How To Quotient A Cpo

Paul Blain Levy

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An exposition of an old result, based on Achim Jung's notes.

1 Admissible Closure

In this section, let A be a cpo (poset with all directed joins) and let $U \subseteq A$. We write $\leqslant$ for the order on A.

Definition 1 We define $\mathrm{adm}_A(U)$ to be least subset of A that is admissible (closed under directed joins) and contains U. $\Box$

Proposition 1 Each $x \in \mathrm{adm}_A(U)$ is a least upper bound of $\{y \in U | y \leqslant x\}$, and this lub is preserved by every continuous function from $\mathrm{adm}_A(U)$. $\Box$

Proof Let $\mathrm{adm}_A(U) \xrightarrow{h} B$ be a continuous function. For each $x \in \mathrm{adm}_A(U)$, it is clear that $h(x)$ is an upper bound for the set $Q(x) \stackrel{\mathrm{def}}{=} \{h(y) | y \in U, y \leqslant x\}$. We wish to show that it is a least upper bound. (Taking h to be the identity on $\mathrm{adm}_A(U)$ then gives that x is the lub of $\{y \in U | y \leqslant x\}$.)

The set $K \stackrel{\mathrm{def}}{=} \{x \in \mathrm{adm}_A(U) \,|\, h(x) \text{ is a lub of } Q(x)\}$ contains U. To see that K is admissible, suppose that D is a directed subset of K. We wish to show that $h(\bigsqcup_{x \in D} x) = \bigsqcup_{x \in D} h(x)$ is a lub of $Q(\bigsqcup_{x \in D} x)$. Suppose that n is an upper bound of $Q(\bigsqcup_{x \in D} x)$; we wish to show that $\bigsqcup_{x \in D} h(x) \leqslant n$. So, for each $x \in D$, we need to show that $h(x) \leqslant n$, i.e. that n is an upper bound for $Q(x)$, which follows from $Q(x) \subseteq Q(\bigsqcup_{x \in D} x)$. $\Box$

2 Quotienting

In this section, let A be a cpo, and let $\mathcal{R}$ be a binary relation on A. We write $\leqslant$ for the order on A. Define $\theta(\mathcal{R})$ to be the full subcategory of the coslice category $A/\mathbf{Cpo}$ consisting of those objects $(B, A \xrightarrow{g} B)$ such that if $x \,\mathcal{R}\, y$ then $g(x) \leqslant_B g(y)$. Our aim is to prove that $\theta(\mathcal{R})$ has an initial object.

Let us say that a set $P \subseteq A$ is $\mathcal{R}$-*compatible* when

- P is $\mathcal{R}$-downclosed (i.e. $x \,\mathcal{R}\, y$ and $y \in P$ implies $x \in P$)

- P is Scott-closed (admissible and $\leqslant$-downclosed).

The poset $\mathcal{L}$ of $\mathcal{R}$-compatible sets (ordered by inclusion) is a complete lattice, with meets given by intersection. In particular, it has directed joins, which we write $\bigsqcup^{\mathcal{L}}$.

For $x \in A$, we define $[x] \in \mathcal{L}$ to be the least $\mathcal{R}$-compatible subset that $\ni x$. Thus for $x \in A$ and $P \in \mathcal{L}$ we have

$$x \in P \quad \Leftrightarrow \quad [x] \subseteq P \tag{1}$$

We then define $\mathcal{K}$ to be $\mathrm{adm}_{\mathcal{L}}(\{[x] \mid x \in A\})$. We are going to show that $X \stackrel{\text{def}}{=} (\mathcal{K},\ A \xrightarrow{[-]} \mathcal{K}\ )$ is an initial object of $\theta(\mathcal{R})$.

To show that X is an object of $\theta(\mathcal{R})$, we reason as follows.

- To show $[-]$ is monotone, suppose $x \leqslant y$. Then $y \in [y]$ gives $x \in [y]$ (since $[y]$ is $\leqslant$-downclosed) which gives $[x] \subseteq [y]$ by (1).
- To show $[-]$ is continuous, suppose that $D \subseteq A$ is directed, and that $P \in \mathcal{L}$ is an upper bound for $\{[d] \mid d \in D\}$. Then $D \subseteq P$, so $\bigsqcup D \in P$ (since P is admissible), so $[\bigsqcup D] \subseteq P$ by (1).
- Suppose $x\mathcal{R}y$. Then $y \in [y]$ gives $x \in [y]$ (since $[y]$ is $\mathcal{R}$-downclosed) which gives $[x] \subseteq [y]$ by (1).

Now suppose that $Y = (B,\ A \xrightarrow{g} B\ )$ is an arbitrary object of $\theta(\mathcal{R})$. We seek a morphism $X \xrightarrow{h} Y$. Prop. 1, together with (1), tells us that h must map each $P \in \mathcal{K}$ to a lub of $g\,"P \stackrel{\text{def}}{=} \{g(y) \mid y \in P\}$, so h is unique. We need to show that $g\,"P$ has a lub, for every $P \in \mathcal{K}$.

Lemma 1

1. Suppose $P, Q \in \mathcal{K}$, and $P \subseteq Q$. If $g\,"P$ has a lub m, and $g\,"Q$ has a lub n, then $m \leqslant_B n$.
2. Suppose $\mathcal{D} \subseteq \mathcal{K}$ is directed. If, for each $P \in \mathcal{D}$, the set $g\,"P$ has a lub m_P, so that the set $\{m_P \mid P \in \mathcal{D}\}$ is directed by part 1, then $m = \bigsqcup_{P \in \mathcal{D}} m_P$ is a lub for $g\,"(\bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P)$.
3. For each $x \in A$, the set $g\,"[x]$ has lub $g(x)$.

$\square$

Proof

1. Because $g\,"P \subseteq g\,"Q$.

2. The set $\{x \in A \,|\, g(x) \leqslant_B m\}$ is $\mathcal{R}$-compatible, and contains P for each $P \in \mathcal{D}$. Therefore it contains $\bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P$. So if $x \in \bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P$, then $g(x) \leqslant_B m$. So m is an upper bound for $g\,"(\bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P)$.

   Suppose n is an arbitrary upper bound for $g\,"(\bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P)$. For each $p \in P$, we have that n is an upper bound for $g\,"P$, so $m_P \leqslant_B n$. Therefore $m \leqslant_B n$. So m is a least upper bound for $g\,"(\bigsqcup^{\mathcal{L}}_{P \in \mathcal{D}} P)$.

3. For each $x \in A$, the set $\{y \in A \,|\, g(y) \leqslant_B g(x)\}$ is $\mathcal{R}$-compatible and contains x, so by (1) it contains $[x]$. Hence $g(x)$ is an upper bound of $g\,"[x]$. Clearly, any upper bound of $g\,"[x]$ is $\geqslant_B g(x)$.

$\square$

Lemma 1 tells us that the set $\{P \in \mathcal{K} \,|\, g\,"P$ has a lub$\}$ is admissible and contains $\{[x] \,|\, x \in A\}$, hence it is the whole of $\mathcal{K}$, as we wanted. Lemma 1 also tells us that the function $\mathcal{K} \xrightarrow{h} B$ that maps $P \in \mathcal{K}$ to the lub of $g\,"[x]$ is monotone and continuous, and satisfies $h([x]) = g(x)$ for all $x \in A$, as required.