Solutions to Walter Rudin's
Principles of Mathematical Analysis

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Abstract. This document contains solutions to the exercises of Walter Rudin's *Principles of Mathematical Analysis*.

Contents

CHAPTER 1

The Real and Complex Number Systems

Exercise 1.1. Let r be a nonzero rational and x an irrational real number. Suppose for sake of contradiction that $r+x$ is rational. Then $x=(r+x)-r$ is rational, a contradiction. Similarly, supposing for sake of contradiction that rx is rational, we see that $x=r^{-1}(rx)$ is rational. This is a contradiction, and so both $r+x$ and rx are irrational.

Exercise 1.2. Suppose for sake of contradiction that there is $p \in \mathbf{Q}$ such that $p^2 = 12$. Then there are integers m, n with $n \neq 0$ and which are not both divisible by 3, such that $p = m/n$. Thus

$$\left(\frac{m}{n}\right)^2 = 12,$$

so

$$m^2 = 12n^2,$$

and then 3 divides m^2. Thus 3 divides m since 3 is prime, and so $12n^2$ is divisible by 9. But 3 is prime and 12 has only one factor of 3 in its factorization, so 3 divides n^2. Thus 3 divides n, contradicting the choice of m and n. Hence no such p exists.

Exercise 1.3 (TODO)**.**

Exercise 1.4. Since E is nonempty, there is some $x \in E$. Then $\alpha \leq x$ and $x \leq \beta$, so $\alpha \leq \beta$.

Exercise 1.5. We show that $\alpha = -\sup(-A)$ is the greatest lower bound of A. If $x \in A$, then $-x \in -A$ and so $-x \leq \sup(-A)$. Hence $\alpha \leq x$, and so α is a lower bound of A. If $\alpha < \beta$, then $-\beta < -\alpha = \sup(-A)$ and so $-\beta$ is not an upper bound of $-A$. Thus there is $x \in A$ such that $-\beta < -x$, and so $x < \beta$. Thus β is not a lower bound of A, and so $\alpha = \inf A$ as desired.

Exercise 1.6. (a) We have that b^m and b^p are positive reals, so $(b^m)^{1/n}$ and $(b^p)^{1/q}$ are well-defined positive reals by Theorem 1.21. We also have that

$$\begin{aligned}((b^m)^{1/n})^{nq} &= (((b^m)^{1/n})^n)^q \\ &= (b^m)^q \\ &= b^{mq}\end{aligned}$$

and

$$\begin{aligned}((b^p)^{1/q})^{nq} &= (((b^p)^{1/q})^q)^n \\ &= (b^p)^n \\ &= b^{pn}.\end{aligned}$$

But $r = m/n = p/q$ implies that $mq = pn$, and so

$$((b^m)^{1/n})^{nq} = ((b^p)^{1/q})^{nq}.$$

By the uniqueness statement in Theorem 1.21,

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence

$$b^r := (b^m)^{1/n}$$

is well-defined. We note also that if $n = 1$, $b^r = (b^m)^{1/1} = b^m$ and so this definition is compatible with usual exponentiation with integer powers.

(b) Let m, n, p, q be integers such that $n, q > 0$ with $r = m/n$ and $s = p/q$. By the definition in part (a),

$$\begin{aligned}(b^r b^s)^{nq} &= ((b^m)^{1/n}(b^p)^{1/q})^{nq} \\ &= (((b^m)^{1/n})^n)^q(((b^p)^{1/q})^q)^n \\ &= (b^m)^q(b^p)^n \\ &= b^{mq}b^{pn} \\ &= b^{mq+pn}.\end{aligned}$$

On the other hand,

$$r + s = \frac{m}{n} + \frac{p}{q} = \frac{mq + pn}{nq}$$

with $nq > 0$, so

$$\begin{aligned}(b^{r+s})^{nq} &= ((b^{mq+pn})^{1/(nq)})^{nq} \\ &= b^{mq+pn}.\end{aligned}$$

Hence

$$(b^r b^s)^{nq} = (b^{r+s})^{nq}.$$

Since $b^r b^s$ and b^{r+s} are positive reals, the uniqueness statement in Theorem 1.21 yields $b^{r+s} = b^r b^s$ as desired.

(c) Since $b^r \in B(x)$, to prove $b^r = \sup B(r)$ it suffices to show that b^r is an upper bound of $B(x)$. Let $t \in \mathbf{Q}$ with $t \leq r$. If m, n, p, q are integers with $n, q > 0$ with $r = m/n$ and $t = p/q$, then $t \leq r$ implies $pn \leq mq$. Then

$$\begin{aligned}(b^t)^{nq} &= ((b^p)^{1/q})^{nq} \\ &= (((b^p)^{1/q})^q)^n \\ &= b^{pn} \\ &\leq b^{mq} \\ &= (((b^m)^{1/n})^n)^q \\ &= ((b^m)^{1/n})^{nq} \\ &= (b^r)^{nq}\end{aligned}$$

where $b^{pn} \leq b^{mq}$ follows by induction from $1 < b$. Thus $b^t \leq b^r$ since nq is a positive integer, proving the claim.

[TODO]

(d) We wish to show that $b^x b^y = \sup B(x+y)$. If $v \in \mathbf{Q}$ such that $v \leq x+y$, let t be any rational for which $v - y \leq t \leq x$, by Theorem 1.20(b). Then $t \leq x$ and $v - t \leq y$, with $t, v-t \in \mathbf{Q}$. Hence $t \in B(x)$ and $v - t \in B(y)$, so $b^t \leq b^x$ and $b^{v-t} \leq b^y$ by the definition of b^x and b^y in part (c). Then by part (b),

$$b^v = b^t b^{v-t} \leq b^x b^y$$

so $b^x b^y$ is an upper bound of $B(x+y)$. On the other hand, suppose $\beta < b^x b^y$. Then $\beta(b^y)^{-1} < b^x$, so $\beta(b^y)^{-1}$ is not an upper bound of $B(x)$. Hence there is $t \in \mathbf{Q}$ such that $t \leq x$ and $\beta(b^y)^{-1} < b^t$. Then $\beta < b^t b^y$, so $(b^t)^{-1}\beta < b^y$. Hence $(b^t)^{-1}\beta$ is not an upper bound of $B(y)$, so there is $s \in \mathbf{Q}$ with $s \leq y$ such that $(b^t)^{-1}\beta < b^s$. Then $\beta < b^t b^s = b^{t+s}$, by part (b). But $t + s \in \mathbf{Q}$ with $t + s < x + y$, and so β is not an upper bound of $B(x+y)$. Hence we have shown that $b^x b^y$ is the least upper bound of $B(x+y)$, as desired.

Exercise 1.7. (a) We prove the claim by induction on n. For $n = 1$, we clearly have an equality. If $n \in \mathbf{N}$, we have from $b > 1$ that $b^n > 1$ and so $b(b^n - 1) > b^n - 1$. Thus if $b^n - 1 \geq n(b-1)$, we have

$$\begin{aligned} b^{n+1} - 1 &= b(b^n - 1) + (b - 1) \\ &> (b^n - 1) + (b - 1) \\ &\geq n(b-1) + (b-1) \\ &= (n+1)(b-1), \end{aligned}$$

proving the claim.

(b) If $b^{1/n} \leq 1$, then $b = (b^{1/n})^n \leq 1$, a contradiction. Thus $b^{1/n} > 1$, and so by part (a),

$$b - 1 = (b^{1/n})^n - 1 \geq n(b^{1/n} - 1)$$

for all positive integers n.

(c) We saw in part (b) that $b^{1/n} > 1$ for all positive integers n. Thus $b^{1/n} - 1 > 0$, and so by part (b), we have for any positive integer $n > (b-1)/(t-1)$ that

$$b - 1 \geq n(b^{1/n} - 1) > \frac{b-1}{t-1}(b^{1/n} - 1).$$

Then since $b - 1 > 0$ and $t - 1 > 0$, rearranging yields

$$t - 1 > b^{1/n} - 1.$$

Hence $b^{1/n} < t$.

(d) From $b^w < y$ and $b^w > 0$, we have $(b^w)^{-1} y > 1$. Thus by part (c), if n is a positive integer with

$$n > \frac{b-1}{(b^w)^{-1}y - 1},$$

then

$$b^{1/n} < (b^w)^{-1} y.$$

Thus since $b^w > 0$,

$$b^{w+1/n} < y$$

for sufficiently large n, by Exercise 1.6(d). (We note that by Theorem 1.20(a), such sufficiently large n exist.)

(e) If $b^w > y$, then $y^{-1} b^w > 1$ as $y > 0$. Hence by part (c), if n is a positive integer for which

$$n > \frac{b-1}{y^{-1}b^w - 1},$$

we have

$$b^{1/n} < y^{-1}b^w.$$

Then from $y > 0$,

$$yb^{1/n} < b^w.$$

We have by Exercise 1.6(b) that

$$b^{w-1/n}b^{1/n} = b^w,$$

and so

$$b^{w-1/n} = b^w(b^{1/n})^{-1} > y,$$

proving the claim. (Again, by Theorem 1.20(a), such sufficiently large n exist.)

(f) We first show that b^x is a strictly increasing function of x, as this will be used throughout and in part (g). Indeed, let $x_1 < x_2$. Then $x_2 - x_1 > 0$, and so by Theorem 1.20(b), there exists a positive rational r such that $r \leq x_2 - x_1$. Let m, n be integers for which $n > 0$ and $r = m/n$. Then $m = rn$ is positive. By the definitions in Exercise 1.6(a, c), we have that $(b^m)^{1/n} \leq b^{x_2 - x_1}$. From $b > 1$, we have that $b^m > 1$ and thus also $(b^m)^{1/n} > 1$. Hence $1 < b^{x_2 - x_1}$, so $b^{x_1} < b^{x_2}$ as desired.

If n is a positive integer for which $n(b-1) > 1/y - 1$ (such an n exists by Theorem 1.20(a) since $b > 1$), then by part (a),

$$b^n - 1 \geq n(b-1) > 1/y - 1.$$

Hence $1/y < b^n$, so $b^{-n} < y$. Thus $-n \in A$, so A is nonempty. On the other hand, if n is a positive integer such that $n(b-1) \geq y - 1$,

$$b^n - 1 \geq n(b-1) \geq y - 1$$

by part (a). Thus if $w \geq n$, $b^w \geq b^n \geq y$ and so $w \notin A$. Hence A is bounded above. Then $x = \sup A$ exists.

Now suppose for sake of contradiction that $b^x \neq y$. If $b^x < y$, then by part (d), $b^{x+1/n} < y$ for some positive integer n. Since $x < x + 1/n$, this contradicts that x is an upper bound of A. Otherwise, $b^x > y$ and so by part (e), $b^{x-1/n} > y$ for some positive integer n. Then if $w > x - 1/n$, we have $b^w > b^{x-1/n} > y$ and thus $w \notin A$. Hence $x - 1/n$ is an upper bound of A with $x - 1/n < x$, a contradiction. Thus $b^x = y$ as desired.

(g) Suppose x_1 and x_2 are distinct reals. WLOG, $x_1 < x_2$. Then since b^x is a strictly increasing function of x (as shown in part (f)), $b^{x_1} < b^{x_2}$ and so there is a unique x for which $b^x = y$.

Exercise 1.8. Suppose for sake of contradiction that there is an order $<$ on $\mathbf{C}$ under which $\mathbf{C}$ is an ordered field. Then by Proposition 1.18(d), we have that $1 > 0$ and $-1 > 0$ since $1 = 1^2$ and $-1 = i^2$. Thus by Proposition 1.18(a), $1 > 0$ and $1 < 0$, a contradiction.

Exercise 1.9 (TODO).

Exercise 1.10. We have that $|u| \leq |w|$ by Theorem 1.33(d), so $|w| + u \geq 0$ and $|w| - u \geq 0$. Then by the Corollary to Theorem 1.21,

$$\begin{aligned}
z^2 &= \left(\left(\frac{|w|+u}{2}\right)^{1/2} + \left(\frac{|w|-u}{2}\right)^{1/2} i\right)^2 \\
&= \frac{|w|+u}{2} + 2\left(\frac{|w|+u}{2}\right)^{1/2}\left(\frac{|w|-u}{2}\right)^{1/2} i - \frac{|w|-u}{2} \\
&= u + 2\left(\frac{|w|^2-u^2}{4}\right)^{1/2} i \\
&= u + |v|i.
\end{aligned}$$

Thus if $v \geq 0$, we have $z^2 = w$, and if $v \leq 0$, then $z^2 = \overline{w}$. In the latter case, we have

$$(\overline{z})^2 = \overline{(z^2)} = \overline{\overline{w}} = w$$

by Theorem 1.31(b).

If z is a nonzero complex number with $z^2 = 0$, then z has a multiplicative inverse in $\mathbf{C}$ and so $z = 0$, a contradiction. Thus 0 has only one complex square root, 0. Now let w be a nonzero complex number. By the argument above, there exists $z \in \mathbf{C}$ such that $z^2 = w$. Since $w \neq 0$, we have that $z \neq 0$ and so $z \neq -z$. Thus since $(-z)^2 = z^2 = w$, we have that w has at least two complex square roots. But if $z_1, z_2 \in \mathbf{C}$ such that $z_1^2 = z_2^2$, then $(z_1 - z_2)(z_1 + z_2) = 0$ and so $z_2 = \pm z_1$. Hence w has exactly two complex square roots, proving the claim.

Exercise 1.11 (TODO).

Exercise 1.12. We prove the claim by induction on n. For $n = 1$, we have a trivial equality. Now suppose that the claim is proven for any collection of n complex numbers (with $n \in \mathbf{N}$), and let $z_1, \ldots, z_{n+1} \in \mathbf{C}$. Then by Theorem 1.33(e),

$$\begin{aligned}
|z_1 + z_2 + \cdots + z_{n+1}| &= |(z_1 + z_2 + \cdots + z_n) + z_{n+1}| \\
&\leq |z_1 + z_2 + \cdots + z_n| + |z_{n+1}| \\
&\leq |z_1| + |z_2| + \cdots + |z_{n+1}|,
\end{aligned}$$

which proves the claim.

Exercise 1.13. We have by Theorem 1.33(e) that

$$|x| = |(x-y) + y| \leq |x-y| + |y|$$

and

$$|y| = |x + (-x+y)| \leq |x| + |x-y|.$$

Rearranging,

$$|x| - |y| \leq |x-y|$$

and

$$-|x| + |y| \leq |x-y|.$$

Since $||x| - |y|| = |x| - |y|$ or $-|x| + |y|$, it thus follows that

$$||x| - |y|| \leq |x-y|.$$

Exercise 1.14. We have by Theorem 1.31(a) that

$$\begin{aligned}
|1+z|^2 + |1-z|^2 &= (1+z)\overline{(1+z)} + (1-z)\overline{(1-z)} \\
&= (1+z)(1+\overline{z}) + (1-z)(1-\overline{z}) \\
&= 1 + (z+\overline{z}) + z\overline{z} + 1 - (z+\overline{z}) + z\overline{z} \\
&= 2 + |z|^2.
\end{aligned}$$

Exercise 1.15 (TODO)**.**

Exercise 1.16. TODO

(a) Let $2r = d$. For $z \in \mathbf{R}^k$, we have

$$\begin{aligned}
|2z-(x+y)|^2 &= (2z-(x+y)) \cdot (2z-(x+y)) \\
&= (2z)\cdot(2z) - (2z)\cdot(x+y) - (x+y)\cdot(2z) + (x+y)\cdot(x+y) \\
&= 4z\cdot z - 4z\cdot x - 4z\cdot y + x\cdot x + 2x\cdot y + y\cdot y
\end{aligned}$$

[TODO]

(b) Suppose that $z \in \mathbf{R}^k$ such that

$$|z-x| = |z-y| = r.$$

Then by Theorem 1.37(f),

$$\begin{aligned}
d &= |x-y| \\
&= |y-x| \\
&\leq |y-z| + |z-x| \\
&= |z-x| + |z-y| \qquad = 2r,
\end{aligned}$$

Thus $d \leq 2r$, and so if $2r < d$, no such z exist.

[TODO: $k = 2$ or 1]

Exercise 1.17. The proof of this equality is essentially identical to Exercise 1.14: for $x, y \in \mathbf{R}^k$, we have

$$\begin{aligned}
|x+y|^2 + |x-y|^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y) \\
&= x\cdot x + (x\cdot y + y\cdot x) + y\cdot y + x\cdot x - (x\cdot y + y\cdot x) + y\cdot y \\
&= 2|x|^2 + 2|y|^2.
\end{aligned}$$

This is the classical parallelogram identity ("the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its side lengths") applied to a parallelogram lying in $\mathbf{R}^k$ with vertices at 0, x, y, and $x+y$.

Exercise 1.18. Suppose first that $k \geq 2$ and $x \in \mathbf{R}^k$. If $x = 0$, let y be any nonzero vector in $\mathbf{R}^k$. Then $y \neq 0$ but $x \cdot y = 0$. Otherwise $x \neq 0$, and assume WLOG that $x_1 \neq 0$. Then let

$$y = (-x_2, x_1, 0, \ldots, 0).$$

Since $x_1 \neq 0$, we have that $y \neq 0$. Moreover,

$$x \cdot y = (x_1)(-x-2) + (x_2)(x_1) + (x_3)(0) + \cdots + (x_k)(0) = 0$$

as desired.

Now suppose that $k = 1$, so $x \in \mathbf{R}$. If $y \in \mathbf{R}$ is nonzero with $x \cdot y = 0$, then $xy = 0$ (in the sense of real multiplication) and thus $x = 0$ as y has a multiplicative inverse. Since there exist nonzero real numbers (e.g., 1), we thus have that the claim fails for $k = 1$.

Exercise 1.19. If $a = b$, then $|x - a| = 2|x - b|$ if and only if $|x - a| = 0$, that is, $x = a$. If there exist $c \in \mathbf{R}^k$ and $r > 0$ as in the problem statement, then $c - (r, 0, \dots, 0)$ and $c + (r, 0, \dots, 0)$ are distinct values of x satisfying $|x - a| = 2|x - b|$, a contradiction. Thus we assume that $a \neq b$.

As in the hint, let $c = (4b - a)/3$ and $r = 2|b - a|/3$. Then $c \in \mathbf{R}^k$, and $r > 0$ since $a \neq b$. Then

$$\begin{aligned}
9|x - c|^2 &= |3x - (4b - a)|^2 \\
&= (3x - (4b - a)) \cdot (3x - (4b - a)) \\
&= (3x) \cdot (3x) - (3x) \cdot (4b - a) - (4b - a) \cdot (3x) + (4b - a) \cdot (4b - a) \\
&= 9x \cdot x - 24x \cdot b - 6x \cdot a + 16b \cdot b - 8b \cdot a + a \cdot a \\
&= 3(4(x \cdot x - 2x \cdot b + b \cdot b) - (x \cdot x - 2x \cdot a + a \cdot a)) + 4(b \cdot b - 2b \cdot a + a \cdot a) \\
&= 3(4|x - b|^2 - |x - a|^2) + 4|b - a|^2 \\
&= 3(4|x - b|^2 - |x - a|^2) + 9r^2.
\end{aligned}$$

Hence $9|x - c|^2 = 9r^2$ if and only if $4|x - b|^2 - |x - a|^2 = 0$, that is, $|x - a| = 2|x - b|$ if and only if $|x - c| = r$.

Exercise 1.20 (TODO)**.**

CHAPTER 2

Basic Topology

Exercise 2.1. Let S be a set. Then there is no $x \in \varnothing$, so vacuously we have that $x \in S$ whenever $x \in \varnothing$. Hence $\varnothing \subseteq S$.

Exercise 2.2 (TODO)**.**

Exercise 2.3 (TODO)**.**

Exercise 2.4 (TODO)**.**

Exercise 2.5 (TODO)**.**

Exercise 2.6. Suppose that $x \notin E'$. Then there is an open neighborhood G of x for which $G \cap E$ is empty or equal to $\{x\}$. Let $y \in G \setminus \{x\}$. Then $y \notin G \cap E$, and so y is not a limit point of E. Hence no point of G other than x is in E', and so x is not a limit point of E'. Thus E' contains all its limit points and so is closed.

Since $E \subseteq \overline{E}$, it is clear that any limit point of E is also a limit point of $\overline{E}$. Thus suppose x is a limit point of $\overline{E}$, and let G be an open neighborhood of x. Then G contains some $y \in \overline{E}$ other than x. If $y \in E'$, then letting r be a positive real such that $r < d(x,y)$ and $B_r(y) \subseteq G$, we have that $B_r(y)$ contains a point z of E other than x and y. Then G contains a point of E other than x, and hence x is a limit point of E. This completes the proof that the set of limit points of E equals that of $\overline{E}$.

It is not the case that E and E' always have the same limit points. For example, let $E = \{1/n \mid n \in \mathbf{N}\}$. Then $E' = \{0\}$, which has now limit point since it is finite (Corollary to Theorem 2.20).

Exercise 2.7. (a) By Theorem 2.27(a), each $\overline{A_i}$ is closed, and thus by Theorem 2.24(d), $\bigcup_{i=1}^n \overline{A_i}$. Then from

$$B = \bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^n \overline{A_i},$$

we have by Theorem 2.27(c) that $\overline{B} \subseteq \bigcup_{i=1}^n \overline{A_i}$. Conversely, for each $i = 1, \ldots, n$, we have $A_i \subseteq B \subseteq \overline{B}$ and hence $\overline{A_i} \subseteq \overline{B}$ by Theorem 2.27(a, c). Thus

$$\overline{B} = \bigcup_{i=1}^n \overline{A_i}$$

as desired.

(b) For each $i \in \mathbf{N}$, we have $A_i \subseteq B \subseteq \overline{B}$. Hence by Theorem 2.27(a, c), $\overline{A_i} \subseteq \overline{B}$, and so

$$\bigcup_{i=1}^\infty \overline{A_i} \subseteq \overline{B}.$$

Let our metric space be the real line $\mathbf{R}$ with the Euclidean metric. For each $i \in \mathbf{N}$, let $A_i = [1/i, 1]$. Then each A_i is closed and so $A_i = \overline{A_i}$ (Theorem 2.27(b)), and so

$$B = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right] = (0, 1]$$

and also

$$\bigcup_{i=1}^{\infty} \overline{A_i} = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right] = (0, 1].$$

But 0 is a limit point of B, and hence

$$\overline{B} \supsetneq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

Exercise 2.8. Let E be an open subset of $\mathbf{R}^k$ for some $k \in \mathbf{N}$, and suppose $x \in E$. Then there is $r > 0$ such that $B_r(x) \subseteq E$. If $s > 0$, then $B_{\min(r,s)}(x) \subseteq B_s(x) \cap E$ contains a point other than x (e.g., $x + (\min(r, s)/2, 0, \ldots, 0)$) and hence x is a limit point of E.

The corresponding claim for closed subsets of $\mathbf{R}^k$ is false for all $k \in \mathbf{N}$. Indeed, any nonempty finite subset of $\mathbf{R}^k$ is vacuously closed by the Corollary to Theorem 2.20 but, for the same reason, is not contained in its set of limit points.

Exercise 2.9. (a) Let $x \in E^\circ$; then there is an open neighborhood G of x for which $G \subseteq E$. Since G is open, we have for all $y \in G$ that G is an open neighborhood of y that is contained in E. Hence $G \subseteq E^\circ$ and thus E° is open.

(b) Since E° is open by part (a), we have that if $E^\circ = E$ then E is open. Conversely, suppose that E is open. Then by definition, $E \subseteq E^\circ$. On the other hand, we always have $E^\circ \subseteq E$, and hence $E^\circ = E$.

(c) Since G is open with $G \subseteq E$, we have that every point of G is an interior point of E and so $G \subseteq E^\circ$.

(d) Let $x \notin E^\circ$. Then for all open neighborhoods G of x, we have $G \not\subseteq E$ and hence G intersects E^c. Thus $x \in \overline{E^c}$. Conversely, suppose $x \in \overline{E^c}$. Then every open neighborhood G of x intersects E^c, and so x is not an interior point of E. Hence $x \notin E^\circ$, and so $(E^\circ)^c = \overline{E^c}$ as desired.

(e) No; for example, let $E = \mathbf{Q}$ in $\mathbf{R}$ (with the Euclidean metric). Then $E^\circ = \varnothing$ since every segment contains an irrational. But $\overline{E} = \mathbf{R}$ has interior $\mathbf{R}$.

(f) No; again, let $E = \mathbf{Q}$ in $\mathbf{R}$ (with the Euclidean metric). Then $\overline{E} = \mathbf{R}$ but $E^\circ = \varnothing$ has empty closure.

Exercise 2.10. Part (a) of Definition 2.15 is clear. Moreover, $p = q$ if and only if $q = p$ and so part (b) holds as well. Now let $p, q, r \in X$. If $p = q = r$, then $d(p, q) = d(p, r) = d(r, q) = 0$ so $d(p, q) = d(p, r) + d(r, q)$. If $p = q$ and r is distinct from p, q, then $d(p, q) = 0$ while $d(p, r) = d(r, q) = 1$ so $d(p, q) < d(p, r) + d(r, q)$. If $p = r$ and q is distinct from p, r, then $d(p, q) = 1$, $d(p, r) = 0$, and $d(r, q) = 1$ so $d(p, q) = d(p, r) + d(r, q)$. Similarly in the case that $q = r$ and p is distinct from q, r. Finally if p, q, and r are distinct, then $d(p, q) = d(p, r) = d(r, q) = 1$ and so $d(p, q) < d(p, r) + d(r, q)$. This proves part (c) of Definition 2.15, and so d is a metric on X.

[TODO]

Exercise 2.11. (a) This is not a metric on $\mathbf{R}$. For example,

$$d_1(0, 2) = (0 - 2)^2 = 4,$$

$$d_1(0, 1) = (0 - 1)^2 = 1,$$

and

$$d_1(1, 2) = (1 - 2)^2 = 1$$

so
$$d_1(0,2) \not\leq d_1(0,1) + d_1(1,2).$$
Then d_1 fails to satisfy part (c) of Definition 2.15.

(b) This is a metric on $\mathbf{R}$. For all $x, y \in \mathbf{R}$, we have $d_2(x,y) \geq 0$ by the definition of square roots (Theorem 1.21). Moreover, $d_2(x,y) = 0$ if and only if $|x-y| = 0$, that is, $x = y$. For $x, y \in \mathbf{R}$, we also have
$$\begin{aligned} d_2(x,y) &= \sqrt{|x-y|} \\ &= \sqrt{|y-x|} \\ &= d_2(y,x). \end{aligned}$$
Finally, if $x, y, z \in \mathbf{R}$ then
$$\begin{aligned} d_2(x,y)^2 &= |x-y| \\ &\leq |x-z| + |z-y| \\ &\leq d_2(x,z)^2 + d_2(z,y)^2 \\ &\leq (d_2(x,z) + d_2(z,y))^2. \end{aligned}$$
Hence
$$d_2(x,y) \leq d_2(x,z) + d_2(z,y),$$
so d_2 is a metric on $\mathbf{R}$.

(c) This is not a metric on $\mathbf{R}$. We observe that
$$d_3(-1,1) = \left|(-1)^2 - 1^2\right| = 0$$
but $-1 \neq 1$, so part (a) of Definition 2.15 is not satisfied by d_3.

(d) This is not a metric on $\mathbf{R}$. For example, we have that
$$d_4(0,1) = |0 - 2(1)| = 2$$
but
$$d_4(1,0) = |1 - 2(0)| = 1.$$
Thus part (b) of Definition 2.15 is not satisfied by d_4.

(e) This is a metric on $\mathbf{R}$. We have for all $x, y \in \mathbf{R}$ that $|x-y| \geq 0$ and $1 + |x-y| > 0$, so $d_5(x,y) \geq 0$. Also $d_5(x,y) = 0$ if and only if $|x-y| = 0$, that is, $x = y$. Moreover,
$$\begin{aligned} d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \\ &= \frac{|y-x|}{1+|y-x|} \\ &= d_5(y,x). \end{aligned}$$
Finally, for $x, y, z \in \mathbf{R}$,
$$|x-y| \leq |x-z| + |z-y| + 2|z-y||x-z| + |z-y||x-y||x-z|,$$
so
$$|x-y|(1+|x-z|)(1+|z-y|) \leq |x-z|(1+|x-y|)(1+|z-y|) + |z-y|(1+|x-y|)(1+|x-z|).$$
Thus
$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|},$$

that is,

$$d_5(x,y) \le d_5(x,z) + d_5(z,y),$$

so d_5 is a metric on $\mathbf{R}$.

Exercise 2.12. Let $\{G_\alpha\}_{\alpha\in A}$ be an open cover of K. Then there is $\alpha_0 \in A$ such that $0 \in G_{\alpha_0}$. Since G_{α_0} is an open subset of $\mathbf{R}$, there is $r > 0$ such that $B_r(0) \subseteq G_{\alpha_0}$. Then for $n > 1/r$, we have that $1/n \in G_{\alpha_0}$. If N denotes the largest natural number less than or equal to $1/r$, then for each $n = 1, \ldots, N$, let $\alpha_n \in A$ such that $1/n \in G_{\alpha_n}$. Then $\{G_{\alpha_n}\}_{n=0}^{N}$ is a finite subcover of $\{G_\alpha\}_{\alpha\in A}$ for K, so K is compact.

Exercise 2.13 (TODO)**.**

Exercise 2.14. For each natural number $n \ge 2$, let $G_n = (1/n, 1)$. Then $\{G_n\}_{n\ge 2}$ is an open cover of $(0,1)$. Suppose for sake of contradiction that there is a finite subcover of $\{G_n\}_{n\ge 2}$ for $(0,1)$. Let N be the largest natural number for which G_N is in this subcover. Then since $(1/n, 1) \supseteq (1/(n+1), 1)$ for all $n \ge 2$, we have that $(0,1) = (1/N, 1)$, a contradiction. Hence $\{G_n\}_{n\ge 2}$ has no finite subcover for $(0,1)$, so $(0,1)$ is not compact.

Exercise 2.15 (TODO)**.**

Exercise 2.16 (TODO)**.**

Exercise 2.17 (TODO)**.**

Exercise 2.18 (TODO)**.**

Exercise 2.19. (a) By Theorem 2.27(b), we have $A = \overline{A}$ and $B = \overline{B}$. Hence

$$A \cap \overline{B} = A \cap B = \varnothing$$

and

$$\overline{A} \cap B = A \cap B = \varnothing.$$

Thus A and B are separated.

(b) Let A, B be disjoint open subsets of a metric space X. Then $B \subseteq A^c$ where A^c is closed (Theorem 2.23), so by Theorem 2.27(c), $\overline{B} \subseteq A^c$. Thus $A \cap \overline{B} = \varnothing$. Similarly, $A \subseteq B^c$ with B^c closed (Theorem 2.23), so by Theorem 2.27(c), $\overline{A} \subseteq B^c$. Then $\overline{A} \cap B = \varnothing$, so A and B are separated.

(c) It is clear that A and B are disjoint, and $A = B_\delta(p)$ is open by Theorem 2.19. We claim that B is also open. Let $q \in B$. Then $d(p,q) > \delta$, and so $d(p,q) - \delta > 0$. If $r \in B_{d(p,q)-\delta}(q)$, then $d(r,q) < d(p,q) - \delta$ so

$$d(p,r) \ge d(p,q) - d(r,q) > \delta.$$

Thus $B_{d(p,q)-\delta}(q) \subseteq B$, so q is an interior point of B. Hence B is open. Now by part (b), A and B are separated.

(d) Let X be a connected metric space and suppose that there is $\delta > 0$ such that there is no $q \in X$ with $d(p,q) = \delta$. Then if A and B are defined as in part (c), we have that $X = A \cup B$ with A and B separated. Since $p \in A$, we thus have that $B = \varnothing$ as X is connected.

Hence if $|X| \ge 2$ and p, q are distinct points of X, then for every $\delta \in [0, d(p,q)]$ there exists $r \in X$ with $d(p,r) = \delta$. Then the cardinality of X is at least that of $[0, d(p,q)]$, which by the Corollary to Theorem 2.43 is uncountable.

Exercise 2.20 (TODO)**.**

Exercise 2.21.

Exercise 2.22. As in the hint, we show that $\mathbf{Q}^k$ is a dense subset of $\mathbf{R}^k$ (Theorem 2.13 and its Corollary, $\mathbf{Q}^k$ is countable). Let $x \in \mathbf{R}^k$ and r a positive real. Then by Theorem 1.20(b), for each $i = 1, \ldots, k$ there exists $p_i \in \mathbf{Q}$ such that

$$x_i - r/\sqrt{k} < p_i < x_i + r/\sqrt{k}.$$

Then $p = (p_1, \ldots, p_k) \in \mathbf{Q}^k$ with

$$|x_i - p_i| < \frac{r}{\sqrt{k}}$$

for each i, so

$$\begin{aligned} |x - p| &< \sqrt{\left(\frac{r}{\sqrt{k}}\right)^2 + \cdots + \left(\frac{r}{\sqrt{k}}\right)^2} \\ &= \sqrt{\frac{r^2}{k} + \cdots + \frac{r^2}{k}} \\ &= \sqrt{r^2} \\ &= r. \end{aligned}$$

Hence $p \in B_r(x)$ and so $\mathbf{Q}^k$ is dense in $\mathbf{R}^k$.

Exercise 2.23. Let C be a countable dense subset of X. As in the hint, we show that $\{B_r(p)\}_{r \in \mathbf{Q}_{>0}, p \in C}$ is a base for X. By the Corollary to Theorem 2.12, this set is at most countable, and by Theorem 2.19, it consists of open subsets of X. Now let $x \in X$ and suppose G is an open neighborhood of x. Then there is $\delta > 0$ such that $B_\delta(x) \subseteq G$. By Theorem 1.20(b), there is a positive rational $r < \delta/2$. Since C is dense in X, there is $p \in C$ such that $d(x, p) < r$. Then

$$x \in B_r(p) \subseteq B_\delta(x) \subseteq G$$

as desired.

Exercise 2.24. (The claim is false; for example, let X be any finite metric space. Rudin likely meant to define a separable metric space as one with an *at most* countable dense subset. This is the definition I use in the solution below.)

We follow the hint. Let $\delta > 0$ and suppose for sake of contradiction that there are $x_i \in X$ indexed by $i \in \mathbf{N}$ for which $d(x_i, x_j) \geq \delta$ for all distinct $i, j \in \mathbf{N}$. Then the x_i are all distinct and so $\{x_i\}_{i \in \mathbf{N}}$ is an infinite subset of X, and hence has a limit point x. By Theorem 1.20, there are then infinitely many $i \in \mathbf{N}$ for which $x_i \in B_{\delta/2}(x)$. Suppose i, j are distinct natural numbers for which $x_i, x_j \in B_{\delta/2}(x)$; then

$$d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) < \delta,$$

a contradiction.

Thus for any $\delta > 0$, there are $x_1, \ldots, x_k \in X$ such that $d(x_i, x_j) \geq \delta$ for distinct $i, j = 1, \ldots, N$ and for which there is no $x \in X$ with $d(x_i, x) \geq \delta$ for $i = 1, \ldots, N$. Then $\{B_\delta(x_i)\}_{i=1}^N$ covers X. In particular, for each $n \in \mathbf{N}$, there is $N_n \in \mathbf{N}$ and $x_{1,n}, \ldots, x_{N_n,n} \in X$ such that $\{B_{1/n}(x_{i,n})\}_{i=1}^{N_n}$. We claim that $\bigcup_{n \in \mathbf{N}} \{x_{i,n}\}_{i=1}^{N_n}$ is a dense subset of X; by the Corollary to Theorem 2.12, this set is at most countable. Let $p \in X$ and $\delta > 0$. Then by Theorem 1.20(a), there is $n \in \mathbf{N}$ such that $n > 1/\delta$, so $1/n < \delta$. Then there is $i = 1, \ldots, N_n$ such that $p \in B_{1/n}(x_{i,n})$ and hence $x_{i,n} \in B_\delta(p)$, proving the claim.

Exercise 2.25. For any $n \in \mathbf{N}$, $\{B_{1/n}(p)\}_{p \in K}$ is an open cover of K (Theorem 2.19). Then since K is compact, there are $p_1, \ldots, p_{N_n} \in K$ such that $\{B_{1/n}(p_i)\}_{i=1}^{N_n}$ covers K. We claim that $\bigcup_{n \in \mathbf{N}} \{B_{1/n}(p_i)\}_{i=1}^{N_n}$ is a base for K; by the Corollary to Theorem 2.12, it is at most countable. Suppose $x \in X$ and let G be an open neighborhood of x. Then there is $\delta > 0$ such that $B_\delta(x) \subseteq G$. By Theorem 1.20(a), there exists $n \in \mathbf{N}$ with $n > 2/\delta$, so $1/n < \delta/2$. Then since $\{B_{1/n}(p_i)\}_{i=1}^{N_n}$ covers K, there is $i = 1, \ldots, N_n$ such that

$$x \in B_{1/n}(p_i) \subseteq B_\delta(x) \subseteq G.$$

Now we show that a metric space X with an at most countable base $\{V_\alpha\}_{\alpha \in A}$ is separable, using the definition as in the solution to Exercise 2.24. We may assume WLOG that each V_α is nonempty. For every $\alpha \in A$, let $x_\alpha \in V_\alpha$. Let $p \in X$ and $\delta > 0$. Then by Theorem 2.19, $B_\delta(p)$ is an open neighborhood of p and so there is $\alpha \in A$ such that $x \in V_\alpha \subseteq B_\delta(p)$. Then $x_\alpha \in B_\delta(p)$, and so $\{x_\alpha\}_{\alpha \in A}$ is a dense subset of X.

Exercise 2.26. We follow the hint. By Exercise 2.24, X is separable, and so by Exercise 2.23, X has a countable base $\{V_\alpha\}_{\alpha \in A}$. Now suppose $\{G_\beta\}_{\beta \in B}$ is an open cover of X. Let A' consist of the $\alpha \in A$ such that $V_\alpha \subseteq G_\beta$ for some $\beta \in B$. For each $\alpha \in A'$, let $\beta_\alpha \in B$ such that $V_\alpha \subseteq G_{\beta_\alpha}$. Then $\{G_{\beta_\alpha}\}_{\alpha \in A'}$ is at most countable by Theorem 2.8. For any $x \in X$, there is $\beta \in B$ such that $x \in G_\beta$ and thus $\alpha \in A'$ such that $x \in V_\alpha \subseteq G_\beta$. Then $x \in G_{\beta_\alpha}$ and so $\{G_{\beta_\alpha}\}_{\alpha \in A'}$ is an at most countable subcover of $\{G_\beta\}_{\beta \in B}$ for X.

Now suppose for sake of contradiction that $\{G_i\}_{i \in \mathbf{N}}$ is a countable open cover of X which has no finite subcover. Then for each $n \in \mathbf{N}$,

$$F_n := (G_1 \cup \cdots \cup G_n)^c$$

is nonempty and $F_n \supseteq F_{n+1}$ for all $n \in \mathbf{N}$. But

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} G_n^c = \varnothing.$$

Then if $x_n \in F_n$ for each $n \in \mathbf{N}$, we have that $\{x_n\}_{n \in \mathbf{N}}$ is infinite. Thus it has a limit point x. Since $\bigcap_{n=1}^{\infty} F_n = \varnothing$, there is $N \in \mathbf{N}$ such that $x \notin F_N$. Since F_N^c is open (Theorem 2.24(a)), F_N^c is then an open neighborhood of x. By Theorem 2.20, F_N^c thus contains infinitely many of the x_n. But $x_n \in F_N$ for $n \geq N$, a contradiction. Thus every open cover of X has a finite subcover, so X is compact.

Exercise 2.27. (Note: The hypothesis that E is uncountable is unnecessary.)

We follow the hint. By Exercise 2.22 and Exercise 2.23, $\mathbf{R}^k$ has a countable base $\{V_n\}_{n \in \mathbf{N}}$. Let W be the union of all V_n such that $V_n \cap E$ is at most countable. Suppose $x \in P$. Then every open neighborhood of x contains uncountably many points of E, and so $V_n \cap E$ is uncountable for any $n \in \mathbf{N}$ such that $x \in V_n$. Thus $x \notin W$. Conversely, suppose $x \notin W$ and let G be an open neighborhood of x. Then there is $n \in \mathbf{N}$ such that $x \in V_n \subseteq G$. Since $x \notin W$, we have that $V_n \cap E$ is uncountable and hence $G \cap E$, which contains $V_n \cap E$, is also uncountable (Theorem 2.8). Hence $x \in P$, and so we have shown that $P = W^c$. Now we have, letting S denote the set of natural numbers n for which $V_n \cap E$ is at most countable,

$$\begin{aligned} P^c \cap E &= W \cap E \\ &= \left(\bigcup_{n \in S} V_n \right) \cap E \\ &= \bigcup_{n \in S} (V_n \cap E) \end{aligned}$$

is at most countable by Theorem 2.8 and the Corollary to Theorem 2.12.

Now we show that P is perfect. Since $P = W^c$, we have by Theorem 2.24(a) and Theorem 2.23 that P is closed. Let x be a point of X which is not a limit point of P. Then there is an open neighborhood G of x such that $G \cap P \subseteq \{x\}$. Then for all $y \in G$ distinct from x, we have $y \in W$ and thus there is $n_y \in \mathbf{N}$ such that $x \in V_{n_y}$ and $V_{n_y} \cap E$ is at most countable. Then

$$\begin{aligned} G \cap E &\subseteq \left(\left(\bigcup_{y \in G \setminus \{x\}} V_{n_y} \right) \cup \{x\} \right) \cap E \\ &\subseteq \left(\bigcup_{y \in G \setminus \{x\}} (V_n \cap E) \right) \cup \{x\} \end{aligned}$$

is at most countable by the Corollary to Theorem 2.12. Thus $x \notin P$, and so P is perfect.

Exercise 2.28. Let F be a closed subset of a separable metric space X, and let P be the set of condensation points of F. Every point of P is a limit point of F, and so $P \subseteq F$ since F is closed. By Exercise 2.23, X has a countable base, and so by Exercise 2.27 (note that the solution to Exercise 2.27 applies to any metric space with a countable base), P is a perfect set and $P^c \cap F$ is at most countable. Since

$$F = (P \cup P^c) \cap F = P \cup (P^c \cap F),$$

we thus have that F is the union of a perfect set and an at most countable set.

Now let F be a countable closed subset of $\mathbf{R}^k$. By Theorem 2.43, F is not perfect, and thus it contains an isolated point.

Exercise 2.29 (TODO)**.**

Exercise 2.30. We prove the equivalent statement, including the fact that $\bigcap_{n=1}^\infty G_n$ is dense in $\mathbf{R}^k$. Clearly since $\mathbf{R}^k$ is nonempty, any dense subset of $\mathbf{R}^k$ is nonempty. Thus it is sufficient to show only that $\bigcap_{n=1}^\infty G_n$ is dense. Let V_0 be any nonempty open subset of $\mathbf{R}^k$. Suppose inductively that we have chosen nonempty open subsets $V_0, V_1, \ldots, V_n$ of $\mathbf{R}^k$ such that $\overline{V_i}$ is compact and contained in $V_{i-1} \cap G_i$ for each $i = 1, \ldots, n$. Then since G_{n+1} is dense in $\mathbf{R}^k$, we have that $V_n \cap G_{n+1}$ is nonempty, and it is open by Theorem 2.24(c). If $x_{n+1} \in V_n \cap G_{n+1}$, there is $r_{n+1} > 0$ such that $\overline{B}_{r_{n+1}}(x_{n+1}) \subseteq V_n \cap G_{n+1}$. Let $V_{n+1} = \overline{B}_{r_{n+1}}(x_{n+1})$. In this way, we construct nonempty open sets $\{V_n\}_{n=0}^\infty$ such that $\overline{V_n}$ is compact and contained in $V_{n-1} \cap G_n$ for $n \in \mathbf{N}$. Then by the Corollary to Theorem 2.36, $\bigcap_{n=1}^\infty \overline{V_n}$ is nonempty. Since

$$\bigcap_{n=1}^\infty \overline{V_n} \subseteq V_0 \cap \left(\bigcap_{n=1}^\infty G_n \right),$$

it follows that V_0 intersects $\bigcap_{n=1}^\infty G_n$. Hence $\bigcap_{n=1}^\infty G_n$ is dense in $\mathbf{R}^k$ as desired.

CHAPTER 3

Numerical Sequences and Series

Exercise 3.1. Suppose $\{s_n\}_{n\in\mathbf{N}}$ converges. Then by Theorem 3.11(a), $\{s_n\}_{n\in\mathbf{N}}$ is Cauchy. Hence for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that $|s_n - s_m| < \varepsilon$ for $n, m \geq N$. Then

$$||s_n| - |s_m|| \leq |s_n - s_m| < \varepsilon$$

for $n, m \geq N$, and so $\{|s_n|\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Thus by Theorem 3.11(c), $\{|s_n|\}_{n\in\mathbf{N}}$ converges.

We provide also a direct proof which does not rely on Cauchy sequences. Suppose $\lim_{n\to\infty} s_n = s$. If $s > 0$, there is $N \in \mathbf{N}$ such that $s_n > 0$ for all $n \geq N$. Hence $|s_n| = s_n$ for $n \geq N$, and so $\lim_{n\to\infty} |s_n| = s$. Similarly, if $s < 0$, there is $N \in \mathbf{N}$ such that $s_n < 0$ for $n \geq N$. Then $|s_n| = -s_n$ for $n \geq N$, and so $\lim_{n\to\infty} |s_n| = -s$. Finally, suppose $s = 0$. Then for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that $|s_n| < \varepsilon$ for $n \geq N$. That is, $\lim_{n\to\infty} |s_n| = 0$.

The converse is false; for example, let $s_n = (-1)^n$ for all $n \in \mathbf{N}$. Then $\{s_n\}_{n\in\mathbf{N}}$ diverges but $\{|s_n|\}_{n\in\mathbf{N}}$ is constant and thus converges.

Exercise 3.2. We observe that for all $n \in \mathbf{N}$,

$$\left(\sqrt{n^2+n}+n\right)\left(\sqrt{n^2+n}-n\right) = \left(\sqrt{n^2+n}\right)^2 - n^2 = n$$

and thus

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

But we have for all $n \in \mathbf{N}$ that

$$1 < \sqrt{1+\frac{1}{n}} < 1 + \frac{1}{2n}$$

with $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty}(1 + 1/(2n)) = 1$. Hence $\lim_{n\to\infty} \sqrt{1+1/n} = 1$. Then

$$\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}+1\right) = 2$$

by Theorem 3.3(b) with

$$\sqrt{1+\frac{1}{n}}+1 > 0$$

for all $n \in \mathbf{N}$, so that

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

by Theorem 3.3(d). Hence

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right) = \frac{1}{2}.$$

Exercise 3.3. We first note that each s_n is positive; this will be used implicitly throughout the solution. By Theorem 3.14, it suffices to show that $\{s_n\}_{n\in\mathbf{N}}$ is monotonically increasing and bounded above. We first show by induction that $s_n < 2$ for all $n \in \mathbf{N}$. For $n = 1$, this claim is true since

$$s_1^2 = 2 < 4 = 2^2.$$

Now suppose $s_n < 2$ for some $n \in \mathbf{N}$. Then

$$s_{n+1}^2 = 2 + \sqrt{s_n} < 2 + \sqrt{2} < 4,$$

so $s_{n+1} < 2$ as desired.

Now we show by induction that $s_n < s_{n+1}$ for all $n \in \mathbf{N}$. We have

$$s_1^2 = 2 < 2 + \sqrt{s_1} = s_2^2,$$

so $s_1 < s_2$. Now suppose $s_n < s_{n+1}$ for some $n \in \mathbf{N}$. Then

$$s_{n+1}^2 = 2 + \sqrt{s_n} < 2 + \sqrt{s_{n+1}} = s_{n+2}^2,$$

so

$$s_{n+1} < s_{n+2},$$

which proves the claim.

Exercise 3.4. We prove by induction that for all $m \in \mathbf{N}$,

$$s_{2m-1} = 1 - \frac{1}{2^{m-1}}$$

and

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m}.$$

We have by definition that $s_1 = 0$ and

$$s_2 = \frac{s_1}{2} = \frac{0}{2} = 0,$$

so the claim holds for $m = 1$. Now suppose it holds for some $m \in \mathbf{N}$; then

$$\begin{aligned}
s_{2(m+1)-1} &= s_{2m+1} \\
&= \frac{1}{2} + s_{2m} \\
&= \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2^m}\right) \\
&= 1 - \frac{1}{2^m} \\
&= 1 - \frac{1}{2^{(m+1)-1}}
\end{aligned}$$

and

$$\begin{aligned}
s_{2(m+1)} &= s_{2m+2} \\
&= \frac{s_{2m+1}}{2} \\
&= \frac{1 - \frac{1}{2^m}}{2} \\
&= \frac{1}{2} - \frac{1}{2^{m+1}},
\end{aligned}$$

proving the claim. Then since $\lim_{m\to\infty} 1/2^{m-1} = 0$, we have by Theorem 3.3(b) that

$$\lim_{m\to\infty} s_{2m-1} = \lim_{m\to\infty}\left(1 - \frac{1}{2^{m-1}}\right) = 1,$$

so $\{s_n\}_{n\in\mathbf{N}}$ has a subsequence convergening to 1. But also $s_n < 1$ for all $n \in \mathbf{N}$, and so

$$\limsup_{n\to\infty} s_n = 1.$$

Moreover, from $\lim_{m\to\infty} 1/2^m = 0$, we have by Theorem 3.3(b) that

$$\lim_{m\to\infty} s_{2m} = \lim_{m\to\infty}\left(\frac{1}{2} - \frac{1}{2^m}\right) = \frac{1}{2}$$

and hence $\{s_n\}_{n\in\mathbf{N}}$ has a subsequence converging to $1/2$. Let $x < 1/2$. Then $1/2 - x > 0$, so there is $M \in \mathbf{N}$ for which

$$\frac{1}{2} - x > \frac{1}{2^M}.$$

Then for any $m \in \mathbf{N}$, we have

$$\frac{1}{2} - x > \frac{1}{2^m}$$

and so

$$\frac{1}{2} - \frac{1}{2^m} > x.$$

Thus $s_{2m} > x$, and so also $s_{2m-1} > x$ since

$$s_{2m-1} = 2s_{2m} > s_{2m}$$

as $s_{2m} > 0$. Then with $N = 2M - 1$, we have that $s_n > x$ for all $n \geq N$. Hence by Theorem 3.17,

$$\liminf_{n\to\infty} s_n = \frac{1}{2}.$$

Exercise 3.5. Let E_{a+b} be the set of all $x \in \overline{\mathbf{R}}$ such that $a_{n_k} + b_{n_k} \to x$ for a subsequence $\{a_{n_k} + b_{n_k}\}_{k\in\mathbf{N}}$ of $\{a_n + b_n\}_{n\in\mathbf{N}}$; let E_a and E_b be defined similarly for the sequences $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$. Then $E_{a+b} \subseteq E_a + E_b$, and so

$$\sup E_{a+b} \leq \sup(E_a + E_b).$$

If $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are both real, then $\sup E_a + \sup E_b$ is an upper bound for $E_a + E_b$. Thus

$$\sup(E_a + E_b) \leq \sup E_a + \sup E_b.$$

Hence

$$\sup E_{a+b} \leq \sup E_a + \sup E_b,$$

that is,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

If $\limsup_{n\to\infty} a_n$ or $\limsup_{n\to\infty} b_n$ is ∞, then there is nothing to prove. Finally, suppose WLOG that $\limsup_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} b_n \neq \infty$. Then by Theorem 3.17(a), $E_a = \{-\infty\}$ and $\{b_n\}_{n\in\mathbf{N}}$ is bounded above. Let $M \in \mathbf{R}$ such that $b_n \leq M$ for all $n \in \mathbf{N}$. By Theorem 3.6(b), it follows from $E_a = \{-\infty\}$ that every subsequence of $\{a_n\}_{n\to\infty}$ is unbounded below. Hence if $\{a_{n_k} + b_{n_k}\}_{k\in\mathbf{N}}$ is any subsequence of $\{a_n + b_n\}_{n\in\mathbf{N}}$, we see from $a_{n_k} + b_{n_k} \leq a_{n_k} + M$ that $\{a_{n_k} + b_{n_k}\}_{k\in\mathbf{N}}$ is unbounded below. Thus $E_{a+b} = -\infty$, so again we have that $\sup E_{a+b} \leq \sup E_a + \sup E_b$. Then

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

in all cases.

Exercise 3.6. (a) It is clear that the nth partial sum of $\sum_{n=1}^{\infty} a_n$ is $\sqrt{n+1}$. Since $\{\sqrt{n+1}\}_{n\in\mathbf{N}}$ is unbounded above, we thus have by Theorem 3.2(c) that $\sum_{n=1}^{\infty} a_n$ diverges.

(b) For each $n \in \mathbf{N}$, we observe that

$$\begin{aligned} |a_n| &= \frac{\sqrt{n+1}-\sqrt{n}}{n} \\ &= \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \\ &< \frac{1}{n\sqrt{n}} \\ &= \frac{1}{n^{3/2}}. \end{aligned}$$

By Theorem 3.28, $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges since $3/2 > 1$. Thus by Theorem 3.25(a), $\sum_{n=1}^{\infty} a_n$ converges.

(c) We have for all $n \in \mathbf{N}$ that

$$\begin{aligned} \sqrt[n]{|a_n|} &= \sqrt[n]{\left|(\sqrt[n]{n}-1)^n\right|} \\ &= \sqrt[n]{n}-1. \end{aligned}$$

But $\lim_{n\to\infty}(\sqrt[n]{n}-1) = 0$ by Theorem 3.20(c) and Theorem 3.3(b). Thus by Theorem 3.33(a), $\sum_{n=1}^{\infty} a_n$ converges.

(d) If $|z| \le 1$, then for any $n \in \mathbf{N}$,

$$\begin{aligned} |a_n| &= \left|\frac{1}{1+z^n}\right| \\ &= \frac{1}{|1+z^n|} \\ &\ge \frac{1}{1+|z|^n} \\ &\ge \frac{1}{2}. \end{aligned}$$

Thus $\{a_n\}_{n\in\mathbf{N}}$ does not converge to 0, and so by Theorem 3.23, $\sum_{n=1}^{\infty} a_n$ diverges.

Now suppose $|z| > 1$. Then for any $n \in \mathbf{N}$,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{1}{1+z^{n+1}}}{\frac{1}{1+z^n}}\right| \\ &= \frac{|1+z^n|}{|1+z^{n+1}|} \\ &\le \frac{1+|z|^n}{|z|^{n+1}-1} \\ &= \frac{1}{|z|} + \frac{1-\frac{1}{|z|}}{|z|^{n+1}-1}. \end{aligned}$$

But since $|z| > 1$, we have that $|z|^{n+1} - 1 \to \infty$ as $n \to \infty$. Hence by Theorem 3.3(b), we have

$$\lim_{n\to\infty} \left(\frac{1}{|z|} + \frac{1 - \frac{1}{|z|}}{|z|^{n+1} - 1} \right) = \frac{1}{|z|} < 1.$$

Thus by Theorem 3.19,

$$\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \limsup_{n\to\infty} \left(\frac{1}{|z|} + \frac{1 - \frac{1}{|z|}}{|z|^{n+1} - 1} \right) < 1,$$

and so by Theorem 3.34(a), $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 3.7. We have for all $n \in \mathbf{N}$ that

$$\left(\sqrt{a_n} - \frac{1}{n} \right)^2 \geq 0$$

and thus

$$a_n - \frac{2\sqrt{a_n}}{n} + \frac{1}{n^2} \geq 0.$$

Rearranging,

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \left(a_n + \frac{1}{n^2} \right).$$

By Theorem 3.28, $\sum_{n=1}^{\infty} 1/n^2$ converges and thus by Theorem 3.47,

$$\sum_{n=1}^{\infty} \frac{1}{2} \left(a_n + \frac{1}{n^2} \right)$$

converges. Now since $\sqrt{a_n}/n \geq 0$ for all $n \in \mathbf{N}$, we conclude from Theorem 3.25(a) that $\sum_{n=1}^{\infty} \sqrt{a_n}/n$ converges.

Exercise 3.8.

Exercise 3.9. (a) We have by Theorem 3.3 and Theorem 3.20(c) that

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|n^3|} &= \lim_{n\to\infty} (\sqrt[n]{n})^3 \\ &= \left(\lim_{n\to\infty} \sqrt[n]{n} \right)^3 \\ &= 1. \end{aligned}$$

Hence by Theorem 3.39, the radius of convergence of $\sum_{n=0}^{\infty} n^3 z^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|n^3|}} = 1.$$

(b) We observe for all $z \in \mathbf{C}$ and $n \in \mathbf{N}$ that

$$\left| \frac{\frac{2^{n+1}}{(n+1)!} z^{n+1}}{\frac{2^n}{n!} z^n} \right| = \frac{2|z|}{n+1}.$$

By Theorem 3.3(b) and Theorem 3.20(a),

$$\lim_{n\to\infty} \frac{2|z|}{n+1} = 2|z| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Then by Theorem 3.34(a), $\sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$ converges for all $z \in \mathbf{C}$. Thus the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n!} z^n$ is ∞.

(c) For all $n \in \mathbf{N}$, we see by Theorem 3.3(b, c, d) and Theorem 3.20(c) that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^2}\right|} &= \lim_{n\to\infty} \frac{2}{(\sqrt[n]{n})^2} \\ &= 2\frac{1}{(\lim_{n\to\infty} \sqrt[n]{n})^2} \\ &= 2.\end{aligned}$$

Hence by Theorem 3.39, the radius of convergence of $\sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^2}\right|}} = \frac{1}{2}.$$

(d) This computation is almost identical to that of part (c). By Theorem 3.3(b, c) and Theorem 3.20(c), we have that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|} &= \lim_{n\to\infty} \frac{(\sqrt[n]{n})^n}{3} \\ &= \frac{1}{3}\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^3 \\ &= \frac{1}{3}.\end{aligned}$$

Then by Theorem 3.39, the radius of convergence of $\sum \frac{n^3}{3^n} z^n$ is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|}} = 3.$$

Exercise 3.10. By Theorem 3.39, we wish to show that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$. Since infinitely many of the a_n are distinct from zero, there is a subsequence $\{a_{n_k}\}_{k\in\mathbf{N}}$ of $\{a_n\}_{n\in\mathbf{N}}$ for which $\sqrt[n]{|a_{n_k}|} \geq 1$ for all $k \in \mathbf{N}$. Then since every subsequence of $\{a_{n_k}\}_{k\in\mathbf{N}}$ and by Theorem 3.19,

$$\limsup_{n\to\infty} a_n \geq \limsup_{k\to\infty} a_{n_k} \geq 1.$$

Exercise 3.11. (a) Suppose for sake of contradiction that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. Then by Theorem 3.23,

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0.$$

Hence there is $N \in \mathbf{N}$ such that

$$\frac{a_n}{1+a_n} < \frac{1}{2}$$

for all $n \geq N$. But then

$$2a_n < 1 + a_n$$

and so $a_n < 1$ for $n \geq N$. Hence

$$|a_n| = a_n < \frac{2a_n}{1+a_n}$$

for all $n \geq N$, so by Theorem 3.25(a), $\sum_{n=1}^{\infty} a_n$ converges. This is a contradiction and thus $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges whenever $\sum_{n=1}^{\infty} a_n$ diverges.

(b) Fix $N, k \in \mathbf{N}$. Since $a_n > 0$ for all $n \in \mathbf{N}$, we have $s_{N+m} < s_{N+k}$ for all $m = 1, \ldots, k$ and hence

$$\begin{aligned}\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \\ &= \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} \\ &= \frac{s_{N+k} - s_N}{s_{N+k}} \\ &= 1 - \frac{s_N}{s_{N+k}}.\end{aligned}$$

We have by Theorem 3.24 that $s_n \to \infty$ as $n \to \infty$. Hence for fixed $N \in \mathbf{N}$, since $s_n > 0$ for all $n \in \mathbf{N}$,

$$\lim_{k\to\infty} \left(1 - \frac{s_N}{s_{N+k}}\right) = 1.$$

By the inequality established above, along with Theorem 3.19, it follows that

$$\limsup_{k\to\infty} \sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \geq \limsup_{k\to\infty} \left(1 - \frac{s_N}{s_{N+k}}\right) = 1.$$

By Theorem 3.22, if $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ converges then there exists $N \in \mathbf{N}$ such that

$$\limsup_{k\to\infty} \sum_{n=N+1}^{N+k} \frac{a_n}{s_n} < 1.$$

Hence $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges.

(c) From $a_n > 0$ for all $n \in \mathbf{N}$, we have $s_{n-1} < s_n$ and thus

$$\begin{aligned}\frac{a_n}{s_n^2} &= \frac{s_n - s_{n-1}}{s_n^2} \\ &< \frac{s_n - s_{n-1}}{s_{n-1}s_n} \\ &= \frac{1}{s_{n-1}} - \frac{1}{s_n}\end{aligned}$$

for $n \geq 2$. The nth partial sum of $\sum_{n=2}^{\infty} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right)$ is

$$\frac{1}{s_1} - \frac{1}{s_n} = \frac{1}{a_1} - \frac{1}{s_n}.$$

But as explained in part (b), $s_n \to \infty$ as $n \to \infty$ and so

$$\sum_{n=2}^{\infty} \left(\frac{1}{s_{n-1}} - \frac{1}{s_n}\right) = \frac{1}{a_1}.$$

Now by Theorem 3.25(a), since $\frac{a_n}{s_n^2} > 0$ for all $n \in \mathbf{N}$, $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges.

(d) It is clear that if $a_n = 1$ for all $n \in \mathbf{N}$ then $\lim_{n\to\infty} a_n = 1$, so by Theorem 3.23, $\sum_{n=1}^{\infty}$ diverges. In this case,

$$\frac{a_n}{1+na_n} = \frac{1}{1+n} \geq \frac{1}{2n}$$

for all $n \in \mathbf{N}$. But $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by Theorem 3.47 and Theorem 3.28, so by Theorem 3.25(a),

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$$

diverges. On the other hand, let S denote the set of positive square numbers and suppose

$$a_n = \begin{cases} 1 & n \in S \\ 0 & n \notin S. \end{cases}$$

Then since S is infinite, $\{a_n\}_{n\in\mathbf{N}}$ does not converge to 0. Thus by Theorem 3.23, $\sum_{n=1}^{\infty} a_n$ diverges. But

$$\frac{a_n}{1+na_n} = \begin{cases} \frac{1}{1+n} & n \in S \\ 0 & n \notin S. \end{cases}$$

Then by Theorem 3.28 and Theorem 3.24,

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$$

converges since $\sum_{n=1}^{\infty} 1/n^2$ converges.

Now let $\{a_n\}_{n\in\mathbf{N}}$ be any sequence of positive real numbers. For any $n \in \mathbf{N}$, we have that

$$\frac{a_n}{1+n^2a_n} < \frac{a_n}{n^2a_n} = \frac{1}{n^2}.$$

Thus by Theorem 3.25(a) and Theorem 3.28 (with $p = 2$), we have that

$$\sum_{n=1}^{\infty} \frac{a_n}{1+n^2a_n}$$

converges.

Exercise 3.12. We first note that $\sum_{k=n}^{\infty} a_k$ converges for all $n \in \mathbf{N}$ by Theorem 3.25(a), and so each r_n is well-defined. Moreover, $\{r_n\}_{n\in\mathbf{N}}$ is monotonically decreasing sequence of positive reals since $a_n > 0$ for all $n \in \mathbf{N}$. Finally, by Theorem 3.22, $\lim_{n\to\infty} r_n = 0$.

(a) We observe for $m < n$,

$$\begin{aligned} \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} &> \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} \\ &= \frac{a_m + \cdots + a_n}{r_m} \\ &= \frac{r_m - r_{n+1}}{r_m} \\ &> \frac{r_m - r_n}{r_m} \\ &= 1 - \frac{r_n}{r_m}. \end{aligned}$$

Since $\lim_{n\to\infty} r_n = 0$, we have for fixed $m \in \mathbf{N}$ that

$$\lim_{n\to\infty}\left(1-\frac{r_n}{r_m}\right)=0.$$

Thus by Theorem 3.19, we have that

$$\limsup_{n\to\infty}\sum_{k=m}^{n}\frac{a_k}{r_k}\geq \limsup_{n\to\infty}\left(1-\frac{r_n}{r_m}\right)=1$$

for all $m \in \mathbf{N}$. But if $\sum_{n=1}^{\infty}\frac{a_n}{r_n}$ converges, then by Theorem 3.22, there is $m \in \mathbf{N}$ such that

$$\limsup_{n\to\infty}\sum_{k=m}^{n}\frac{a_k}{r_k}<1.$$

Hence $\sum_{n=1}^{\infty}\frac{a_n}{r_m}$ diverges.

(b) We observe that for any $n \in \mathbf{N}$,

$$(\sqrt{r_n}+\sqrt{r_{n+1}})(\sqrt{r_n}-\sqrt{r_{n+1}})=r_n-r_{n+1}=a_n.$$

Thus

$$\begin{aligned}\sqrt{r_n}-\sqrt{r_{n+1}} &= \frac{a_n}{\sqrt{r_n}+\sqrt{r_{n+1}}}\\ &> \frac{a_n}{\sqrt{r_n}+\sqrt{r_n}}\\ &= \frac{a_n}{2\sqrt{r_n}}.\end{aligned}$$

Rearranging,

$$\frac{a_n}{\sqrt{r_n}}<2(\sqrt{r_n}-\sqrt{r_{n+1}})$$

for all $n \in \mathbf{N}$. Then the nth partial sum of $\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{r_n}}$ is bounded above by

$$2\sqrt{r_1}=2\sqrt{\sum_{k=1}^{\infty}a_k}.$$

By Theorem 3.24, it follows that $\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{r_n}}$ converges.

Exercise 3.13. Let $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ be two absolutely convergent series, and let

$$c_n=\sum_{k=0}^{n}a_kb_{n-k}$$

for each $n \geq 0$. Then $\sum_{n=0}^{\infty}|a_n|$ and $\sum_{n=0}^{\infty}|b_n|$ converge absolutely, and so by Theorem 3.45 and Theorem 3.50, we have that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_k||b_{n-k}|$$

converges. But

$$|c_n|=\left|\sum_{k=0}^{n}a_kb_{n-k}\right|\leq\sum_{k=0}^{n}|a_k||b_{n-k}|.$$

Thus by Theorem 3.25(a), $\sum_{n=0}^{\infty} |c_n|$ converges, that is, the Cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges absolutely.

Exercise 3.14. (a) Let $\varepsilon > 0$. Then there is $N \in \mathbf{N}$ such that $|s_n - s| < \varepsilon$ for $n \geq N$. Then for all $n \geq N$, we have

$$\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + s_1 + \cdots + s_n}{n+1} - s \right| \\
&\leq \frac{|s_0 + s_1 + \cdots + s_{N-1}| + N|s| + |s_N + \cdots + s_n - (n-N+1)s|}{n+1} \\
&\leq \frac{|s_0 + s_1 + \cdots + s_{N-1}| + |s_N - s| + \cdots + |s_n - s|}{n+1} \\
&< \frac{|s_0 + s_1 + \cdots + s_{N-1}| + (n-N+1)\varepsilon}{n+1} \\
&= \frac{|s_0 + s_1 + \cdots + s_{N-1}| - N\varepsilon}{n+1} + \varepsilon.
\end{aligned}$$

Since $\lim_{n\to\infty} 1/(n+1) = 0$, by Theorem 3.3(b) we have that

$$\lim_{n\to\infty} \frac{|s_0 + s_1 + \cdots + s_{N-1}| - N\varepsilon}{n+1} = 0.$$

Hence by Theorem 3.19,

$$\limsup_{n\to\infty} |\sigma_n - s| \leq \limsup_{n\to\infty} \left(\frac{|s_0 + s_1 + \cdots + s_{N-1}| - N\varepsilon}{n+1} + \varepsilon \right) = \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have that $\limsup_{n\to\infty} |\sigma_n - s| = 0$ and hence $\lim_{n\to\infty} \sigma_n = s$ as desired.

(b) Let $s_n = (-1)^n$ for all $n \geq 0$. Then

$$\sigma_n = \begin{cases} \frac{1}{n+1} & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$

Then since $\lim_{n\to\infty} 1/(n+1) = 0$, we have that $\lim_{n\to\infty} \sigma_n = 0$. But clearly $\{s_n\}_{n\in\mathbf{N}}$ does not converge since it is not Cauchy (Theorem 3.11(a)).

TODO
TODO
TODO

Exercise 3.15.

Exercise 3.16. (a) It is clear by induction that $x_n > 0$ for all $n \in \mathbf{N}$. We first show by induction that $x_n > \sqrt{\alpha}$ for all $n \in \mathbf{N}$. The claim holds for $n = 1$ by assumption. If $x_n > \sqrt{\alpha}$ for some $n \in \mathbf{N}$, then $x_n \neq \sqrt{\alpha}$ and hence

$$\left(x_n - \frac{\alpha}{x_n} \right)^2 > 0.$$

Thus

$$4x_{n+1}^2 = \left(x_n + \frac{\alpha}{x_n} \right)^2 \geq 4\alpha,$$

and so $x_{n+1} > \sqrt{\alpha}$, proving the claim.

Now for any $n \in \mathbf{N}$, we have

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \\ &< \frac{1}{2}\left(x_n + \frac{\alpha}{\sqrt{\alpha}}\right) \\ &= \frac{1}{2}(x_n + \sqrt{\alpha}) \\ &< \frac{1}{2}(2x_n) \\ &= x_n. \end{aligned}$$

In particular, $\{x_n\}_{n\in\mathbf{N}}$ is monotonically decreasing. [TODO: limit]

(b) We observe for any $n \in \mathbf{N}$ that

$$\begin{aligned} \frac{\varepsilon_n^2}{2x_n} &= \frac{(x_n - \sqrt{\alpha})^2}{2x_n} \\ &= \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} \\ &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\ &= x_{n+1} - \sqrt{\alpha} \\ &= \varepsilon_{n+1}. \end{aligned}$$

Thus since $x_n > \sqrt{\alpha}$ for all $n \in \mathbf{N}$ (as shown in part (a)), we have

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

Now let $\beta = 2\sqrt{\alpha}$; we prove by induction that

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$$

for all $n \in \mathbf{N}$. For $n = 1$, we have by the above that

$$\begin{aligned} \varepsilon_2 &< \frac{\varepsilon_1^2}{2\sqrt{\alpha}} \\ &= 2\sqrt{\alpha}\left(\frac{\varepsilon_1}{2\sqrt{\alpha}}\right)^{2^1} \\ &= \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^1}. \end{aligned}$$

Now suppose for some $n \in \mathbf{N}$ that

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}.$$

Then

$$\begin{aligned}\varepsilon_{n+2} &< \frac{\varepsilon_{n+1}^2}{2\sqrt{\alpha}} \\ &< \frac{1}{\beta}\left(\beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}\right)^2 \\ &= \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n+1}},\end{aligned}$$

proving the claim.

(c) We observe that for $\alpha = 3$ and $x_1 = 2$,

$$\frac{\varepsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{1}{2}.$$

Then since

$$\begin{aligned}\left(\frac{1}{\sqrt{3}}\right)^2 &= \frac{1}{3} \\ &< \frac{9}{25} \\ &= \left(\frac{3}{5}\right)^2 \\ &= \left(\frac{1}{2}+\frac{1}{10}\right)^2,\end{aligned}$$

we have

$$\frac{1}{\sqrt{3}} - \frac{1}{2} < \frac{1}{10}$$

and so $\varepsilon_1/\beta < 1/10$.

Since $\sqrt{3} < 2$ (as $3 < 4$), we have $\beta = 2\sqrt{3} < 4$. Thus by part (b),

$$\begin{aligned}\varepsilon_{n+1} &< \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \\ &< 4\left(\frac{1}{10}\right)^{2^n} \\ &= 4\cdot 10^{-2^n}.\end{aligned}$$

For example,

$$\varepsilon_5 < 4\cdot 10^{-16}$$

and

$$\varepsilon_6 < 4\cdot 10^{-32}.$$

Exercise 3.17. (a) We prove that $x_{2n-1} > x_{2n+1}$ for all $n \in \mathbf{N}$. For any $n \in \mathbf{N}$, we see that

$$\begin{aligned} x_{2n+1} &= x_{2n} + \frac{\alpha - x_{2n}^2}{1 + x_{2n}} \\ &= x_{2n-1} + \frac{\alpha - x_{2n-1}^2}{1 + x_{2n-1}} + \frac{\alpha - x_{2n}^2}{1 + x_{2n}} \end{aligned}$$

(b)
(c)
(d)

Exercise 3.18. We assume that α is a positive real number and $x_1 > \sqrt[p]{\alpha}$. [TODO]

Exercise 3.19.

Exercise 3.20. Suppose $\{p_{n_k}\}_{k\in\mathbf{N}}$ is a subsequence of $\{p_n\}_{n\in\mathbf{N}}$ which converges to p. Then for any $\varepsilon > 0$, there is $K \in \mathbf{N}$ such that $d(p_{n_k}, p) < \varepsilon/2$ for all $k \geq K$. Since $\{p_n\}_{n\in\mathbf{N}}$ is Cauchy, there also exists $N \in \mathbf{N}$ such that $d(p_n, p_m) < \varepsilon/2$ for $n, m \geq N$. For any $n \geq N$, we may choose $k \geq K$ such that $n_k \geq N$. Then

$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_k}) + d(p_{n_k}, p) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon, \end{aligned}$$

and hence $\lim_{n\to\infty} p_n = p$.

Exercise 3.21. As in the proof of Theorem 3.10(b), we have from $E \subseteq E_n$ for all $n \in \mathbf{N}$ and

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0$$

that $\bigcap_{n=1}^{\infty} E_n$ contains at most one point, so it suffices to show it is nonempty. For each $n \in \mathbf{N}$, let $p_n \in E_n$ (since E_n is nonempty). Then for each $N \in \mathbf{N}$, we have $\{p_n\}_{n\geq N} \subseteq E_N$ and hence

$$\lim_{N\to\infty} \operatorname{diam}\{p_n\}_{n\geq N} = 0$$

since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$. Thus $\{p_n\}_{n\in\mathbf{N}}$ is Cauchy, and so it converges to some point p since X is complete. For any $N \in \mathbf{N}$, the sequence $\{p_n\}_{n\geq N}$ in E_N also converges to p. Hence p is a limit point of E_N for all $N \in \mathbf{N}$, and so $p \in \bigcap_{n=1}^{\infty} E_n$ since each E_n is closed. This proves that $\bigcap_{n=1}^{\infty} E_n$ is nonempty as desired.

Exercise 3.22.

Exercise 3.23. As in the hint, we have for all $m, n \in \mathbf{N}$ that

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n).$$

If $\varepsilon > 0$, then there is $N \in \mathbf{N}$ such that

$$d(p_n, p_m) < \frac{\varepsilon}{2}$$

and

$$d(q_n, q_m) < \frac{\varepsilon}{2}$$

for $n, m \geq N$. Then

$$\begin{aligned} d(p_n, q_n) - d(p_m, q_m) &\leq d(p_n, p_m) + d(q_m, q_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

and so

$$|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$$

(by interchanging n and m) for $n, m \geq N$. Thus $\{d(p_n, q_n)\}_{n\in\mathbf{N}}$ is a Cauchy sequence. By Theorem 3.11(c), it follows that $\{d(p_n, q_m)\}_{n\in\mathbf{N}}$ converges.

Exercise 3.24. (a) For any Cauchy sequence $\{p_n\}_{n\in\mathbf{N}}$ in X, we have $d(p_n, p_n) = 0$ for all $n \in \mathbf{N}$ and hence $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{p_n\}_{n\in\mathbf{N}}$. Suppose $\{p_n\}_{n\in\mathbf{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ are Cauchy sequences such that $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{q_n\}_{n\in\mathbf{N}}$. Then

$$\lim_{n\to\infty} d(p_n, q_n) = 0,$$

and so also

$$\lim_{n\to\infty} d(q_n, p_n) = \lim_{n\to\infty} d(p_n, q_n) = 0.$$

That is, $\{q_n\}_{n\in\mathbf{N}}$ is equivalent to $\{p_n\}_{n\in\mathbf{N}}$. Finally, suppose $\{p_n\}_{n\in\mathbf{N}}$, $\{q_n\}_{n\in\mathbf{N}}$, and $\{r_n\}_{n\in\mathbf{N}}$ are Cauchy sequences in X such that $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{q_n\}_{n\in\mathbf{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ is equivalent to $\{r_n\}_{n\in\mathbf{N}}$. Then for any $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that

$$d(p_n, q_n) < \frac{\varepsilon}{2}$$

and

$$d(q_n, r_n) < \frac{\varepsilon}{2}$$

for $n \geq N$. Thus

$$\begin{aligned} d(p_n, r_n) &\leq d(p_n, q_n) + d(q_n, r_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for $n \geq N$, and so $\lim_{n\to\infty} d(p_n, r_n) = 0$. Hence $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{r_n\}_{n\in\mathbf{N}}$, and so equivalence of Cauchy sequences in X is an equivalence relation.

(b) Let $P, Q \in X^*$ and supose $\{p_n\}_{n\in\mathbf{N}}$ and $\{p'_n\}_{n\in\mathbf{N}}$ are representatives of P and $\{q_n\}_{n\in\mathbf{N}}$ and $\{q'_n\}_{n\in\mathbf{N}}$ are representatives of Q. Then we have for all $n \in \mathbf{N}$ that

$$d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n)$$

so

$$d(p_n, q_n) - d(p'_n, q'_n) \leq d(p_n, p'_n) + d(q_n, q'_n)$$

and similarly

$$d(p'_n, q'_n) - d(p_n, q_n) \leq d(p_n, p'_n) + d(q_n, q'_n).$$

Hence

$$|d(p_n, q_n) - d(p'_n, q'_n)| \leq d(p_n, p'_n) + d(q_n, q'_n).$$

But $\lim_{n\to\infty} d(p_n, p'_n) = 0$ and $\lim_{n\to\infty} d(q_n, q'_n) = 0$ since $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{p'_n\}_{n\in\mathbf{N}}$ and $\{q_n\}_{n\in\mathbf{N}}$ is equivalent to $\{q'_n\}_{n\in\mathbf{N}}$. Thus

$$\lim_{n\to\infty} (d(p_n, q_n) - d(p'_n, q'_n)) = 0,$$

and so

$$\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$$

by Theorem 3.3(a) and Exercise 3.23. Hence $\Delta(P, Q)$ is well-defined.

Now we show that Δ is a metric on X^*. Suppose $P, Q \in X^*$, and let $\{p_n\}_{n\in\mathbf{N}}$ be a representative of P and $\{q_n\}_{n\in\mathbf{N}}$ a representative of Q. Then we have by Theorem 3.19 that

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) \geq 0$$

since $d(p_n, q_n) \geq 0$ for all $n \in \mathbf{N}$. Moreover, $\Delta(P, Q) = 0$ if and only if $\lim_{n\to\infty} d(p_n, q_n) = 0$, that is, $\{p_n\}_{n\in\mathbf{N}}$ is equivalent to $\{q_n\}_{n\in\mathbf{N}}$. Thus $\Delta(P, Q) = 0$ if and only if $P = Q$, so part (a) of Definition 2.15 is established. We also have that $d(p_n, q_n) = d(q_n, p_n)$ for all $n \in \mathbf{N}$, and so

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = \Delta(Q, P).$$

This proves part (b) of Definition 2.15. Finally, suppose also that $R \in X^*$ and $\{r_n\}_{n\in\mathbf{N}}$ is a representative of R. Then we have

$$d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$$

for all $n \in \mathbf{N}$ and thus by Theorem 3.19 and Theorem 3.3(a),

$$\begin{aligned}
\Delta(P, R) &= \lim_{n\to\infty} d(p_n, r_n) \\
&\leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n)) \\
&= \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) \\
&= \Delta(P, Q) + \Delta(Q, R).
\end{aligned}$$

This is part (c) of Definition 2.15, and so Δ is a metric on X^*.

(c) Let $\{P_k\}_{k\in\mathbf{N}}$ be a Cauchy sequence in X^*. Let $\{p_{n,k}\}_{n\in\mathbf{N}}$ be a representative of P_k for each $k \in \mathbf{N}$. For all $n, m \in \mathbf{N}$, we have

$$d(p_{n,n}, p_{m,m}) \leq d(p_{n,n}, p_{m,n}) + d(p_{m,n}, p_{m,m}).$$

For any $\varepsilon > 0$, there is $K \in \mathbf{N}$ such that

$$\Delta(P_k, P_l) < \frac{\varepsilon}{2}$$

for $k, l \geq K$.

(d) We have that $\{p\}_{n\in\mathbf{N}}$ is Cauchy since $d(p, p) = 0$; thus the class $P_p \in X^*$ is well-defined. By definition, for any $p, q \in X$, we have

$$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p, q) = d(p, q).$$

That is, if $\varphi : X \to X^*$ is given by $\varphi(p) = P_p$, we have

$$\Delta(\varphi(p), \varphi(q)) = d(p, q)$$

for all $p, q \in X$. Then φ is an isometric embedding of X into X^* (note that by part (a) of Definition 2.15, a distance-preserving map of metric spaces is necessarily injective).

(e) Let $P \in X^*$ and $\varepsilon > 0$. Suppose $\{p_n\}_{n \in \mathbf{N}}$ is a representative of P. Then $\{p_n\}_{n \in \mathbf{N}}$ is Cauchy, and so there is $N \in \mathbf{N}$ such that $d(p_n, p_m) < \varepsilon/2$ for $n, m \geq N$. Thus

$$\lim_{n \to \infty} d(p_n, p_N) \leq \frac{\varepsilon}{2} < \varepsilon$$

by Theorem 3.19, and so

$$\Delta(P, P_{p_N}) < \varepsilon.$$

But $P_{p_N} = \varphi(p_N) \in \varphi(X)$, and hence $\varphi(X)$ is dense in X^*.

Now suppose X is complete, and let $P \in X^*$. Let $\{p_n\}_{n \in \mathbf{N}}$ be a representative of P. Then $\{p_n\}_{n \in \mathbf{N}}$ is a Cauchy sequence in X, and so since X is complete, there is $p \in X$ such that $\{p_n\}_{n \in \mathbf{N}}$ converges to p. Thus

$$\lim_{n \to \infty} d(p_n, p) = 0,$$

and so $\{p_n\}_{n \in \mathbf{N}}$ is equivalent to $\{p\}_{n \in \mathbf{N}}$. Then $P = \varphi(p)$, and so $\varphi(X) = X^*$.

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