Solutions to Qing Liu's *Algebraic Geometry and Arithmetic Curves*

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Abstract. This document contains solutions to the exercises of Qing Liu's *Algebraic Geometry and Arithmetic Curves*.

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