Solutions to Haynes Miller's
Lectures on Algebraic Topology

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Abstract. This document contains solutions to the exercises of Haynes Miller's *Lectures on Algebraic Topology*.

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CHAPTER 1

Singular homology

1. Introduction: singular simplices and chains

Exercise 1.8. (a)
(b)
(c)
(d)
(e)
(f)
(g)

2. Homology

Exercise 2.2.

Exercise 2.3.

3. Categories, functors, and natural transformations

Exercise 3.7.

Exercise 3.8.

4. Categorical language

Exercise 4.10. (a)
(b)
(c)
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5. Homotopy, star-shaped regions

Exercise 5.15. (a)
(b)

Exercise 5.16.

6. Homotopy invariance of homology

Exercise 6.3.

7. Homology cross product

Exercise 7.3.

Exercise 7.4.

8. Relative homology

Exercise 8.8.

Exercise 8.9.

9. Homology long exact sequence

Exercise 9.8.

Exercise 9.9.

Exercise 9.10.

Exercise 9.11.

10. Excision and applications

Exercise 10.10. (a)
(b)
(c)
(d)
(e)
(f)

Exercise 10.11. (a)
(b)
(c)
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11. Eilenberg-Steenrod axioms and the locality principle

Exercise 11.7.

Exercise 11.8. (a)
(b)

Exercise 11.9.

12. Subdivision

Exercise 12.2. (a)
(b)

13. Proof of the locality principle

Exercise 13.6.

CHAPTER 2

Computational methods

14. CW complexes I

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Exercise 14.11.

15. CW complexes II

Exercise 15.7.

Exercise 15.8.

16. Homology of CW complexes

Exercise 16.7.

Exercise 16.8.

17. Real projective space

Exercise 17.2.

18. Euler characteristic and homology approximation

Exercise 18.7.

19. Coefficients

Exercise 19.2.

Exercise 19.3.

20. Tensor Product

Exercise 20.12.

Exercise 20.13.

22. Fundamental theorem of homological algebra

Exercise 22.5.

23. Hom and Lim

Exercise 23.16.

Exercise 23.17. (a)
(b)
(c)
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24. Universal coefficient theorem

Exercise 24.3.

Exercise 24.4.

Exercise 24.5.

25. Künneth and Eilenberg-Zilber

Exercise 25.18. (a)
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Exercise 25.19.

CHAPTER 3

Cohomology and duality

26. Coproducts, cohomology

Exercise 26.9.

27. Ext and UCT

Exercise 27.6.

Exercise 27.7.

Exercise 27.8.

28. Products in cohomology

Exercise 28.3.

29. Cup product, continued

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Exercise 29.7.

30. Surfaces and nondegenerate symmetric bilinear forms

Exercise 30.9.

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31. Local coefficients and orientations

Exercise 31.15.

Exercise 31.16.

Exercise 31.17.

33. A plethora of products

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34. Cap product and Čech cohomology theory

Exercise 34.6.

36. Fully relative cap product

Exercise 36.3.

37. Poincaré duality

Exercise 37.8.

38. Applications

Exercise 38.14.

CHAPTER 4

Basic homotopy theory

39. Limits, colimits, and adjunctions

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Exercise 39.15.

Exercise 39.16.

Exercise 39.17.

Exercise 39.18.

40. Cartesian closure and compactly generated spaces

Exercise 40.8.

Exercise 40.9.

Exercise 40.10. (a)
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Exercise 40.11.

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41. Basepoints and the homotopy category

Exercise 41.5.

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42. Fiber bundles

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43. Fibrations, fundamental groupoid

Exercise 43.10.

Exercise 43.11.

44. Cofibrations

Exercise 44.5.

45. Cofibration sequences and coexactness

Exercise 45.4. (a)
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46. Weak equivalences and Whitehead's theorems

Exercise 46.12.

Exercise 46.13.

Exercise 46.14.

47. Homotopy long exact sequence and homotopy fibers

Exercise 47.8.

Exercise 47.9.

Exercise 47.10. (a)
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Exercise 47.11.

CHAPTER 5

The homotopy theory of CW complexes

49. Connectivity and approximation

Exercise 49.9.

51. Hurewicz, Eilenberg, Mac Lan, and Whitehead

Exercise 51.5.

Exercise 51.6.

Exercise 51.7. (a)
(b)

Exercise 51.8.

53. Obstruction theory

Exercise 53.7.

Exercise 53.8.

Exercise 53.9. (a)
(b)

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CHAPTER 6

Vector bundles and principal bundles

54. Vector bundles

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Exercise 54.12.

Exercise 54.13.

Exercise 54.14.

55. Principal bundles, associated bundles

Exercise 55.8.

Exercise 55.9. (a)
(b)

56. G-CW complexes and the I-invariance of Bun_G

Exercise 56.4. (a)
(b)
(c)

Exercise 56.5.

57. The classifying space of a group

Exercise 57.6.

Exercise 57.7.

58. Simplicial sets and classifying spaces

Exercise 58.7.

Exercise 58.8. (a)
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59. The Čech category and classifying maps

Exercise 59.6.

CHAPTER 7

Spectral sequences and Serre classes

61. Spectral sequence of a filtered complex

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(c)

Exercise 61.6.

62. Serre spectral sequence

Exercise 62.3. (a)
(b)

Exercise 62.5.

Exercise 62.6.

Exercise 62.7.

63. Exact couples

Exercise 63.3.

Exercise 63.4.

Exercise 63.5.

Exercise 63.6. (a)
(b)
(c)
(d)

64. Gysin sequence, edge homomorphisms, and transgression

Exercise 64.6. (a)
(b)

65. Serre exact sequence and the Hurewicz theorem

Exercise 65.8. (a)
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66. Double complexes and the Dress spectral sequence

Exercise 66.1.

67. Cohomological spectral sequences

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Characteristic classes, Steenrod operations, and cobordism

71. Chern classes, Stiefel-Whitney classes, and the Leray-Hirsch theorem

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72. $H^*(BU(n))$ and the splitting principle

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73. Thom class and Whitney sum formula

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74. Closing the Chern circle, and Pontryagin classes

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Exercise 74.8. (a)
(b)

Exercise 74.9.

Exercise 74.10.

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75. Steenrod operations

Exercise 75.12.

76. Cobordism

Exercise 76.10.

77. Hopf algebras

Exercise 77.10.

Exercise 77.11.

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78. Applications of cobordism

Exercise 78.4.

Exercise 78.5.

Exercise 78.6.

Exercise 78.7.