Time & Freq. responses of 1st-order lowpass and highpass filters (Linear systems 1)

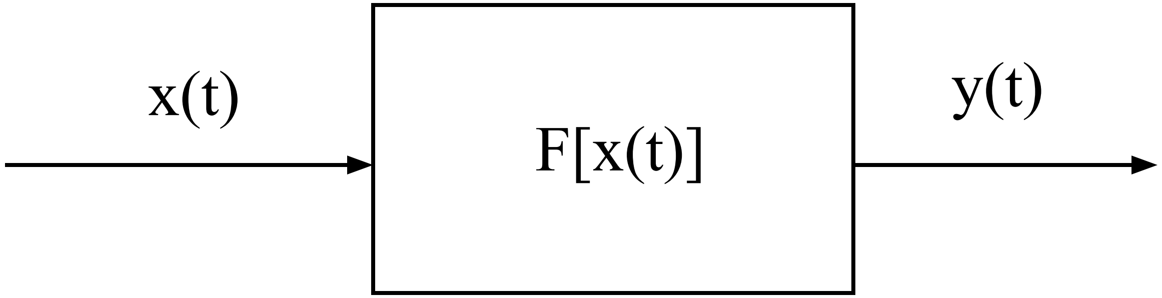
T Delbruck/SC Liu/G Indiveri

- Linear systems
  - Linear time-invariant systems
  - Convolution and impulse response
  - Complex numbers and transfer functions
- RC integrators and differentiators
  - Frequency analysis
- VLSI Integrators
  - Follower integrator
  - Translinear principle
  - Log-domain current-mode integrator

Linear Systems, Small Signal Analysis, and Integrators
(WS07-08)

- Linear systems
  - Linear time-invariant systems
  - Convolution and impulse response
  - Complex numbers and transfer functions
- RC integrators and differentiators
  - Frequency analysis
- VLSI Integrators
  - Follower integrator
  - Translinear principle
  - Log-domain current-mode integrator

Linear Systems (I)



If a linear time-invariant system has not internally stored energy, then the output of the system y(t) is the forced response due entirely to the input x(t): y(t)=F[x(t)].

A system is said to be linear if it obeys the two fundamental principles of **homogeneity** and **additivity**.

Homogeneity and Additivity

Homogeneity If

$$\hat{x}(t) = \alpha x(t)$$

then

$$\hat{y}(t) = F[\alpha x(t)] = \alpha F[x(t)]$$

Additivity If

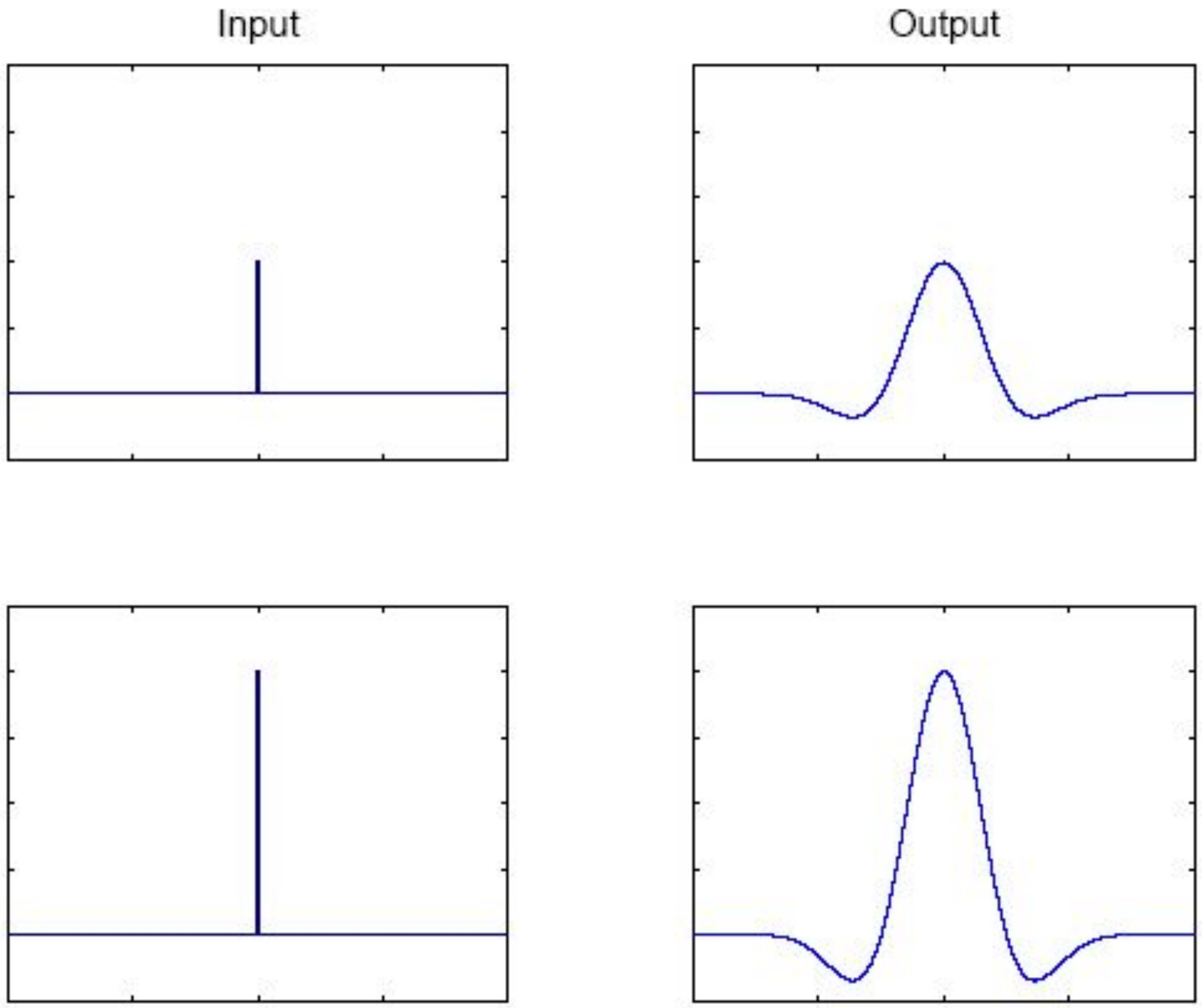
$$x(t) = \sum_k a_k x_k(t)$$

with a_k constant $\forall k$

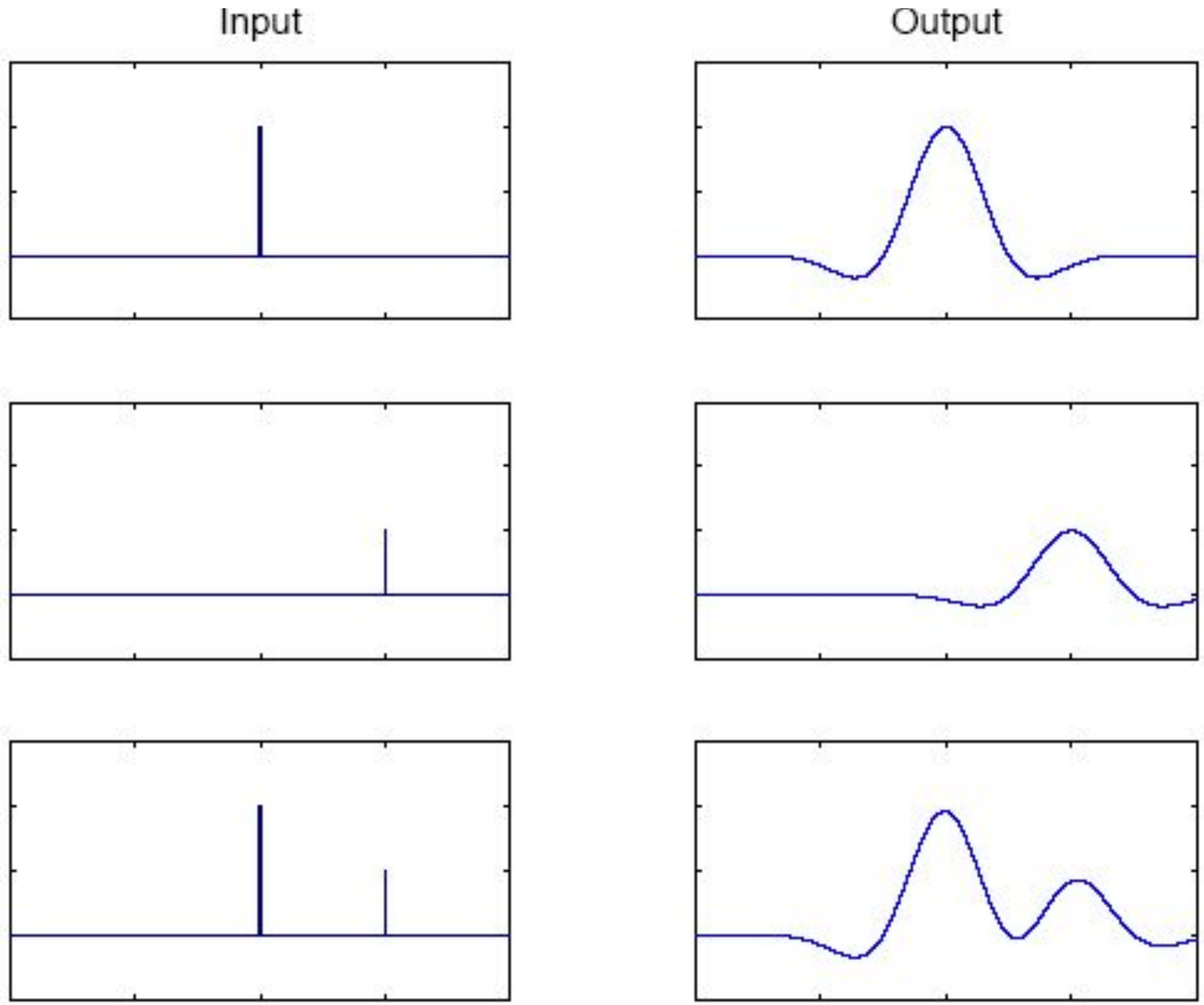
then

$$y(t) = \sum_k a_k F[x_k(t)]$$

Homogeneity



Additivity



Time Invariance

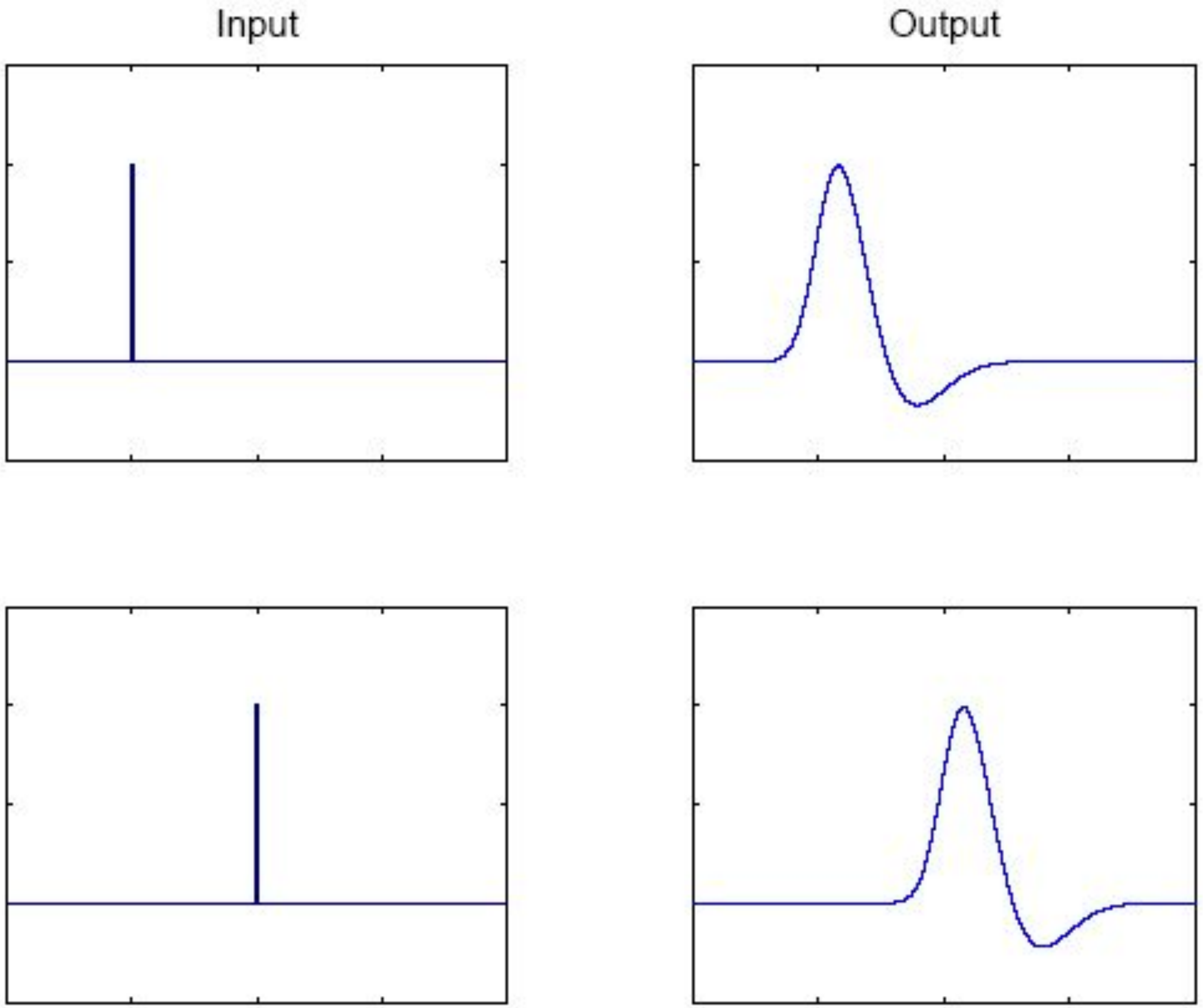
A time-invariant system obeys the following time-shift invariance property: If the input signal $\hat{x}(t)$ is

$$\hat{x}(t) = x(t-\tau)$$

then for any real constant τ,

$$\hat{y}(t) = F[x(t-\tau)] = y(t-\tau)$$

Time Invariance



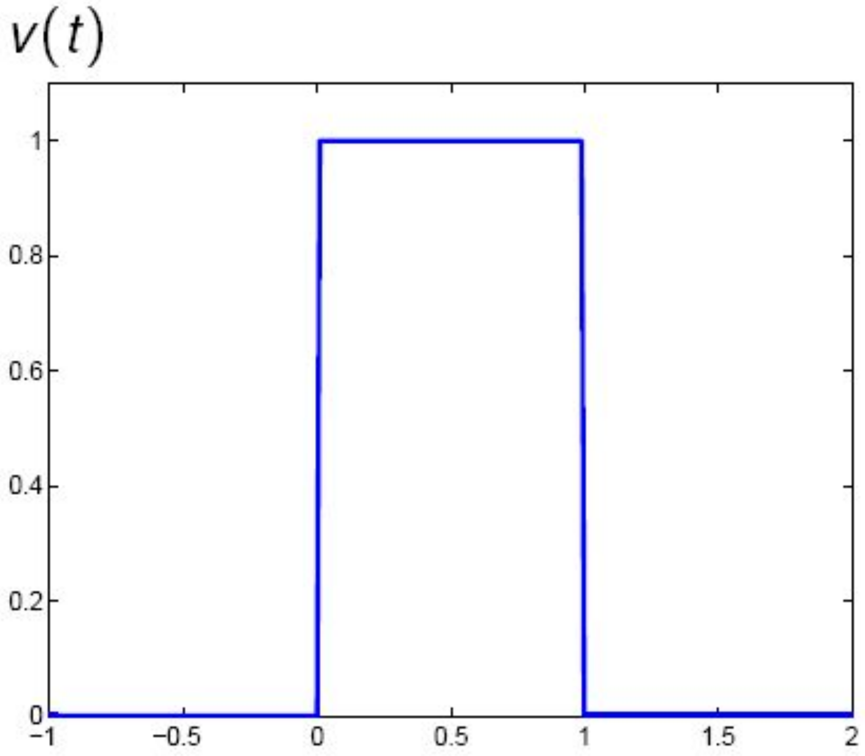
Convolution

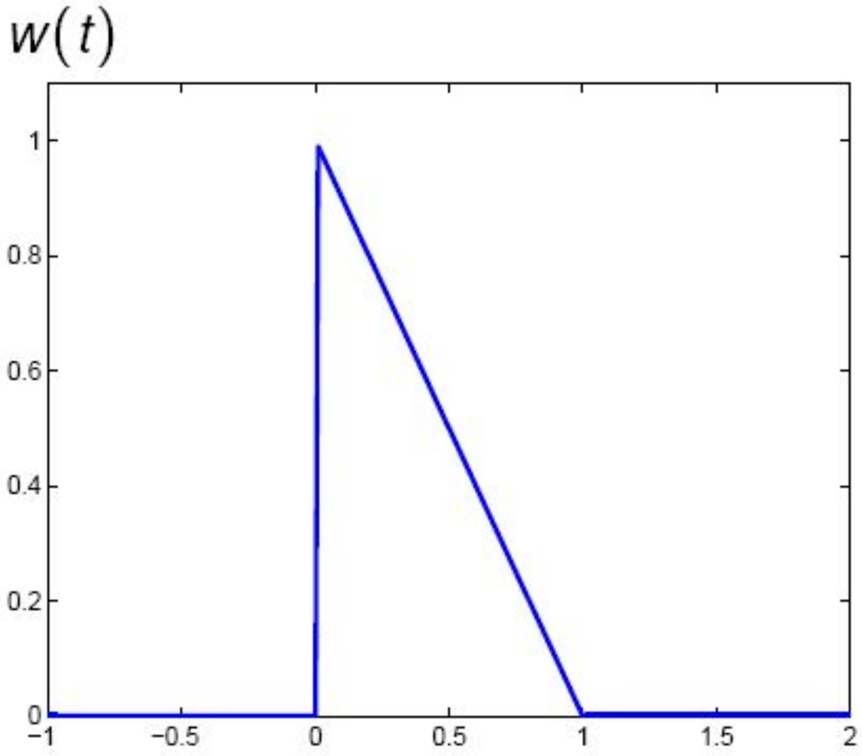
It is a useful mathematical tool for analyzing linear systems.

$$
\begin{aligned}
v(t) * w(t) &\doteq \int_{-\infty}^{+\infty} v(\lambda) w(t-\lambda) d\lambda \\
&= \int_{-\infty}^{+\infty} v(t-\lambda) w(\lambda) d\lambda
\end{aligned}
$$

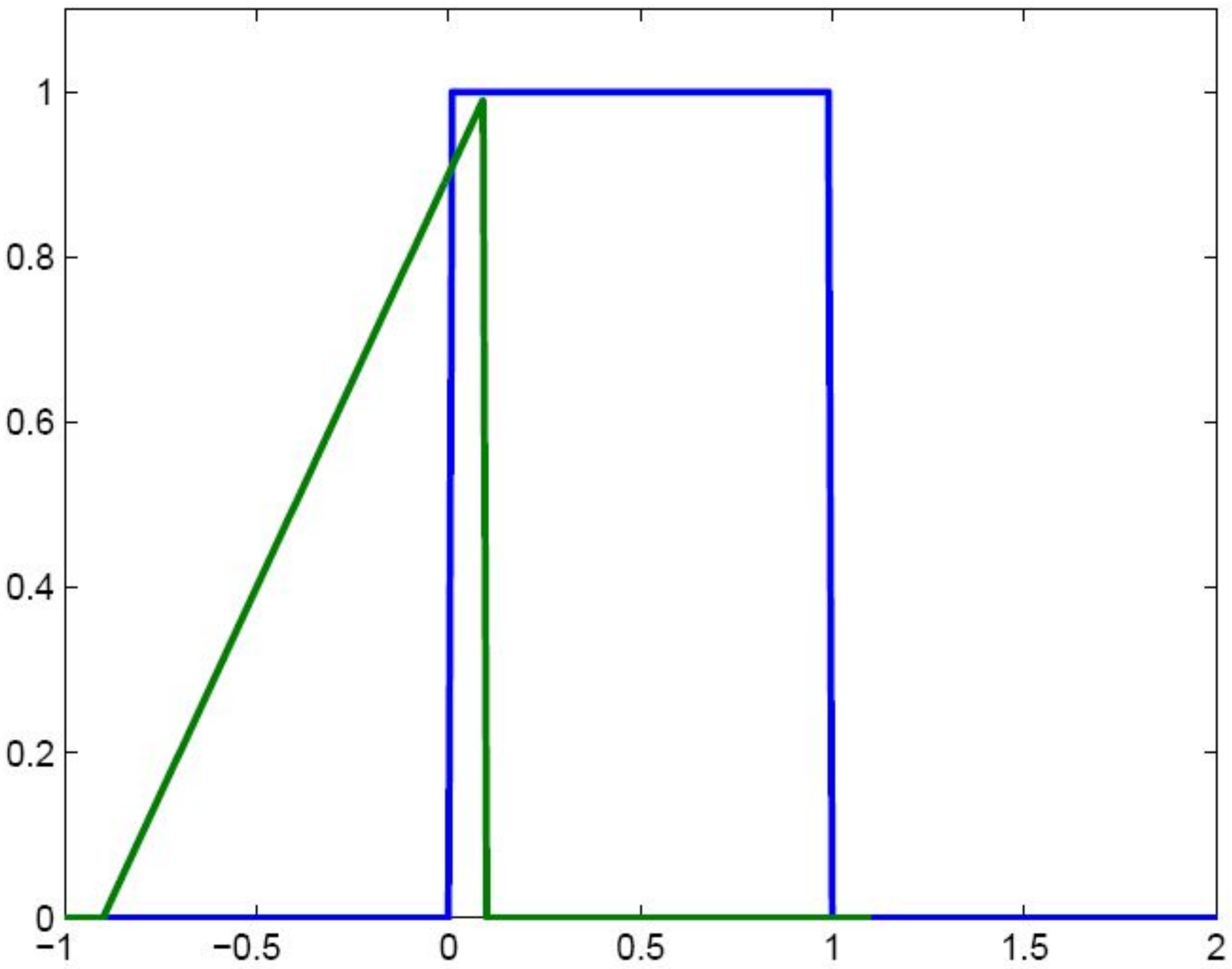
Note that in the integral t is the *independent* variable, and λ is the *integration* variable.
Insofar as the integral is concerned, t is constant.

Graphical Example

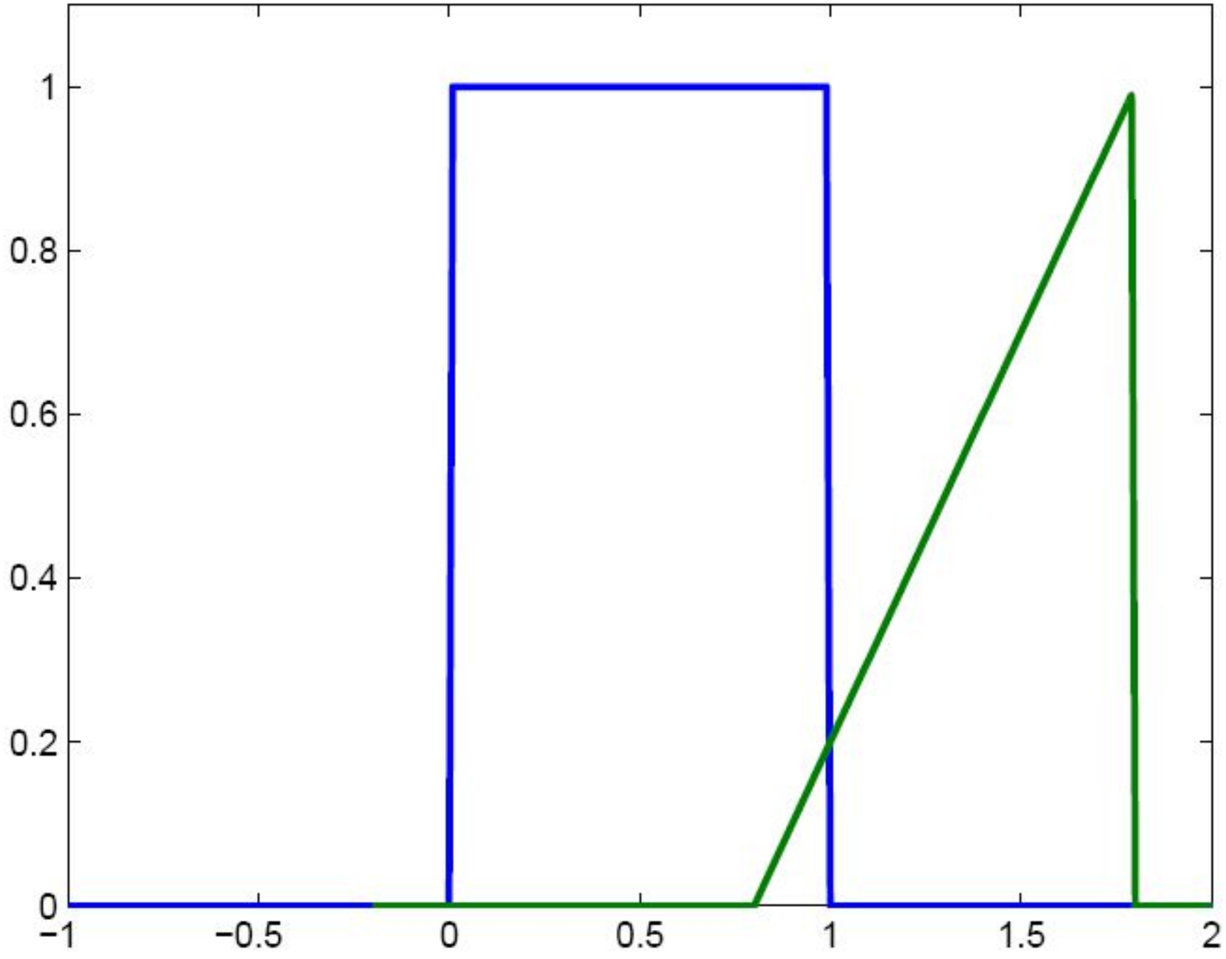




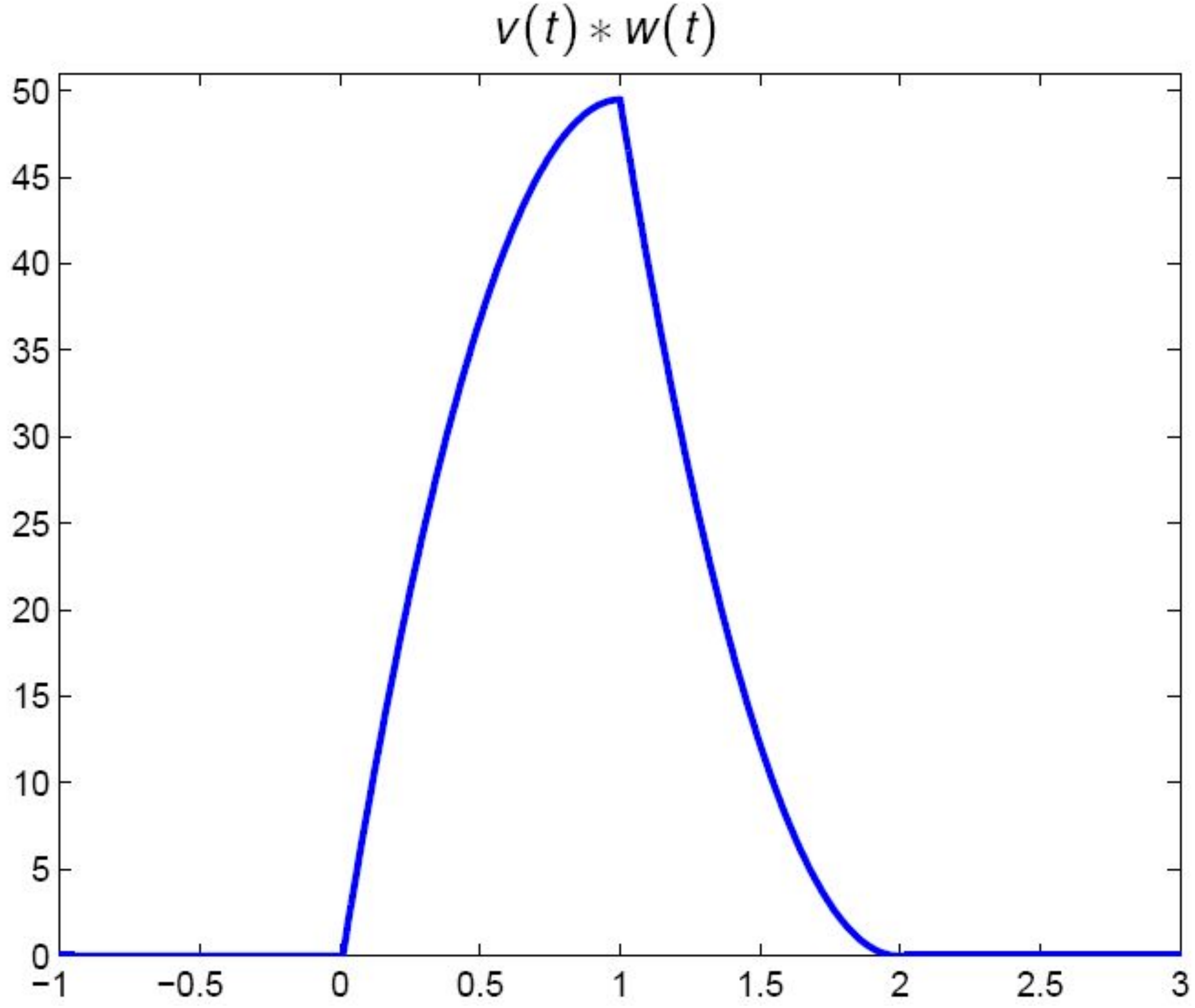
Graphical Example (II)



Graphical Example (III)



Graphical Example (IV)



Impulse

The *Dirac-delta function*, or *unit impulse* $\delta(t)$ is not a strict function in the mathematical sense. It is only defined by a set of assignment rules. If $v(t)$ is continuous at time $t = 0$,

$$\int_{t_1}^{t_2} v(t)\delta(t)dt = \begin{cases} v(0) & \text{if } t_1 < 0 < t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\varepsilon}^{+\varepsilon} \delta(t)dt = 1$$

The *Dirac-delta function*, $\delta(t)$ has unit area at $t = 0$, is null for $t \neq 0$, and has no mathematical or physical meaning, unless it appears under the operation of integration.

Impulse Integration Properties

Replication operation

$$v(t) * \delta(t - \tau) = v(t - \tau)$$

Sampling operation

$$\int_{-\infty}^{+\infty} v(t)\delta(t - \tau)dt = v(\tau)$$

Linear System's Impulse Response

We *define* the impulse response of a system characterized by $y(t) = F[x(t)]$ as:

$$\boxed{h(t) \doteq F[\delta(t)]}$$

If $x(t)$ is a continuous function, $x(t) = x(t) * \delta(t)$.

$$y(t) = F[x(t)] = F\left[\int_{-\infty}^{+\infty} x(\lambda)\delta(t-\lambda)d\lambda\right]$$

If the system is linear

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)F[\delta(t-\lambda)]d\lambda$$

If the system is shift-invariant

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda \Rightarrow \boxed{y(t) = x(t) * h(t)}$$

Step and Impulse Response

If $u(t)$ is a step function such that

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

the system's *step response* is defined as $g(t) \doteq F[u(t)]$.
Exploiting the system's impulse response properties:

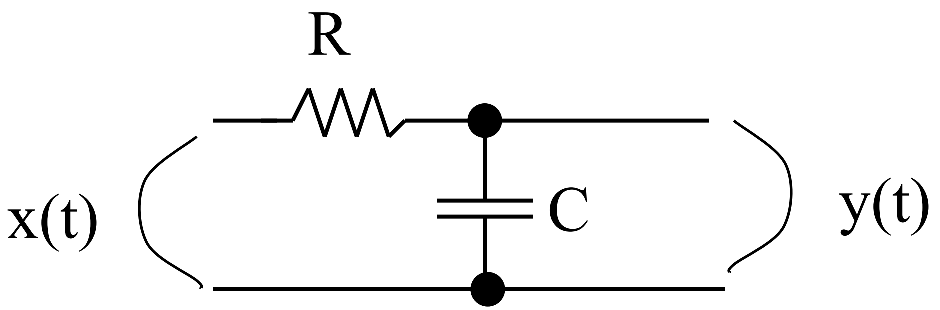
$$g(t) = h(t) * u(t)$$

$$\Rightarrow \frac{d}{dt} g(t) = h(t) * \frac{d}{dt} u(t) = h(t) * \delta(t)$$

$$\boxed{h(t) = \frac{d}{dt} g(t)}$$

A system's impulse response is the derivative of its step response.

Resistor-Capacitor Circuit



- 1st order linear differential equation:

$$RC\frac{d}{dt}y(t) + y(t) = x(t)$$

Solving for $x(t) = \delta(t)$:

$$h(t) = \frac{1}{RC}e^{-t/RC}\ .u(t)$$

In general: $y(t) = x(t) * h(t)$

$$= \int_0^\infty \frac{1}{RC}e^{-t/RC}x(t-\lambda)d\lambda$$

Complex Exponentials

All solutions to linear homogeneous equations are of the form e^{st} where $s \doteq \sigma + j\omega = M\cos(\phi) + jM\sin(\phi)$, and $j = \sqrt{-1}$.
σ represents the real part of the complex number, ω its imaginary part, M represents its *magnitude* and ϕ its *phase*.
Magnitude and phase of a complex number obey to the following relationships:

$$M = \sqrt{\sigma^2 + \omega^2} \qquad e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\phi = \arctan\left(\frac{\omega}{\sigma}\right) \qquad e^{-j\phi} = \cos(\phi) - j\sin(\phi)$$

It follows that s can be also written as $\boxed{s = Me^{j\phi}}$

Complex Exponentials

- Example of second order linear differential equation:

$$\frac{d^2}{dt^2}V(t) + \alpha\frac{d}{dt}V(t) + \beta V(t) = 0$$

Substitute e^{st} for V: $s^2e^{st} + \alpha s e^{st} + \beta e^{st} = 0$

Solving for s: $s = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$

If $\alpha^2 - 4\beta \geq 0$, s is real, otherwise s is complex

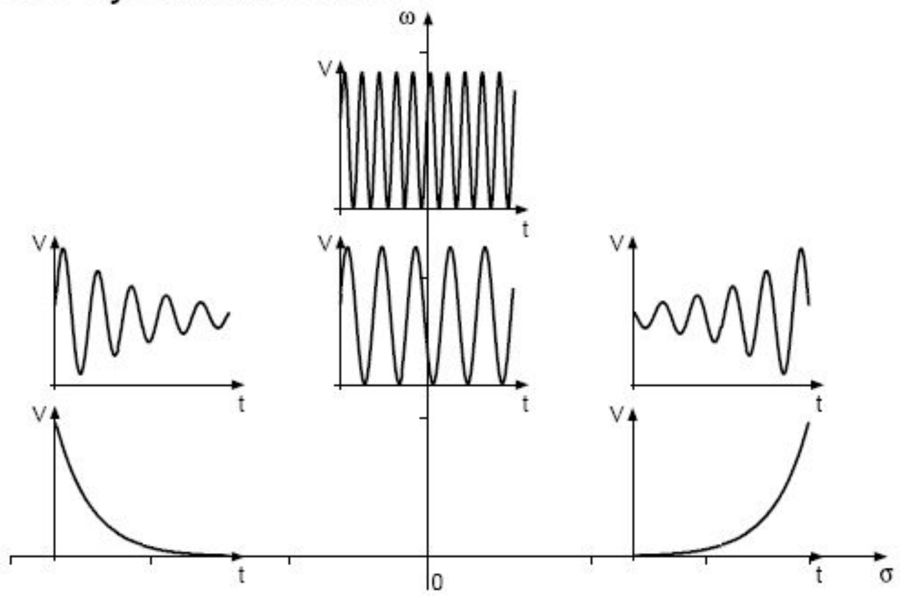
Measured Responses

In practice, we measure responses of the type $V = e^{st}$ with *real* instruments, so if e^{st} were a complex exponential, we would be able to measure only

$$Re\{e^{st}\} = Re\left\{e^{(\sigma+j\omega)t}\right\} = e^{\sigma t} Re\{e^{j\omega t}\}$$

So the measured response of the system would be

$$\boxed{V_{meas} = e^{\sigma t} \cos \omega t}$$



Heaviside-Laplace Transform

Oliver Heaviside, in analyzing analog circuits made the following observation:

$$\frac{d^n}{dt^n} e^{st} = s^n e^{st}$$

we can consider s as an operator meaning derivative with respect to time. Similarly, we can view $\frac{1}{s}$ as the operator for integration with respect to time.

This observation was formalized by mathematicians, when they realized that Heaviside's method was using polynomials in *s* that were *Laplace transforms*. The Laplace transform is an operator that allows to link functions that operate in the time domain with functions of complex variables:

- $$L[y(t)] = Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt$$

Transfer Function

- Transfer function is defined as $H(s) = \dfrac{Y(s)}{X(s)}$

If $x(t) = \delta(t)$, then

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = 1$$

and

$$Y(s) = \int_{-\infty}^{\infty} y(t) e^{-st} dt = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

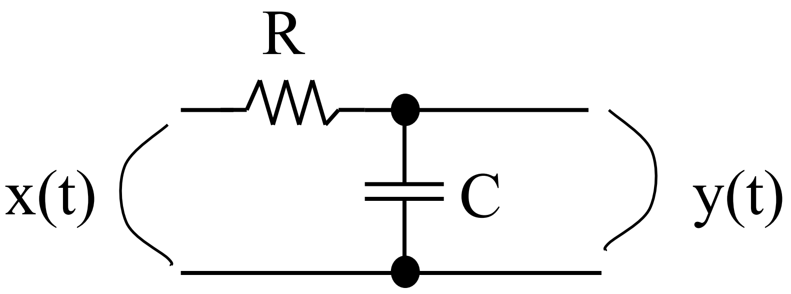
Therefore

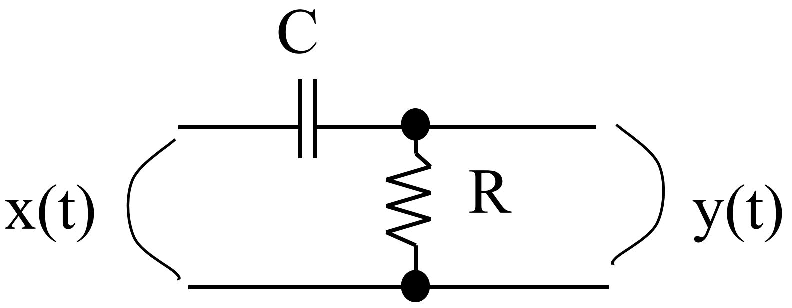
$$H(s) = \int_{-\infty}^{\infty} h(\lambda) e^{-\lambda s} d\lambda = L[h(t)]$$

Linear Systems, Small Signal Analysis, and Integrators
(WS07-08)

- Linear systems
  - Linear time-invariant systems
  - Convolution and impulse response
  - Complex numbers and transfer functions
- **RC integrators and differentiators**
  - **Frequency analysis**
- VLSI Integrators
  - Follower integrator
  - Translinear principle
  - Log-domain current-mode integrator

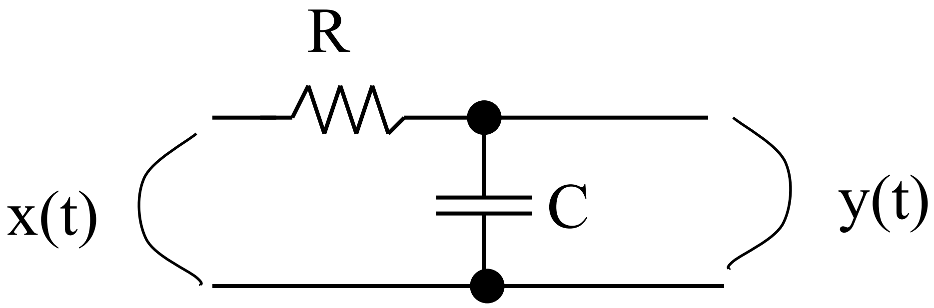
Resistor-Capacitor Circuits





- First order, linear time-invariant systems
- Input is x(t), output is y(t)
- RC integrator circuit
- CR differentiator circuit

R-C Integrator



- Time domain:

$$RC\frac{d}{dt}y(t) + y(t) = x(t)$$

Solving for $x(t) = \delta(t)$:

$$h(t) = \frac{1}{RC}e^{-t/RC}\ u(t)$$

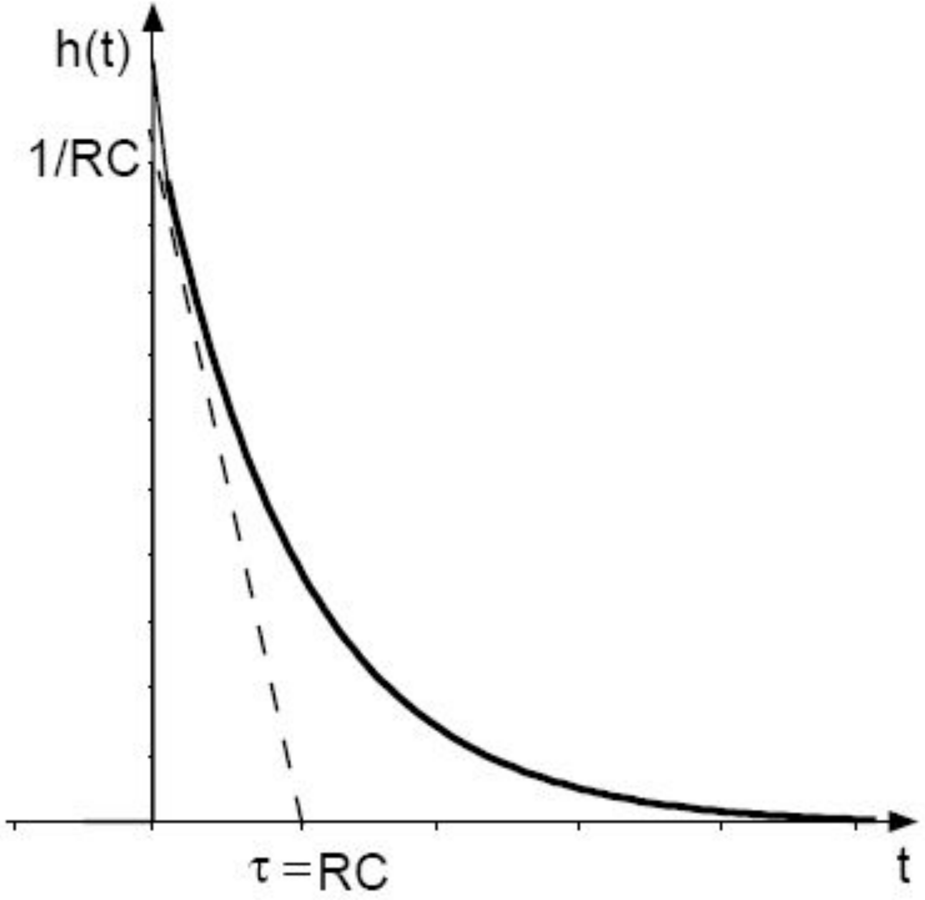
- Laplace domain:

$$RCsY(s) + Y(s) = X(s)$$

Solving for $X(s) = 1$:

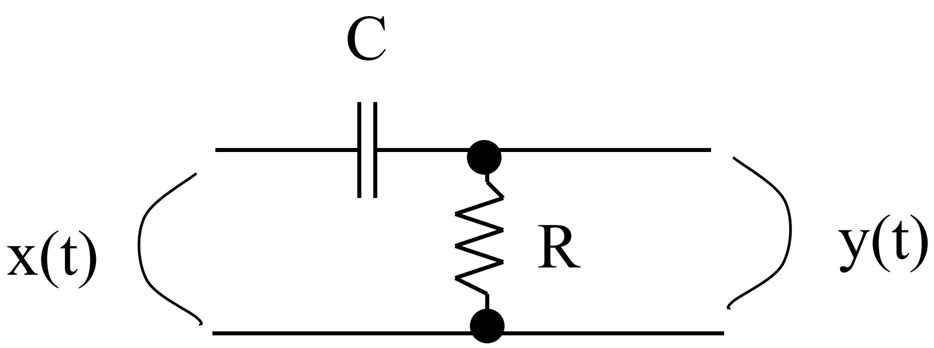
$$H(s) = \frac{1}{1 + RCs}$$

RC Integrator Impulse Response



The circuit's *time constant* $\tau = RC$ is the time required to discharge the capacitor, through the resistor, to 36.8% ($1/e$) of its final steady state value.

C-R Differentiator



- Time domain:

$$\tau \frac{d}{dt} y(t) + y(t) = \tau \frac{d}{dt} x(t)$$

For a step input, the derivative of
the input is an impulse at t=0.

- Laplace domain:

$$\tau s Y(s) + Y(s) = \tau X(s)$$

Solving for $X(s) = 1$:

$$H(s) = \frac{\tau s}{1 + \tau s}$$

Frequency Domain Analysis

A system's response in the frequency domain can be determined by using pure sinusoids as input signals. For time-invariant linear systems, the amplitude and phase of the input sinusoid will be altered by the system's transfer function.

Input: $x(t) = \sin(\omega t)$
Output: $y(t) = A \sin(\omega t+\phi)$

where A and ϕ determine the amount of scaling and phase shift

RC-Integrator Frequency Domain Analysis

In this domain $s = j\omega$ and the RC-circuit's transfer function is:

$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

- $\omega\tau \ll 1 \Rightarrow Y(j\omega) \approx X(j\omega)$
- $\omega\tau \gg 1 \Rightarrow Y(j\omega) \approx \frac{1}{j\omega\tau} X(j\omega)$

The frequency $f_c = \frac{\omega}{2\pi} = \frac{1}{2\pi \cdot \tau}$ is defined to be the cutoff frequency.

In electronics, cutoff frequency (fc) is the frequency either above which or below which the power output of a circuit, such as a line, amplifier, or filter, is reduced to 1/2 of the passband power; the half-power point. This is equivalent to a voltage (or amplitude) reduction to 70.7% of the passband, because voltage V^2 is proportional to power P. This happens to be close to -3 decibels, and the cutoff frequency is frequently referred to as the -3 dB point.

Magnitude, Phase, Bode Plots

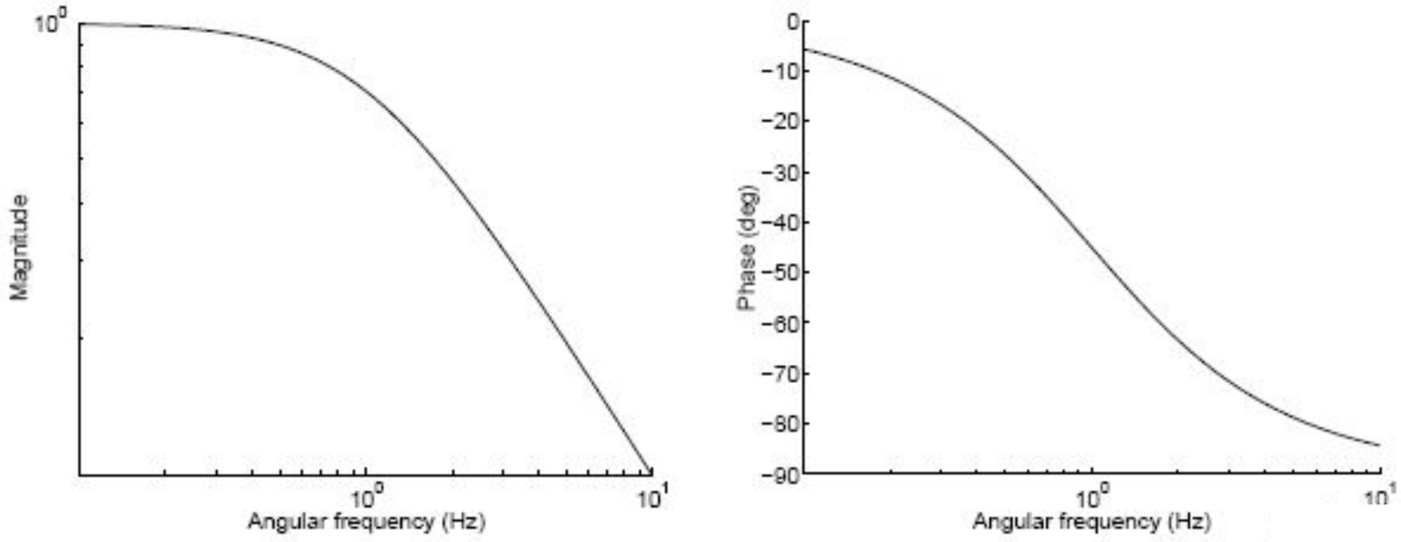
The magnitude of the RC-integrator's transfer function is:

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

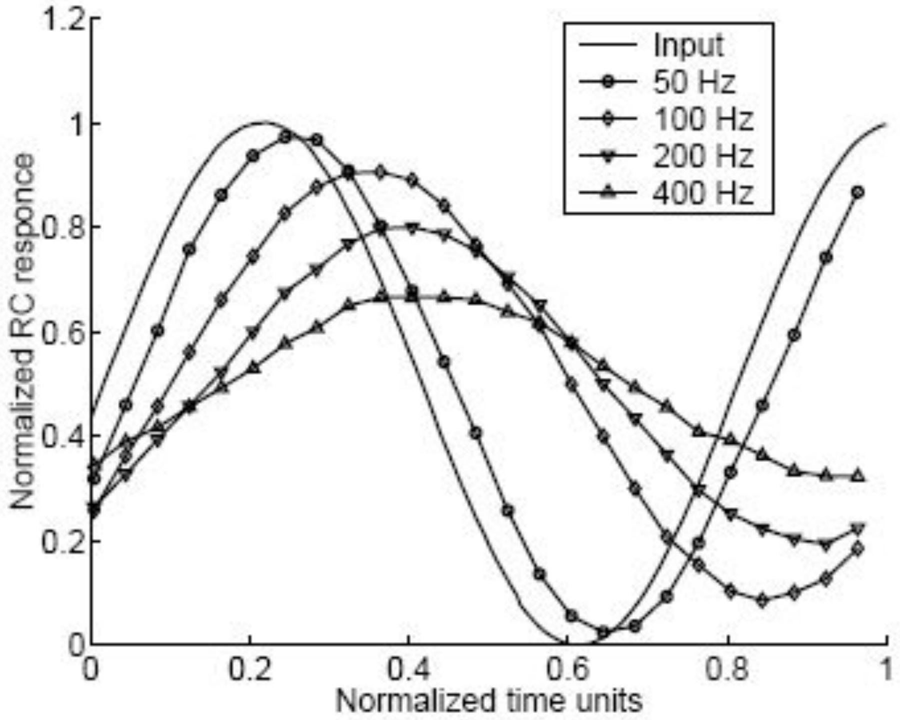
Its phase is

$$\phi = \arctan(-\omega\tau)$$

The plots of a transfer function's magnitude and phase versus input frequency are called *Bode* plots (after Hendrik Wade Bode).



RC-Integrator Frequency Response



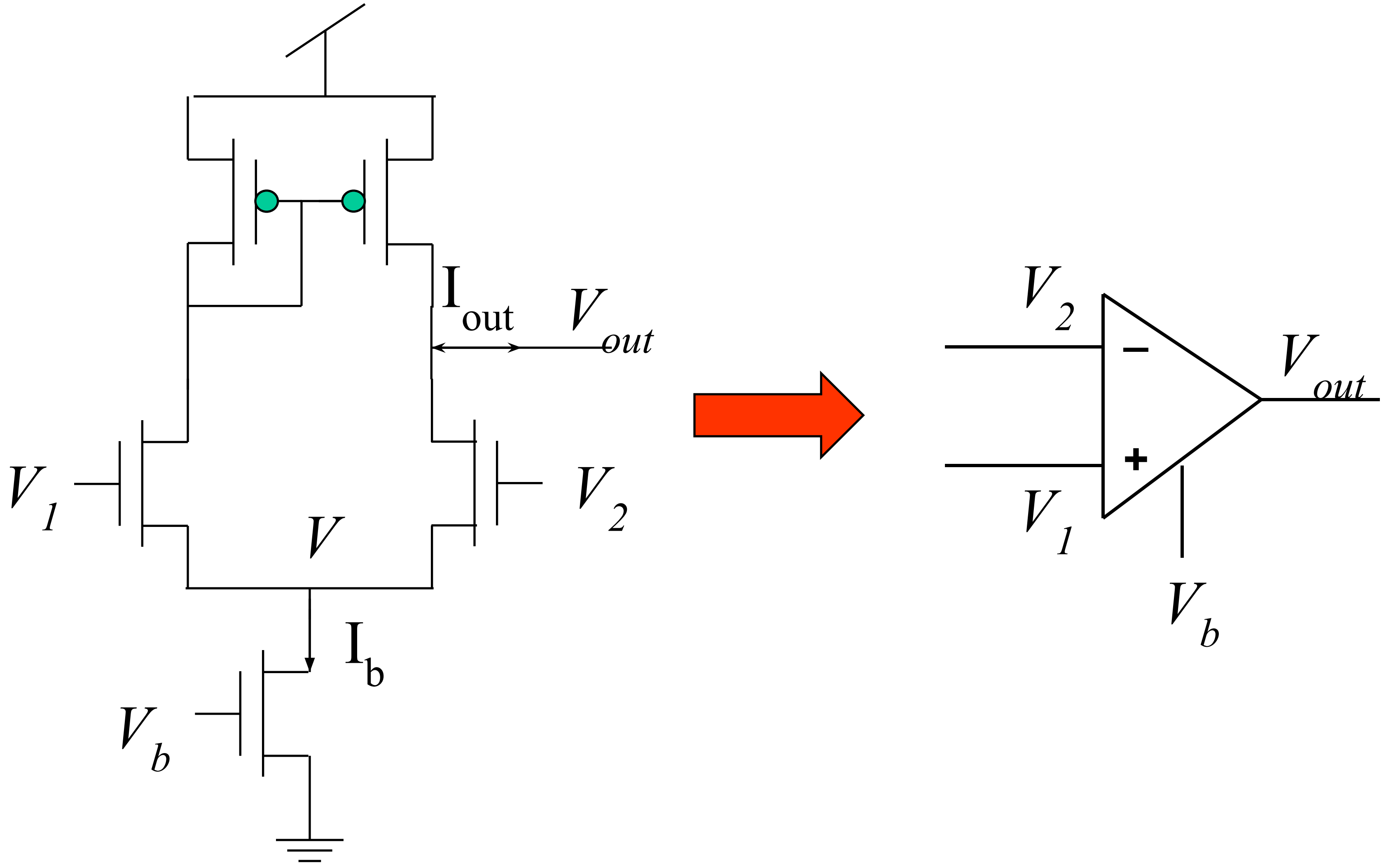
Responses of an RC lowpass filter with $R = 10M\Omega$ and $C = 1nF$.

All data is plotted on a normalized scale. The signals' amplitudes have been normalized with respect to the input and the time-base has been normalized to unity.

Linear Systems, Small Signal Analysis, and Integrators (WS07-08)

- Linear systems
  - Linear time-invariant systems
  - Convolution and impulse response
  - Complex numbers and transfer functions
- RC integrators and differentiators
  - Frequency analysis
- VLSI Integrators
  - Follower integrator
  - Translinear principle
  - Log-domain current-mode integrator

Transconductance Amplifier



Transconductance Amplifier Specs

$$\boxed{I_{out} = I_b \tanh\left(\frac{\kappa}{2U_T}(V_1 - V_2)\right)}$$

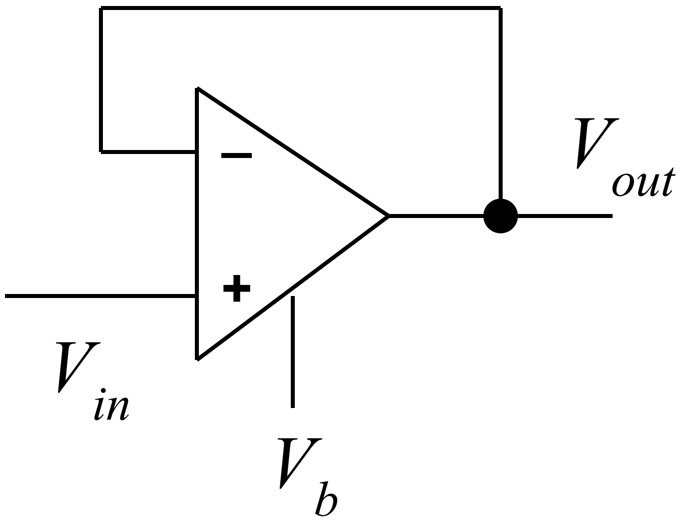
Transconductance: $$\boxed{g_m = \frac{\partial I_{out}}{\partial (V_1 - V_2)} = \frac{\kappa I_b}{2U_T}}$$

Output conductance: $$\boxed{g_d = -\frac{\partial I_{out}}{\partial V_{out}} \approx \frac{I_b}{V_E}}$$

Input conductance: $$\boxed{g_{in} = \frac{\partial I_{in}}{\partial V_{out}} \approx 0}$$

Voltage gain: $$\boxed{A = \frac{\partial V_{out}}{\partial (V_1 - V_2)} = \frac{g_m}{g_d}}$$

Unity-Gain Follower



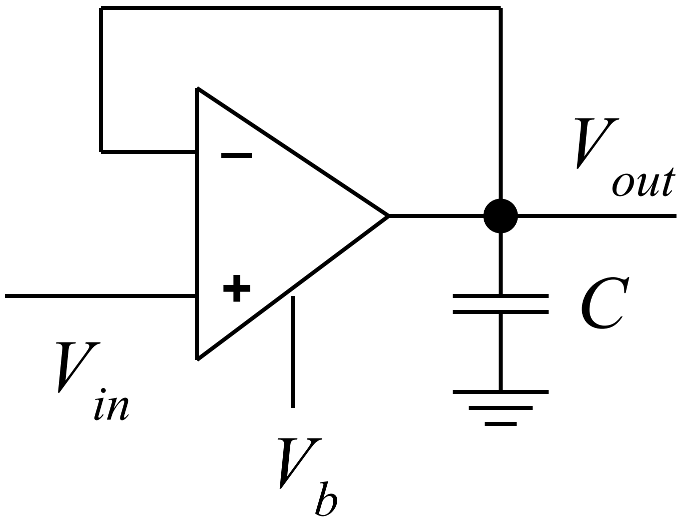
- $\partial V_{out} = A(\partial V_{in} - \partial V_{out})$

- Transfer function: $\dfrac{\partial V_{out}}{\partial V_{in}} = \dfrac{A}{1+A}$

  Input impedance: $Z_{in} = \dfrac{\partial V_{in}}{\partial I_{in}} \to \infty$

  Output impedance: $Z_{out} = -\dfrac{\partial V_{out}}{\partial I_{out}} = \dfrac{1}{g_d} \approx \dfrac{V_E}{I_b}$

Follower Integrator



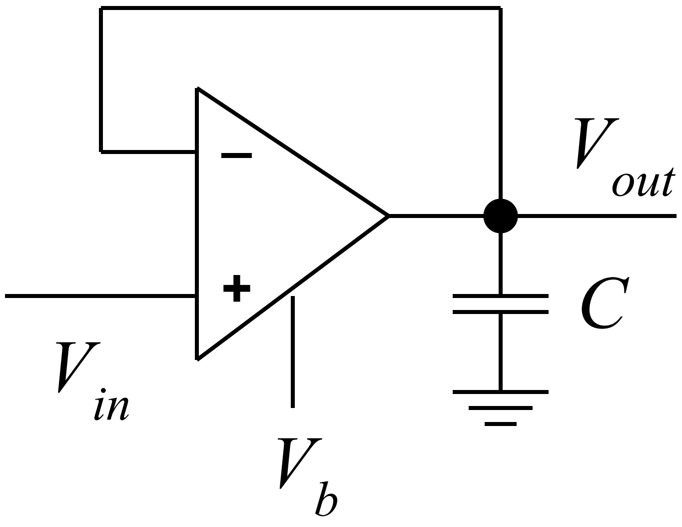
$$C\frac{dV_{out}}{dt} = I_b \tanh\left(\frac{\kappa(V_{in} - V_{out})}{2U_T}\right)$$

In the small−signal regime

$$C\frac{dV_{out}}{dt} = G(V_{in} - V_{out})$$

where $G = \dfrac{kI_b}{2U_T}$ is the amplifier transconductance

Follower Integrator (Laplace domain)



- $$C\frac{dV_{out}}{dt} = G(V_{in} - V_{out}) \quad \Rightarrow \quad sCV_{out} = G(V_{in} - V_{out})$$

  Transfer function: $\dfrac{V_{out}}{V_{in}} = \dfrac{1}{s\tau + 1}$ where $\tau = \dfrac{C}{G}$

- Amplifier operates in linear region only if

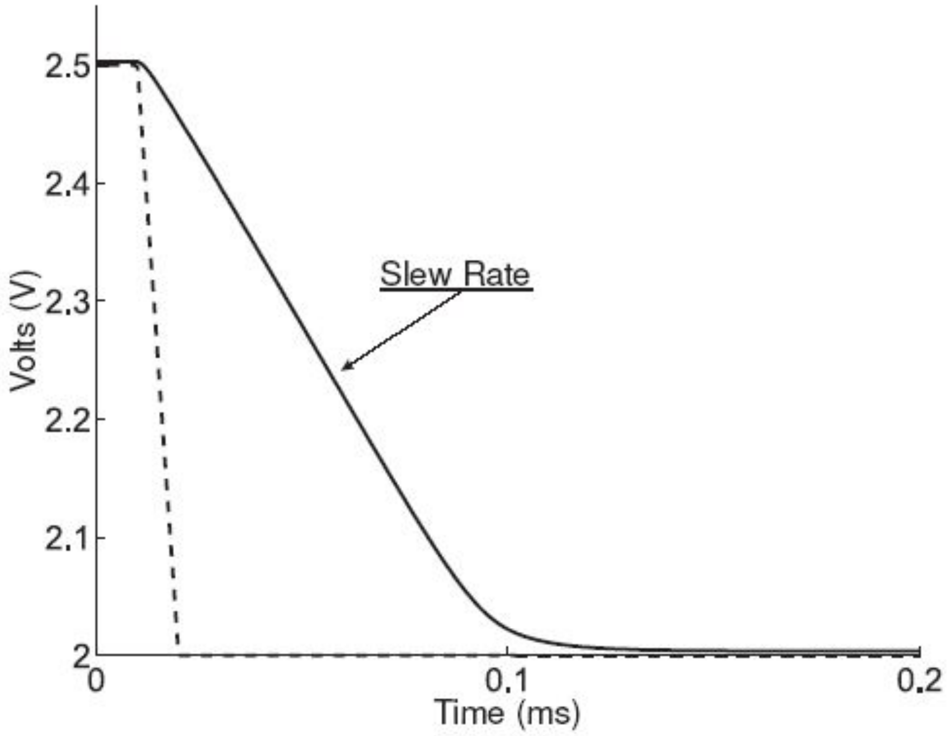
  $$V_{in} - V_{out} < 4U_T$$

Follower Integrator
(Large Signal Behavior)

For large variations of V_{in}, the output current of the transconductance amplifier saturates to I_b or $-I_b$.

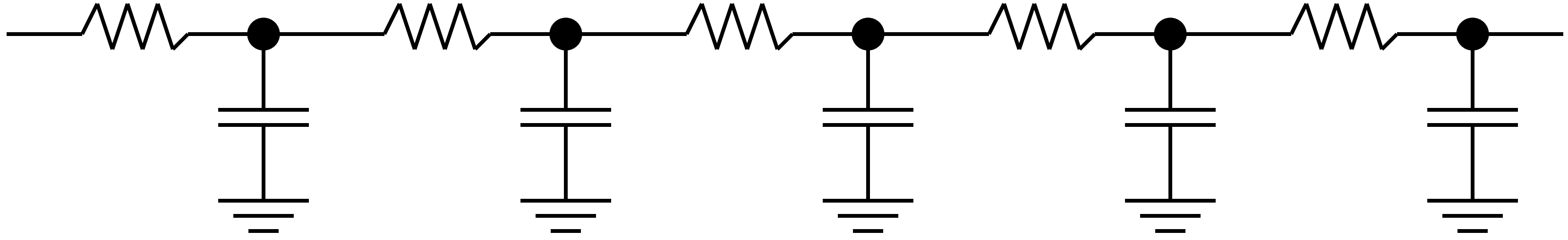
As long as $|V_{out} - V_{in}| > 4\,U_T$, V_{out} changes linearly over time. This region is also called the **slew rate** of an amplifier.

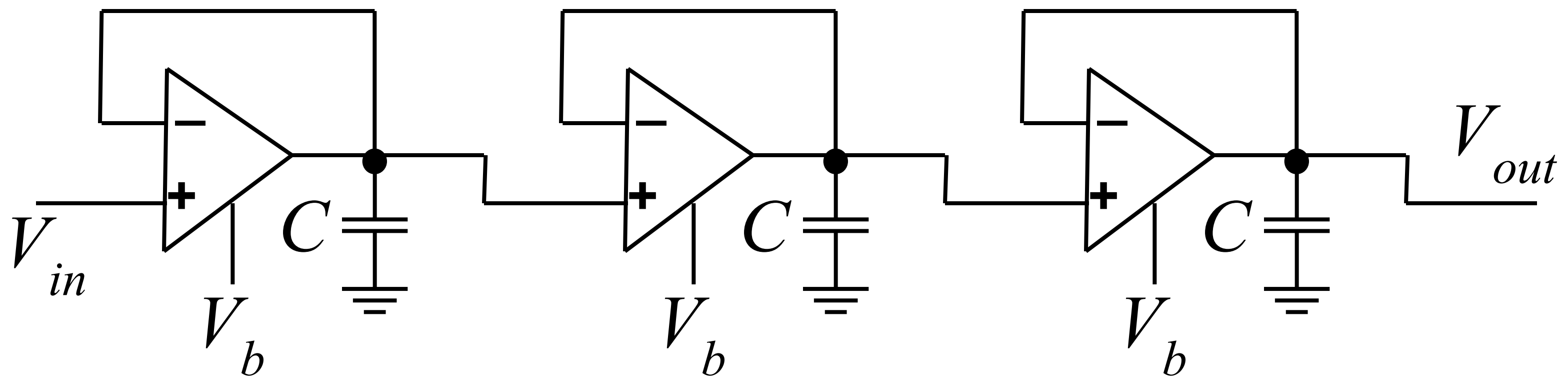
If $|V_{out} - V_{in} < 4\,U_T|$, the amplifier behaves like a linear conductance and V_{out} changes exponentially.



Delay Line

Cascade of RC integrators and follower-integrators





Follower-Integrator Composition Property

- Transfer function of delay line with n delay elements:

$$\frac{V_{out}}{V_{in}} = \left(\frac{1}{s\tau + 1}\right)^{n}$$

- We can write the transfer function directly in terms of
  magnitude and phase valid for $(j\omega\tau) << 1$

$$\frac{V_{out}}{V_{in}} \approx \left(\frac{1}{1+\frac{n}{2}(j\omega\tau)^2}\right)e^{(-jn\omega\tau)}$$

Translinear Principle

- Coined by Barrie Gilbert in 1975, *translinear* means that the bipolar junction transistor's *trans*conductance is *linear* in its collector current.
- In a bipolar transistor, the collector current is exponential in the base-emitter voltage. This exponential dependence is also captured in the subthreshold domain of a MOSFET.

Translinear Principle

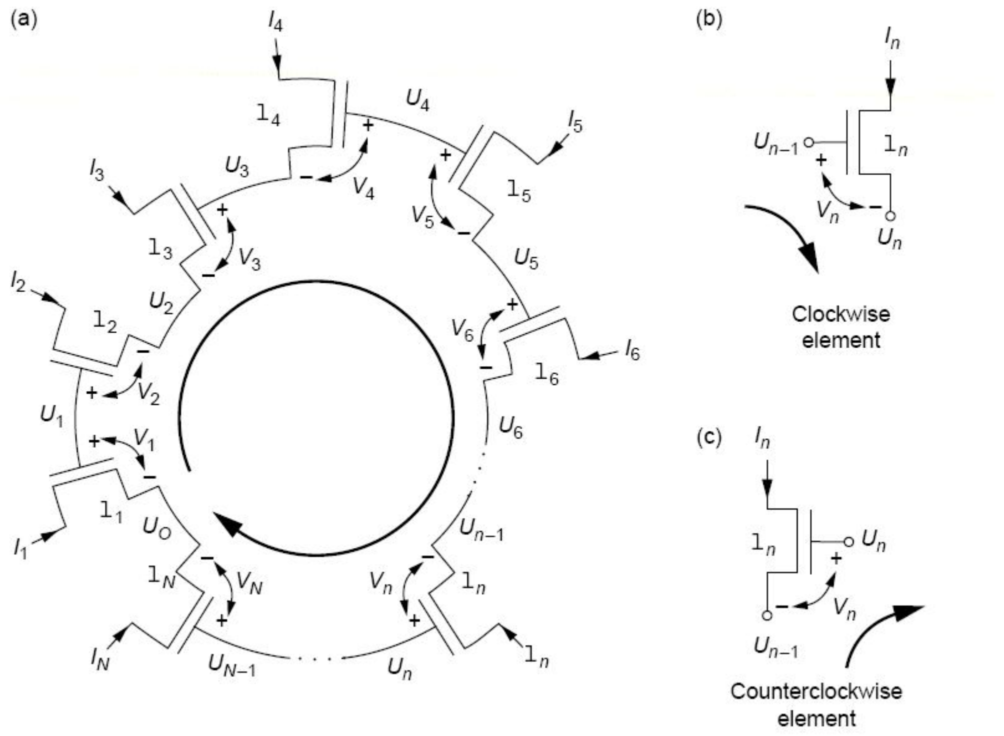
- Kirchoff's Voltage Law around loop:

$$\sum_{n\in CCW} V_n = \sum_{n\in CW} V_n \quad (1)$$

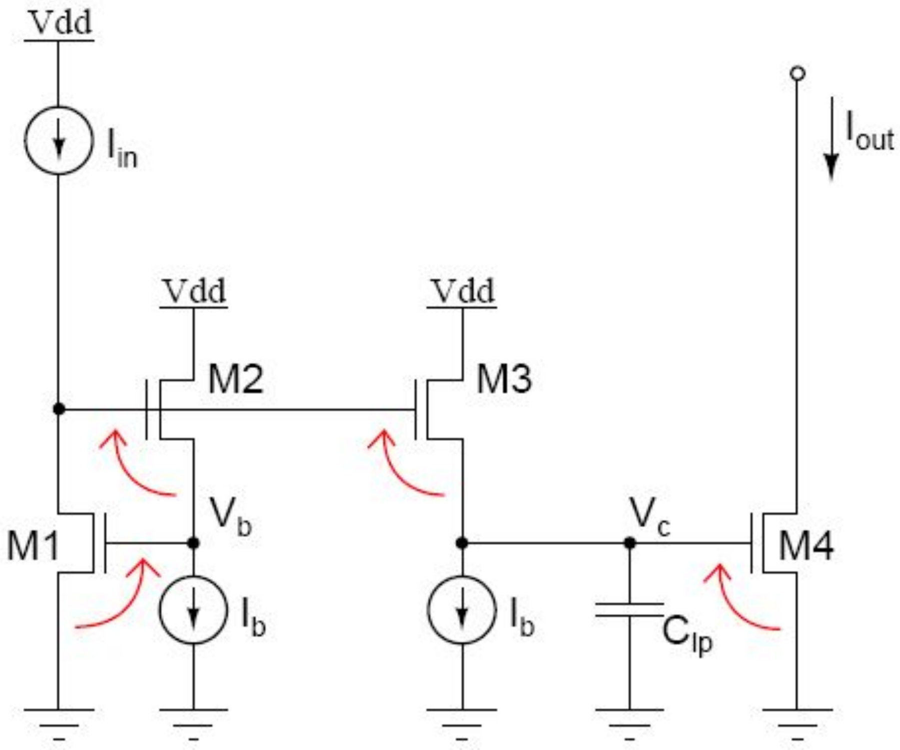
Substituting for I in (1) :

$$\sum_{n\in CCW} U_T \log\left(\frac{I_n}{I_0}\right) = \sum_{n\in CW} U_T \log\left(\frac{I_n}{I_0}\right)$$

$$\rightarrow \prod_{n\in CCW} I_n = \prod_{n\in CW} I_n$$



Current-Mode Low-Pass Filter



$$V_{gs}^{M1} + V_{gs}^{M2} - V_{gs}^{M3} - V_{gs}^{M4} = 0$$

$$I_{ds} = I_0 e^{\frac{\kappa V_g - V_s}{U_T}}$$

$$I_{ds}^{M1} \cdot I_{ds}^{M2} = I_{ds}^{M3} \cdot I_{ds}^{M4}$$

$$I_{ds}^{M3} = I_b + C_{lp} \frac{d}{dt} V_c$$

$$\frac{d}{dt} I_{out} = \frac{\kappa}{U_T} I_{out} \frac{d}{dt} V_c$$

$$I_{in} \cdot I_b = \left(I_b + \frac{C_{lp} U_T}{\kappa} \frac{\dot{I_{out}}}{I_{out}}\right) \cdot I_{out}$$

$$\boxed{\tau \dot{I_{out}} + I_{out} = I_{in}}$$

with $\tau = \frac{C_{lp} U_T}{\kappa I_b}$

(Frey, 1982)

THE END

Next week: Current-mode and winner-take-all circuits