Solutions to John L. Kelley's
General Topology

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Abstract. This document contains solutions to the problems of John L. Kelley's *General Topology*.

Contents

CHAPTER 1

Topological Spaces

Problem 1.A (Largest and Smallest Topologies)**.** (a) Let $\{\mathscr{T}_\alpha\}_{\alpha\in I}$ be a family of topologies for X, and let $\mathscr{T} = \bigcap_{\alpha\in I}\mathscr{T}_\alpha$. Then $\mathscr{T}$ is a family of subsets of X and $\varnothing, X \in \mathscr{T}$ since $\varnothing, X \in \mathscr{T}_\alpha$ for all $\alpha \in I$. An arbitrary union of $\mathscr{T}$-open sets is also an arbitrary union of $\mathscr{T}_\alpha$-open sets for all $\alpha \in I$, and hence is $\mathscr{T}$-open. Similarly, any finite intersection of $\mathscr{T}$-open sets is a finite intersection of $\mathscr{T}_\alpha$-open sets for all $\alpha \in I$, and so is $\mathscr{T}$-open. Thus $\mathscr{T}$ is a topology on X.

(b)

(c) Let $\{\mathscr{T}_\alpha\}_{\alpha\in I}$ be a collection of topologies for X. By part (a), $\bigcap_{\alpha\in I}\mathscr{T}_\alpha$ is a topology for X which is contained in every $\mathscr{T}_\alpha$. If $\mathscr{U}$ is another topology for X contained in every $\mathscr{T}_\alpha$, then $\mathscr{U} \subseteq \bigcap_\alpha \mathscr{T}_\alpha$ and so $\bigcap_\alpha \mathscr{T}_\alpha$ is the (unique) largest topology for X contained in $\mathscr{T}_\alpha$ for all $\alpha \in I$. On the other hand, by Theorem 1.12, there is a unique smallest topology for X containing each $\mathscr{T}_\alpha$ since

$$\bigcup \bigcup_{\alpha\in I} \mathscr{T}_\alpha = \bigcup_{\alpha\in I} \bigcup \mathscr{T}_\alpha = X.$$

Problem 1.B (Topologies From Neighborhood Systems)**.** (a) (i) This holds by definition of $\mathscr{U}_x$.

(ii) Let $x \in U' \subseteq U$ and $x \in V' \subseteq V$ with U' and V' open. Then $x \in U' \cap V' \subseteq U \cap V$ with $U' \cap V'$ open, so $U \cap V \in \mathscr{U}_x$. (This is part of Theorem 1.2.)

(iii) Let U' be an open set for which $x \in U' \subseteq V$. Then $x \in U' \subseteq V$ and so $V \in \mathscr{U}_x$. (This is also part of Theorem 1.2.)

(iv) Let V be an open set for which $x \in V \subseteq U$, so in particular $V \in \mathscr{U}_x$ and $V \subseteq U$. Moreover, V is a neighborhood of each of its points, and so $V \in \mathscr{U}_y$ for all $y \in V$.

(b) Suppose that conditions (i), (ii), and (iii) above are satisfied. If $U \in \mathscr{T}$, then for all $x \in U$ we have $U \in \mathscr{U}_x$ and hence $x \in X$. Thus $\mathscr{T}$ is a family of subsets of X. We have that $\varnothing \in \mathscr{T}$ vacuously. From (iii) and the fact that each $\mathscr{U}_x$ is nonempty, $X \in \mathscr{U}_x$ for all $x \in X$ and thus $X \in \mathscr{T}$. Let $\mathscr{U}$ be a subfamily of $\mathscr{T}$. Then for all $x \in \bigcup\mathscr{U}$, there is $U \in \mathscr{U}$ such that $x \in U$ and so $U \in \mathscr{U}_x$. Thus $\bigcup\mathscr{U} \in \mathscr{U}_x$ by (iii), and so $\bigcup\mathscr{U} \in \mathscr{T}$. Finally if $U, V \in \mathscr{T}$ then for all $x \in U \cap V$, we have $U \in \mathscr{U}_x$ and $V \in \mathscr{U}_x$. Then by (ii), $U \cap V \in \mathscr{U}_x$ and so $U \cap V \in \mathscr{T}$. (This part does not actually use (i).)

Now suppose also that (iv) holds, and fix $x \in X$. If U is a $\mathscr{T}$-neighborhood of x, then there is $V \in \mathscr{T}$ such that $x \in V \subseteq U$. Thus $V \in \mathscr{U}_x$, and so $U \in \mathscr{U}_x$ by (iii) (this implication does not require (iv)). Conversely, if $U \in \mathscr{U}_x$ then by (iv), there is $V \in \mathscr{U}_x$ such that $V \subseteq U$ and $V \in \mathscr{U}_y$ for all $y \in V$. Thus $V \in \mathscr{T}$. By (i), $x \in V \subseteq U$ so U is a $\mathscr{T}$-neighborhood of x. Hence $\mathscr{U}_x$ is the $\mathscr{T}$-neighborhood system of x.

Problem 1.C (Topologies From Interior Operators)**.** Suppose first that $\mathscr{T}$ is a topology for X. Then the following statements are satisfied by the interior operator ^{i} associated to $\mathscr{T}$:

(a) $X^i = X$. (Proof: This is clear since $X \in \mathscr{T}$, and hence X is a neighborhood of every $x \in X$.)

(b) For each $A \subseteq X$, $A^i \subseteq A$. (Proof: By definition, A^i consists only of points of A.)

(c) For each $A \subseteq X$, $A^{ii} = A^i$. (Proof: We know by Theorem 1.9 that A^i is open, and hence also $A^{ii} = A^i$.)

(d) For each $A, B \subseteq X$, $(A \cap B)^i = A^i \cap B^i$. (Proof: Let $x \in (A \cap B)^i$. Then $A \cap B$ is a neighborhood of x, and so from $A \cap B \subseteq A, B$ we have that A and B are neighborhoods of x by Theorem 1.2. Hence $x \in A^i \cap B^i$. On the other hand, if $x \in A^i \cap B^i$ then A and B are neighborhoods of x. Thus by Theorem 1.2, $A \cap B$ is a neighborhood of x, so $x \in (A \cap B)^i$.)

(Note that these are "dual" to the Kuratowski closure axioms of Theorem 1.8, using that $X \setminus A^i = \overline{X \setminus A}$ (Theorem 1.9). This provides an alternative proof of (a)-(d), and also shows that the claim below is equivalent to Theorem 1.8; in fact, the arguments below are dual to the proof of Theorem 1.8.)

We claim that, conversely, if ${}^i : 2^X \to 2^X$ is any function satisfying conditions (a)-(d) above then the family $\mathscr{T}$ of all $A \subseteq X$ for which $A^i = A$ is a topology for X such that ^{i} is the interior operator associated to $\mathscr{T}$. We have from (a) that $X \in \mathscr{T}$, and from (b) that $\varnothing \in \mathscr{T}$. If $U, V \in \mathscr{T}$, then

$$(U \cap V)^i = U^i \cap V^i = U \cap V$$

by (d), and so $U \cap V \in \mathscr{T}$. Observe that if $A \subseteq B \subseteq X$, then $A = A \cap B$ so

$$A^i = (A \cap B)^i = A^i \cap B^i$$

by (d). Thus $A^i \subseteq B^i$. Now suppose $\mathscr{U}$ is a subfamily of $\mathscr{T}$. Then for all $U \in \mathscr{U}$, we have $U \subseteq \bigcup \mathscr{U}$ and so

$$U = U^i \subseteq \left(\bigcup \mathscr{U}\right)^i.$$

Hence $\bigcup \mathscr{U} \subseteq \bigcup \mathscr{U}^i$. By (b), it follows that $(\bigcup \mathscr{U})^i = \bigcup \mathscr{U}$ and so $\bigcup \mathscr{U} \in \mathscr{T}$. Then $\mathscr{T}$ is a topology for X. Let ${}^\circ$ denote the interior operator associated to $\mathscr{T}$. Then for all $A \subseteq X$, A° is open by Theorem 1.9 and so $(A^\circ)^i = A^\circ$. Then $A^\circ \subseteq A^i$ since $A^\circ \subseteq A$. On the other hand, by (c), A^i is open and thus $A^i \subseteq A^\circ$ by Theorem 1.9. Hence $A^i = A^\circ$; that is, ^{i} is the interior operator associated to $\mathscr{T}$.

Problem 1.D (Accumulation Points in T_1-Spaces)**.** (a) By Theorem 1.12, there is a smallest topology $\mathscr{T}$ for X which contains $X \setminus \{x\}$ for all $x \in X$, which is also the smallest topology for X for which every singleton is closed. That is, $\mathscr{T}$ it is the smallest topology such that $(X, \mathscr{T})$ is a T_1-space. (Alternatively, we may apply Problem 1.A(a): an arbitrary intersection of T_1-topologies for X is again a T_1-topology for X.)

(b) We have by part (a) that $\mathscr{T}$ consists of arbitrary unions of finite intersections of complements of singletons in X, so $\mathscr{T}$ consists of $\varnothing$ and the complements of finite subsets of X. Now suppose $A \subseteq X$ is not $\varnothing$ or X. Then A is open if and only if $X \setminus A$ is finite and A is closed if and only if A is finite. Since $X = (X \setminus A) \cup A$ is infinite, it follows that A cannot be both open and closed. Thus X is connected.

(c) Let $A \subseteq X$, and let A' denote the set of accumulation points of A. Let $x \in X \setminus A'$. Then there is an open neighborhood U of x such that $U \cap (A \setminus \{x\}) = \varnothing$. Suppose for sake of contradiction that $U \cap A' \neq \varnothing$. Then for $y \in U \cap A'$, we have $U \cap (A \setminus \{y\}) \neq \varnothing$. But $U \cap (A \setminus \{y\}) \subseteq U \cap A$ and $U \cap (A \setminus \{x\}) = \varnothing$, so $U \cap (A \setminus \{y\}) = \{x\}$. Then since $(X, \mathscr{T})$ is a T_1-space, $U \setminus \{x\}$ is a neighborhood of y such that $(U \setminus \{x\}) \cap (A \setminus \{y\})$. This contradicts that $y \in A'$, and so $U \cap A' = \varnothing$. Thus x is not an accumulation point of A'. By Theorem 1.5, it follows that A' is closed.

Problem 1.E (Kuratowski Closure and Complement Problem)**.**

Problem 1.F (Exercise on Spaces With a Countable Base)**.** Let $\mathscr{B}$ be a countable base for a topological space $(X, \mathscr{T})$, and suppose $\mathscr{A}$ is another base. Let S be the set of all pairs (B, C) such that $B, C \in \mathscr{B}$ and there is $A \in \mathscr{A}$ such that $B \subseteq A \subseteq C$. Then S is countable by Theorem 0.17, and for all $(B, C) \in S$, let $A_{B,C} \in \mathscr{A}$ such that $B \subseteq A_{B,C} \subseteq C$. By Theorem 0.16, $\mathscr{A}' = \{A_{B,C} \mid (B, C) \in S\}$ is a countable subfamily of $\mathscr{A}$. Now let U be open and $x \in U$. Since $\mathscr{A}$ and $\mathscr{B}$ are bases for $\mathscr{T}$, there are $A \in \mathscr{A}$ and $B, C \in \mathscr{B}$ such that

$$x \in B \subseteq A \subseteq C \subseteq U.$$

Then $(B, C) \in S$ and so $x \in A_{B,C} \subseteq U$ with $A_{B,C} \in \mathscr{A}'$. Thus $\mathscr{A}'$ is also a base for $\mathscr{T}$, proving the claim.

Problem 1.G (Exercise on Dense Sets)**.** Let $x \in U$, and suppose V is a neighborhood of x. Then by Theorem 1.1 and Theorem 1.2, $U \cap V$ is a neighborhood of x and so $U \cap V$ intersects A as A is dense in X. Then since $U \cap V \subseteq U$, we also have that $U \cap V$ intersects $A \cap U$ and so V intersects $A \cap U$. Thus $x \in (A \cap U)^-$, which shows that $U \subseteq (A \cap U)^-$.

Problem 1.H (Accumulation Points)**.** For each $x \in A \setminus B$, let U_x be an open neighborhood of x for which $U_x \cap A$ is countable. Then $\{U_x \cap (A \setminus B)\}_{x \in A \setminus B}$ is an open cover of $A \setminus B$ in the subspace topology. Then since $A \setminus B$ is Lindelöf, there is a countable subset C of $A \setminus B$ for which $\{U_x \cap (A \setminus B)\}_{x \in C}$ is also an open cover of $A \setminus B$. But each $U_x \cap (A \setminus B)$ is countable by Theorem 0.15, and so $A \setminus B$ is countable by Theorem 0.17. Now let $x \in B$ and U be a neighborhood of x. If U intersects only countably many points of B, then $U \cap A = (U \cap B) \cup (U \cap (A \setminus B))$ is countable by Theorems 0.15 and 0.17. But $U \cap A$ is uncountable since $x \in B$, so this is a contradiction. Thus $U \cap B$ is uncountable; that is, any neighborhood of a point of B intersects uncountably many points of B.

Problem 1.I (The Order Topology)**.** (Note: We assume that X consists of at least two points, for else the given "subbase" does not have union equal to X and is hence not a subbase of any topology for X.)

(a) Let $\mathscr{T}$ denote the order topology for X associated to $<$. Then if $a, b \in X$ with $a < b$, let $U = \{x \in X \mid x < a\}$ and $V = \{x \in X \mid a < x\}$. Then $U, V \in \mathscr{T}$ and if $x \in U$ and $y \in V$, we have $x < a < y$ and so $x < y$. On the other hand, suppose $\mathscr{U}$ is a topology for X such that for all $a, b \in X$ with $a < b$ there are neighborhoods U of a and V of b such that for $x \in U$ and $y \in V$, $x < y$. Fix $a \in X$. Then if $x \in X$ such that $x < a$, there is a $\mathscr{U}$-open neighborhood U of x such that $U \subseteq \{x \in X \mid x < a\}$. Thus $\{x \in X \mid x < a\}$ is $\mathscr{U}$-open. Similarly, if $x \in X$ with $a < x$, there is a $\mathscr{U}$-open neighborhood V of x such that $V \subseteq \{x \in X \mid a < x\}$ so $\{x \in X \mid a < x\}$ is $\mathscr{U}$-open. Then $\mathscr{T} \subseteq \mathscr{U}$ by definition of $\mathscr{T}$.

(b)

(c) Let A be a nonempty subset of X which has an upper bound. Let $U = \bigcup_{a \in A}\{x \in X \mid x < a\}$. Then U is open by definition of $\mathscr{T}$. If $U = \varnothing$, then there are no $a, b \in A$ for which $a < b$. It follows that A is a singleton and thus has a supremum. Now suppose $U \neq \varnothing$. Since $<$ is antisymmetric, any upper bound of A is not in U, and so $U \neq X$. Thus since X is connected, it follows that U is not closed. By Theorem 1.5, there is an accumulation point x of U which is not contained in U. Then $x \in \bigcap_{a \in A}\{x \in X \mid x \not< a\}$ and so for all $a \in A$, $x > a$ or $x = a$. Thus x is an upper bound for A. If $y < x$, then $\{x \in X \mid y < x\}$ is an open neighborhood of x, which must intersect A. Thus there is $a \in A$ such that $y < a$, and so y is not an upper bound of A since $<$ is antisymmetric. Hence x is the supremum of A.

(d) Let $A = \{x \in X \mid a < x\}$ and $B = \{x \in X \mid x < b\}$. Then A and B are nonempty open subsets of X (as $b \in A$ and $a \in B$), and they are disjoint by the assumption on a and b. If x is any point of X, then $a < x$, $a = x$, or $x < a$. If $a < x$, then $x \in A$. If $a = x$ or $x < a$, then $x \in B$ since $a < b$. Thus $X = A \cup B$ is a separation of X, so X is not connected.

Together with part (c), we have that if X is connected under the order topology then X is order-complete and has no gaps. Conversely, suppose that X is order-complete with no gaps. Then

Problem 1.J (Properties of the Real Numbers)**.** (a)

(b)

(c)

(d)

(e)

Problem 1.K (Half-Open Interval Space)**.** (a)

(b)

(c)
(d)
(e)

Problem 1.L (Half-Open Rectangle Space). (a)
(b)
(c)

Problem 1.M (Example (the Ordinals) on 1st and 2nd Countability). (a)
(b)
(c)

Problem 1.N (Countable Chain Condition). Let $(X, \mathscr{T})$ be a separable topological space and let $\mathscr{U}$ be a disjoint family of open subsets of X. Let $\mathscr{U}'$ denote the nonempty elements of $\mathscr{U}$, and let C be a countable dense subset of X. For each $U \in \mathscr{U}'$, let $x_U \in U \cap C$. If $U, V \in \mathscr{U}'$ such that $x_U = x_V$, then $(U \cap C) \cap (V \cap C)$ is nonempty. Hence $U \cap V \neq \varnothing$, so $U = V$. Thus $U \mapsto x_U$ is an injective function $\mathscr{U}' \to C$, and so $\mathscr{U}'$ is countable by Theorem 0.15. Then $\mathscr{U} \subseteq \{\varnothing\} \cup \mathscr{U}'$ is countable by Theorems 0.15 and 0.17.

Now let X be an uncountable set and $\mathscr{T}$ be the collection of complements of countable subsets of X, along with the empty set. By Theorems 0.15, 0.17, and 1.4, $\mathscr{T}$ is a topology for X. Suppose U and V are disjoint open subsets of X. Then $(X \setminus U) \cup (X \setminus V) = X$, so U or V is the empty set by Theorem 0.17. Thus any disjoint subfamily $\mathscr{U}$ of $\mathscr{T}$ consists of at most two sets, and so $(X, \mathscr{T})$ satisfies the countable chain condition. But for any countable subset C of X, $X \setminus C$ is a nonempty open subset of X which is disjoint from C. Hence X is not separable.

Problem 1.O (The Euclidean Plane). (a)
(b)

Problem 1.P (Example on Components).

Problem 1.Q (Theorem on Separated Sets).

Problem 1.R (Finite Chain Theorem for Connected Sets). Let $C = \bigcup \mathscr{A}$, and suppose D is a subset of C which is both open and closed in C. Then for all $A \in \mathcal{A}$, we have that $D \cap A$ is both open and closed in A. Since A is connected, it follows that $D \cap A = A$ or $D \cap A = \varnothing$. Thus $A \subseteq D$ or $A \subseteq C \setminus D$. Suppose for sake of contradiction that there are $A, B \in \mathscr{A}$ such that $A \subseteq D$ and $B \subseteq C \setminus D$. Let $A_0, \ldots, A_n \in \mathscr{A}$ such that $A_0 = A$, $A_n = B$, and for all $i = 0, \ldots, n-1$, A_i and A_{i+1} are not separated. Then there is i such that $A_i \subseteq D$ and $A_{i+1} \subseteq C \setminus D$. Thus A_i and A_{i+1} are disjoint, and since D and $C \setminus D$ are closed in C we have that $A_i = (A_i \cap A_{i+1}) \cap D$ and $A_{i+1} = (A_i \cap A_{i+1}) \cap (C \setminus D)$ are closed in $A_i \cup A_{i+1}$. Hence A_i and A_{i+1} are separated, a contradiction. Thus either $A \subseteq D$ for all $A \in \mathscr{A}$ or $A \subseteq C \setminus D$ for all $A \in \mathscr{A}$, so $D = C$ or $D = \varnothing$. Then C is connected.

If $\mathcal{A}$ is a family of connected subsets of a topological space such that no two members of $\mathcal{A}$ are separated, then it is clear that the finite chain condition is satisfied by $\mathcal{A}$. Hence $\bigcup \mathcal{A}$ is connected, which proves Theorem 1.21.

Problem 1.S (Locally Connected Spaces). (a) Let U be an open set and C a component of U. Then for all $x \in C$, C is the component of U containing x and hence is a neighborhood of x. Then C is a neighborhood of each of its points, and is thus open by Theorem 1.1.

(b) Suppose that $(X, \mathscr{T})$ is a locally connected topological space. Then if U is open and $x \in U$, we have by part (a) that the connected component C of U containing x is open. In particular, C is an open connected subset of X for which $x \in C \subseteq U$. Thus the open connected subsets form a base for $\mathscr{T}$. Conversely, suppose the family of open connected subsets is a base for $\mathscr{T}$. Let $x \in X$, and let U be a neighborhood of x. Then there is an open connected set C such that $x \in C \subseteq U$. By the proof of Theorem 1.22, the connected component of U containing x also contains C, and so is a neighborhood of x. Hence X is locally connected.

(c) Let A be the component of X containing x. By part (a), A is open, and by Theorem 1.22, A is closed. Then setting $B = X \setminus A$, we have that $X = A \cup B$ is a separation of X such that $x \in A$ and $y \in B$.

Problem 1.T (The Brouwer Reduction Theorem)**.** (a) We restate the theorem as follows: Let $(X, \mathscr{T})$ be a topological space such that every subspace of X is Lindelöf (in particular, this holds for any second countable space by Theorem 1.15 since any subspace of a second countable space is second countable). Let $\mathscr{P}$ be a family of subsets of X which is closed under intersections of countable nests of closed sets. Then for any closed set $A \in \mathscr{P}$, there is a closed subset B of A such that $B \in \mathscr{P}$ and no proper closed subset of B is in $\mathscr{P}$.

Proof: Suppose A is a closed subset of X, and let $\mathscr{A}$ be the family of closed subsets of A which lie in $\mathscr{P}$. Let $\mathscr{N} \subseteq \mathscr{A}$ be a nest. Then

$$X \setminus \left(\bigcap \mathscr{N}\right) = \bigcup_{N \in \mathscr{N}} (X \setminus N).$$

Since $\mathscr{A}$ consists only of closed subsets of X, we thus have that $\{X \setminus N \mid N \in \mathscr{N}\}$ is an open cover of $X \setminus (\bigcap \mathscr{N})$. But $X \setminus (\bigcap \mathscr{N})$ is Lindelöf, and so there is a countable subnest $\mathscr{M}$ of $\mathscr{N}$ such that

$$X \setminus \left(\bigcap \mathscr{N}\right) = \bigcup_{M \in \mathscr{M}} (X \setminus M).$$

Taking complements,

$$\bigcap \mathscr{N} = \bigcap \mathscr{M}.$$

But $\bigcap \mathscr{M} \in \mathscr{P}$ by the assumption on $\mathscr{P}$, and hence $\bigcap \mathscr{N} \in \mathscr{P}$. If $\mathscr{N}$ is nonempty, then since $\mathscr{N} \subseteq \mathscr{A}$ we have that $\bigcap \mathscr{N}$ is a closed subset of A. Hence $\bigcap \mathscr{N}$ is a lower bound of $\mathscr{N}$ in $\mathscr{A}$. Otherwise $\mathscr{N}$ is empty, and so A is a lower bound of $\mathscr{N}$ in $\mathscr{A}$. Hence every nest in $\mathscr{A}$ has a lower bound, and so by Theorem 0.25(b) (the Minimal Principle), it follows that $\mathscr{A}$ has a minimal element. That is, there is a closed subset B of A in $\mathscr{P}$ for which no proper closed subset of B lies in $\mathscr{P}$.

(b)

CHAPTER 2

Moore–Smith Convergence

Problem 2.A (Exercise on Sequences)**.**

Problem 2.B (Example: Sequences are Inadequate)**.**

Problem 2.C (Exercise on Hausdorff Spaces: Door Spaces)**.** Let $(X, \mathscr{T})$ be a Hausdorff door space, and let s be an accumulation point of X. Then by Theorem 2.2(a), there is a net in $X \setminus \{s\}$ converging to s. Hence by Theorem 2.2(c), $X \setminus \{s\}$ is not closed and thus $\{s\}$ is not open. Since X is a door space, it follows that $\{s\}$ is open and so $X \setminus \{s\}$ is closed. Now if $t \in X \setminus \{s\}$ is also an accumulation point of X, we have that $\{t\}$ is open and hence $X \setminus \{s,t\}$ is closed.

Problem 2.D (Exercise on Subsequences)**.** If $m \in \omega$, then $\{i \in \omega \mid N_i < m\}$ is finite and so it has a maximal element n. Then if $p \in \omega$ with $p \geq n+1$, we have $p \notin \{i \in \omega \mid N_i < m\}$ and hence $N_p \geq m$. Thus for any sequence S, we have that $S \circ N$ is a subsequence of S.

Now suppose N is a sequence of nonnegative integers such that $S \circ N$ is not a subsequence of S. Then by definition of a subsequence, there is $m \in \omega$ for which the set of $i \in \omega$ with $N_i \geq m$ is cofinal in ω. By well-ordering of ω, there is a least such m. Then by choice of m, $\{i \in \omega \mid N_i \leq m-1\}$ is not cofinal in ω, and so it is bounded above. Thus $\{i \in \omega \mid N_i = m\}$ is cofinal in ω, so also $\{i \in \omega \mid S_{N_i} = S_m\}$ is cofinal in ω. Then S_m is a cluster point of $S \circ N$.

Problem 2.E (Example: Cofinal Subsets are Inadequate)**.** (a) Since every point of X other than $(0,0)$ is open,it suffices to show that if $x \in X \setminus \{(0,0)\}$ then x and $(0,0)$ can be separated by disjoint neighborhoods. But $\{x\}$ is open and also $X \setminus \{x\}$ is an open neighborhood of $(0,0)$ since for all $m \in \omega$, $\{n \in \omega \mid (m,n) \in \{x\}\}$ has at most one element.

(b) We showed in part (a) that if $x \in X \setminus \{(0,0)\}$ then $\{x\}$ is closed and so the claim holds for x. On the other hand, $\{x\}$ is open for all $x \in X \setminus \{(0,0)\}$ by definition of the topology and so $\{(0,0)\}$ is the intersection of the closed neighborhoods $X \setminus \{x\}$ for $x \in X \setminus \{(0,0)\}$. Since X is countable by Theorem 0.17, we have by Theorem 0.15 that $X \setminus \{(0,0)\}$ is countable and so $\{(0,0)\}$ is a countable intersection of closed neighborhoods.

(c) Let $\mathscr{U}$ be an open cover of X, and for each $x \in X$ let $U_x \in \mathscr{U}$ such that $x \in U_x$. Since X is countable, $\{U_x \mid x \in X\}$ is countable by Theorem 0.16. Then $\{U_x \mid x \in X\}$ is a countable subcover of $\mathscr{U}$, so X is Lindelöf.

(d)

(e)

Problem 2.F (Monotone Nets)**.** (Note: We assume that $>$ is antisymmetric, as in Problem 1.I.)

(a) Let $\{S_n, n \in D, \succ\}$ be a monotone increasing net with bounded range, and let s be the supremum of the range of S. Let $a, b \in X$ such that $a < s < b$. Then a is not an upper bound of the range of S since $<$ is antisymmetric. Thus there is $n \in D$ for which $a < S_n$. Then for $m \in D$ with $m \succ n$, $S_m \geq S_n$. Hence $a < S_m$. Since s is an upper bound of the range of S, we have $a < S_m \leq s$ and so $a < S_m < b$

for $m \succ n$. Then S is eventually in $\{x \in X \mid a < x < b\}$. But the collection of $\{x \in X \mid a < x < b\}$ for $a, b \in X$ is a base for the order topology on X and so the collection of $\{x \in X \mid a < x < b\}$ for $a, b \in X$ with $a < s < b$ is a local base at s. Then S converges to s.

(b)

Problem 2.G (Integration Theory, Junior Grade)**.** (Note: Since $\mathbf{R}$ is Hausdorff, we have by Theorem 2.3 that nets in $\mathbf{R}$ converge to at most one point. Thus if f is summable over A, we can unambiguously write $\sum_A f$ for the unique point to which S converges.)

(a) Suppose that f is nonnegative and that $\{S_F \mid F \in \mathscr{A}\}$ is bounded above. The net S is monotone when $\mathbf{R}$ is linearly ordered by $>$, for if $G \supseteq F$ with $F, G \in \mathscr{A}$ then $S_F \geq S_G$. Since $(\mathbf{R}, >)$ is order-complete, we thus have by Problem 2.F(a) that S converges to the supremum of $\{S_F \mid F \in \mathscr{A}\}$. Conversely, suppose for sake of contradiction that f is nonnegative and summable but that $\{S_F \mid F \in \mathscr{A}\}$ is not bounded above. Let S converge to $s \in \mathbf{R}$. Let $a, b \in \mathbf{R}$ such that $a < s < b$. There exists $F \in \mathscr{A}$ such that $S_F \geq b$, since $\{S_F \mid F \in \mathscr{A}\}$ is not bounded above. For any $G \in \mathscr{A}$ with $G \supseteq F$, we have since f is nonnegative that $S_G \geq S_F$ and thus $S_G \geq b$. Since $(\mathscr{A}, \supseteq)$ is a directed set, it follows that S is not eventually in (a, b). But (a, b) is a neighborhood of s, and so this contradicts the convergence of S to s. Hence $\{S_F \mid F \in \mathscr{A}\}$ must be bounded above.

We now obtain the analogous result for nonpositive f by replacing the linear order $>$ on $\mathbf{R}$ with $<$. Let f be nonpositive and suppose that $\{S_F \mid F \in \mathscr{A}\}$ is bounded below. We have that S is a monotone net when $\mathbf{R}$ is linearly ordered by $<$: if $F, G \in \mathscr{A}$ such that $G \supseteq F$, then $S_G \leq S_F$. Thus by Problem 2.F(a), S converges to the infimum of $\{S_F \mid F \in \mathscr{A}\}$. Conversely, let f be nonpositive and summable but suppose for sake of contradiction that $\{S_F \mid F \in \mathscr{A}\}$ is not bounded below. Let S converge to $s \in \mathbf{R}$. If $a, b \in \mathbf{R}$ such that $a < s < b$, there exists $F \in \mathscr{A}$ such that $S_F \leq a$ since $\{S_F \mid F \in \mathscr{A}\}$ is not bounded below. Then for $G \in \mathscr{A}$ with $G \supseteq F$, we have $S_G \leq S_F$ so $S_G \leq a$. Now since $\mathscr{A}$ is directed by $\supseteq$, S is not eventually in (a, b), a contradiction. Hence $\{S_F \mid F \in \mathscr{A}\}$ is bounded below.

(b) Let $\mathscr{A}_+$ denote the family of finite subsets of A_+ and $\mathscr{A}_-$ the family of finite subsets of $\mathscr{A}_-$. Then $\sum_F f \geq 0$ for all $F \in \mathscr{A}_+$ and $\sum_F f \leq 0$ for all $F \in \mathscr{A}_-$. Suppose first that f is summable over A, and suppose for sake of contradiction that $\{S_{F_+} \mid F_+ \in \mathscr{A}_+\}$ is not bounded above. For $a \in \mathbf{R}$ such that $\sum_A f < a$, there is $F \in \mathscr{A}$ such that $S_G < a$ for all $G \in \mathscr{A}$ with $G \supseteq F$. Since F is finite, it has a finite number of subsets and hence there is $b \in \mathbf{R}$ such that $b \leq S_{F'}$ for all $F' \subseteq F$. Then there is $F_+ \in \mathscr{A}_+$ such that $S_{F_+} \geq a - b$. We have that $F_+ \cup F \in \mathscr{A}$ contains F, so $S_{F_+ \cup F} < a$. Since

$$S_{F_+ \cup F} = S_{F_+} + S_{F \setminus F_+}$$

and $F \setminus F_+ \subseteq F$, it follows that

$$a = (a - b) + b \leq S_{F_+} + S_{F \setminus F_+} < a,$$

a contradiction. Hence $\{S_{F_+} \mid F_+ \in \mathscr{A}_+\}$ is bounded above. If $\{S_{F_-} \mid F_- \in \mathscr{A}_-\}$ is not bounded below, let $a \in \mathbf{R}$ such that $a < \sum_A f$ and let $b \in \mathbf{R}$ such that $S_{F'} \leq b$ for all $F' \subseteq F$. Then there is $F \in \mathscr{A}$ such that $a < S_G$ for all $G \in \mathscr{A}$ with $G \supseteq F$, and $F_- \in \mathscr{A}_-$ such that $S_{F_-} \leq a - b$. Then $F_- \cup F \in \mathscr{A}$ contains F, so $a < S_{F_- \cup F}$. But

$$S_{F_- \cup F} = S_{F_-} + S_{F \setminus F_-}$$

and $F \setminus F_- \subseteq F$ implies

$$a < S_{F_-} + S_{F \setminus F_-} \leq (a - b) + b = a.$$

This is a contradiction, so $\{S_{F_-} \mid F_- \in \mathscr{A}_-\}$ is bounded below. By part (a), we thus have that f is summable on A_+ and A_-.

Now suppose f is summable over A_+ and A_-. For convenience, let $s_+ = \sum_{A_+} f$ and $s_- = \sum_{A_-} f$. Let $a, b \in \mathbf{R}$ such that $a < s_+ + s_- < b$. We have

$$\frac{a + s_+ - s_-}{2} = s_+ + \frac{a - s_+ - s_-}{2} < s_+ < s_+ + \frac{b - s_+ - s_-}{2} = \frac{b + s_+ - s_-}{2}$$

and

$$\frac{a - s_+ + s_-}{2} = s_- + \frac{a - s_+ - s_-}{2} < s_- < s_- + \frac{b - s_+ - s_-}{2} = \frac{b - s_+ + s_-}{2}.$$

Then since $\{S_F, F \in \mathscr{A}_+, \supseteq\}$ converges to s_+ and $\{S_F, F \in \mathscr{A}_-, \supseteq\}$ converges to s_-, are $F_+ \in \mathscr{A}_+$ and $F_- \in \mathscr{A}_-$ such that for $G_+ \in \mathscr{A}_+$ with $G_+ \supseteq F_+$ and $G_- \in \mathscr{A}_-$ with $G_- \supseteq F_-$,

$$\frac{a + s_+ - s_-}{2} < S_{G_+} < \frac{b + s_+ - s_-}{2}$$

and

$$\frac{a - s_+ + s_-}{2} < S_{G_-} < \frac{b - s_+ + s_-}{2}.$$

Now let $F = F_+ \cup F_-$. Then for $G \in \mathscr{A}$ with $G \supseteq F$, let

$$G_+ = \{a \in G \mid f(a) \geq 0\}$$

and

$$G_- = \{a \in G \mid f(a) < 0\}.$$

Then $G_+ \subseteq A_+$ and $G_- \subseteq A_-$, and from $G_+, G_- \subseteq G$ we have that G_+ and G_- are finite. Thus $G_+ \in \mathscr{A}_+$ and $G_- \in \mathscr{A}_-$. Moreover, f is nonnegative on F_+ and $F_+ \subseteq F \subseteq G$, so $G_+ \supseteq F_+$. Similarly, $G_- \supseteq F_-$. Then

$$\frac{a + s_+ - s_-}{2} < S_{G_+} < \frac{b + s_+ - s_-}{2}$$

and

$$\frac{a - s_+ + s_-}{2} < S_{G_-} < \frac{b - s_+ + s_-}{2}$$

by the choice of F_+ and F_-. But G is the disjoint union of G_+ and G_-, so $S_G = S_{G_+} + S_{G_-}$. Then from

$$\frac{a + s_+ - s_-}{2} + \frac{a - s_+ + s_-}{2} = a$$

and

$$\frac{b + s_+ - s_-}{2} + \frac{b - s_+ + s_-}{2},$$

we conclude that $a < S_G < b$. Hence S is eventually in (a, b), so S converges to $s_+ + s_-$. That is, f is summable over A and

$$\sum_A f = \sum_{A_+} f + \sum_{A_-} f.$$

In particular, if f is summable over A, then f is summable over A_+ and A_- and so

$$\sum_A f = \sum_{A_+} f + \sum_{A_-} f.$$

(c) Suppose f is summable over A. Then f is summable over A_+ and A_- by part (b). We have

$$|f|(a) = \begin{cases} f(a) & a \in A_+ \\ -f(a) & a \in A_- \end{cases}$$

(d)

(e)
(f)
(g)
(h) (i)
(ii)
(iii)

Problem 2.H (Integration Theory, Utility Grade)**.** (a)
(b)
(c)
(d)
(e)
(f)

Problem 2.I (Maximal Ideals in Lattices)**.** (a) Let $\mathscr{A}$ denote the family of ideals which contain A and are disjoint from B, and let $\mathscr{N}$ be a nest in $\mathscr{A}$. If $\mathscr{N}$ is empty, then since $A \in \mathscr{A}$, A is an upper bound of $\mathscr{N}$ in $\mathscr{A}$. If $\mathscr{N}$ is nonempty, we claim that $\bigcup \mathscr{N} \in \mathscr{A}$. Clearly since $A \subseteq N$ for any $N \in \bigcup \mathscr{N}$, we have $A \subseteq \bigcup \mathscr{N}$. Moreover,

$$\left(\bigcup \mathscr{N}\right) \cap B = \bigcup_{N \in \mathscr{N}} (N \cap B) = \bigcup_{N \in \mathscr{N}} \varnothing = \varnothing$$

so $\bigcup \mathscr{N}$ is disjoint from B. Suppose $y \in \bigcup \mathscr{N}$ and $x \in X$ such that $y \geq x$. Then if $N \in \mathscr{N}$ such that $y \in N$, we have also that $x \in N$ since N is an ideal of X. If $y, z \in \bigcup \mathscr{N}$, let $N, M \in \mathscr{N}$ such that $y \in N$ and $z \in M$. Then since $\mathscr{N}$ is a nest, WLOG $M \subseteq N$. Hence $y, z \in N$, and so $y \vee z \in N$. Thus $y \vee z \in \bigcup \mathscr{N}$, so that $\bigcup \mathscr{N}$ is an ideal of X. Now since $N \subseteq \bigcup \mathscr{N}$ for all $N \in \mathscr{N}$, we have that $\bigcup \mathscr{N}$ is an upper bound for $\mathscr{N}$ in $\mathscr{A}$. Hence by Theorem 0.25(a) (the Maximal Principle), there is a maximal element A' of $\mathscr{A}$; that is, A' is maximal among the ideals of X containing A and disjoint from B.

The existence of B' is proven in exactly the same manner. Let $\mathscr{B}$ denote the family of dual ideals of X which contain B and are disjoint from A'. Since B is a dual ideal of X disjoint from A', we have $B \in \mathscr{B}$. Thus the empty nest is bounded above in $\mathscr{B}$. Now suppose that $\mathscr{N}$ is a nonempty nest in $\mathscr{B}$; we wish to show that $\bigcup \mathscr{N} \in \mathscr{B}$. For any $N \in \mathscr{N}$, we have that $B \subseteq N$ and so $B \subseteq \bigcup \mathscr{N}$. Moreover,

$$\left(\bigcup \mathscr{N}\right) \cap A' = \bigcup_{N \in \mathscr{N}} (N \cap A') = \bigcup \varnothing = \varnothing,$$

so $\bigcup \mathscr{N}$ is disjoint from A'. Now let $y \in \bigcup \mathscr{N}$ and suppose $x \in X$ with $x \geq y$. Then if $N \in \mathscr{N}$ such that $y \in N$, we have $x \in N$ since N is a dual ideal of X and thus $x \in \bigcup \mathscr{N}$. For $y, z \in \bigcup \mathscr{N}$, there are $N, M \in \mathscr{N}$ such that $y \in N$ and $z \in M$. WLOG, since $\mathscr{N}$ is a nest, $M \subseteq N$. Then $y, z \in N$ and so since N is a dual ideal, $y \wedge z \in N$. Thus $y \wedge z \in \bigcup \mathscr{N}$, and so $\bigcup \mathscr{N}$ is a dual ideal of X. Then $\bigcup \mathscr{N} \in \mathscr{B}$. Now by Theorem 0.25(a) (the Maximal Principle), there is a maximal element B' of $\mathscr{B}$. This B' is maximal among the dual ideals of X containing B and disjoint from A'.

(b) For $c \in X$, let

$$C = \{x \in X \mid x \leq c \text{ or } x \leq c \vee y \text{ for some } y \in A'\}.$$

We claim that C is an ideal of X containing A' and c. [TODO] If C' is any ideal of X containing A' and c, then $c \vee y \in C'$ for any $y \in A'$. Then if $x \in X$ such that $x \leq c$ or $x \leq c \vee y$ for some $y \in A'$, we have $x \in C'$ since C' is an ideal. Thus $C \subseteq C'$, and so C is the smallest such ideal.

If C is disjoint from B, then C is an ideal of X containing A' (and thus also A) which is disjoint from B. Hence by maximality of A', we have $C \subseteq A'$. In particular, $c \in A'$. Now suppose c is in neither

A' nor B; then C intersects B. Let $y \in B \cap C$. From $y \in C$, either $y \leq c$ or there exists $x \in A'$ such that $y \leq c \vee x$. But B is a dual ideal and $c \notin B$, so $y \leq c$ is impossible. Thus there is $x \in A'$ such that $y \leq c \vee x$, and so since B is a dual ideal, $c \vee x \in B$.

(c) Since $B \subseteq B'$, if c is in neither A' nor B' then by part (b), there is $x \in A'$ such that $c \vee x \in B$. By an argument entirely analogous to part (b), there is $y \in B'$ such that $c \wedge y \in A'$. Then $(c \vee x) \wedge y \in B'$ since B' is a dual ideal containing $c \vee x$ and y. We also have that

$$(c \vee x) \wedge y = (c \wedge y) \vee (x \wedge y)$$

since X is distributive. From $x \geq x \wedge y$, we have $x \wedge y \in A'$ and thus $(c \wedge y) \vee (x \wedge y) \in A'$. But B' is disjoint from A' by construction, and so we have a contradiction. Then $A' \cup B' = X$, proving the claim.

Problem 2.J (Universal Nets)**.** (a) Suppose $\{S_n, n \in D\}$ is a universal net in X which is frequently in $A \subseteq X$. Then S is not eventually in $X \setminus A$, and so S is eventually in A since it is a universal net.

(b) We first show that if a net $\{S_n, n \in D\}$ in X is eventually in $A \subseteq X$, then any subnet of S is eventually in A. Indeed, let $N : E \to D$ be a function of directed sets as in the definition of a subnet of S. Since S is eventually in A, there is $m \in D$ such that $S_n \in A$ for $n \geq m$. Let $n \in E$ such that for $p \geq n$, $N_p \geq m$. Then $S_{N_p} \in A$ for $p \geq n$, and so $S \circ N$ is eventually in A. Now if S is a universal net in X, for any subset $A \subseteq X$, S is eventually in A or eventually in $X \setminus A$. Hence any subnet of S is eventually in A or eventually in $X \setminus A$, and so is a universal net in X.

Suppose $\{S_n, n \in D\}$ is a universal net in X and that $f : X \to Y$ is a function. Then $f \circ S$ is a net in Y. For any subset $B \subseteq Y$, S is eventually in $f^{-1}(B)$ or $X \setminus f^{-1}(B)$. Thus there is $n \in D$ such that either for all $m \geq n$, $S_n \in f^{-1}(B)$, or for all $m \geq n$, $S_n \in X \setminus f^{-1}(B)$. But $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$ and hence for all $m \geq n$, $f(S_n) \in B$ or for all $m \geq n$, $f(S_n) \in Y \setminus B$. Hence $f \circ S$ is eventually in B or $Y \setminus B$, so $f \circ S$ is a universal net in Y.

(c) Let S be a net in X and let $\mathscr{A}$ be the family of all subsets $A \subseteq X$ for which S is frequently in A, and let $\mathscr{P}$ be the family of subsets of $\mathscr{A}$ which are closed under finite intersections. Suppose that $\mathscr{N}$ is a nest in $\mathscr{P}$; we claim that $\bigcup \mathscr{N} \in \mathscr{P}$. Let $A, B \in \bigcup \mathscr{N}$. Then there are $N, M \in \mathscr{N}$ such that $A \in N$ and $B \in M$, and WLOG we assume that $M \subseteq N$. Then $A, B \in N$ and so $A \cap B \in N$. Hence $A \cap B \in \bigcup \mathscr{N}$, so $\bigcup \mathscr{N} \in \mathscr{P}$ (clearly $\bigcup \mathscr{N} \subseteq \mathscr{A}$). Thus $\mathscr{N}$ has an upper bound in $\mathscr{P}$, and so by Theorem 0.25(a) (the Maximal Principle), there is a maximal element of $\mathscr{P}$. Let $\mathscr{C}$ be a maximal element of $\mathscr{P}$. Suppose for sake of contradiction that there is $A \subseteq X$ such that A and $X \setminus A$ are not in $\mathscr{C}$. To yield a contradiction, it suffices by definition of $\mathscr{C}$ to show that $\mathscr{C} \cup \{A\}$ or $\mathscr{C} \cup \{X \setminus A\}$ is in $\mathscr{P}$. [TODO: finish]

[TODO: other method]

(d) Let S be a net in X, and let $\mathscr{C}$ be as in part (c). By Lemma 2.5, there is a subnet T of S which is eventually in each member of $\mathscr{C}$. But for all $A \subseteq X$, we have $A \in \mathscr{C}$ or $X \setminus A \in \mathscr{C}$ and hence T is eventually in A or $X \setminus A$. Then T is a universal subnet of S.

Problem 2.K (Boolean Rings: There are Enough Homomorphisms)**.** (a) We have for $r, s \in R$ that

$$\begin{aligned} r + s &= (r+s)^2 \\ &= r^2 + rs + sr + s^2 \\ &= r + rs + sr + s, \end{aligned}$$

and so $0 = rs + sr$. Adding rs to both sides yields $rs = sr$, and so R is commutative.

(b) Since R is a ring, it has the usual $\mathbf{Z}$-algebra structure. For all $r \in R$, we have $2r = r + r = 0$ and thus this $\mathbf{Z}$-algebra structure factors through a $\mathbf{Z}/2\mathbf{Z}$-algebra structure.

(c) If $A \in \mathscr{A}$, then

$$\begin{aligned} A\Delta\varnothing &= (A \cup \varnothing) \setminus (A \cap \varnothing) \\ &= A \setminus \varnothing \\ &= A. \end{aligned}$$

For $A, B \in \mathscr{A}$, it is clear that $A\Delta B = B\Delta A$. Moreover, we notice that

$$\begin{aligned} A\Delta B &= (A \cup B) \setminus (A \cap B) \\ &= (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \\ &= (A \setminus B) \cup (B \setminus A). \end{aligned}$$

Now for $A, B, C \in \mathscr{A}$,

$$\begin{aligned} (A\Delta B)\Delta C &= ((A\Delta B) \setminus C) \cup (C \setminus (A\Delta B)) \\ &= (((A \setminus B) \cup (B \setminus A)) \setminus C) \cup (C \setminus ((A \cup B) \setminus (A \cap B))) \\ &= (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \cup (A \cap B \cap C) \\ &= (A \setminus (B \cup C)) \cup (A \cap B \cap C) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \\ &= (A \setminus ((B \cup C) \setminus (B \cap C))) \cup (((B \setminus C) \setminus A) \cup ((C \setminus B) \setminus A)) \\ &= (A \setminus ((B \cup C) \setminus (B \cap C))) \cup (((B \setminus C) \cup (C \setminus B)) \setminus A) \\ &= (A \setminus (B\Delta C)) \cup ((B\Delta C) \setminus A) \\ &= A\Delta(B\Delta C). \end{aligned}$$

Thus $(\mathscr{A}, \Delta)$ is an abelian group. It is clear that $\cap$ is associative, and so we show that $\cap$ distributes over Δ. For all $A, B, C \in \mathscr{A}$, we have

$$\begin{aligned} A \cap (B\Delta C) &= A \cap ((B \cup C) \setminus (B \cap C)) \\ &= (A \cap (B \cup C)) \setminus (B \cap C) \\ &= ((A \cap B) \cup (A \cap C)) \setminus (B \cap C) \\ &= ((A \cap B) \setminus (B \cap C)) \cup ((A \cap C) \setminus (B \cap C)) \\ &= ((A \cap B) \setminus (A \cap C)) \cup ((A \cap C) \setminus (A \cap B)) \\ &= (A \cap B)\Delta(A \cap C), \end{aligned}$$

as desired. Moreover, for all $A \in \mathscr{A}$, we have $A \cap X = A$ and $X \cap A = A$ so $(\mathscr{A}, \Delta, \cap)$ is a ring with additive unit $\varnothing$ and multiplicative unit X. Finally, we have that $(\mathscr{A}, \Delta, \cap)$ is also a Boolean ring: for any $A \in \mathscr{A}$, it is clear that $A \cap A = A$ and

$$A\Delta A = (A \cup A) \setminus (A \cap A) = A \setminus A = A.$$

TODO

(d) Let $r, s, t \in R$ such that $r \geq s$ and $s \geq t$. Then

$$r \cdot t = r \cdot (s \cdot t) = (r \cdot s) \cdot t = s \cdot t = t,$$

so $r \geq t$. Thus $\geq$ partially orders R. For $r, s \in R$, let $r \vee s = r + s + r \cdot s$ and $r \wedge s = r \cdot s$. Then (note that by part (a), R is commutative) $r \vee s = s \vee r$, $r \wedge s = s \wedge r$, and

$$\begin{aligned}(r \vee s) \cdot r &= (r + s + r \cdot s) \cdot r \\ &= r + s \cdot r + (r \cdot s) \cdot r \\ &= r + s \cdot r + (s \cdot r) \cdot r \\ &= r + s \cdot r + s \cdot r^2 \\ &= r + s \cdot r + s \cdot r \\ &= r\end{aligned}$$

and

$$\begin{aligned}(r \wedge s) \cdot r &= (r \cdot s) \cdot r \\ &= r^2 \cdot s \\ &= r \cdot s \\ &= r \wedge s.\end{aligned}$$

Then $r \vee s \geq r, s$ and $r, s \geq r \wedge s$. On the other hand, if $a \geq r, s$ and $r, s \geq b$, then

$$\begin{aligned}a \cdot (r \vee s) &= a \cdot (r + s + r \cdot s) \\ &= a \cdot r + a \cdot s + a \cdot (r \cdot s) \\ &= r + s + (a \cdot r) \cdot s \\ &= r + s + r \cdot s\end{aligned}$$

and

$$\begin{aligned}(r \wedge s) \cdot b &= (r \cdot s) \cdot b \\ &= r \cdot (s \cdot b) \\ &= r \cdot b \\ &= b.\end{aligned}$$

Thus $a \geq r \vee s$ and $r \wedge s \geq b$, so $r \vee s$ is the join of r, s and $r \wedge s$ is the meet of r, s. For $r, s, t \in R$,

$$\begin{aligned}(r \vee s) \vee t &= (r + s + r \cdot s) \vee t \\ &= (r + s + r \cdot s) + t + (r + s + r \cdot s) \cdot t \\ &= r + s + r \cdot s + t + r \cdot t + s \cdot t + (r \cdot s) \cdot t \\ &= r + (s + t + s \cdot t) + r \cdot (s + t + s \cdot t) \\ &= r \vee (s + t + s \cdot t) \\ &= r \vee (s \vee t)\end{aligned}$$

and

$$\begin{aligned}(r \wedge s) \wedge t &= (r \cdot s) \cdot t \\ &= r \cdot (s \cdot t) \\ &= r \wedge (s \wedge t).\end{aligned}$$

Then $\vee$ and $\wedge$ are associative. Moreover, we have for $r, s, t \in R$ that

$$\begin{aligned} r \wedge (s \vee t) &= r \cdot (s + t + s \cdot t) \\ &= r \cdot s + r \cdot t + r \cdot (s \cdot t) \\ &= r \cdot s + r \cdot t + r^2 \cdot (s \cdot t) \\ &= r \cdot s + r \cdot t + (r \cdot s) \cdot (r \cdot t) \\ &= (r \cdot s) \vee (r \cdot t) \\ &= (r \wedge s) \vee (r \wedge t) \end{aligned}$$

and

$$\begin{aligned} r \vee (s \wedge t) &= r \vee (s \cdot t) \\ &= r + s \cdot t + r \cdot (s \cdot t) \\ &= r^2 + r \cdot t + r \cdot t + s \cdot r + s \cdot t + r \cdot (s \cdot t) + r \cdot (s \cdot t) + r \cdot (s \cdot t) \\ &= r^2 + r \cdot t + r^2 \cdot t + s \cdot r + s \cdot t + s \cdot (r \cdot t) + (r \cdot s) \cdot r + (r \cdot s) \cdot t + (r \cdot s) \cdot (r \cdot t) \\ &= (r + s + r \cdot s) \cdot (r + t + r \cdot t) \\ &= (r \vee s) \wedge (r \vee t) \end{aligned}$$

Thus $(R, \geq)$ is a distributive latttice.

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Problem 2.L (Filters)**.** (a) Suppose U is an open subset of X, $x \in U$, and $\mathscr{F}$ is a filter in X which converges to x. Then U is a neighborhood of x, so $U \in \mathscr{F}$.

Conversely, suppose that $U \subseteq X$ such that U belongs to every filter which converges to a point of U. By Theorem 1.2, for any $x \in U$ the neighborhood system $\mathscr{U}_x$ of x is a dual ideal of the Boolean ring $(2^X, \Delta, \cap)$. Moreover, every neighborhood of x is nonempty since it contains x, and hence $\mathscr{U}_x$ is a filter in X. It is immediate that $\mathscr{U}_x$ converges to x, and hence $U \in \mathscr{U}_x$. Then U is a neighborhood of x, and so by Theorem 1.1, U is open.

(b) Suppose $A \setminus \{x\}$ belongs to some filter $\mathscr{F}$ in X which converges to x. Then for all neighborhoods U of x¡ we have $U \in \mathscr{F}$ and hence $U \cap (A \setminus \{x\}) \in \mathscr{F}$. Since $\varnothing \notin \mathscr{F}$, it follows that U intersects $A \setminus \{x\}$ and so x is an accumulation point of A.

Conversely, suppose that x is an accumulation point of A. As shown in part (a), the neighborhood system $\mathscr{U}_x$ of x is a filter in X. By Problem 2.I(b), the smallest dual ideal of $(2^X, \Delta, \cap)$ containing $\mathscr{U}_x$ and $A \setminus \{x\}$ is

$$\mathscr{F} = \{B \subseteq X \mid A \setminus \{x\} \subseteq B \text{ or } U \cap (A \setminus \{x\}) \subseteq B \text{ for some } U \in \mathscr{U}_x\}.$$

But any neighborhood U of x intersects $A \setminus \{x\}$ since x is an accumulation point of A, and thus $\mathscr{F}$ consists of nonempty sets. Hence $\mathscr{F}$ is a filter in X which converges to x and contains $A \setminus \{x\}$.

(c) By definition, every $\mathscr{F} \in \varphi_x$ contains $\mathscr{U}_x$. Thus $\mathscr{U}_x \subseteq \bigcap \varphi_x$. Conversely, we saw in part (a) that $\mathscr{U}_x$ is a filter. Then $\mathscr{U}_x \in \varphi_x$ and so $\bigcap \varphi_x \subseteq \mathscr{U}_x$, proving the claim.

(d) Since $\mathscr{F}$ converges to x, we have $\mathscr{U}_x \subseteq \mathscr{F}$. Then from $\mathscr{F} \subseteq \mathscr{G}$, we also have $\mathscr{U}_x \subseteq \mathscr{G}$ and so $\mathscr{G}$ converges to x.

(e) Suppose for sake of contradiction that $A, B \subseteq X$ such that $A, B \notin \mathscr{F}$ but $A \cup B \in \mathscr{F}$. Then since $\mathscr{F}$ is an ultrafilter, we must have that the smallest dual ideal of $(2^X, \Delta, \cap)$ containing $\mathscr{F}$ and A is 2^X and the smallest dual ideal containing $\mathscr{F}$ and B is 2^X. Then by Problem 2.I(b), there exist $F, G \in \mathscr{F}$ such that $A \cap F = B \cap G = \varnothing$. Then $(A \cup B) \cap (F \cap G) \in \mathscr{F}$, but

$$\begin{aligned}(A \cup B) \cap (F \cap G) &= (A \cap (F \cap G)) \cup (B \cap (F \cap G)) \\ &= ((A \cap F) \cap G) \cup ((B \cap G) \cap F) \\ &= \varnothing\end{aligned}$$

since $(A \cap F) \cap G \subseteq A \cap F = \varnothing$ and $(B \cap G) \cap F \subseteq B \cap G = \varnothing$. This is a contradiction (since $\varnothing \notin \mathscr{F}$), so A or B is in $\mathscr{F}$.

If $X = \varnothing$, then the empty filter is the only ultrafilter in X and so the second claim in the problem statement is false. Now suppose $X \neq \varnothing$. Then $\{X\}$ is a filter in X which contains $\varnothing$ and so $\varnothing$ is not an ultrafilter in X. Thus if $\mathscr{F}$ is an ultrafilter in X, $\mathscr{F}$ is nonempty and so it follows that $X \in \mathscr{F}$ as X contains every subset of X. Then for any $A \subseteq X$, we have $A \cup (X \setminus A) = X \in \mathscr{F}$ so that A or $X \setminus A$ lies in $\mathscr{F}$.

(f) (a) Since the order on D is reflexive, $\{x_n, n \in D\}$ is not eventually in $\varnothing$. Thus $\varnothing \notin \mathscr{F}$. If $A, B \in \mathscr{F}$, let $n, m \in D$ such that $x_p \in A$ for all $p \geq n$ and $x_p \in B$ for all $p \geq m$. Since D is directed, there is $p \in D$ such that $p \geq n, m$. Then for $q \geq p$, we have $q \geq n, m$ so $x_q \in A \cap B$. Then $A \cap B \in \mathscr{F}$. Finally, suppose $A \in \mathscr{F}$ and that B is a subset of X containing A. Then there is $n \in D$ such that $x_m \in A$ for $m \geq n$, and so also $x_m \in B$ for $m \geq n$. Thus $B \in \mathscr{F}$, so that $\mathscr{F}$ is a filter in X.

(b) (Note: We assume $\mathscr{F}$ is nonempty so that D is nonempty.) We first show that D is directed by $\geq$. Since $\mathscr{F}$ is nonempty, there is $F \in \mathscr{F}$. But $\mathscr{F}$ is a filter and so $F \neq \varnothing$. Thus there is $x \in F$, so $(x, F) \in D$; that is, D is nonempty. If $(x, F), (y, G), (z, H) \in \mathscr{F}$ such that $(z, H) \geq (y, G)$ and $(y, G) \geq (x, F)$, then $H \subseteq G$ and $G \subseteq F$. Thus $(z, H) \geq (x, F)$, so $\geq$ is a partial order on D. For any $(x, F) \in D$, we clearly have $F \subseteq F$ and so $(x, F) \geq (x, F)$. Finally, let $(x, F), (y, G) \in D$. Then $F, G \in \mathscr{F}$ so $F \cap G \in \mathscr{F}$ with $F \cap G \subseteq F, G$. Letting $z \in F \cap G$ ($F \cap G$ is nonempty since $\mathscr{F}$ is a filter), we then have $(z, F \cap G) \in D$ with $(z, F \cap G) \geq (x, F), (y, G)$. Hence D is directed by $\geq$. Now we have that $\{f(x, F), (x, F) \in D\}$ is a net in X. Suppose $A \in \mathscr{F}$, and let $x \in A$. Then for any $(y, B) \in D$ such that $(y, B) \geq (x, A)$, we have $y \in B$ and $B \subseteq A$. Hence $f(y, B) = y \in A$, so that $\{f(x, F), (x, F) \in D\}$ is eventually in A. Conversely, let $A \subseteq X$ such that $\{f(x, F), (x, F) \in D\}$ is eventually in A. Then there is $(x, F) \in D$ such that $y \in A$ whenever $(y, G) \geq (x, F)$. Since $(y, F) \geq (x, F)$ for all $y \in F$, we thus have $y \in A$ for all $y \in F$. Hence $F \subseteq A$, and so $A \in \mathscr{F}$ since $F \in \mathscr{F}$.

CHAPTER 3

Product and Quotient Spaces

Problem 3.A (Connected Spaces)**.** Let $f : X \to Y$ be a continuous map of topological spaces and suppose Y is not connected. Then there are disjoint closed subsets A, B of Y such that $Y = A \cup B$. Thus

$$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),$$

where $f^{-1}(A)$ and $f^{-1}(B)$ are closed subsets of X. Moreover, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint since A and B are disjoint. Hence X is not connected, which shows the contrapositive of the required statement.

Problem 3.B (Theorem on Continuity)**.** Let Y be a topological space and $f : X \to Y$ a function which is continuous on A and on B. Suppose V is an open subset of Y. Then since $f|_A : A \to Y$ is continuous, we have that

$$(f|_A)^{-1}(V) = f^{-1}(V) \cap A$$

is open in A. Similarly, since $f|_B : B \to Y$ is continuous,

$$(f|_B)^{-1}(V) = f^{-1}(V) \cap B$$

is open in B. Thus by Corollary 1.19, $f^{-1}(V)$ is open in X as desired.

Problem 3.C (Exercise on Continuous Functions)**.** Let E denote the subset of X on which f and g agree, and let $x \in X \setminus E$. Then $f(x)$ and $g(x)$ are distinct points of Y, so there are disjoint neighborhoods U and V of $f(x)$ and $g(x)$, respectively. Then $f^{-1}(U)$ and $g^{-1}(V)$ are neighborhoods of x by Theorem 3.1(d), and so $f^{-1}(U) \cap g^{-1}(V)$ is a neighborhood of x by Theorem 1.2. If $y \in f^{-1}(U) \cap g^{-1}(V)$, then $f(y) \in U$ while $g(y) \in V$, so $f(y) \neq g(y)$ as U and V are disjoint. Thus $f^{-1}(U) \cap g^{-1}(V)$ is contained in $X \setminus E$, and hence $X \setminus E$ is open by Theorem 1.1. Now E is a closed subset of X.

If E is also dense in X, then $E = \overline{E} = X$ and so $f = g$.

Problem 3.D (Continuity at a Point; Continuous Extension)**.** (a) Suppose first that f is continuous at x. Then $x \in \overline{X_0}$ by definition. Let $y \in Y$ such that every neighborhood of y has inverse image under f equal to the intersection of X_0 with a neighborhood of x. Let S be a net in X_0 converging to x. Let V be a neighborhood of y; by choice of y there exists a neighborhood U of x for which $f^{-1}(V) = X_0 \cap U$. But since S is in X_0 and converges to x, it is eventually in $X_0 \cap U$. Hence $f \circ S$ is eventually in V, and thus $f \circ S$ converges to y. If T is another net in X_0 converging to x, then $f \circ S$ and $f \circ T$ both converge to y.

Conversely, suppose that $x \in \overline{X_0}$ and that for any two nets S and T in X_0 converging to x, we have that $f \circ S$ and $f \circ T$ converge to the same point of Y. By Theorem 2.3, the limit of a net in Y is unique if it exists. Hence there is $y \in Y$ such that $f \circ S$ converges to y whenever S is a net in X_0 converging to x. Let V be a neighborhood of y, and suppose for sake of contradiction that there is no neighborhood U of x for which $X_0 \cap U \subseteq f^{-1}(V)$. We can then define a net S in X_0 with domain $\mathscr{U}_x$ (directed by $\subseteq$) as follows: for all $U \in \mathscr{U}_x$, let S_U be an element of $X_0 \cap U$ which is not in $f^{-1}(V)$. Then for all $U \in \mathscr{U}_x$, S is eventually in U. Hence S is a net in X_0 converging to x and thus $f \circ S$ converges to y. But by definition of S, $f \circ S$ is in $Y \setminus V$ and hence cannot converge to y. This is a contradiction, and so there is a neighborhood U of x for which $X_0 \cap U \subseteq f^{-1}(V)$. Then $U \cup f^{-1}(V)$ is a neighborhood of x (by Theorem 1.2) and

$$f^{-1}(V) = (X_0 \cap U) \cup f^{-1}(V) = X_0 \cap (U \cup f^{-1}(V))$$

since $f^{-1}(V) \subseteq X_0$. Thus f is continuous at x.

TODO

Problem 3.E (Exercise on Real-Valued Continuous Functions)**.** (a) As explained in the hint, it suffices to show that the function $\mathbf{R} \to \mathbf{R}$ given by multiplication by a is continuous. If $a = 0$, then h is the constant function with value zero and is hence continuous. Now suppose $a \neq 0$. Since $\mathbf{R}$ is equipped with the order topology, it suffices by Theorem 3.1(c) to show that $\{y \in \mathbf{R} \mid ay < b\}$ and $\{y \in \mathbf{R} \mid b < ay\}$ are open for each $b \in \mathbf{R}$. Indeed, if $a > 0$, then

$$\{y \in \mathbf{R} \mid ay < b\} = \{y \in \mathbf{R} \mid y < b/a\}$$

and

$$\{y \in \mathbf{R} \mid b < ay\} = \{y \in \mathbf{R} \mid b/a < y\}$$

are open; if $a < 0$, then

$$\{y \in \mathbf{R} \mid ay < b\} = \{y \in \mathbf{R} \mid b/a < y\}$$

and

$$\{y \in \mathbf{R} \mid b < ay\} = \{y \in \mathbf{R} \mid y < b/a\}$$

are open.

(b) Like in part (a), it suffices to show that $|\cdot| : \mathbf{R} \to \mathbf{R}$ is continuous. Let $A = \{y \in \mathbf{R} \mid y \leq 0\}$ and $B = \{y \in \mathbf{R} \mid 0 \leq y\}$. Then

$$\mathbf{R} \setminus A = \{y \in \mathbf{R} \mid 0 < y\}$$

and

$$\mathbf{R} \setminus B = \{y \in \mathbf{R} \mid y < 0\}$$

are open, so A and B are closed. Thus by Theorem 1.17, $A \setminus B$ and $B \setminus A$ are separated. [TODO]

(c)

(d)

(e)

Problem 3.F (Upper Semi-Continuous Functions)**.** (a) (Note: in the problem statement, $\geq$ should be replaced by $\leq$.) Suppose that $\{S_n, n \in D\}$ is a net of real numbers which converges to s relative to $\mathscr{U}$. Then for all $a \in \mathbf{R}$ with $s < a$, we have that $U = \{t \in \mathbf{R} \mid t < a\}$ is a $\mathscr{U}$-neighborhood of s and so S is eventually in U. Hence there is $p \in D$ such that $S_m \in U$ for $m \geq p$. Now for all $n \in D$ with $n \geq p$, we have that

$$\{S_m \mid m \in D \text{ and } m \geq n\} \subseteq U$$

and hence

$$\sup\{S_m \mid m \in D \text{ and } m \geq n\} \leq a.$$

[TODO]

(b) By Theorem 3.1(b), f is $\mathscr{U}$-continuous if and only if the inverse image of each $\mathscr{U}$-closed set is closed in X. But $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(\mathbf{R}) = X$ are of course closed in X and so f is $\mathscr{U}$-continuous if and only if $f^{-1}(\{t \in \mathbf{R} \mid t \geq a\})$ is closed in X for all $a \in \mathbf{R}$. That is, f is $\mathscr{U}$-continuous if and only if f is upper semicontinuous.

We know from Theorem 3.1(f) that f is $\mathscr{U}$-continuous if and only if for all nets $\{x_n, n \in D\}$ in X converging to x, $\{f(x_n), n \in D\}$ converges to $f(x)$ relative to $\mathscr{U}$. By part (a), the $\mathscr{U}$-convergence of $\{f(x_n), n \in D\}$ to $f(x)$ is equivalent to $\limsup\{f(x_n), n \in D\} \leq f(x)$, proving the claim.

TODO

(c) Let $a \in \mathbf{R}$. Then for $x \in X$, $i(x) \geq a$ if and only if $f(x) \geq a$ for all $f \in F$. Hence

$$\{x \in X \mid i(x) \geq a\} = \bigcap \{\{x \in X \mid f(x) \geq a\} f \in F\}$$

is closed in X since each $\{x \in X \mid f(x) \geq a\}$ is closed by upper semicontinuity of f.

(d) As described in the problem statement, we define f^- as follows: for each neighborhood U of x, let $S_U = \sup\{f(y) \mid y \in U\}$ (which exists since f is bounded). If $U, V \in \mathscr{U}_x$ with $U \subseteq V$ then $S_U \leq S_V$. Thus by Problem 2.F(a), $\lim\{S_U, U \in \mathscr{U}_x, \subseteq\}$ exists and is equal to $\inf\{S_U \mid U \in \mathscr{U}_x\}$. Then set $f^-(x) = \lim\{S_U, U \in \mathscr{U}_x, \subseteq\}$. For all $U \in \mathscr{U}_x$, we have that $x \in U$ and so $S_U \geq f(x)$ by definition of S_U. Hence $f^- \geq f$.

Let $a \in \mathbf{R}$. Then for $x \in X$, $f^-(x) \geq a$ if and only if $S_U \geq a$ for all $U \in \mathscr{U}_x$. Equivalently, for all $U \in \mathscr{U}_x$, there exists $y \in U$ for which $f(y) \geq a$, that is, $x \in \{y \in X \mid f(y) \geq a\}^-$. Hence

$$\{x \in X \mid f^-(x) \geq a\} = \{y \in X \mid f(y) \geq a\}^-$$

is closed and so f^- is upper semicontinuous.

Now suppose $g : X \to \mathbf{R}$ is another upper semicontinuous function with $g \geq f$. Then for all $x \in X$, $\{y \in X \mid g(y) \geq f^-(x)\}$ is a closed set containing $\{y \in X \mid f(y) \geq f^-(x)\}$. Hence

$$\{y \in X \mid g(y) \geq f^-(x)\} \supseteq \{y \in X \mid f^-(y) \geq f^-(x)\}.$$

Since x lies in the latter set, it follows that $g(x) \geq f^-(x)$. Thus $g \geq f^-$, which shows that f^- is the unique smallest upper semicontinuous function greater than or equal to f.

(e) We note that if f is bounded, then $-f$ is also bounded. Hence $f_- = -(-f)^-$ is well-defined and so is $Q_f = f^- - f_-$. By definition, we have that

$$Q_f = f^- - (-(-f)^-) = f^- + (-f)^-$$

is the sum of two upper semicontinuous real-valued functions on X. Hence by part (c), Q_f is also upper semicontinuous. We also observe that by part (e), for all $x \in X$,

$$\begin{aligned} f_-(x) &= -(-f)^-(x) \\ &= -\inf_{U \in \mathscr{U}_x} \sup_{y \in U}(-f(y)) \\ &= \sup_{U \in \mathscr{U}_x} \inf_{y \in U} f(y). \end{aligned}$$

Now we show that f is continuous if and only if $Q_f = 0$. By Theorem 3.1(d), it suffices to show that f is continuous at $x \in X$ if and only if $Q_f(x) = 0$. We have that $f^- \geq f$ and $(-f)^- \geq -f$, so $f_- \leq f$. Then

$$Q_f = f^- - f_- \geq f - f = 0.$$

Thus if $Q_f(x) \neq 0$, we have $Q_f(x) > 0$ and so $f^-(x) > f_-(x)$. From $f^-(x) \geq f(x) \geq f_-(x)$, it follows that either $f^-(x) > f(x)$ or $f_-(x) < f(x)$. WLOG, suppose that $f^-(x) > f(x)$. Then $(f^-(x) + f(x))/2 > f(x)$, and so $\{t \in \mathbf{R} \mid t < (f^-(x) + f(x))/2\}$ is a neighborhood of $f(x)$ in $\mathbf{R}$. Hence if f were continuous at x, there exists $U \in \mathscr{U}_x$ such that $f(y) < (f^-(x) + f(x))/2$ for all $y \in U$. Then

$$\sup_{y \in U} f(y) < f^-(x)$$

and so

$$\inf_{U \in \mathscr{U}_x} \sup_{y \in U} f(y) < f^-(x).$$

But by part (e),

$$\inf_{U\in\mathscr{U}_x}\sup_{y\in U} f(y) = f^-(x),$$

a contradiction. Hence f is not continuous at x.

Conversely, suppose that $Q_f(x) = 0$. Then from $f^-(x) \geq f(x) \geq f_-(x)$, we have that $f^-(x) = f(x) = f_-(x)$. If $a \in \mathbf{R}$ such that $f(x) < a$, then $f^-(x) < a$ and thus by part (e), $\sup_{y\in U} f(y) < a$ for some neighborhood U of x. Thus $f(y) < a$ for all $y \in U$, and so $f(U) \subseteq \{t \in \mathbf{R} \mid t < a\}$. Similarly, if $a \in \mathbf{R}$ such that $a < f(x)$, then $a < f_-(x)$ and so there exists a neighborhood U of x for which $a < \inf_{y\in U} f(y)$. Then $a < f(y)$ for all $y \in U$ and so $f(U) \subseteq \{t \in \mathbf{R} \mid a < t\}$. Since the collection of all $\{t \in \mathbf{R} \mid t < a\}$ for $f(x) < a$ and $\{t \in \mathbf{R} \mid a < t\}$ for $a < f(x)$ is a local subbase at x, it thus follows from Theorem 1.2 and Theorem 3.1(e) that f is continuous at x.

(f) By Theorem 3.12, the projection $P : G \to \mathscr{D}$ is closed. For any $a \in \mathbf{R}$, $(X \times \{t \in \mathbf{R} \mid t < a\}) \cap G$ is an open subset of G. For $x \in X$, we have that $(\{x\} \times \mathbf{R}) \cap G$ is contained in $(X \times \{t \in \mathbf{R} \mid t < a\}) \cap G$ if and only if $f(x) < t$. Hence the union of all elements of $\mathscr{D}$ contained in $(X \times \{t \in \mathbf{R} \mid t < a\}) \cap G$ is equal to $(\{x \in X \mid f(x) < a\} \times \mathbf{R}) \cap G$. By Theorem 3.10, this is an open subset of G. [TODO]

Problem 3.G (Exercise on Topological Equivalence). (a)
(b)
(c)
(d)

Problem 3.H (Homeomorphisms and One-to-One Continuous Maps).

Problem 3.I (Continuity in Each of Two Variables).

Problem 3.J (Exercise on Euclidean n-Space).

Problem 3.K (Exercise on Closure, Interior, and Boundary in Products). (a)
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Problem 3.L (Exercise on Product Spaces).

Problem 3.M (Product of Spaces with Countable Bases).

Problem 3.N (Example on Products and Separability). (a)
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Problem 3.O (Product of Connected Spaces).

Problem 3.P (Exercise on T_1-Spaces).

Problem 3.Q (Exercise on Quotient Spaces).

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(b)
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Problem 3.S (Topological Groups). (a)
(b)
(c)

(d)
(e) (a)
(b)
(c)
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(h)
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Problem 3.T (Subgroups of a Topological Group). (a)
(b)
(c)
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Problem 3.U (Factor Groups and Homeomorphisms). (a)
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Problem 3.V (Box Spaces). (a)
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Problem 3.W (Functionals on Real Linear Spaces). (a)
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Problem 3.X (Real Linear Topological Spaces). (a)
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CHAPTER 4

Embedding and Metrization

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Problem 4.C (Problem on Metrics).

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Problem 4.E (Example (the Ordinals) on the Product of Normal Spaces). (a)
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Problem 4.F (Example (the Tychonoff Plank) on Subspaces of Normal Spaces).

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(b)
(c)

Problem 4.H (Hereditary, Productive, and Divisible Properties).

Problem 4.I (Half-Open Interval Space). (a)
(b)
(c)

Problem 4.J (The Set of Zeros of a Real Continuous Function). (a)
(b)

Problem 4.K (Perfectly Normal Spaces). (a)
(b)

Problem 4.L (Characterization of Completely Regular Spaces).

Problem 4.M (Upper Semi-Continuous Decomposition of a Normal Space).

CHAPTER 5

Compact Spaces

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Problem 5.B (Compact Subsets). (a)
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(d)

Problem 5.C (Compactness Relative to the Order Topology).

Problem 5.D (Isometries of Compact Metric Spaces).

Problem 5.E (Countably Compact and Sequentially Compact Spaces). (a)
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Problem 5.F (Compactness; the Intersection of Comapct Connected Sets). (a)
(b)

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Problem 5.H (Nest Characterization of Compactness).

Problem 5.I (Complete Accumulation Points).

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Problem 5.K (Example (the Ordinals) on Normality and Products).

Problem 5.L (The Transfinite Line).

Problem 5.M (Example: The Helly Space). (a)
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Problem 5.N (Exmaples on Closed Maps and Local Compactness). (a)
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Problem 5.O (Cantor Spaces). (a)
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Problem 5.Q (Example (the Ordinals) on Compactification).

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Problem 5.V (Point Finite Covers and Metacompact Spaces). (a)
(b)
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Problem 5.W (Partition of Unity).

Problem 5.X (The Between Theorem for Semi-Continuous Functions).

Problem 5.Y (Paracompact Spaces). (a)
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CHAPTER 6

Uniform Spaces

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Problem 6.D (Exercise: Uniform Spaces with a Nested Base).

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Problem 6.H (Uniform Neighborhood Systems). (a)
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Problem 6.R (Continuity and Openness of Homomorphisms: The Closed Graph Theorem). (a)
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Problem 6.S (Summability). (a)
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Problem 6.T (Uniformly Locally Compact Spaces). (a)
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Problem 6.U (The Uniform Boundedness Theorem). (a)
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Problem 6.V (Boolean σ-Rings). (a)
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CHAPTER 7

Function Spaces

Problem 7.A (Exercise on the Topology of Pointwise Convergence).

Problem 7.B (Exercise on Convergence of Functions).

Problem 7.C (Pointwise Convergence on a Dense Subset).

Problem 7.D (The Diagonal Process and Sequential Compactness). (a)
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Problem 7.E (Dini's Theorem).

Problem 7.F (Continuity of an Induced Map).

Problem 7.G (Uniform Equicontinuity). (a)
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Problem 7.H (Exercise on the Uniformity $\mathscr{U} \mid \mathscr{A}$).

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Problem 7.J (Subspaces, Products, and Quotients of k-Spaces). (a)
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Problem 7.P (Density Lemma for Linear Subspaces of $C(X)$)**.**

Problem 7.Q (The Square Root Lemma for Banach Algebras)**.** (a)
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(b)

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(b)
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(e)
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(b)
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