bicategory of coproduct embeddings
to $\mathrm{Oles}(\mathcal{C})$.

Complements of an Oles embedding

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Does every Oles embedding have

- a complement? Not necessarily
- an essentially unique complement? If $\mathcal{C}$ is extensive.
- a terminal complement? If $\mathcal{C}$ has equalizers preserved by $- + X$
- an initial complement? Not necessarily

The dual story: Oles expansions

Let $\mathcal{C}$ have binary products and a terminal object.

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Let $\mathcal{C}$ have binary products and a terminal object.

An Oles expansion $A \to B$ is an Oles embedding in $\mathcal{C}^{\mathrm{op}}$

- a morphism $p : B \to A$ (the projection)
- a morphism $\bullet : B \times A \to B$ (the overwriter)

satisfying

$$\begin{array}{lrcl} \forall b \in B, a \in A. & p(b \bullet a) & = & a \\ \forall b \in B. & b \bullet p(b) & = & b \\ \forall b \in B, a, a,' \in A. & (b \bullet a) \bullet a' & = & b \bullet a' \end{array}$$

Also called a very well-behaved lens.

Oles expansions in **Set**

In **Set**, an Oles expansion $A \to B$ can be described as

- a map $p : B \to A$ (the projection)
- an equivalence relation $\sim$ on B

such that for every $b \in B$ and $a \in A$
there is unique $c \in B$ such that $c \sim b$ and $p(c) = a$.

Quotients of an Oles expansion

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A product expansion $A \rightarrow B$ consists of an object X and $X \times A \cong B$. We take 2-cells to be $\mathcal{C}$-morphisms.

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Oles proved: in **Set**, every expansion has an initial quotient.

Disjoint embeddings

A pair of Oles embeddings
$$\begin{tikzcd} B \arrow[dr, "f"'] & & C \arrow[dl, "g"] \\ & D & \end{tikzcd}$$
is disjoint when

$$\begin{tikzcd} D \arrow[rr, "f^{c}"] \arrow[d, "g^{c}"'] & & D+B \arrow[d, "g^{c}+B"] \\ D+C \arrow[r, "f^{c}+C"'] & (D+B)+C \arrow[r, "\cong"'] & (D+C)+B \end{tikzcd}$$

In **Set**, this just says the ranges are disjoint.

Oles intersection square

A square of Oles embeddings
$$\begin{tikzcd} A \arrow[r, tail, "h"] \arrow[d, tail, "k"'] & C \arrow[d, tail, "g"] \\ B \arrow[r, tail, "f"'] & D \end{tikzcd}$$
is an Oles intersection square

when

$$\begin{tikzcd} D \arrow[rrr, "f^{c}"] \arrow[d, "g^{c}"'] & & & D+B \arrow[d, "g^{c}+k^{c}"] \\ D+C \arrow[r, "f^{c}+h^{c}"'] & (D+B)+(C+A) \arrow[rr, "\cong"'] & & (D+C)+(B+A) \end{tikzcd}$$

In **Set**, this is just an intersection square.

The case $A = 0$ says that f and g are disjoint.

Are they pullbacks?

An Oles intersection square
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It's also a pullback in $\mathrm{Oles}(\mathcal{C})$, provided $- + Y$ preserves pullbacks.

Oles intersection squares: basic properties

- The identity square $\begin{tikzcd} A \arrow[r, tail, "\mathrm{id}"] \arrow[d, tail, "f"'] & A \arrow[d, tail, "f"] \\ B \arrow[r, tail, "\mathrm{id}"'] & B \end{tikzcd}$ is an Oles intersection.
- Closed under composition.
- Closed under transpose.
- Closed under $+$.
- The square $\begin{tikzcd} A \arrow[r, tail, "\mathrm{id}"] \arrow[d, tail, "\mathrm{id}"'] & A \arrow[d, tail, "f"] \\ A \arrow[r, tail, "f"'] & B \end{tikzcd}$ is an Oles intersection.

Oles intersection diagrams

Let I be a finite meet semilattice.
An I-shaped Oles intersection diagram is a functor $I \to \mathrm{Oles}\mathcal{C}$ where

$$\begin{tikzcd} A_{j \wedge k} \arrow[r, tail] \arrow[d, tail] & A_j \arrow[d, tail] \\ A_k \arrow[r, tail] & A_i \end{tikzcd}$$

is an Oles intersection square for every $j, k \leqslant i$.

Alternative formulation

A coalgebra for the comonad L on $[I, \mathbf{Set}]$.
$(LA)_i = \sum_{j \leqslant i} A_j$.

Properties of disjointness

The cocone
$$\begin{tikzcd}[column sep=small] A_0 \arrow[drr, "\mathsf{in}_0"'] & & \cdots & & A_{n-1} \arrow[dll, "\mathsf{in}_{n-1}"] \\ & & \sum_{i<n} A_i & & \end{tikzcd}$$
is pairwise disjoint.

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is pairwise disjoint.

For any pairwise disjoint cocone
$$\begin{tikzcd} A_0 \arrow[dr, "g_0"'] & \cdots & A_{n-1} \arrow[dl, "g_{n-1}"] \\ & B & \end{tikzcd}$$
there's a unique Oles embedding $\sum_{i<n} A_i \to B$ that's a morphism of cocones.

Covering intersection squares

Given two Oles intersection squares

$$\begin{tikzcd}
A \arrow[r, tail, "h"] \arrow[d, tail, "k"'] & B \arrow[d, tail, "g"'] \arrow[ddr, tail, "g'"] & \\
C \arrow[r, tail, "f"] \arrow[drr, tail, "f'"'] & D & \\
& & E
\end{tikzcd}$$

if the inner one is covering then there is a unique Oles embedding $D \rightarrowtail E$ that's a morphism of cocones.

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This may be generalized to other diagram shapes.

Base for a monad

A monad T on a category $\mathcal{D}$ gives a comonad $F^T U^T$ on the Eilenberg-Moore category $\mathcal{C}^T$.

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A coalgebra for this comonad is called a T-base. Lack, Taylor, Jacobs ...

This consists of an object P and maps $\theta : TP \to P$ and $\phi : P \to TP$, satisfying 5 equations, of which 2 are redundant.

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This consists of an object P and maps $\theta : TP \to P$ and $\phi : P \to TP$, satisfying 5 equations, of which 2 are redundant.

In fact θ is redundant: a T-base can be described as $\phi : P \to TP$ such that

$$P \xrightarrow{\phi} TP \underset{T\eta_P}{\overset{T\phi}{\rightrightarrows}} T^2 P$$

is an equalizer.

Monoidal actions

A monoidal action of a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ on a category $\mathcal{D}$ is a map $\oslash\ :\ \mathcal{D} \times \mathcal{C} \longrightarrow \mathcal{D}$ and isomorphisms

$$\begin{aligned} P \oslash (B \otimes C) &\cong (P \oslash B) \oslash C \\ P \oslash I &\cong P \end{aligned}$$

satisfying the pentagon and the triangle.

Oles embedding across a monoidal action

Suppose $\mathcal{C}$ has binary coproducts and an initial object,
and acts monoidally on $\mathcal{D}$.

Any A in $\mathcal{C}$ gives a monad $P \mapsto P \oslash A$ on $\mathcal{D}$.

A base structure on P for this monad is called an Oles embedding $A \rightarrowtail P$.

Oles embedding across a monoidal action

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We can compose Oles embeddings

$$A \rightarrowtail B \rightarrowtail P$$

and speak of disjoint embeddings and intersection squares into P.

Examples: category acting on itself

$\mathcal{C}$ acts monoidally on itself.

This gives Oles embeddings in $\mathcal{C}$.

Lookup/update algebras

$\mathbf{Set}^{op}$ acts monoidally on $\mathbf{Set}$ via exponentiation.

An Oles embedding from $S \rightarrowtail P$ is a lookup/update algebra structure on P. Plotkin and Power

A lookup function $P^S \longrightarrow P$ and an update function $P \longrightarrow P^S$ satisfying 5 equations.

Think: P is the set of computations of a given type.

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Disjoint embedding indicate lookup/update for separate cells.

Handling exceptions and reading

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Disjoint embeddings indicate that the effect handling is independent.

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An Oles embedding $H_\Sigma \rightarrowtail \mathbf{T}$ says how $\mathbf{T}$ models effect handling for I/O.

Summary

In a category

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- Covering Oles intersections are initial.

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In a category

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Oles embeddings across an action includes many structures in the semantics of state, exceptions and I/O.

Disjoint embeddings indicate that effects are treated independently.