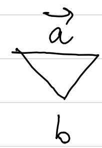
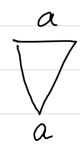
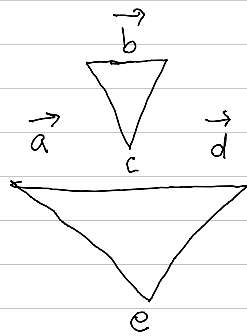
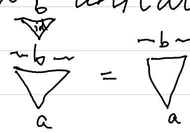
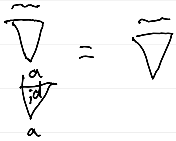
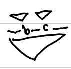
A multicategory $\mathcal{C}$ is

- a set of objects
- for each object list $\vec{a}$ and object b, a set of morphisms  written $\mathcal{C}(\vec{a}; b)$
- for each object a, an identity 
- composition 

(upper & lower in our pictures)

satisfying four equations: left & right unitality  

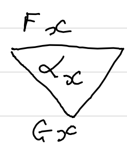
 associativity

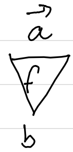
 interchange

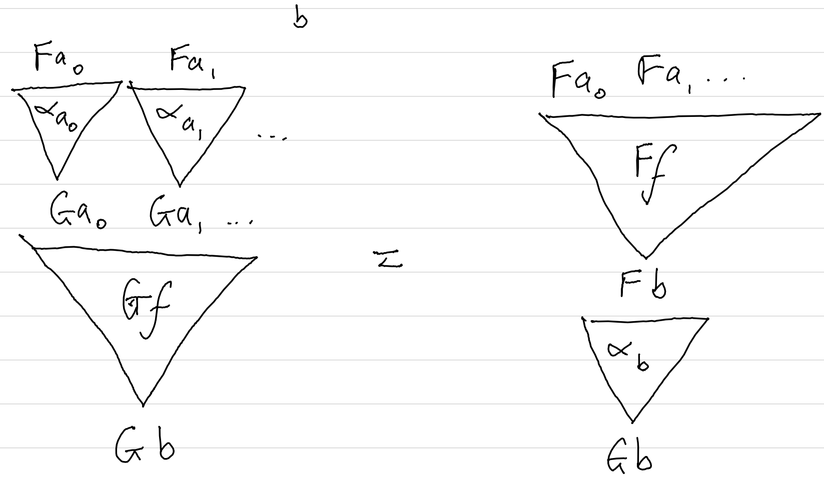
Multicategories form a 2-category (ahem, size)

Multifunctor $F : \mathcal{C} \to \mathcal{D}$ is obvious

Given $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

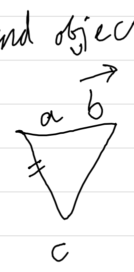
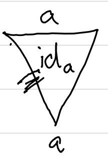
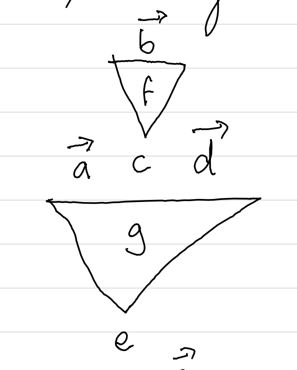
sends each $x \in \mathcal{C}$ to unary 

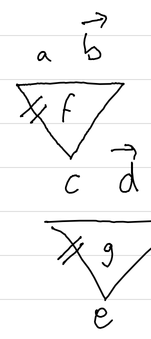
s.t. for any $\mathcal{C}$-morphism  we have

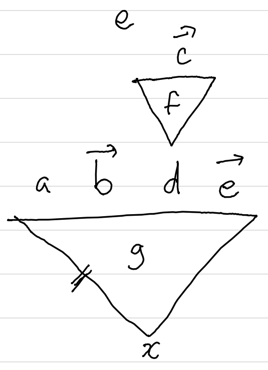


Left-skew multicategory $\mathcal{C}$ (Bourke & Lack) is

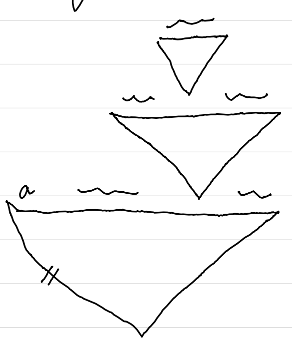
nlab post by Will Offord

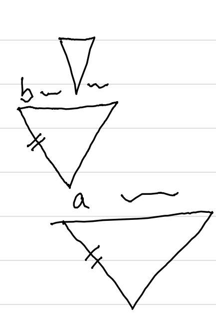
- a set of objects
- for each object list $\vec{a}$ and object b, a set of loose morphisms  written $\mathcal{C}(\vec{a}, b)$
- for each object a, object list $\vec{b}$ and object c a set of tight morphisms  written $\mathcal{C}(\lfloor a, \vec{b}\,; c)$
- a loosening operation $\mathcal{C}(\lfloor a, \vec{b}\,; c) \to \mathcal{C}(a, \vec{b}\,; c)$, $f \mapsto \acute{f}$
- for each object a, a tight identity 
- loose composition 

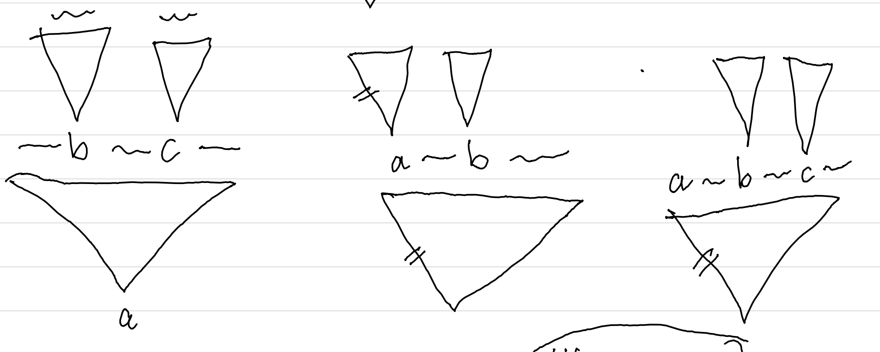
- tight composition 

- mixed composition 

satisfying

- mixed composition commutes with loosening
- tight composition commutes with loosening
- left unitality for each kind of composition
- right unitality for loose & tight composition
- associativity for loose
- associativity for tight
- associativity 

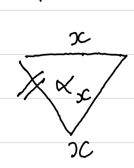
- associativity 



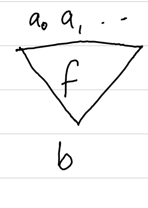
14 equations

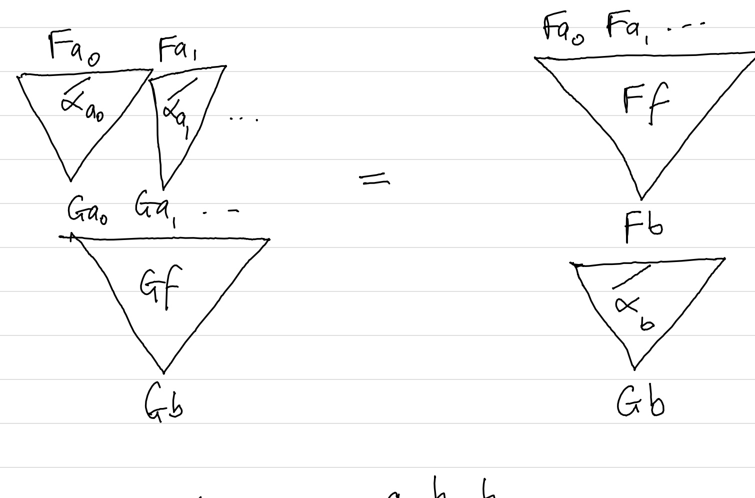
These form a 2-category.

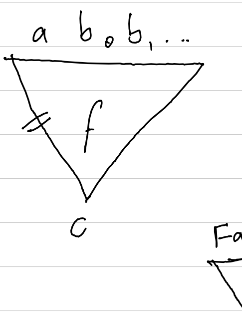
Given $F, G : \mathcal{C} \longrightarrow \mathcal{D}$, a natural transf. $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$

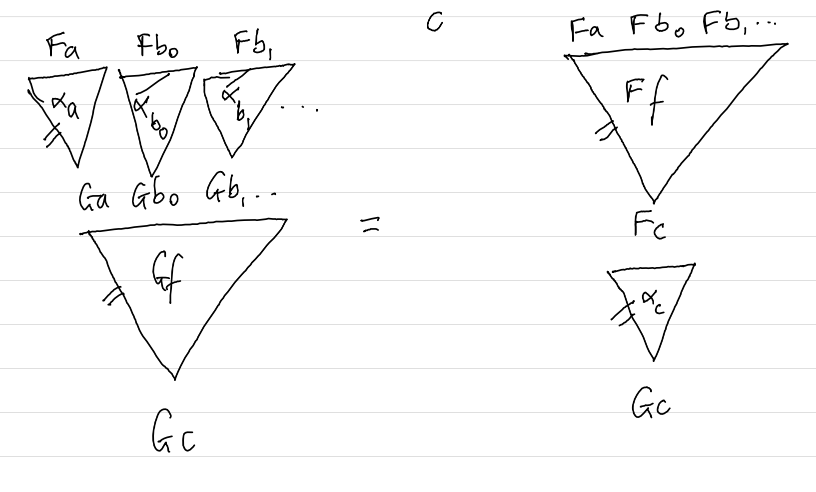
Sends each $\mathcal{C}$-object x to a tight unary morphism 

such that

- for each loose morphism 

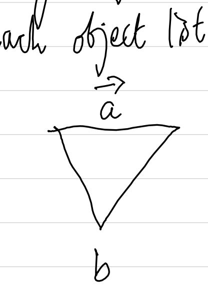
we have 

- for each tight morphism 

we have 

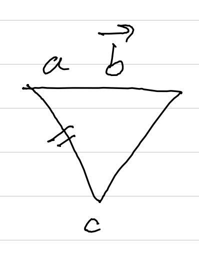
A biskew multicategory $\mathcal{C}$ consists of

- a set of objects $\mathcal{C}$
- for each object list $\vec{a}$ and object b, a set of loose-loose morphisms



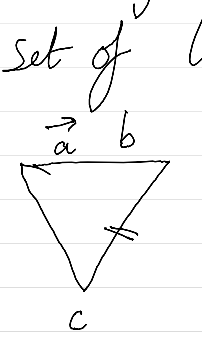
written $\mathcal{C}(\vec{a}\,;b)$

- for each object a, object list $\vec{b}$ and object c, a set of tight-loose morphisms



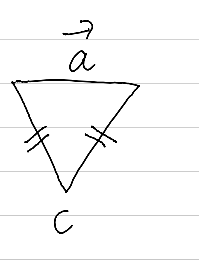
written $\mathcal{C}(\lfloor a, \vec{b}\,;c)$

- for each object list $\vec{a}$, object b and object c, a set of loose-tight morphisms



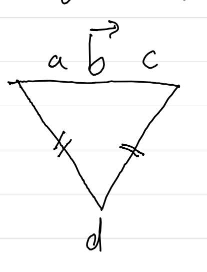
written $\mathcal{C}(\vec{a}, b\rfloor\,;c)$

- for each nonempty object list $\vec{a}$ and object b, a set of tight-tight morphisms

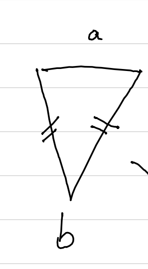


written $\mathcal{C}(\lfloor\vec{a}\rfloor\,;b)$

Note that there are two cases:



and



omitted in the CT 2019 slides

- left-loosening operations

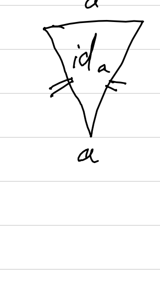
$$\mathcal{C}(\lfloor\vec{a}\,;b) \longrightarrow \mathcal{C}(\vec{a}\,;b)$$
$$\mathcal{C}(\lfloor\vec{a}\rfloor;b) \longrightarrow \mathcal{C}(\vec{a}\rfloor;b) \qquad f \mapsto \acute{f}$$

- right-loosening operations

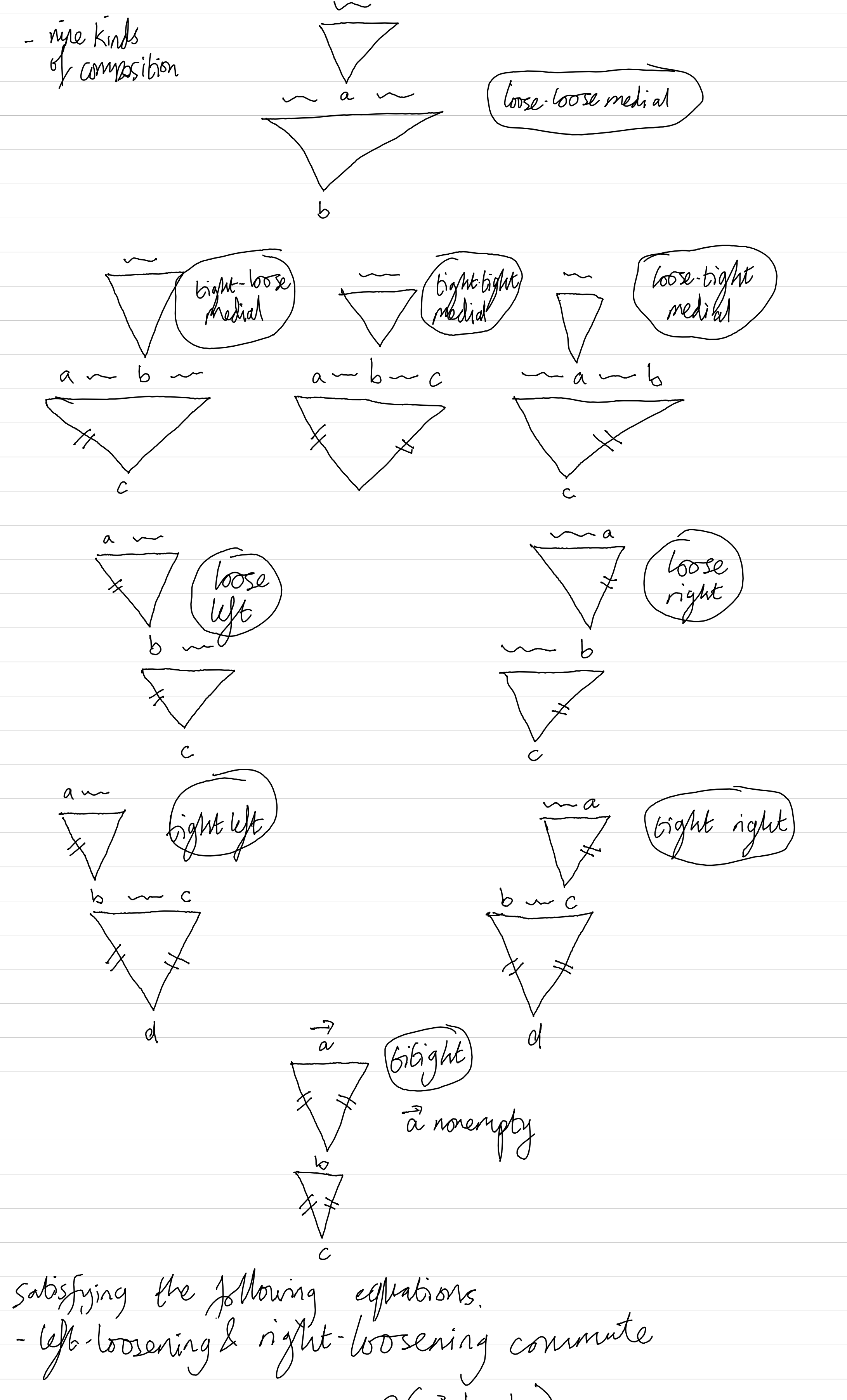
$$\mathcal{C}(\vec{a}\rfloor;b) \longrightarrow \mathcal{C}(\vec{a}\,;b)$$
$$\mathcal{C}(\lfloor\vec{a}\rfloor;b) \longrightarrow \mathcal{C}(\lfloor\vec{a}\,;b) \qquad f \mapsto \grave{f}$$

($\vec{a}$ is nonempty in the above)

- for each object a, a tight-tight identity



- nine kinds of composition



satisfying the following equations.

- left-loosening & right-loosening commute

$$\begin{tikzcd}
 & \mathcal{C}(\vec{a}\rfloor;b) \arrow[dr] & \\
\mathcal{C}(\lfloor\vec{a}\rfloor;b) \arrow[ur] \arrow[dr] & & \mathcal{C}(\vec{a}\,;b) \\
 & \mathcal{C}(\lfloor\vec{a}\,;b) \arrow[ur] &
\end{tikzcd}$$

written $f \mapsto \hat{f}$ $\qquad$ $\vec{a}$ nonempty

- each kind of composition resulting in a tight-loose or tight-tight morphism

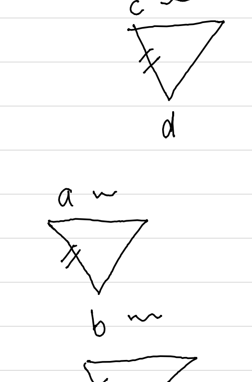
viz. $\left\{\begin{array}{l}\text{tight-loose medial}\\ \text{tight-tight medial}\\ \text{loose left}\\ \text{tight left}\\ \text{bitight}\end{array}\right.$

commutes with left-loosening

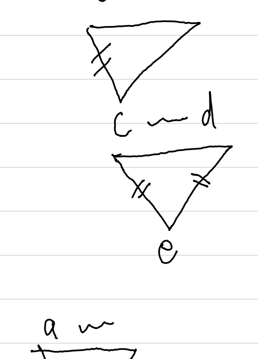
- each kind of composition resulting in a loose-tight or tight-tight morphism (5 kinds) commutes with right-loosening
- each kind of composition is left-unital
- each kind of morphism is right-unital
- sixteen associativity laws:

nine consisting of medial composition above and any kind of composition below

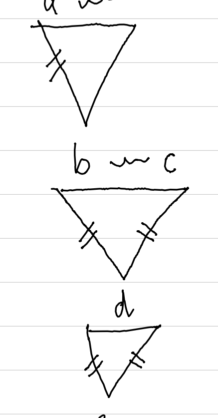
seven more:



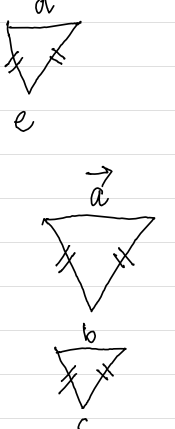
and mirror image



and mirror image

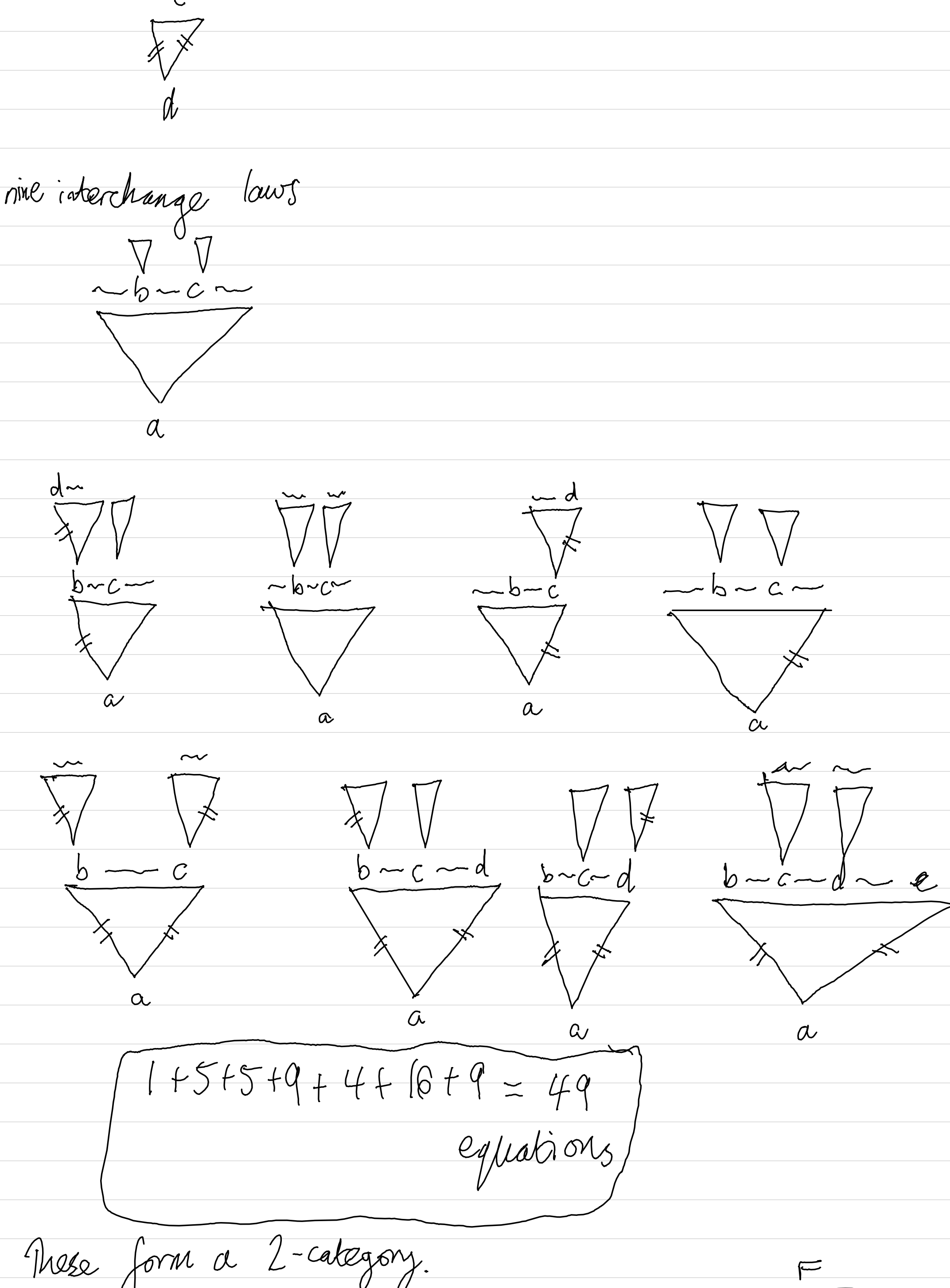


and mirror image



$\vec{a}$ nonempty

nine interchange laws



$$\boxed{\begin{array}{r}1+5+5+9+4+16+9 = 49\\ \text{equations}\end{array}}$$

These form a 2-category.

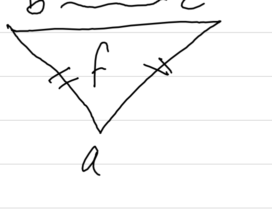
Given $F, G : \mathcal{C} \to \mathcal{D}$, a natural trans

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}
\end{tikzcd}$$

sends each $x \in \mathcal{C}$ to a tight-tight identity

agreeing with $\mathcal{C}$-morphisms of each kind.

The tight-tight kind splits into two cases



and

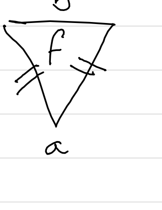


Diagram of 2-categories

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
& \text{Multicat} \arrow[dl, bend right, "\text{all left-tight}"'] \arrow[dr, bend left, "\text{all right-tight}"] \arrow[dd, shift right=2, "\text{all tight}"'] & \\
\text{Left-skew Multicat} \arrow[ur, bend right=15, "\text{remove left-tight}"'] \arrow[dr, bend right, "\text{all right-tight}"'] \arrow[r, phantom, "\perp"] & & \text{Right-skew multicat} \arrow[ul, bend left=15, "\text{remove right-tight}"] \arrow[dl, bend left, "\text{all left-tight}"] \arrow[l, phantom, "\perp"] \\
& \text{Biskew multicat} \arrow[ul, bend left=15, "\text{remove right-tight}"] \arrow[ur, bend right=15, "\text{remove left-tight}"'] \arrow[uu, shift right=2, "\text{remove tight}"'] &
\end{tikzcd}
$$

Diagram commutes

Counits are identities

- Given a multicategory $\mathcal{C}$

$$\text{All left-tight}(\mathcal{C})(\vec{a}\,;b) \overset{def}{=} \mathcal{C}(\vec{a}\,;b)$$

$$\text{All left-tight}(\mathcal{C})([a,\vec{b}\,;c) \overset{def}{=} \mathcal{C}(a,\vec{b}\,;c)$$

Loosening is identity.
Identity on a is id_a

- Given a left skew-multicategory $\mathcal{D}$

$$\text{Remove left-tight}(\mathcal{D})(\vec{a}\,;b) \overset{def}{=} \mathcal{D}(\vec{a}\,;b)$$

Identity on a is id_a

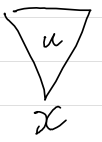
- Given a left-skew multicategory $\mathcal{D}$ and category $\mathcal{C}$ we have

$$\text{Multicat}\left(\begin{matrix}\text{Remove}\\ \text{left-tight}\end{matrix}(\mathcal{C}),\ \mathcal{D}\right) \cong \begin{matrix}\text{Left-skew}\\ \text{multicat}\end{matrix}\left(\mathcal{C},\ \begin{matrix}\text{All}\\ \text{left-tight}\end{matrix}(\mathcal{D})\right)$$

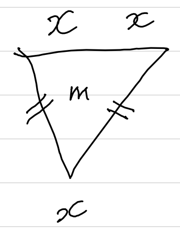
In a biskew multicategory $\mathcal{C}$,

a **monoid** consists of

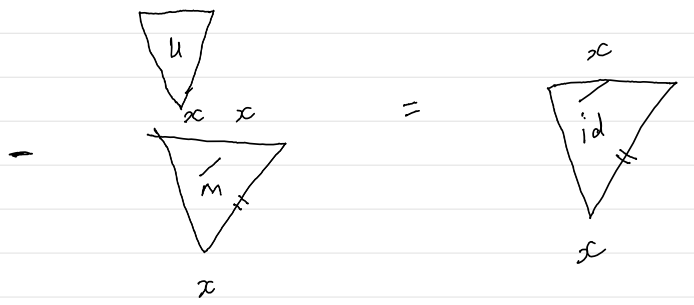
- an object x
- a loose-loose morphism

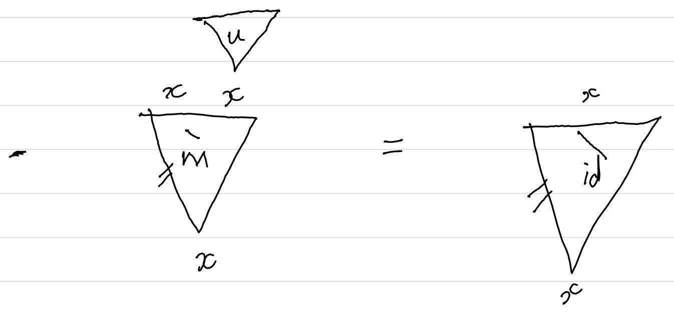


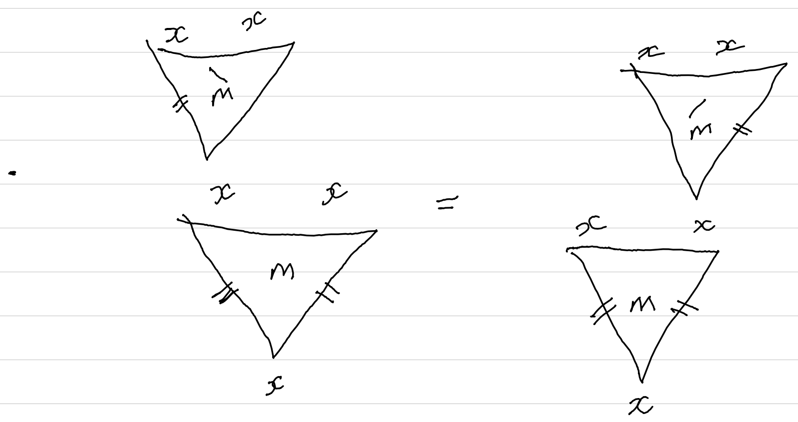
- a tight-tight morphism



satisfying equations:

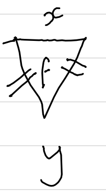




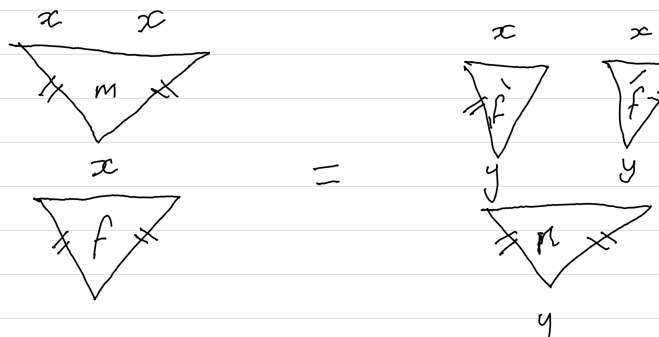


A homomorphism $(x, u, m) \longrightarrow (y, v, n)$

is a tight-tight morphism



such that



Hence we obtain a notion of monoid & homomorphism

in a multicategory

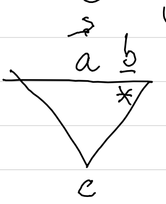
left-skew multicat

right-skew multicat

Let $\mathcal{C}$ be a multicategory

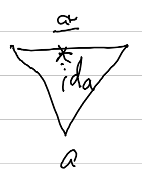
A left $\mathcal{C}$-multiactegory $\mathcal{D}$ consists of

- a set of objects
- for each $\mathcal{C}$-object list $\vec{a}$ and $\mathcal{D}$-objects $\underline{b}$ and $\underline{c}$, a set of morphisms

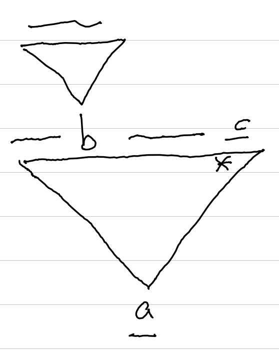


written $\mathcal{D}(\vec{a}, \underset{*}{\underline{b}}; \underline{c})$

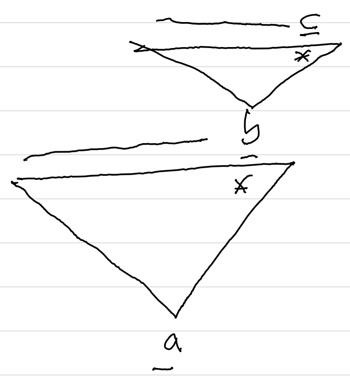
- for each $\mathcal{D}$-object $\underline{a}$, an identity morphism



- two kinds of composition



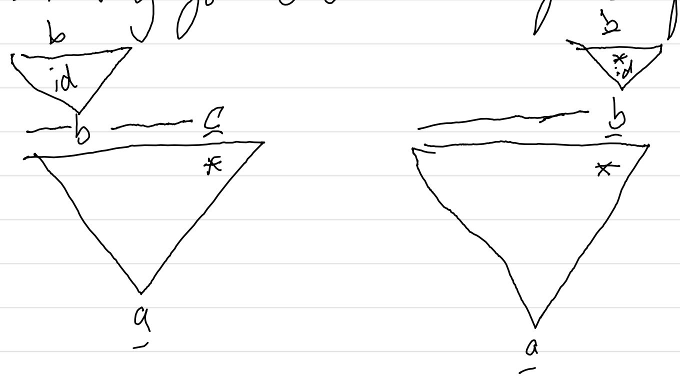
outer



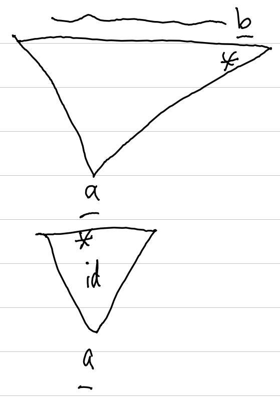
inner

satisfying the following equations:

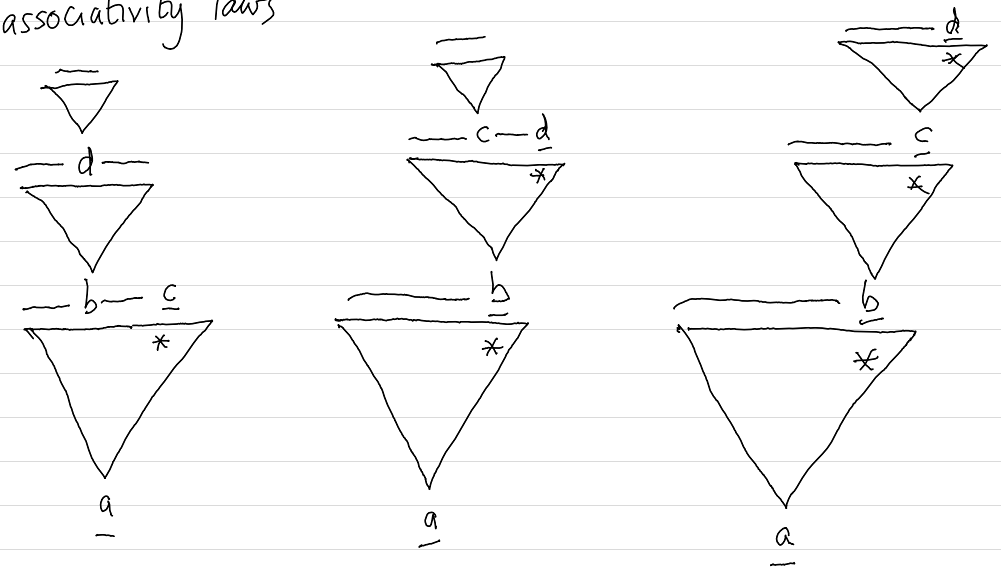
left-identity for each kind of composition



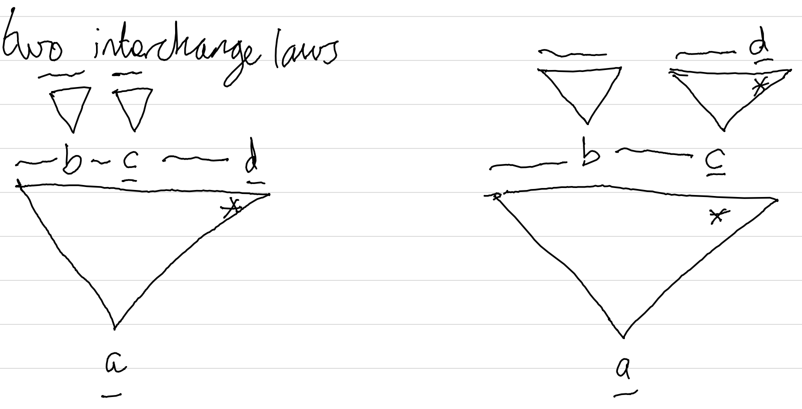
right identity



three associativity laws



two interchange laws



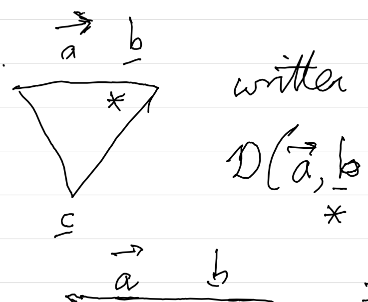
8 equations

Let $\mathcal{C}$ be a left-skew multicategory.

A left-skew $\mathcal{C}$-multiactegory $\mathcal{D}$ is

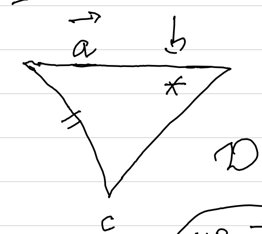
- a set of objects
- for each $\mathcal{C}$-object list $\vec{a}$ and $\mathcal{D}$-objects $\underline{b}$, $\underline{c}$

a set of loose morphisms



written $\mathcal{D}(\vec{a}, \underset{*}{\underline{b}}; \underline{c})$

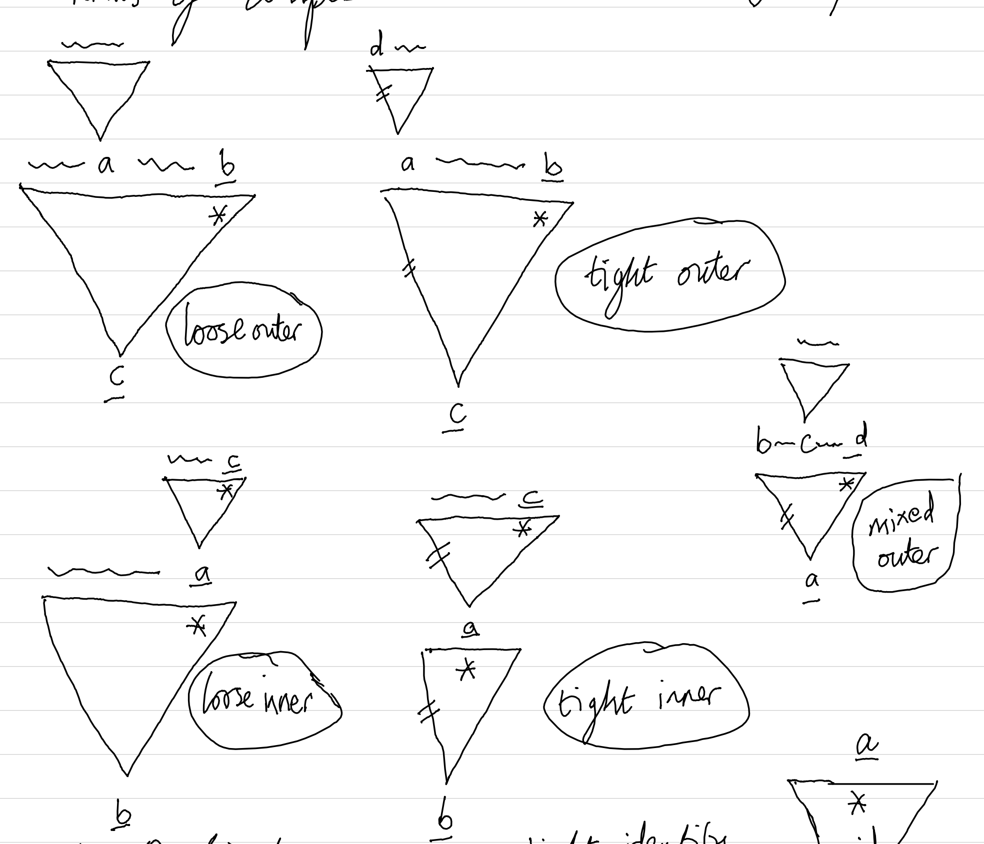
and a set of tight morphisms



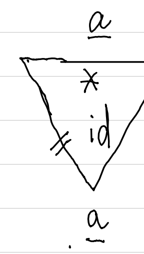
written $\mathcal{D}(\lfloor\vec{a}, \underset{*}{\underline{b}}; \underline{c})$

NB $\vec{a}$ can be empty

- a loosening operation $\mathcal{D}(\lfloor\vec{a}, \underset{*}{\underline{b}}; \underline{c}) \longrightarrow \mathcal{D}(\vec{a}, \underset{*}{\underline{b}}; \underline{c})$ written $f \mapsto \bar{f}$
- four kinds of composition

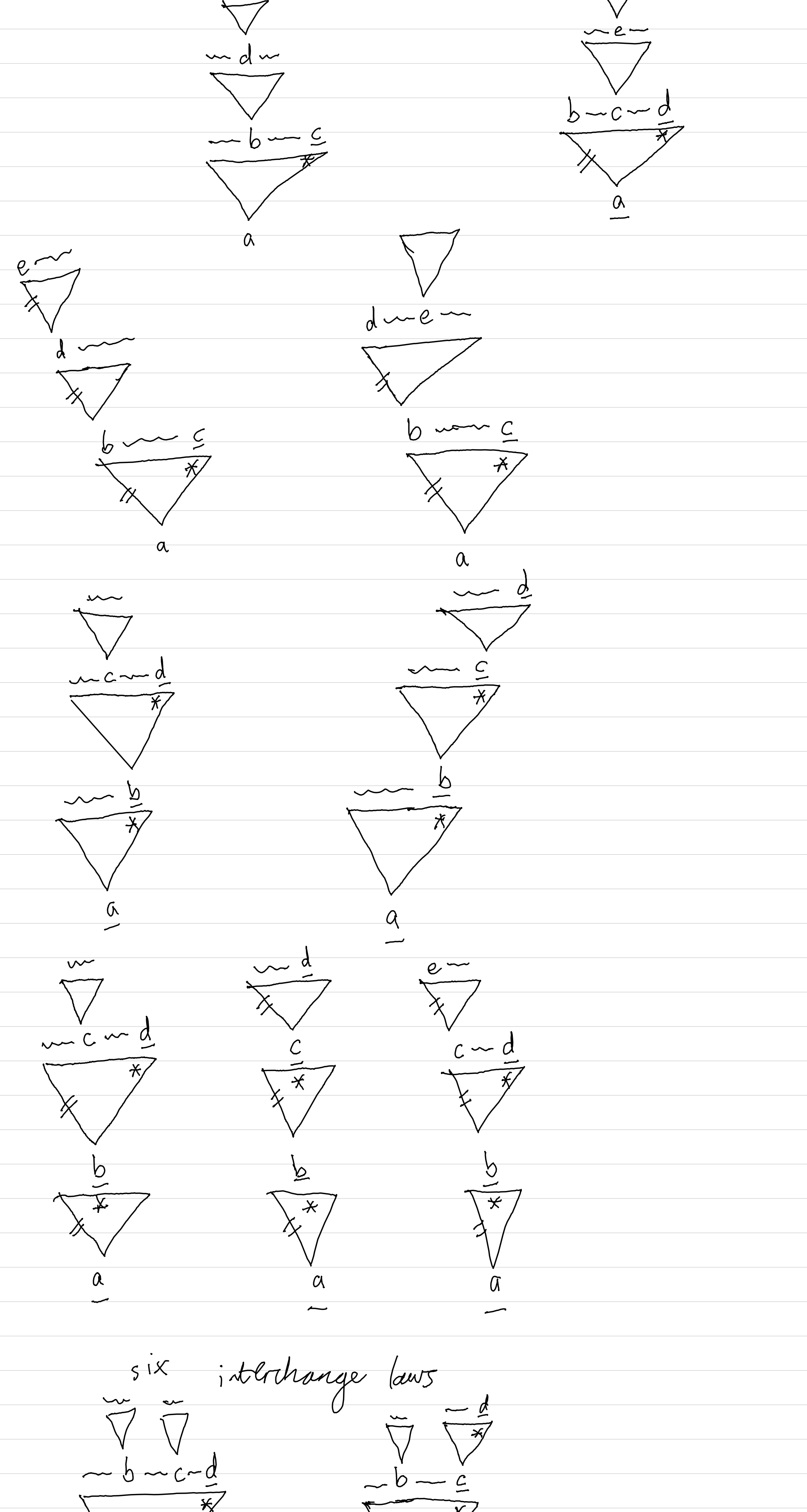


- for each $\mathcal{D}$-object $\underline{a}$, a tight identity

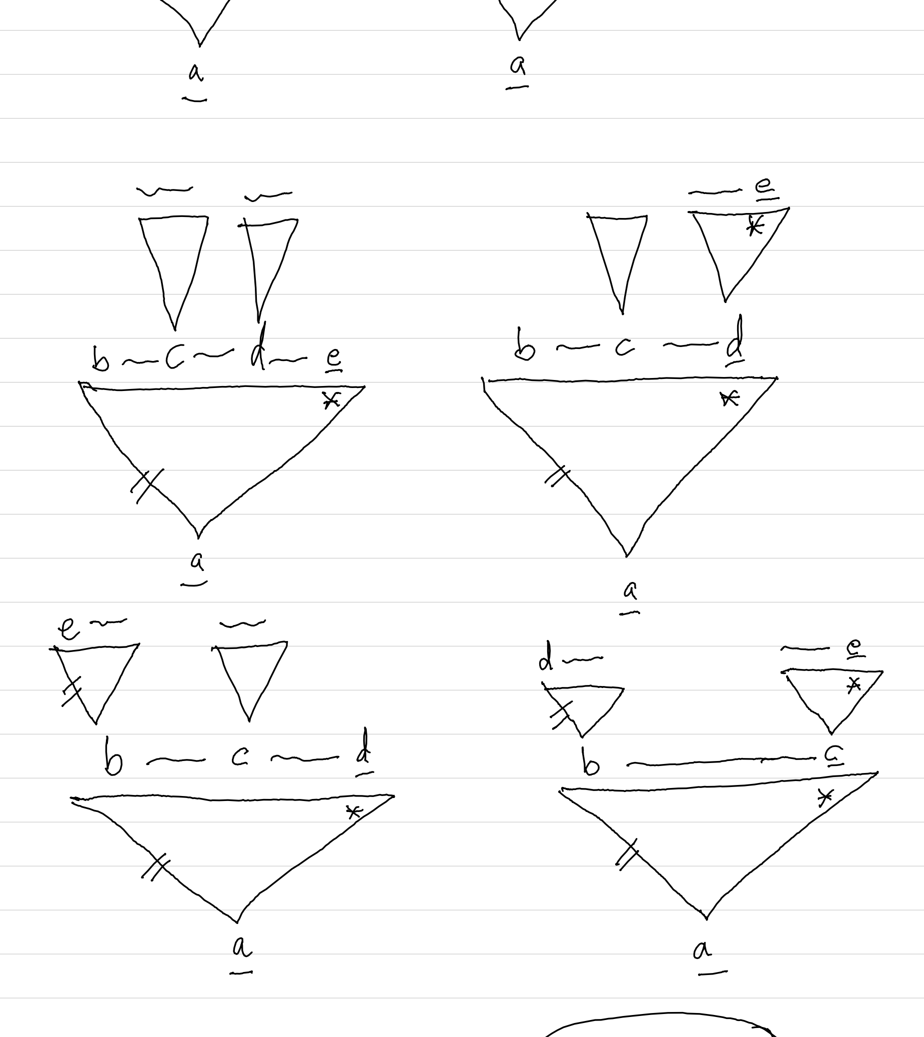


satisfying the following equations:

- tight outer composition commutes with loosening
- tight inner composition commutes with loosening
- mixed outer composition commutes with loosening
- left unitality for each kind of composition
- right unitality for each kind of morphism (with inner composition)
- nine associativity laws



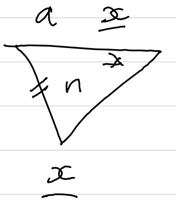
six interchange laws



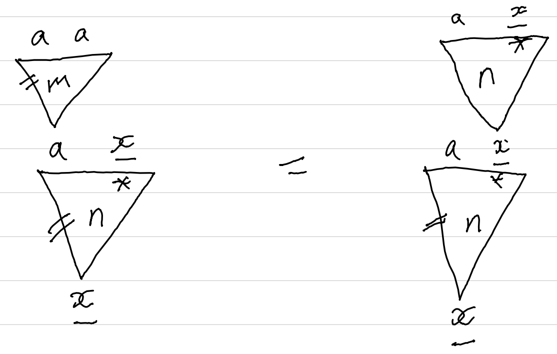
25 equations

Given a left-skew multicategory $\mathcal{C}$
and a left-skew $\mathcal{C}$-multicategory $\mathcal{D}$
and a monoid $M = (a, u, m)$ in $\mathcal{C}$

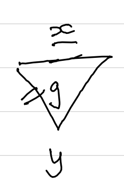
a left M-set is
- a $\mathcal{D}$-object $\underline{x}$
- a tight morphism



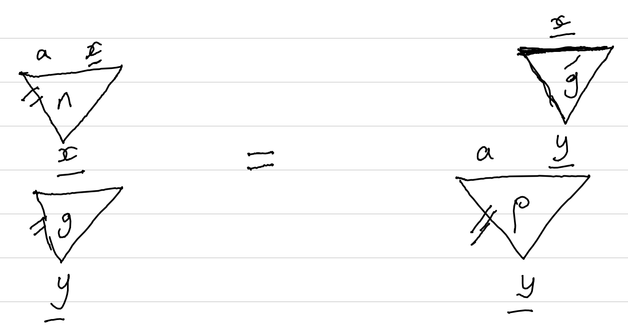
satisfying the following equations



A homomorphism $(\underline{x}, n) \longrightarrow (\underline{y}, p)$ is
a tight morphism



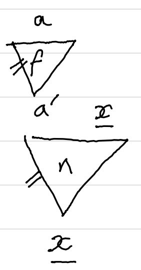
satisfying the following equation



A monoid homomorphism $\begin{matrix} M \\ \parallel \\ (a, m, u) \end{matrix} \xrightarrow{f} \begin{matrix} M' \\ \parallel \\ (a', m', u') \end{matrix}$

gives a faithful functor Left M'-set $\longrightarrow$ Left M-set

It sends $(\underline{x}, n)$ to $(\underline{x},$



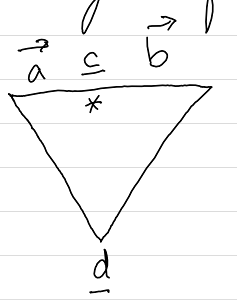
$)$

and $f : (\underline{x}, n) \longrightarrow (\underline{y}, p)$ to f.

Let $\mathcal{C}$ and $\mathcal{D}$ be multicategories.

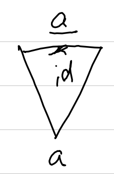
A $(\mathcal{C},\mathcal{D})$-multicategory $\mathcal{E}$ is

- a set of objects
- for each $\mathcal{C}$-object list $\vec{a}$, $\mathcal{D}$-object list $\vec{b}$ and $\mathcal{E}$-objects $\underset{\sim}{c}$ and $\underset{\sim}{d}$, a set of morphisms

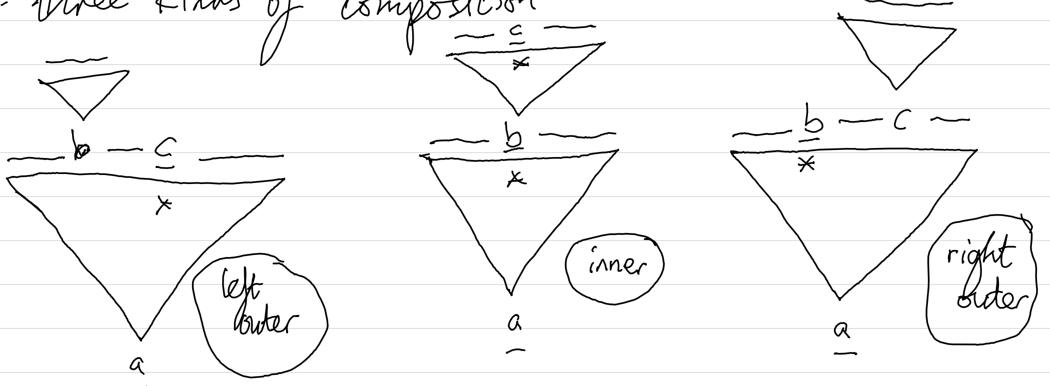


written $\mathcal{E}(\vec{a}, \underset{*}{\underset{\sim}{c}}, \vec{b} ; \underset{\sim}{d})$

- for each $\mathcal{E}$-object $\underset{\sim}{a}$, an identity morphism

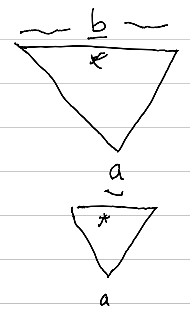


- three kinds of composition

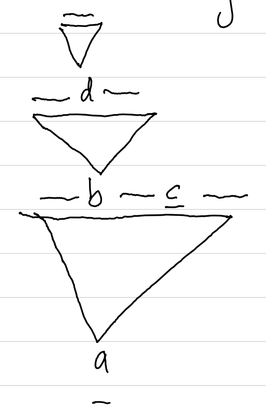


satisfying the following equations:

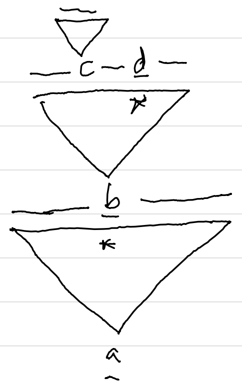
- left-unitality for each kind of composition
- right-unitality



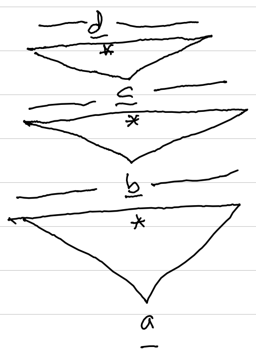
- five associativity laws



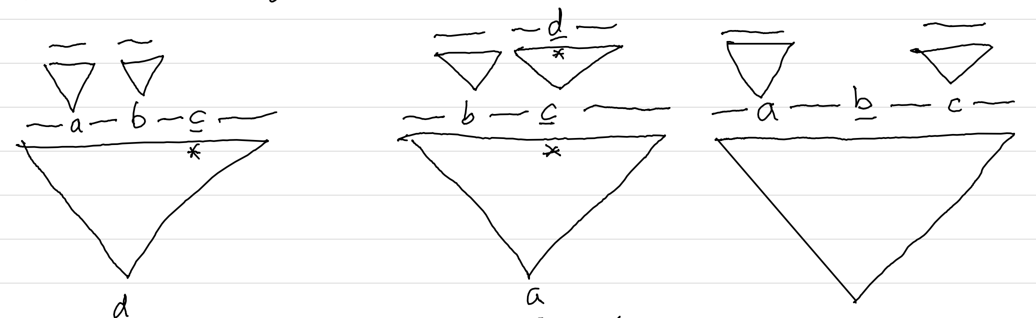
and mirror



and mirror



- five interchange laws



and mirror

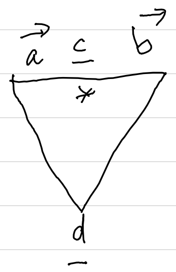
14 equations

Let $\mathcal{C}$ be a left-skew multicategory and $\mathcal{D}$ a multicategory.

A left-skew $(\mathcal{C}, \mathcal{D})$-multiactegory $\mathcal{E}$ is

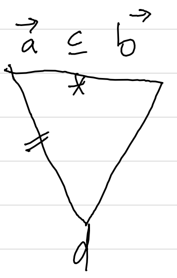
- a set of objects
- for each $\mathcal{C}$-object list $\vec{a}$, $\mathcal{D}$-object list $\vec{b}$ and $\mathcal{E}$-objects $\underline{c}$ and $\underline{d}$

  a set of loose morphisms

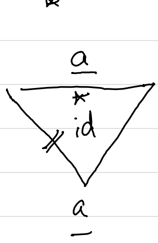
  written $\mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$

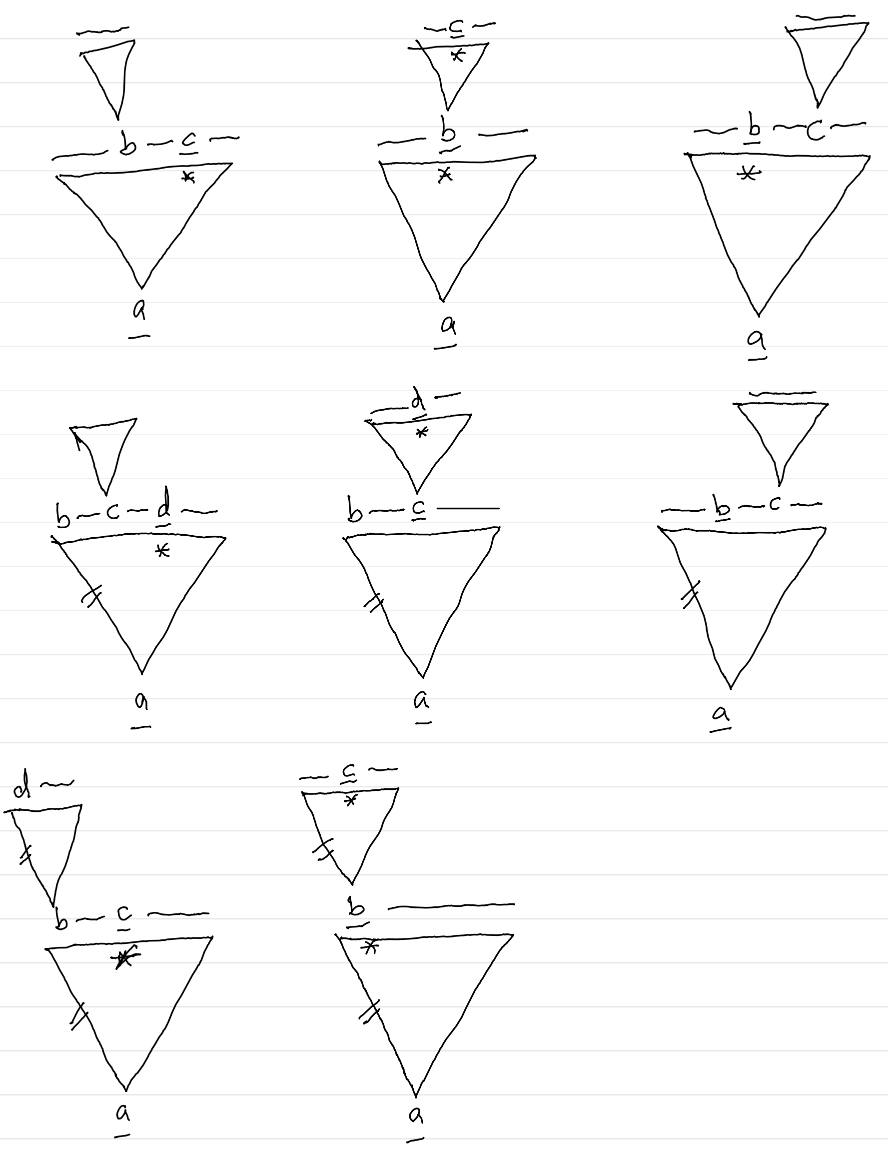
  and a set of tight morphisms

  written $\mathcal{E}([\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$
- loosening operations $\mathcal{E}([\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d}) \to \mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$

  written $f \mapsto f$
- for each $\mathcal{E}$-object $\underline{a}$, a tight identity morphism

  
- eight kinds of composition



satisfying the following equations

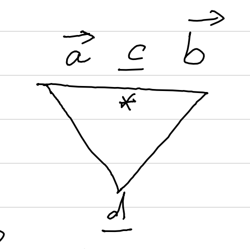
- composition commutes with loosening
- left-unitality for each kind of composition
- right unitality for each kind of morphism
- associativity
- interchange

Let $\mathcal{C}$ be a left-skew multicategory and $\mathcal{D}$ a right-skew multicategory.

A biskew $(\mathcal{C}, \mathcal{D})$-multicategory $\mathcal{E}$:

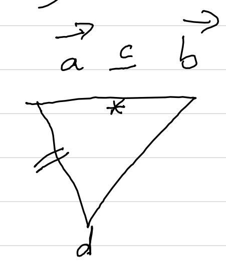
- a set of objects
- for each $\mathcal{C}$-object list $\vec{a}$ and $\mathcal{D}$-objects $\vec{b}$ and $\mathcal{E}$-objects $\underline{c}$, $\underline{d}$

a set of loose-loose morphisms



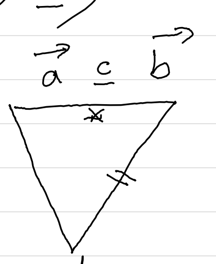
written $\mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$

a set of tight-loose morphisms



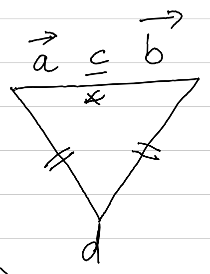
written $\mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$

a set of loose-tight morphisms



written $\mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d})$

a set of tight-tight morphisms



written $\mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d})$

- left-loosening operations

$$\mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d}) \to \mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$$

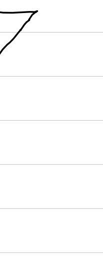
$$\mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d}) \to \mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d})$$

- right-loosening operations

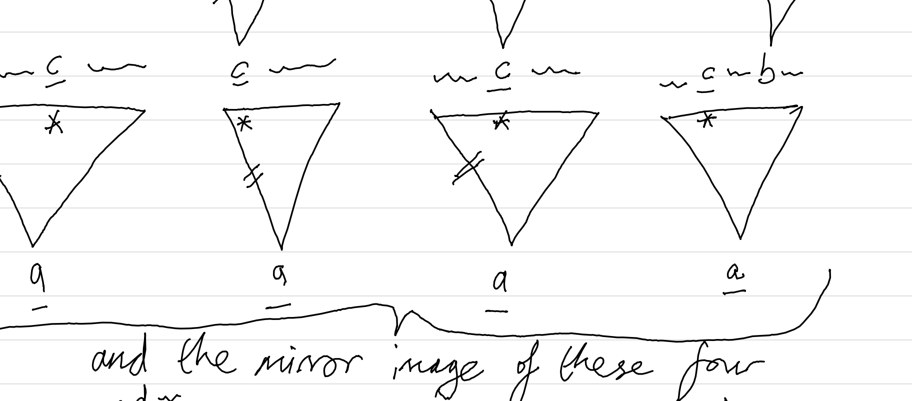
$$\mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d}) \to \mathcal{E}(\vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$$

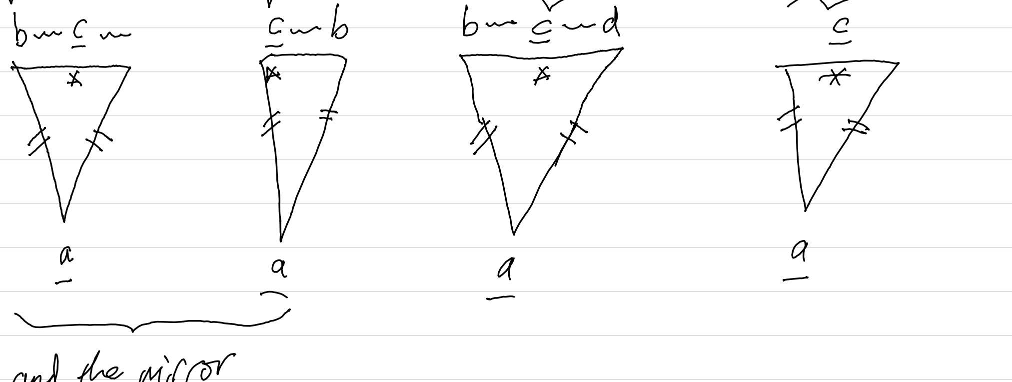
$$\mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b} \rfloor; \underline{d}) \to \mathcal{E}(\lfloor \vec{a}, \underset{*}{\underline{c}}, \vec{b}; \underline{d})$$

- for each $\mathcal{E}$-object $\underline{a}$, a tight-tight identity

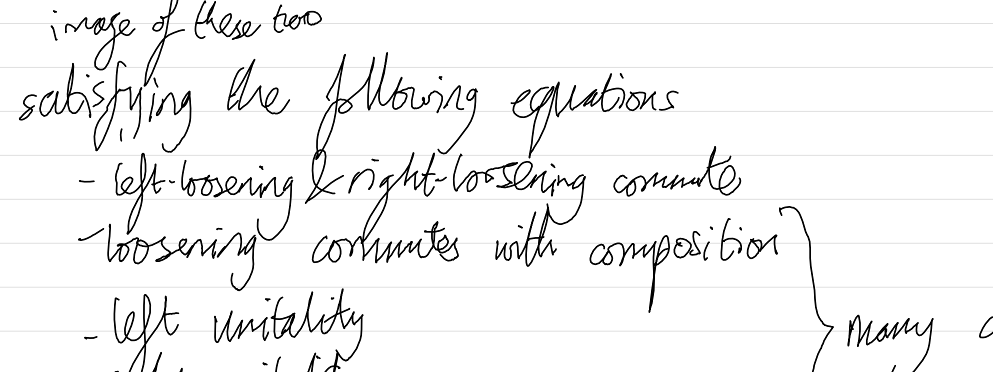


- seventeen kinds of composition





and the mirror image of these four



and the mirror image of these two

satisfying the following equations

- left-loosening & right-loosening commute
- loosening commutes with composition
- left unitality
- right unitality
- associativity
- interchange

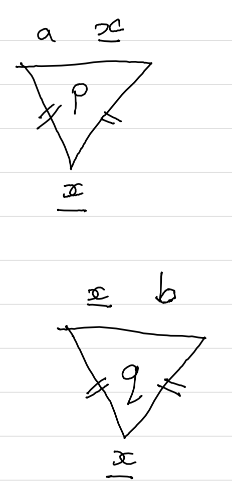
many cases

Let $\mathcal{C}$ be a left-skew multicategory
and $\mathcal{D}$ a right-skew multicategory
and $\mathcal{E}$ a biskew $(\mathcal{C},\mathcal{D})$-multicategory.

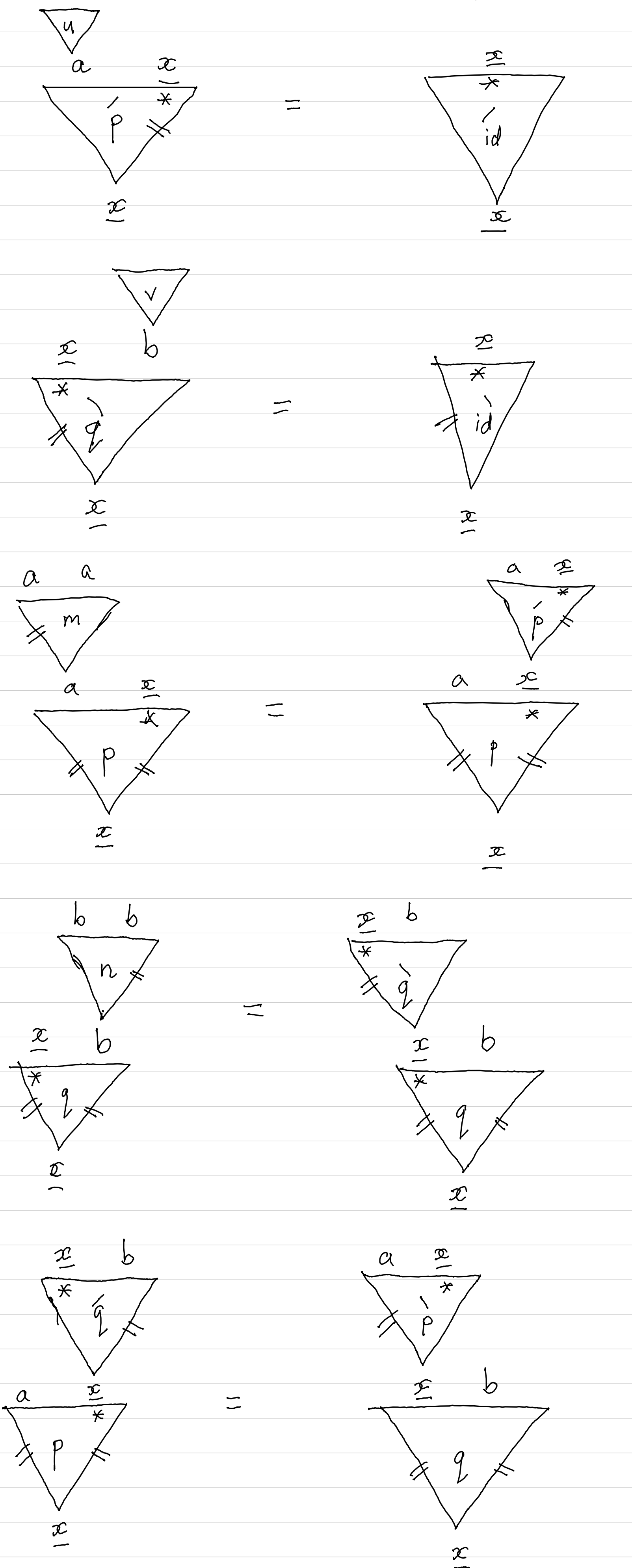
Let $M=(a,u,m)$ be a monoid in $\mathcal{C}$
and $N=(b,v,n)$ a monoid in $\mathcal{D}$.

An (M,N)-set is

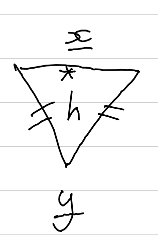
- an $\mathcal{E}$-object x
- a tight-tight morphism
- a tight-tight morphism



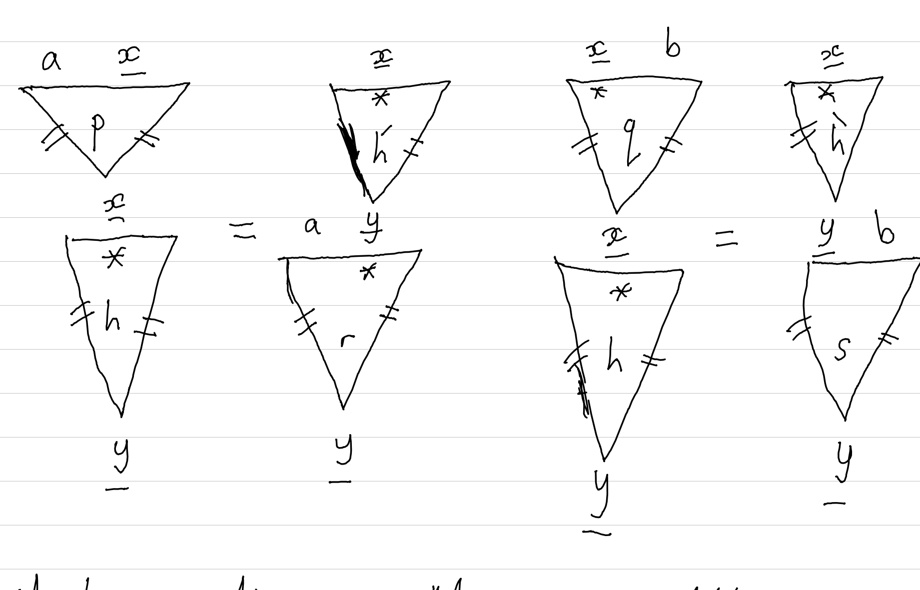
satisfying the following 5 equations.



A morphism $(x,p,q) \longrightarrow (y,r,s)$
is a tight-tight morphism



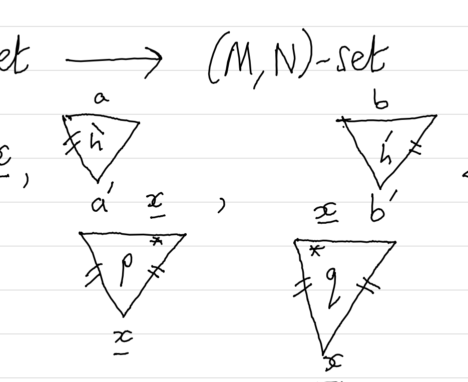
satisfying



Monoid homomorphisms $f\colon \underset{(a,u,m)}{M} \longrightarrow \underset{(a',u',m')}{M'}$

and $g\colon \underset{(b,v,n)}{N} \longrightarrow \underset{(b',v',n')}{N'}$

give a faithful functor (M',N')-set $\longrightarrow$ (M,N)-set
It sends (x,p,q) to $(x,$ ... $,$ ... $)$



and $h\colon (x,p,q) \longrightarrow (y,r,s)$ to h.