The Price of Mathematical Scepticism

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Outline

Classical positions

The "classical" view of mathematics claims the following:

- The Continuum Hypothesis is bivalent,
  i.e. either objectively true or objectively false.
- The Banach-Tarski (sphere duplication) theorem is objectively true.

Some people are sceptical towards these claims.

My thesis

I argue that such scepticism, while legitimate, comes at a price:

- Not being able to rely on mathematics developed in a strong theory such as higher-order arithmetic or ZF.
- Having to entertain the possibility that these theories are inconsistent.
- Having to work in a weaker theory in order to get reliable theorems.

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We shall largely ignore Gödel's theorems, Löwenheim-Skolem, forcing, topos theory etc.

We shall focus on simple intuitions.

List of questionable totalities

- $\mathbb{N}_\mathsf{G}$ consists of all Googolplex-small numbers,
  i.e. natural numbers $< 10^{10^{100}}$.
  According to current scientific theories,
  all physically realizable numbers are Googolplex-small.
- $\mathbb{N}$ consists of all natural numbers $0, 1, 2, \ldots$
- $2^{\mathbb{N}}$ consists of all bitstreams, e.g. 100111. . .
- $2^{2^{\mathbb{N}}}$ consists of all functions $2^{\mathbb{N}} \to \{0, 1\}$.
- Ord consists of all ordinals.
- 2^{Ord} consists of all transfinite bitstreams.

Goldbach variations

Goldbach conjecture

Every even number greater than 2
is a sum of two primes.

Googolplex Goldbach

Every even Googolplex-small number greater than 2
is a sum of two primes.

Liminal Goldbach

$4.01 \times 10^{18} + 4$ is a sum of two primes.
This is the least number that hasn't been checked.

Two forms of choice

Axiom of Choice (AC)

For any sets A and B, and entire relation from A to B,
there's a function $f : A \to B$ such that $\forall x \in A.\, x\, R\, f(x)$.

Dependent Choice (DC)

For any set A, and entire endorelation R on A, and element $a \in A$,
there's a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $x_0 = a$ and $\forall n \in \mathbb{N}.\, x_n\, R\, x_{n+1}$.

DC is a weak form of AC that doesn't yield Banach-Tarski
but is still very useful.

Doubt

In this talk, the words "doubt" and "scepticism" mean a lack of belief.

Example usage

"Since Liminal Goldbach has not yet been checked,
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Of course, there is no obligation to believe that Liminal Goldbach is likely to be false. All the speaker means is that we must entertain the possibility of it being false.

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Mathematicians care about the boundary between established and doubtful propositions.

Our question is: where is that boundary?

Principles of justified belief

- For any proposition, our default attitude is doubt.
- Only intuition and/or proof can move us to belief.
- We must decide which intuitions to accept.
- We don't accept inductive evidence; heuristic arguments; extrinsic justification; inference to the best explanation; geometric or probabilistic intuitions; argument from utility, beauty, indispensability.

Examples

- The inductive/heuristic evidence for Liminal Goldbach is insufficient.
- The inductive evidence for $\mathrm{P} \neq \mathrm{NP}$ is insufficient.
- The extrinsic justification for analytic determinacy is insufficient.
- The usefulness of AC and DC is insufficient.

Engaging the audience

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Please consider what you think and why.
(Not what some hypothetical person could possibly think.)

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Bivalence ambivalence is allowed,
i.e. different parts of your mind can answer differently.

Non-mathematical sentence

Cleopatra hypothesis

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Why the pessimistic assumption?

The question is intended to (crudely) measure your belief in objective reality, even when finding the answer is impossible.
This will not be achieved if you are optimistic.
So, while answering, you must force yourself to be pessimistic.

Computational sentence

Do you think that Googolplex Goldbach is bivalent?
Assume pessimistically that the truth value can't be obtained in the universe's lifetime, even with new plausible axioms.

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If you answered No, end here.

Arithmetical sentence, one quantifier

The Goldbach conjecture is $\forall n \in \mathbb{N}.\, \phi(n)$ ϕ is computational.

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Arithmetical sentence, two quantifiers

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Higher order arithmetic

Second order, one quantifier

The Littlewood conjecture is $\forall x \in 2^{\mathbb{N}}.\, \phi(x)$ $\quad$ ϕ is arithmetical.

Second order, two quantifiers

The Toeplitz conjecture is $\forall x \in 2^{\mathbb{N}}.\, \exists y \in 2^{\mathbb{N}}. \phi(x, y)$ $\quad$ ϕ is arithmetical.

Third order, one quantifier

The Continuum Hypothesis is $\exists x \in 2^{2^{\mathbb{N}}}.\, \phi(x)$ $\quad$ ϕ is second-order.

Third order, two quantifiers

The Suslin Hypothesis is $\forall x \in 2^{2^{\mathbb{N}}}.\, \exists y \in 2^{2^{\mathbb{N}}}.\, \phi(x, y)$ $\quad$ ϕ is second-order.

Sets and classes

Set-theoretic, one quantifier

The Generalized Continuum Hypothesis is

$\forall \kappa \in \mathsf{Ord}.\, \phi(\kappa)$ ϕ is restricted to particular sets.

Set-theoretic, two quantifiers

The Eventually Generalized Continuum Hypothesis is

$\exists \lambda \in \mathsf{Ord}.\, \forall \kappa \in \mathsf{Ord}.\, \phi(\lambda, \kappa)$ ϕ is restricted to particular sets.

Class-theoretic, one quantifier

The Club-Failure Hypothesis is

$\forall X \in 2^{\mathsf{Ord}}.\, \phi(X)$ ϕ is set-theoretic.

Class-theoretic, two quantifiers

The Ord-Suslin Hypothesis is

$\forall X \in 2^{\mathsf{Ord}}.\, \exists Y \in 2^{\mathsf{Ord}}.\, \phi(X, Y)$ ϕ is set-theoretic.

Preview of the taxonomy

- Ultrafinitist: Bivalence of Googolplex Goldbach is doubtful.
- Finitist: Computational sentences are bivalent,
  but bivalence of the Goldbach conjecture is doubtful.
- Countabilist: Arithmetical sentences are bivalent,
  but bivalence of the Littlewood conjecture is doubtful.
- Sequentialist: Second-order arithmetical sentences are bivalent and DC is true,
  but bivalence of the Continuum Hypothesis is doubtful.
- Particularist : Higher-order arithmetical sentences are bivalent and AC is true,
  but bivalence of the Generalized Continuum Hypothesis is doubtful.
- Totalist: Set-theoretic sentences are bivalent,
  but bivalence of the Club-Failure Hypothesis is doubtful.
- Beyond the scope of this talk: Class-theoretic sentences are bivalent.

Bivalence beliefs from intuition

According to our principles, beliefs should come from proof and/or intuition.

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So what are the intuitions that underlie our bivalence beliefs?

My intuitions and yours

I am going to present several intuitions that I experience.

Hopefully these verbal descriptions will resonate with you.

But please remember that words cannot fully capture an intuition.

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We postpone the key question of which intuitions should be accepted.

The intuitions

Googolplex

"I intuit the notion of Googolplex-small number. Since this is a clearly defined notion, quantification over the set $\mathbb{N}_{\mathsf{G}}$ yields an objective truth value."

Arbitrary Natural Number

"I intuit the notion of a natural number, given by zero and successor. This is a clearly defined notion, as restrictive as possible. So quantification over the set $\mathbb{N}$ yields an objective truth value."

The intuitions (continued)

Arbitrary Sequence

"Given a set B, I intuit the notion of a sequence $(x_n)_{n \in \mathbb{N}}$ in B, which consists of successive arbitrary choices of an element of B. This is a clearly defined notion, as liberal as possible. So quantification over the set $B^{\mathbb{N}}$ yields an objective truth value. Since a sequence consists of successive arbitrary choices, DC holds."

Arbitrary Function

"Given sets A and B, I intuit the notion of a function $f : A \to B$, which consists of independent arbitrary choices $f(a) \in B$, one for each $a \in A$. This is a clearly defined notion, as liberal as possible. So quantification over the set B^A yields an objective truth value. Since a function consists of independent arbitrary choices, AC holds."

The intuitions (continued)

Arbitrary Ordinal

"I intuit the notion of an ordinal. This is a clearly defined notion, as liberal as possible. So quantification over the class Ord yields an objective truth value."

Taxonomy

Each school draws a line (the platonic boundary) between the credible and doubtful intuitions.

School	Accepts
Ultrafinitism	Nothing
Finitism	Googolplex
Countabilism	Arbitrary Natural Number
Sequentialism	Arbitrary Sequence
Particularism 	Arbitrary Function
Totalism	Arbitrary Ordinal

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This taxonomy is crude and ignores finer distinctions, e.g. between finitists and constructivists/intuitionists, who accept higher-order constructions.

One quantifier vs two

Some authors (e.g. Kahrs 1999) espouse positivism:
falsifiable sentences are bivalent,
but the bivalence of the Twin Prime conjecture is doubtful.

Our taxonomy does not allow this. For if Arbitrary Natural Number is not accepted, then even the bivalence of the Goldbach conjecture is in doubt.

Likewise, throughout the questionnaire, one-quantifier and two-quantifier examples have the same bivalence status.

Realism and scepticism: possible misconceptions

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while being sceptical beyond the boundary.

To avoid misunderstanding:

- Realism is not knowledge optimism.
- Realism is not essentialism.
- Realism is not mere truth value realism.
- Bivalence doubt is not indeterminism.

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But others consider this unlikely or even impossible.

They still believe there's an objectively correct answer—just not one that can be found.

Likewise for the Cleopatra hypothesis, the twin prime conjecture etc.

Absence (or even impossibility) of knowledge does not entail an absence of objective fact.

Realism is not essentialism

- Essentialism is the idea that something exists "out there" with the essential property of being the number 23.
- Although my verbal description of the intuitions may suggest that realists believe this, they don't.
- The evidence for this is that representation suffices: whenever totality X can be represented in totality Y, belief in Y entails belief in X.

Representation suffices

- People sometimes say: "I believe in the natural numbers but have doubts about the reals."
- Nobody ever says: "I believe in the natural numbers but have doubts about the rationals."
- Rationals can be represented in natural numbers, and vice versa. Therefore, realism about $\mathbb{N}$ and realism about $\mathbb{Q}$ are the same belief.
- Likewise, realism about $2^{\mathbb{N}}$, realism about $\mathbb{R}$ and realism about $\mathbb{C}$ are the same belief.
- Any account of mathematical realism that doesn't recognize this is a misunderstanding.

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That's why the questionnaire works as a realism measuring device.

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To see this, consider a one-quantifier sentence just beyond the platonic boundary.

Example: finitism

A finitist's doubt in Goldbach conjecture bivalence stems from a fear that $\mathbb{N}$ may be unreal.

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According to finitists, this would make the conjecture simply true, assuming that version of $\mathbb{N}$ means at least a model of Robinson arithmetic.

Example: sequentialism

A sequentialist's doubt in CH bivalence stems from a fear that $2^{2^{\mathbb{N}}}$ may be unreal.

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According to sequentialists, this would make CH simply true, assuming that version of $2^{2^{\mathbb{N}}}$ means at least a collection of functions $2^{\mathbb{N}} \to 2$.

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- Totalists argue similarly for doubting 2^{Ord}.

Consistency

- A theory is a system of formal proof for propositions. Example: ZF.
- We define each theory $\mathcal{T}$ so that every proof has alength.
- $\mathrm{Con}(\mathcal{T})$ says that $\mathcal{T}$ is consistent,
  i.e. False has no proof.
  This assertion is falsifiable.
- $\mathrm{Con}_{\mathrm{G}}(\mathcal{T})$ says that $\mathcal{T}$ is Googolplex-consistent,
  i.e. False has no proof whose length is Googolplex-small.
  This assertion is computational.

Classical theories

$$\mathrm{ERA} \subseteq \mathrm{PRA} \subseteq \mathrm{PA} \subseteq \mathrm{Z}_2 \subseteq \mathrm{Z}_3 \subseteq \mathrm{ZF}$$

- Elementary Recursive Arithmetic (ERA) is a theory of exponentiation.
- Primitive Recursive Arithmetic (PRA) is a theory of primitive recursive functions.
- Peano Arithmetic (PA) is a theory of natural numbers.
- Second-order arithmetic (Z_2) is a theory of $\mathbb{N}$ and $2^{\mathbb{N}}$
- Third-order arithmetic (Z_3) is a theory of $\mathbb{N}$ and $2^{\mathbb{N}}$ and $2^{2^{\mathbb{N}}}$.
- Zermelo-Fraenkel (ZF) is a general theory of sets.

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$$\mathrm{ERA} \;\subseteq\; \mathrm{PRA} \;\subseteq\; \mathrm{HA} \;\subseteq\; \mathrm{IIZ}_2 \;\subseteq\; \mathrm{IIZ}_3$$

- If PA is inconsistent, then so is its intuitionistic subsystem, called Heyting arithmetic (HA).
- If Z_2 is inconsistent, then so is its intuitionistic intensional subsystem, called IIZ_2.
- If Z_3 is inconsistent, then so is its intuitionistic intensional subsystem, called IIZ_3.

The consistency question

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Only proof and/or intution will move us to belief.

Gödel's second theorem does not justify relaxing this standard.

The price list

- An ultrafinitist must doubt $\mathsf{Con_G}(\mathrm{ERA})$.
- A finitist without higher-order constructions must doubt $\mathsf{Con_G}(\mathrm{HA})$.
- A countabilist must doubt $\mathsf{Con_G}(\mathrm{IIZ}_2)$.
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To justify this, we argue that each known consistency argument is unavailable to the relevant school.

Consistency via soundness

- ERA is sound and therefore consistent.
  Unavailable to the ultrafinitist.
- PA is sound for $\mathbb{N}$ and therefore consistent.
  Unavailable to the finitist.
- Z_2 is sound for $2^{\mathbb{N}}$ and therefore consistent.
  Unavailable to the countabilist.
- Z_3 is sound for $2^{2^{\mathbb{N}}}$ and therefore consistent.
  Unavailable to the sequentialist.

Other consistency arguments

- Induction up to a suitable ordinal, e.g. ε_0 for $\mathrm{Con}(\mathrm{PA})$.
  Generally unavailable.
- Interpret HA using higher-order constructions.
  Unavailable to a finitist without higher-order constructions.
- Prove $\mathrm{Con}(\mathrm{IIZ}_2)$ via higher-typed bar recursion (Spector).
  Surely unavailable to a countabilist?
- Prove $\mathrm{Con}(\mathrm{IIZ}_2)$ via impredicative definition over $2^{\mathbb{N}}$.
  Unavailable to a countabilist.
- Prove $\mathrm{Con}(\mathrm{IIZ}_3)$ via impredicative definition over $2^{2^{\mathbb{N}}}$.
  Unavailable to a sequentialist.

Future work

- A particularist  doubts the reality of Ord.
  Suggested price: doubting $\mathsf{Con}_{\mathsf{G}}$(Broad ZF).
- A totalist doubts the reality of 2^{Ord}.
  Suggested price: doubting $\mathsf{Con}_{\mathsf{G}}$(Broad Kelley-Morse).

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In particular:

Doubting CH bivalence or AC comes at the price of doubting $\mathsf{Con}_{\mathsf{G}}(\mathrm{IIZ}_3)$.