Size restrictions on higher categories

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Working within ZFC set theory, we shall use the following notions of universe.

Definition 1

1. *A* regular universe *is a set $\mathfrak{R}$ such that:*
   - *If $x \in \mathfrak{R}$ and $y \in x$ then $y \in \mathfrak{R}$.*
   - *$\emptyset \in \mathfrak{R}$.*
   - *If $x \in \mathfrak{R}$ then $\{x\} \in \mathfrak{R}$.*
   - *$2 \in \mathfrak{R}$.*
   - *If $I \in \mathfrak{R}$ and $(x_i)_{i \in I}$ is a family of elements of $\mathfrak{R}$ then $\bigcup_{i \in I} x_i \in \mathfrak{R}$.*
2. *A* Grothendieck universe *is a regular universe $\mathfrak{U}$ such that $x \in \mathfrak{U}$ implies $\mathcal{P}x \in \mathfrak{U}$.*

Thus a regular universe models ZFC without the powerset axiom whereas a Grothendieck universe models the whole of ZFC. Examples:

- The set of hereditarily finite sets, written $\mathcal{H}_{<\aleph_0}$, is a Grothendieck universe.
- The set of hereditarily countable sets, written $\mathcal{H}_{\leqslant\aleph_0} = \mathcal{H}_{<\aleph_1}$, is a regular but not a Grothendieck universe.

We often exclude $\mathcal{H}_{<\aleph_0}$ by requiring $\mathbb{N} \in \mathfrak{R}$ in Definition 1(1).

For any cardinal κ, let $\mathcal{H}_{<\kappa}$ be the set of hereditarily $< \kappa$-sized sets. Regular infinite cardinals correspond to regular universes, and strongly inaccessible cardinals to Grothendieck universes, via $\kappa \mapsto \mathcal{H}_\kappa$.

In this note, "n-category" means "strict n-category", and $-1 \leqslant n \leqslant \omega$. There is just one (-1)-category. For the case $n = \omega$ we must assume $\mathbb{N} \in \mathfrak{R}$ and read "$n+1$" and "$n+2$" as ω. I expect a similar story to hold for various notions of weak n-category.

We consider two size restrictions on higher categories.

Definition 2

1. *Let $\mathfrak{R}$ be a regular universe. An n-category $\mathcal{C}$ is* $\mathfrak{R}$-small *when* $\mathsf{ob}\ \mathcal{C}$*, all the homsets, all the 2-homsets, etc., are $\mathfrak{R}$-sets (elements of $\mathfrak{R}$). This is equivalent to the condition $\mathcal{C} \in \mathfrak{R}$. In particular the sole (-1)-category is $\mathfrak{R}$-small.*
2. *Let $\mathfrak{U}$ be a Grothendieck universe. An $(n+1)$-category $\mathcal{C}$ is* $\mathfrak{U}$-included *when* $\mathsf{ob}\ \mathcal{C}$ *is a $\mathfrak{U}$-class (subset of $\mathfrak{U}$) and each hom-n-category is $\mathfrak{U}$-small.*

Thus, for $n \geqslant 0$, any $\mathfrak{U}$-small n-category is $\mathfrak{U}$-included. We abbreviate "$\mathfrak{U}$-included 1-category" by "$\mathfrak{U}$-category", as these occur so frequently. Variations of this are widespread in the literature.

Definition 3 *For any Grothendieck universe $\mathfrak{U}$, let $\mathfrak{U}^{\oplus}$ be the least regular universe that has $\mathcal{P}\mathfrak{U}$ as an element.*

Explicitly, we have $(\mathcal{H}_{<\kappa})^{\oplus} = \mathcal{H}_{\leqslant 2^\kappa} = \mathcal{H}_{<(2^\kappa)^+}$.

We now list the essential properties of smallness and includedness, to see the interplay between the two notions.

Proposition 1 *Let $\mathfrak{U}$ be a Grothendieck universe.*

1. *For any n-categories $\mathcal{C}$ and $\mathcal{D}$, the functor n-category $\mathcal{D}^{\mathcal{C}}$ is*
   - *$\mathfrak{U}$-small if $\mathcal{C}$ and $\mathcal{D}$ are $\mathfrak{U}$-small*
   - *$\mathfrak{U}$-included if $n \geqslant 0$ and $\mathcal{C}$ is $\mathfrak{U}$-small and $\mathcal{D}$ is $\mathfrak{U}$-included*
   - *$\mathfrak{U}^{\oplus}$-small if $n \geqslant 0$ and $\mathcal{C}$ and $\mathcal{D}$ are $\mathfrak{U}$-included.*
2. *The $(n+1)$-category $n\mathbf{Cat}_{\mathfrak{U}}$ of $\mathfrak{U}$-small n-categories is $\mathfrak{U}$-included.*
3. *For $n \geqslant 0$, the $(n+1)$-category $n\mathbf{CAT}_{\mathfrak{U}}$ of $\mathfrak{U}$-included n-categories is $\mathfrak{U}^{\oplus}$-small.*

A final remark. One might have expected that treating higher categories would necessitate a plethora of size restrictions at various levels. Happily this appears not to be the case.

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