

Oles Embeddings (work in progress)

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November 12, 2015

Background

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It’s connected to numerous structures in the semantics of effects:

- Power and Plotkin’s lookup-update algebras
- Power and Plotkin’s and Melliès’ redundancy theorems for lookup equations
- Power and Shkaravska’s account of arrays as comodels
- Hyland, Plotkin and Power’s combination of a functor and a monad
- My account and Mossakowski and Schröder’s account of monads supporting exception handling
- Hermida and Tennent’s account of monoidal indeterminates
- Johnson et al’s account of lenses.

Three levels of generality

- ① Oles embedding in a category
- ② Oles embedding across an action
- ③ Base for a monad

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Most of the talk will be about (1).

- Oles embeddings and their complements
- Oles expansions and their quotients
- Oles intersections.

Complementors

The **complementor** of an injection $f : A \rightarrowtail B$ is the function $f^c : B \rightarrow B + A$ sending

- $f(a) \mapsto \text{inr } a$
- $b \mapsto \text{inl } b$ if $b \notin \text{range}(f)$.

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We then have the equations:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \text{inr} & \downarrow f^c \\ & & B + A \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{f^c} & B + A \\ & \searrow \text{id} & \downarrow [\text{id}, f] \\ & & B \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{f^c} & B + A \\ f^c \downarrow & & \downarrow f^c + A \\ B + A & \xrightarrow{\text{inl} + A} & (B + A) + A \end{array}$$

Basic definition

Let \mathcal{C} be a category with binary coproducts and initial object.
We form a category $\mathbf{Oles}(\mathcal{C})$ with the same objects as \mathcal{C} .

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An **Oles embedding** $f : A \rightarrowtail B$ consists of

- an map $f^i : A \rightarrow B$ (the **injection**)
- a map $f^c : B \rightarrow B + A$ (the **complementor**)

satisfying the equations:

$$\begin{array}{ccc} A & \xrightarrow{f^i} & B \\ & \searrow \text{inr} & \downarrow f^c \\ & & B + A \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{f^c} & B + A \\ & \searrow \text{id} & \downarrow [\text{id}, f^i] \\ & & B \end{array}$$

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In **Set**, f^i is an injection and f^c is its complementor.

Alternative: don't mention the injector

An Oles embedding $A \hookrightarrow B$ can be described as a map $g : B \rightarrow B + A$ such that

$$B \xrightarrow{g} B + A \begin{array}{c} \xrightarrow{g+A} \\ \xrightarrow{\text{inl}+A} \end{array} (B + A) + A$$

is an equalizer.

Making a category

The identity on A has injector id_A and complementor

$$\text{inr} : A \rightarrow A + A$$

The composite of $f : A \multimap B$ and $g : B \multimap C$ has injector

$$A \xrightarrow{f^i} B \xrightarrow{g^i} C$$

and complementor

$$C \xrightarrow{g^c} C + B \xrightarrow{C+f^c} C + (B + A) \xrightarrow{[\text{inl}, g^i + A]} C + A$$

Basic properties

- $(\text{Oles}(\mathcal{C}), 0, +)$ is a symmetric monoidal category.
- Its groupoid of isomorphisms is the same as that of \mathcal{C} .
- 0 is a strict initial object. (**Strict** means: any morphism to it is an isomorphism.)

Oles embeddings in extensive categories

In an extensive category:

- The forgetful functor $-^i : \mathbf{Oles}\mathcal{C} \rightarrow \mathcal{C}$ is faithful, so complementors are redundant. (Not true in \mathbf{Set}^{op})
- Injectors are monic. (Not true in \mathbf{Set}^{op})

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In an extensive category with 1, an Oles embedding $A \multimap B$ can be described as

- a map $\chi_f : B \rightarrow 1 + 1$ (the **characteristic map**)
- and an isomorphism $A \cong \chi_f^* \top$.

From coproduct embeddings to Oles embeddings

A **coproduct embedding** $A \multimap B$ consists of an object X and $\alpha : X + A \cong B$.

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These give a symmetrical monoidal bicategory with the same objects as \mathcal{C} .

A 2-cell from (X, α) to (Y, β) is $h : X \rightarrow Y$ such that

$$\begin{array}{ccc} X + A & \xrightarrow{\alpha} & B \\ h+A \downarrow & \nearrow \beta & \\ Y + A & & \end{array}$$

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Every coproduct embedding gives rise to an Oles embedding, so there's a symmetric monoidal functor from the bicategory of coproduct embeddings to $\text{Oles}(\mathcal{C})$.

Complements of an Oles embedding

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These form a category.

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Does every Oles embedding have

- a complement? **Not necessarily**
- an essentially unique complement? **If \mathcal{C} is extensive.**
- a terminal complement? **If \mathcal{C} has equalizers preserved by $- + X$**
- an initial complement? **Not necessarily**

The dual story: Oles expansions

Let \mathcal{C} have binary products and a terminal object.

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An **Oles expansion** $A \rightarrow B$ is an Oles embedding in \mathcal{C}^{op}

- a morphism $p : B \rightarrow A$ (the **projection**)
- a morphism $\bullet : B \times A \rightarrow B$ (the **overwriter**)

satisfying

$$\forall b \in B, a \in A. \quad p(b \bullet a) = a$$

$$\forall b \in B. \quad b \bullet p(b) = b$$

$$\forall b \in B, a, a' \in A. \quad (b \bullet a) \bullet a' = b \bullet a'$$

Also called a **very well-behaved lens**.

Oles expansions in **Set**

In **Set**, an Oles expansion $A \rightarrow B$ can be described as

- a map $p : B \rightarrow A$ (the **projection**)
- an equivalence relation \sim on B

such that for every $b \in B$ and $a \in A$

there is unique $c \in B$ such that $c \sim b$ and $p(c) = a$.

Quotients of an Oles expansion

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Oles proved: in **Set**, every expansion has an initial quotient.

Disjoint embeddings

A pair of Oles embeddings B C is **disjoint** when

$$\begin{array}{ccc}
 B & & C \\
 \searrow f & & \swarrow g \\
 & D &
 \end{array}$$

$$\begin{array}{ccc}
 D & \xrightarrow{f^c} & D + B \\
 \downarrow g^c & & \downarrow g^c + B \\
 D + C & \xrightarrow{f^c + C} (D + B) + C \xrightarrow{\cong} (D + C) + B
 \end{array}$$

In **Set**, this just says the ranges are disjoint.

Oles intersection square

A square of Oles embeddings $A \xrightarrow{h} C$ is an **Oles intersection square**

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ k \downarrow & & \downarrow g \\ B & \xrightarrow{f} & D \end{array}$$

when

$$\begin{array}{ccc} D & \xrightarrow{f^c} & D + B \\ g^c \downarrow & & \downarrow g^c + k^c \\ D + C & \xrightarrow{f^c + h^c} (D + B) + (C + A) \xrightarrow{\cong} (D + C) + (B + A) \end{array}$$

In **Set**, this is just an intersection square.

The case $A = 0$ says that f and g are disjoint.

Are they pullbacks?

An Oles intersection square $A \rightrightarrows B$ is not a pullback in general.

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It's also a pullback in $\mathbf{Oles}(\mathcal{C})$, provided $- + Y$ preserves pullbacks.

Oles intersection squares: basic properties

- The identity square $A \xrightarrow{\text{id}} A$ is an Oles intersection.

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ f \downarrow & & \downarrow f \\ B & \xrightarrow{\text{id}} & B \end{array}$$

- Closed under composition.
- Closed under transpose.
- Closed under $+$.

- The square $A \xrightarrow{\text{id}} A$ is an Oles intersection.

$$\begin{array}{ccc} A & \xrightarrow{\text{id}} & A \\ \text{id} \downarrow & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

Oles intersection diagrams

Let I be a finite meet semilattice.

An I -shaped **Oles intersection diagram** is a functor $I \rightarrow \mathbf{OlesC}$ where

$$\begin{array}{ccc} A_{j \wedge k} & \rightrightarrows & A_j \\ \downarrow & & \downarrow \\ A_k & \rightrightarrows & A_i \end{array}$$

is an Oles intersection square for every $j, k \leq i$.

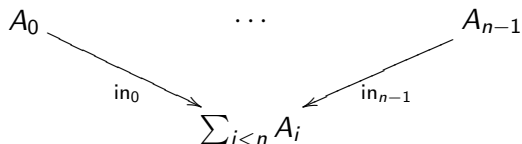
Alternative formulation

A coalgebra for the comonad L on $[I, \mathbf{Set}]$.

$$(LA)_i = \sum_{j \leq i} A_j.$$

Properties of disjointness

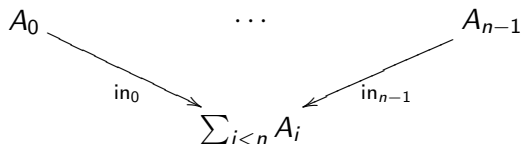
The cocone



is pairwise disjoint.

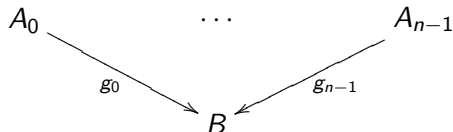
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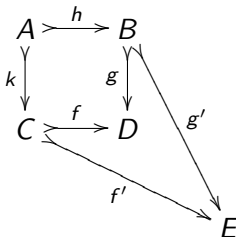
For any pairwise disjoint cocone



there's a unique Oles embedding $\sum_{i < n} A_i \rightarrow B$ that's a morphism of cocones.

Covering intersection squares

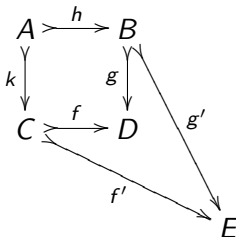
Given two Oles intersection squares



if the inner one is **covering** then there is a unique Oles embedding $D \rightarrowtail E$ that's a morphism of cocones.

Covering intersection squares

Given two Oles intersection squares



if the inner one is **covering** then there is a unique Oles embedding $D \multimap E$ that's a morphism of cocones.

This may be generalized to other diagram shapes.

Base for a monad

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A coalgebra for this comonad is called a **T -base**. Lack, Taylor, Jacobs ...

This consists of an object P and maps $\theta : TP \rightarrow P$ and $\phi : P \rightarrow TP$, satisfying 5 equations, of which 2 are redundant.

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In fact θ is redundant: a T -base can be described as $\phi : P \rightarrow TP$ such that

$$P \xrightarrow{\phi} TP \xrightarrow[T\eta_P]{T\phi} T^2P$$

is an equalizer.

Monoidal actions

A **monoidal action** of a symmetric monoidal category $(\mathcal{C}, I, \otimes)$ on a category \mathcal{D} is a map $\otimes : \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{D}$ and isomorphisms

$$\begin{aligned} P \otimes (B \otimes C) &\cong (P \otimes B) \otimes C \\ P \otimes I &\cong P \end{aligned}$$

satisfying the pentagon and the triangle.

Oles embedding across a monoidal action

Suppose \mathcal{C} has binary coproducts and an initial object, and acts monoidally on \mathcal{D} .

Any A in \mathcal{C} gives a monad $P \mapsto P \otimes A$ on \mathcal{D} .

A base structure on P for this monad is called an **Oles embedding** $A \multimap P$.

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We can compose Oles embeddings

$$A \multimap B \multimap P$$

and speak of disjoint embeddings and intersection squares into P .

Examples: category acting on itself

\mathcal{C} acts monoidally on itself.

This gives Oles embeddings in \mathcal{C} .

Lookup/update algebras

Set^{op} acts monoidally on **Set** via exponentiation.

An Oles embedding from $S \multimap P$ is a **lookup/update algebra** structure on P . Plotkin and Power

A lookup function $P^S \longrightarrow P$ and an update function $P \longrightarrow P^S$ satisfying 5 equations.

Think: P is the set of computations of a given type.

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Disjoint embedding indicate lookup/update for separate cells.

Handling exceptions and reading

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- **Set** acts on **MonadSet** via

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Disjoint embeddings indicate that the effect handling is independent.

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It's the **free extension** of \mathbf{T} by H .

Hyland, Plotkin, Power

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An Oles embedding $H_\Sigma \rightarrow \mathbf{T}$ says how \mathbf{T} models effect handling for I/O.

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Oles embeddings across an action includes many structures in the semantics of state, exceptions and I/O.

Disjoint embeddings indicate that effects are treated independently.