

The SPINOR object of the CP-PAW code

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Chapter 1

SPINOR object

1.1 Spin orbitals

An electron is specified by a position and a spin. We combine position and spin into a pseudo-fourdimensional vector

$$\vec{x} = (\vec{r}, \sigma) \quad (1.1)$$

where $\sigma \in \{\uparrow, \downarrow\}$.

An electron wave function naturally obtains a two-component form

$$\psi(\vec{x}) = \psi(\vec{r}, \sigma) = \langle \vec{r}, \sigma | \psi \rangle = \langle \vec{x} | \psi \rangle \quad (1.2)$$

Similarly, we combine sum over spin indices and integration over position into a quasi-fourdimensional integration

$$\int d^4x = \sum_{\sigma} \int d^3r \quad (1.3)$$

The identity operator has the form

$$\hat{1} = \int d^4x |\vec{x}\rangle \langle \vec{x}| \quad (1.4)$$

1.2 Pauli matrices and observables

All hermitean matrices in the two-dimensional spinor space can be represented as a superposition of the unit matrix and the three Pauli matrices. In other words, the Pauli matrices including the unit element are a complete basis in the space of all complex 2×2 matrices. All hermitean matrices 2×2 matrices are a superposition of Pauli matrices with real coefficients.

$$\begin{aligned} \mathbf{1} = \sigma^{(0)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{and} & \sigma_x = \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y = \sigma^{(2)} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \text{and} & \sigma_z = \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (1.5)$$

The total density can be expressed by the unit matrix as

$$\rho(\vec{r}) = -e^2 \sum_{\sigma, \sigma'} \langle \psi | \vec{r}, \sigma \rangle \sigma_{\sigma, \sigma'}^{(0)} \langle \vec{r}, \sigma' | \psi \rangle = -e^2 \langle \psi | \left[|\vec{r}\rangle \langle \vec{r}| \circ \hat{\sigma}^{(0)} \right] | \psi \rangle$$

where $\hat{\sigma}^{(0)}$ is an operator in the two-dimensional spinor state. With the symbol “ \circ ” we denote the product of two operators, where each operator acts in its own Hilbert space.

Similarly, we obtain the spin density in the form

$$S_j(\vec{r}) = \frac{\hbar}{2} \sum_{\sigma, \sigma'} \langle \psi | \vec{r}, \sigma \rangle \sigma_{\sigma, \sigma'}^{(j)} \langle \vec{r}, \sigma' | \psi \rangle = \frac{\hbar}{2} \sum_{\sigma, \sigma'} \langle \psi | \left[|\vec{r}\rangle \langle \vec{r}| \circ \hat{\sigma}^{(j)} \right] | \psi \rangle$$

Eigenvectors of Pauli matrices

The eigenvalue equation is for each $j \in \{x, y, z\}$

$$\sigma^{(j)} \xi^{(+j)} = \xi^{(+j)} \quad \text{and} \quad \sigma^{(j)} \xi^{(-j)} = -\xi^{(-j)} \quad (1.6)$$

Thus the eigenvalues are $+1$ and -1 .

The eigenvectors of the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ are

$$\xi^{(\pm x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad \text{and} \quad \xi^{(\pm y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \quad \text{and} \quad \xi^{(\pm z)} = \frac{1}{2} \begin{pmatrix} 1 \pm 1 \\ 1 \mp 1 \end{pmatrix} \quad (1.7)$$

for the eigenvalues ± 1 .

More explicitly,

$$\begin{aligned} \xi^{(+x)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{and} & \quad \xi^{(+y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix} & \text{and} & \quad \xi^{(+z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi^{(-x)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \text{and} & \quad \xi^{(-y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} & \text{and} & \quad \xi^{(-z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \quad (1.8)$$

1.3 Representation of a matrices in terms of Pauli matrices

Definition

Now we introduce a transformation between the two-dimensional matrix representation and the total-spin-vector representation for matrices

TRANSFORMATION OF MATRICES BETWEEN (\uparrow, \downarrow) AND (T, X, Y, Z)
REPRESENTATION

$$\underbrace{\rho_{\alpha, \beta, \sigma, \sigma'} = \frac{1}{2} \sum_{j=0}^3 \bar{\rho}_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)}}_{\text{back transform}} \quad \xleftrightarrow{\text{Eqs. A.4, A.7}} \quad \underbrace{\bar{\rho}_{\alpha, \beta}^{(j)} \stackrel{\text{def}}{=} \sum_{\sigma, \sigma'} \rho_{\alpha, \beta, \sigma, \sigma'} \left(\sigma_{\sigma, \sigma'}^{(j)} \right)^*}_{\text{forward transform}} \quad (1.9)$$

Conversion of a matrix from \uparrow, \downarrow into t, x, y, z representation and vice versa

We transform a matrix using to Eq. 1.9, which yields for the forward transformation

$$\begin{aligned}
 \rho_{a,b}^{(0)} &= \rho_{a,b,\uparrow,\uparrow} + \rho_{a,b,\downarrow,\downarrow} \\
 \rho_{a,b}^{(x)} &= \rho_{a,b,\downarrow,\uparrow} + \rho_{a,b,\uparrow,\downarrow} \\
 \rho_{a,b}^{(y)} &= -i(\rho_{a,b,\downarrow,\uparrow} - \rho_{a,b,\uparrow,\downarrow}) \\
 \rho_{a,b}^{(z)} &= \rho_{a,b,\uparrow,\uparrow} - \rho_{a,b,\downarrow,\downarrow}
 \end{aligned} \tag{1.10}$$

and for the backward transformation

$$\begin{aligned}
 \rho_{a,b,\uparrow,\uparrow} &= \frac{1}{2}(\rho_{a,b}^{(0)} + \rho_{a,b}^{(z)}) \\
 \rho_{a,b,\downarrow,\uparrow} &= \frac{1}{2}(\rho_{a,b}^{(x)} + i\rho_{a,b}^{(y)}) \\
 \rho_{a,b,\uparrow,\downarrow} &= \frac{1}{2}(\rho_{a,b}^{(x)} - i\rho_{a,b}^{(y)}) \\
 \rho_{a,b,\downarrow,\downarrow} &= \frac{1}{2}(\rho_{a,b}^{(0)} - \rho_{a,b}^{(z)})
 \end{aligned} \tag{1.11}$$

In practice we distinguish the three cases, namely non spin-polarized, collinear spin-polarized, and non-collinear.

The same in matrix form

$$\begin{pmatrix} \rho_{ab}^{(0)} \\ \rho_{ab}^{(z)} \\ \rho_{ab}^{(y)} \\ \rho_{ab}^{(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{ab,\uparrow,\uparrow} \\ \rho_{ab,\downarrow,\uparrow} \\ \rho_{ab,\uparrow,\downarrow} \\ \rho_{ab,\downarrow,\downarrow} \end{pmatrix}$$

$$\begin{pmatrix} \rho_{ab,\uparrow,\uparrow} \\ \rho_{ab,\downarrow,\uparrow} \\ \rho_{ab,\uparrow,\downarrow} \\ \rho_{ab,\downarrow,\downarrow} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{ab}^{(0)} \\ \rho_{ab}^{(z)} \\ \rho_{ab}^{(y)} \\ \rho_{ab}^{(x)} \end{pmatrix} \tag{1.12}$$

Motivation

The convention to apply the factor $1/2$ on the back transformation of Eq. 1.9 is motivated as follows: For a collinear spin density in z -direction, the total density is defined as $n_t = n_{\uparrow,\uparrow} + n_{\downarrow,\downarrow}$ and the spin density is defined as $n_s = n_{\uparrow,\uparrow} - n_{\downarrow,\downarrow}$. In that case, the density matrix would have the form

$$n_{\sigma,\sigma'} = \frac{1}{2}n^{(0)}\sigma_{\sigma,\sigma'}^{(0)} + \frac{1}{2}n^{(4)}\sigma_{\sigma,\sigma'}^{(4)} = \begin{pmatrix} \frac{1}{2}n^{(0)} + \frac{1}{2}n^{(4)} & 0 \\ 0 & \frac{1}{2}n^{(0)} - \frac{1}{2}n^{(4)} \end{pmatrix}$$

which allows to identify $n^{(0)} = n_t$ with the total density and $n^{(4)} = n_s$ with the spin density.

This yields

SPIN DEPENDENCE OF THE DENSITY MATRIX

$$\rho(\vec{r}, \vec{r}') = \sum_{\alpha, \beta} \rho_{\alpha, \beta, \sigma, \sigma'} \bar{\chi}_{\alpha}(\vec{r}) \bar{\chi}_{\beta}(\vec{r}') \quad (1.13)$$

$$= \frac{1}{2} \sum_{j=0}^3 \sum_{\alpha, \beta} \rho_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)} \bar{\chi}_{\alpha}(\vec{r}) \bar{\chi}_{\beta}(\vec{r}') \quad (1.14)$$

1.3.1 Identity in a spinor representation

The identity matrix in up-down representation¹ is

$$\begin{pmatrix} \rho_{\uparrow, \uparrow} & \rho_{\uparrow, \downarrow} \\ \rho_{\downarrow, \uparrow} & \rho_{\downarrow, \downarrow} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{\uparrow, \uparrow} & \mathbf{0}_{\uparrow, \downarrow} \\ \mathbf{0}_{\downarrow, \uparrow} & \mathbf{1}_{\downarrow, \downarrow} \end{pmatrix} \quad (1.15)$$

After conversion into the spinor representation we obtain

$$\rho^{(0)} = 2 \cdot \mathbf{1}; \quad \rho^{(1)} = \mathbf{0}; \quad \rho^{(2)} = \mathbf{0}; \quad \rho^{(3)} = \mathbf{0} \quad (1.16)$$

1.3.2 Hermitean conjugate in a spinor representation

The identity matrix in up-down representation is

$$\begin{pmatrix} (\rho^{\dagger})_{\uparrow, \uparrow} & (\rho^{\dagger})_{\uparrow, \downarrow} \\ (\rho^{\dagger})_{\downarrow, \uparrow} & (\rho^{\dagger})_{\downarrow, \downarrow} \end{pmatrix} = \begin{pmatrix} (\rho_{\uparrow, \uparrow})^{\dagger} & (\rho_{\downarrow, \uparrow})^{\dagger} \\ (\rho_{\uparrow, \downarrow})^{\dagger} & (\rho_{\downarrow, \downarrow})^{\dagger} \end{pmatrix} \quad (1.17)$$

After conversion into the spinor representation we obtain

$$(\rho^{\dagger})^{(0)} = (\rho^{(0)})^{\dagger}; \quad (\rho^{\dagger})^{(x)} = (\rho^{(x)})^{\dagger}; \quad (\rho^{\dagger})^{(y)} = (\rho^{(y)})^{\dagger}; \quad (\rho^{\dagger})^{(z)} = (\rho^{(z)})^{\dagger} \quad (1.18)$$

1.3.3 Multiplication of matrices in a spinor representation

Consider two matrices in spinor representation

$$f_{a, b, \sigma, \sigma'} = \frac{1}{2} \sum_{j=0}^3 f_{a, b}^{(j)} \sigma_{\sigma, \sigma'}^{(j)} \quad \text{and} \quad g_{a, b, \sigma, \sigma'} = \frac{1}{2} \sum_{j=0}^3 g_{a, b}^{(j)} \sigma_{\sigma, \sigma'}^{(j)}$$

¹Here each matrix element is itself considered a matrix in the space of spatial orbitals.

We wish to perform a matrix multiplication

$$\begin{aligned}
\sum_{c,\sigma''} f_{a,c,\sigma,\sigma''} g_{c,b,\sigma'',\sigma'} &= \sum_{c,\sigma''} \left(\frac{1}{2} \sum_{j=0}^3 f_{a,c}^{(j)} \sigma_{\sigma,\sigma''}^{(j)} \right) \left(\frac{1}{2} \sum_{j=0}^3 g_{c,b}^{(j)} \sigma_{\sigma'',\sigma'}^{(j)} \right) \\
&= \frac{1}{4} \sum_{i,j=0}^3 \left(\sum_c f_{a,c}^{(i)} g_{c,b}^{(j)} \right) \left(\sum_{\sigma''} \sigma_{\sigma,\sigma''}^{(i)} \sigma_{\sigma'',\sigma'}^{(j)} \right) \\
&\stackrel{\text{Eq. A.11}}{=} \frac{1}{2} \left[\frac{1}{2} \sum_{j=0}^3 \left(\mathbf{f}^{(j)} \mathbf{g}^{(j)} \right)_{a,b} \right] \sigma_{\sigma,\sigma'}^{(0)} \\
&\quad + \frac{1}{2} \sum_{k=1}^3 \left\{ \frac{1}{2} \left(\mathbf{f}^{(0)} \mathbf{g}^{(k)} \right)_{a,b} + \frac{1}{2} \left(\mathbf{f}^{(k)} \mathbf{g}^{(0)} \right)_{a,b} \right. \\
&\quad \left. + \frac{i}{2} \sum_{i,j=1}^3 \epsilon_{i,j,k} \left(\mathbf{f}^{(i)} \mathbf{g}^{(j)} \right)_{a,b} \right\} \sigma_{\sigma,\sigma'}^{(k)}
\end{aligned}$$

Thus, if we denote the multiplication as defined above with the symbol \square , we obtain

$$(\mathbf{f} \square \mathbf{g})_{a,b}^{(0)} = \frac{1}{2} \sum_{j=0}^3 \left(\mathbf{f}^{(j)} \mathbf{g}^{(j)} \right)_{a,b} \quad (1.19)$$

$$(\mathbf{f} \square \mathbf{g})_{a,b}^{(k)} = \frac{1}{2} \left(\mathbf{f}^{(0)} \mathbf{g}^{(k)} \right)_{a,b} + \frac{1}{2} \left(\mathbf{f}^{(k)} \mathbf{g}^{(0)} \right)_{a,b} + \frac{i}{2} \sum_{i,j=1}^3 \epsilon_{i,j,k} \left(\mathbf{f}^{(i)} \mathbf{g}^{(j)} \right)_{a,b} \quad \text{for } j > 0 \quad (1.20)$$

This expression requires 16 matrix multiplication in the a, b, c, \dots space, just as if the operations would be done in the \uparrow, \downarrow representation.

1.3.4 Inversion of a matrix in a spinor representation

The inversion is done by first bringing the matrix into the \uparrow, \downarrow representation using Eq. 1.9.

The problem can then be formulated as a matrix inversion in the (orbital/spin) space

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad (1.21)$$

In components, we obtain

$$\begin{aligned}
\mathbf{A}_{11} \mathbf{B}_{11} + \mathbf{A}_{12} \mathbf{B}_{21} &= \mathbf{1} \\
\mathbf{A}_{11} \mathbf{B}_{12} + \mathbf{A}_{12} \mathbf{B}_{22} &= \mathbf{0} \\
\mathbf{A}_{21} \mathbf{B}_{11} + \mathbf{A}_{22} \mathbf{B}_{21} &= \mathbf{0} \\
\mathbf{A}_{21} \mathbf{B}_{12} + \mathbf{A}_{22} \mathbf{B}_{22} &= \mathbf{1}
\end{aligned} \quad (1.22)$$

which leads to

$$\begin{aligned}
 B_{12} &= -\underbrace{A_{11}^{-1}A_{12}}_{a_{12}} B_{22} \\
 B_{21} &= -\underbrace{A_{22}^{-1}A_{21}}_{a_{21}} B_{11} \\
 B_{11} &= \left(A_{11} - A_{12} \underbrace{A_{22}^{-1}A_{21}}_{-a_{21}} \right)^{-1} \\
 B_{22} &= \left(A_{22} - A_{21} \underbrace{A_{11}^{-1}A_{12}}_{a_{12}} \right)^{-1}
 \end{aligned}$$

The operations are done in the following order

$$\begin{aligned}
 C_{11} &= A_{11}^{-1} \\
 C_{12} &= -C_{11}A_{12} = -A_{11}^{-1}A_{12} \\
 C_{22} &= A_{22} + A_{21}C_{12} = A_{22} - A_{21}A_{11}^{-1}A_{12} \\
 B_{22} &= C_{22}^{-1} = \left(A_{22} - A_{21}A_{11}^{-1}A_{12} \right)^{-1} \\
 B_{12} &= C_{12}B_{22} \\
 C_{22} &= A_{22}^{-1} \\
 C_{21} &= -C_{22}A_{21} = -A_{22}^{-1}A_{21} \\
 C_{11} &= A_{11} + A_{12}C_{21} = A_{11} - A_{12}A_{22}^{-1}A_{21} \\
 B_{11} &= C_{11}^{-1} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21} \right)^{-1} \\
 B_{21} &= C_{21}B_{11}
 \end{aligned}$$

The matrix A_{11}^{-1} can be overlayed with B_{11} , A_{22}^{-1} can be overlayed with B_{22} , C_{12} can be overlayed with B_{12} , and C_{21} can be overlayed with B_{21} .

The operations can be done with 4 matrix inversions and 6 matrix multiplications for a general non-collinear problem. (Operations that scale better than the cube behavior are ignored.)

It may be interesting to analyze the scaling behavior. Consider that the computational effort for an inversion of a matrix with dimension n is an^3 . The effort for a matrix multiplication shall be bn^3 . Thus the scaling will be

$$a(2n)^3 = 4a^3 + 6bn^3 + cn^2 \quad \rightarrow \quad a = \frac{3}{2}b + \frac{c}{4n} \quad (1.23)$$

This implies that a matrix inversion takes about 1.5 times the computational effort of a matrix multiplication.

1.3.5 Hermitean Matrices

A matrix that is hermitean in spin-up-down representation, that is

$$\rho_{a,b,\sigma,\sigma'} = \rho_{b,a,\sigma',\sigma}^* \Leftrightarrow \rho_{a,b}^{(j)} = \left(\rho_{b,a}^{(j)} \right)^* \quad (1.24)$$

has hermitean matrices in total-spin representation, and vice versa

This is derived in the following. First we show that

$$\rho_{a,b}^{(j)} = \left(\rho_{b,a}^{(j)} \right)^* \quad (1.25)$$

can be obtained from the hermitean property in spin space, namely

$$\rho_{a,b,\sigma,\sigma'} = \rho_{b,a,\sigma',\sigma}^* \quad (1.26)$$

This is shown as follows

$$\begin{aligned} \rho_{a,b}^{(j)} &\stackrel{\text{Eq. 1.9}}{=} \sum_{\sigma\sigma'} \rho_{a,b,\sigma,\sigma'} \left(\sigma_{\sigma,\sigma'}^{(j)} \right)^* \stackrel{\text{Eq. 1.26}}{=} \sum_{\sigma\sigma'} \rho_{b,a,\sigma',\sigma}^* \left(\sigma_{\sigma,\sigma'}^{(j)} \right)^* \stackrel{\sigma^{(j)} = \sigma'^{(j),\dagger}}{=} \sum_{\sigma\sigma'} \rho_{b,a,\sigma',\sigma}^* \left(\sigma_{\sigma',\sigma}^{(j)} \right) \\ &\stackrel{\text{Eq. 1.9}}{=} \left(\rho_{b,a}^{(j)} \right)^* \end{aligned} \quad (1.27)$$

Now we derive the opposite direction:

$$\begin{aligned} \rho_{a,b,\sigma,\sigma'} &\stackrel{\text{Eq. 1.9}}{=} \frac{1}{2} \sum_{j=0}^3 \rho_{a,b,\sigma,\sigma'}^{(j)} \sigma_{\sigma,\sigma'}^{(j)} \stackrel{\text{Eq. 1.25}}{=} \frac{1}{2} \sum_{j=0}^3 \left(\rho_{b,a}^{(j)} \right)^* \sigma_{\sigma,\sigma'}^{(j)} \stackrel{\sigma^{(j)} = \sigma'^{(j),\dagger}}{=} \left(\frac{1}{2} \sum_{j=0}^3 \rho_{b,a,\sigma',\sigma}^{(j)} \sigma_{\sigma',\sigma}^{(j)} \right)^* \\ &\stackrel{\text{Eq. 1.9}}{=} \left(\rho_{b,a,\sigma',\sigma} \right)^* \end{aligned} \quad (1.28)$$

1.4 Density matrices and spin orbitals with defined spin

Let us choose a basis set $\{|\chi_\alpha\rangle\}$ with states that are product states of a spatial orbital $\bar{\chi}_\alpha(\vec{r})$ and a spin orbital ξ_α , such as

$$\chi_\alpha(\vec{x}) = \bar{\chi}_\alpha(\vec{r}) \xi_\alpha(\sigma) \quad (1.29)$$

Typically, the spin orbitals are eigenstates to σ_z so that $\xi_\alpha(\sigma) = \delta_{\sigma,\sigma_\alpha}$ and $\sigma_\alpha \in \{\uparrow, \downarrow\} = \{(1,0), (0,1)\}$.

In that case we can write the density matrix

$$\begin{aligned} \rho(\vec{x}, \vec{x}') &= \sum_{\alpha,\beta} \chi_\alpha(\vec{x}) \rho_{\alpha,\beta} \chi_\beta^*(\vec{x}') \\ &= \sum_{\alpha,\beta} \bar{\chi}_\alpha(\vec{r}) \underbrace{\xi_\alpha(\sigma) \rho_{\alpha,\beta} \xi_\beta^*(\sigma')}_{\rho_{\alpha,\beta,\sigma,\sigma'}} \bar{\chi}_\beta^*(\vec{r}') \\ &= \sum_{\alpha,\beta} \bar{\chi}_\alpha(\vec{r}) \rho_{\alpha,\beta,\sigma,\sigma'} \bar{\chi}_\beta^*(\vec{r}') \end{aligned} \quad (1.30)$$

Here we have defined the density matrix with explicit spin dependence

$$\rho_{\alpha,\beta,\sigma,\sigma'} \stackrel{\text{def}}{=} \xi_\alpha(\sigma) \rho_{\alpha,\beta} \xi_\beta^*(\sigma') \quad (1.31)$$

The density matrices $\rho_{\sigma,\sigma'} = \xi(\sigma) \xi^*(\sigma')$ for the spin eigenstates Eq. 1.7 are

$$\rho(\pm x) = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho(\pm y) = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} \quad \text{and} \quad \rho(\pm z) = \frac{1}{2} \begin{pmatrix} 1 \pm 1 & 0 \\ 0 & 1 \mp 1 \end{pmatrix}$$

They obey $\text{Tr}[\rho(\pm j) \sigma^{(j')}] = \pm \delta_{jj'}$ for $j, j' \in \{x, y, z\}$ and $\text{Tr}[\rho(\pm j) \sigma^{(0)}] = 1$.

1.5 Potentials and spin orbitals

General definition of a potential for a complex matrix quantity

POTENTIAL AS ENERGY DERIVATIVE

The potential of a matrix quantity must be written in the following form

$$V = \frac{dE}{dn^*}$$

For Hermitean quantities, this yields

$$V_{\alpha,\beta} = \frac{dE}{d\rho_{\alpha,\beta}^*} = \frac{dE}{d\rho_{\beta,\alpha}}$$

This has the following reasons

- The trace formula comes from

$$dE = \text{Tr}[\hat{V}d\hat{\rho}] = \sum_{\alpha,\beta} \langle \alpha | \hat{V} | \beta \rangle \langle \beta | d\hat{\rho} | \alpha \rangle = \sum_{\alpha,\beta} V_{\alpha,\beta} d\rho_{\beta,\alpha} \stackrel{\rho=\rho^\dagger}{=} \sum_{\alpha,\beta} V_{\alpha,\beta} d\rho_{\alpha,\beta}^* \Rightarrow \frac{\partial E}{\partial \rho_{\alpha,\beta}^*} = V_{\alpha,\beta}$$

- Another form

$$\begin{aligned} E &= F\left[\underbrace{\sum_{\alpha,\beta} \chi_\alpha(\vec{x}) \rho_{\alpha,\beta} \chi_\beta^*(\vec{x}')}_{\rho(\vec{x},\vec{x}')}\right] \\ V_{\alpha,\beta} &= \frac{\partial E}{\partial \rho_{\alpha,\beta}^*} = \left(\frac{\partial E}{\partial \rho_{\alpha,\beta}}\right)^* \stackrel{\text{Eq. 1.30}}{=} \left(\int d^4x \int d^4x' \frac{\partial E}{\partial \rho(\vec{x},\vec{x}')} \chi_\alpha(\vec{x}) \chi_\beta^*(\vec{x}')\right)^* \\ &= \int d^4x \int d^4x' \chi_\alpha^*(\vec{x}) \underbrace{\frac{\partial E}{\partial \rho^*(\vec{x},\vec{x}')} \chi_\beta(\vec{x})}_{v(\vec{x},\vec{x}')} = \int d^4x \int d^4x' \chi_\alpha^*(\vec{x}) v(\vec{x},\vec{x}') \chi_\beta(\vec{x}') \\ &= \langle \chi_\alpha | \hat{V} | \chi_\beta \rangle \end{aligned}$$

- Similarly, we obtain

$$\begin{aligned} E &= F\left[\sum_n \langle \pi_\alpha | \psi_n \rangle f_n \langle \psi_n | \pi_\beta \rangle\right] \\ \frac{\partial E}{\partial \langle \psi_n |} &= \sum_{\alpha,\beta} |\pi_\beta \rangle \frac{\partial F}{\partial \rho_{\alpha,\beta}} \langle \pi_\alpha | \psi_n \rangle f_n = \sum_{\alpha,\beta} |\pi_\beta \rangle \frac{\partial F}{\partial \rho_{\beta,\alpha}^*} \langle \pi_\alpha | \psi_n \rangle f_n \\ &= \sum_{\alpha,\beta} |\pi_\beta \rangle V_{\beta,\alpha} \langle \pi_\alpha | \psi_n \rangle f_n \end{aligned}$$

- with k-points: The density matrix is defined as

$$\rho_{\alpha,\beta}(\vec{t}_\beta) = \frac{1}{N_k} \sum_{\vec{k}} \langle \pi_\alpha | \psi_n(\vec{k}) \rangle f_n(\vec{k}) \langle \psi_n(\vec{k}) | \pi_\beta \rangle e^{i\vec{k}\vec{t}_\beta}$$

The density matrix connects an orbital $|\chi_\alpha\rangle$ at \vec{R}_α with an orbital $|\chi_\beta\rangle$ at $\vec{R}_\beta + \vec{t}_\beta$.

$$\begin{aligned}
 \frac{dF}{d\langle\psi_n(\vec{k})|} &= \sum_{\alpha,\beta,\vec{t}_\beta} \frac{dF}{d\rho_{\alpha,\beta}(\vec{t}_\beta)} \frac{d}{d\langle\psi_n(\vec{k})|} \left[\frac{1}{N_k} \sum_{\vec{k}} \sum_n \langle\pi_\alpha|\psi_n(\vec{k})\rangle f_n(\vec{k}) \langle\psi_n(\vec{k})|\pi_\beta\rangle e^{i\vec{k}\vec{t}_\beta} \right] \\
 &= \frac{1}{N_k} \sum_{\vec{k}} \sum_{\alpha,\beta,\vec{t}_\beta} |\pi_\beta\rangle \left[\frac{dF}{d\rho_{\alpha,\beta}(\vec{t}_\beta)} e^{i\vec{k}\vec{t}_\beta} \langle\pi_\alpha|\psi_n(\vec{k})\rangle f_n(\vec{k}) \right] \\
 &= \frac{1}{N_k} \sum_{\vec{k}} \sum_{\beta} |\pi_\beta\rangle \left[\sum_{\alpha,\vec{t}_\beta} \underbrace{\frac{dF}{d\rho_{\alpha,\beta}(\vec{t}_\beta)}}_{=V_{\beta,\alpha}(\vec{t}_\alpha)} e^{i\vec{k}\vec{t}_\beta} \langle\pi_\alpha|\psi_n(\vec{k})\rangle \right] f_n(\vec{k})
 \end{aligned}$$

In the implementation, I am using the variable hamil sometimes as V and sometimes as derivative of the functional. The two are hermitean adjuncts of each other, i.e. $V_{\alpha,\beta}(\vec{t}) = V_{\beta,\alpha}^*(-\vec{t})$.

Spin potentials

Let us now return to the potentials obtained as derivative with respect to the different forms of the density matrix.

$$\begin{aligned}
 \bar{V}_{\alpha,\beta}^{(j)} &\stackrel{\text{def}}{=} \frac{\partial E}{\partial \bar{\rho}_{\alpha,\beta}^{(j)*}} \\
 V_{\alpha,\beta,\sigma,\sigma'} &= \frac{\partial E}{\partial \rho_{\alpha,\beta,\sigma,\sigma'}^*} = \sum_j \frac{\partial E}{\partial \bar{\rho}_{\alpha,\beta}^{(j)*}} \frac{\partial \bar{\rho}_{\alpha,\beta}^{(j)*}}{\partial \rho_{\alpha,\beta,\sigma,\sigma'}^*} = \sum_j \bar{V}_{\alpha,\beta}^{(j)} \left(\frac{\partial \bar{\rho}_{\alpha,\beta}^{(j)}}{\partial \rho_{\alpha,\beta,\sigma,\sigma'}} \right)^* \\
 &\stackrel{\text{Eq. 1.9}}{=} \sum_j \bar{V}_{\alpha,\beta}^{(j)} \left(\sigma_{\sigma,\sigma'}^{(j)} \right)^*
 \end{aligned}$$

$$\begin{aligned}
dE &= \sum_{a,b,\sigma,\sigma'} \frac{\delta E}{\delta \rho_{a,b,\sigma,\sigma'}} \delta \rho_{a,b,\sigma,\sigma'} = \sum_{a,b,\sigma,\sigma'} V_{b,a,\sigma',\sigma} \delta \rho_{a,b,\sigma,\sigma'} \\
&\stackrel{\text{Eq. 1.10}}{=} \sum_{a,b} \left\{ \underbrace{\frac{\delta E}{\delta \rho_{a,b,\uparrow,\uparrow}}}_{V_{b,a,\uparrow,\uparrow}} \underbrace{\frac{1}{2} \left(\delta \rho_{a,b}^{(t)} + \delta \rho_{a,b}^{(z)} \right)}_{\delta \rho_{a,b,\uparrow,\uparrow}} + \underbrace{\frac{\delta E}{\delta \rho_{a,b,\downarrow,\uparrow}}}_{V_{b,a,\uparrow,\downarrow}} \underbrace{\frac{1}{2} \left(\delta \rho_{a,b}^{(x)} + i \delta \rho_{a,b}^{(y)} \right)}_{\delta \rho_{a,b,\downarrow,\uparrow}} \right. \\
&\quad \left. + \underbrace{\frac{\delta E}{\delta \rho_{a,b,\uparrow,\downarrow}}}_{V_{b,a,\downarrow,\uparrow}} \underbrace{\frac{1}{2} \left(\delta \rho_{a,b}^{(x)} - i \delta \rho_{a,b}^{(y)} \right)}_{\delta \rho_{a,b,\uparrow,\downarrow}} + \underbrace{\frac{\delta E}{\delta \rho_{a,b,\downarrow,\downarrow}}}_{V_{b,a,\downarrow,\downarrow}} \underbrace{\frac{1}{2} \left(\delta \rho_{a,b}^{(t)} - \delta \rho_{a,b}^{(z)} \right)}_{\delta \rho_{a,b,\downarrow,\downarrow}} \right\} \\
&= \sum_{a,b} \left\{ \underbrace{\frac{1}{2} \left(\frac{\delta E}{\delta \rho_{a,b,\uparrow,\uparrow}} + \frac{\delta E}{\delta \rho_{a,b,\downarrow,\downarrow}} \right)}_{\bar{V}_{b,a}^{(t)}} \delta \rho^{(t)} + \underbrace{\frac{1}{2} \left(\frac{\delta E}{\delta \rho_{a,b,\downarrow,\uparrow}} + \frac{\delta E}{\delta \rho_{a,b,\uparrow,\downarrow}} \right)}_{\bar{V}_{b,a}^{(x)}} \delta \rho_{a,b}^{(x)} \right. \\
&\quad \left. + \underbrace{\frac{i}{2} \left(\frac{\delta E}{\delta \rho_{a,b,\downarrow,\uparrow}} - \frac{\delta E}{\delta \rho_{a,b,\uparrow,\downarrow}} \right)}_{\bar{V}_{b,a}^{(y)}} \delta \rho_{a,b}^{(y)} + \underbrace{\frac{1}{2} \left(\frac{\delta E}{\delta \rho_{a,b,\uparrow,\uparrow}} - \frac{\delta E}{\delta \rho_{a,b,\downarrow,\downarrow}} \right)}_{\bar{V}_{b,a}^{(z)}} \delta \rho^{(z)} \right\} \\
&= \sum_{j=0}^3 \sum_{a,b} \bar{V}_{b,a}^{(j)} \delta \rho_{a,b}^{(j)} \tag{1.32}
\end{aligned}$$

Thus

$$\begin{aligned}
\bar{V}_{b,a}^{(t)} &= \frac{1}{2} (V_{b,a,\uparrow,\uparrow} + V_{b,a,\downarrow,\downarrow}) \\
\bar{V}_{b,a}^{(x)} &= \frac{1}{2} (V_{b,a,\uparrow,\downarrow} + V_{b,a,\downarrow,\uparrow}) \\
\bar{V}_{b,a}^{(y)} &= \frac{i}{2} (V_{b,a,\uparrow,\downarrow} - V_{b,a,\downarrow,\uparrow}) \\
\bar{V}_{b,a}^{(z)} &= \frac{1}{2} (V_{b,a,\uparrow,\uparrow} - V_{b,a,\downarrow,\downarrow}) \tag{1.33}
\end{aligned}$$

TRANSFORMATION FROM A TOTAL-SPIN TO AN UP-DOWN REPRESENTATION

$$\rho^{(t)} = \rho_{\uparrow,\uparrow} + \rho_{\downarrow,\downarrow}$$

$$\rho^{(x)} = \rho_{\uparrow,\downarrow} + \rho_{\downarrow,\uparrow}$$

$$\rho^{(y)} = i(\rho_{\uparrow,\downarrow} - \rho_{\downarrow,\uparrow})$$

$$\rho^{(z)} = \rho_{\uparrow,\uparrow} - \rho_{\downarrow,\downarrow}$$

$$\rho_{\uparrow,\uparrow} = \frac{1}{2}(\rho^{(t)} + \rho^{(z)})$$

$$\rho_{\uparrow,\downarrow} = \frac{1}{2}(\rho^{(x)} - i\rho^{(y)})$$

$$\rho_{\downarrow,\uparrow} = \frac{1}{2}(\rho^{(x)} + i\rho^{(y)})$$

$$\rho_{\downarrow,\downarrow} = \frac{1}{2}(\rho^{(t)} - \rho^{(z)})$$

For the potentials $v = \left(\frac{\partial E}{\partial \rho}\right)^*$ we obtain

$$v^{(t)} = \frac{1}{2}(v_{\uparrow,\uparrow} + v_{\downarrow,\downarrow})$$

$$v^{(x)} = \frac{1}{2}(v_{\uparrow,\downarrow} + v_{\downarrow,\uparrow})$$

$$v^{(y)} = \frac{i}{2}(v_{\uparrow,\downarrow} - v_{\downarrow,\uparrow})$$

$$v^{(z)} = \frac{1}{2}(v_{\uparrow,\uparrow} - v_{\downarrow,\downarrow})$$

$$v_{\uparrow,\uparrow} = v^{(t)} + v^{(z)}$$

$$v_{\uparrow,\downarrow} = v^{(x)} - i v^{(y)}$$

$$v_{\downarrow,\uparrow} = v^{(x)} + i v^{(y)}$$

$$v_{\downarrow,\downarrow} = v^{(t)} - v^{(z)}$$

These transformations are used in LDAPLUSU_edft, LDAPLUSU_SPINDENMAT, LMTO_NTBDENMAT, LMTO_NTBDENMATDER, WAVES_DENMAT, WAVES_DENSITY.

1.6 Description of Subroutines

[1]

We consider a Hilbert space of two-component spinor wave functions. A real-space-spin basis is $|\vec{r}, \sigma\rangle$. Instead of the real space position we may also use a set of orbitals $|\alpha, \sigma\rangle$, which are spin eigenstates with the spatial dependence defined by α , that is $\langle \vec{r}, \sigma | \alpha, \sigma \rangle = \langle \vec{r}, \sigma' | \alpha, \sigma' \rangle$ and $\langle \vec{r}, \sigma | \alpha, \sigma' \rangle = 0$ for $\sigma \neq \sigma'$.

In this basis a matrix element has the form

$$A_{\alpha,\beta,\sigma,\sigma'} = \langle \alpha, \sigma | \hat{A} | \beta, \sigma' \rangle$$

$$\hat{A} = \sum_{\alpha,\beta,\sigma,\sigma'} |\alpha, \sigma\rangle A_{\alpha,\beta,\sigma,\sigma'} \langle \beta, \sigma'|$$

An expectation value is obtained by

$$\begin{aligned}
 \langle A \rangle &= \sum_n f_n \langle \psi_n | \hat{A} | \psi_n \rangle \\
 &= \sum_{\alpha, \beta, \sigma, \sigma'} \underbrace{\left(\sum_n \langle \beta, \sigma' | \psi_n \rangle f_n \langle \psi_n | \alpha, \sigma \rangle \right)}_{\rho_{\beta, \alpha, \sigma', \sigma}} A_{\alpha, \beta, \sigma, \sigma'} \\
 &= \sum_{\alpha, \beta, \sigma, \sigma'} \rho_{\beta, \alpha, \sigma', \sigma} A_{\alpha, \beta, \sigma, \sigma'} \tag{1.34}
 \end{aligned}$$

This defines the matrix elements of the density matrix as

$$\rho_{\alpha, \beta, \sigma, \sigma'} = \langle \beta, \sigma' | \psi_n \rangle f_n \langle \psi_n | \alpha, \sigma \rangle \tag{1.35}$$

(t,x,y,z) representation

Let us transform the matrix elements

$$\begin{aligned}
 A_{\alpha, \beta}^{(j)} &= \sum_{\sigma, \sigma'} A_{\alpha, \beta, \sigma, \sigma'} \sigma_{\sigma', \sigma}^{(j)} \\
 \rho_{\alpha, \beta}^{(j)} &= \sum_{\sigma, \sigma'} \rho_{\alpha, \beta, \sigma, \sigma'} \sigma_{\sigma', \sigma}^{(j)} \tag{1.36}
 \end{aligned}$$

The back transformation is correspondingly

$$\begin{aligned}
 A_{\alpha, \beta, \sigma, \sigma'} &= \frac{1}{2} \sum_{j=0}^3 A_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)} \\
 \rho_{\alpha, \beta, \sigma, \sigma'} &= \frac{1}{2} \sum_{j=0}^3 \rho_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)} \tag{1.37}
 \end{aligned}$$

Proof:

$$\begin{aligned}
 \frac{1}{2} \sum_{j=0}^3 \rho_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)} &= \frac{1}{2} \sum_{j=0}^3 \underbrace{\left(\sum_{\sigma'', \sigma'''} \rho_{\alpha, \beta, \sigma'', \sigma'''} \sigma_{\sigma''', \sigma''}^{(j)} \right)}_{\rho_{\alpha, \beta}^{(j)}} \sigma_{\sigma, \sigma'}^{(j)} \\
 &= \sum_{\sigma'', \sigma'''} \rho_{\alpha, \beta, \sigma'', \sigma'''} \underbrace{\left(\frac{1}{2} \sum_{j=0}^3 \sigma_{\sigma''', \sigma''}^{(j)} (\sigma_{\sigma', \sigma}^{(j)})^* \right)}_{\delta_{\sigma''', \sigma'} \delta_{\sigma'', \sigma}} \\
 &= \rho_{\alpha, \beta, \sigma, \sigma'} \tag{1.38}
 \end{aligned}$$

Expectation value by trace

Now we need the expression for the expectation value

$$\begin{aligned}
 \langle A \rangle &= \text{Tr}(\hat{\rho} \hat{A}) = \sum_{\alpha, \beta, \sigma, \sigma'} \rho_{\alpha, \beta, \sigma, \sigma'} A_{\beta, \alpha, \sigma', \sigma} = \sum_{\alpha, \beta, \sigma, \sigma'} \underbrace{\frac{1}{2} \sum_{j=0}^3 \bar{\rho}_{\alpha, \beta}^{(j)} \sigma_{\sigma, \sigma'}^{(j)}}_{\rho_{\alpha, \beta, \sigma, \sigma'}} A_{\beta, \alpha, \sigma', \sigma} \\
 &= \frac{1}{2} \sum_{j=0}^3 \sum_{\alpha, \beta} \bar{\rho}_{\alpha, \beta}^{(j)} \underbrace{\left(\sum_{\sigma, \sigma'} A_{\beta, \alpha, \sigma', \sigma} \sigma_{\sigma, \sigma'}^{(j)} \right)}_{A_{\beta, \alpha}^{(j)}} = \frac{1}{2} \sum_{j=0}^3 \sum_{\alpha, \beta} \bar{\rho}_{\alpha, \beta}^{(j)} A_{\beta, \alpha}^{(j)}
 \end{aligned}$$

Physical

Total density

$$\rho_t = \rho^{(0)} = \rho_{\uparrow, \uparrow} + \rho_{\downarrow, \downarrow} \quad (1.39)$$

$$\rho_x = \rho^{(0)} = \rho_{\uparrow, \downarrow} + \rho_{\downarrow, \uparrow} \quad (1.40)$$

$$\rho_y = \rho^{(0)} = -i(\rho_{\uparrow, \downarrow} - \rho_{\downarrow, \uparrow}) \quad (1.41)$$

$$\rho_z = \rho^{(0)} = \rho_{\uparrow, \uparrow} - \rho_{\downarrow, \downarrow} \quad (1.42)$$

An expectation value is

$$\begin{aligned}
 A_{\uparrow, \uparrow} &= \frac{1}{2} (A^{(t)} + A^{(z)}) \\
 A_{\uparrow, \downarrow} &= \frac{1}{2} (A^{(x)} - iA^{(y)}) \\
 A_{\downarrow, \uparrow} &= \frac{1}{2} (A^{(x)} + iA^{(y)}) \\
 A_{\downarrow, \downarrow} &= \frac{1}{2} (A^{(t)} - A^{(z)})
 \end{aligned} \quad (1.43)$$

$$\begin{aligned}
 dE &= \sum_{\alpha, \beta, \sigma, \sigma'} \frac{dE}{d\rho_{\beta, \alpha, \sigma', \sigma}} d\rho_{\beta, \alpha, \sigma', \sigma} \\
 &= \sum_{\alpha, \beta, \sigma, \sigma'} \sum_{j=0}^3 \frac{dE}{d\rho_{\beta, \alpha}^{(j)}} \frac{d\rho_{\beta, \alpha}^{(j)}}{d\rho_{\beta, \alpha, \sigma', \sigma}} d\rho_{\beta, \alpha, \sigma', \sigma} \\
 &= \sum_{j=0}^3 \sum_{\alpha, \beta} \frac{dE}{d\rho_{\beta, \alpha}^{(j)}} \left(\sum_{\sigma, \sigma'} \sigma_{\sigma, \sigma'}^{(j)} d\rho_{\beta, \alpha, \sigma', \sigma} \right) \\
 &= \sum_{j=0}^3 \sum_{\alpha, \beta} \left(\frac{dE}{d\rho_{\beta, \alpha}^{(j)}} \right) d\rho_{\beta, \alpha}^{(j)}
 \end{aligned} \quad (1.44)$$

There is an ambiguity because of the trace formula

$$A = \frac{1}{2} \sum_{j=0}^3 \rho^{(j)} A^{(j)} \quad (1.45)$$

$$dE = \sum_{j=0}^3 \frac{dE}{d\rho^{(j)}} d\rho^{(j)} \quad (1.46)$$

$$v^{(j)} = 2 \frac{dE}{d\rho^{(j)}} \quad (1.47)$$

Using the transformation equation for expectation values

$$v_{\sigma,\sigma'} = \frac{1}{2} \sum_{j=0}^3 v^{(j)} \sigma_{\sigma,\sigma'}^{(j)} = \sum_{j=0}^3 \frac{dE}{d\rho^{(j)}} \sigma_{\sigma,\sigma'}^{(j)} \quad (1.48)$$

$$\begin{aligned} v_{\uparrow,\uparrow} &= \frac{1}{2}(v_t + v_z) \\ v_{\uparrow,\downarrow} &= \frac{1}{2}(v_x - i v_y) \\ v_{\downarrow,\uparrow} &= \frac{1}{2}(v_x + i v_y) \\ v_{\downarrow,\downarrow} &= \frac{1}{2}(v_t - v_z) \end{aligned} \quad (1.49)$$

1.6.1 SPINOR\$CONVERT

Converts a density matrix from the (t,x,y,z) into the (uu,ud,du,dd) representation. Converting a

$$\begin{aligned} A_{\alpha,\beta}^{(j)} &= \frac{1}{2} \sum_{\sigma,\sigma'} A_{\alpha,\beta,\sigma,\sigma'} \sigma_{\sigma',\sigma}^{(j)} \\ \rho_{\alpha,\beta}^{(j)} &= \sum_{\sigma,\sigma'} \rho_{\alpha,\beta,\sigma,\sigma'} \sigma_{\sigma',\sigma}^{(j)} \end{aligned} \quad (1.50)$$

- ID='FWRD': (TOUPDN=.false.) transforms the density matrix from (uu,ud,du,dd)→(t,x,y,z)
- ID='BACK': (TOUPDN=.true.) (t,x,y,z)→(uu,ud,du,dd)

Appendix A

Vector representation of Pauli matrices

Pauli matrices can be represented as vectors in four dimensions.

$$\boldsymbol{\sigma}^{(j)} \triangleq \vec{\sigma}^{(j)} := \begin{pmatrix} \sigma_{11}^{(j)} \\ \sigma_{12}^{(j)} \\ \sigma_{21}^{(j)} \\ \sigma_{22}^{(j)} \end{pmatrix} \quad (\text{A.1})$$

The usefulness of this representation is that the scalar product of two such vectors can be related to the trace of the corresponding Pauli matrices

$$\left(\vec{\sigma}^{(j)}\right)^* \cdot \vec{\sigma}^{(j')} = \sum_{\sigma, \sigma'} \left(\sigma_{\sigma, \sigma'}^{(j)}\right)^* \sigma_{\sigma, \sigma'}^{(j')} = \sum_{\sigma, \sigma'} \sigma_{\sigma', \sigma}^{(j)} \sigma_{\sigma, \sigma'}^{(j')} = \text{Tr} \left[\boldsymbol{\sigma}^{(j)} \boldsymbol{\sigma}^{(j')} \right] \quad (\text{A.2})$$

We have exploited that a complex conjugation of the Pauli matrices is identical to a transposition, which follows directly from their being hermitean.

The vector representation of the Pauli matrices is

$$\boldsymbol{\sigma}^{(0)} \triangleq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{\sigma}^{(1)} \triangleq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{\sigma}^{(2)} \triangleq \begin{pmatrix} 0 \\ -i \\ i \\ 0 \end{pmatrix} \quad \boldsymbol{\sigma}^{(3)} \triangleq \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

These vectors have length $\sqrt{2}$ and they are mutually orthogonal to each other in the sense

$$\frac{1}{2} \left(\vec{\sigma}^{(j)}\right)^* \cdot \vec{\sigma}^{(j')} = \delta_{j, j'} \quad (\text{A.3})$$

The orthonormality Eq. A.3 of these vectors together with the expression Eq. A.2 for their scalar product establishes

$$\frac{1}{2} \text{Tr} \left[\boldsymbol{\sigma}^{(j)} \boldsymbol{\sigma}^{(j')} \right] = \delta_{j, j'} \quad (\text{A.4})$$

The expression for the scalar products can be generalized to dyadic products in the vector representation. Let us consider the Product

$$\begin{aligned} \sum_{\sigma, \sigma', \bar{\sigma}, \bar{\sigma}'} A_{\sigma, \sigma'} \sigma_{\sigma, \sigma'}^{(j)} \left(\sigma_{\bar{\sigma}, \bar{\sigma}'}^{(j')} \right)^* B_{\bar{\sigma}, \bar{\sigma}'} &= [\vec{A} \cdot \vec{\sigma}^{(j)}] [(\vec{\sigma}^{(j')})^* \cdot \vec{B}] = \vec{A} [(\vec{\sigma}^{(j)})^* \otimes \vec{\sigma}^{(j')}] \vec{B} \\ \Rightarrow \sigma_{\sigma, \sigma'}^{(j)} \left(\sigma_{\bar{\sigma}, \bar{\sigma}'}^{(j')} \right)^* &= [(\vec{\sigma}^{(j)})^* \otimes \vec{\sigma}^{(j')}]_{\sigma, \sigma'; \bar{\sigma}, \bar{\sigma}'} \end{aligned} \quad (\text{A.5})$$

The sum of the outer products of the Pauli matrices in the vector representation Eq. A.1 gives the identity matrix.

$$\frac{1}{2} \sum_{j=0}^3 \vec{\sigma}^{(j)} \otimes (\vec{\sigma}^{(j)})^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.6})$$

Together with Eq. A.5, the above result Eq. A.6 provides the second important relation

$$\frac{1}{2} \sum_j \sigma_{\sigma, \sigma'}^{(j)} \left(\sigma_{\bar{\sigma}, \bar{\sigma}'}^{(j)} \right)^* = \delta_{\sigma, \bar{\sigma}} \delta_{\sigma', \bar{\sigma}'} \quad (\text{A.7})$$

Product table of Pauli matrices

The product table of the Pauli matrices including the unit matrix as element with $j = 0$.

$$\begin{aligned} \sigma^{(i)} \sigma^{(j)} &= \begin{pmatrix} \sigma^{(0)} & \sigma^{(x)} & \sigma^{(y)} & \sigma^{(z)} \\ \sigma^{(x)} & \sigma^{(0)} & i\sigma^{(z)} & -i\sigma^{(y)} \\ \sigma^{(y)} & -i\sigma^{(z)} & \sigma^{(0)} & i\sigma^{(x)} \\ \sigma^{(z)} & i\sigma^{(y)} & -i\sigma^{(x)} & \sigma^{(0)} \end{pmatrix} \\ &= \sum_k \left(\delta_{i,j} \delta_{k,0} + \delta_{i,0} \delta_{j,k} + \delta_{i,k} \delta_{j,0} - 2\delta_{i,0} \delta_{j,0} \delta_{k,0} \right. \\ &\quad \left. + i(1 - \delta_{i,0})(1 - \delta_{j,0})(1 - \delta_{k,0}) \epsilon_{i,j,k} \right) \sigma^{(k)} \end{aligned} \quad (\text{A.8})$$

$$+ i(1 - \delta_{i,0})(1 - \delta_{j,0})(1 - \delta_{k,0}) \epsilon_{i,j,k} \sigma^{(k)} \quad (\text{A.9})$$

Do not get confused, because i is used as index and as $\sqrt{-1}$.

In the more intuitive notation with three-dimensional vectors we obtain

$$\left(\frac{(\sigma^{(0)})^2}{\sigma^{(0)} \vec{\sigma}} \middle| \frac{\sigma^{(0)} \vec{\sigma}}{\sigma^{(i)} \sigma^{(j)}} \right) = \left(\frac{\sigma^{(0)}}{\vec{\sigma}} \middle| \frac{\vec{\sigma}}{\delta_{i,j} \sigma^{(0)} + i \sum_{k=1,3} \epsilon_{i,j,k} \sigma^{(k)}} \right) \quad (\text{A.10})$$

Thus we obtain

$$\sum_{i,j=0}^3 A_{i,j} \boldsymbol{\sigma}^{(i)} \boldsymbol{\sigma}^{(j)} = \left(\sum_{k=0}^3 A_{k,k} \right) \boldsymbol{\sigma}^{(0)} + \sum_{j=1,3} \left(A_{0,j} + A_{j,0} + i \sum_{n,m=1}^3 \epsilon_{j,k,l} A_{k,l} \right) \boldsymbol{\sigma}^{(j)} \quad (\text{A.11})$$

Take care of the extent of the sum indices. Some run over $[0, 1, 2, 3]$, others over $[1, 2, 3]$ only.

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