



Scale-free networks

Lecture 6

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Lecture Objectives

1. Learn what is a scale-free distribution
2. Learn what are the mechanisms that lead to the formation of scale-free distributions
3. Learn how to practically determine the parameters of a scale-free distribution



Contents

- 1 Distributions
- 2 Barabási-Albert model
- 3 A long story
 - Yule models
 - The Matthew Effect
 - The Price Model
- 4 Model extensions
 - Fitness model
 - Intrinsic fitness



*Large-scale phenomena that
have their roots in
micro-behaviour*

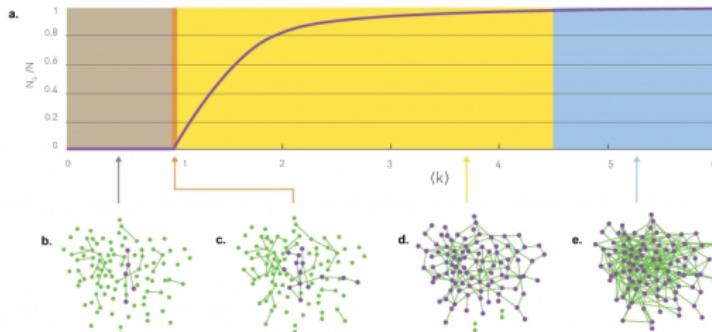


*Global order emerges
(appears by itself) bottom-up
and not enforced (**top-down**)
by a designer or system
planner*



Distributions

Random Networks: ER Model

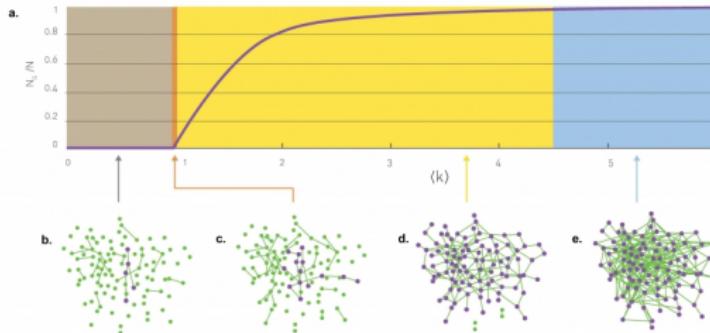


The network is static

Mechanism under study

Random (non-preference) link between nodes

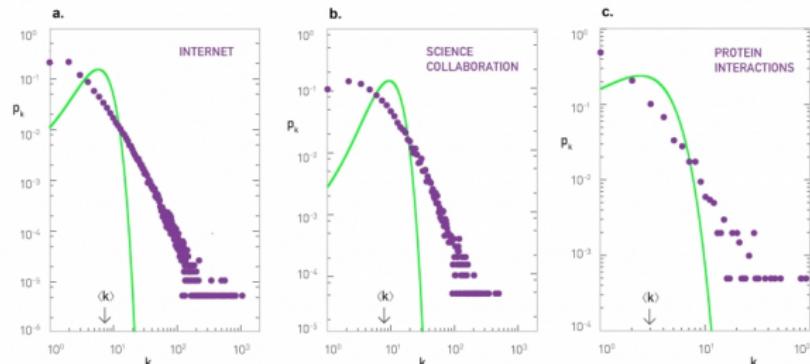
Random Networks: ER Model



It produces Poisson degree distributions (and well-defined average degree), which are unrealistic

Real networks are not Poisson

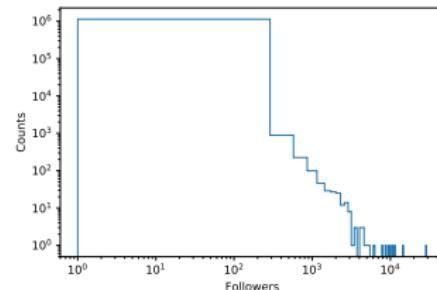
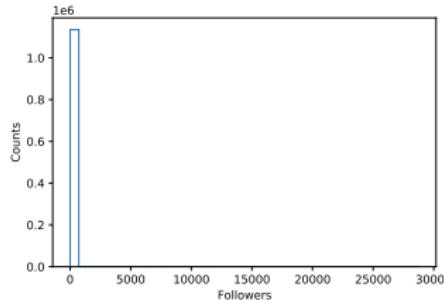
Figure: The green line corresponds to the Poisson prediction, obtained by measuring $\langle k \rangle$ for the real network and then plotting Poisson curve.



Random network model:

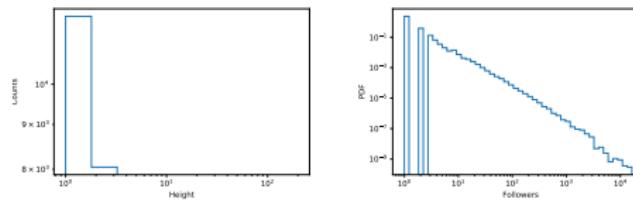
- + *underestimates* size and frequency of high degree nodes;
- + *underestimates* the number of low degree nodes.
- + *overestimates* number of nodes in the vicinity of $\langle k \rangle$.

Heavy-tailed distributions



- + In many situations (and in networks it happens very often), the distributions involved are *heavy-tailed*
- + These distributions when plotted in linear scales do not convey information

Distributions in log-scale



- + When plotting distributions in logarithmic scale, you bin logarithmically the data. I.e. $x_2 = a x_1$, $x_2 = a^2 x_1$, ... $x_N = a^N x_1$
- + Then, you must plot the distributions, i.e.

$$p(a \leq x \leq b) \approx \text{count}(a \leq x \leq b) / (b - a)$$

- + *Bins should be of similar size in the scale in which they are plotted*

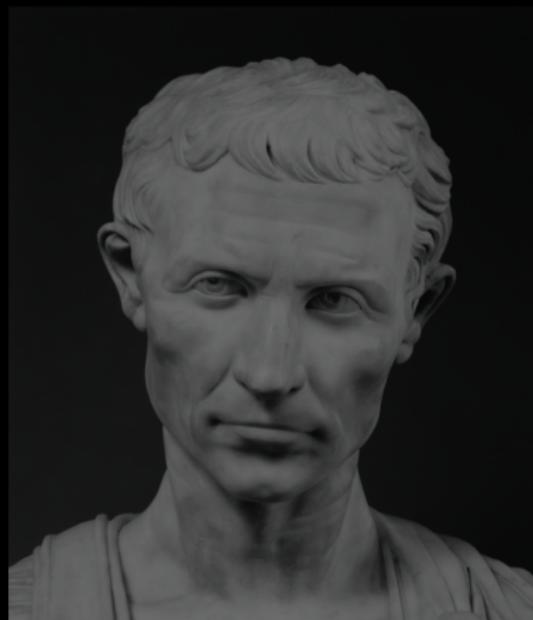


When plotting data distributions, they have their natural scales, and the bins should accompany them



Questions...

How tall was Julius Caesar?





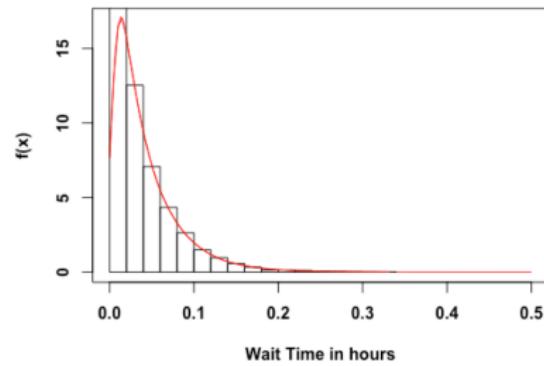
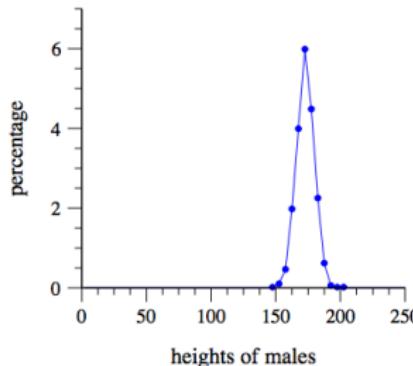
Questions...

If you go now to take a bus to the train station,
how long do you think you would need to wait?

Abschnitt	Linie	Von	Nach	Durchfahrt	Gleis	Bemerkung
20.05	IR	Schaffhausen			12	
20.06	IR	Baden Brugg	Aarau	Otten	14	
20.07	IR	Flughafen +	Winterthur		11	
20.08	IR	Lenzburg	Aarau		18	
20.09	ICN	Flughafen +	Zug	Winterthur	10	
20.10	IR		Thalwil	Arth-Goldau	8	
20.12	RE RegioExpress		Otten	Pfäffikon SZ	7	
20.30	IR	Bern		Solothurn	14	
20.32	IS	Basel		Lausanne	15	
20.34	IR		Thalwil	Zug	9	
20.35	IR		Baden Brugg	Basel	5	
20.36	IR		Sargans	Landquart	16	
20.37	IS		Flughafen +	Winterthur	6	
20.38	IR	Lenzburg	Aarau		11	
20.39	IS	Flughafen +	Winterthur	St.Gallen	13	

S-Bahn	
Abfahrt	Zeit
3-3	20.57
3-4	20.58
3-5	20.59
3-6	20.10
3-7	20.11
3-8	20.12
3-9	20.13
3-10	20.14
3-11	20.14
3-12	20.14
3-13	20.15
3-14	20.15
3-15	20.16
3-16	20.17
3-17	20.18
Informationen	

Empirical distributions



Many empirical quantities cluster around a typical value



Single-scale distribution: the underlying processes that generate these distributions fall into the general class well-described by the central limit theorem



Questions...

How many connections does this person have in LinkedIn?



Connections



Questions...

Which is the wealth of this person?



Degree distribution

$$P(k) = \text{Probability}(k_i = k)$$

In many contexts of interest, the degree distribution does not have a characteristic scale

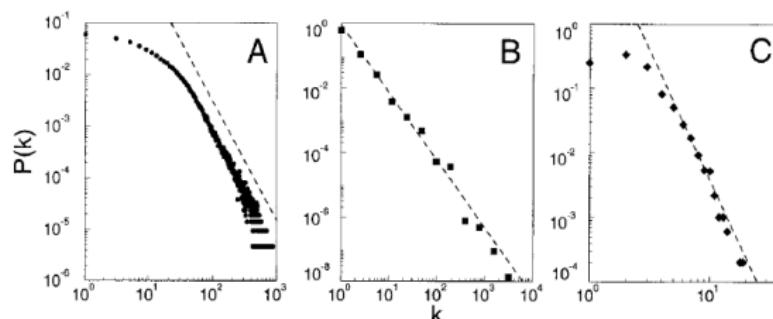
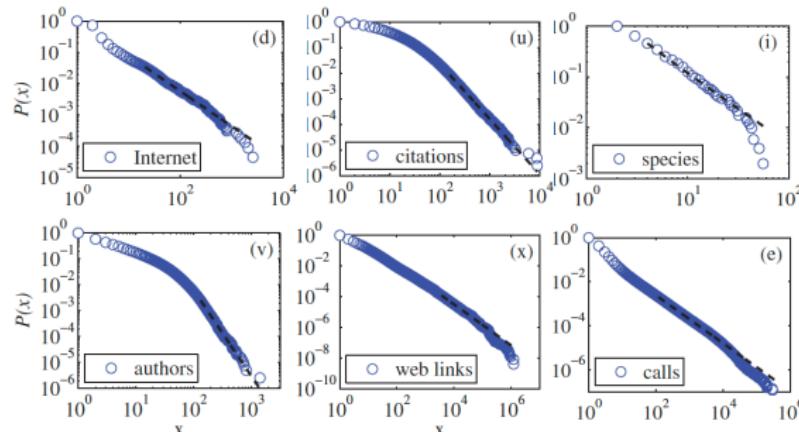


Fig. 1. The distribution function of connectivities for various large networks. (A) Actor collaboration graph with $N = 212,250$ vertices and average connectivity $\langle k \rangle = 28.78$. (B) WWW, $N = 325,729$, $\langle k \rangle = 5.46$ (6). (C) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{\text{actor}} = 2.3$, (B) $\gamma_{\text{www}} = 2.1$ and (C) $\gamma_{\text{power}} = 4$.

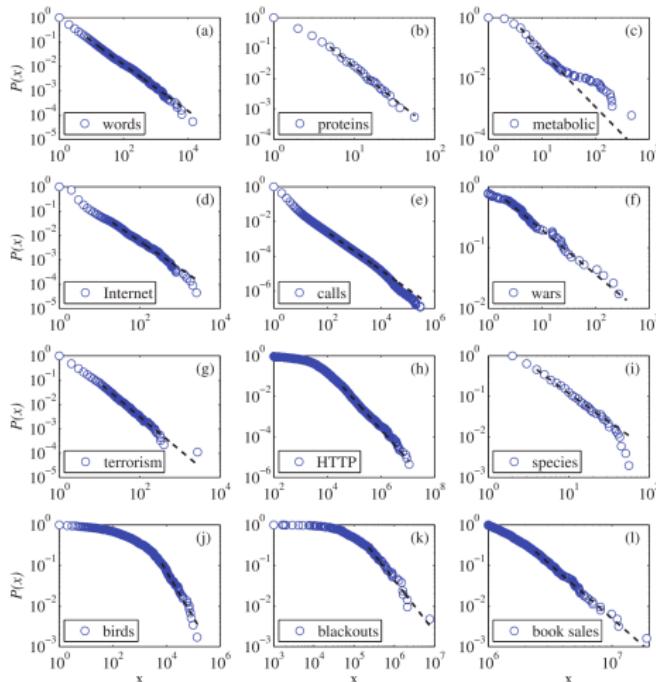
Degree distribution

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Scale-free property



Scale-free distributions are ubiquitous in social and economic systems



Scale-free property

- + Given a function $f(x)$
- + ...Scaling its argument, means computing $c x$ for all x

A function is said to be scale-free iff $f(cx) = bf(x), \forall c, x$

- + Let us take the power-law function $f(x) = a x^\gamma$.

$$f(cx) = a(cx)^\gamma = c^\gamma f(x) = bf(x)$$

- + Regardless of the value of c , the function evaluated at $c x$ looks the same as the function at x (because all the derivatives are the same!)



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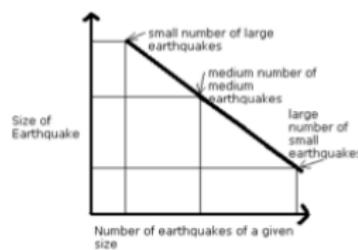
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Scale Invariance



Let us consider x to be the size of an earthquake.
 $p(x)$ is the PDF of their frequency, which is compatible with a power-law

$$p(x) = N x^{-\alpha}$$

The transformation
 $x \rightarrow c x$ implies

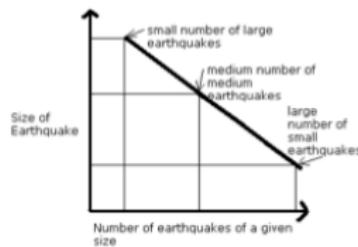
$$p(cx) = c^{-\alpha} N x^{-\alpha} \propto p(x)$$

We can also compute

$$\frac{p(100x)}{p(10x)} = \frac{p(10x)}{p(x)} = \frac{p(x)}{p(x/10)}$$

for the same scaling,
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Preferential attachment

This process is any in which some quantity, typically some form of wealth or size, is distributed among a number of elements according to how much they already have. Therefore, those who are already wealthy receive more than those who are not.

Other names:

- + rich get richer
- + cumulative advantage
- + Yule process
- + Matthew effect



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Barabási-Albert model



Barabási-Albert model

- + Was proposed in 1999, by Barabási and Albert for studying of the growth of the World Wide Web; (but it is not the first to report the mechanism!)
- + *Proposes the **preferential attachment** mechanism to generate random scale-free networks*
- + Degree distribution follows a **power-law**, unlike random (ER-model) or small-world (WS-model) graphs.



Applicability of BA model

A number of natural and artificial systems are found to be approximately scale-free:

- + Internet
- + World-Wide-Web
- + Citation networks
- + Social networks
- + etc.



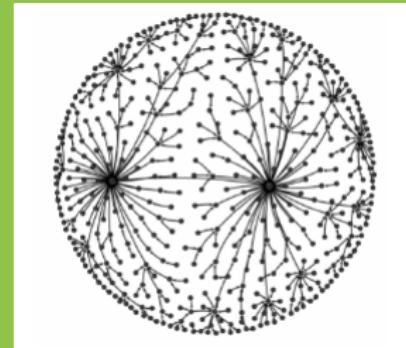
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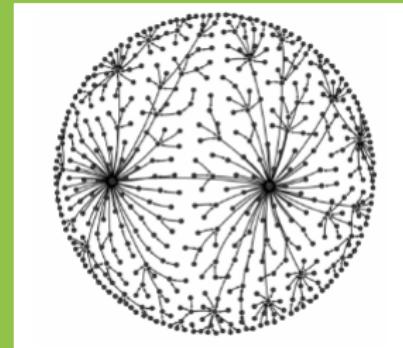
Barabási-Albert model explained

- + Model incorporates two mechanisms: **network growth** and **preferential attachment rule**
- + **Network Growth:** the network grows in discrete time-steps
- + At each time step t , a new node i is added to the network
- + The new node is connected to m existing nodes.
- + Node degrees are a function of time $k_i(t)$



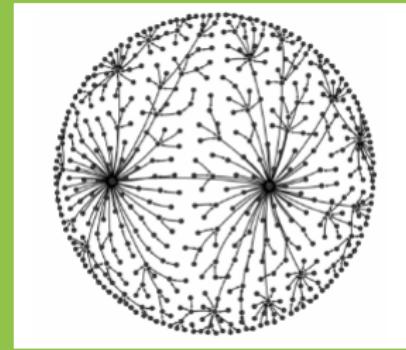
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Limiting cases

- + **Model A:** only growth, no preferential attachment
 - For a new node, probability of connecting to any pre-existing node is equal
 - Resulting degree distribution - geometric
 - **Thus, only growth is not sufficient for a scale-free property**
- + **Model B:** only preferential attachment, no growth
 - Model with a set of existing nodes, adding only links
 - Probability of new connection to node j is $\propto k_j$
 - Degree distribution is scale-free early in simulation, but eventually becomes nearly Gaussian
 - **Thus, only preferential attachment is not sufficient for a scale-free property**

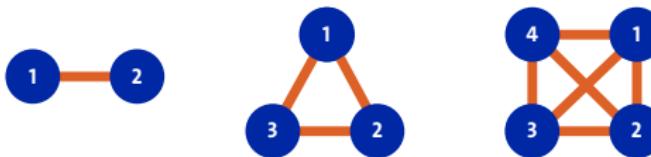


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Initial Condition



At time $t = 0$ a clique of m_0 nodes is created and serves as initial condition



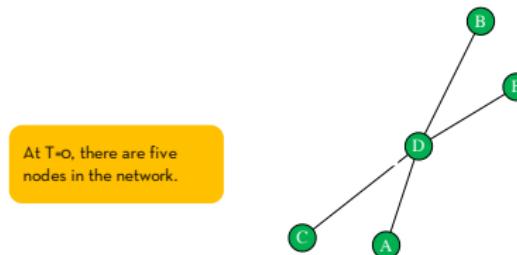
Preferential attachment

- + **Preferential attachment:** The probability that the new node i establishes links to an existing one j is:

$$p(i \leftrightarrow j) \propto k_j(t) = \frac{k_j(t)}{\sum_{j'} k_{j'}(t)} \quad (1)$$



Toy example

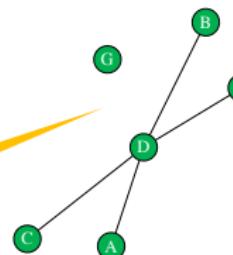




Toy example

$$p(G \leftrightarrow D) = \frac{4}{4 + 1 + 1 + 1 + 1} = 50\%$$

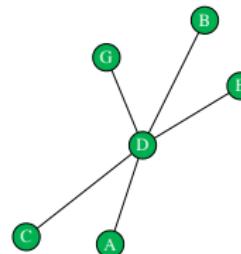
The probability that G establishes a link to D is 50%.



The probabilities that G establishes a link to any other node is 12.5%.

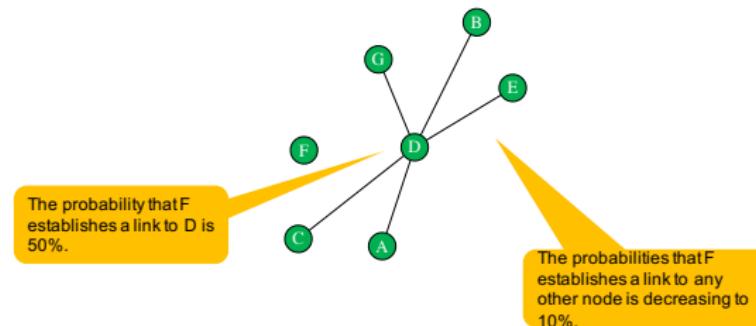
$$p(G \leftrightarrow E) = \frac{1}{4 + 1 + 1 + 1 + 1} = 12.5\%$$

Toy example



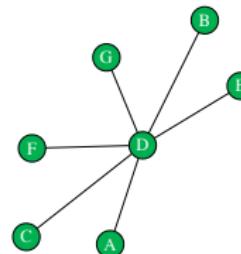
We assume that node **G** connects to the most likely one, **D**

Toy example





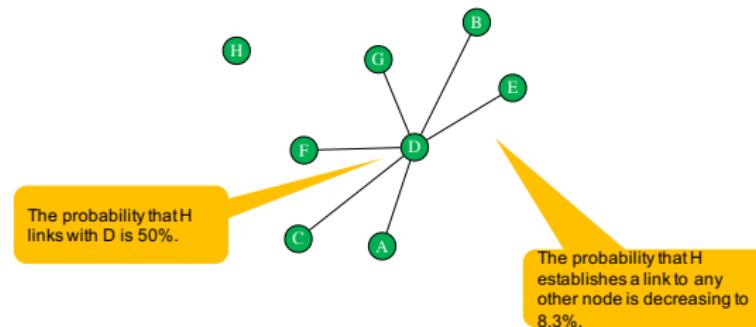
Toy example



We assume that node F connects to the most likely one, D

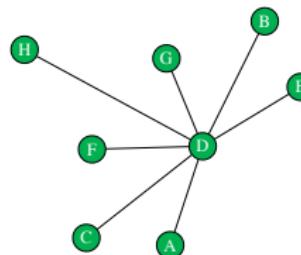


Toy example



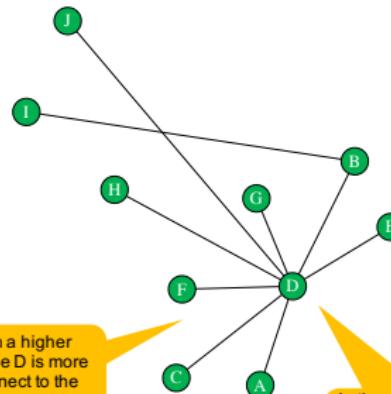


Toy example



We assume that node H connects to the most likely one, D

Toy example





Node degree distribution: For large t and k , this process produces a scale-free degree distribution $p(k) \propto 1/k^3$



Power-law distribution of node degrees

Node degree

Preferential attachment process generates a “long-tailed” distribution following a Pareto distribution or **power-law** in its tail

$$P(k) \propto k^{-\gamma} \quad (2)$$

Mathematical derivation in Barabási's book

We will now study several models for generation of scale-free networks (networks with power-law distribution of the node degree).



Power-law distribution of node degrees

Node degree

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Some characteristics of PA process

- + Earlier appearing nodes have more connections
- + Heavily connected nodes, **hubs**, quickly accumulate even more connections
- + Nodes with few links are unlikely to be chosen for new connections
- + New nodes tend to connect themselves to more ‘popular’ nodes



Some characteristics of PA process

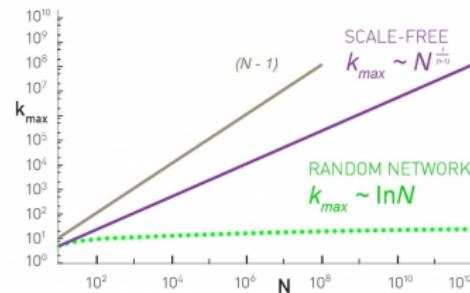
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Some characteristics of PA process



The typical maximum degree in the network grows with network size!



Some notes on realism of the PA model

Preferential attachment captures a mechanism: The more acquaintances the users already have, the more likely is that they will be known (or be “attractive” to others).

- + Model suggests that users come one at a time and do not change the connections over time
- + This is not correct description of reality
- + The origin of the mechanism is unexplained. May be due to:
 1. new users are more likely to know someone who has several acquaintances;
 2. an extroverted person will be signaled by a large social neighborhood and will be more likely to increase it even further.



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Extensions of BA model

- + Plenty of extensions are built on top of BA model
- + They were verified on different types of real data
- + Some extensions are:
 - Popularity and similarity (Papadopoulos et al, 2012)
Nodes connect to popular ones and similar ones
 - Homophily (Centola, 2011)
People who are connected in a social network tend to like similar things
 - Influentiality and susceptability (Aral and Walker, 2012)
If influential and susceptible users are connected into scale-free network, information spreads efficiently
 - Positive and negative links
Friends and foes in a social network
 - Bipartite networks (Li et.al, 2013)
Users and items connected in a network with scale-free property



A long story

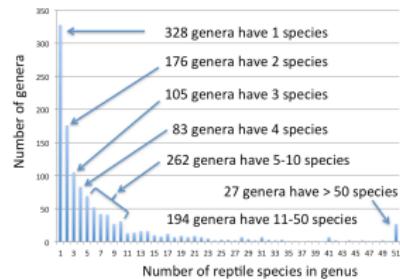


Yule model

Yule model: Species and Taxonomies



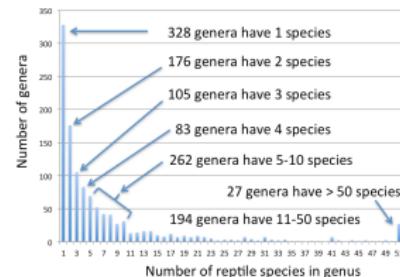
- + Species are grouped in genera
- + The distribution of number of species that belong to different genera is very skewed



Yule model: Species and Taxonomies



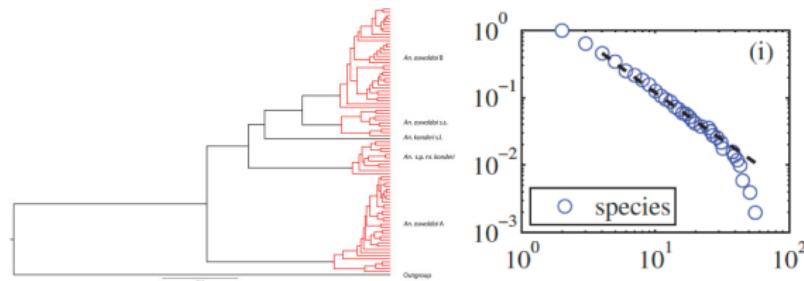
- + Species are grouped in *genera*
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Yule model

[Yule, *Phyl. Trans. Royal Soc. B* (1925)]

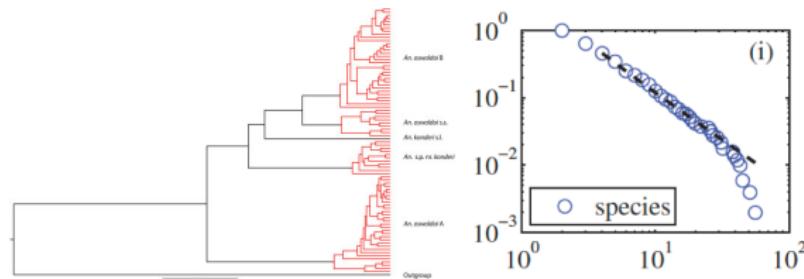
- + First example of a proportionate growth process
- + Intended to explain biological evolution
- + Based on the observation of the highly skewed distribution of number of species (of flowers) that compose different genera



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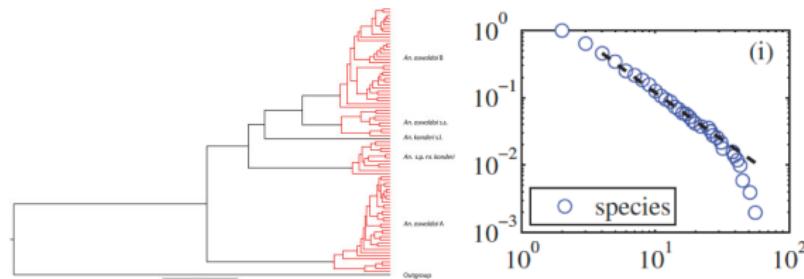
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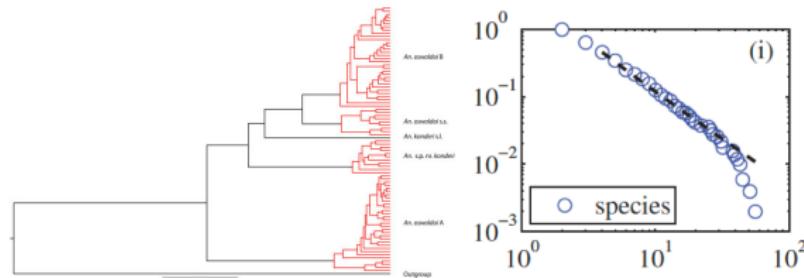
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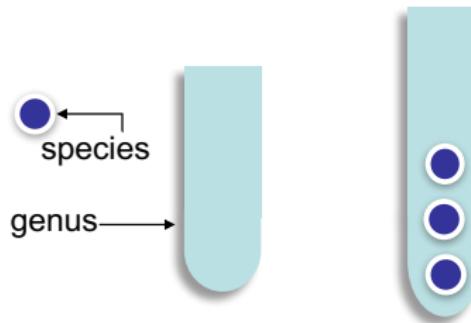
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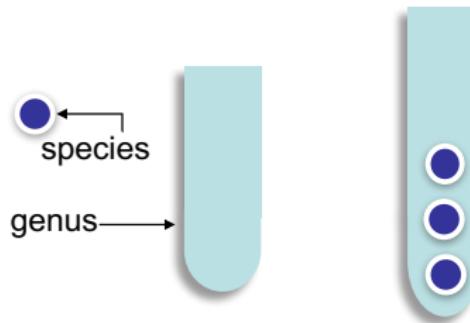


Yule model: species and genera



- + Species are grouped in genera
- + Over time, new species appear by the mutation of existing ones
- + At rate $1 - p$, the new species belongs to the same genus
- + Otherwise, at rate p , new species generates a new genus

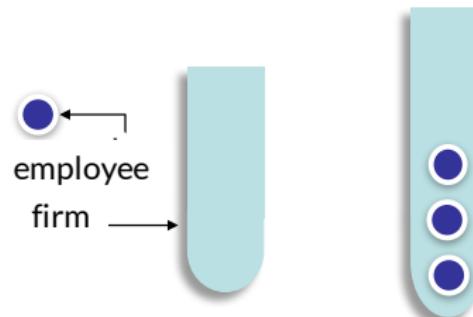
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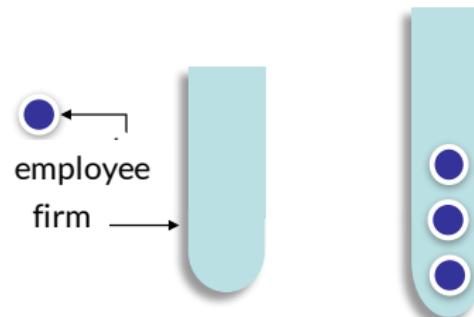


Yule-Simon model: Firm-size distribution



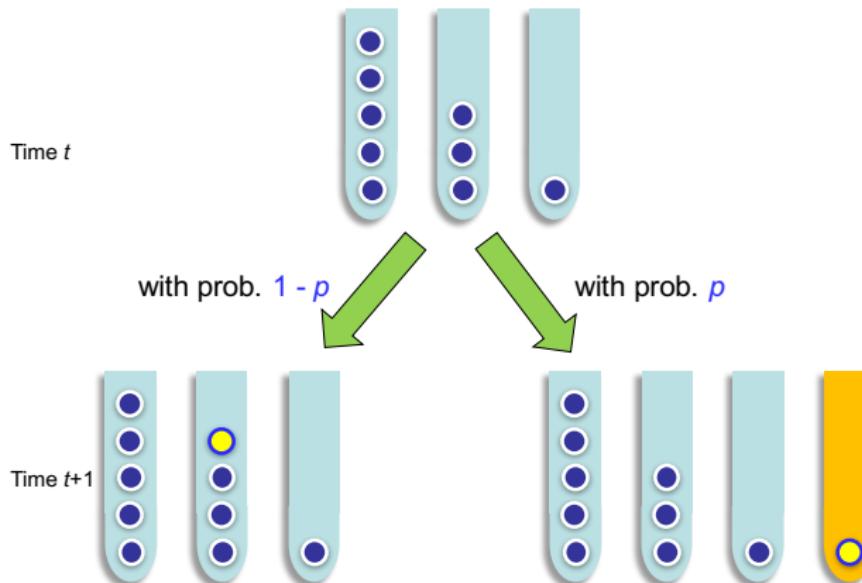
- + Employees work in firms
- + Over time, new employees enter into the market
- + With probability $1 - p$, the new employee works in an existing firm
- + Otherwise, with probability p , the new employee opens a new firm

Yule-Simon model: Firm-size distribution



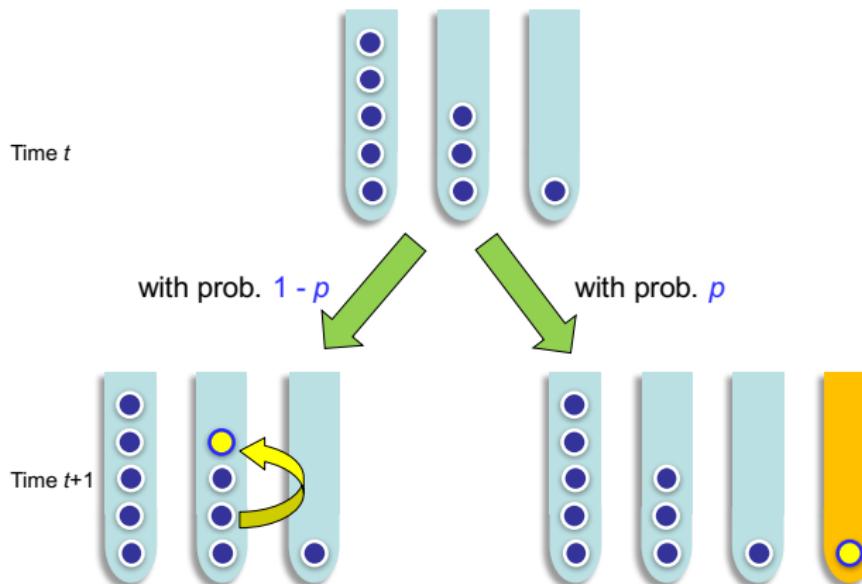
- + Employees work in firms
- + Over time, new employees enter into the market
- + With probability $1 - p$, the new employee works in an existing firm
- + Otherwise, with probability p , the new employee opens a new firm

Yule-Simon process sketch



Growth of a firms is proportional to its size

Yule-Simon process sketch



Firms size distribution

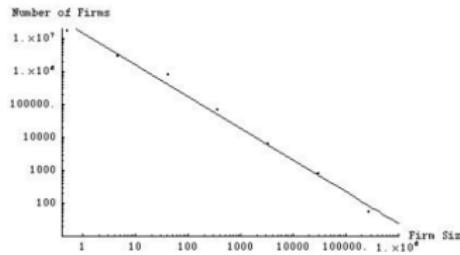


Figure 2: Distribution of U.S. firm sizes (by employees) for 1997, data combined from Census/SBA and Compustat together with self-employment data

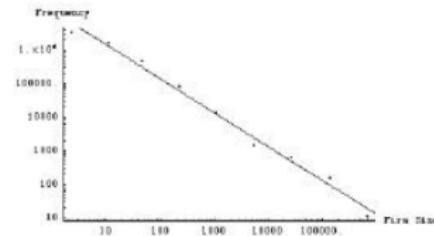
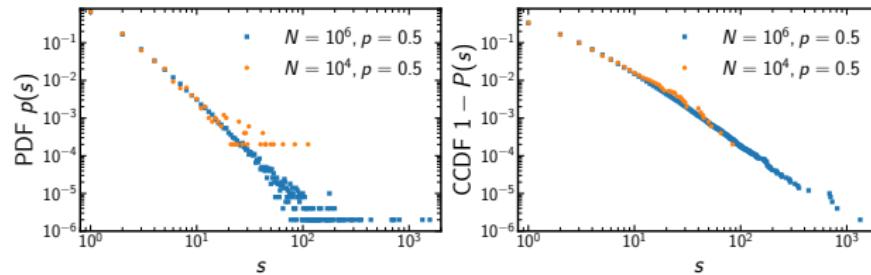


Figure 3: Distribution of U.S. firm sizes (by revenue, in \$ million) for 1997, data from Census

The empirical observation is that firms have broad size distributions, by size or revenue [Axtel, Science (2001)]

Results of the model

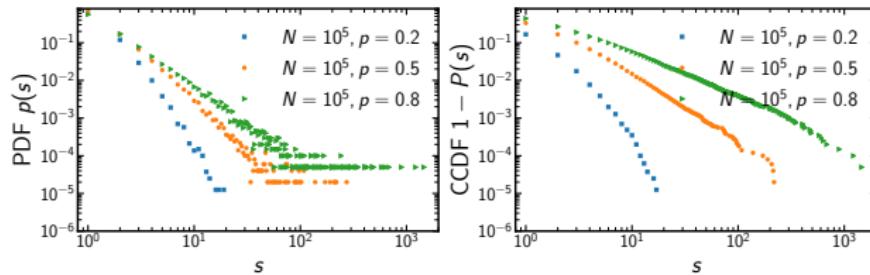


$$p(s) = c \frac{1}{s^\gamma} \Rightarrow 1 - P(s) = 1 - \int_0^s p(\sigma) d\sigma = c_2 \frac{1}{s^{\gamma-1}}$$

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The Matthew Effect



Citations in scientific publications

The Matthew Effect in Science



The reward and communication systems of science are considered.

Robert K. Merton

This paper develops a conception of ways in which certain psychosocial processes affect the allocation of rewards to scientists for their contributions—an allocation which in turn affects the flow of ideas and findings through the communication networks of science. The conception is based

image and the public image of scientists are largely shaped by the communally validating testimony of significant others that they have variously lived up to the exacting institutional requirements of their roles.

A number of workers, in empirical studies, have investigated various as-

reers are more productive later on than those who do not. And the Coles have also found that, at least in the case of contemporary American physics, the reward system operates largely in accord with institutional values of the science, inasmuch as quality of research is more often and more substantially rewarded than mere quantity.

In science as in other institutional realms, a special problem in the workings of the reward system turns up when individuals or organizations take on the job of gauging and suitably rewarding lofty performance on behalf of a large community. Thus, that ultimate accolade in 20th-century science, the Nobel prize, is often assumed to mark off its recipients from all the other scientists of the time. Yet this assumption is at odds with the well-known fact that a good number of scientists who have not received the prize and will not receive it have contributed as much to the advancement

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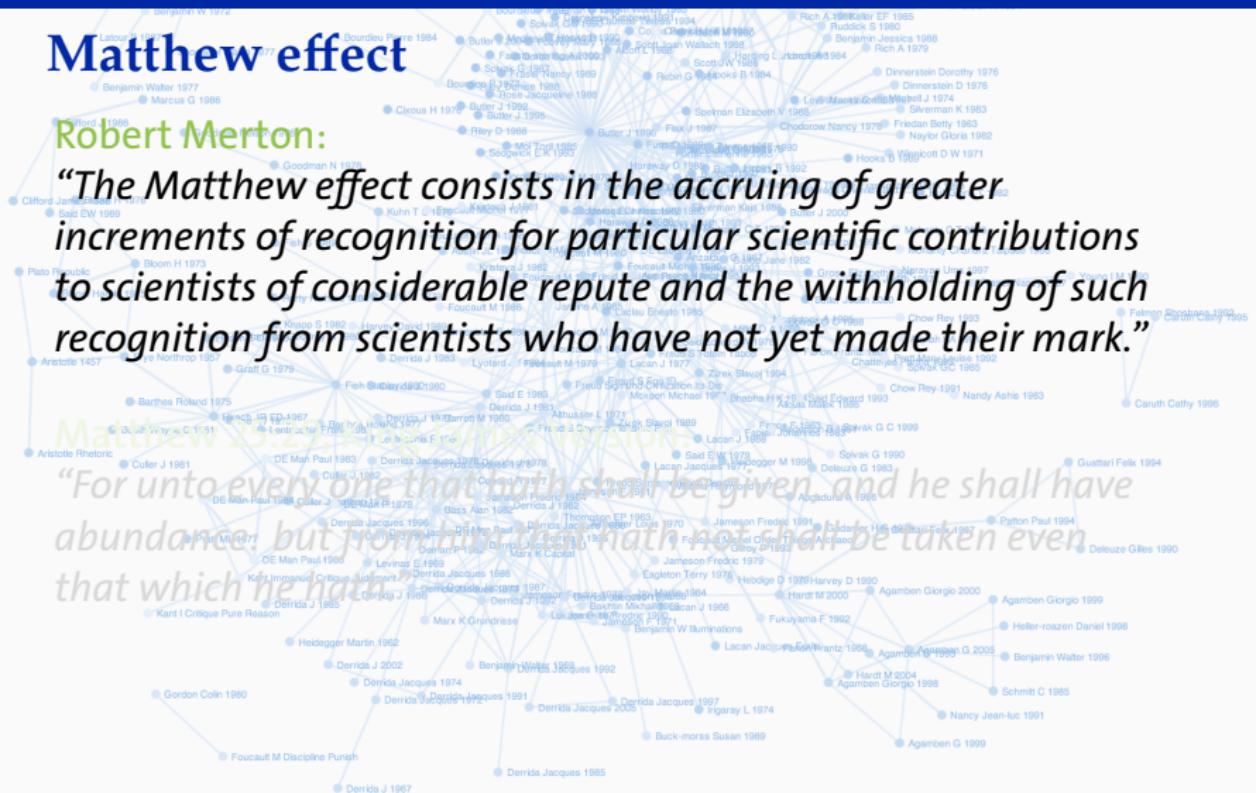
January 5, 1968, Vol. 159, No. 3810, pages 56-63

Matthew effect

Robert Merton:

"The Matthew effect consists in the accruing of greater increments of recognition for particular scientific contributions to scientists of considerable repute and the withholding of such recognition from scientists who have not yet made their mark."

"For unto every one that hath shall be given, and he shall have abundance: but from him that hath not, shall not be taken even that which he hath."





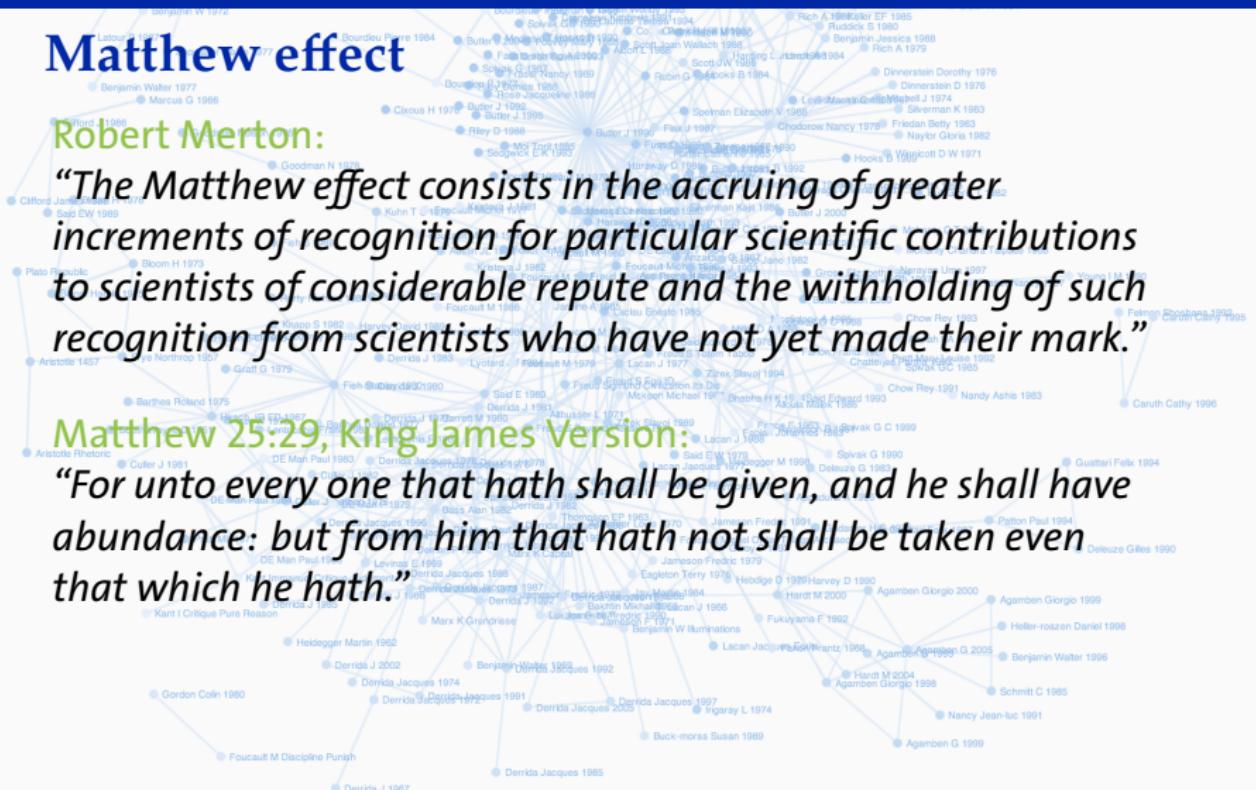
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Matthew 25:29, King James Version:

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Examples of Matthew effect

- + **Academia**, researchers tend to cite well-known scholars
- + **Consumer activity** follows the apparent popularity: books and music from bestseller lists are sold more
- + **Career progress**: more experienced and more reputable individuals possess more competitive advantage in obtaining new career opportunities
- + **Internet, social networks**



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The Price model

A General Theory of Bibliometric and Other Cumulative Advantage Processes*

A Cumulative Advantage Distribution is proposed which models statistically the situation in which success breeds success. It differs from the Negative Binomial Distribution in that lack of success, being a non-event, is not punished by increased chance of failure. It is shown that such a stochastic law is governed by the Beta Function, containing only one free parameter, and this is approximated by a skew or hyperbolic distribution of the type that is widespread in bibliometrics and diverse social science phenomena. In particular, this is shown to

be an appropriate underlying probabilistic theory for the Bradford Law, the Lotka Law, the Pareto and Zipf Distributions, and for all the empirical results of citation frequency analysis. As side results one may derive also the obsolescence factor for literature use. The Beta Function is peculiarly elegant for these manifold purposes because it yields both the actual and the cumulative distributions in simple form, and contains a limiting case of an inverse square law to which many empirical distributions conform.

Derek de Solla Price
*Department of History of Science and Medicine
Yale University
New Haven, CT 06520*

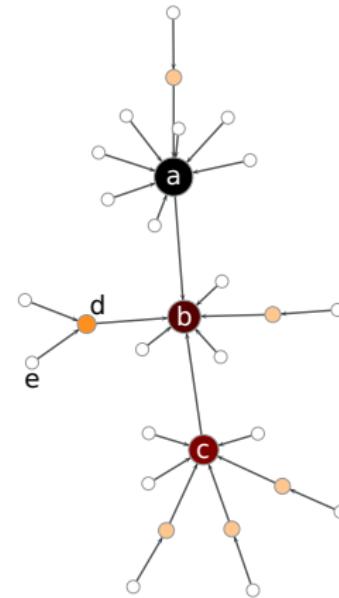
[De Solla Price, 1976 (J. Assoc. Inf. Sci.)]

The Price model

- + **Nodes:** scientific papers
- + **Links:** citations
- + Probability of a new node i points to an existing one j :

$$p(i \rightarrow j) \propto k_j^{(in)} + a$$

- + $p(\rightarrow d) = 2p(\rightarrow e),$
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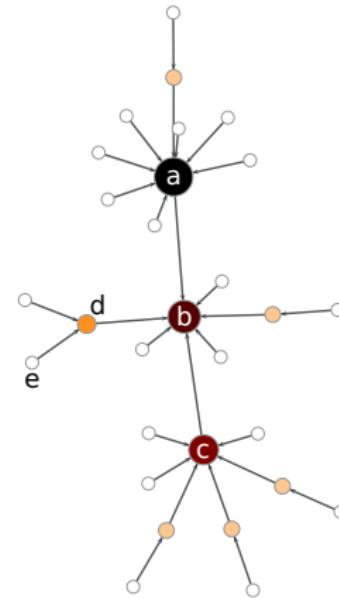


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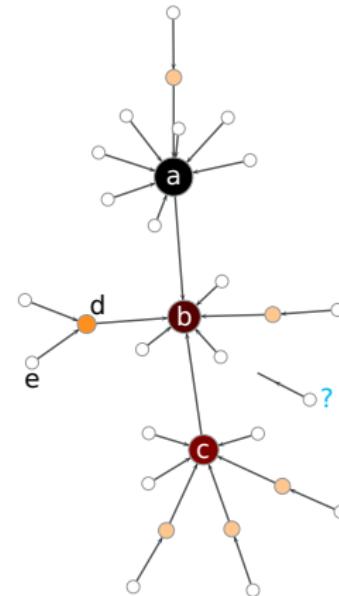


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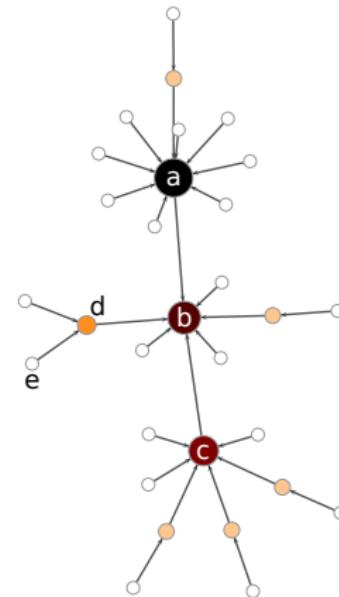


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The Price Model explained

- + Let p_k to denote the fraction of nodes with degree k
... so $\sum_k p_k = 1$
- + Each paper cites on average c papers
- + This value is constant over time, and:
 $c = \sum_k kp_k$
- + Probability to link to a specific node is:

$$\text{Prob}(\text{attach to } i, t) = \frac{q_i + a}{\sum_j q_j + a} = \frac{q_i + a}{t(c + a)} \quad (3)$$

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Let $P(q, t)$ be the degree distribution of nodes with in-degree q at time t

At time t , there are $tP(q, t)$ nodes with in-degree q

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The Price Model properties

The stationary degree distribution is given by:

$$P(q, t \rightarrow \infty) = \frac{B(q + a, 2 + \frac{a}{c})}{B(a, 1 + \frac{a}{c})} \quad (6)$$

For large degree, this reduces to:

$$P(q, t \rightarrow \infty) \propto q^{2 + \frac{a}{c}} \quad (7)$$

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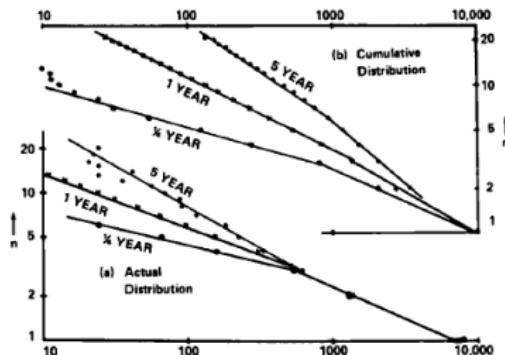
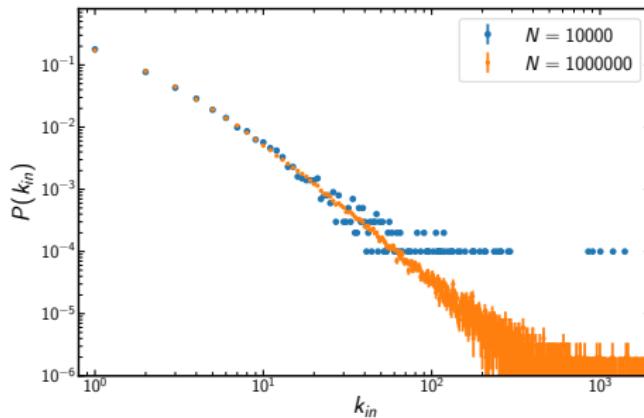


Fig. 1. Number of papers with (a) exactly and (b) at least n citations in ¼, 1, and 5-year indexes.

Cummulative distribution of number of citations

Notice that both scales are logarithmic!

Properties





Model extensions



Fitness model

[Bianconi, Ginestra, Barabási, Albert-László, *Bose-Einstein condensation in complex networks* (2001)]

- + Extension of BA model
- + *Fitness is an inherent time independent factor* that makes particular nodes more attractive
- + Fitness works together with the degree of nodes creating overall attractiveness of a node



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Fitness model explained

- + Consider a network that grows in discrete time, with initial size m_0
- + **Each node is endowed with a fitness factor η_i**
- + At each time step, a new node is added to the network
- + The new node i is connected to $m \leq m_0$ existing nodes
- + The probability that the new node i establishes a link to an existing one j is:

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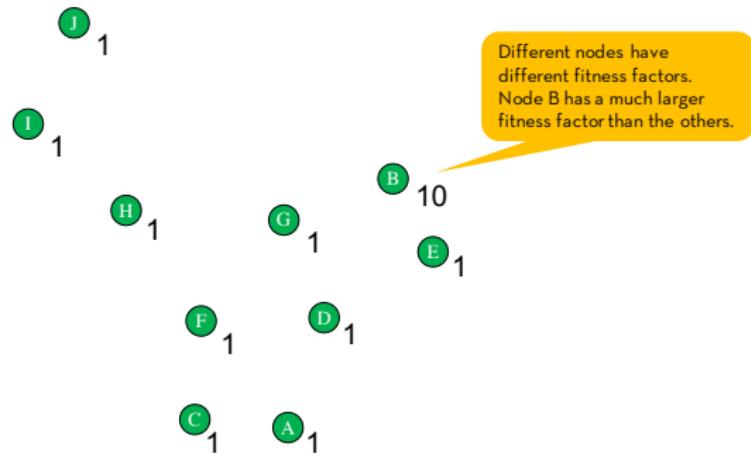
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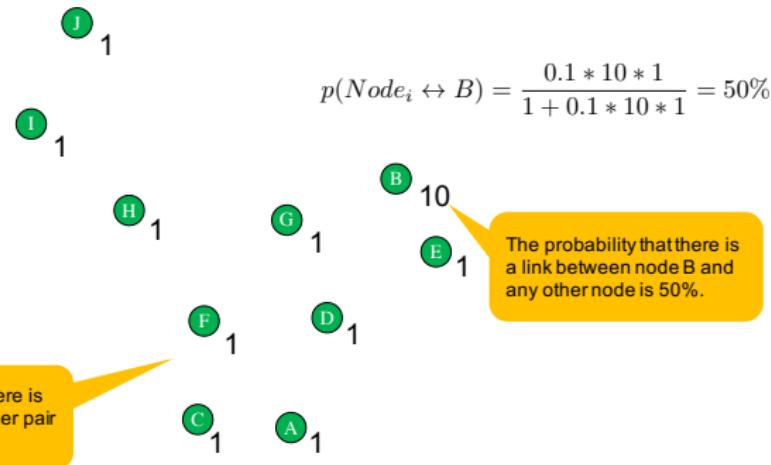


Fitness model illustration





Fitness model illustration





Some properties of fitness model

- + Reflects basic properties of many real systems;
- + A node with strong competitive power (higher fitness factor) may enter the network at a later time and overcome 'older' nodes (unlike in plain BA model);
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Intrinsic fitness introduction

[G. Caldarelli, *et al.*, *Scale-Free Networks from Varying Vertex Intrinsic Fitness* (2002)]

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Intrinsic fitness explained

[Diego Garlaschelli, Maria I Loffredo, *Fitness-dependent topological properties of the world trade web* (2004)]

- + Consider network with N nodes
- + Each node is endowed with a fitness z_i
- + Every link is created with a probability p_{ij}

$$p_{ij} = f(z_i, z_j) = \frac{z_i z_j}{1 + z_i z_j} \quad (9)$$

- + Probability of existence of an edge between two nodes depends uniquely on the actual nodes characteristics!



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Intrinsic fitness model in real data

- + *Network modelling*: aimed at understanding mechanisms yielding global properties
- + *Intrinsic fitness model*: aimed to relate node intrinsic properties of the observed network

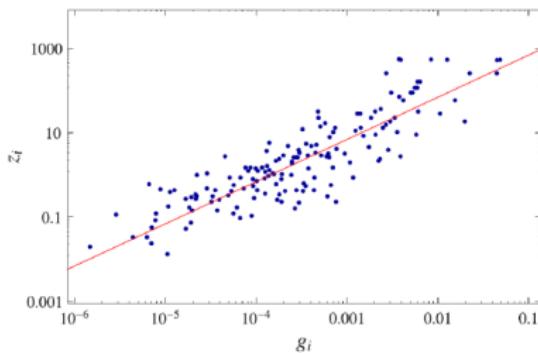
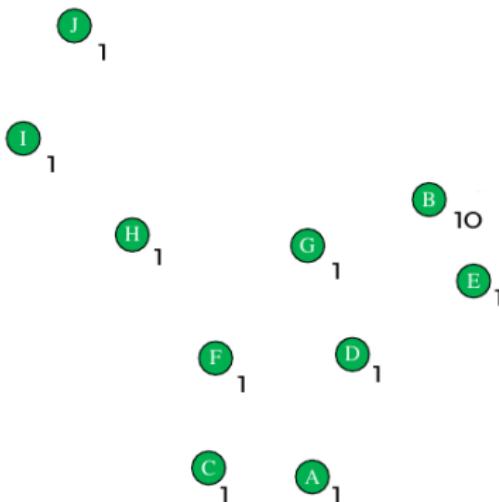


Figure: GDP vs. fitness in the World Trade Web



Intrinsic fitness visualisation

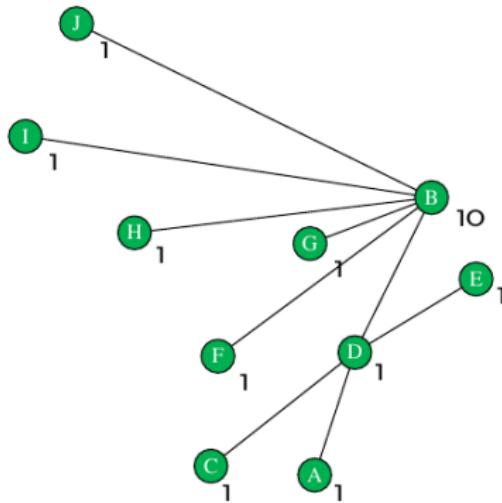


We start with a graph with no edges.
For every pair of nodes, there is a probability of connection:

$$P(\text{Node}_i \leftrightarrow B) = \frac{0.1 * 10 * 1}{1 + 0.1 * 10 * 1} = \frac{1}{2}$$

$$P(\text{Node}_i \leftrightarrow \text{Node}_j) = \frac{0.1 * 1 * 1}{1 + 0.1 * 1 * 1} = \frac{1}{11}$$

Intrinsic fitness visualisation



One possible network
might look like this



Intrinsic fitness model: properties

- + Can successfully reproduce power-law degree distribution
- + Shows the *good-get-richer* rather than the rich get richer
- + Intrinsic fitness factor can be obtained from other sources (e.g. GDP), but can also be unobserved or change over time
- + Cannot be used to model network evolution



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Intrinsic fitness model: interpretation

- + We start with a network $G^*, \{a_{ij}\}, \{k_i^*\}$
- + Each node is endowed with a fitness x_i
- + Probability of creating a link between two nodes is:
$$p_{ij} = f(x_i, x_j) = \frac{x_i x_j}{1 + x_i x_j}$$
- + The probability to observe a network G^* , given these fitness values, is then:
$$P(G^* | \{x_i\}) = \prod_{i < j} p_{ij}^{a_{ij}} (1 - p_{ij})^{1 - a_{ij}}$$
- + Log-likelihood of this observation:
$$L(G^* | \{x_i\}) = \sum_{i < j} [a_{ij} \log(p_{ij}) + (1 - a_{ij}) \log(1 - p_{ij})]$$



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Maximisation of the log-likelihood w.r.t. x_i yields:

$$\langle k_i \rangle = \sum_{i \neq j} \frac{x_i x_j}{1 + x_i x_j} = k_i^*, \forall i \quad (10)$$

Interpretation of x_i : expected degrees $\langle k_i \rangle$ match the observed ones $k_i^* = k_i(G^*)$

This is equivalent to maximising entropy under constraint: $k_i^* = \langle k_i \rangle$, and entropy is given by:

$$\mathbb{H}(G^*) = - \sum_{i=1}^N \log(x_i) k_i^* \quad (11)$$



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