Chapter 8

The configuration model and the Molloy-Reed condition

8.1 Introduction

Until now we have considered generative models like the small-world network model, the BA model or other evolving network model that try to explain the emerge of the complexity of network from simple dynamic rules. Another approach is instead to consider maximally random networks that display specific properties observed in complex network data, as for example the degree distribution of a network. Here we describe the main results regarding the configuration model that is able to generate maximally random networks with given degree sequence. Interestingly, with this setting, it is possible to fully explore the consequences that the degree distribution of the network can have on other structural properties such as the presence and the size of the giant component, the clustering coefficient and the average distance of the network. It this way it is shown clearly that scale-free networks display a rich a novel scenario for almost every structural property of the network, affecting in a very relevant way the emergence of the giant component, the average distance and the clustering coefficient of the network. It is therefore to be also expected that these topologies will behave in a non trivial way beyond random damage, or targeted attack and they will have special properties when dynamical processes will be defined on them. Although generative network model have the power to explain why we do observe some non trivial properties of complex networks, it is undoubtedly clear that random complex networks like the one generated by the configuration model are especially useful when one wants to characterize the effect of some given structural property of the network. For example they are very precious way for analysing the effect of the the degree distribution on the dynamical processes defined on the networks or on other structural properties.

8.2 The configuration model

Definition 1. *The* configuration model *is the set (or ensemble) of networks with N nodes and with given degree sequence $\{k_i\} = (k_1, k_2, k_3, \dots k_N)$ where k_i is the degree of node $i = 1, 2 \dots, N$.*

A network in the configuration model can be generated by the following recursive procedure: Given a degree sequence $\{k_i\} = (k_1, k_2, \dots, k_N)$ with an even $\sum_j k_j$

- *Step a)*
  We place k_i stubs on each node i of the network.
- *Step b)*
  We match each stub of the network with another stub of the network.
- *Step c)*
  We repeat step b) until all the stubs of the network are matched. *Step d)* If the network constructed in this way contains multiedges and tadpoles repeat step b and step c.

8.3 Uncorrelated networks

Definition 2. *A network with degree distribution $\{k_i\}$ is* uncorrelated *if the probability p_{ij} that a node i is connected to a node j is given by*

$$p_{ij} = \frac{k_i k_j}{\langle k \rangle N}, \tag{8.1}$$

for any pair of nodes (i, j) of the network. Moreover, the probability q_j that one link of any node $i$of the network is linked to a node j is given by

$$q_j = \frac{k_j}{\langle k \rangle N} \tag{8.2}$$

Definition 3. *The* structural cutoff *of a network is a maximal allowed degree given by*

$$K = \sqrt{\langle k \rangle N}. \tag{8.3}$$

Therefore if a network has a structural cutoff we have

$$k_i \leq K = \sqrt{\langle k \rangle N} \tag{8.4}$$

for every node $i = 1, 2 \dots, N$.

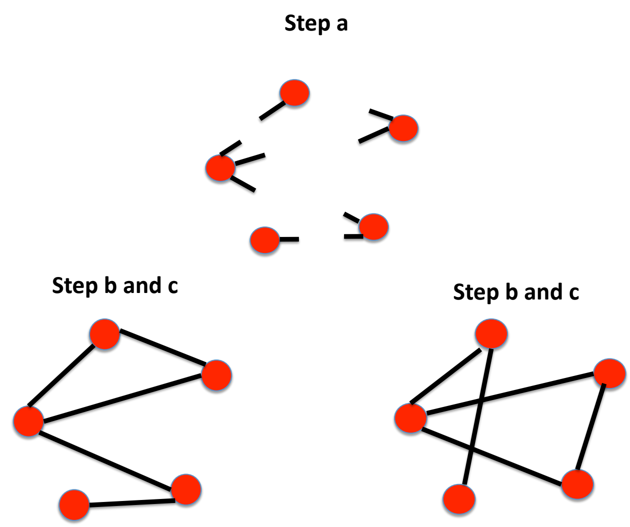

Figure 8.1: Construction of a network in the configuration model. In step a) the k_i stubs are placed on each node i of the network. In step b) and c) all the stubs are repeatedly matched until the full network if formed. In the figure we show two possible networks generated by the configuration model starting from the same degree distribution. Starting from the degree distribution in this figure, one can construct 6 different simple networks, here we just show two network realizations.

Proposition 8.3.1. *The networks generated with the configuration model are uncorrelated if and only if they maximal degree*

$$K \leq \sqrt{\langle k \rangle N}. \tag{8.5}$$

Therefore networks generated by the configuration model with structural cutoff are uncorrelated.

Proof. Here we will prove only the necessary condition, i.e. a necessary condition for generating uncorrelated networks with the configuration model is that the degree sequence has a cutoff $K \leq \sqrt{\langle k \rangle N}$. In fact if the networks are uncorrelated then the probability that a node i is connected to a node j is given by

$$p_{ij} = \frac{k_i k_j}{\langle k \rangle N}. \tag{8.6}$$

by putting $k_i = K$ and $k_j = K$ where K is the maximal degree of the network and imposing that p_{ij} is a probability, i.e. $p_{ij} \leq 1$ we have

$$p_{ij} = \frac{K^2}{\langle k \rangle N} \leq 1, \tag{8.7}$$

which imply that the maximal degree must satisfy

$$K \leq \sqrt{\langle k \rangle N}. \tag{8.8}$$

One can also prove that the presence of a structural cutoff in the networks is a sufficient condition for generating uncorrelated networks with the configuration model. □

Proposition 8.3.2. *In an uncorrelated network the probability $q(k)$ that by following a link in the network we reach a node of degree k is given by*

$$q(k) = \frac{k}{\langle k \rangle} P(k). \tag{8.9}$$

Proof. In fact, assume that we follow a link of a generic node i of the network, the probability q_{ij} that we reach a node j with degree $k_j = k$ is given by

$$q_{ij} = \frac{k}{\langle k \rangle N}, \tag{8.10}$$

and only depend on the degree $k_j = k$ of the target node. The number of nodes with the degree k is given by $NP(k)$.
Therefore the probability $q(k)$ that by following a link of the network we reach a node of degree k is given by

$$q(k) = \frac{k}{\langle k \rangle N} NP(k) = \frac{k}{\langle k \rangle} P(k). \tag{8.11}$$

□

Proposition 8.3.3. *The average degree of the neighbours of a random node in the configuration model with structural cutoff is given by*

$$\sum_k kq(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}. \tag{8.12}$$

Proof. In fact, $q(k)$ is the probability that a neighbour of a node has degree k, therefore the average degree of a neighbour of a node is

$$\sum_k kq(k) = \sum_k k \frac{k}{\langle k \rangle} P(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}. \tag{8.13}$$

□

Note that in such networks the neighbours of a random node have in average more links that the starting node. This phenomenon can be expressed by the sentence that applies to social networks:
Your friends have more friends than you do!
Or, more properly speaking, given a random person in a social network
His/her friends have more friends than him/her! In fact let us assume that the social network is uncorrelated *(note that this assumption does not hold since social networks tend to have hubs more likely connected to hubs than to nodes of small degree, but this effects remains also in presence of correlations most of the cases).* In this network the average number of friends of a node is given by $\frac{\langle k^2 \rangle}{\langle k \rangle}$, while the average degree of a random node is given by $\langle k \rangle$. But we always have

$$\frac{\langle k^2 \rangle}{\langle k \rangle} > \langle k \rangle, \tag{8.14}$$

as soon as the network is not a regular network with all the nodes having the same degree. In fact we have always

$$\langle k^2 \rangle - \langle k \rangle^2 = \left\langle (k - \langle k \rangle)^2 \right\rangle \geq 0, \tag{8.15}$$

where the equality holds only if all the nodes has the same degree.

8.4 Birth of the giant component and Molloy-Reed criterion

We have already seen that the giant component emerges in Poisson networks at a critical value of the average degree $\langle k \rangle = c = 1$. Poisson networks with average degree $c < 1$ do not have a giant component, i.e. the fraction of nodes in the giant component is $S = 0$ in the limit, $N \to \infty$ while Poisson networks with average degree $c > 1$ have a giant component. Therefore if $c = \langle k \rangle > 1$ Poisson networks the have a positive fraction of nodes S in the giant component,

i.e. $S > 0$. This drastic change in the structure of the network, is characterized with the same tools used to study phase transitions in condensed matter, (ex. the transition between a ferromagnetic and a paramagnetic material as a function of the temperature). In this chapter we will study how to characterize the emergence of the giant component in sparse networks with generic degree distribution $P(k)$ and finite average degree $\langle k\rangle$. Interestingly we will show that the main parameter that determines weather or not there is a giant component in the network is not given in general by the average degree but it given by $\frac{\langle k(k-1)\rangle}{\langle k\rangle}$. We start by defining a recursive criterion for determining is a node of the network is in the giant component. Applying this definition we will first find the equation for the fraction S of nodes in the giant component of a network with degree distribution $P(k)$, and secondly we will show that a network has a giant component $S > 0$ if and only if $\frac{\langle k(k-1)\rangle}{\langle k\rangle} > 1$ or, equivalently if and only if $\frac{\langle k^2\rangle}{\langle k\rangle} > 2$.
Let us stat with a recursive algorithm to define is a node is in the giant component of a network.

Definition 4. *A node is in the giant component of the network if, at least one of its links reach a node that is also in the giant component of the network.*
A node reached by following a link of a network is in the giant component if at least one of the nodes reached by following one of the other links of the node is also in the giant component.

Proposition 8.4.1. *The probability S' that by following a link, in a locally tree-like network with degree distribution $P(k)$ we reach a node in the giant component, needs to satisfy the following equation:*

$$S' = 1 - \sum_k \frac{k}{\langle k\rangle} P(k)(1-S')^{k-1}. \tag{8.16}$$

The fraction of nodes S that are in the giant component of the same network is given by

$$S = 1 - \sum_k P(k)(1-S')^k, \tag{8.17}$$

*where S' is the solution of Eq. (**??**).*

Proof. To find the equation Eq. (8.16) for S' we use the recursive rule for determining is a node reached by following a link in the network is in the giant component. By following a link we reach a node of degree k with probability $q_k = kP(k)/\langle k\rangle$, the probability that at least one of the remaining $k-1$ links of this node reach a node in the giant component is

$$1-(1-S')^{k-1}, \tag{8.18}$$

where we have assumed that the network is locally tree-like and neglected any possible correlations between the fact that two or more neighbours of the same node are/(are not) in the giant component. Therefore summing over all the possible degrees k of the node reached by following a link, we have

$$\begin{aligned} S' &= \sum_k \frac{k}{\langle k\rangle} P(k)\left[1-(1-S')^{k-1}\right] \\ S' &= 1-\sum_k \frac{k}{\langle k\rangle} P(k)(1-S')^{k-1}. \end{aligned} \tag{8.19}$$

To find the expression for S, the fraction of nodes in the giant component of the network, we first notice that S indicates also the probability that a random node is in the giant component, when we consider the limit $N \to \infty$. A random node of the network has degree k with probability $P(k)$. The probability that a node of degree k is not in the giant component is given by the probability that all this links reach nodes that are not in the giant component, therefore we have

$$1-S=\sum_k P(k)(1-S')^k. \tag{8.20}$$

Finally the fraction S of nodes in the giant component can be written as

$$S=1-\sum_k P(k)(1-S')^k. \tag{8.21}$$

□

Proposition 8.4.2. *The Molloy-Reed criterion for having a giant component is the following: a sparse random network with degree distribution $P(k)$ has a giant component if and only if*

$$\frac{\langle k^2\rangle}{\langle k\rangle} > 2. \tag{8.22}$$

Proof. the fraction of nodes S in the giant component satisfies Eq. (8.17), i.e.

$$S=1-\sum_k P(k)(1-S')^k, \tag{8.23}$$

therefore there is a giant component in the network ($S>0$) if and only if $S'>0$. The probability S' satisfies Eq. (8.16) given by

$$S'=1-\sum_k \frac{k}{\langle k\rangle} P(k)(1-S')^{k-1}. \tag{8.24}$$

This equation is always satisfied for $S'=0$, but, depending on the properties of the degree distribution $P(k)$ it can have another non-trivial solution $S'>0$. Unfortunately this equation cannot be solved analytically for arbitrary value of

S'. For this reason we will make use of some graphical argument. The solution of Eq. (8.16) can be seen as the value of S' where the two functions $y = f(S')$ and $y = g(S')$ with

$$\begin{aligned} f(S') &= S' \\ g(S') &= 1 - \sum_k \frac{k}{\langle k\rangle} P(k)(1-S')^{k-1} \end{aligned} \tag{8.25}$$

cross.

Since the function $g(S')$ is an increasing function of S', the non trivial solution $S' > 0$ emerges when the functions $y = f(S')$ and $y = g(S')$ are tangent to each other at $S' = 0$.

In order to detect when this new solution emerges, we impose therefore

$$\begin{aligned} \left.\frac{dS'}{dS'}\right|_{S'=0} &= \left.\frac{d(1-\sum_k \frac{k}{\langle k\rangle} P(k)(1-S')^{k-1})}{dS'}\right|_{S'=0}, \\ 1 &= \left.\sum_k \frac{k(k-1)}{\langle k\rangle} P(k)\right|_{S'=0}, \\ 1 &= \frac{\langle k(k-1)\rangle}{\langle k\rangle} \end{aligned} \tag{8.26}$$

Therefore a random network generated with the configuration model will have a giant component if an only if

$$\frac{\langle k(k-1)\rangle}{\langle k\rangle} > 1, \tag{8.27}$$

or

$$\frac{\langle k^2\rangle}{\langle k\rangle^2} > 2. \tag{8.28}$$

□

8.4.1 Giant component in Poisson and scale-free networks

Proposition 8.4.3. *The Molloy-Reed condition for having a giant component in a Poisson network reduced to the already obtained necessary and sufficient condition*

$$c = \langle k\rangle > 1. \tag{8.29}$$

Proof. In fact for a Poisson degree distribution $P(k) = c^k e^{-c}/k!$ we have $\langle k(k-1)\rangle = c^2$ and $\langle k\rangle = c$. Therefore the Molloy-Reed condition can be written as

$$\begin{aligned} \frac{\langle k(k-1)\rangle}{\langle k\rangle} &> 1 \\ \frac{c^2}{c} = c = \langle k\rangle &> 1. \end{aligned} \tag{8.30}$$

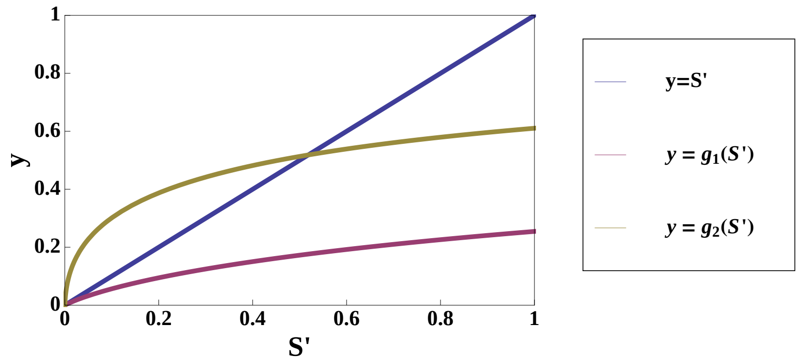

Figure 8.2: The graphical solution of Eq. (8.16).

□

In a Poisson network we found a critical value of the average degree $\langle k\rangle = c = 1$ necessary for having a giant component in the network. In scale-free network the situation is significantly different. In fact it is not the average degree that is determining if the network has a giant component, but the ratio $\langle k(k-1)\rangle/\langle k\rangle$. As we will see then scale-free networks with $\gamma \in (2,3]$ have always a non vanishing giant component independently on their average degree. This is one of the most important signals that these structures are also more robust to random damage.

Proposition 8.4.4. *Uncorrelated sparse scale-free networks with degree distribution $P(k) = Ck^{-\gamma}$ and $\gamma \in (2,3]$ have always a giant component in the limit $N \to \infty$.*

Proof. Let us consider uncorrelated power-law networks with degree distribution $P(k) = Ck^{-\gamma}$ with $\gamma > 2$ and $k \in [k_{min}, \sqrt{\langle k\rangle N}]$. For these network the average degree $\langle k\rangle$ is finite in the limit $N \to \infty$. Let us evaluate $\langle k^2\rangle$ in the continuous limit approximation. We have

$$\begin{aligned}\langle k^2\rangle &= \int_{k_{min}}^{K} dk k^2 P(k) = C\int_{k_{min}}^{K} dk k^{2-\gamma} \\ &= \begin{cases} C\frac{1}{3-\gamma}\left[K^{3-\gamma} - k_{min}^{3-\gamma}\right] & \text{for} \quad \gamma \neq 3 \\ C\left[\ln K - \ln k_{min}\right] & \text{for} \quad \gamma = 3 \end{cases}\end{aligned}$$

Using the expression for the structural cutoff $K = \sqrt{\langle k\rangle N}$ and evaluating $\langle k^2\rangle$ at the leading term for $N \gg 1$ we get

$$\langle k^2\rangle = \begin{cases} C\frac{1}{3-\gamma}\left[(\langle k\rangle N)^{(3-\gamma)/2}\right] & \text{for} \quad \gamma < 3 \\ \frac{C}{2}\left[\ln\langle k\rangle N\right] & \text{for} \quad \gamma = 3 \\ C\frac{1}{\gamma-3}\left[k_{min}^{(3-\gamma)/2}\right] & \text{for} \quad \gamma > 3 \end{cases}$$

Therefore for $\gamma \in (2,3]$, the ratio $\frac{\langle k^2 \rangle}{\langle k \rangle}$ diverges for $N \to \infty$ and the Molloy-Reed condition $\frac{\langle k^2 \rangle}{\langle k \rangle} > 2$ is always satisfied. This means that a scale-free network has always a giant component, independently on the value of the average degree $\langle k \rangle$ (fixed by the power-law exponent γ and the minimal degree of the network k_{min}). □

8.5 Local clustering coefficient of the uncorrelated configuration model

The local clustering coefficient of a node of a network is the probability that two nearest neighbours of a node are connected together. In an uncorrelated network the average local clustering coefficient is independent on the degree of the starting node. In fact the average local clustering coefficient can be easily calculated.

Proposition 8.5.1. *The average local clustering coefficient of a node in a uncorrelated random network with degree distribution $P(k)$ is equal to the Watts-and Strogatz clustering coefficient and is given by*

$$C_{WS} = \frac{1}{\langle k \rangle N} \left(\frac{\langle k(k-1) \rangle}{\langle k \rangle} \right)^2 \tag{8.31}$$

Proof. Suppose that two nearest neighbours of a given node i are called node ℓ and node m. Node ℓ has degree k_ℓ with probability $q(k_\ell)$, node m has degree k_m with probability $q(k_m)$. These two nodes have each one link linked to the node i. Therefore node ℓ has $k_\ell - 1$ remaining links and node m has $k_m - 1$ remaining links. In the configuration model stubs are randomly matched, therefore the probability that node ℓ and node m are connected, given that they are both connected to node i is given by

$$\frac{(k_\ell - 1)(k_m - 1)}{\langle k \rangle N}. \tag{8.32}$$

If we want to evaluate the average clustering coefficient of a node in this network (equal to the Watts and Strogatz clustering coefficient C_{WS} of the network) we have to evaluate the probability that any two nearest neighbours of a generic node i are connected therefore we have

$$
\begin{aligned}
C_{WS} &= \sum_{k_\ell,k_m} q(k_\ell)q(k_m)\frac{(k_\ell-1)(k_m-1)}{\langle k\rangle N} \\
&= \sum_{k_\ell,k_m} \frac{k_\ell}{\langle k\rangle}P(k_\ell)\frac{k_m)}{\langle k\rangle}P(k_m)\frac{(k_\ell-1)(k_m-1)}{\langle k\rangle N} = \\
&= \frac{1}{\langle k\rangle N}\left(\sum_k \frac{k(k-1)}{\langle k\rangle}P(k)\right)^2 \\
&= \frac{1}{\langle k\rangle N}\left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^2 \qquad (8.33)
\end{aligned}
$$

□

Let us now consider different types of sparse uncorrelated network, i.e. Poisson networks and scale-free networks with $\gamma > 2$.

- *Poisson networks*
  In the case of Poisson networks with average degree $\langle k\rangle = c$ we have $\langle k(k-1)\rangle = c^2$, therefore the general result in Eq. (**??**) reduces to the known results for the Poisson network graph, i.e.

$$C_{WS} = \frac{c}{N}. \qquad (8.34)$$

- *Power-law networks with* $\gamma > 3$
  In the case of power-law networks with $\gamma > 3$, we have that $\langle k(k-1)\rangle$ converges to a finite number independent on N as $K = \sqrt{\langle k\rangle N} \to \infty$therefore we find

$$C_{WS} = \frac{1}{\langle k\rangle N}\left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^2 = \mathcal{O}(1/N). \qquad (8.35)$$

  In this case the average number of triangles in the network remain finite in the limit $N \to \infty$ and we say that the network is *locally tree-like.*

- *Power-law networks with* $\gamma \in (2, 3]$
  In the case of power-law networks with $\gamma \in (2, 3]$, we have that $\langle k(k-1)\rangle \propto N^{(3-\gamma)/2}$ when we impose a structural cutoff $K = \sqrt{\langle k\rangle N} \ll 1$. Therefore we find

$$C_{WS} = \frac{1}{\langle k\rangle N}\left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^2 = \mathcal{O}(N^{2-\gamma}). \qquad (8.36)$$

  Therefore the clustering coefficient is vanishing in the limit $N \to \infty$ but the average number of triangles in the network is diverging in the limit $N \to \infty$.

8.6 Average distance of an uncorrelated network

Let us consider uncorrelated networks that are locally tree-like. In this networks the average number of short loops is finite in the limit $N \to \infty$. Therefore we must have $\langle k(k-1)\rangle/\langle k\rangle$ finite in the limit $N \to \infty$. In this case we can evaluate the average distance of the network following the same procedure that we have used for random networks and Cayley trees in chapter 5. In particular we will evaluate the average branching ratio of a node reached by following a random link of the network, and we will express the typical distance in the network as a function of this average branching ratio.

Definition 5. *The branching ratio b_k of a node of degree k is given by*

$$b_k = (k-1), \tag{8.37}$$

expressing the number of reaming links of the node if we reach the node by following a link.

Proposition 8.6.1. *The average branching ratio in an uncorrelated network, is calculated as*

$$\langle b\rangle = \sum_k b_k q(k) = \sum_k (k-1)\frac{k}{\langle k\rangle}P(k) = \frac{\langle k(k-1)\rangle}{\langle k\rangle}. \tag{8.38}$$

Proposition 8.6.2. *In a random uncorrelated network with finite average branching ratio $\langle b\rangle = \langle k(k-1)\rangle/\langle k\rangle$ the average number N_d of nodes at distance d from a given node i of degree k_i can be approximated to be*

$$N_d \sim \begin{cases} k_i\left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^{d-1} & \text{for} \quad d>0 \\ 1 & \text{for} \quad d=0 \end{cases} \tag{8.39}$$

Proof. The above relation is valid for the nodes at distance $d=0$. In fact the number of nodes at distance zero from the node i is given by one, i.e. it is only the node i itself. Moreover the relation given by Eq. (8.39) is also valid for distances $d=1$ from the node i, since the degree of the node i is k_i.
Let us now show that if the relation (8.39) is valid for the nodes at distance $d \geq 1$, then it must be valid for the nodes at distance $d+1$.

Since the uncorrelated random network with finite average branching ratio we can neglect loops, we can assume that the branching ratio of each node of degree k is $b_k = k-1$. It follows that each node of degree k at distance d will branch into $b_k = k-1$ nodes at distance $d+1$. These nodes will have degree k with probability $q(k)$.
Moreover since we are neglecting the loops in the network any node at distance $d+1$ from the central node can be reached only by one node at distance d. Therefore we will have

$$N_{d+1} = N_d\langle b\rangle = k_i\langle b\rangle^{d-1} = k_i\left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^{d-1} \tag{8.40}$$

$\square$

Proposition 8.6.3. *The* average distance $\langle d\rangle = \ell$ *of a uncorrelated network with finite average branching ration* $\langle k(k-1)\rangle/\langle k\rangle$ *can be approximated in the limit of large network sizes* $N \gg 1$ *to*

$$\ell \sim \frac{\ln N}{\ln \frac{\langle k(k-1)\rangle}{\langle k\rangle}} \tag{8.41}$$

Proof. We start from the expression given by Eq. (8.39) for the node at distance d from a given node i of degree i. Assuming that the node i is choose at random in the network we can evaluate the number of nodes at distance $d > 0$ from a random node as

$$N_d \sim k_i \langle b\rangle^{d-1} \sim \langle k\rangle \langle b\rangle^{d-1}. \tag{8.42}$$

Therefore we can evaluate the number of nodes at distance $d \leq d'$ from a random node as

$$N_{d\leq d'} = 1 + \sum_{d=1}^{d'} N_d = 1 + \langle k\rangle \sum_{d=1}^{d'} \langle b\rangle^{d-1} = 1 + \langle k\rangle \frac{\langle b\rangle^{d'} - 1}{\langle b\rangle - 1}. \tag{8.43}$$

The average distance between the nodes of the network can be estimated by imposing that the number of nodes at distance $d \leq \ell$ must be equal to the total number of nodes N. Therefore we have

$$\begin{aligned} N &= 1 + \langle k\rangle \frac{\langle b\rangle^{\ell} - 1}{\langle b\rangle - 1} \\ \langle b\rangle^{\ell} &= 1 + (N-1)\frac{\langle b\rangle - 1}{\langle k\rangle} \\ \ell &= \frac{\ln[1 + (N-1)\frac{\langle b\rangle - 1}{\langle k\rangle}]}{\ln\langle b\rangle}. \end{aligned} \tag{8.44}$$

It follows that in the limit of large network sizes $N \gg 1$

$$\ell \sim \frac{\ln N}{\ln\langle b\rangle} = \frac{\ln N}{\frac{\langle k(k-1)\rangle}{k}}. \tag{8.45}$$

□

Therefore any network with finite average branching ratio has the small-world distance property.
For scale-free networks with diverging branching ratio, other methods should be adopted to evaluate the average distance. In particular by these other methods it has been found that these networks have an average distance that increases with N *slower than* $\ln N$*. In fact we have*

Proposition 8.6.4. *Scale-free network with power-law exponent* $\gamma \in (2,3]$ *have a typical distance that for large network sizes* $N \ll 1$ *scales like*

$$\ell = \begin{cases} \frac{\ln N}{\ln\ln N} & \textit{for} \quad \gamma = 3 \\ \frac{\ln\ln N}{\ln(\gamma-1)} & \textit{for} \quad \gamma \in (2,3). \end{cases}$$

The very small average distance of scale-free networks is an effect of the presence of the highly connected nodes, or hubs in this networks. In fact most shortest paths go through the hub nodes of the network, and in this way the average distance between nodes of the network is greatly reduced.