



# Random Graph Models

*Network Science - Lecture 5*

**Carlo Campajola**

**Blockchain & Distributed Ledger Technologies Group**

*Claudio J. Tessone*



*LAST WARNING:  
from Assignment 3 (today) we  
will NOT accept  
non-standardised submissions*



## Assignment submission

- + only ONE .ipynb or .html file
- + name must be LASTNAME\_FIRSTNAME\_EX#.ipynb
- + we do NOT run your code: all results must be clearly shown in the notebook

*Not following these guidelines results in a FAIL*



## Lecture Objectives

1. Learn how to generate synthetic random networks
2. Get acquainted with the concept of network connectedness regimes
3. Learn the most important network randomisation algorithms



# *What is a random network?*

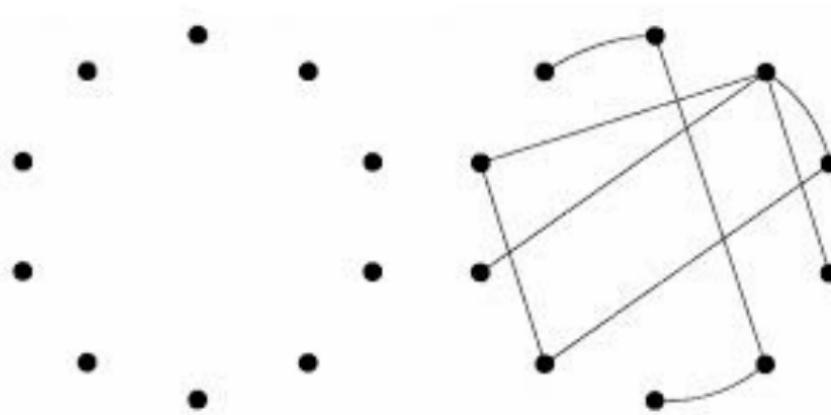


*A network resulting from a stochastic process*

## Example

A simple stochastic rule:

- +  $N$  nodes;
- + randomly select  $L$  pairs of nodes (without replacement)
- + add a link on each of the pairs





# *Why random networks?*



## Motivations

- + Statistical testing (null models): how likely is an observation if the network is formed with some rule?
- + Network generation: sometimes you just need to decide how to connect some nodes - e.g. p2p networks, agent-based models, ...

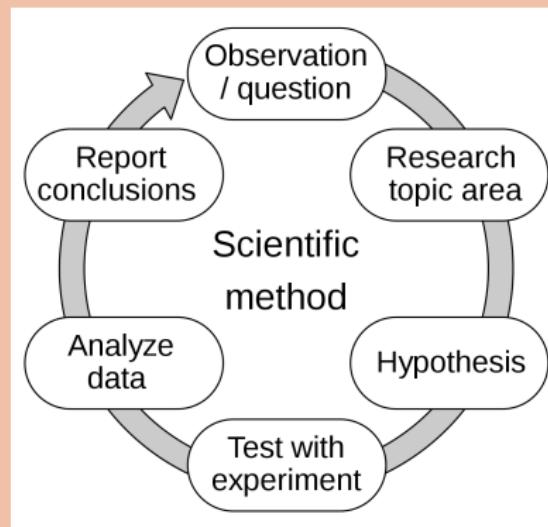


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## Refresher: Hypothesis testing

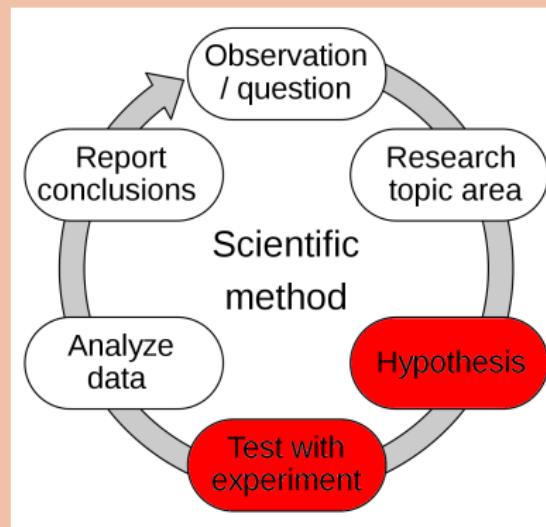
The scientific method is based on **falsification**





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1. Make a simple null hypothesis
2. What are the consequences of your hypothesis?
3. Identify a probability distribution for observations if your null hypothesis was true
4. Calculate the probability of observing something as extreme as the data, given your hypothesis
5. If the probability is low, the hypothesis is rejected and an alternative should be considered



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The scientific method is based on **falsification**

1. Null hypothesis  $\mathbb{H}_0$ : nodes are linked by random selection of pairs
2. Consequence: the maximum degree follows from the rule above
3. Find the null max degree probability  $\mathbb{P}(k_{max}|\mathbb{H}_0)$
4. Measure  $\hat{k}_{max}$  on data and calculate  $p = \mathbb{P}(k_{max} \geq \hat{k}_{max}|\mathbb{H}_0)$
5. If  $p < \alpha$  then reject  $\mathbb{H}_0$  at confidence level  $\alpha$ : random linkage is not realistic



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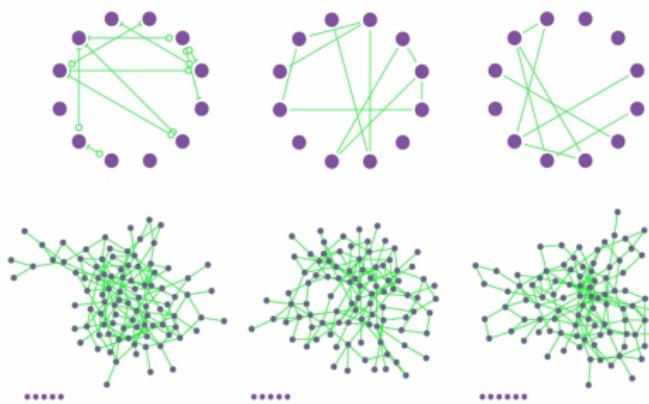


## The Erdos-Renyi Model

- + First attempt to approximate a real network (null model in multiple scenarios)
- + **Network density is fixed**
- + Connections are placed just randomly between every pair of nodes

- + *G(N, L) Model: N labeled nodes are connected with L randomly placed links*
- + *G(N, p) Model: each pair of N labeled nodes is connected with edge creation probability p*

## Erdos-Renyi is “homogeneous”



- + Links are equally likely between pairs of nodes
- + There are no nodes more important than others



## Degree distribution

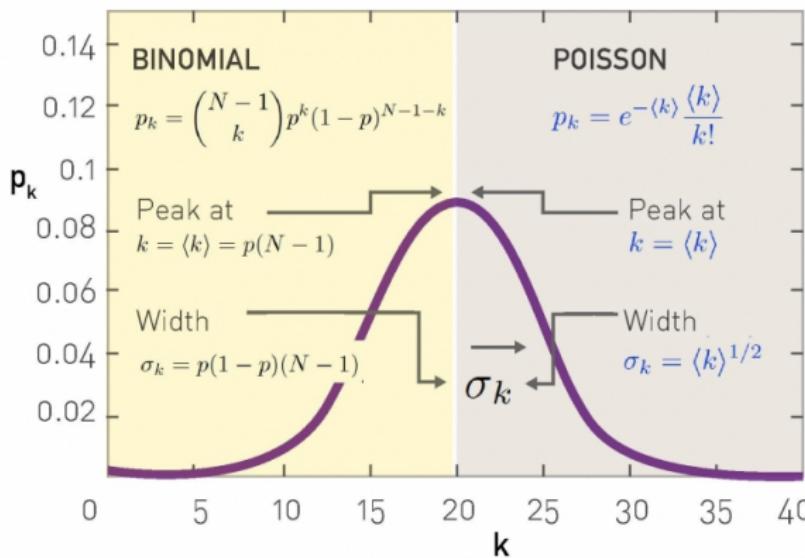
- + The binomial distribution describes the number of successes in  $N$  independent experiments with two possible outcomes, in which the probability of one outcome is  $p$ , and of the other is  $1 - p$ .
- + Exact distribution: Binomial:  $P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$
- + Depends on the network size
- + Approximation: Poisson:  $P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$
- + Does not depend on the network size, only on the average degree
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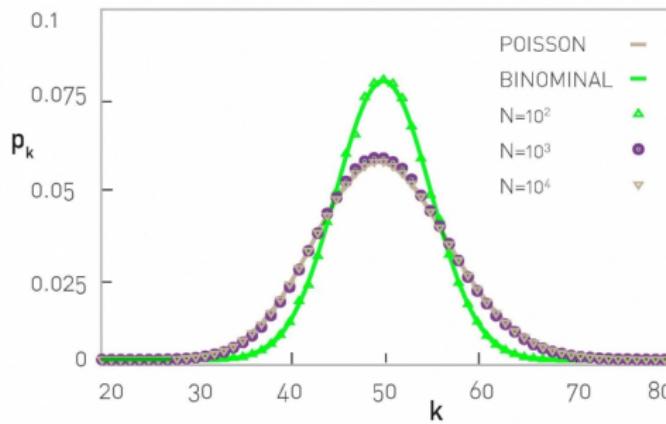
## Binomial vs. Poisson



## Poisson is an approximation

As network size  $N$  goes to  $\infty$ , for fixed  $\langle k \rangle = pN$ :

$$\binom{N-1}{k} p^k (1-p)^{N-1-k} \longrightarrow e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$





# Giant component



## Connectedness

- + The network behind the phone or the Internet must be capable of establishing a path between any two nodes
- + Key utility of most networks: they ensure connectedness
- + In an undirected network nodes  $i$  and  $j$  are connected if there is (at least) a path between them
- + They are disconnected if such a path does not exist, in which case we have  $d_{ij} = \infty$



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## Connectedness

- + A network is **connected** if all pairs of nodes in the network are connected
- + A network is **disconnected** if there is at least one pair with  $d_{ij} = \infty$
- + A **component** is a subset of nodes in a network, so that there is a path between any two nodes that belong to the component, but one cannot add any more nodes to it that would have the same property.



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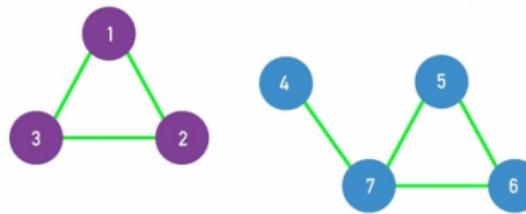


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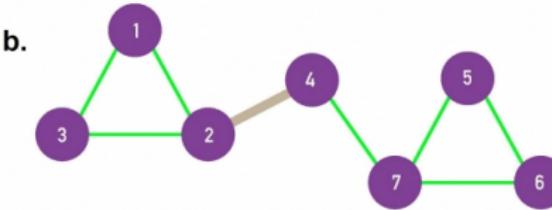
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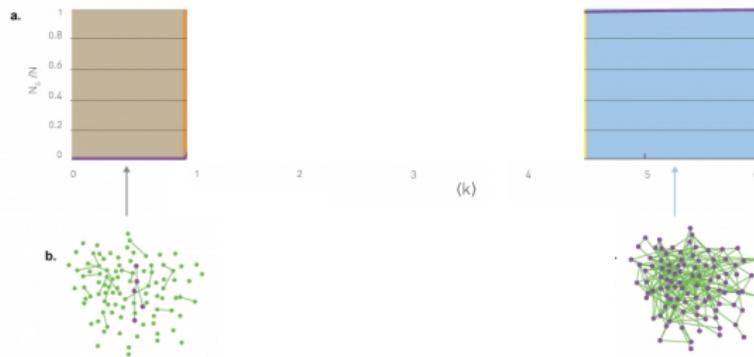
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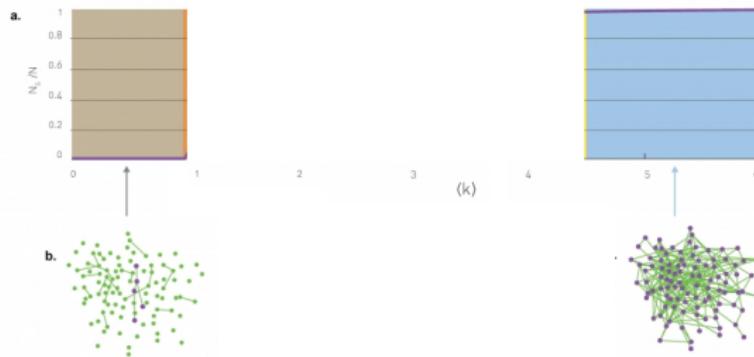
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## Evolution of random networks



- +  $N_G$  = size of the largest connected component
- + If  $p \sim 0 \Rightarrow \langle k \rangle \sim 0, N_G \sim 1, \frac{N_G}{N} \rightarrow 0$  for large  $N$ ;
- + If  $p \sim 1 \Rightarrow \langle k \rangle \sim N, N_G \sim N, \frac{N_G}{N} \rightarrow 1$  for large  $N$ ;
- + What happens in between?

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## Giant Component

- + We call a **giant component** a network component whose size grows *proportionally* with  $N$
- + The size of giant component does not grow linearly with  $p$
- + Instead, there is a *phase transition* from constant size to extensive size
- + For a Poisson random graph, the size of the giant component is defined by the equation:

*Size of the giant component*

Equation given by Erdos and Rényi in 1959:

$$N_G = 1 - e^{-\langle k \rangle N_G}$$



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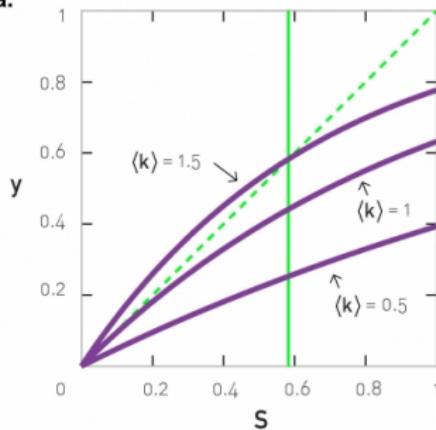
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## Erdos-Rényi equation

- + There is no analytical solution to this equation
- + We can take a look at the graphical solution for  $S = N_G/N$

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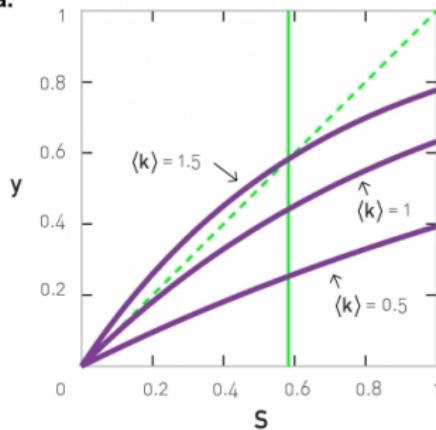


- + Depending on value of  $\langle k \rangle$ , there can be one or two solutions.
- +  $\langle k \rangle = 1$  is the threshold between the regimes with and without a non-trivial solution.

## Erdos-Rényi equation

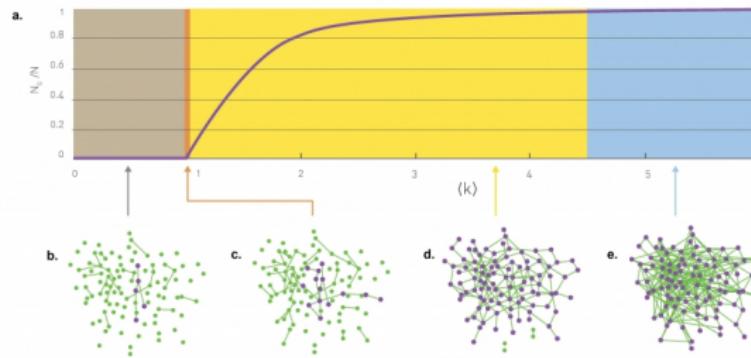
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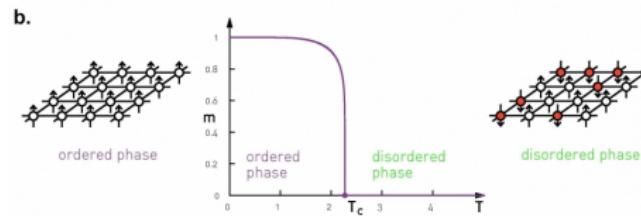
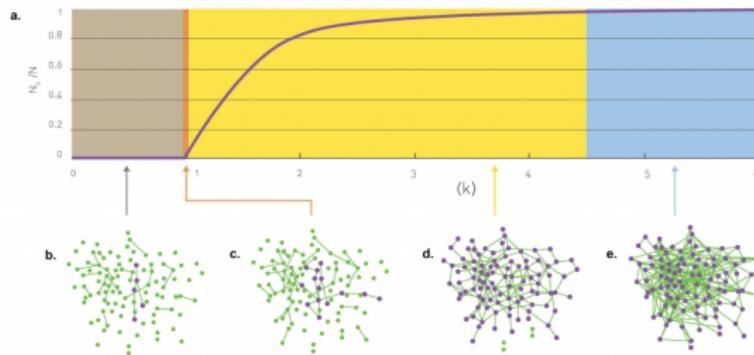
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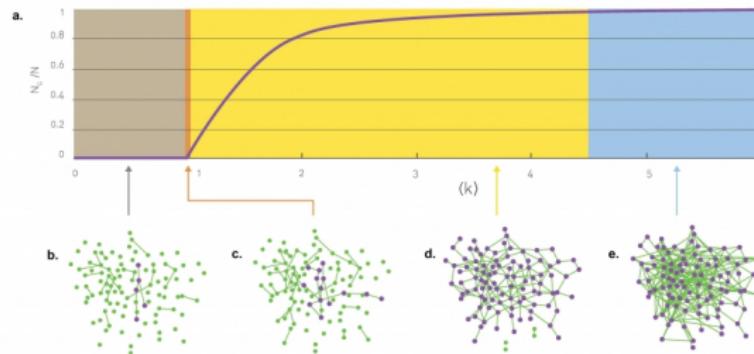


- +  $\frac{N_G}{N}$  as function of average degree  $\langle k \rangle$ .
- + Phase transition at  $\langle k \rangle = 1$ , emergence of a giant component.
- + Sample network and its properties in the four regimes that characterise a random network.

## Network regimes

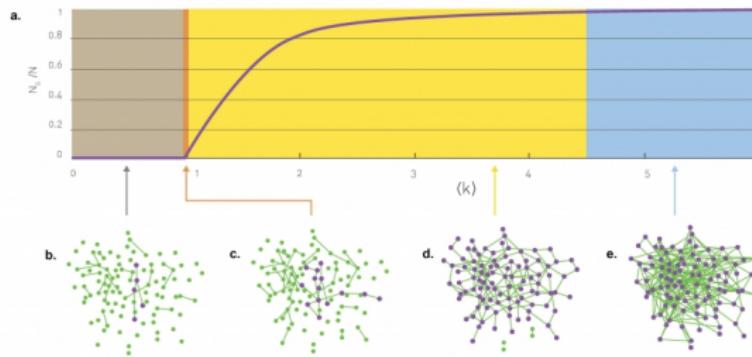


## Network regimes



- + **Subcritical Regime:**  $\langle k \rangle < 1 \iff p < \frac{1}{N}$ : numerous tiny components, fig. b

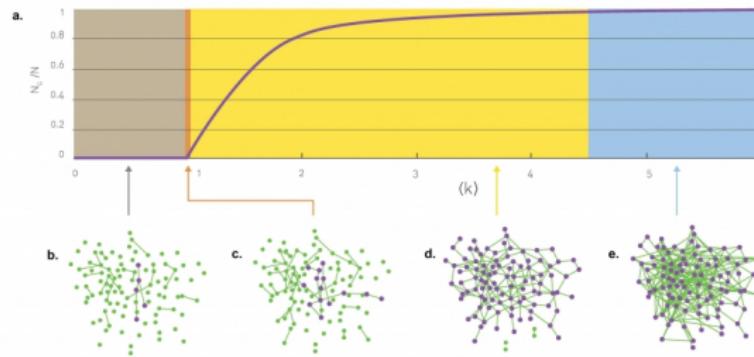
## Network regimes



- + **Critical point:**  $\langle k \rangle = 1 \iff p = \frac{1}{N} = p_{crit}$ :

Still no giant component, the size of the largest component is  $N_G \propto N^{\frac{2}{3}}$ , fig. c

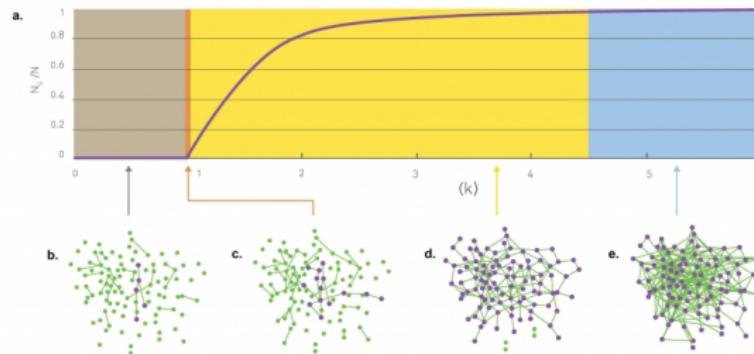
## Network regimes



+ **Supercritical Regime:**  $\langle k \rangle > 1 \iff p > \frac{1}{N}$

Giant component size is  $N_G \propto (p - p_{crit})N$ , fig. d

## Network regimes



+ **Connected Regime:**  $\langle k \rangle > \log(N)$

All components are absorbed by the giant component, fig. e

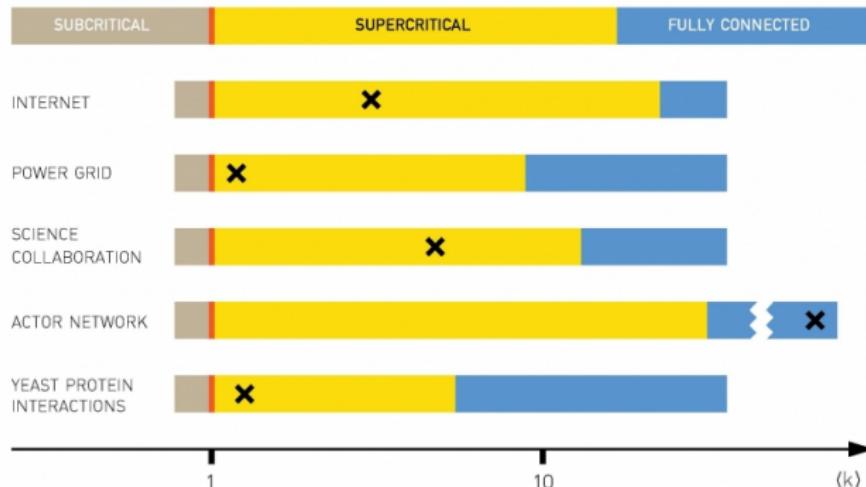


## Network regimes

Video: <http://networksciencebook.com/images/ch-03/video-3-2.mp4>

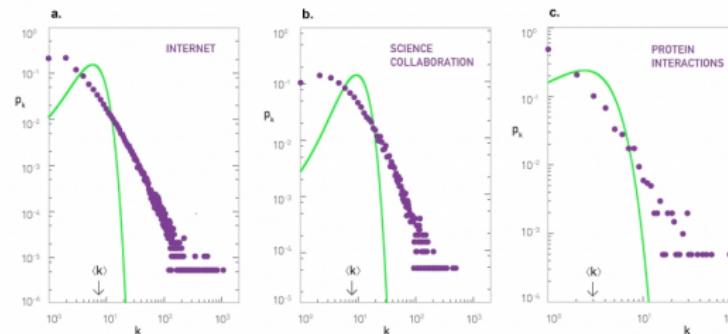
## Real networks regimes: predictions

Note: Only valid if the networks are described by ER model.



## Real networks are not Poisson

**Figure:** The green line corresponds to the Poisson prediction, obtained by measuring  $\langle k \rangle$  for the real network and then plotting Poisson curve.



Random network model:

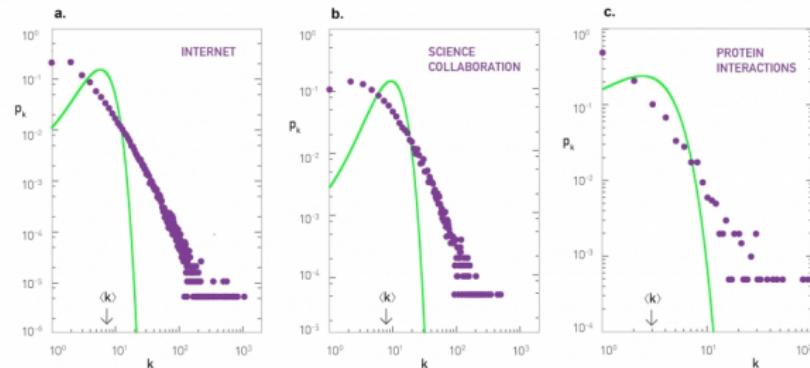
- + *underestimates* size and frequency of high degree nodes;
- + *underestimates* the number of low degree nodes.
- + *overestimates* number of nodes in the vicinity of  $\langle k \rangle$ .



# Configuration Model

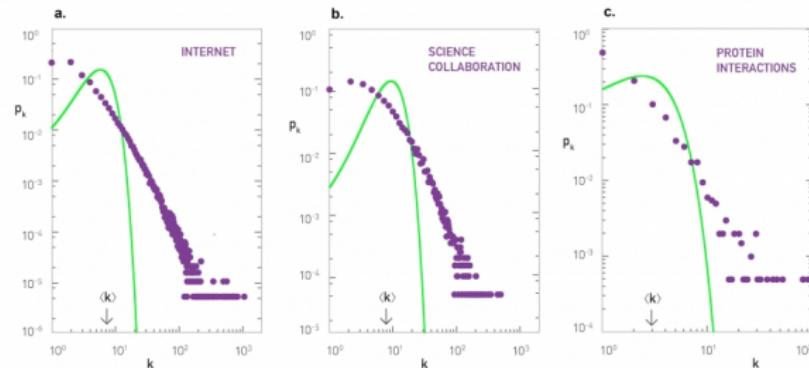
## ER versus configuration model

- + Erdos-Rényi model only replicated network density
- + What happens if we want to generate networks with the same degree distribution than one observation?
- + Final question is: How much of the network properties are a result of the degree distribution?



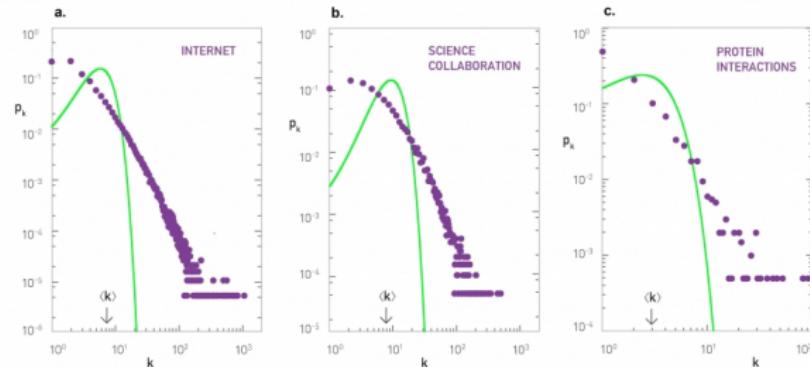
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## Configuration model algorithm

### Algorithm:

1. Generate the degree sequence either analytically from a preselected  $P(k)$  distribution or extracted from a real network
2. Assign a degree to each node, represented as *stubs* or *half-links*
3. Start from an even number of stubs
4. Randomly select a stub pair and connect them
5. Then randomly choose another pair from the remaining  $2L - 2$  stubs and connect them
6. This procedure is repeated until all stubs are paired up
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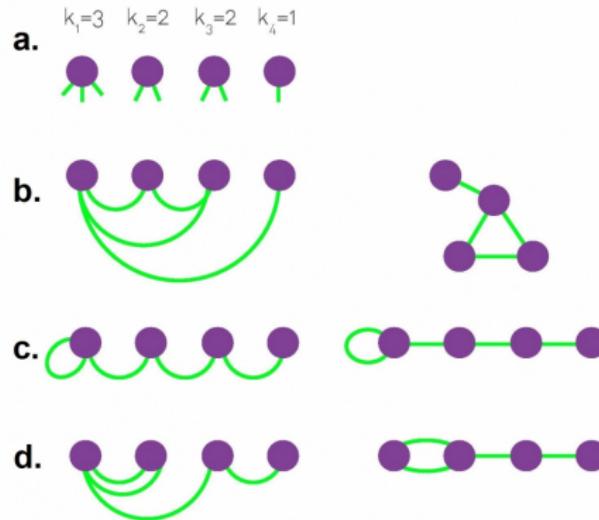


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## Configuration model illustration





## Configuration model properties

- + Helps to build a network with a pre-defined degree sequence
- + This fixes the number of edges in the network (like ER models)
- + This is an extension to the  $G(N, L)$  model, which also fixes the number of edges  $L$
- + Generates each possible matching of stubs with equal probability
- + Each stub is equally likely to be connected to any other.
- + All generated networks are equally likely (with self-loops or multiedges)
- + *Requirement:* start with an even total number of stubs
- + *Note:* self-loops and multiedges cannot be avoided



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## Configuration model: edge probability

- + Assume  $k_i, k_j > 0$
- +  $L$  edges  $\rightarrow 2L$  stubs
- + Any stub out of  $k_i$  stubs of node  $i$  can lead to any  $2L - 1$  remaining ones
- + There are  $k_j$  possibilities to connect to node  $j$ .
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## Hidden Parameter Model

- + Generalisation to  $G(N, p)$  model, which fixes probability of an edge.
- + We define a *hidden parameter*  $\eta_i \sim \rho(\eta)$  for each vertex.
- + Each node pair is connected with probability

$$p(\eta_i, \eta_j) = \frac{\eta_i \eta_j}{\langle \eta \rangle N}$$

- + The degree distribution of the obtained network is:
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$$p_k = \int \frac{e^{-\eta} \eta^k}{k!} \rho(\eta) d\eta$$

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# Degree-preserving randomisation



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- + Used when we check if a certain network property is *predicted by its degree distribution alone*, or if it represents some additional property not contained in  $P(k)$ .
- + Generates networks that are wired randomly, but whose  $P(k)$  is **identical** to the original network.
- + Idea: we randomly select two links and swap them, if this does not lead to multilink.
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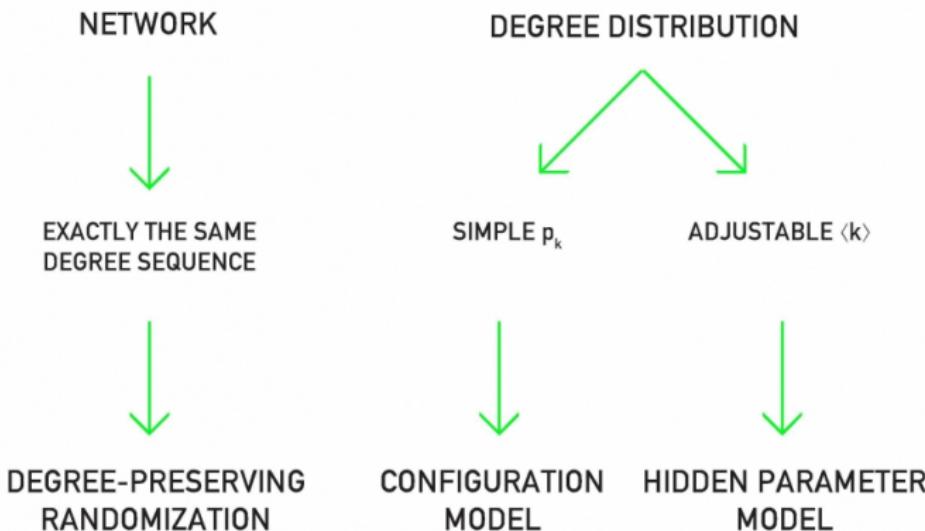


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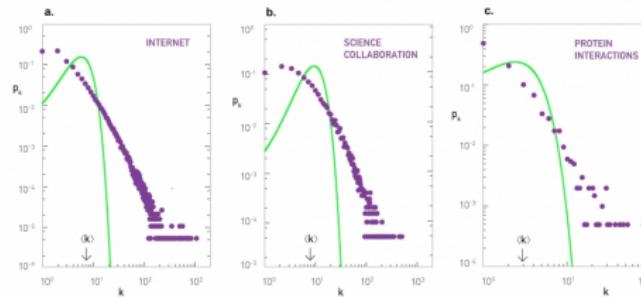




# Small World

## Real networks are not Poisson

**Figure:** The green line corresponds to the Poisson prediction, obtained by measuring  $\langle k \rangle$  for the real network and then plotting Poisson curve.



Random network model:

- + *underestimates* size and frequency of high degree nodes;
- + *underestimates* the number of low degree nodes.
- + *overestimates* number of nodes in the vicinity of  $\langle k \rangle$ .



## Theory of 6 handshakes

- + Pool and Kochen, 1950:

*What is the probability that two strangers will have a mutual friend? What about when there is no mutual friend – how long would the chain of intermediaries be?*

- + Milgram, 1969:

*Let's conduct an experiment!*

- + John Guare, 1990:

*There are six degree of separation between you and everyone else on this planet*

- + Backstrom et al., 2012:

*Four Degrees of Separation*



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## Milgram's experiment



- + 300 initial senders in the central states of the USA: *Letters and business reply cards*
- + The letters should reach a specific person in Boston
- + *If a sender does not know how to reach this person, she should forward the letter to some acquaintance she thought may know how to deliver it*
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- + Around  $\frac{1}{5}$  of all letters reached the target
- + Among the arrived letters, the average number of intermediate steps was between 5 and 6 (over many trials)
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- + The small-world phenomenon is a fundamental property of social networks



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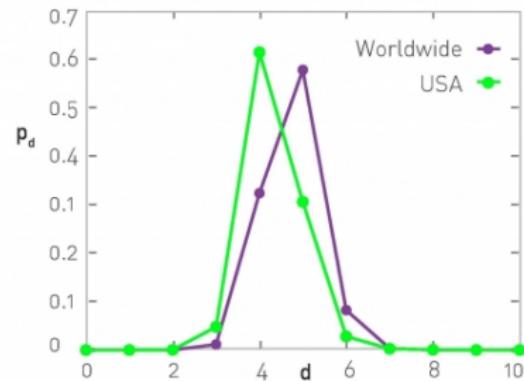
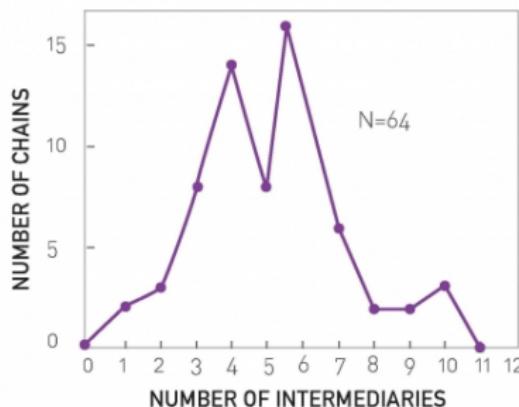


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[Albert- László Barabási, *Network Science* (Cambridge University Press, 2015)]





## Small-world phenomenon

Imagine a network where a node has on average  $\langle k \rangle$  connections.

Then:

- + There are  $\langle k \rangle$  nodes at distance 1;  $10^3$  acquaintances
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For the human acquaintances network, the estimate for  $\langle k \rangle$  is approximately 1000.

*This means that we have  $10^6$  acquaintances at distance 2,  
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Expected number of nodes up to distance  $d$  from a starting node:

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which gives:

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Short compared to what?

Typically, the small-world property is defined as:

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*Short distance means that the average path length (or diameter) depends logarithmically on the system size. This is a result by Manfred Kochen and Ithiel de Sola Pool (1950), which has inspired the Milgram experiment (1969).*



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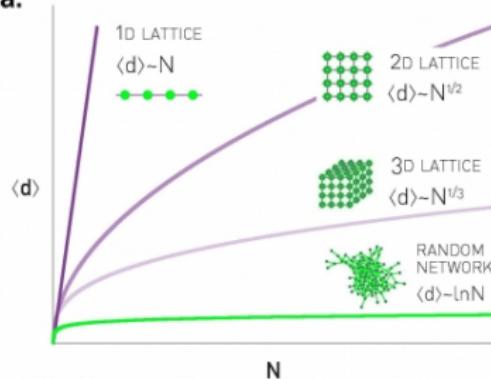
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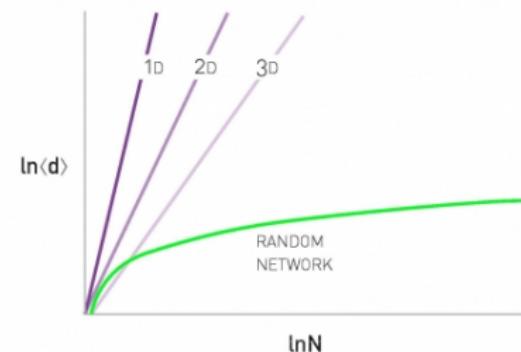


## Small-world comparing to lattice

a.



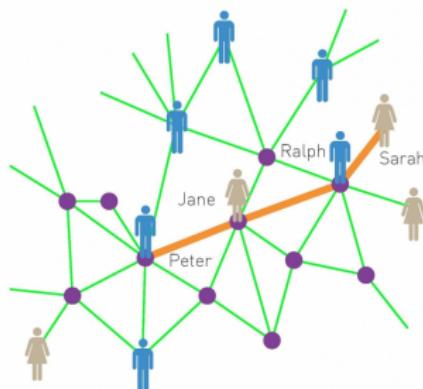
b.



## Small-world in social networks

For all individuals on Earth ( $N \approx 7 \cdot 10^9$ ), assuming  $\langle k \rangle \approx 10^3$  :

$$\langle d \rangle = \frac{\log(7 \cdot 10^9)}{\log(10^3)} \approx 3.28$$





## Watts and Strogatz model

[Watts and Strogatz, *Collective dynamics of 'small-world' networks* (1998)]

- + A random graph generation model that produces graphs with small-world properties
- + Starting from arbitrary topology, adding just few random links connecting distant nodes helps decrease average distance
- + The resulting network features many clusters of connected people
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## WS model: changing edge target

1. Start with regular ring lattice with  $N$  nodes and  $2\kappa$  neighbours of each node
2. For each node  $i$  consider edges  $(i, j), i < j$  and rewire it with certain probability  $p$
3. *Rewiring* means replacing edge  $(i, j)$  with another edge  $(i, \ell)$ ,  $\ell$  is selected at random
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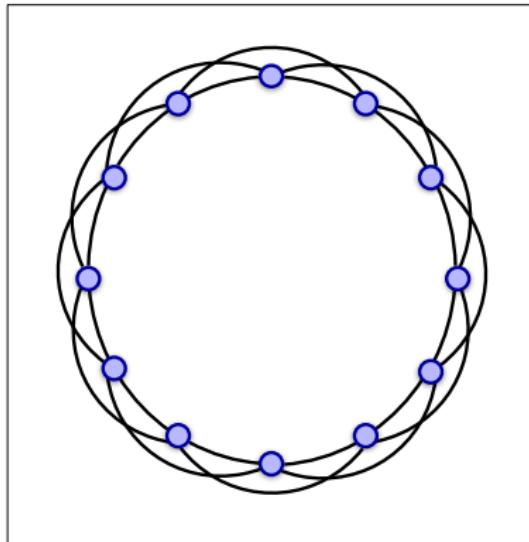


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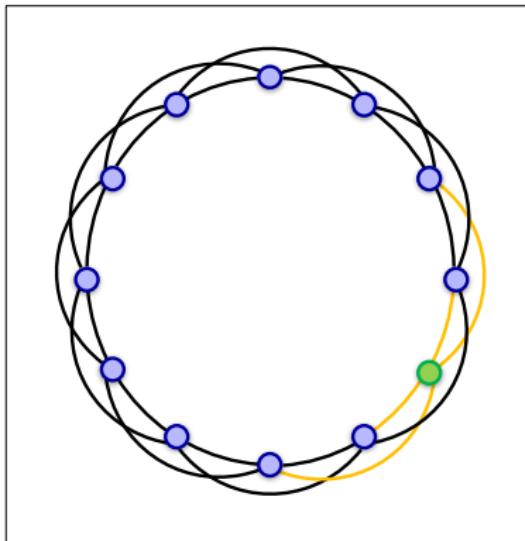


## Visual explanation



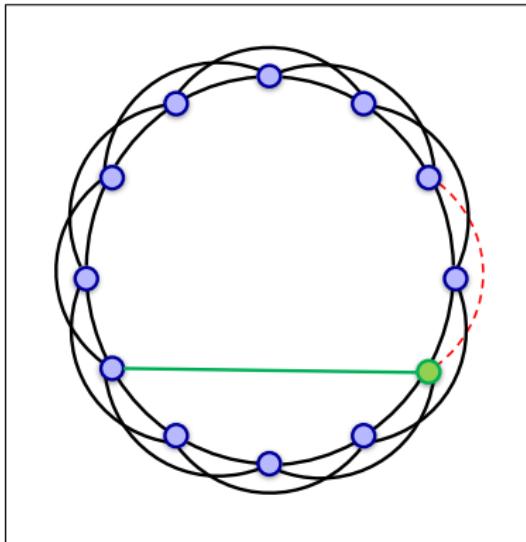
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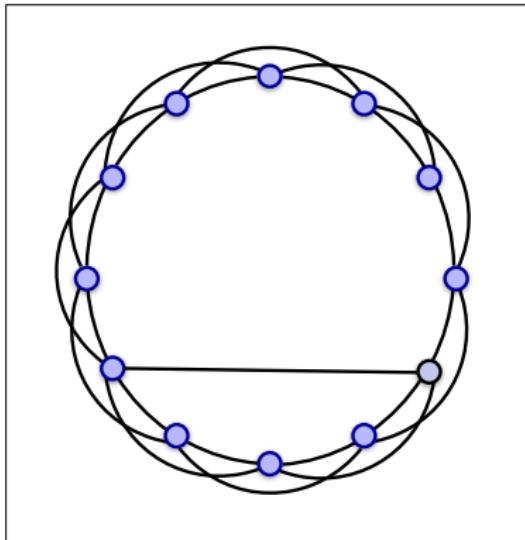
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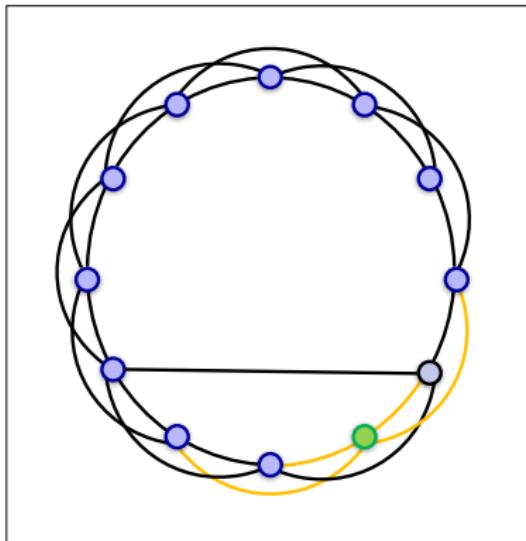
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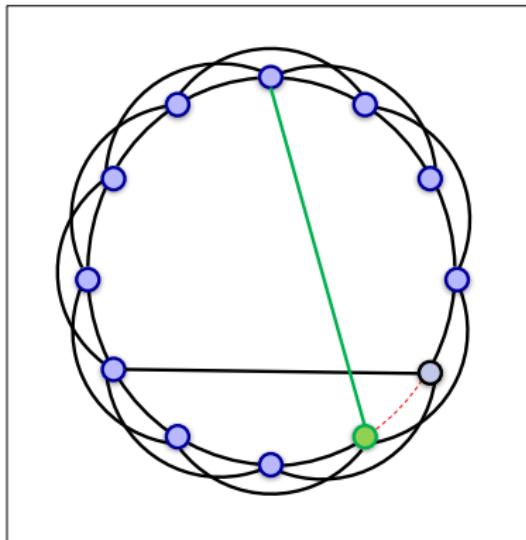
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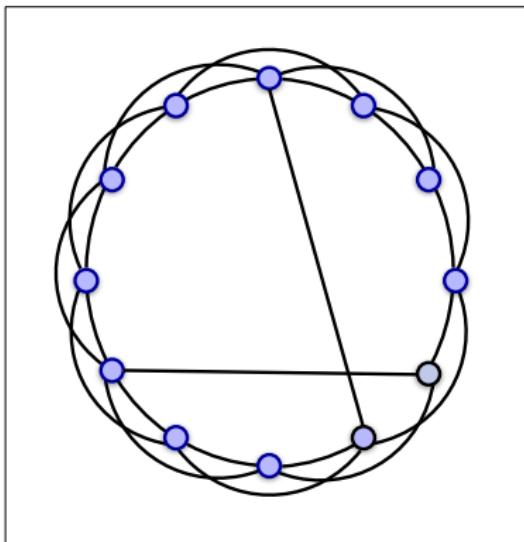
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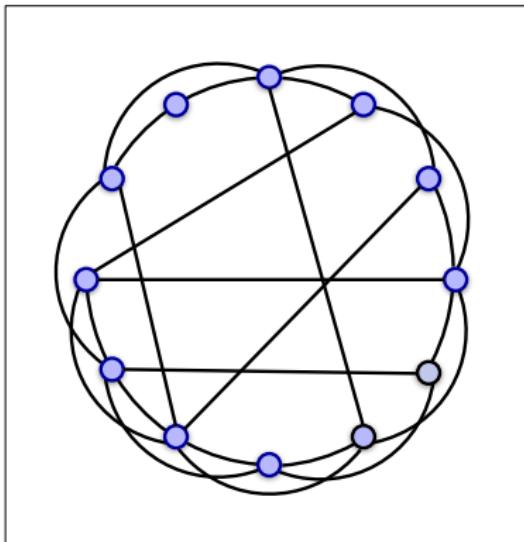
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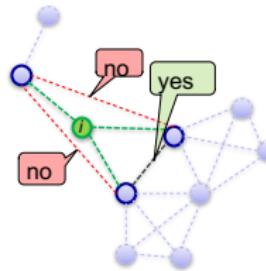
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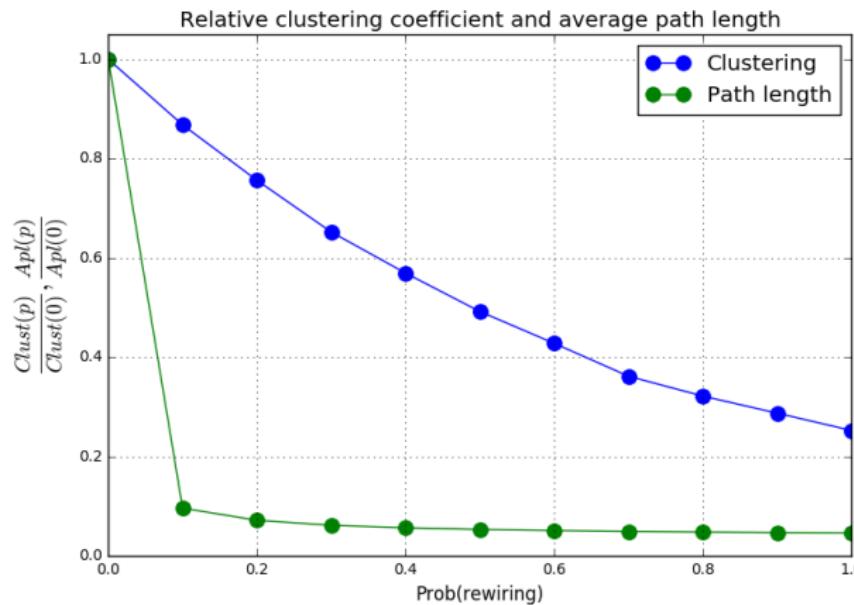


How likely is it that if two nodes are common neighbours of another, they are connected to each other?

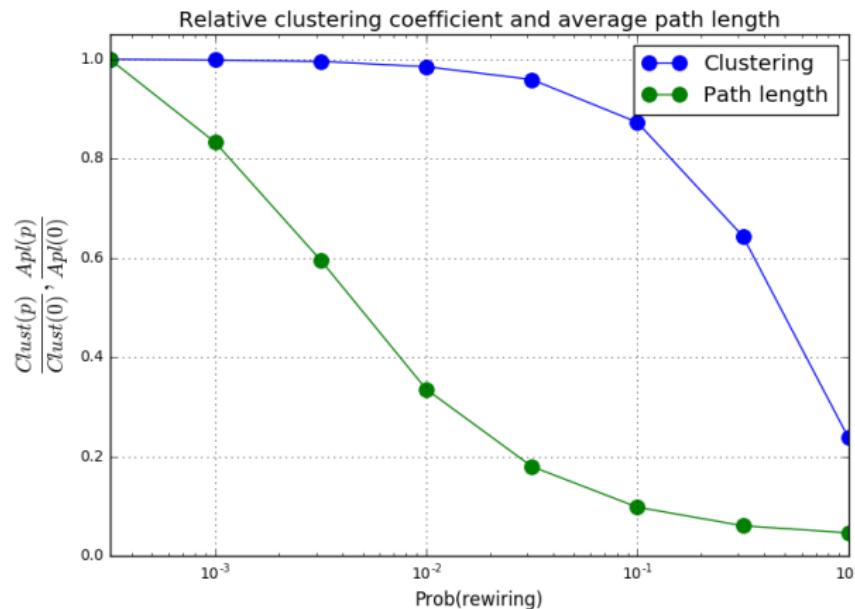
$$C(i) = \frac{\sum_{u,v \in N_i} A_{u,i} A_{v,i} A_{u,v}}{\sum_{u,v \in N_i} A_{u,i} A_{v,i}} = \frac{2 \sum_{u,v \in N_i} A_{u,v}}{k_i(k_i - 1)} \quad (5)$$

*number of closed triangles divided over number of all possible triangles*

## WS model: changing edge target



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## Small-world network in a nutshell

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- + The relative number of necessary shortcuts vanishes for larger networks
- + Average path length similar to a random network.  
Clustering is much larger



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## WS model: swapping edges

[Maslov, Sneppen, *Specificity and stability in topology of protein networks* (2002)]

1. Start with regular ring lattice with  $N$  nodes and  $2\kappa$  neighbours of each node;
2. Select a pair of edges  $(i,j)$  and  $(k,l)$  and rewire them with certain probability  $p$
3. Rewiring means replacing the edges targets, putting edges from  $(i,j)$  and  $(k,l)$  to  $(i,k)$  and  $(j,l)$ .
  - + Degree distribution is preserved
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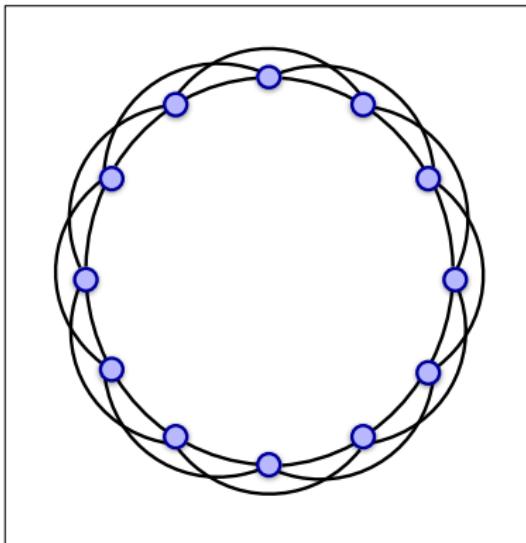


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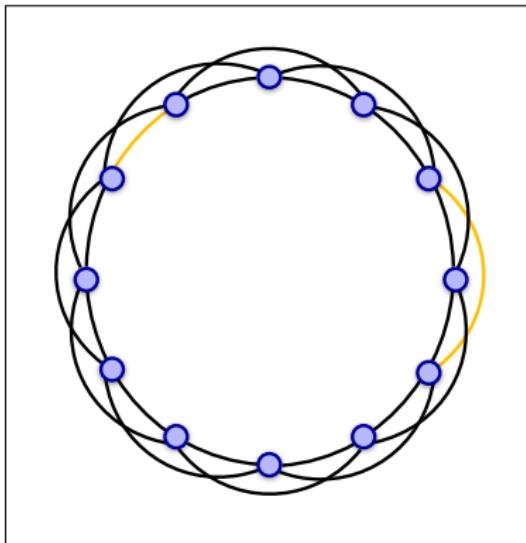
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## Visual explanation



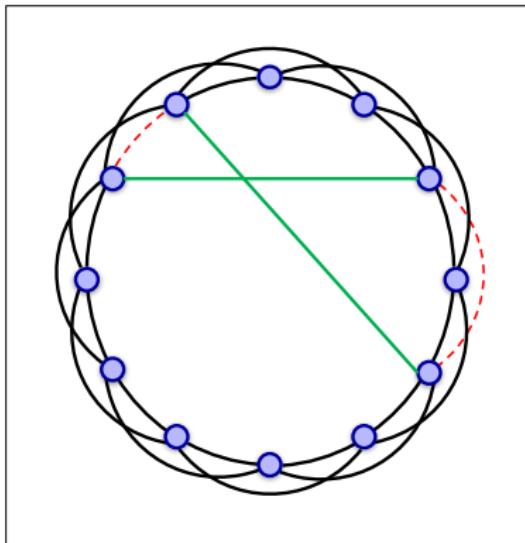
- + Start from a one dimensional lattice
- + Select two edges at a time
- + Exchange ends with probability  $p$
- + Avoid creation of self-loops or parallel edges

## Visual explanation



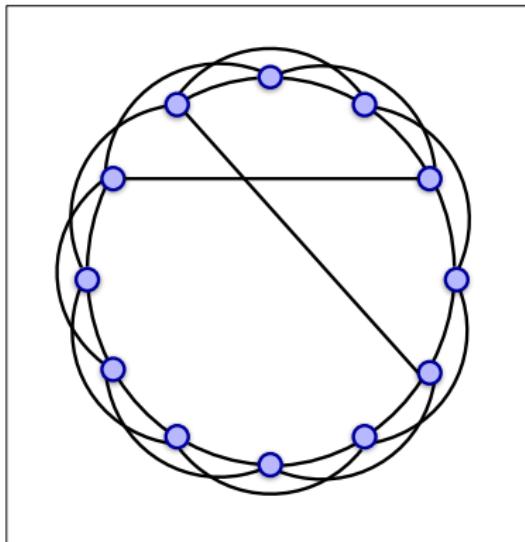
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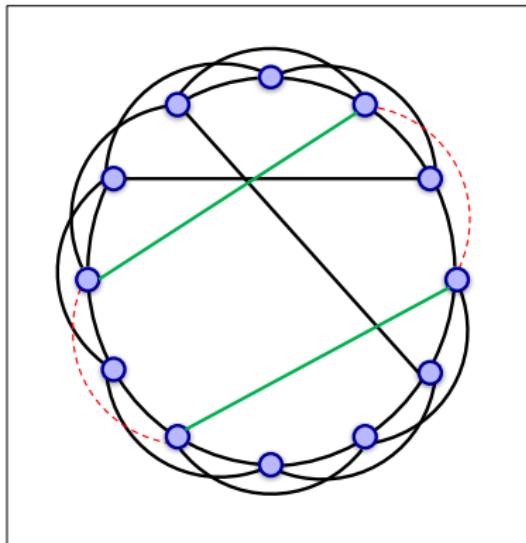
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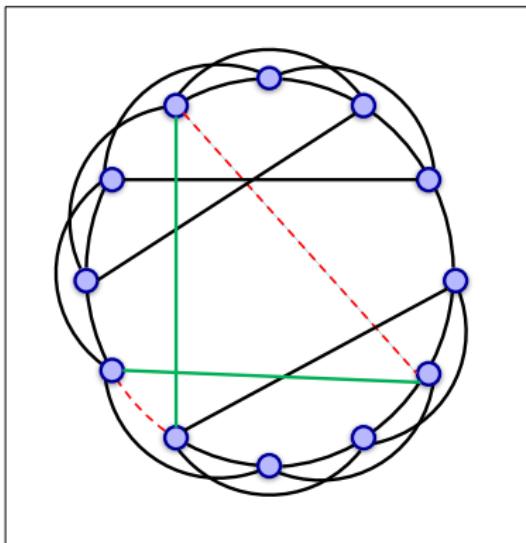
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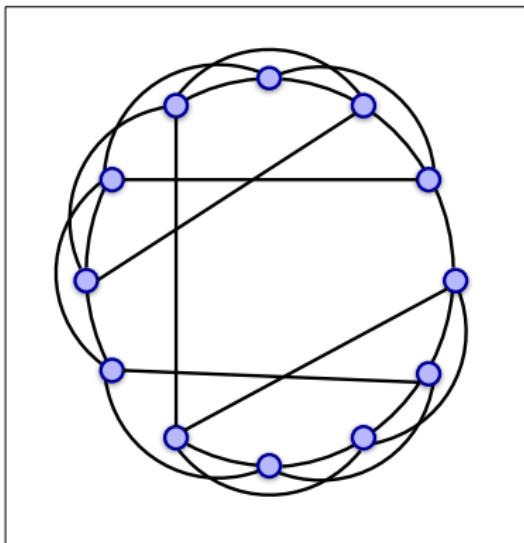
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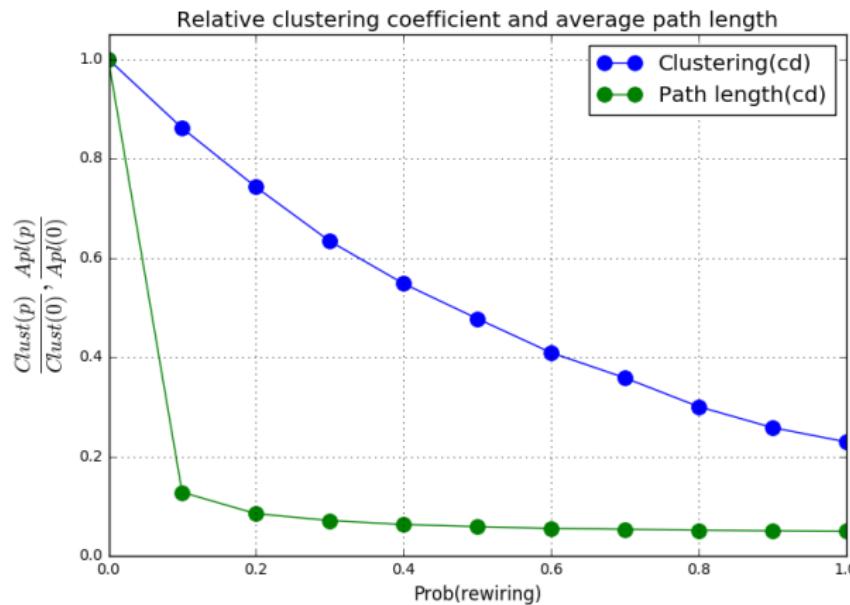
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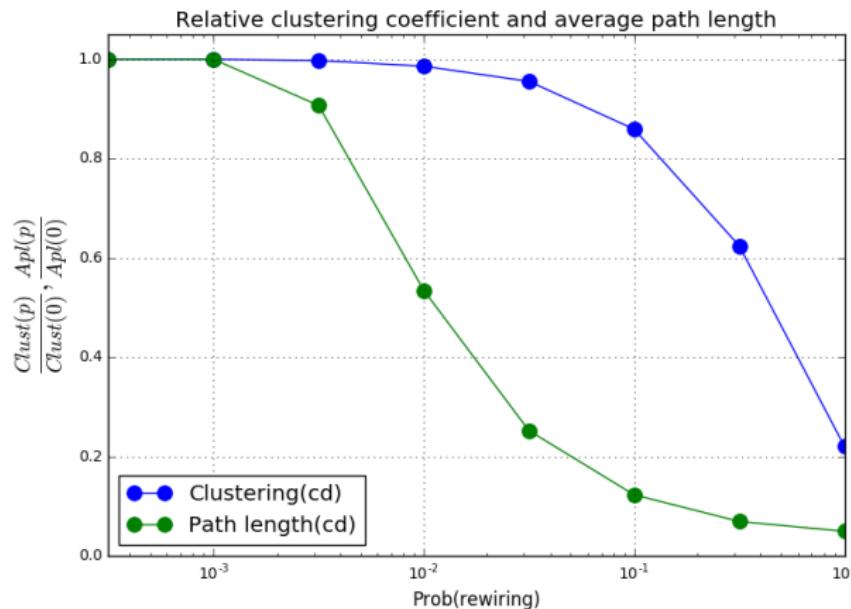


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## WS model: swapping edges

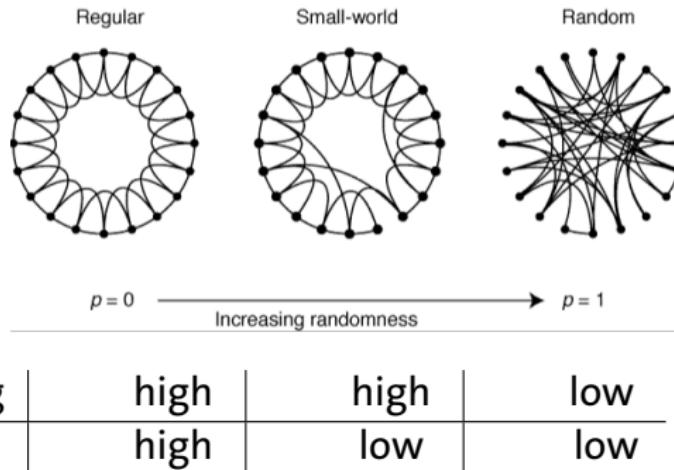


## WS model: swapping edges





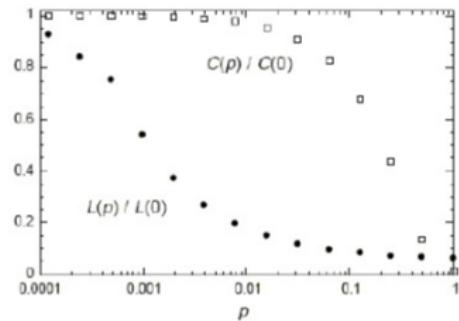
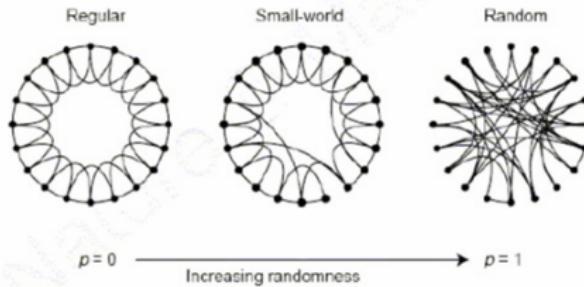
## From regular to random network





## Small-world network summary

- + Short path length (shortcuts)
- + High clustering
- + Unrealistic degree distribution
- + Cannot be used to model network growth





## References I

- ▶ Albert- Laszlo Barabasi, *Network Science*, Cambridge University Press, 2015.
- ▶ M.E.J. Newman *Networks: an Introduction*, Oxford University Press Inc., New York, 2010.
- ▶ Duncan J. Watts and Steven H. Strogatz, *Collective dynamics of ‘small-world’ networks*, in *Nature*, 1998.



## Carlo Campajola

Blockchain & Distributed Ledger Technologies Group

✉ campajola@ifi.uzh.ch

↗ <https://www.ifi.uzh.ch/en/bdlt.html>