Solutions to Jürgen Neukirch's
Algebraic Number Theory

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ABSTRACT. This document contains solutions to the exercises of Jürgen Neukirch's *Algebraic Number Theory*.

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