1 Trace Processes

For a monad T on **Set**, a *submonad* of T is a subset $SX \subseteq TX$ for each set X, such that

- $\eta_X x \in SX$ for all $x \in X$
- $f^* p \in SY$ for all $p \in SX$ and $X \xrightarrow{f} SY$

For any class $\mathcal{K}$ of cardinals, we write $\mathbf{Set}_{\mathcal{K}}$ for the class of sets with cardinality in $\mathcal{K}$. For any set X, we write

$$\mathcal{P}^{\mathcal{K}} X \stackrel{\text{def}}{=} \mathcal{P}X \cap \mathbf{Set}_{\mathcal{K}} = \{U \subseteq X \mid |U| \in \mathcal{K}\}$$

Then the submonads of the powerset monad are $\mathcal{P}^{[o,\kappa)}$, where $o \in \{0,1\}$ and κ is either 2 or a regular infinite cardinal or ∞. We call $[o,\kappa)$ a *range of nondeterminism.*

We recall a basic concept of universal algebra.

Definition 1 *1. A* signature Σ *consists of a set* $|\Sigma|$ *of* function symbols k *each equipped with a set* A_k *called its* arity*. Those* k *for which* A_k *is empty are the* constants *of* Σ*.*

2. The endofunctor H_Σ *on* **Set** *maps* X *to* $\sum_{k \in \Sigma} X^{A_k}$*.*

3. A Σ-transition system *is a set* X *together with a function* $\zeta : X \longrightarrow \mathcal{P}H_\Sigma X$*, i.e. a* $\mathcal{P}H_\Sigma$*-coalgebra. It is* $\mathcal{K}$*-branching, for a range of nondeterminism* $\mathcal{K}$*, when* $|\zeta(x)| \in \mathcal{K}$ *for all* $x \in X$*, i.e. when it is a* $\mathcal{P}^{\mathcal{K}} H_\Sigma$*-coalgebra. For* $x \in X$ *and* $k \in |\Sigma|$ *and* $p \in X^{A_k}$ *we write* $x \overset{k}{\rightsquigarrow} p$ *when* $(k,p) \in \zeta(x)$*.*

4. For $p \in X^A$ *and* $j \in A$ *and* $x \in X$ *we write* $p \xmapsto{j} x$ *when* $x = pj$ *(i.e. when application-to-*j *sends* p *to* x*).*

We can understand a Σ-transition system (X,ζ) in computational terms. Each $x \in X$ is a state that nondeterministically outputs $k \in |\Sigma|$ and then stops and waits for input $j \in A_k$. If input is received, a new state is entered. For example $|\Sigma|$ might consist of four messages,

- "Please enter your name", whose arity is the set of strings
- "Please enter your age", whose arity is $\mathbb{N}$
- "Successful transaction. Hit Return to continue", whose arity is singleton
- "The computer has crashed.", whose arity is the empty set.

When a state outputs the last message, no input is possible. We shall be concerned with traces that record the interaction between the system and the user.

Definition 2 *1. A* Σ-trace *is a finite or infinite sequence* $k_0, p_0, k_1, p_1, \ldots$, *where* k_i *is a function symbol and* $p_i \in A_{k_i}$. *A finite* Σ*-trace is said to be* active *or* passive *according as its length is even or odd.*

2. *We write . for concatenation of traces and* ε *for the empty trace.*

3. *A* Σ-trace process *is a set of passive* Σ*-traces such that* $s.p.k \in \tau$ *implies* $s \in \tau$, *or equivalently every passive prefix of* $s \in \tau$ *is in* τ.

4. *For any* Σ*-trace process* τ, *we set*

$$\tau_{\mathsf{active}} \stackrel{\text{def}}{=} \{\varepsilon\} \cup \{s.k.p \mid s.k \in \tau, p \in A_k\}$$

i.e. those active Σ*-traces whose passive prefixes are all in* τ.

5. *We write* PC *for the set of all* Σ*-trace processes.*

Definition 3 *Let* (X, ζ) *be a* Σ*-transition system.*

1. *Let* $x \in X$. *A (finite or infinite)* Σ*-trace* $k_0, j_0, k_1, j_1, \ldots$ *is a* trace of x *when*

$$x = x_0 \overset{k_0}{\rightsquigarrow} p_0 \overset{j_0}{\longmapsto} x_1 \overset{k_1}{\rightsquigarrow} p_1 \overset{j_1}{\longmapsto} \cdots$$

for some $x = x_0 \in X, p_0 \in X^{A_{k_0}}, x_1 \in X, p_1 \in X^{A_{k_1}}, \ldots$. *We write* $\mathsf{traces}(x)$ *for the set of passive traces of* x. *Clearly* $\mathsf{traces}(x)$ *is a* Σ*-trace process and* $\mathsf{traces}(x)_{\mathsf{active}}$ *is the set of active traces of* x.

2. Trace equivalence *is the kernel of* traces, *i.e. the equivalence relation on* X *given by* $\{(x, y) \mid \mathsf{traces}(x) = \mathsf{traces}(y)\}$.

3. Trace inclusion *is the preorder on* X *given by* $\{(x, y) \mid \mathsf{traces}(x) \subseteq \mathsf{traces}(y)\}$.

Definition 4 *Let* τ *be a* Σ*-trace process.*

1. *For any* $s \in \tau_{\mathsf{active}}$ *we write* $\tau(s)$ *for the set of* $k \in |\Sigma|$ *such that* $s.k \in \tau$, *and* τ/s *for the* Σ*-trace process consisting of* Σ*-traces* t *such that* $s.t \in \tau$.

2. *For any range of nondeterminism* $\mathcal{K}$, *we write* $\mathsf{PC}_{\mathcal{K}}$ *for the set of prefix-closed sets* τ *of passive* Σ*-traces such that* $|\tau(s)| \in \mathcal{K}$ *for all* $s \in \tau_{\mathsf{active}}$. *In particular, we call a* $\tau \in \mathsf{PC}_{[0,2)}$ *is a* partial Σ-tree *and we call a* $\tau \in \mathsf{PC}_{[1,2)}$ *a* total Σ-tree.

Theorem 1 *Let* $\mathcal{K}$ *be a range of nondeterminism. For any* Σ*-trace process* τ, *the following are equivalent:*

- $\tau \in \mathsf{PC}_{\mathcal{K}}$
- *there is a* $\mathcal{K}$*-branching transition system* (X, ζ) *and* $x \in X$ *such that* $\tau = \mathsf{traces}(x)$.

Proof For ($\Leftarrow$): for $x \in X$ and active trace $s = k_0, j_0, \ldots, k_{n-1}, j_{n-1} \in \mathsf{traces}(x)_{\mathsf{active}}$, let $\mathsf{end}_x(s)$ be the set of $y \in X$ such that

$$x = x_0 \overset{k_0}{\rightsquigarrow} p_0 \overset{j_0}{\longmapsto} x_1 \overset{k_1}{\rightsquigarrow} \cdots \overset{k_{n-1}}{\rightsquigarrow} p_{n-1} \overset{j_{n-1}}{\longmapsto} x_n = y$$

for some $x = x_0 \in X, p_0 \in X^{A_{k_0}}, \ldots p_{n-1} \in A^{k_{n-1}}, x_n = y \in X$. We see that $|\mathsf{end}_x(s)| \in \mathcal{K}$ by induction on s, since

$$\begin{array}{rcl} \mathsf{end}_x(\varepsilon) & = & \{x\} \\ \mathsf{end}_x(s.k.j) & = & \displaystyle\bigcup_{y \in \mathsf{end}_x(s)} \{pj \mid (k,p) \in \zeta(y)\} \end{array}$$

Then $\mathsf{traces}(x)(s) = \bigcup_{y \in \mathsf{end}_x(s)} \{k \in |\Sigma| \mid \exists p \in A_k.(k,p) \in \zeta(y)\}$ which must have size in $\mathcal{K}$.

For ($\Rightarrow$), we take $\mathsf{PC}_{\mathcal{K}}$ to be a Σ-transition system, with behaviour

$$\tau \mapsto \{(k, (\tau/k.j \mid j \in A_k)) \mid k \in \tau(\varepsilon)\}$$

Clearly this is $\mathcal{K}$-bounded. For any $\tau \in \mathsf{PC}_{\mathcal{K}}$ and Σ-trace s, we see that s is a trace of τ iff $s \in \tau$. This is proved by induction on the length of s. Therefore $\mathsf{traces}(\tau) = \tau$. $\square$

Definition 5 1. *Let (X, ζ) be a Σ-transition system. A node $x \in X$ is* well-founded *when it has no infinite trace. A node of a Σ-transition system (X, ζ) is* well-founded *when no $x \in X$ has an infinite trace.*

2. *A Σ-trace process τ is* well-founded *when there is no infinite Σ-trace whose passive prefixes are all in τ.*

Clearly if $\mathsf{traces}(x)$ is well-founded then so is x, but not conversely.

Definition 6 *For any range of nondeterminism $\mathcal{K} = [o, \kappa)$, we define $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}} \subseteq \mathsf{PC}_{\mathcal{K}}$ as follows:*

- *for $\kappa = 2$ or $\kappa = \aleph_0$ we define $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ to consist of all $\tau \in \mathsf{PC}_{\mathcal{K}}$ that are well-founded*
- *for $o = 0$ and $\kappa > \aleph_0$, we define $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ to be $\mathsf{PC}_{\mathcal{K}}$*
- *for $o = 1$ and $\kappa > \aleph_0$ we define $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ to consist of all $\tau \in \mathsf{PC}_{\mathcal{K}}$ such that, for every $t \in \tau_{\mathsf{active}}$, there exists a well-founded total Σ-tree $\sigma \subseteq \tau/t$.*

Theorem 2 *Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism. For any Σ-trace process τ, the following are equivalent:*

- $\tau \in \mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$
- *there is a transition system (X, ζ) and well-founded $x \in X$ such that $\tau = \mathsf{traces}(x)$.*

Proof For ($\Leftarrow$): Suppose $\kappa = 2$ or $\kappa = \aleph_0$. We know $\mathsf{traces}(x) \in \mathsf{PC}_{\mathcal{K}}$. To show $\mathsf{traces}(x)$ well-founded (a form of König's Lemma), let $s = k_0, j_0, \ldots$ be an infinite Σ-trace whose passive prefixes are in $\mathsf{traces}(x)$. For a contradiction, we shall construct a sequence

$$x = x_0 \overset{k_0}{\rightsquigarrow} p_0 \overset{j_0}{\longmapsto} x_1 \overset{k_1}{\rightsquigarrow} \cdots \tag{1}$$

so that s is a trace of x, contradicting well-foundedness of x. We shall inductively construct (1) up to x_n, with every passive prefix of $s_n \stackrel{\text{def}}{=} k_n, j_n, k_{n+1}, j_{n+1}, \ldots$ is a trace of x_n. For $n = 0$ this is given by assumption. If we have done this for n, then ζx_n is finite, say $\{(k^0, p^0), \ldots, (k^{r-1}, p^{r-1})\}$. For each $b \in \mathbb{N}$, let $s_{n,b}$ be the prefix of s_{n+1} of length $2b$, and let R_b be the set of $i < r$ such that $k^i = k_n$ and $s_{n,b}$ is a trace of $p^i j_n$. Then $R_0, R_1, R_2, \ldots$ is a decreasing sequence of finite sets, so eventually reaches its minimum R_N. Since $k_n.p_n.s_{n,N}$ is a prefix of s_n, it is a trace of x_n, so R_N is inhabited by some i, and we then set $p_n \stackrel{\text{def}}{=} p^{\hat{\imath}}$ and $x_{n+1} \stackrel{\text{def}}{=} p^{\hat{\imath}} j_n$.

Suppose $\kappa > \aleph_0$ and $o = 1$. (If $o = 0$ there is nothing to prove.) For each $y \in X$ pick an element (k_y, p_y) of $\zeta(x)$, and let $\zeta' : X \to \mathcal{P} H_\Sigma X$ map $y \mapsto \{(k_y, p_y)\}$. This is $\{1\}$-bounded, i.e. deterministic, and well-founded because (X, ζ) is well-founded. Now for $t \in \mathsf{traces}(x)$, define the set $\mathsf{end}_x(t)$ as in the proof of Prop. 1($\Leftarrow$). We showed it to be nonempty so pick an element y. Let σ be the set of traces of y in (X, ζ'), which must be a well-founded total Σ-tree. Finally, if $s \in \sigma$ then $t.s \in \mathsf{traces}(x)$ as required.

For ($\Rightarrow$), suppose $\kappa = 2$ or $\kappa = \aleph_0$. Then take $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ to be a Σ-transition system, with behaviour

$$\tau \mapsto \{(k, (\tau/k.j \mid j \in A_k)) \mid k \in \tau(\varepsilon)\}$$

Clearly this is $\mathcal{K}$-bounded. For any $\tau \in \mathsf{PC}_{\mathcal{K}}$ and Σ-trace s, we see that s is a trace of τ iff $s \in \tau$. This is proved by induction on the length of s. Therefore $\mathsf{traces}(\tau) = \tau$. It must be well-founded because if there is an infinite sequence of transitions (**??**) from τ then every passive prefix of $k_0, p_0, \ldots$ is a trace of τ therefore in τ, a contradiction.

Suppose $\kappa > \aleph_0$ and $o = 1$. For each $\tau \in \mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ pick a well-founded total Σ-tree $\tilde{\tau}$ contained in τ. Informally, we implement τ as follows:

1. choose $n \in \mathbb{N}$
2. implement τ for n pairs of output and input transitions, giving a trace s of length $2n$
3. implement $\widetilde{\tau/s}$.

The states of our system are

$$\begin{aligned} V &\stackrel{\text{def}}{=} \textstyle\sum_{n \in \mathbb{N}} \begin{cases} \mathsf{PC}^{\mathsf{well}}_{1} & \text{if } n = 0 \\ \mathsf{PC}^{\mathsf{well}}_{\mathcal{K}} & \text{otherwise} \end{cases} \\ U &\stackrel{\text{def}}{=} \mathsf{PC}^{\mathsf{well}}_{\mathcal{K}} + V \end{aligned}$$

Informally, V is the set of states with a given quota, whereas U includes the initial states where a quota has not yet been chosen. We define

$$\begin{array}{rcccl} f & : & V & \longrightarrow & \mathcal{P}^{\mathcal{K}} H_\Sigma U \\ & & (0,\tau) & \mapsto & \{(k,(\mathsf{inr}\ (0,\tau/k.j) \mid j \in A_k))\} \text{ where } \tau(\varepsilon)=\{k\} \\ & & (n+1,\tau) & \mapsto & \{(k,(\mathsf{inr}\ (n,\left\{\begin{array}{ll} \widetilde{\tau/k.j} & \text{if } n=0 \\ \tau/k.j & \text{otherwise} \end{array}\right\})) \mid j \in A_k) \mid k \in \tau(\varepsilon)\} \\ \zeta & : & U & \longrightarrow & \mathcal{P}^{\mathcal{K}} H_\Sigma U \\ & & \mathsf{inl}\ \tau & \mapsto & \bigcup_{n\in\mathbb{N}} f(n,\left\{\begin{array}{ll} \widetilde{\tau} & \text{if } n=0 \\ \tau & \text{otherwise} \end{array}\right\}) \\ & & \mathsf{inr}\ (n,\tau) & \mapsto & f(n,\tau) \end{array}$$

giving a $\mathcal{K}$-bounded Σ-transition system (U,ζ). For any $\tau \in \mathsf{PC}_\tau^{\mathsf{well}}$, a passive Σ-trace s is a trace of $\mathsf{inl}\ (0,\tau)$ iff $s \in \tau$, by induction on s. Hence $\mathsf{traces}(\mathsf{inr}\ (0,\tau)) = \tau$ and so $\mathsf{inr}\ (0,\tau)$ is well-founded. Next we show by induction on $n \in \mathbb{N}$ that $\mathsf{traces}(\mathsf{inr}\ (n,\tau))$ is contained in τ and contains all $s \in \tau$ of length $\leqslant 2n$, and $\mathsf{inr}\ (n,\tau)$ is well-founded. Finally we show that $\mathsf{traces}(\mathsf{inl}\ \tau) = \tau$, and $\mathsf{inl}\ \tau$ is well-founded.

For $\kappa > \aleph_0$ and $o = 0$, we use the same algorithm but step (3) is replaced by "do nothing". We accordingly define

$$\begin{array}{rcl} V & \stackrel{\text{def}}{=} & \sum_{n\in\mathbb{N}} \left\{\begin{array}{ll} \{\emptyset\} & \text{if } n=0 \\ \mathsf{PC}_{\mathcal{K}}^{\mathsf{well}} & \text{otherwise} \end{array}\right. \\ U & \stackrel{\text{def}}{=} & \mathsf{PC}_{\mathcal{K}}^{\mathsf{well}} + V \end{array}$$

and we set

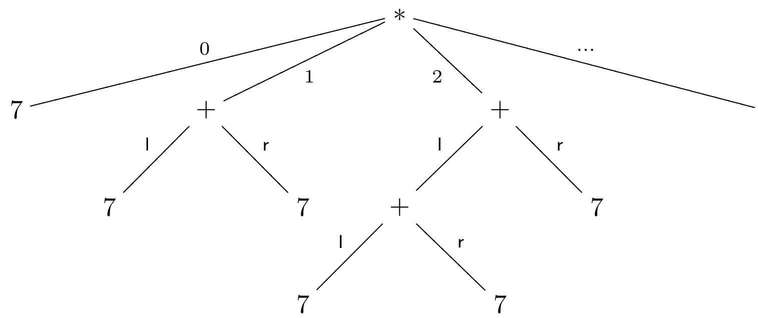
$$\begin{array}{rcccl} f & : & V & \longrightarrow & \mathcal{P}^{\mathcal{K}} H_\Sigma U \\ & & (0,\emptyset) & \mapsto & \emptyset \\ & & (n+1,\tau) & \mapsto & \{(k,(\mathsf{inr}\ (n,\left\{\begin{array}{ll} \emptyset & \text{if } n=0 \\ \tau/k.j & \text{otherwise} \end{array}\right\})) \mid j \in A_k) \mid k \in \tau(\varepsilon)\} \\ \zeta & : & U & \longrightarrow & \mathcal{P}^{\mathcal{K}} H_\Sigma U \\ & & \mathsf{inl}\ \tau & \mapsto & \bigcup_{n\in\mathbb{N}} f(n,\left\{\begin{array}{ll} \emptyset & \text{if } n=0 \\ \tau & \text{otherwise} \end{array}\right\}) \\ & & \mathsf{inr}\ (n,\tau) & \mapsto & f(n,\tau) \end{array}$$

The rest of the proof is unchanged. $\square$

2 Algebras

A Σ-*algebra* is a set Y equipped with a function $\theta_k : Y^{A_k} \longrightarrow Y$ for all $k \in \Sigma$. Equivalently it is an algebra for the endofunctor H_Σ on **Set** mapping Y to $\sum_{k\in\Sigma} Y^{A_k}$.

It is well known that an initial Σ-algebra can be constructed as a set of abstract syntax trees. For example, suppose Σ has an operation $+$ with arity $\{\mathsf{l}, \mathsf{r}\}$, an operation $*$ with arity $\mathbb{N}$, and a constant 7. Then the following is an abstract syntax tree:



To put this in the terms of Sect. 1:

Theorem 3 *1. the set of all well-founded total Σ-trees is an initial Σ-algebra*

2. the set of all total Σ-trees is a terminal H_Σ-coalgebra
In each case the Σ-algebra structure is given by

$$\theta_k : (\tau_i \mid i \in A_k) \mapsto \{k\} \cup \{k.i.s \mid s \in \tau_i\} \quad \text{for } k \in |\Sigma| \tag{2}$$

and the (inverse, by Lambek's Lemma) H_Σ-coalgebra structure maps τ to $\tau \mapsto (k, (\tau/k.j \mid j \in A_{k(\tau)}))$ where $\tau(\varepsilon) = \{k\}$.

Definition 7 *Let T be a monad on* **Set** *and Σ a signature.*

1. *A T, Σ-algebra is a Σ-algebra $(X, (\theta_k)_{k \in |\Sigma|})$ equipped with a T-algebra structure $TX \xrightarrow{\phi} X$.*
2. *For any set Y, a T, Σ-algebra* on *Y is a T, Σ-algebra (X, ϕ, θ) together with a function $Y \xrightarrow{f} X$.*

In general, a T, Σ-algebra on Y corresponds to a $T, \Sigma + Y$-algebra, where $\Sigma + Y$ is the signature Σ extended with Y constants, and therefore a free T, Σ-algebra on Y is the same thing as an initial $T, \Sigma + Y$-algebra. This necessarily exists if T is accessible, by Sect. **??**.

In particular, consider the monad $\mathsf{Maybe} : X \mapsto X + 1$, whose algebras are pointed sets. Then an initial Maybe, Σ-algebra is given by the set of well-founded partial Σ-trees, with Σ-structure given by (2) and point given by the empty set. Equivalently we could take the set of well-founded total $\Sigma + 1$-trees.

3 Nondeterminism and Traces

For any range of nondeterminism $\mathcal{K} = [o, \kappa)$ with $\kappa \geqslant \aleph_0$, an algebra for the monad $\mathcal{P}^{\mathcal{K}}$ is a complete lattice with I-indexed joins for all $I \in \mathbf{Set}_{\mathcal{K}}$.

Definition 8 *Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism with $\kappa \geqslant \aleph_0$, and let $R = (X, \leqslant, \theta)$ be a $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra. R is* quasicommuting *when*

$$\theta_k(\bigvee_{i \in I} x_{i,j} \mid j \in A_k) = \bigvee_{i \in I} \theta_k(x_{i,j} \mid j \in A_k) \tag{3}$$

for every nonempty $I \in \mathbf{Set}_{\mathcal{K}}$ and every $k \in |\Sigma|$. Equivalently[1] *R is quasicommuting when $\theta_k : (X, \leqslant)^{A_k} \longrightarrow (X, \leqslant)$ is monotone for all $k \in |\Sigma|$, and*

$$\theta_k(\bigvee_{i \in I} x_{i,j} \mid j \in A_k) \leqslant \bigvee_{i \in I} \theta_k(x_{i,j} \mid j \in A_k) \tag{4}$$

for every nonempty $I \in \mathbf{Set}_{\mathcal{K}}$ and every $k \in |\Sigma|$.

Lemma 1 *The set PC of prefix-closed sets of passive Σ-traces is a quasicommuting $\mathcal{P}, \Sigma$-algebra, with order given by inclusion and Σ-operations by (2).*

Proof Clearly it has arbitrary suprema given by union, and θ_k is monotone for all $k \in |\Sigma|$. For any nonempty set I and $k \in |\Sigma|$ we have

$$\begin{aligned}
\theta_k(\bigcup_{i \in I} \tau_{i,j} \mid j \in A_k) &= \{k\} \cup \{k.j.s \mid j \in \bigcup_{i \in I} \tau_{i,j}\} \\
&\subseteq \bigcup_{i \in I} (\{k\} \cup \{k.j.s \mid j \in A_k, s \in \tau_{i,j}\}) \text{ since } I \neq \emptyset \\
&= \bigcup_{i \in I} \theta_k(\tau_{i,j} \mid j \in A_k)
\end{aligned}$$

□

In the sequel, we want to construct quasicommuting algebras from arbitrary algebras by quotienting. We therefore require the following concept.

Definition 9 *Let $R = (X, \leqslant, \theta)$ be a $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra.*

1. *A* quasicommuting congruence *on R is an equivalence relation $\equiv$ such that*
   - *$\bigvee_{i \in I}$ preserves $\equiv$, for every $I \in \mathbf{Set}_{\mathcal{K}}$*
   - *θ_k preserves $\equiv$, for every $k \in |\Sigma|$*

[1] For ($\Rightarrow$), to show monotonicity of $k \in |\Sigma|$, if $x_j \leqslant y_j$ for all $j \in A_k$, then

$$\begin{aligned}
\theta_k(x_j \mid j \in A_k) &\leqslant \theta_k(x_j \mid j \in A_k) \vee \theta_k(y_j \mid j \in A_k) \\
&= \theta_k(x_j \vee y_j \mid j \in A_k) \\
&= \theta_k(y_j \mid j \in A_k)
\end{aligned}$$

($\Leftarrow$) is immediate.

- $$\theta_k(\bigvee_{i\in I} x_{i,j} \mid j \in A_k) \equiv \bigvee_{i\in I} \theta_k(x_{i,j} \mid j \in A_k) \tag{5}$$

  for every nonempty $|I| \in \mathbf{Set}_{\mathcal{K}}$ *and every* $k \in |\Sigma|$.

2. *A* quasicommuting precongruence *on* R *is a preorder* $\sqsubseteq$ *such that*

   - $\bigvee_{i\in I} x_i$ *is a* $\sqsubseteq$*-supremum of* $\{x_i \mid i \in I\}$ *for every* $I \in \mathbf{Set}_{\mathcal{K}}$*, or equivalently*[2] *when* $\sqsubseteq$ *contains* $\leqslant$ *and* $\bigvee_{i\in I}$ *preserves* $\sqsubseteq$ *for every* $I \in \mathbf{Set}_{\mathcal{K}}$
   - θ_k *preserves* $\sqsubseteq$*, for every* $k \in |\Sigma|$
   - $$\theta_k(\bigvee_{i\in I} x_{i,j} \mid j \in A_k) \sqsubseteq \bigvee_{i\in I} \theta_k(x_{i,j} \mid j \in A_k) \tag{6}$$

     for every nonempty $|I| \in \mathbf{Set}_{\mathcal{K}}$ *and every* $k \in |\Sigma|$.

Lemma 2 *Let* $R = (X, \leqslant, \theta)$ *be a* $\mathcal{P}^{\mathcal{K}}, \Sigma$*-algebra. Then we have a complete lattice isomorphism* α *from the quasicommuting precongruences on* R *to the quasicommuting congruences on* R *mapping* $\sqsubseteq$ *to its symmetrization. The precongruence corresponding to* $\equiv$ *is given by*

$$\sqsubseteq^{\equiv} \stackrel{\text{def}}{=} \{(x,y) \in X \mid x \vee y \equiv y\}. \tag{7}$$

Proof Suppose $\sqsubseteq$ is a quasicommuting precongruence on R. Then its symmetrization $\equiv$ is clearly a quasicommuting congruence. To show $\sqsubseteq^{\equiv}$ is the same as $\sqsubseteq$, suppose $x \sqsubseteq y$; then y is a $\sqsubseteq$-upper bound of $\{x, y\}$ so $x \vee y \sqsubseteq y$; and automatically $y \sqsubseteq x \vee y$, so $x \vee y \equiv y$. Conversely if $x \sqsubseteq^{\equiv} y$ then $x \sqsubseteq x \vee y \sqsubseteq y$.

Suppose $\equiv$ is a quasicommuting congruence on R. Then the symmetrization of $\sqsubseteq^{\equiv}$ is $\equiv$, because $x \equiv y$ implies $x \vee y \equiv y \vee y = y$ and likewise $y \vee x \equiv x$, and conversely if x and y are related by the symmetrization of $\sqsubseteq^{\equiv}$ then $x \equiv x \vee y \equiv y$. To show $\sqsubseteq^{\equiv}$ is a quasicommuting precongruence:

- if $x \leqslant y$ then $x \vee y = y \equiv y$ so $x \sqsubseteq^{\equiv} y$
- if $x_i \sqsubseteq^{\equiv} y_i$ for all $i \in I$ then

$$\begin{aligned} \bigvee_{i\in I} x_i \vee \bigvee_{i\in I} y_i &= \bigvee_{i\in I} (x_i \vee y_i) \\ &\equiv \bigvee_{i\in I} y_i \end{aligned}$$

[2] To show ($\Rightarrow$), if $x \leqslant y$ then $x \sqsubseteq x \vee y = y$; and if $x_i \sqsubseteq y_i$ for all $i \in I$ then $\bigvee_{i\in I} y_i$ is a $\sqsubseteq$-upper bound for $\{x_i \mid i \in I\}$ so is $\sqsupseteq \bigvee_{i\in I} x_i$. To show ($\Leftarrow$), the element $\bigvee_{i\in I} x_i$ is a $\leqslant$-upper bound and hence a $\sqsubseteq$-upper bound of $\{x_i \mid i \in I\}$, and for any $\sqsubseteq$-upper bound y we have

$$\begin{aligned} \bigvee_{i\in I} x_i &\leqslant (\bigvee_{i\in I} x_i) \vee y \\ &\sqsubseteq (\bigvee_{i\in I} y) \vee y \\ &= y \end{aligned}$$

so $\bigvee_{i\in I} x_i \sqsubseteq^{\equiv} \bigvee_{i\in I} y_i$.

- if $k \in |\Sigma|$ and $x_j \sqsubseteq^{\equiv} y_j$ for all $j \in A_k$ then

$$\begin{array}{rcl} \theta_k(x_j \mid j \in A_k) \vee \theta_k(y_j \mid j \in A_k) & \equiv & \theta_k(x_j \vee y_j \mid j \in A_k) \\ & \equiv & \theta_k(y_j \mid j \in A_k) \\ \text{so } \theta_k(x_j \mid j \in A_k) & \sqsubseteq^{\equiv} & \theta_k(y_j \mid j \in A_k) \end{array}$$

- for $k \in |\Sigma|$ and nonempty $I \in \mathbf{Set}_{\mathcal{K}}$ (6) follows from (5) since $\sqsubseteq$ contains its symmetrization $\equiv$.

Finally it is clear that both symmetrization and $\equiv \mapsto \sqsubseteq^{\equiv}$ are monotone. $\square$

Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism with $\aleph_0 \leqslant \kappa < \infty$. We shall first construct an initial $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra and then quotient by the least quasicommuting congruence to obtain an initial quasicommuting $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra.

Lemma 3 *Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism with $\aleph_0 \leqslant \kappa < \infty$, and let (M, b) be an initial $\mathcal{P}^{\mathcal{K}} H_\Sigma$-algebra.*

1. *M is an initial $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra, with*

$$\begin{array}{rcccl} \theta_k & : & (x_i \mid i \in A_k) & \mapsto & b\{(k, (x_i \mid i \in A_k))\} \\ & & x \leqslant y & \stackrel{\text{def}}{\Leftrightarrow} & b^{-1}x \subseteq b^{-1}y \\ & & \bigvee_{i\in I} x_i & = & b \bigcup_{i\in I} b^{-1} x_i \end{array}$$

2. *Each $x \in M$ is of the form $\bigvee_{i\in I} \theta_{k_i}(x_{i,j} \mid j \in A_k)$ with $I \in \mathbf{Set}_{\mathcal{K}}$.*

Proof

1. This is an instance of Sect. **??**.

2. Put $I = b^{-1}x$. Each $i \in I$ is of the form $(k_i, (x_{i,j} \mid j \in A_{k_i}))$, and we have

$$\begin{array}{rcl} \bigvee_{i\in I} \theta_{k_i}(x_{i,j} \mid j \in A_{k_i}) & = & b \bigcup_{i\in I} b^{-1}\theta_{k_i}(x_{i,j} \mid j \in A_{k_i}) \\ & = & b \bigcup_{i\in I} b^{-1} b\{(k_i, (x_{i,j} \mid j \in A_{k_i}))\} \\ & = & b \bigcup_{i\in I} \{(k_i, (x_{i,j} \mid j \in A_{k_i}))\} \\ & = & b\{(k_i, (x_{i,j} \mid j \in A_{k_i})) \mid i \in I\} \\ & = & b\{i \mid i \in I\} \\ & = & bI \\ & = & x \end{array}$$

$\square$

In fact the least quasicommuting precongruence can be described explicitly.

Lemma 4 *Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism with $\aleph_0 \leqslant \kappa < \infty$. Let (M, b) be an initial $\mathcal{P}^{\mathcal{K}} H_\Sigma$-algebra, and so (M, b^{-1}) is a Σ-transition system.*

1. *The function* $\mathsf{traces} : M \longrightarrow \mathsf{PC}$ *is a homomorphism of $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebras.*
2. *The least quasicommuting precongruence on M is trace inclusion.*
3. *The least quasicommuting congruence on M is trace equivalence.*

Proof

1. Let $k \in |\Sigma|$, and for each $i \in A_k$ let $x_i \in M$. Then

$$\begin{aligned} b^{-1}\theta_k(x_j | j \in A_k) &= \{(k, (x_j \mid j \in A_k))\} \\ \text{so } \mathsf{traces}(\theta_k(x_j | j \in A_k)) &= \{k\} \cup \{k.j.s \mid j \in A_k, s \in \mathsf{traces}(x_i)\} \end{aligned}$$

Let $I \in \mathbf{Set}_{\mathcal{K}}$ and for each $i \in I$ let $x_i \in M$. Then

$$\begin{aligned} b^{-1} \bigvee_{i \in I} x_i &= \bigcup_{i \in I} b^{-1} x_i \\ \text{so } \mathsf{traces}(\bigvee_{i \in I} x_i) &= \bigcup_{i \in I} \mathsf{traces}(x_i) \end{aligned}$$

2. By Lemma 1 and part(1), trace inclusion is a quasicommuting precongruence on M. Given any quasicommuting precongruence $\sqsubseteq$ on M, we show that the set

$$F \stackrel{\text{def}}{=} \{x \in M \mid \forall y \in M.\ \mathsf{traces}(x) \subseteq \mathsf{traces}(y) \Rightarrow x \sqsubseteq y\}$$

is a sub-$\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra of M, and therefore is M, showing that trace inclusion is contained in $\sqsubseteq$. Clearly F is closed under $\bigvee_{i \in I}$, where $I \in \mathbf{Set}_{\mathcal{K}}$. For $k \in |\Sigma|$ and $x_j \in F$ for all $j \in A_k$, we want to show $\theta_k(x_j \mid j \in A_k) \in F$. Suppose $\mathsf{traces}(x) \subseteq \mathsf{traces}(y)$. We have

$$\begin{aligned} \mathsf{traces}(x) &= \mathsf{traces}(\theta_k(x_j \mid j \in A_k)) \\ &= \theta_k(\mathsf{traces}(x_j) \mid j \in A_k) \\ &= \{k\} \cup \{k.j.t \mid j \in A_k, t \in \mathsf{traces}(x_j)\} \end{aligned}$$

By Lemma 3(2) we have $y = \bigvee_{i \in I} \theta_{k_i}(y_{i,j} \mid j \in A_{k_i})$ with $I \in \mathbf{Set}_{\mathcal{K}}$, and then

$$\begin{aligned} \mathsf{traces}(y) &= \mathsf{traces}(\bigvee_{i \in I} \theta_{k_i}(y_{i,j} \mid j \in A_{k_i})) \\ &= \bigcup_{i \in I} \theta_{k_i}(\mathsf{traces}(y_{i,j}) \mid j \in A_{k_i}) \\ &= \bigcup_{i \in I} (\{k_i\} \cup \{k_i.j.t \mid j \in A_{k_i}, t \in \mathsf{traces}(y_{i,j})\}) \end{aligned}$$

Put $I' \stackrel{\text{def}}{=} \{i \in I | k_i = k\}$. The fact that $k \in \mathsf{traces}(y)$ implies I' is nonempty. For all $j \in A_k$, if $t \in \mathsf{traces}(x_j)$ then the fact $k.j.t \in \mathsf{traces}(y)$ implies there exists $i \in I'$ with $t \in \mathsf{traces}(y_{i,j})$. Therefore

$$\begin{aligned} \mathsf{traces}(x_j) &\subseteq \bigcup_{i \in I'} \mathsf{traces}(y_{i,j}) \\ &= \mathsf{traces}(\bigvee_{i \in I'} y_{i,j}) \end{aligned}$$

so $x_j \in F$ gives $x_j \sqsubseteq \bigvee_{i \in I'} y_{i,j}$. Now we reason

$$\begin{aligned} x &= \theta_k(x_j | j \in A_k) \\ &\sqsubseteq \theta_k \left(\bigvee_{i \in I'} y_{i,j} \middle| j \in A_k \right) \\ &= \bigvee_{i \in I'} \theta_k(y_{i,j} | j \in A_k) \\ &\leqslant \bigvee_{i \in I} \theta_{k_i}(y_{i,j} | j \in A_{k_i}) \\ &= y \end{aligned}$$

3. Immediate from part (2) and Lemma 2.

□

Corollary 1 *Let $\mathcal{K} = [o, \kappa)$ be a range of nondeterminism with $\kappa \geqslant \aleph_0$. Then $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$ is an initial quasicommuting $\mathcal{P}^{\mathcal{K}}, \Sigma$-algebra.*

Proof Suppose $\kappa < \infty$. Let (M, b) be an initial $\mathcal{P}^{\mathcal{K}} H_\Sigma$-algebra. By the results of [Tay99], (M, b^{-1}) is a final well-founded $\mathcal{P}^{\mathcal{K}} H_\Sigma$-coalgebra. So the range of $\mathsf{traces} : M \longrightarrow \mathsf{PC}$ is precisely $\mathsf{PC}^{\mathsf{well}}_{\mathcal{K}}$.

Suppose $\mathcal{K} = [o, \infty)$, we can use the same argument where M is a class; alternatively, we can proceed as follows. Pick a regular uncountable cardinal κ such that $||\Sigma|| < \kappa$ and $> |\mathcal{P}\mathsf{PC}| < \kappa$. Then every $\tau \in \mathsf{PC}$ is κ-bounded and so $\mathsf{PC}^{\mathsf{well}}_{[o,\kappa)} = \mathsf{PC}^{\mathsf{well}}_{[o,\infty)}$. For any quasicommuting $\mathcal{P}^{[o,\infty)}, \Sigma$-algebra N, there is a unique $\mathcal{P}^{[o,\kappa)}, \Sigma$-algebra homomorphism $f : \mathsf{PC}^{\mathsf{well}}_{[o,\infty)} \longrightarrow N$. Moreover f preserves the join of every subset $U \subseteq \mathsf{PC}^{\mathsf{well}}_{[o,\infty)}$ since $|U| < \kappa$, so f is a $\mathcal{P}^{[o,\infty)}, \Sigma$-algebra homomorphism. □

References

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