University of Zurich[UZH]

ROBOTICS & PERCEPTION GROUP

Vision Algorithms for Mobile Robotics

Lecture 03
Camera Calibration

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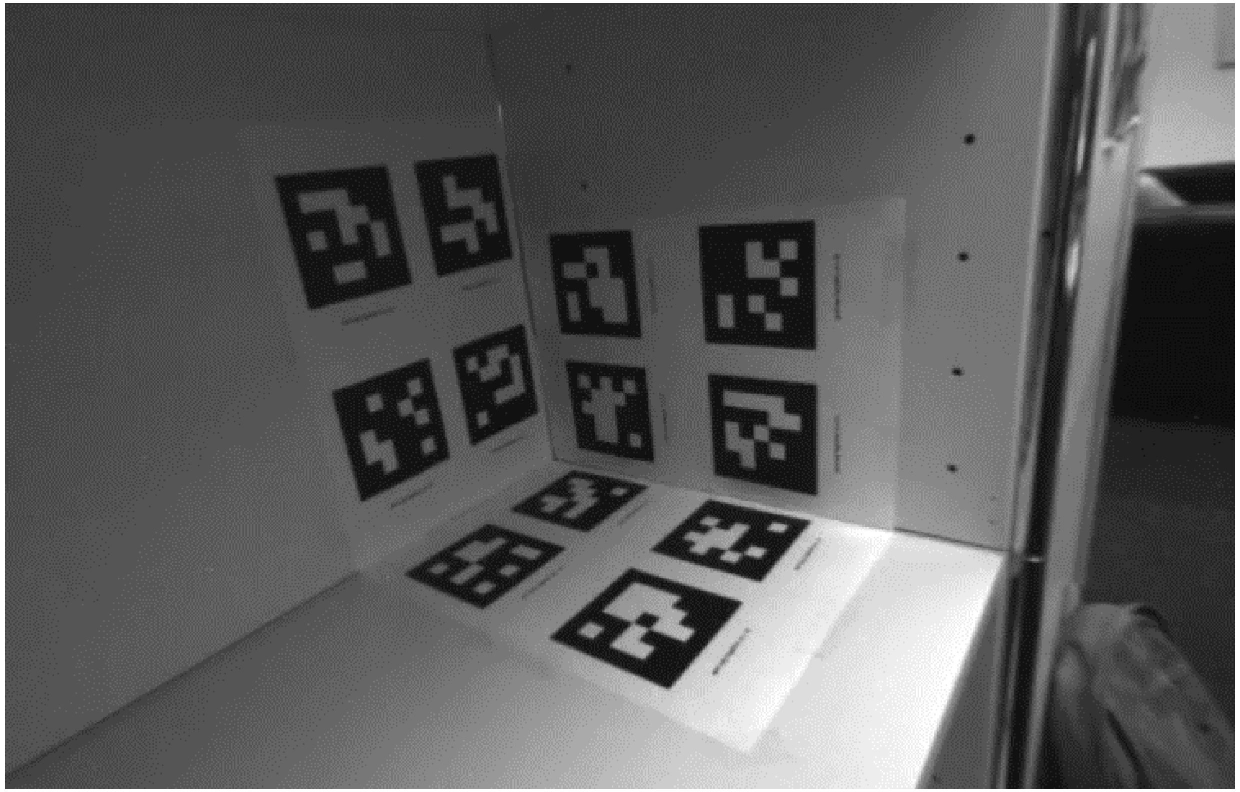
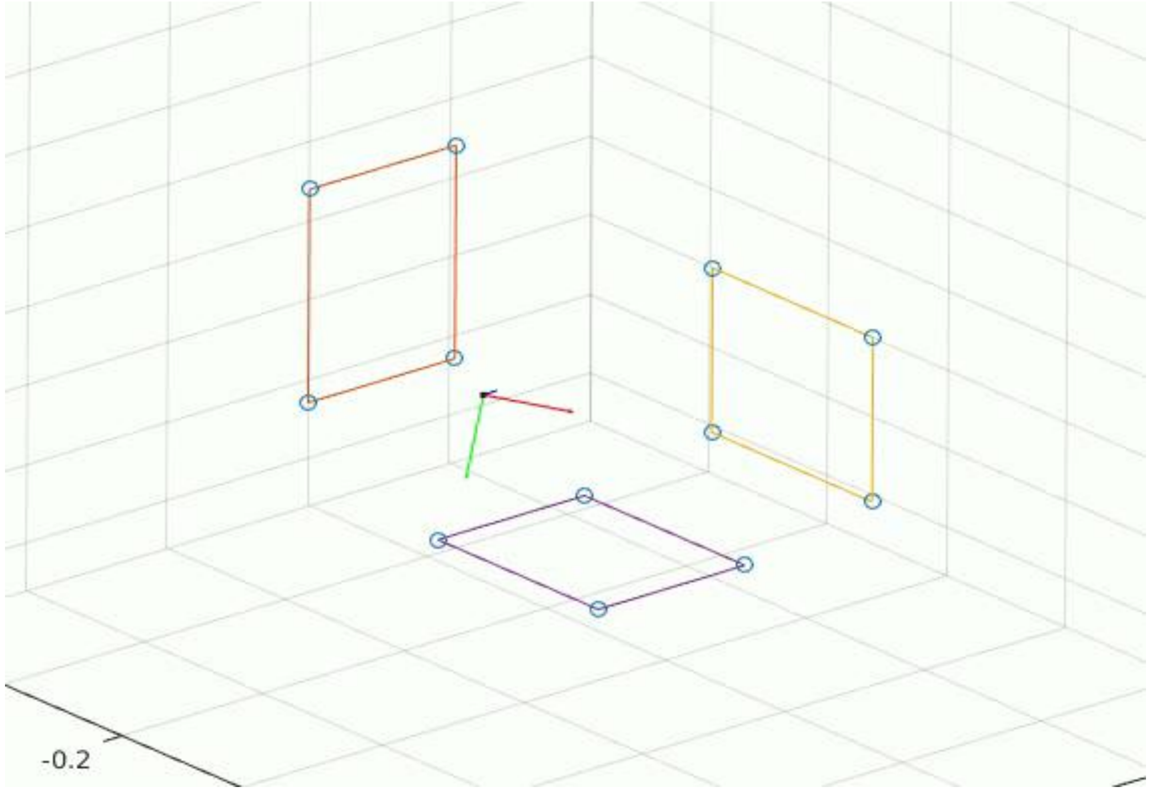
Lab Exercise 2 – This afternoon

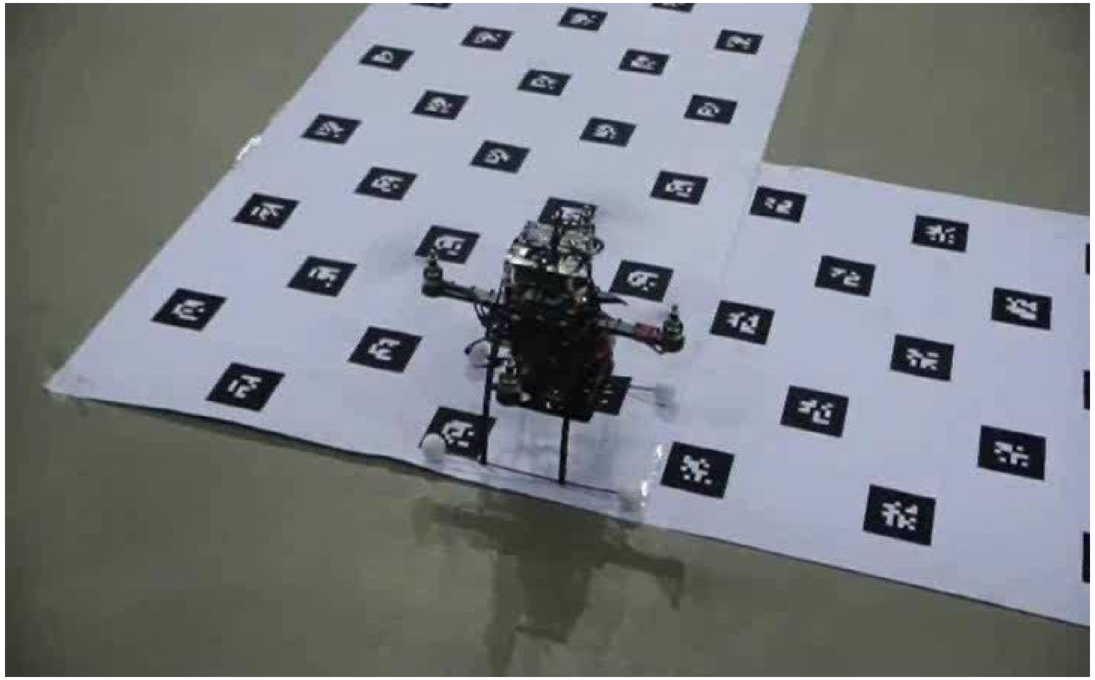
Implement your first camera motion estimator using the DLT algorithm





Goal of today's lecture

- Learn how to calibrate a camera
- Study the foundational algorithms for camera localization



Two applications of the camera localization algorithms covered in this lecture: drone navigation & Microsoft Hololens

Today's Outline

- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Camera Calibration

- Calibration is the process to determine the **intrinsic** (K plus lens distortion) **and extrinsic** (R, T) parameters of a camera. For now, we will **neglect the lens distortion** and see later how it can be determined.

- K, R, T can be **determined by applying the perspective projection** equation to known 3D-2D point correspondences:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \left[R \mid T \right] \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

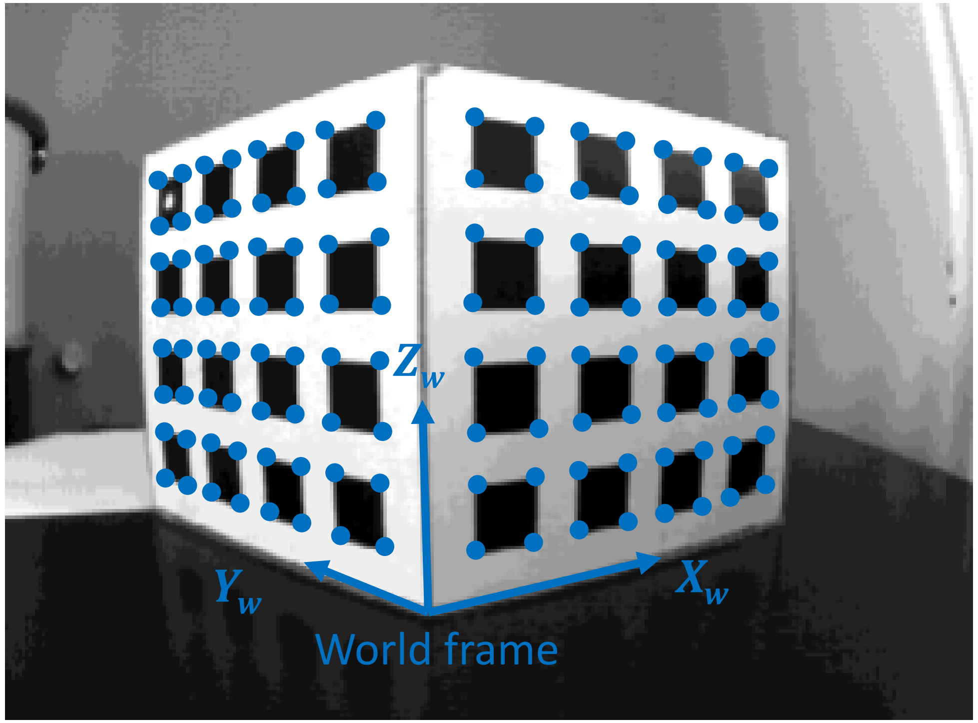
- There are two popular methods:
  - **Tsai's method**: uses 3D objects
  - **Zhang's method**: uses planar grids

Today’s Outline

- Camera calibration
  - Tsai’s method: From 3D objects
  - Zhang’s method: from planar grids
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Tsai's Method: Calibration from 3D Objects

- This method was proposed in 1987 by Tsai and consists of measuring the 3D position of $n \geq 6$ **control points** on a 3D calibration target and the **2D coordinates of their projection** in the image.



Tsai, Roger Y. (1987) "A Versatile Camera Calibration Technique for High-Accuracy 3D Machine Vision Metrology Using Off-the-Shelf TV Cameras and Lenses,'' *IEEE Journal of Robotics and Automation*, 1987. PDF.

Applying the Direct Linear Transform (DLT) algorithm

The idea of the DLT is to rewrite the perspective projection equation as a **homogeneous linear equation** and solve it by standard methods. Let's write the perspective equation for a generic 3D-2D point correspondence:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \left[R \mid T \right] \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u r_{11} + u_0 r_{31} & \alpha_u r_{12} + u_0 r_{32} & \alpha_u r_{13} + u_0 r_{33} & \alpha_u t_1 + u_0 t_3 \\ \alpha_v r_{21} + v_0 r_{31} & \alpha_v r_{22} + v_0 r_{32} & \alpha_v r_{23} + v_0 r_{33} & \alpha_v t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

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$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

What are the assumptions behind this this substitution?

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = M \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

where m_i^{T} is the i-th row of M

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_1^{\mathrm{T}} \\ m_2^{\mathrm{T}} \\ m_3^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} m_1^{\mathrm{T}} \\ m_2^{\mathrm{T}} \\ m_3^{\mathrm{T}} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}}_{P}$$

Conversion back from homogeneous coordinates to pixel coordinates leads to:

$$u = \frac{\lambda u}{\lambda} = \frac{m_1^{\mathrm{T}} \cdot P}{m_3^{\mathrm{T}} \cdot P} \qquad\qquad (m_1^{\mathrm{T}} - u_i m_3^{\mathrm{T}}) \cdot P_i = 0$$

$$\Rightarrow$$

$$v = \frac{\lambda v}{\lambda} = \frac{m_2^{\mathrm{T}} \cdot P}{m_3^{\mathrm{T}} \cdot P} \qquad\qquad (m_2^{\mathrm{T}} - v_i m_3^{\mathrm{T}}) \cdot P_i = 0$$

Applying the Direct Linear Transform (DLT) algorithm

- By re-arranging the terms, we obtain

$$\begin{matrix} (m_1^{\mathrm{T}} - u_i m_3^{\mathrm{T}}) \cdot P_i = 0 \\ (m_2^{\mathrm{T}} - v_i m_3^{\mathrm{T}}) \cdot P_i = 0 \end{matrix} \Rightarrow \begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- For n points, we can stack all these equations into a big matrix:

$$\begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \\ & \vdots & \\ P_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n P_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_n^{\mathrm{T}} & -v_n P_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\underbrace{\begin{pmatrix} X_w^1 & Y_w^1 & Z_w^1 & 1 & 0 & 0 & 0 & 0 & -u_1X_w^1 & -u_1Y_w^1 & -u_1Z_w^1 & -u_1 \\ 0 & 0 & 0 & 0 & X_w^1 & Y_w^1 & Z_w^1 & 1 & -v_1X_w^1 & -v_1Y_w^1 & -v_1Z_w^1 & -v_1 \\ & & & & & & & \vdots & & & & \\ X_w^n & Y_w^n & Z_w^n & 1 & 0 & 0 & 0 & 0 & -u_nX_w^n & -u_nY_w^n & -u_nZ_w^n & -u_n \\ 0 & 0 & 0 & 0 & X_w^n & Y_w^n & Z_w^n & 1 & -v_nX_w^n & -v_nY_w^n & -v_nZ_w^n & -v_n \end{pmatrix}}_{\text{Q (this matrix is known)}} \underbrace{\begin{pmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{pmatrix}}_{\text{M (this matrix is unknown)}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Q} \cdot \mathrm{M} = 0$$

Q (this matrix is **known**)

M (this matrix is **unknown**)

Applying the Direct Linear Transform (DLT) algorithm

$$Q \cdot M = 0$$

Minimal solution

- $Q_{(2n \times 12)}$ should have rank 11 to have a unique (up to a scale) *non-zero* solution M
- Because each 3D-to-2D point correspondence provides 2 independent equations, then $5+\frac{1}{2}$ point correspondences are needed (in practice **6 point** correspondences!)

Over-determined solution

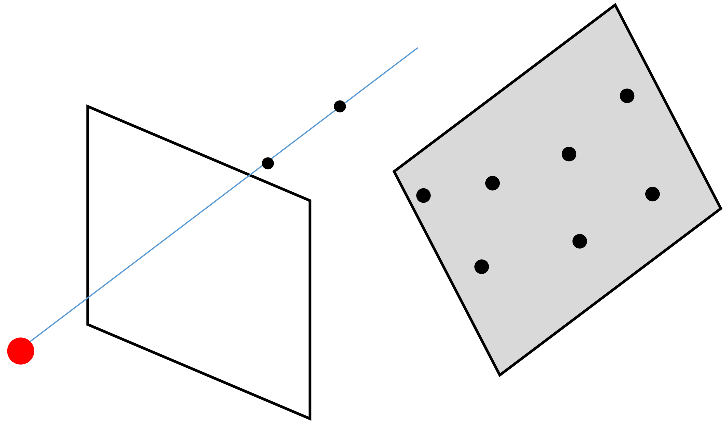
- For $n \geq 6$ points, a solution is the **Least Square solution**, which minimizes the sum of squared residuals, $||QM||^2$, subject to the constraint $||M||^2 = 1$. It can be solved through Singular Value Decomposition (SVD). The solution is the eigenvector corresponding to the smallest eigenvalue of the matrix $Q^T Q$ (because it is the unit vector x that minimizes $||Qx||^2 = x^T Q^T Q x$.
- Matlab instructions:
  - `[U,S,V] = SVD(Q);`
  - `M = V(:,12);`

Applying the Direct Linear Transform (DLT) algorithm

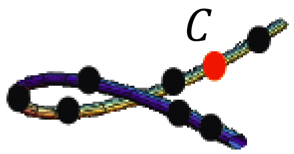
Degenerate configurations

$$Q \cdot M = 0$$

1. Points lying on a **plane** and/or along a single **line** passing through the **center of projection**



2. Camera and points on a **twisted cubic** (i.e., smooth curve in 3D space of degree 3)



Applying the Direct Linear Transform (DLT) algorithm

- Once we have determined M, we can recover the intrinsic and extrinsic parameters by remembering that:

$$\mathrm{M} = \mathrm{K}(\mathrm{R} \mid \mathrm{T})$$

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

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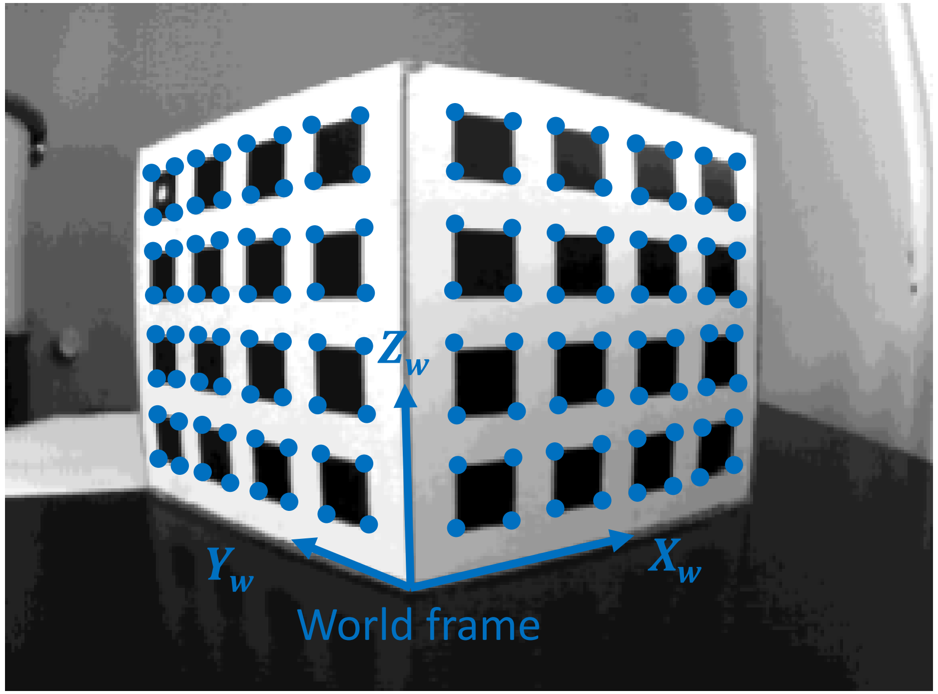
- However, notice that we are not enforcing the constraint that **$\boldsymbol{R}$ is orthogonal, i.e., $\boldsymbol{R} \cdot \boldsymbol{R}^{\boldsymbol{T}} = \boldsymbol{I}$**
- To do this, we can use the so-called **QR factorization of $\boldsymbol{M}$**, which decomposes M into a R (orthogonal), T, and an upper triangular matrix (i.e., K)
- What if K is known (calibrated camera)?

Example of Tsai's Calibration Results

Recommendation: use many **more than 6 points** (ideally more than 20) **and non coplanar**



Corners can be detected with accuracy < 0.1 pixels (see Lecture 5)

α_u	α_u/α_v	K_{12}	u_0	v_0	**Average Reprojection error**
1673.3	1.0063	1.39	379.96	305.78	0.365

Why is this ratio not 1?

What is this?

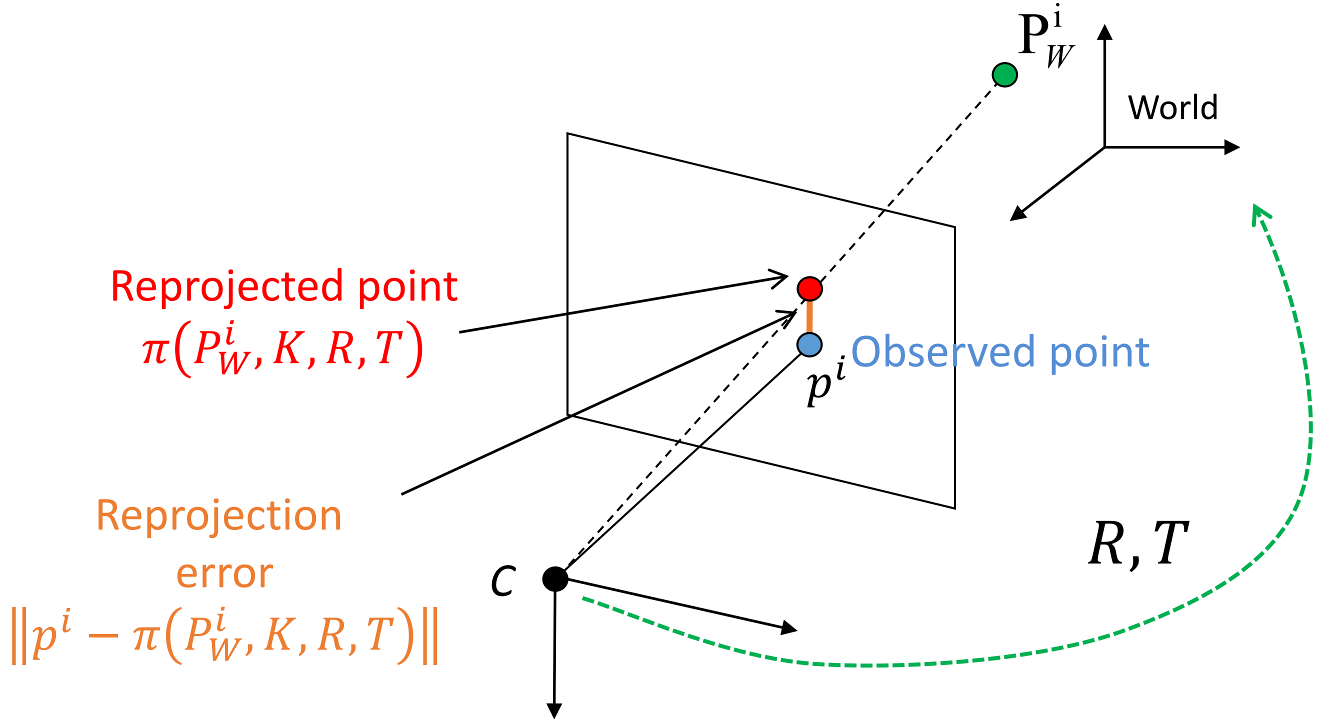
What is this?

How can we estimate the lens distortion parameters?

How can we enforce $\alpha_u = \alpha_v$ and $K_{12} = 0$?

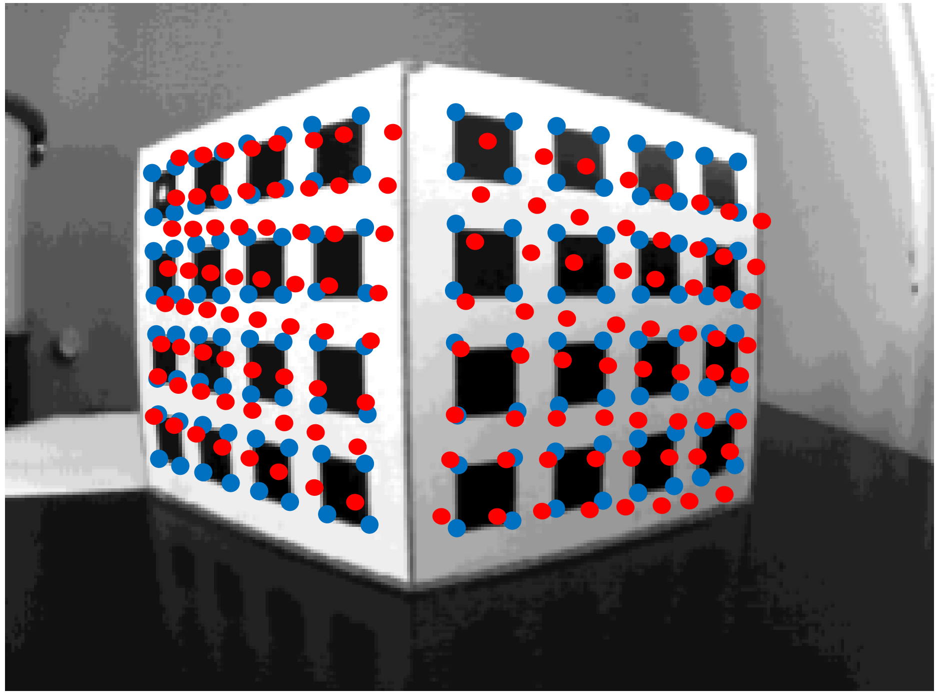
Reprojection Error

- The reprojection error is the **Euclidean distance** (in pixels) between an **observed image point** and the **corresponding** 3D point **reprojected** onto the camera frame.
- The reprojection error gives us a **quantitative measure of the accuracy** of the calibration (**ideally it should be zero**).



Reprojection Error

- The reprojection error can be used to assess the quality of the camera calibration
- What reprojection error is acceptable?
- What are the sources of the reprojection error?
- How can we further improve the calibration parameters?



Control points (observed points)

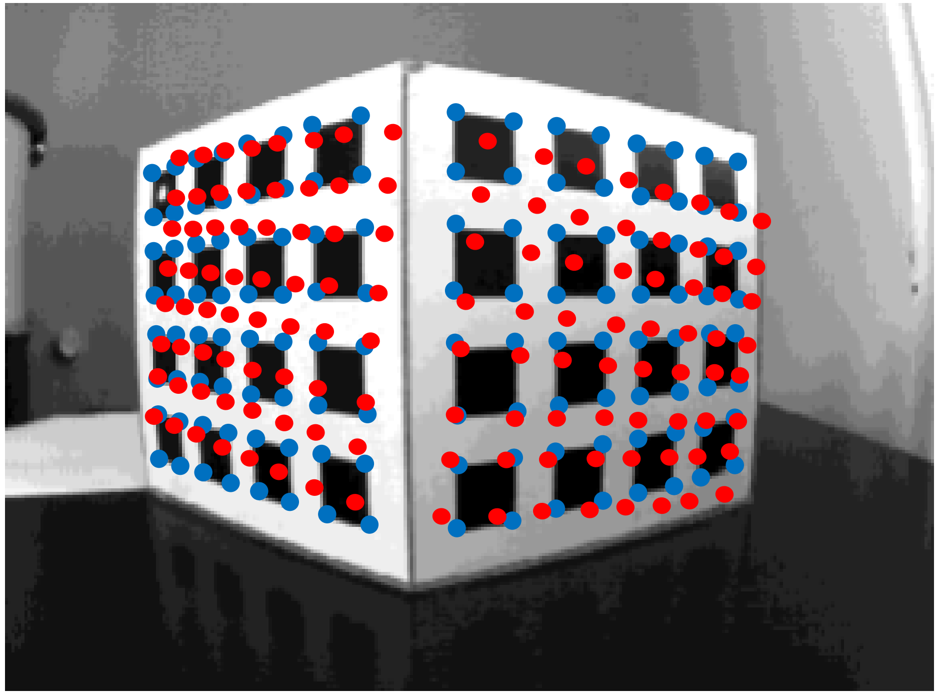
Reprojected points $\pi\left(P_W^i, K, R, T\right)$

Non-Linear Calibration Refinement

- The calibration parameters K, R, T determined by the DLT can be refined by minimizing the following cost:

$$K, R, T, lens\ distortion = argmin_{K,R,T,lens} \sum_{i=1}^{n} \left\| p^i - \pi\left(P_W^i, K, R, T\right) \right\|^2$$

- This time we also include the **lens distortion** (can be set to 0 for initialization)
- Can be minimized using **Levenberg–Marquardt** (more robust than Gauss-Newton to local minima)



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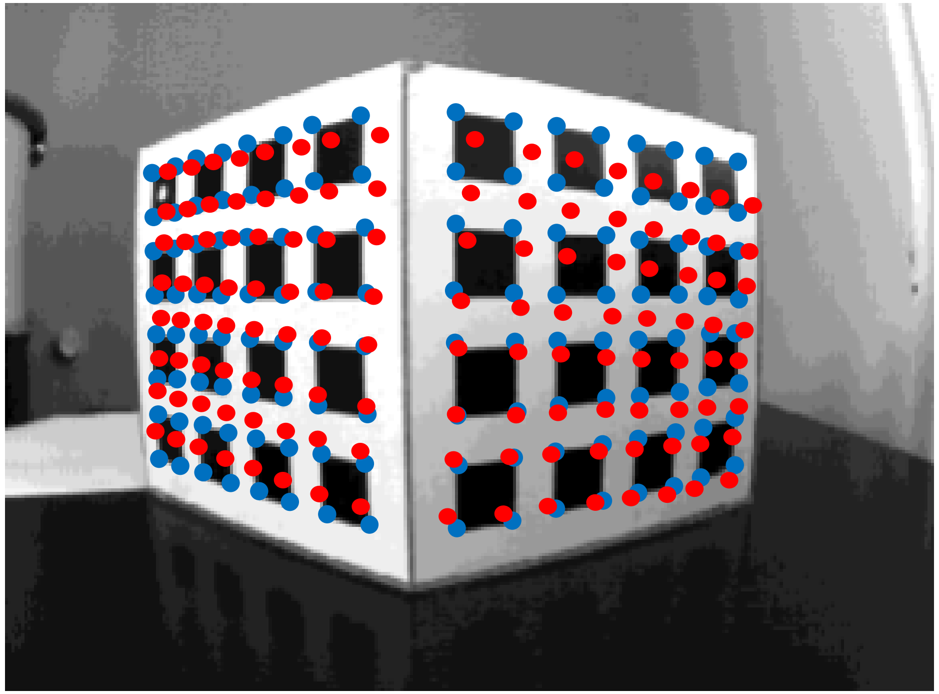

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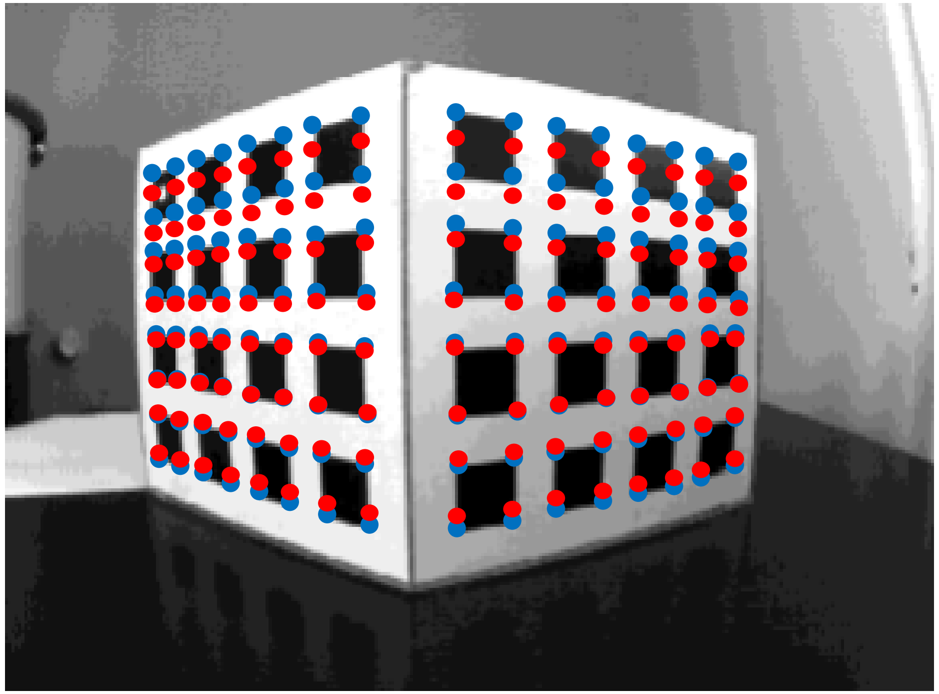

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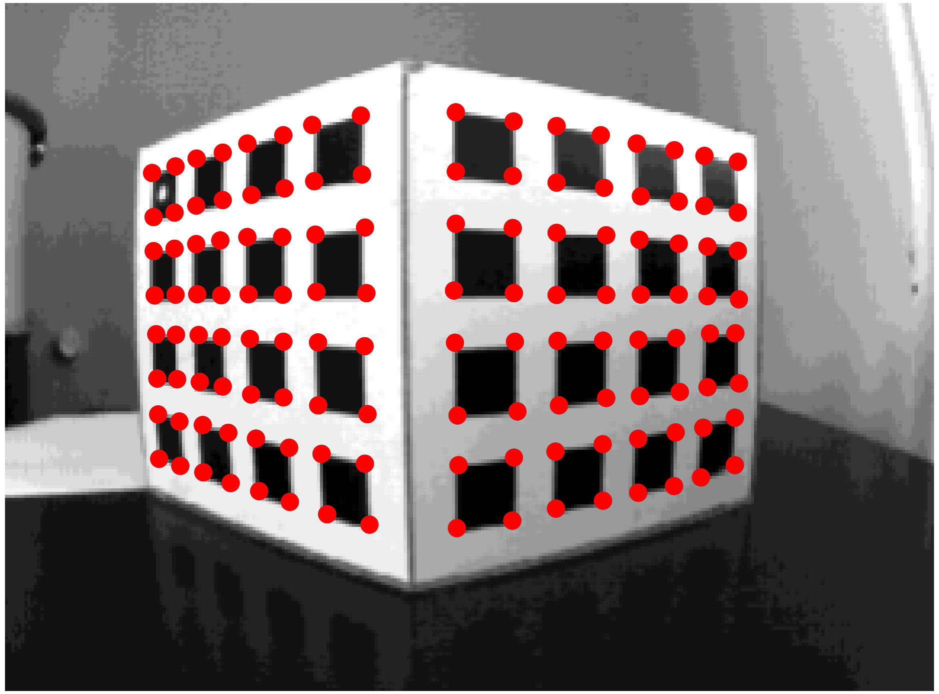


Control points (observed points)

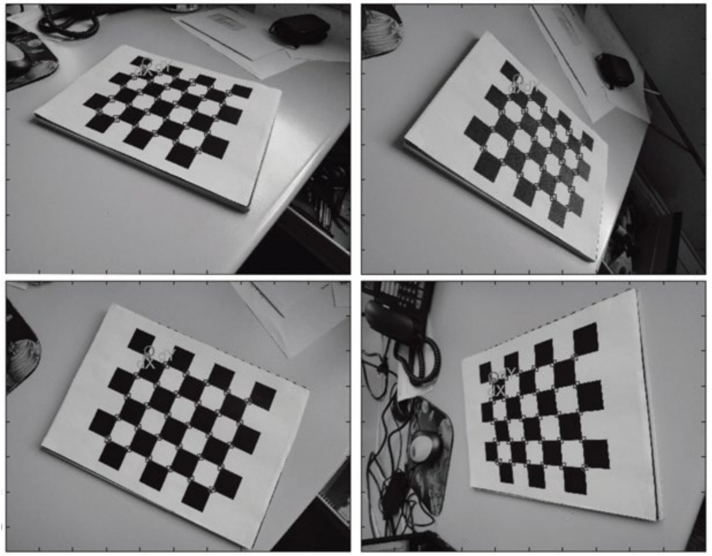
Reprojected points $\pi\left(P_W^i, K, R, T\right)$

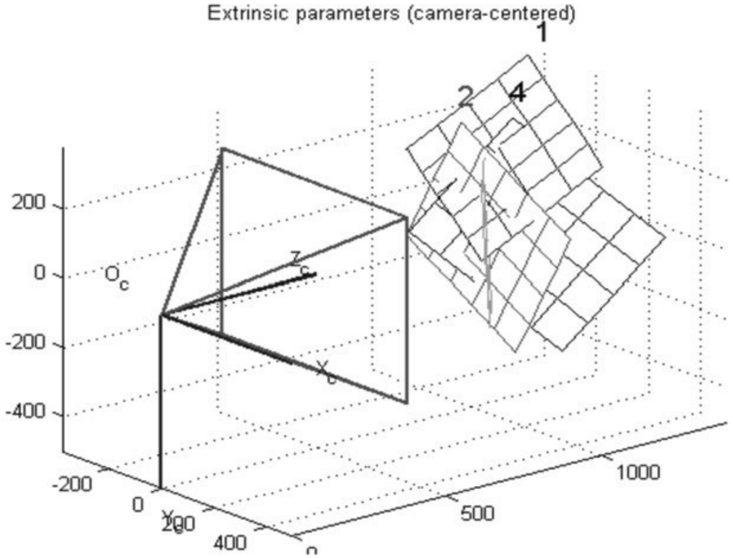
Today’s Outline

- Camera calibration
  - Tsai’s method: From 3D objects
  - Zhang’s method: from planar grids
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Zhang’s Algorithm: Calibration from Planar Grids

- **Tsai’s calibration** requires that the world’s 3D points are non-coplanar, which is **not very practical**
- **Today’s camera calibration toolboxes** (Matlab, OpenCV) use **multiple views** of a **planar grid** (e.g., a checker board)
- They are based on a method developed in 2000 by Zhang (Microsoft Research)





Zhang, A flexible new technique for camera calibration, IEEE Transactions on Pattern Analysis and Machine Intelligence, 2000. PDF.

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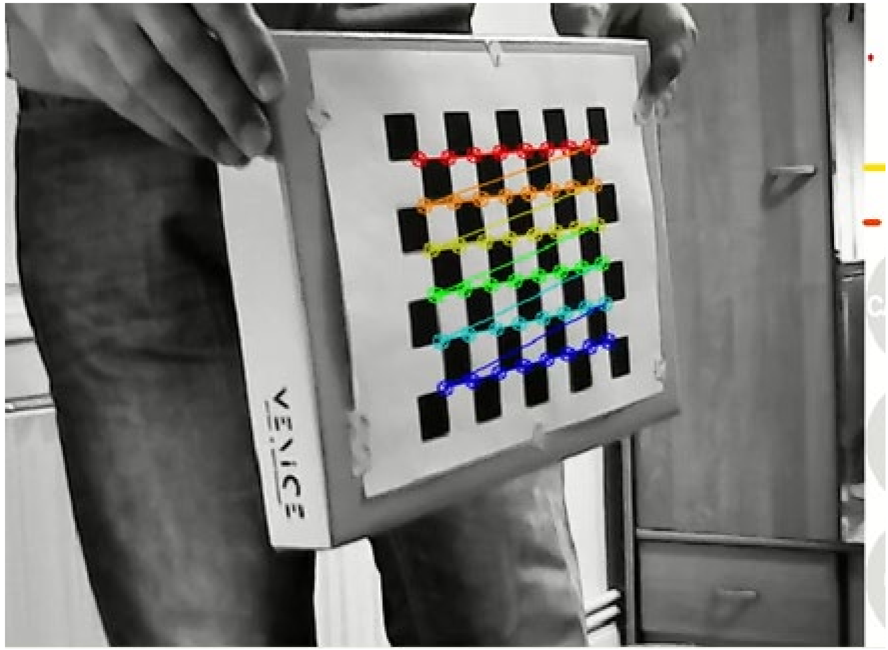

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Applying the Direct Linear Transform (DLT) algorithm

As in Tsai's method, we start by writing the perspective projection equation (again, we neglect the radial distortion). However, in **Zhang's method the points are all coplanar**, i.e., $\boldsymbol{Z_w = 0}$, and thus we can write:

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K\left[R \mid T\right] \cdot \begin{bmatrix} X_w \\ Y_w \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 0 \\ 1 \end{bmatrix}$$

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$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = H \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

This matrix is called Homography

$$\Rightarrow \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_1^{\mathrm{T}} \\ h_2^{\mathrm{T}} \\ h_3^{\mathrm{T}} \end{bmatrix} \cdot \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

where h_i^{T} is the i-*th* row of H

Applying the Direct Linear Transform (DLT) algorithm

$$\Rightarrow \quad \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} h_1^{\mathrm{T}} \\ h_2^{\mathrm{T}} \\ h_3^{\mathrm{T}} \end{bmatrix} \cdot \boxed{\begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}} \rightarrow P$$

Conversion back from homogeneous coordinates to pixel coordinates leads to:

$$u = \frac{\lambda u}{\lambda} = \frac{h_1^{\mathrm{T}} \cdot P}{h_3^{\mathrm{T}} \cdot P} \qquad\qquad (h_1^{\mathrm{T}} - u_i h_3^{\mathrm{T}}) \cdot P_i = 0$$

$$\Rightarrow$$

$$v = \frac{\lambda v}{\lambda} = \frac{h_2^{\mathrm{T}} \cdot P}{h_3^{\mathrm{T}} \cdot P} \qquad\qquad (h_2^{\mathrm{T}} - v_i h_3^{\mathrm{T}}) \cdot P_i = 0$$

Applying the Direct Linear Transform (DLT) algorithm

- By re-arranging the terms, we obtain:

$$
\begin{aligned}
(h_1^{\mathrm{T}} - u_i h_3^{\mathrm{T}}) \cdot P_i &= 0 \\
(h_2^{\mathrm{T}} - v_i h_3^{\mathrm{T}}) \cdot P_i &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
P_i^{\mathrm{T}} \cdot h_1 + 0 \cdot h_2^{\mathrm{T}} - u_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} &= 0 \\
0 \cdot h_1^{\mathrm{T}} + P_i^{\mathrm{T}} \cdot h_2^{\mathrm{T}} - v_i P_i^{\mathrm{T}} \cdot h_3^{\mathrm{T}} &= 0
\end{aligned}
\quad \Rightarrow \quad
\begin{pmatrix} P_i^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_i^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_i^{\mathrm{T}} & -v_1 P_i^{\mathrm{T}} \end{pmatrix}
\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

- For n points (from a **single view**), we can stack all these equations into a big matrix:

$$
\underbrace{\begin{pmatrix} P_1^{\mathrm{T}} & 0^{\mathrm{T}} & -u_1 P_1^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_1^{\mathrm{T}} & -v_1 P_1^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ P_n^{\mathrm{T}} & 0^{\mathrm{T}} & -u_n P_n^{\mathrm{T}} \\ 0^{\mathrm{T}} & P_n^{\mathrm{T}} & -v_n P_n^{\mathrm{T}} \end{pmatrix}}_{\text{Q (this matrix is \textbf{known})}}
\underbrace{\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}}_{\text{H (this matrix is \textbf{unknown})}}
= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}
\quad \Rightarrow \quad \mathrm{Q} \cdot \mathrm{H} = 0
$$

Applying the Direct Linear Transform (DLT) algorithm

$$\mathrm{Q} \cdot \mathrm{H} = 0$$

Minimal solution

- $Q_{(2n\times 9)}$ should have rank 8 to have a unique (up to a scale) non-trivial solution H
- Each point correspondence provides 2 independent equations
- Thus, a minimum of **4 non-collinear points** is required

Solution for $\boldsymbol{n} \geq \boldsymbol{4}$ points

- It can be solved through Singular Value Decomposition (SVD) (same considerations as before)

How to recover K, R, T

- H can be decomposed by recalling that:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ r_{31} & r_{32} & t_3 \end{bmatrix}$$

- Differently from Tsai's, the decomposition of H into K, R, T requires **at least two views**
- In practice **the more views the better**, e.g., 20-50 views spanning the entire field of view of the camera for the best calibration results
- Notice that now **each view j has a different homography** H^j (and so a different R^j and T^j). However, $\boldsymbol{K}$ **is the same for all views**:

$$\begin{bmatrix} h^j_{11} & h^j_{12} & h^j_{13} \\ h^j_{21} & h^j_{22} & h^j_{23} \\ h^j_{31} & h^j_{33} & h^j_{33} \end{bmatrix} = \begin{bmatrix} \alpha_u & 0 & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r^j_{11} & r^j_{12} & t^j_1 \\ r^j_{21} & r^j_{22} & t^j_2 \\ r^j_{31} & r^j_{32} & t^j_3 \end{bmatrix}$$

How to recover K, R, T from H and from multiple views?

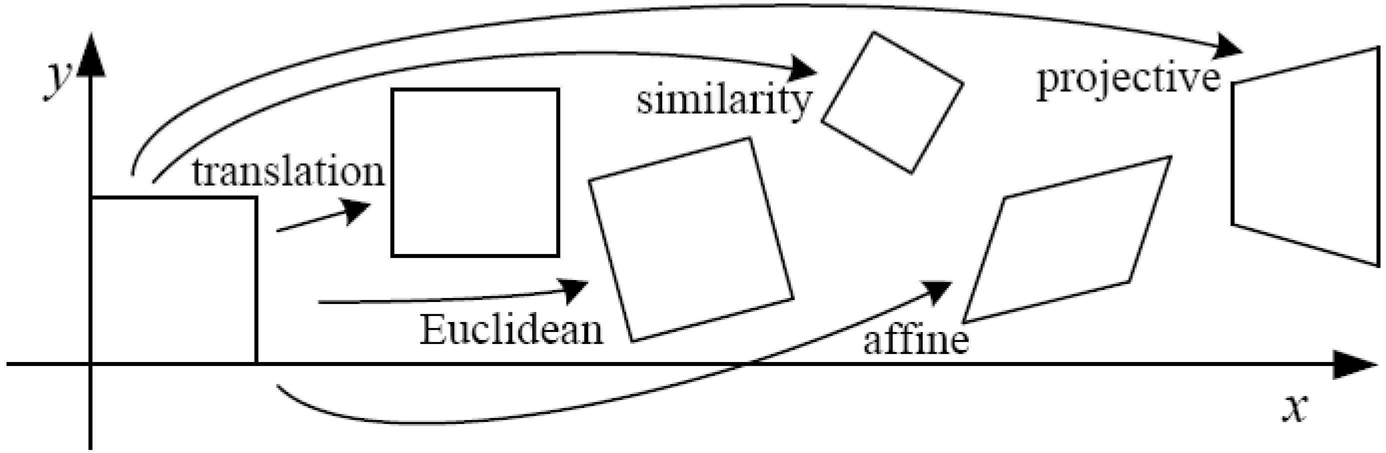
Won't be asked at the exam ☺

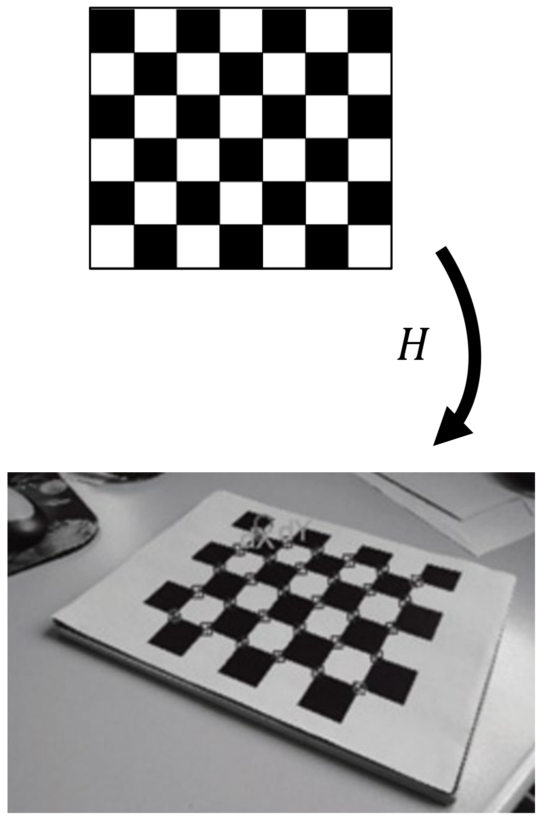
1. Estimate the homography H_i for each i-th view using the DLT algorithm.

2. Determine the intrinsics K of the camera from a set of homographies:
   1. Each homography $H_i \sim K(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{t})$ provides two *linear* equations in the 6 entries of the matrix $\mathrm{B} \coloneqq K^{-\top}K^{-1}$. Letting $\boldsymbol{w}_1 \coloneqq K\boldsymbol{r}_1$, $\boldsymbol{w}_2 \coloneqq K\boldsymbol{r}_2$, the rotation constraints $\boldsymbol{r}_1^\top\boldsymbol{r}_1 = \boldsymbol{r}_2^\top\boldsymbol{r}_2 = 1$ and $\boldsymbol{r}_1^\top\boldsymbol{r}_2 = 0$ become $\boldsymbol{w}_1^\top B\boldsymbol{w}_1 - \boldsymbol{w}_2^\top B\boldsymbol{w}_2 = 0$ and $\boldsymbol{w}_1^\top B\boldsymbol{w}_2 = 0$.
   2. Stack 2*N* equations from *N* views, to yield a linear system $A\boldsymbol{b} = \boldsymbol{0}$. Solve for $\boldsymbol{b}$ (i.e., *B*) using the Singular Value Decomposition (SVD).
   3. Use Cholesky decomposition to obtain *K* from *B*.

3. The extrinsic parameters for each view can be computed using *K*:
   $\boldsymbol{r}_1 \sim \lambda K^{-1}H_i(:,1)$, $\boldsymbol{r}_2 \sim \lambda K^{-1}H_i(:,2)$, $\boldsymbol{r}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2$ and $T_i = \lambda K^{-1}H_i(:,3)$, with $\lambda = 1/K^{-1}H_i(:,1)$.
   Finally, build $R_i = (\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ and enforce rotation matrix constraints.

Types of 2D Transformations





Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\, \boldsymbol{I} \mid \boldsymbol{t} \,\right]_{2\times 3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\left[\, \boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	3	lengths $+ \cdots$	
similarity	$\left[\, s\boldsymbol{R} \mid \boldsymbol{t} \,\right]_{2\times 3}$	4	angles $+ \cdots$	
affine	$\left[\, \boldsymbol{A} \,\right]_{2\times 3}$	6	parallelism $+ \cdots$	
projective	$\left[\, \tilde{\boldsymbol{H}} \,\right]_{3\times 3}$	8	straight lines	

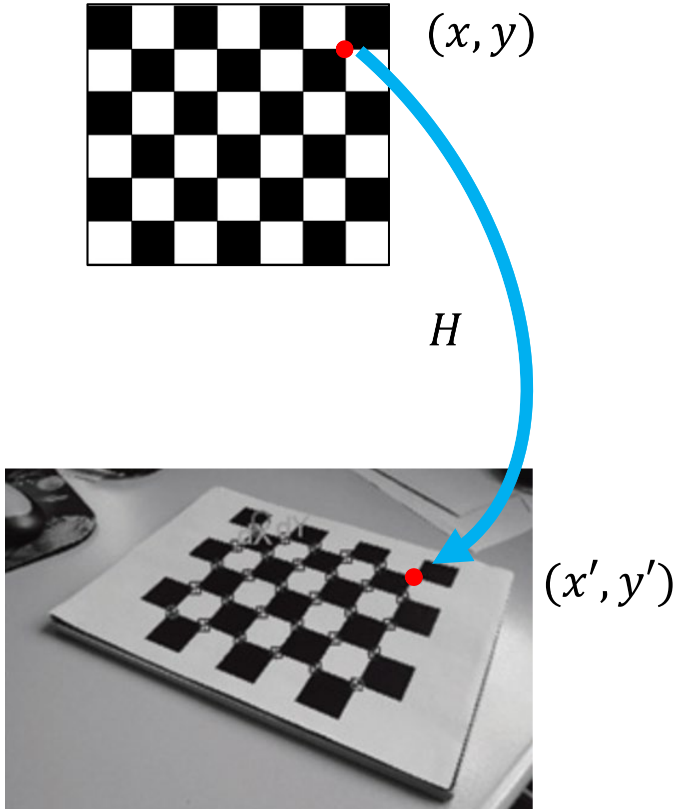
This matrix is called **Homography**

Projective Transformation (Homography)

- A point (x, y) is transformed into (x', y') via:

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}$$

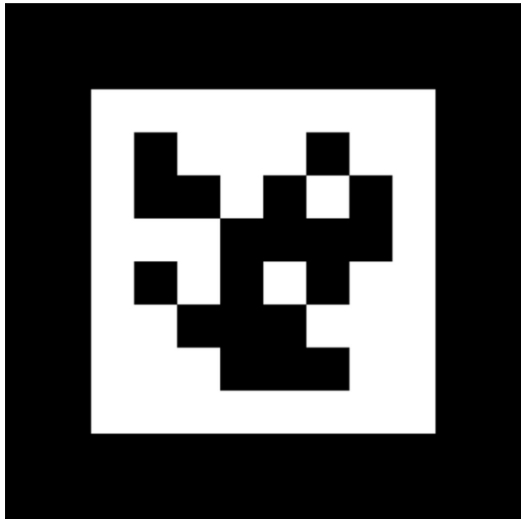


- Homogeneous coordinates:

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

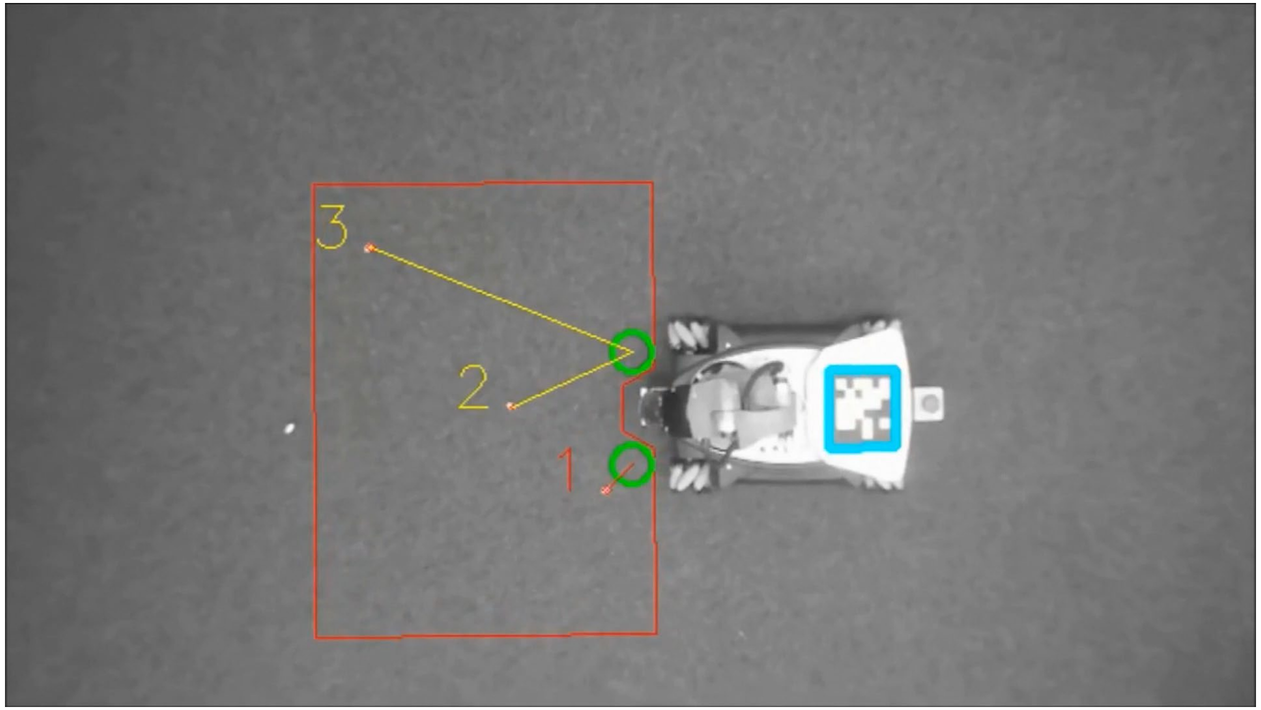
Application to Augmented Reality

- Today, there are thousands of application of Zhang’s algorithm, e.g. Augmented Reality (AR)
- See AprilTag or ARuco Markers

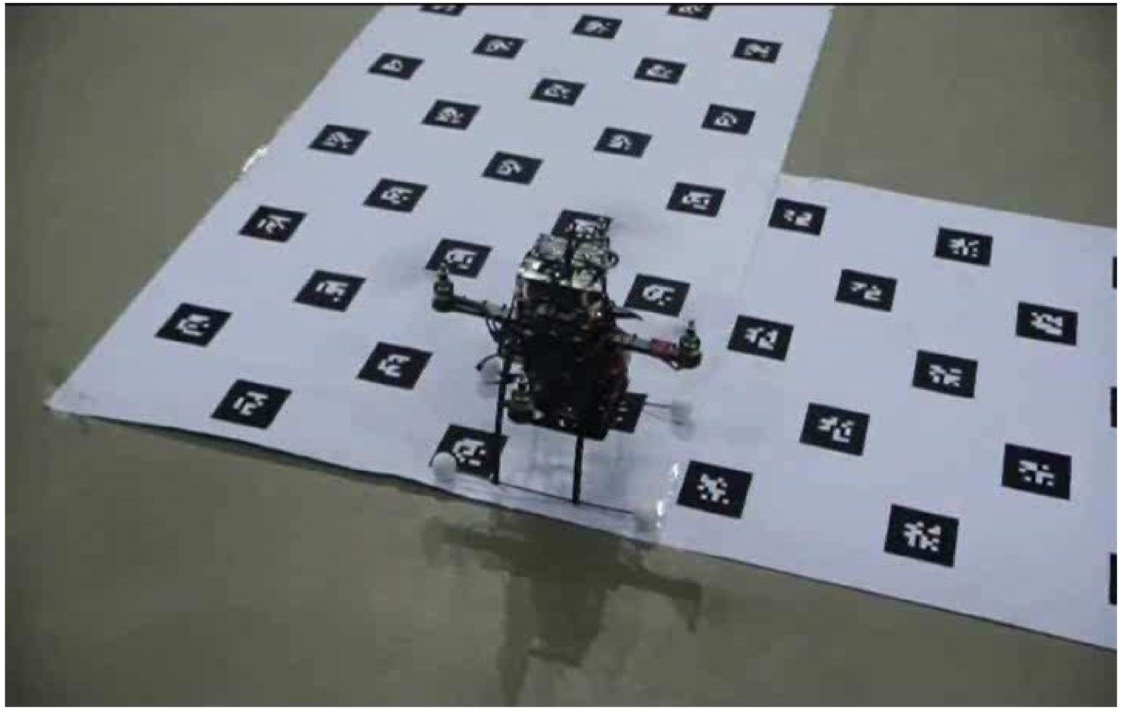


Application to Robotics

- Do we need to know the size of the tag?
  - For Augmented Reality?
  - For Control?



My lab. Video.



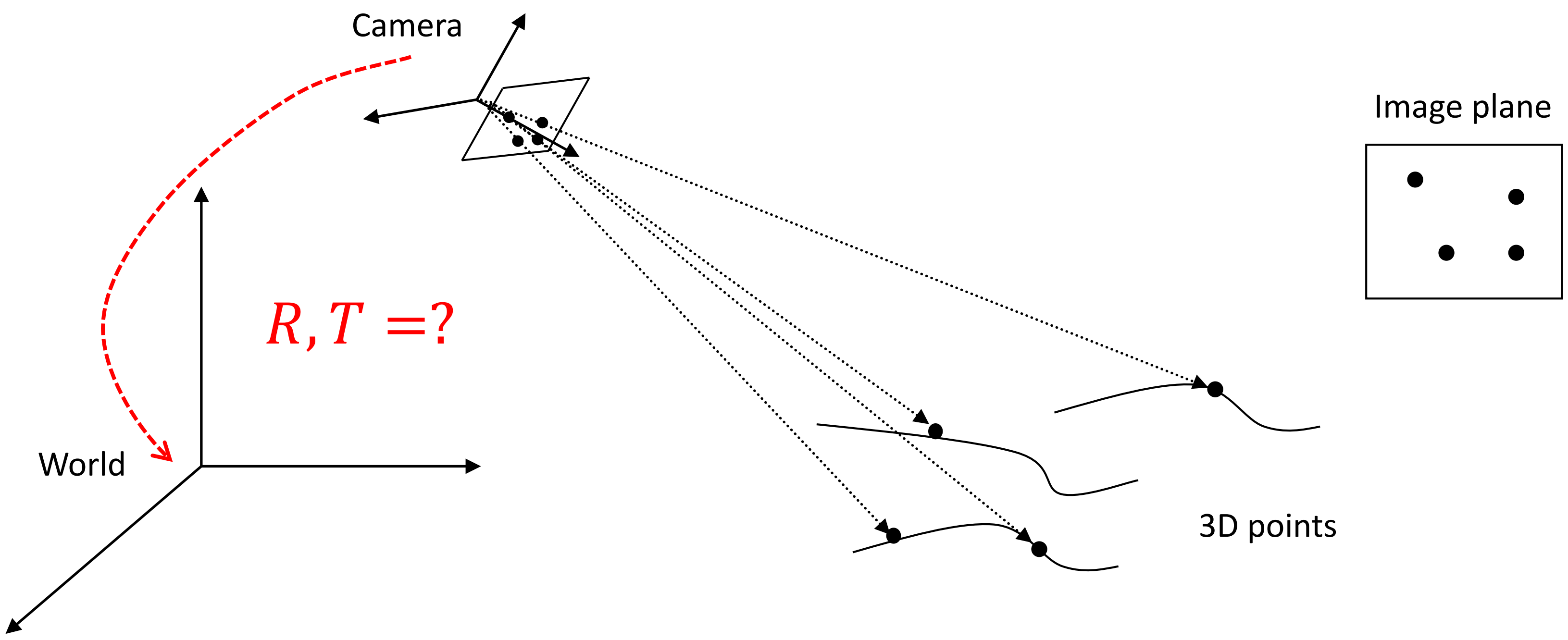
Marc Pollefeys' lab. Video.

Today’s Outline

- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Camera Localization (or Perspective from n Points: PnP)

- This is the problem of determining the **6DoF pose of a camera** (position and orientation) with respect to the world frame **from a set of 3D-2D point correspondences**.
- It assumes that the **camera** is **already calibrated**
- The **DLT can be used** to solve this problem **but is suboptimal**. We want to study **algebraic solutions** to the problem.



How Many Points are Enough?

- **1 Point**:
  - infinite solutions
- **2 Points**:
  - infinitely many solutions, but bounded
- **3 Points** (non collinear):
  - up to 4 solution
- **4 Points**:
  - Unique solution

1 Point

- **1 Point**:
  - infinite solutions

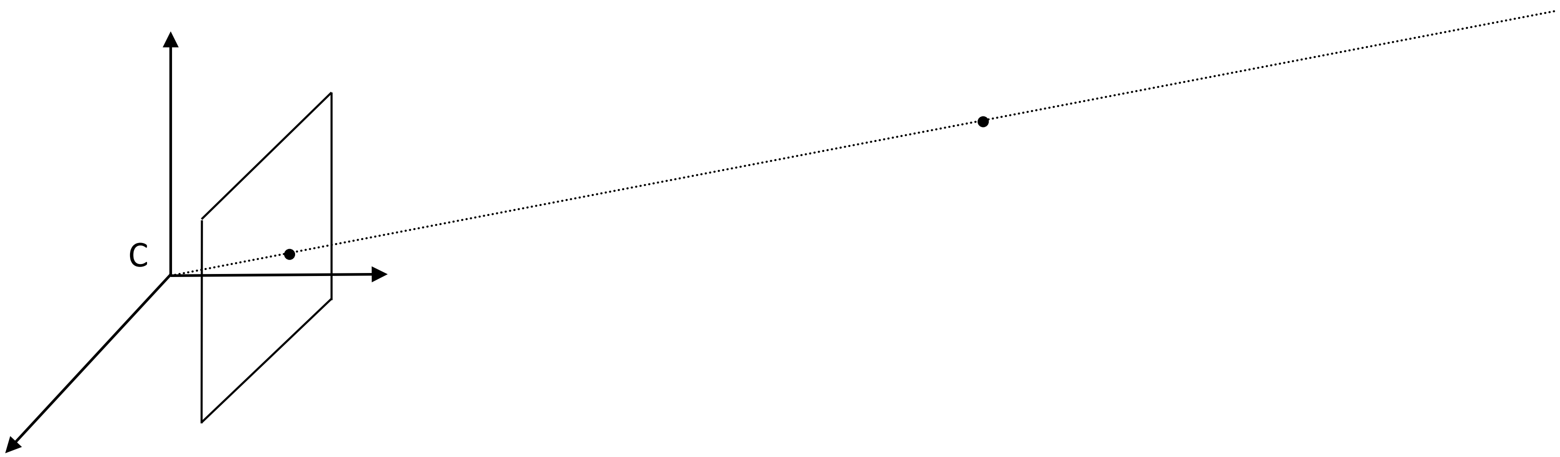
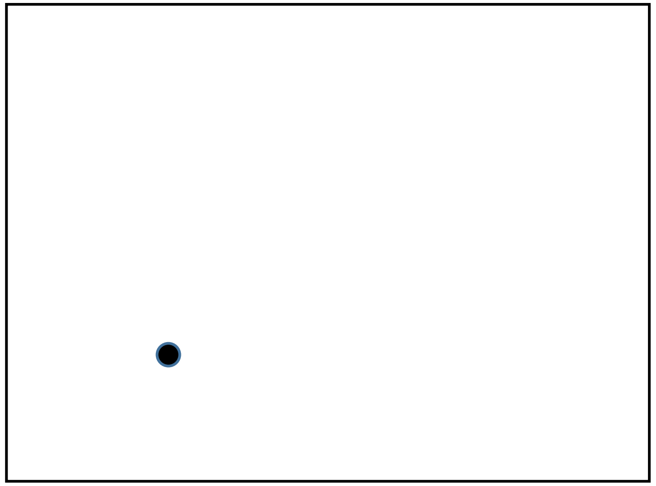
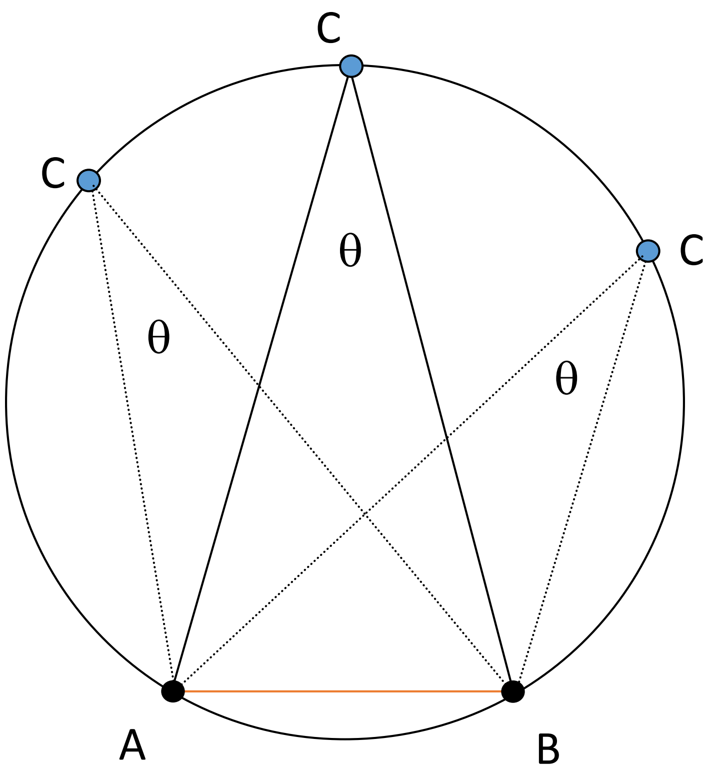


Image plane



2 Points



- **2 Points**:
  - infinite solutions, but bounded

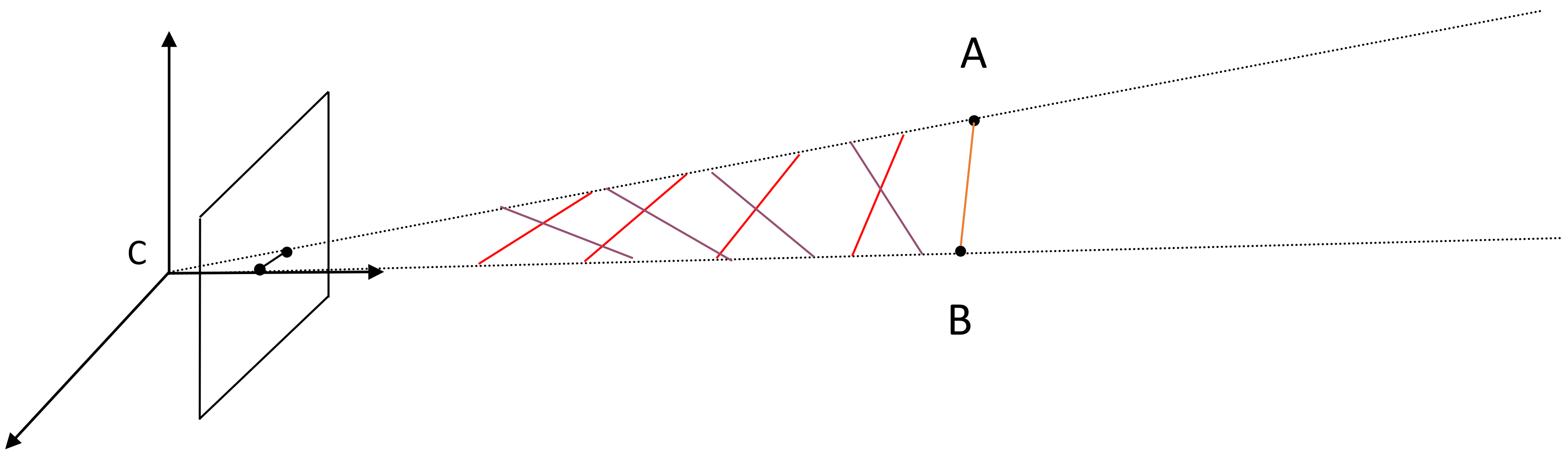
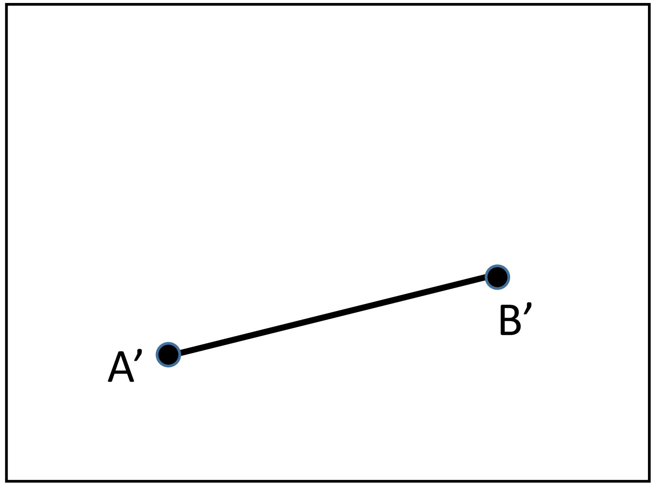
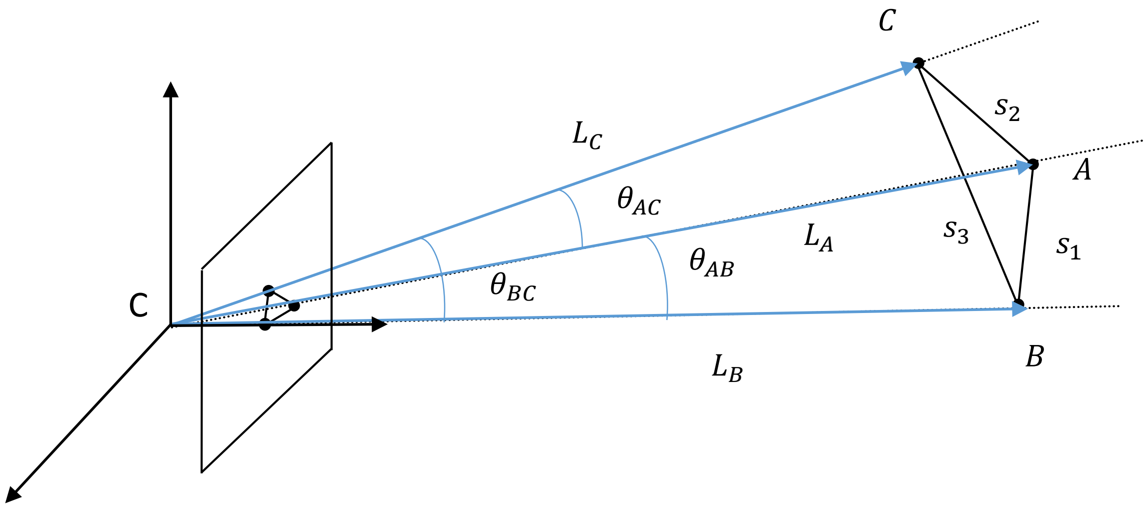


Image plane



3 Points (P3P problem)

- **3 Points** (non collinear):
  - up to 4 solution



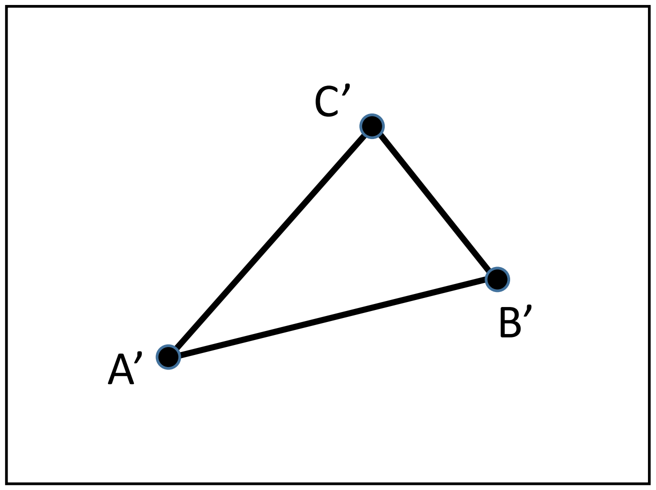
From the law of cosines:

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

Image plane



Algebraic Approach: reduce to 4th order equation

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

- It is known that $\boldsymbol{n}$ **independent polynomial equations**, in $\boldsymbol{n}$ **unknowns**, can have no more **solutions** than the **product of their respective degrees**. Thus, the system can have a maximum of **8** solutions. However, because every term in the system is either a constant or of second degree, for every real positive solution **there is a negative solution**.
- Thus, with 3 points, there are at most **4 valid (positive) solutions**.

M. A.Fischler and R. C.Bolles. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. Graphics and Image Processing, 1981. PDF.

Algebraic Approach: reduce to 4th order equation

$$s_1^2 = L_B^2 + L_A^2 - 2L_B L_A \cos\theta_{AB}$$

$$s_2^2 = L_A^2 + L_C^2 - 2L_A L_C \cos\theta_{AC}$$

$$s_3^2 = L_B^2 + L_C^2 - 2L_B L_C \cos\theta_{BC}$$

- By defining $\boldsymbol{x} = \boldsymbol{L_B}/\boldsymbol{L_A}$, it can be shown that the system can be reduced to a 4th order equation:

$$G_0 + G_1 x + G_2 x^2 + G_3 x^3 + G_4 x^4 = 0$$

How can we disambiguate the 4 solutions? How do we determine R and T?

- A 4th point can be used to disambiguate the solutions. A classification of the four solutions and the determination of R and T from the point distances was given by Gao's algorithm, implemented in OpenCV (*solvePnP_P3P*)

Gao, Hou, Tang, Cheng. Complete Solution Classification for the Perspective-Three-Point Problem. IEEE Transactions on Pattern Analysis and Machine Intelligence, 2003. PDF.

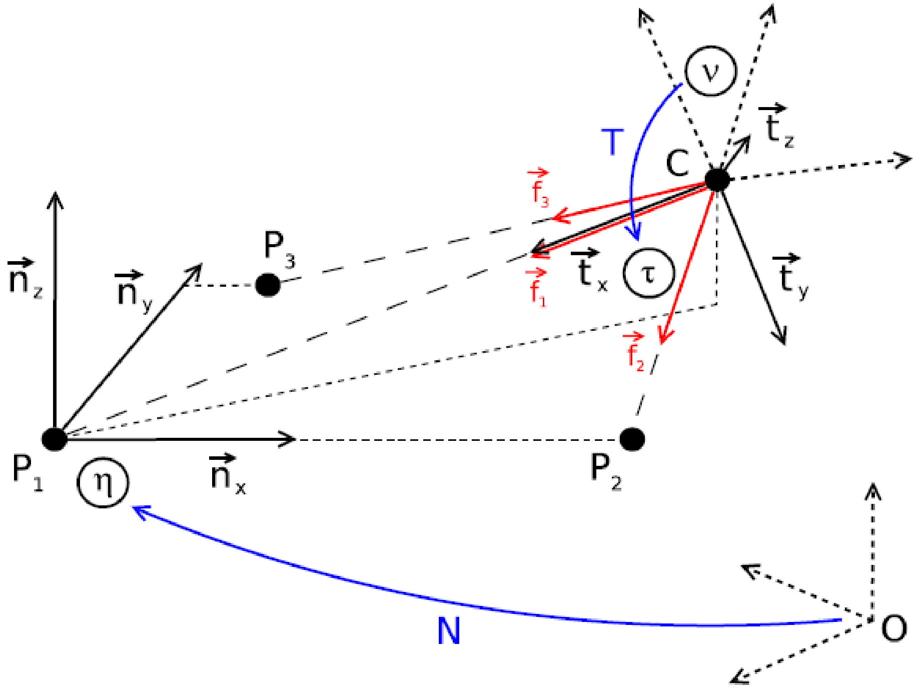
Modern Solution to P3P

Won't be asked at the exam ☺

A more **modern version of P3P** was developed by Kneip in 2011 and **directly solves for the camera's pose** (not distances from the points). This solution inspired the algorithm currently used in OpenCV (*solvePnP_AP3P*), by Ke'17, which consists of two steps:

1. Eliminate the camera's position and the features' distances to yield a system of 3 equations in *the camera's orientation alone*.
2. Successively eliminate two of the unknown 3-DOFs (angles) algebraically and arrive at a ***quartic polynomial equation***.

- Outperforms previous methods in terms of speed, accuracy, and robustness to close-to-singular cases.



Kneip, Scaramuzza, Siegwart. A Novel Parameterization of the Perspective-Three-Point Problem for a Direct Computation of Absolute Camera Position and Orientation. IEEE Conference on Computer Vision and Pattern Recognition (CVPR), 2011. PDF.

Ke, Roumeliotis. An Efficient Algebraic Solution to the Perspective-Three-Point Problem. CVPR'17. PDF.

SWISS MADE

Solution to PnP for $n \geq 4$

Won't be asked at the exam ☺

An efficient algebraic solution to the PnP problem for $n \geq 4$ was developed by Lepetit in 2009 and coined **EPnP** (Efficient PnP) and can be found in OpenCV (solvePnP_EPnP)

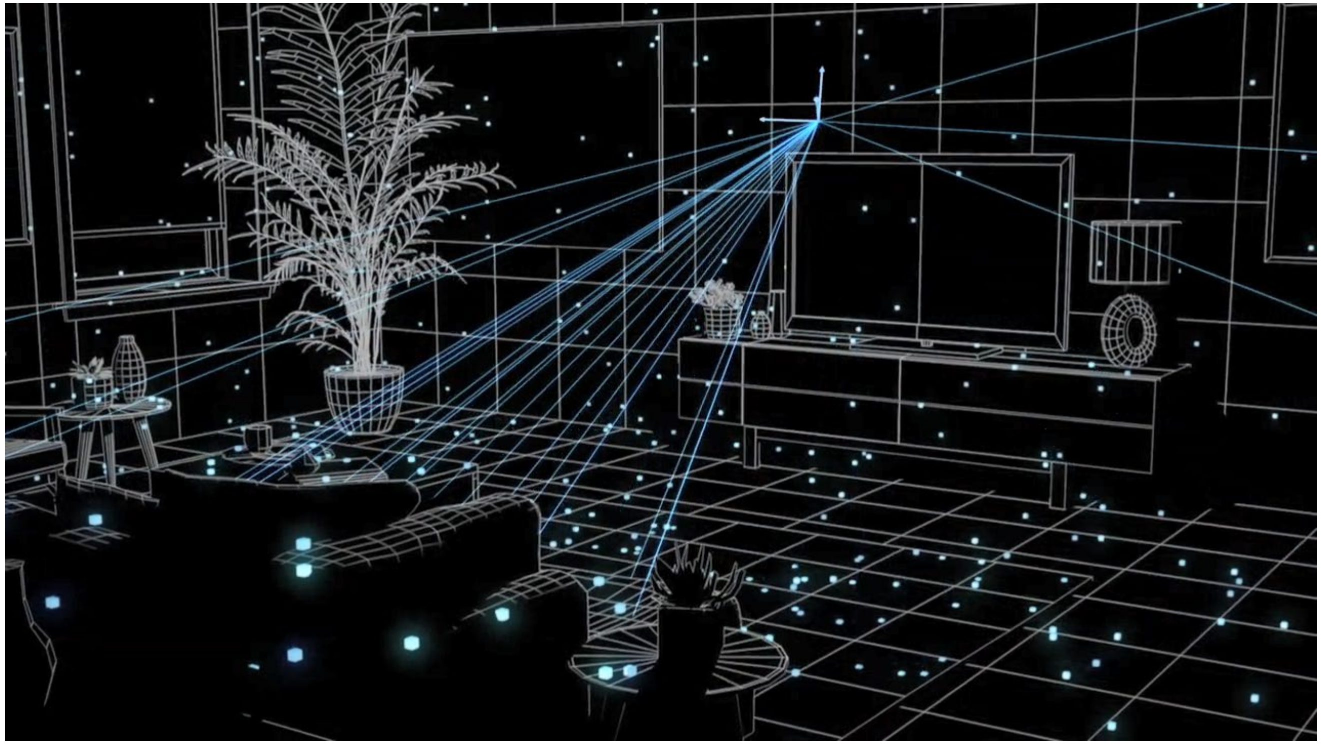
- EPnP expresses the *n* world's points as a weighted sum of **four virtual control points**
- The coordinates of these virtual control points become the **unknowns of the problem**, which can be solved in $O(n)$ time by solving a **constant number** of **quartic polynomial equations**
- The final pose of the camera is then solved from the control points

SWISS MADE

Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. PDF.

Application to Monocular Localization

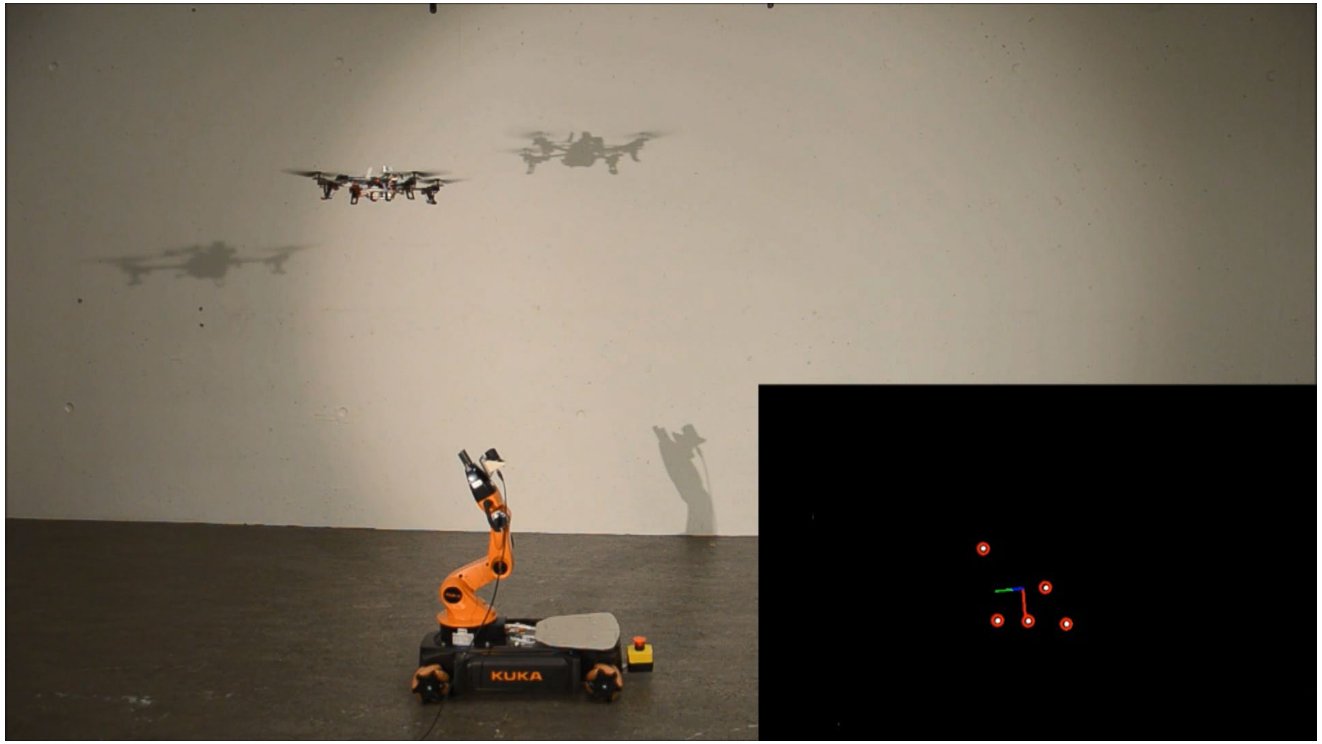
Localization: Given a 3D point cloud (map), determine the pose of the camera



Video of Oculus Insight (the VIO used in Oculus Quest): built by former Zurich-Eye team, today Facebook Zurich.
The story from Zurich-Eye to Facebook Oculus Quest.

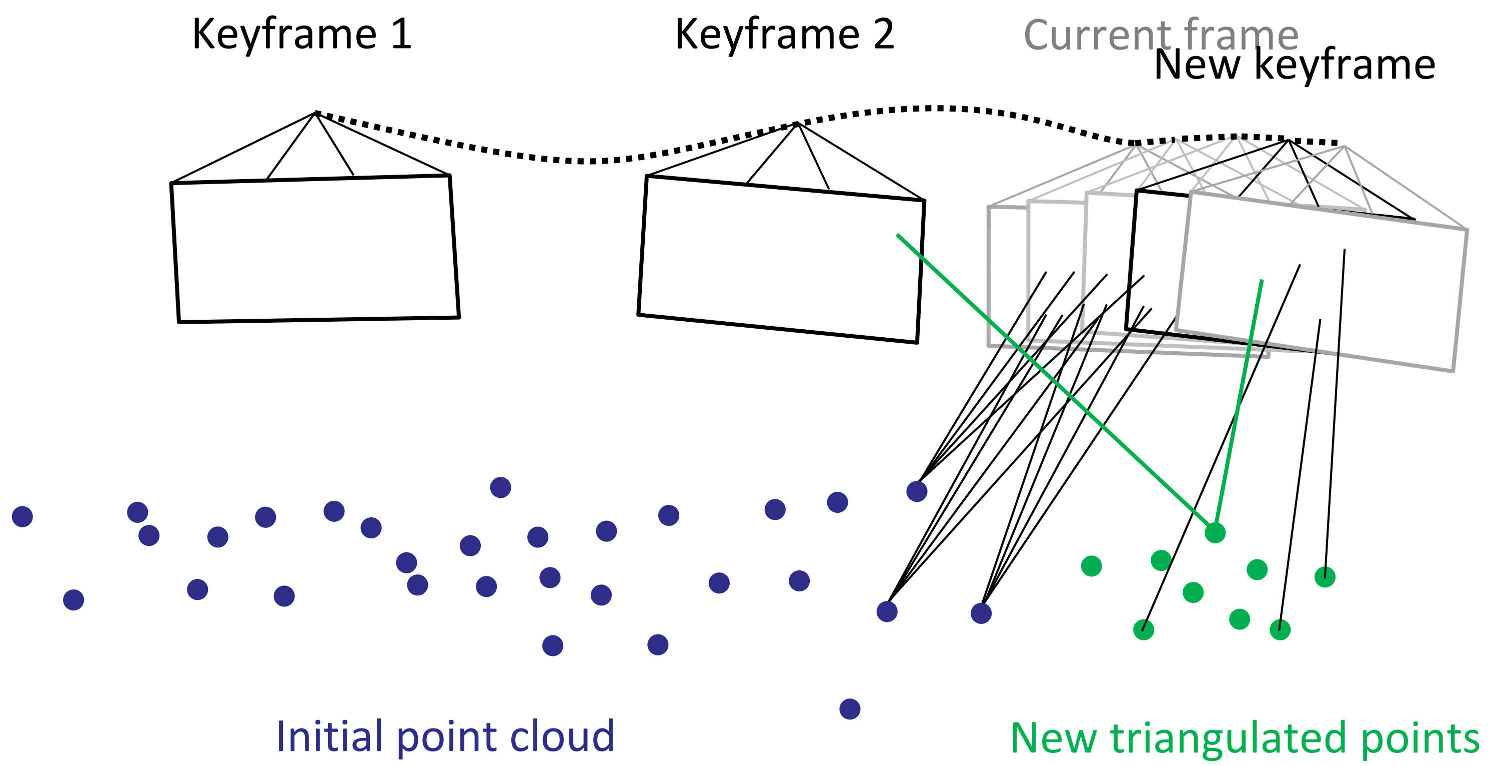
Application to Multi-Robot mutual Localization

Here, the drone carries 5 LEDs that are used by the ground robot to control the drone's position relative to it



Faessler, Mueggler, Schwabe, Scaramuzza. A Monocular Pose Estimation System based on Infrared LEDs.
IEEE International Conference on Robotics and Automation (ICRA), Hong Kong, 2014. [PDF](). [Video]().

Application to Monocular Visual Odometry



Robust Estimation in Presence of Outliers

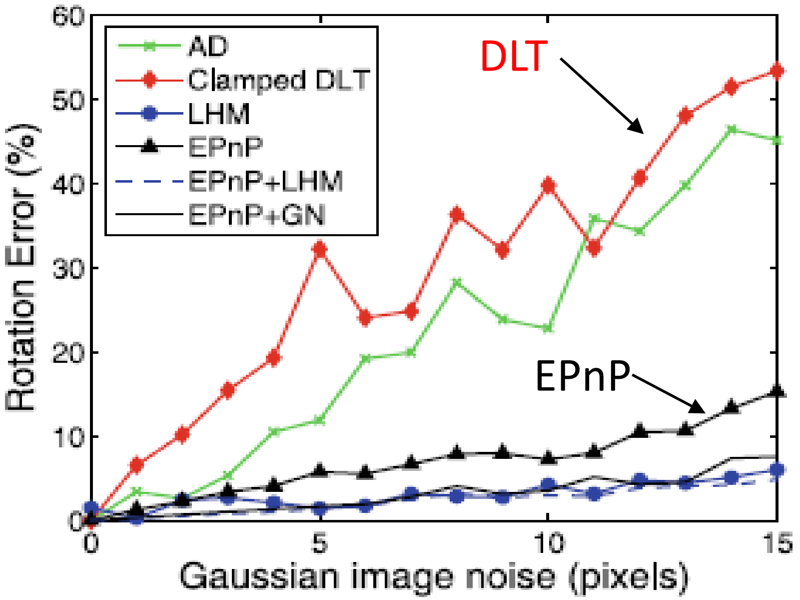
- **All P*n*P problems** (solved by DLT, EPnP, or P3P algorithms) are prone to errors if there are outliers in the set of 3D-2D point correspondences.
- The **RANSAC** algorithm (**Lecture 08**) can be used, in conjunction with the PnP algorithm, to **remove the outliers**.
- P*n*P with RANSAC can be found in OpenCV's (*solvePnPRansac*)

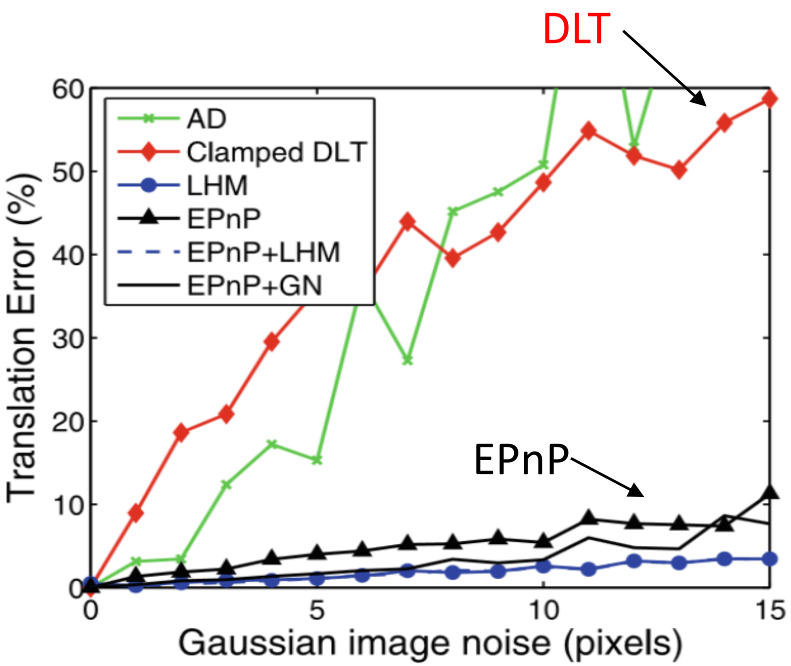
EPnP vs. DLT

If a camera is calibrated, only R and T need to be determined. In this case, **should we use DLT or EPnP**?

EPnP vs. DLT: Accuracy vs. noise

EPnP is up to **10 times more robust to noise** than DLT

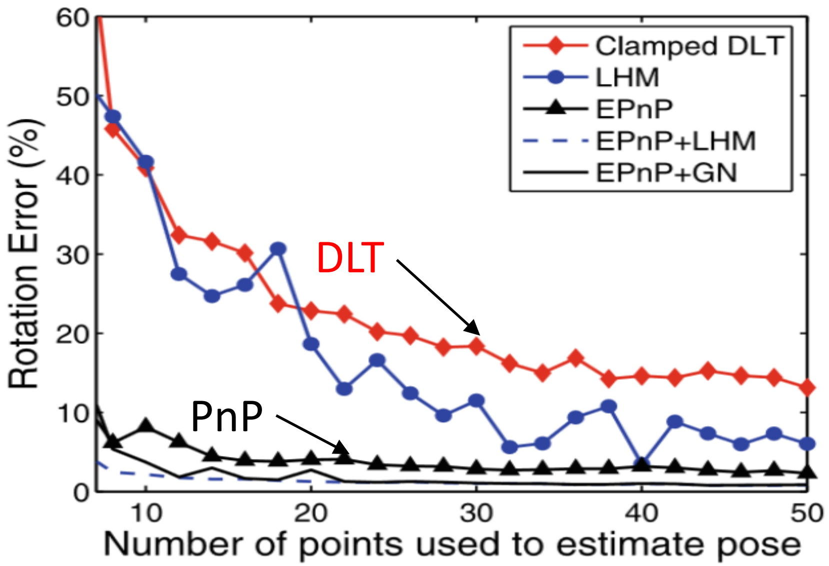


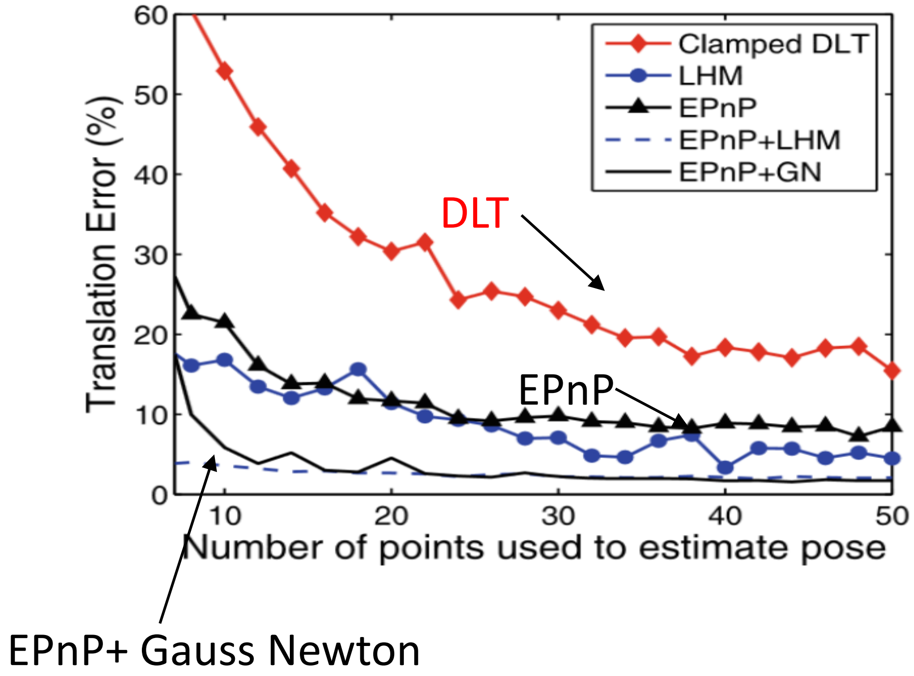


Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. PDF.

EPnP vs. DLT: Accuracy vs. number of points

EPnP is up to **10 times more accurate** than DLT

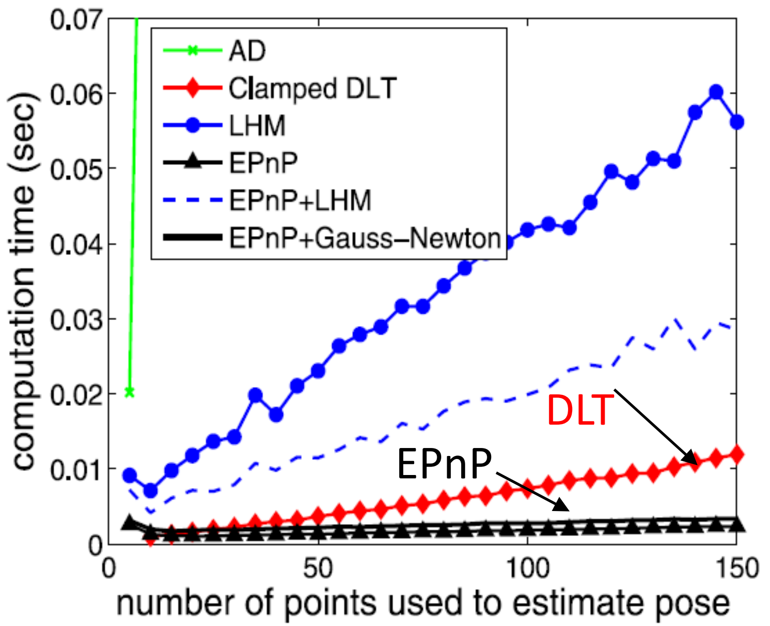




Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. [PDF](#).

EPnP vs. DLT: Timing

EPnP is up to **10 times more efficient** than DLT



Plots from
Lepetit, Moreno Noguer, Fua, EP*n*P: An Accurate *O(n)* Solution to the P*n*P Problem, International Journal of Computer Vision. [PDF](PDF).

PnP problem: Recap

Calibrated camera (i.e., instrinc parameters are known)	**Uncalibrated camera** (i.e., intrinsic parameters unknown)
Either DLT or EPnP can be used	**Only DLT** can be used

EPnP: minimum number of points: **3 (P3P) +1** for disambiguation
DLT: Minimum number of points: **4 if coplanar, 6 if non-coplanar**

The output of both DLT and EPnP can be refined via **non-linear optimization** by minimizing the sum of squared reprojection errors

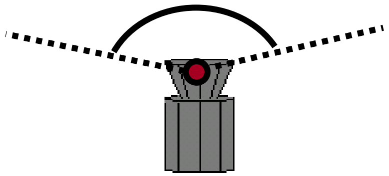
Today’s Outline

- Camera calibration
- Camera localization
- Non conventional camera models: fisheye and catadioptric cameras

Overview on Omnidirectional Cameras

Fisheye

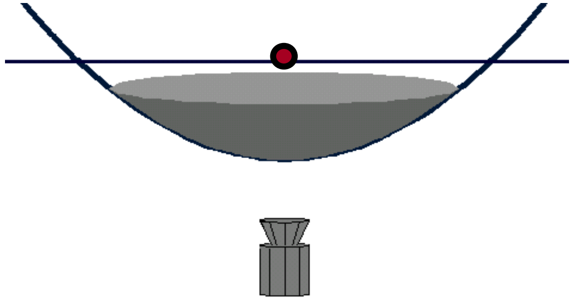
FOV > 130º



Wide FOV dioptric cameras (e.g. fisheye)

Catadioptric

360º all around

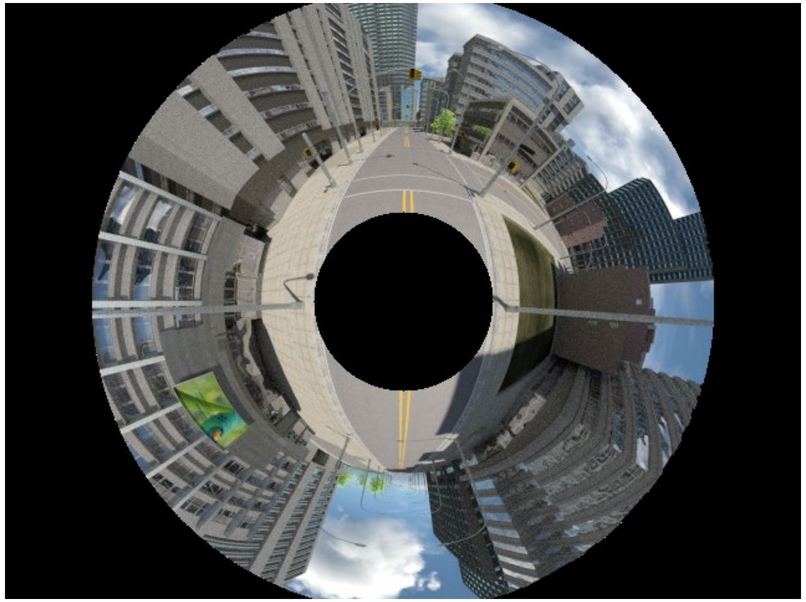


Catadioptric cameras (e.g. cameras and mirror systems)

Camera View Comparison

Perspective

Fisheye

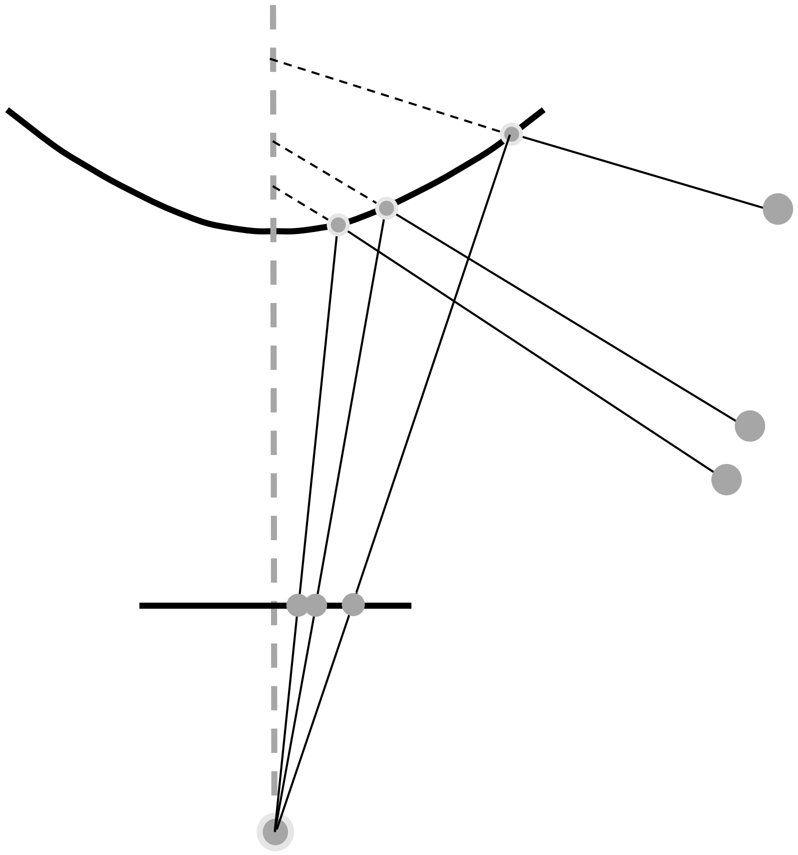


Catadioptric

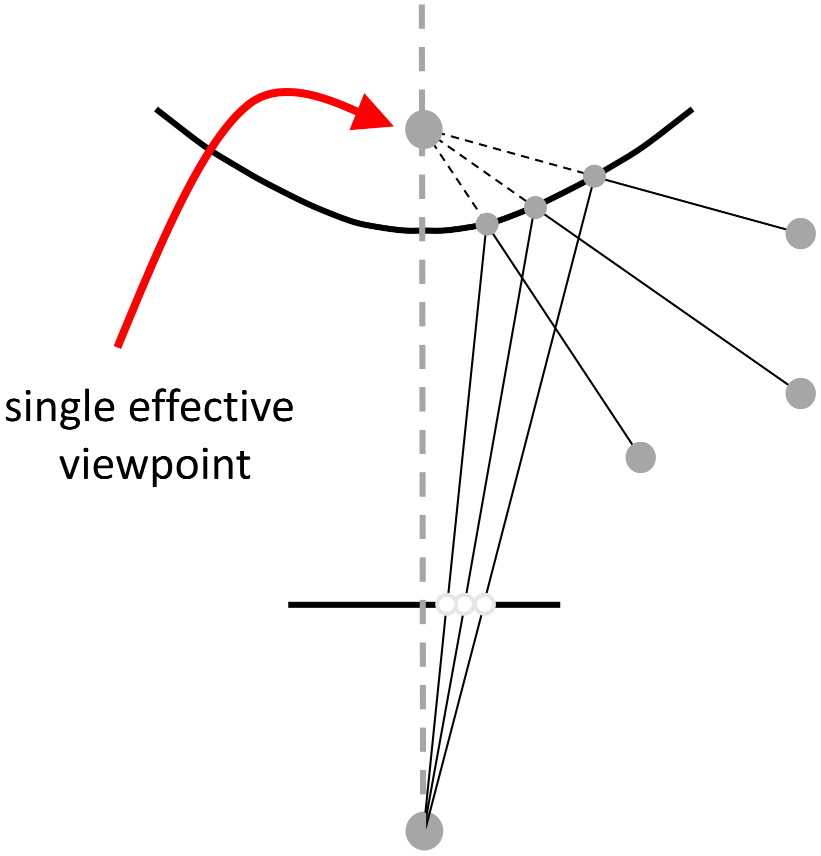
Zhang, Rebecq, Forster, Scaramuzza. Benefit of Large Field-of-View Cameras for Visual Odometry.
IEEE International Conference on Robotics and Automation (ICRA), 2016. PDF.

Central vs Non-Central Omnidirectional Cameras

Non-Central projection system
Rays do not intersect in a single point

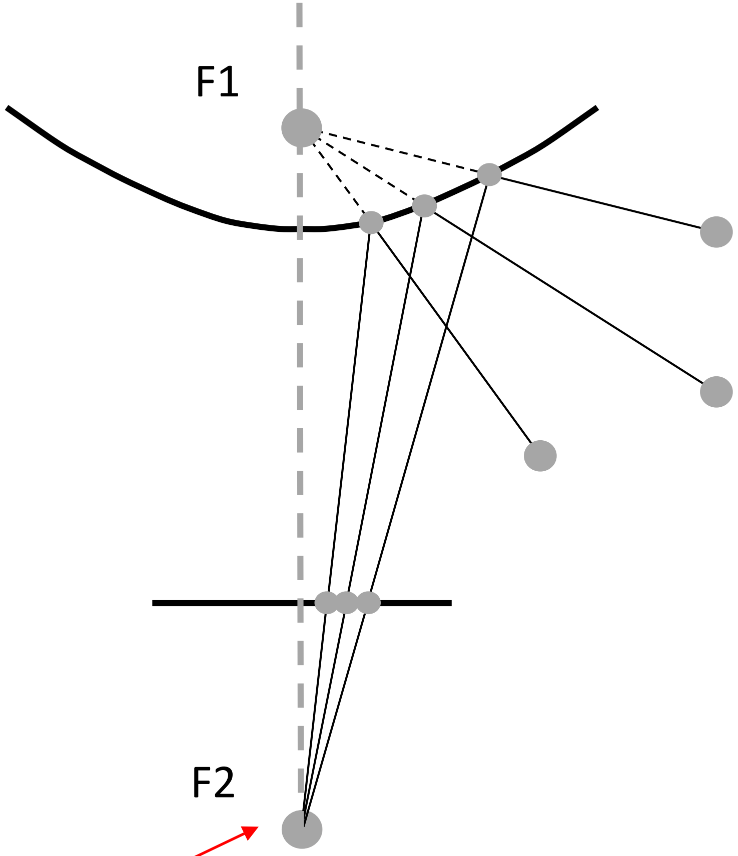


Central projection system
Rays intersect in a single point



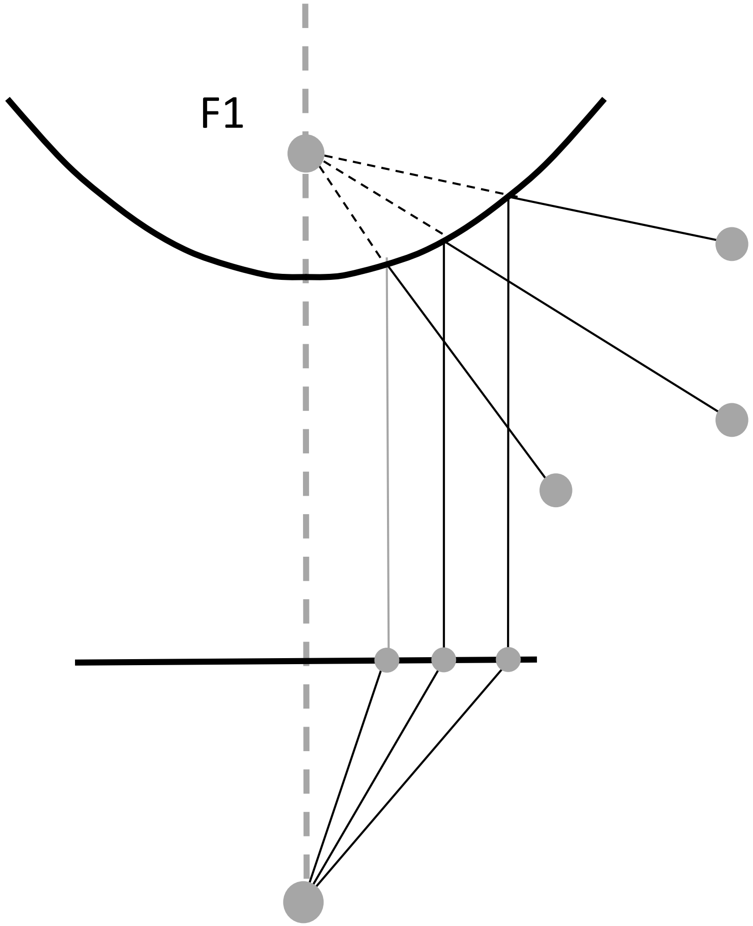
Central Omnidirectional Cameras

Hyperbola + Perspective camera



NB: one of the foci of the hyperbola must lie in the the camera's center of projection

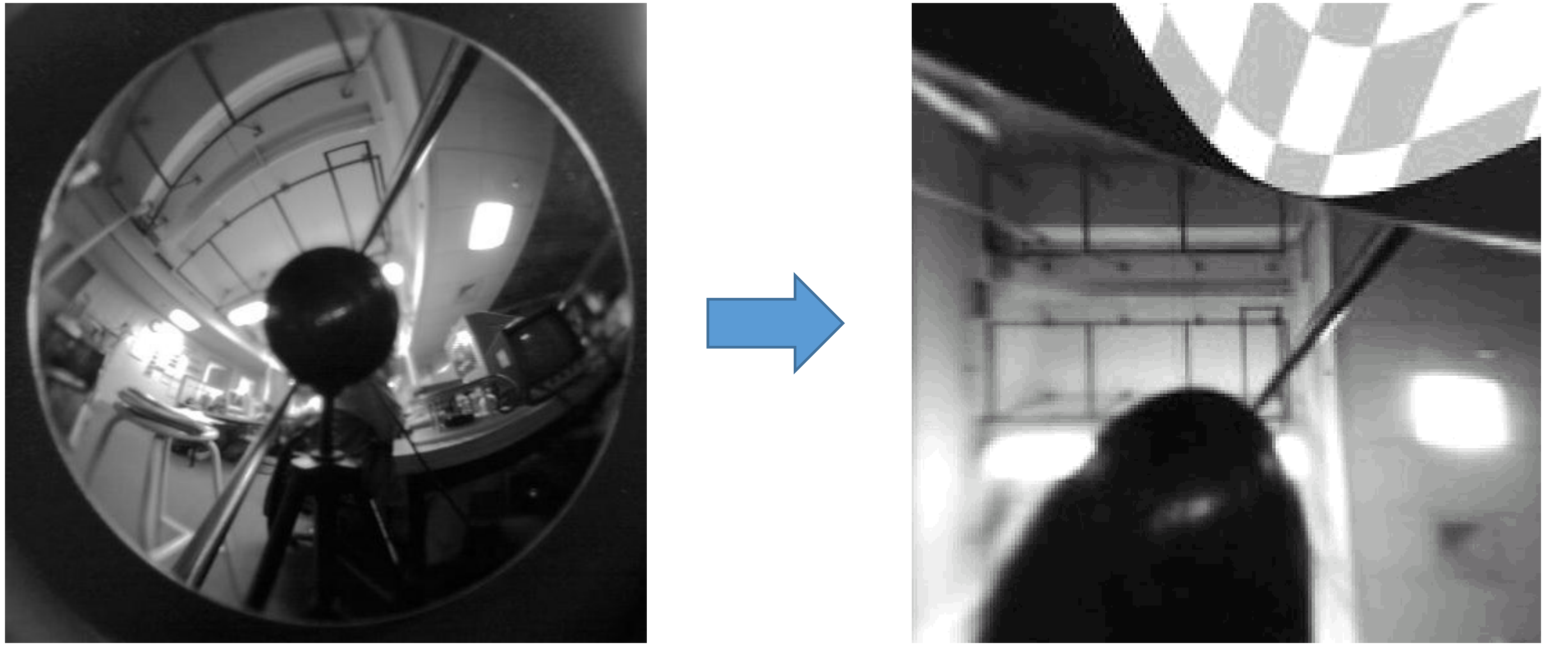
Parabola + Orthographic lens



Why do we prefer central cameras?

Because we can:

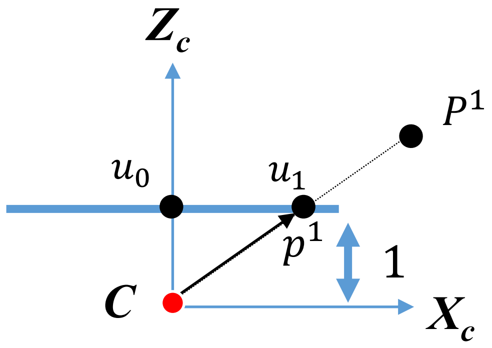
- Apply standard algorithms valid for perspective geometry.
- Unwarp parts of an image into a perspective one
- Transform image points into normalized vectors on the unit sphere



Recall: Normalized Image Coordinates (Lecture 2, slide 62)

- If we pre-multiply both terms of the perspective projection equation in camera frame by K^{-1}, we get:

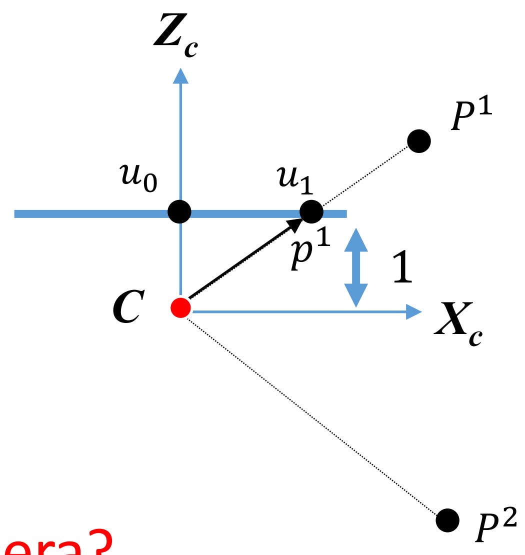
$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = K \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} \quad \Rightarrow \lambda\, K^{-1} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} \quad \Rightarrow \quad \lambda \begin{bmatrix} \dfrac{u - u_0}{\alpha} \\ \dfrac{v - v_0}{\alpha} \\ 1 \end{bmatrix} = \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$



Recall: Normalized Image Coordinates (Lecture 2, slide 62)

- If we pre-multiply both terms of the perspective projection equation in camera frame by K^{-1}, we get:

$$\lambda \begin{bmatrix} \dfrac{u-u_0}{\alpha} \\ \dfrac{v-v_0}{\alpha} \\ 1 \end{bmatrix} = \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$



How do we model world points that lie behind the camera?
The standard pinhole model is not enough.
We need a distortion model.

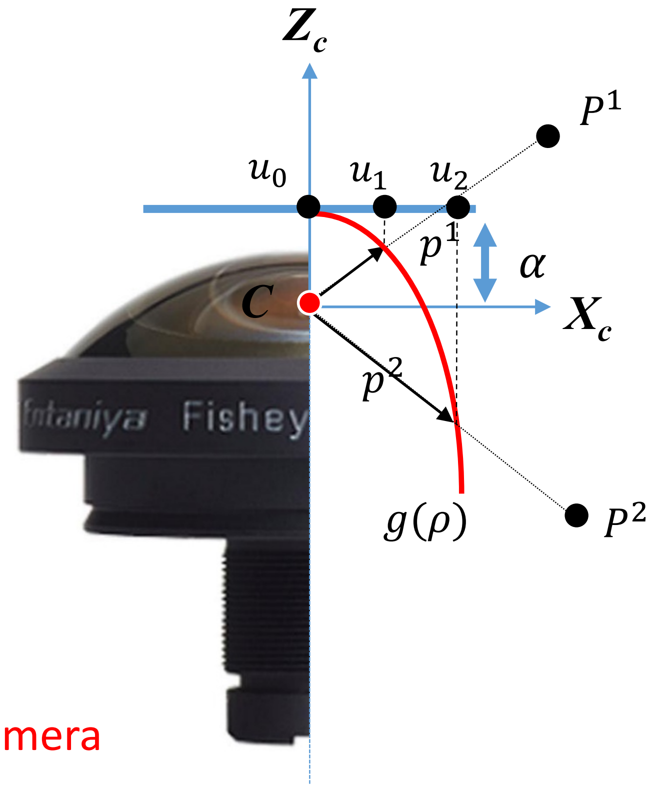
Unified Omnidirectional Camera Model
(for Fisheye and Catadioptric cameras)

- We model the focal length as polynomial function, whose coefficients are the parameters to be estimated
- The coefficients of the polynomial, the intrinsic parameters, and extrinsics are then found via DLT

$$\lambda \begin{bmatrix} \dfrac{u-u_0}{\alpha} \\ \dfrac{v-v_0}{\alpha} \\ g(\rho) \end{bmatrix} = \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix}$$

$$g(\rho) = 1 + a_1\rho + a_2\rho^2 + \ldots + a_N\rho^N$$

$$\rho = \sqrt{(u-u_0)^2 + (v-v_0)^2}$$

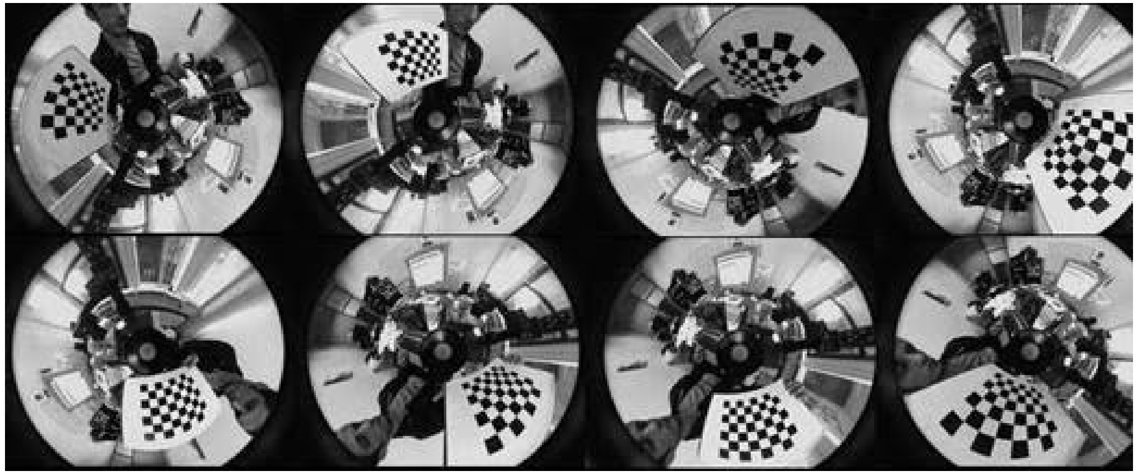


When $a_1, a_2, \ldots, a_N = 0$ then we get a pinhole camera

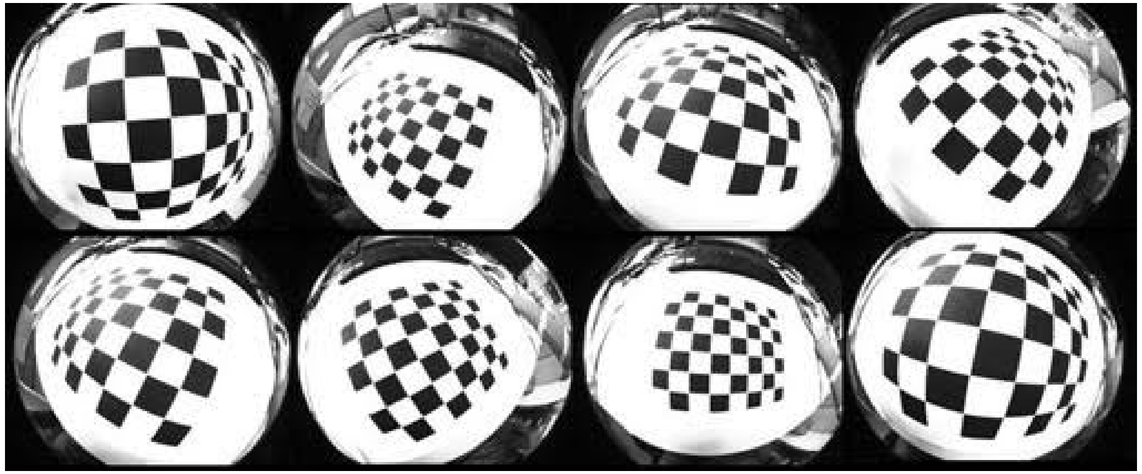
Scaramuzza, Martinelli, Siegwart. A Flexible Technique for Accurate Omnidirectional Camera Calibration and Structure from Motion. ICVS'06. [PDF](#).
Scaramuzza, *Omnidirectional Camera*, chapter of Encyclopedia of Computer Vision, Springer, 2014. [PDF](#)

OCamCalib: Omnidirectional Camera Calibration Toolbox

- Released in 2006, OCamCalib is the standard toolbox for calibrating wide angle cameras (fisheye and catadioptric)
- Since 2015, included in the Matlab Computer Vision Toolbox



Example calibration images of a catadioptric camera

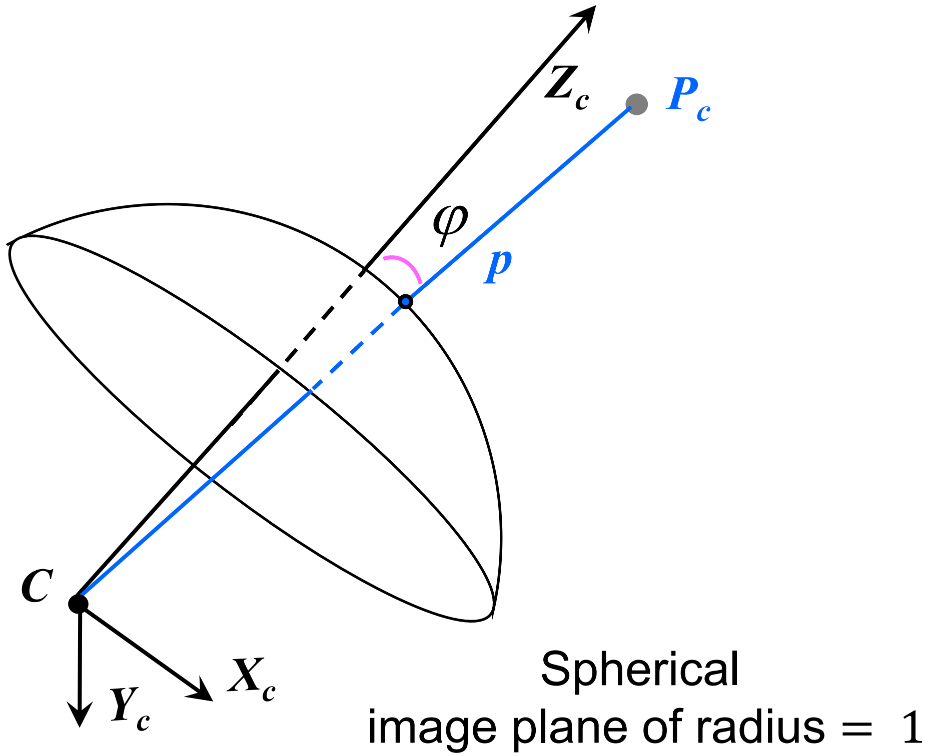


Example calibration images of a fisheye camera

Scaramuzza, Martinelli, Siegwart, *A toolbox for easily calibrating omnidirectional cameras*, IROS'06. PDF

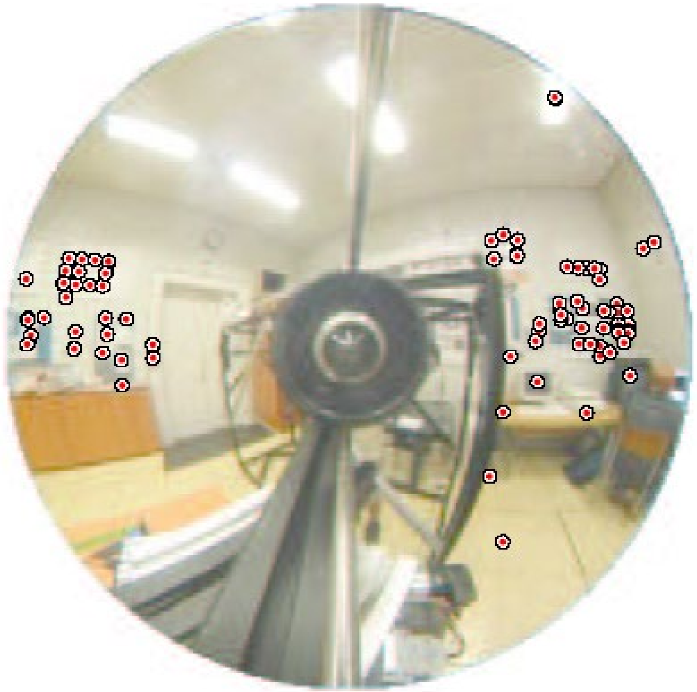
Projection of Image Points on the Unit Sphere

- Always possible after the camera has been calibrated

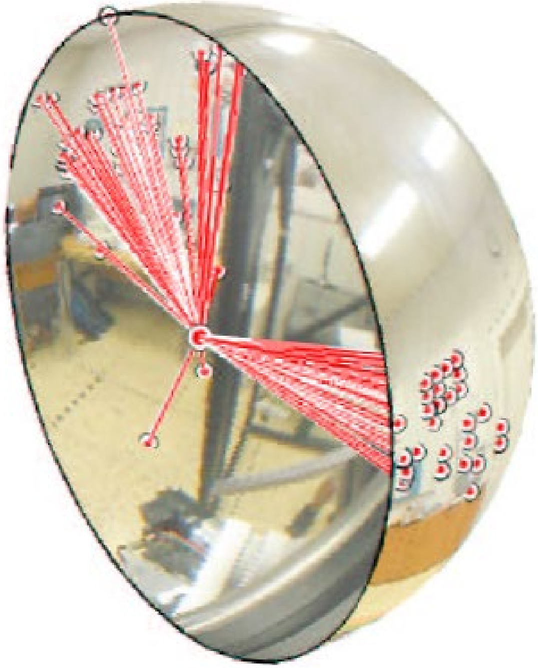


Projection of Image Points on the Unit Sphere

- Always possible after the camera has been calibrated



Points



Rays

Summary (things to remember)

- Calibration from 3D objects: DLT algorithm
- Calibration from planar grids: DLT algorithm using homography projection
- Reprojection Error and non linear optimization
- P3P algorithm
- DLT vs EPNP comparison
- Omnidirectional cameras
    - Central vs non central projection
    - Unified (spherical) model for perspective and omnidirectional cameras

Readings

- Ch. 2.1 of Szeliski book, 2nd Edition
- Chapter 4 of Autonomous Mobile Robots book: link

Understanding Check

Are you able to:

- Describe the differences between Tsai’s and Zhang’s calibration methods
- Explain and derive the DLT in both Tsai’s and Zhang’s methods? What is the minimum number of point correspondences they require?
- Describe the general PnP problem and derive the behavior of its solutions?
- Explain the working principle of the P3P algorithm?
- What is the reprojection error and how is it used for refining the calibration?
- Define central and non central omnidirectional cameras?
- What kind of mirrors ensure central projection?