THE VANISHING TENSOR QUESTION*

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To Di Levine and Dov Stekel on the occasion of their wedding

Abstract. The vanishing tensor question asks when a linear combination of pure tensors is equal to zero. This paper provides two novel answers to this question, each of which highlights the fact that the linear dependency space of the pure tensors is monotonic in the linear dependency spaces of the constituent vectors. One answer expresses each constituent vector in terms of the others, using an "expansion matrix". The other involves a bijection between subspaces of a free vector space and closed subspaces of a topological vector space of functions.

Key words.

Tensor product, Basis independence, Orthogonal complement, Topological vector space.

AMS subject classifications.

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1. Basics. Throughout the paper, all vector spaces are over a fixed field $\mathbb{F}$. We use the following basic nomenclature, see e.g. [2].

- Let I be a set.
  - We write $\mathbb{F}(I)$ for the vector space of finitely supported functions from I to $\mathbb{F}$ (i.e. the vector space freely generated by I).
  - For each $i \in I$, we write $1_i \in \mathbb{F}(I)$ for the function mapping i to 1 and everything else to 0 (i.e. the ith generator).
- Let V be a vector space.
  - A *V-vector* is an element of V.
  - We write V^* for the vector space of linear functions $V \longrightarrow \mathbb{F}$.
  - If W is a subspace of V, the vector space V/W is defined to be the quotient of V by the congruence

$$\{(\mathbf{v}, \mathbf{v}') \in V \times V \mid \mathbf{v} - \mathbf{v}' \in W\}$$

  - If $\mathcal{A}$ is a basis of V, then for any $\mathbf{v} \in V$ and $\mathbf{a} \in \mathcal{A}$ we write $(\overline{\mathcal{A}}\mathbf{v})_{\mathbf{a}}$ for the $\mathbf{a}$-component (with respect to $\mathcal{A}$) of $\mathbf{v}$.

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- Let V and W be vector spaces. A *tensor product* of V and W is a vector space X together with a bilinear function $p : V \times W \longrightarrow X$ enjoying the following property: for any vector space Y and bilinear function $q : V \times W \longrightarrow Y$, there is a unique linear map $f : X \longrightarrow Y$ making the following commute:

$$\begin{tikzcd} V \times W \arrow[r, "p"] \arrow[dr, "q"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

  Clearly if (X, p) and (X', p') are two such, they are uniquely isomorphic. Moreover, the tensor product always exists—there are three well-known ways of constructing it, which we describe in Sect. 2. We therefore write $(V \otimes W, \otimes)$ for a chosen tensor product of V and W.

2. Discussion. The "vanishing tensor question", for vector spaces V and W, is as follows: when does the equation

$$\sum_{i \in I, j \in J} \lambda_{ij} \mathbf{v}_i \otimes \mathbf{w}_j = \mathbf{0} \tag{2.1}$$

hold in the tensor space $V \otimes W$? Here $(\mathbf{v}_i)_{i \in I}$ is a V-vector tuple and $(\mathbf{w}_j)_{j \in J}$ is a W-vector tuple. In the simplest case, the indexing sets I and J are finite. More generally, they might be infinite, and we require $\lambda \in \mathbb{F}(I \times J)$—the finite supportedness of λ makes the LHS of (2.1) well-defined.

In this paper we give five answers to this question. The first three come from the standard constructions of $V \otimes W$ [2].

1. We take the vector space $\mathbb{F}(V \times W)$, and let Vanish be the least subspace containing

$$\begin{array}{rcl} & \{1_{(\mathbf{v}+\mathbf{v}',\mathbf{w})} - 1_{(\mathbf{v},\mathbf{w})} - 1_{(\mathbf{v}',\mathbf{w})} & \mid \mathbf{v}, \mathbf{v}' \in V, \mathbf{w} \in W\} \\ \cup & \{1_{(\lambda\mathbf{v},\mathbf{w})} - \lambda 1_{(\mathbf{v},\mathbf{w})} & \mid \lambda \in \mathbb{F}, \mathbf{v} \in V, \mathbf{w} \in W\} \\ \cup & \{1_{(\mathbf{v},\mathbf{w}+\mathbf{w}')} - 1_{(\mathbf{v},\mathbf{w})} - 1_{(\mathbf{v},\mathbf{w}')} & \mid \mathbf{v} \in V, \mathbf{w}, \mathbf{w}' \in W\} \\ \cup & \{1_{(\mathbf{v},\lambda\mathbf{w})} - \lambda 1_{(\mathbf{v},\mathbf{w})} & \mid \lambda \in \mathbb{F}, \mathbf{v} \in V, \mathbf{w} \in W\} \end{array}$$

   We then construct $V \otimes W$ as the quotient $\mathbb{F}(V \times W)/\mathrm{Vanish}$, with $\mathbf{v} \otimes \mathbf{w}$ defined to be the equivalence class of $1_{(\mathbf{v},\mathbf{w})}$. Therefore (2.1) holds when

$$\sum_{i \in I, j \in J} \lambda_{ij} 1_{(\mathbf{v}_i, \mathbf{w}_j)} \in \mathrm{Vanish}$$

   This answer is utterly unhelpful, because identifying when an element of $\mathbb{F}(V \times W)$ belongs to Vanish is itself an instance of the vanishing tensor question.

2. Let $\mathcal{A}$ be a basis of V and $\mathcal{B}$ a basis of W. Then we construct $V \otimes W$ as $\mathbb{F}(\mathcal{A} \times \mathcal{B})$, with $\mathbf{v} \otimes \mathbf{w}$ defined to be $(\mathbf{a}, \mathbf{b}) \mapsto (\overline{\mathcal{A}}\mathbf{v})_{\mathbf{a}}(\overline{\mathcal{B}}\mathbf{w})_{\mathbf{b}}$. Therefore (2.1) holds when

$$\sum_{i \in I, j \in J} (\overline{\mathcal{A}}\mathbf{v}_i)_{\mathbf{a}}(\overline{\mathcal{B}}\mathbf{w}_j)_{\mathbf{b}} = 0 \text{ for all } \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B}.$$

   This answer is useful for calculation, but it has the conceptual demerit of being basis dependent.
3. [1] Write $\mathrm{Bil}(V, W)$ be the space of bilinear functions $V \times W \longrightarrow \mathbb{F}$. Define a bilinear function

$$\begin{array}{ccccccc} [-,-] & : & V & \times & W & \longrightarrow & \mathrm{Bil}(V,W)^* \\ & & \mathbf{v} & & \mathbf{w} & \mapsto & (f \mapsto f(\mathbf{v}, \mathbf{w})) \end{array}$$

   Then we construct $V \otimes W$ as the least subspace of $\mathrm{Bil}(V,W)^*$ containing $\{[\mathbf{v}, \mathbf{w}] \mid \mathbf{v} \in V, \mathbf{w} \in W\}$, with $\mathbf{v} \otimes \mathbf{w}$ defined to be $[\mathbf{v}, \mathbf{w}]$. Therefore (2.1) holds when

$$\sum_{i \in I, j \in J} f(\mathbf{v}_i, \mathbf{w}_j) = 0 \text{ for all } f \in \mathrm{Bil}(V, W).$$

All these traditional answers ignore an important fact. Let us say that the *linear dependency space* of a V-vector tuple $(\mathbf{v}_i)_{i \in I}$ written $\mathrm{LD}(V, (\mathbf{v}_i)_{i \in I})$, is the set of $\lambda \in \mathbb{F}(I)$ such that $\sum_{i \in I} \lambda_i \mathbf{v}_i = 0$. Our goal is to determine

$$\mathrm{LD}(V \otimes W, (\mathbf{v}_i \otimes \mathbf{w}_j)_{(i,j) \in I \times J})$$

The crucial fact is that this subspace depends monotonically on $\mathrm{LD}(V, (\mathbf{v}_i)_{i \in I})$ and $\mathrm{LD}(W, (\mathbf{w}_j)_{j \in J})$.

To make this claim more precise, let us say that an *I-matrix* is a function $c : I \longrightarrow \mathbb{F}(I)$. For any vector space V, this induces a linear map

$$\begin{array}{ccccc} c[V] & : & V^I & \longrightarrow & V^I \\ & & (\mathbf{v}_i)_{i \in I} & \mapsto & (\sum_{i' \in I} c_{ii'} \mathbf{v}_{i'})_{i \in I} \end{array}$$

We say that c is an *expansion matrix* for a V-vector tuple $(\mathbf{v}_i)_{i \in I}$ when the latter is a fixpoint of $c[V]$. Using this notion, we can state our fourth answer:

PROPOSITION 2.1. *(2.1) holds when there exist expansion matrices c for $(\mathbf{v}_i)_{i \in I}$ and d for $(\mathbf{w}_j)_{j \in J}$ such that*

$$\sum_{i \in I, j \in J} \lambda_{ij} c_{ii'} d_{jj'} = 0 \textit{ for all } i' \in I, j' \in J \tag{2.2}$$

Let us express this in terms of linear dependency spaces. We write Sub $\mathbb{F}(I)$ for the poset of subspaces of $\mathbb{F}(I)$, ordered by inclusion. An I-matrix c is an *expansion matrix* for $A \in \text{Sub}\ \mathbb{F}(I)$ when

$$1_i - c_i \in A \text{ for all } i \in I$$

Thus c is an expansion matrix for a V-vector tuple $(\mathbf{v}_i)_{i \in I}$ iff it is an expansion matrix for $\text{LD}(V, (\mathbf{v}_i)_{i \in I})$. We define a monotone function

$$\begin{array}{cccccll} * & : & \text{Sub}\ \mathbb{F}(I) & \times & \text{Sub}\ \mathbb{F}(J) & \longrightarrow & \text{Sub}\ \mathbb{F}(I \times J) \\ & & A & & B & \mapsto & \{\lambda \in \mathbb{F}(I \times J) \mid \exists \text{ expansion matrices} \\ & & & & & & \quad c \text{ for } A \text{ and } d \text{ for } B \\ & & & & & & \quad \text{such that (2.2) holds } \} \end{array}$$

Thus Prop. 2.1 says

$$\text{LD}(V \otimes W, (\mathbf{v}_i \otimes \mathbf{w}_j)_{(i,j) \in I \times J}) = \text{LD}(V, (\mathbf{v}_i)_{i \in I}) * \text{LD}(W, (\mathbf{w}_j)_{j \in J})$$

Whereas the above definition of $*$ uses existential quantification, we shall give another description that uses universal quantification and is arguably more elegant.

For this, we move to a topological setting. We regard $\mathbb{F}$ and $\mathbb{F}(I)$ as discretely topologized, and we write $\mathbb{F}^I$ for the topological vector space of functions $I \longrightarrow \mathbb{F}$ with the product topology. Explicitly, a subset $U \subseteq \mathbb{F}^I$ is closed when for each $s \in \mathbb{F}^I$, if for each finite $K \subseteq I$ there exists $t \in A$ with the same restriction to K as s, then $s \in A$. We write ClSub $\mathbb{F}^I$ for the poset of closed subspaces of $\mathbb{F}^I$, ordered by inclusion.

We define a continuous and bilinear function

$$\begin{array}{cccccc} \cdot & : & \mathbb{F}(I) & \times & \mathbb{F}^I & \longrightarrow & \mathbb{F} \\ & & \lambda & & s & \mapsto & \sum_{i \in I} \lambda_i s_i \end{array}$$

giving us antitone functions

$$\begin{array}{cccl} (-)^{\acute{\perp}} & : & \text{Sub}\ \mathbb{F}(I) & \longrightarrow & \text{ClSub}\ \mathbb{F}^I \\ & & A & \mapsto & \{s \in \mathbb{F}(I) \mid \forall \lambda \in A.\ \lambda \cdot s = 0\} \end{array}$$

$$\begin{array}{cccl} (-)^{\grave{\perp}} & : & \text{ClSub}\ \mathbb{F}^I & \longrightarrow & \text{Sub}\ \mathbb{F}(I) \\ & & S & \mapsto & \{\lambda \in \mathbb{F}(I) \mid \forall s \in A.\ \lambda \cdot s = 0\} \end{array}$$

Proposition 2.2. *The functions* $\mathrm{Sub}\ \mathbb{F}(I) \underset{(-)^{\grave{\perp}}}{\overset{(-)^{\acute{\perp}}}{\rightleftarrows}} \mathrm{ClSub}\ \mathbb{F}^I$ *are inverse.*

For finite I, this is the well-known fact that orthogonal complementation is an involution on $\mathrm{Sub}\ \mathbb{F}(I)$.

We also define a continuous and bilinear function

$$\begin{array}{ccccccc} \boxtimes & : & \mathbb{F}(I) & \times & \mathbb{F}(J) & \longrightarrow & \mathbb{F}(I \times J) \\ & & s & & t & \mapsto & ((i,j) \mapsto s_i t_j) \end{array}$$

(which is a tensor product if I and J are finite), giving us a monotone function

$$\begin{array}{ccccccl} \boxtimes & : & \mathrm{ClSub}\ \mathbb{F}^I & \times & \mathrm{ClSub}\ \mathbb{F}^J & \longrightarrow & \mathrm{ClSub}\ \mathbb{F}^{I \times J} \\ & & S & & T & \mapsto & \text{the least closed subspace} \\ & & & & & & \quad \text{containing } \{s \boxtimes t \mid s \in S, t \in T\} \end{array}$$

It is easy to see that $(S \boxtimes T)^{\grave{\perp}}$ consists of those $\lambda \in \mathbb{F}(I \times J)$ such that $\lambda \cdot (s \boxtimes t) = 0$ for all $s \in S$ and $t \in T$.

We can now characterize the $*$ operation as follows:

Proposition 2.3. *For any* $A \in \mathrm{Sub}\ \mathbb{F}(I)$ *and* $B \in \mathrm{Sub}\ \mathbb{F}(J)$ *we have*

$$A * B = (A^{\acute{\perp}} \boxtimes B^{\acute{\perp}})^{\grave{\perp}}$$

Prop. 2.3 provides our fifth answer: that (2.1) holds iff $\lambda \cdot (s \boxtimes t) = 0$ for all $s \in \mathrm{LD}(V, (\mathbf{v}_i)_{i \in I})^{\acute{\perp}}$ and $t \in \mathrm{LD}(W, (\mathbf{w}_j)_{j \in J})^{\acute{\perp}}$.

Finally we note that all our results generalize straightforwardly from a pair of vector spaces (V, W) to a finite sequence of vector spaces $(V_k)_{k<n}$.

3. Proofs for Sect. 2. Definition 3.1. *Let* $(\mathbf{v}_{i \in I})$ *be a* V*-vector tuple.*

1. *For* $i \in I$*, an* i-expansion *for* $(\mathbf{v}_{i \in I})$ *is an element* $\lambda \in \mathbb{F}(I)$ *such that*

$$\mathbf{v}_i = \sum_{i' \in I} \lambda_{i'} \mathbf{v}_{i'} \tag{3.1}$$

2. *An index set* $M \subseteq I$ *is* linearly independent *for* $(\mathbf{v}_i)_{i \in I}$ *when, for any* $\mu \in \mathbb{F}(M)$ *satisfying* $\sum_{i \in M} \mu_i \mathbf{v}_i = \mathbf{0}$*, we have* $\mu_i = 0$ *for all* $i \in M$*.*
3. *An* index basis *for* $(\mathbf{v}_{i \in I})$ *is a maximal linearly independent index set.*

Thus an I-matrix c is an expansion matrix for $(\mathbf{v}_i)_{i\in I}$ iff, for all $i \in I$, the row c_i is an i-expansion.

LEMMA 3.2. *Let $M \subseteq I$ be a basis for a V-vector tuple $(\mathbf{v}_i)_{i\in I}$. For each $i \in I$, there is a unique i-expansion $M(V,(\mathbf{v}_i)_{i\in I})_i$ for $(\mathbf{v}_i)_{i\in I}$ that vanishes outside M (i.e. maps each $i \in I \setminus M$ to 0). We thus obtain an expansion matrix $M(V,(\mathbf{v}_i)_{i\in I})$ for $(\mathbf{v}_{i\in I})$.*

Proof. Standard. □

Proof (of Prop. 2.1).

($\Leftarrow$) We calculate

$$\begin{aligned}\sum_{i\in I,j\in J}\lambda_{ij}\mathbf{v}_i\otimes\mathbf{w}_j &= \sum_{i\in I,j\in J}\lambda_{ij}(\sum_{i'\in I}c_{ii'}\mathbf{v}_{i'})\otimes(\sum_{j'\in J}d_{jj'}\mathbf{w}_{j'})\\ &= \sum_{i'\in I,j'\in J}(\sum_{i\in I,j\in J}\lambda_{ij}c_{ii'}d_{jj'})\mathbf{v}_{i'}\otimes\mathbf{w}_{j'}\\ &= \mathbf{0} \text{ by equation (2.2)}\end{aligned}$$

($\Rightarrow$) Using Zorn's Lemma, we pick an index basis $M \subseteq I$ of $(\mathbf{v}_i)_{i\in I}$ and then extend $\{\mathbf{v}_m \mid m \in M\}$ to a basis $\mathcal{A}$ of V. Likewise we pick an index basis $N \subseteq J$ of $(\mathbf{w}_j)_{j\in J}$ and then extend $\{\mathbf{w}_n \mid n \in N\}$ to a basis $\mathcal{B}$ of W. We claim that

$$\sum_{i\in I,j\in J}\lambda_{ij}M(V,(\mathbf{v}_i)_{i\in I})_{ii'}N(W,(\mathbf{w}_j)_{j\in J})_{jj'} = 0 \tag{3.2}$$

so that we can put $c = M(V,(\mathbf{v}_i)_{i\in I})$ and $d = N(W,(\mathbf{w}_j)_{j\in J})$ and obtain (2.2).
(3.2) is immediate if $i' \notin M$ or $j' \notin N$. For $i' \in M$ and $j' \in N$ we have

$$\sum_{i\in I,j\in J}\lambda_{ij}(\overline{\mathcal{A}}\mathbf{v}_i)_{\mathbf{v}_{i'}}(\overline{\mathcal{B}}\mathbf{w}_j)_{\mathbf{w}_{j'}} = 0 \tag{3.3}$$

Since $\mathbf{v}_i = \sum_{i'\in M} M^{(V,\mathbf{v})}_{ii'}\mathbf{v}_{i'}$, we obtain $(\overline{\mathcal{A}}\mathbf{v}_i)_{\mathbf{v}_{i'}} = M(V,(\mathbf{v}_i)_{i\in I})$. Likewise $(\overline{\mathcal{B}}\mathbf{w}_j)_{\mathbf{w}_{j'}} = N(W,(\mathbf{w}_j)_{j\in J})$. So (3.2) is given by equation (3.3). □

We adapt Def. 3.1 from vector tuples to subspaces of $\mathbb{F}(I)$.

DEFINITION 3.3. *Let I be a set and let A be a subspace of $\mathbb{F}(I)$.*

1. *For $i \in I$, an* i-expansion *for A is an element $\lambda \in \mathbb{F}(I)$ such that $1_i - \lambda \in A$.*
2. *An index set $M \subseteq I$ is* linearly independent *for A when any $\mu \in A$ that vanishes outside M is 0.*
3. *An* index basis *for A is a maximal linearly independent index set.*

It is clear that Def. 3.1 is a special case of Def. 3.3: for a vector I-tuple $(V, \mathbf{v})$, an i-expansion (or linearly independent index set, or index basis) for $(V, \mathbf{v})$ is precisely an i-expansion (or linearly independent index set, or index basis) for $\mathrm{LD}(V, (\mathbf{v}_i)_{i \in I})$.

Lemma 3.4. *Let I be a set and let $A \in \mathrm{Sub}\ \mathbb{F}(I)$. Let $M \subseteq I$ be an index basis for A.*

1. *(cf. Lemma 3.2) For each $i \in I$, there is a unique i-expansion $M(A)_i$ for A that vanishes outside M. Hence we obtain an expansion matrix $M(A)$ for A.*
2. *For each $i' \in I$, the row $r^A_{i'} \stackrel{\text{def}}{=} (M(A)_{ii'})_{i \in I}$ is in $A^{\acute{\perp}}$.*

Proof.

1. **uniqueness** Let λ and λ' be two i-expansions for A that vanish outside M. Then

$$\lambda - \lambda' = (1_i - \lambda') - (1_i - \lambda) \in A$$

and $\lambda - \lambda'$ vanishes outside M, because both λ and λ' do. Since M is linearly independent for A, we obtain $\lambda - \lambda' = 0$ i.e. $\lambda = \lambda'$.

existence If $i \in M$, then put $M(A)_i \stackrel{\text{def}}{=} 1_i$. If $i \in I \setminus M$ then $M \cup \{i\}$ is not linearly independent for A, so there exists nonzero $\mu \in A$ that vanishes outside $M \cup \{i\}$. Since I is linearly independent for A, μ cannot vanish outside M, so $\mu_i \neq 0$. Put $M(A)_i \stackrel{\text{def}}{=} (\mu/\mu_i) - 1_i$. This vanishes outside M, and $1_i - M(A)_i = \mu/\mu_i \in A$.

2. Let $\lambda \in A$. Put $s \stackrel{\text{def}}{=} (\lambda \cdot r^A_{i'})_{i' \in I}$. Then $s = \lambda - \sum_{i \in I} \lambda_i (1_i - M(A)_i) \in A$. But s vanishes outside M, so $s = 0$ by linear independence of M for A. Thus, for $i' \in I$, we have $\lambda \cdot r^A_{i'} = 0$ as required. □

If I is a finite set and $A \in \mathbb{F}(I)$, we write $A^{\perp}$ for $A^{\acute{\perp}} = A^{\grave{\perp}}$.

Lemma 3.5. *Let I be a finite set. For $A \in \mathrm{Sub}\ \mathbb{F}(I)$, we have*

$$\dim A^{\perp} = |I| - \dim A \tag{3.4}$$

$$A^{\perp\perp} = A \tag{3.5}$$

Proof. For (3.4), pick a basis $(M_k)_{k < \dim A}$ for A. This gives a $(\dim A) \times I$-matrix M. The rows of M are linearly independent so M has rank $\dim A$. The linear transformation $\mathbb{F}(I) \longrightarrow \mathbb{F}(\dim A)$ mapping $\lambda \mapsto M\lambda$ has kernel $A^{\perp}$, so $\dim A^{\perp}$ is the nullity of M. The rank-nullity theorem then gives us (3.4).

We clearly have $A \subseteq A^{\perp\perp}$, and (3.4) gives $\dim A^{\perp\perp} = \dim A$, whence (3.5). □

Proof (of Prop. 2.2). Let $A \in \mathrm{Sub}\ \mathbb{F}(I)$ and we shall prove that $A^{\dashv\vdash} = A$. The $\supseteq$ requirement is immediate. For $\subseteq$, suppose $\lambda \in A^{\dashv\vdash}$. Using Zorn's Lemma, pick an index basis $M \subseteq I$ for A. For any $i' \in I$, Lemma 3.4(2) gives us $(M(A)_{ii'})_{i \in I} \in A^{\acute{\perp}}$, and so $\lambda \cdot (M(A)_{ii'})_{i \in I} = \sum_{i \in I} \lambda_i M(A)_{ii'} = 0$, and so $\lambda_{i'} = (\sum_{i \in I} \lambda_i (1_i - M(A)_i))_{i'}$. Hence $\lambda = \sum_{i \in I} \lambda_i (1_i - M(A)_i) \in A$ as required.

Let $S \in \mathrm{ClSub}\ \mathbb{F}^I$ and we shall prove that $S^{\vdash\dashv} = S$. The $\supseteq$ requirement is immediate. For $\subseteq$, suppose $t \in S^{\vdash\dashv}$ and let K be a finite subset of I. We define two functions:

- $G : \mathbb{F}^I \longrightarrow \mathbb{F}^K$, where Gs is s restricted to K
- $H : \mathbb{F}(K) \longrightarrow \mathbb{F}(I)$ where $H\alpha$ maps $i \in K$ to α_i and $i \in I \setminus K$ to 0.

We note that these functions are "adjoint" in the sense that

$$\alpha \cdot (Gs) = (H\alpha) \cdot s \text{ for all } \alpha \in \mathbb{F}(K) \text{ and } s \in \mathbb{F}^I.$$

Let $GS \overset{\text{def}}{=} \{Gs \mid s \in S\}$. Our task is to prove that $Gt \in GS$. By Lemma 3.5, it suffices to prove $Gt \in (GS)^{\perp\perp}$. Suppose $\alpha \in (GS)^{\perp}$. Then for any $s \in S$ we have

$$(H\alpha) \cdot s = \alpha \cdot (Gs) = 0$$

Thus $H\alpha \in S^{\grave{\perp}}$ and so

$$\alpha \cdot (Gt) = (H\alpha) \cdot t = 0$$

as required. $\square$

Proof (of Prop. 2.3). Suppose $\lambda \in A * B$, i.e. there exists an expansion matrix a for A and an expansion matrix b for B satisfying equation (2.2). Suppose $s \in A^{\acute{\perp}}$ and $t \in B^{\acute{\perp}}$. For each $i \in I$ we have $1_i - a_i \in A$ so $0 = (1 - a_i) \cdot s = s_i - \sum_{i' \in I} a_{ii'} s_{i'}$, so $s_i = \sum_{i' \in I} a_{ii'} s_{i'}$. Likewise for each $j \in J$ we have $t_j = \sum_{j' \in J} b_{jj'} t_{j'}$. Then

$$\begin{aligned}
\lambda \cdot (s \otimes t) &= \sum_{i \in I, j \in J} \lambda_{ij} s_i t_j \\
&= \sum_{i \in I, j \in J} \lambda_{ij} (\sum_{i' \in I} a_{ii'} s_{i'})(\sum_{j' \in J} b_{jj'} t_{j'}) \\
&= \sum_{i' \in I, j' \in J} (\sum_{i \in I, j \in J} \lambda_{ij} a_{ii'} b_{jj'}) s_{i'} t_{j'} \\
&= \sum_{i' \in I, j' \in J} 0 s_{i'} t_{j'} \text{ by equation (2.2)} \\
&= 0
\end{aligned}$$

as required.

Conversely, suppose $\lambda \in (A^{\acute{\perp}} \otimes B^{\acute{\perp}})^{\grave{\perp}}$. Using Zorn's Lemma, pick an index basis $M \subseteq I$ for A and an index basis $N \subseteq J$ for B. For any $i' \in I$ and $j' \in J$, Lemma 3.4(2) gives us $(M(A)_{ii'})_{i\in I} \in A^{\acute{\perp}}$ and $(N(B)_{jj'})_{j\in J} \in B^{\acute{\perp}}$, and hence

$$0 = \lambda \cdot ((M(A)_{ii'})_{i\in I} \otimes (N(B)_{jj'})_{j\in J}) = \sum_{i\in I, j\in J} \lambda_{ij} M(A)_{ii'} N(B)_{jj'}$$

We thus set $c = M(A)$ and $d = N(B)$ to obtain equation (2.2). □

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