Solutions to Manfredo do Carmo's
Differential Geometry of Curves and Surfaces

Patrick Borse

Abstract. This document contains solutions to the exercises of Manfredo do Carmo's *Differential Geometry of Curves and Surfaces*.

Contents

CHAPTER 1

Curves

2. Parameterized Curves

Exercise 2.1. Define $\alpha : \mathbf{R} \to \mathbf{R}$ by $\alpha(t) = (\sin t, -\cos t)$. Then α is differentiable since sin and cos are differentiable, and we have

$$\alpha(0) = (\sin 0, -\cos 0) = (0, 1).$$

For all $t \in \mathbf{R}$,

$$\|\alpha(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1,$$

so the trace of α lies in the unit circle in $\mathbf{R}^2$. Moreover, for all $t \in \mathbf{R}$ we have that

$$\alpha'(t) = (\cos t, \sin t)$$

so α traverses the unit circle clockwise. If $(x, y) \in \mathbf{R}^2$ with $x^2 + y^2 = 1$, let

$$t = \begin{cases} \pi + \arctan \frac{x}{y} & \text{if } y > 0 \\ -\arctan \frac{x}{y} & \text{if } y < 0 \\ \frac{\pi}{2} & \text{if } (x, y) = (1, 0) \\ -\frac{\pi}{2} & \text{if } (x, y) = (-1, 0). \end{cases}$$

Then we may verify that in each case, $\alpha(t) = (x, y)$:

(i) If $y > 0$, then $t \in (\pi/2, 3\pi/2)$ with $\tan t = -x/y$ and so

$$\begin{aligned} \alpha(t) &= (\sin t, -\cos t) \\ &= \left(\frac{x/y}{\sqrt{1 + (-x/y)^2}}, -\frac{-1}{\sqrt{1 + (-x/y)^2}} \right) \\ &= \left(\frac{x/y}{1/y}, \frac{1}{1/y} \right) \\ &= (x, y). \end{aligned}$$

(ii) If $y < 0$, then $t \in (-\pi/2, \pi/2)$ with $\tan t = -x/y$, so

$$\begin{aligned} \alpha(t) &= (\sin t, -\cos t) \\ &= \left(\frac{-x/y}{\sqrt{1 + (-x/y)^2}}, -\frac{1}{\sqrt{1 + (-x/y)^2}} \right) \\ &= \left(\frac{-x/y}{-1/y}, -\frac{1}{-1/y} \right) \\ &= (x, y). \end{aligned}$$

(iii) If $(x, y) = (1, 0)$, then $t = \pi/2$ yields

$$\alpha(t) = \left(\sin\frac{\pi}{2}, -\cos\frac{\pi}{2}\right) = (1, 0) = (x, y).$$

(iv) If $(x, y) = (-1, 0)$, then $t = -\pi/2$ and so

$$\alpha(t) = \left(\sin\left(-\frac{\pi}{2}\right), -\cos\left(-\frac{\pi}{2}\right)\right) = (-1, 0) = (x, y).$$

Hence the trace of α is exactly the unit circle in $\mathbf{R}^2$.

Exercise 2.2. We are given that t_0 attains the minimum of the differentiable function $t \mapsto \|\alpha(t)\|^2 = \alpha(t) \cdot \alpha(t)$. Then

$$\left.\frac{d}{dt}\right|_{t=t_0} \|\alpha(t)\|^2 = 2\alpha(t_0) \cdot \alpha'(t_0)$$

vanishes. Thus from $\alpha(t_0), \alpha'(t_0) \neq 0$, we have that $\alpha(t_0)$ is geometrically orthogonal to $\alpha'(t_0)$.

Exercise 2.3. If $\alpha''(t) = 0$ for all $t \in I$, we have that the tangent vector $\alpha'(t)$ is constant. Let v be this constant. Then for any $t_0 \in I$,

$$\alpha(t) = \alpha(t_0) + \int_{t_0}^{t} \alpha'(t)\, dt = \alpha(t_0) + \int_{t_0}^{t} v\, dt = \alpha(t_0) + (t - t_0)v$$

for all $t \in I$. Thus α is a straight line with constant velocity.

Exercise 2.4. Observe that

$$\frac{d}{dt}(\alpha(t) \cdot v) = \alpha'(t) \cdot v = 0$$

for all $t \in I$. Then $\alpha(t) \cdot v$ is constant, so since $\alpha(0) \cdot v = 0$ we conclude that $\alpha(t) \cdot v = 0$ for all $t \in I$. Then $\alpha(t)$ is orthogonal to v for all $t \in I$ as long as $\alpha(t) \neq 0$.

Exercise 2.5. We have that $t \mapsto \|\alpha(t)\|^2 = \alpha(t) \cdot \alpha(t)$ is differentiable with

$$\frac{d}{dt}\|\alpha(t)\|^2 = 2\alpha(t) \cdot \alpha'(t),$$

and $\|\alpha(t)\|^2$ is constant if and only if $\|\alpha(t)\|$ is constant. Thus $\|\alpha(t)\|$ is constant if and only if $\alpha(t) \cdot \alpha'(t) = 0$ for all $t \in I$. Moreover, $\|\alpha(t)\|$ is nonzero for all $t \in I$ if and only if $\alpha(t) \neq 0$ for all $t \in I$, and so from $\alpha'(t) \neq 0$ for all $t \in I$ we conclude that $\|\alpha(t)\|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

3. Regular Curves; Arc Length

Exercise 3.1. The line in $\mathbf{R}^3$ defined by $y = 0$ and $z = x$ is the $\mathbf{R}$-span of $(1, 0, 1)$. Thus it suffices to show that $\alpha'(t)$ makes a constant angle with $(1, 0, 1)$. Indeed, we have

$$\alpha'(t) = (3, 6t, 6t^2)$$

for all $t \in \mathbf{R}$ so if $\theta(t)$ is the angle between $\alpha'(t)$ and $(1,0,1)$ then $0 \le \theta(t) \le \pi$ with

$$\begin{aligned}
\cos\theta(t) &= \frac{\alpha'(t)\cdot(1,0,1)}{\|\alpha'(t)\|\|(1,0,1)\|} \\
&= \frac{3+6t^2}{\sqrt{3^2+(6t)^2+(6t^2)^2}\sqrt{2}} \\
&= \frac{3+6t^2}{3\sqrt{1+4t^2+4t^4}\sqrt{2}} \\
&= \frac{1+2t^2}{\sqrt{(1+2t^2)^2}\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

since $1+2t^2 \ge 0$ for all $t \in \mathbf{R}$. Then $\theta(t) = \pi/4$ is constant.

Exercise 3.2. a. As in the diagram, let t be the angle swept out by the radius to the given point on the circumference. Then at time t, the center of the disk lies at $(t,1)$ while the position of the point relative to the center is $(-\sin t, -\cos t)$. Hence $\alpha : \mathbf{R} \to \mathbf{R}^2$ given by

$$\alpha(t) = (t-\sin t, 1-\cos t)$$

is a parameterized curve which traces the cycloid. For all $t \in \mathbf{R}$,

$$\alpha'(t) = (1-\cos t, \sin t).$$

Then $\alpha'(t) = 0$ if and only if $t \in 2\pi\mathbf{Z}$, so the singular points are $(2\pi n, 0)$ for $n \in \mathbf{Z}$.

b. By the choice of parameterization, a complete rotation of the disk occurs from $t=0$ to $t=2\pi$. The arc length of the cycloid on this interval is

$$\begin{aligned}
\int_0^{2\pi} |\alpha'(t)|\,dt &= \int_0^{2\pi} \sqrt{(1-\cos t)^2+\sin^2 t}\,dt \\
&= \int_0^{2\pi} \sqrt{2-2\cos t}\,dt \\
&= \int_0^{2\pi} \sqrt{4\sin^2(t/2)}\,dt \\
&= \int_0^{2\pi} 2\sin(t/2)\,dt \\
&= \int_0^{\pi} 4\sin u\,du \\
&= (-4\cos u)\big|_0^{\pi} \\
&= 8.
\end{aligned}$$

Exercise 3.3. a. At any point on the circumference of a circle, the tangent line is perpendicular to the radius. Thus AV is the line $x = 2a$. Then the ray r making angle $\theta \in (-\pi/2, \pi/2)$ with the positive x-axis intersects AV at $B = (2a, 2a\tan\theta)$ and intersects S^1 at $C = (t\cos\theta, t\sin\theta)$ for $t > 0$ such that

$$(t\cos\theta - a)^2 + (t\sin\theta)^2 = a^2.$$

Then

$$t^2 - 2at\cos\theta = 0$$

so $t = 2a\cos\theta$, and thus

$$C = (2a\cos^2\theta, 2a\cos\theta\sin\theta).$$

Then

$$\overline{CB} = (2a, 2a\tan\theta) - (2a\cos^2\theta, 2a\cos\theta\sin\theta) = (2a\sin^2\theta, 2a\sin^2\theta\tan\theta),$$

so

$$p = (2a\sin^2\theta, 2a\sin^2\theta\tan\theta).$$

We have that

$$\frac{\tan^2\theta}{1+\tan^2\theta} = \frac{\tan^2\theta}{\sec^2\theta} = \sin^2\theta,$$

so

$$p = \left(\frac{2a\tan^2\theta}{1+\tan^2\theta}, \frac{2a\tan^2\theta}{1+\tan^2\theta}\right).$$

Thus since $\tan : (-\pi/2, \pi/2) \to \mathbf{R}$ is a diffeomorphism, we see that the parameterized curve $\alpha : \mathbf{R} \to \mathbf{R}^2$ given by

$$\alpha(t) = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right)$$

traces the cissoid of Diocles.

b. Using the parameterized curve $\alpha : \mathbf{R} \to \mathbf{R}^2$ from part a., we have that

$$\begin{aligned}\alpha'(t) &= \left(\frac{(4at)(1+t^2) - (2at^2)(2t)}{(1+t^2)^2}, \frac{(6at^2)(1+t^2) - (2at^3)(2t)}{(1+t^2)^2}\right) \\ &= \left(\frac{4at}{(1+t^2)^2}, \frac{6at^2 + 2at^4}{(1+t^2)^2}\right).\end{aligned}$$

Since $(1+t^2)^2 > 0$ for all $t \in \mathbf{R}$, we thus have $\alpha'(t) = 0$ if and only if $4at = 0$ and $6at^2 + 2at^4 = 0$. Then the only singular point of the cissoid is $\alpha(0) = (0, 0)$.

c. We have that

$$\lim_{t\to\pm\infty} \frac{2at^2}{1+t^2} = 2a$$

so $\alpha(t)$ approaches the line $x = 2a$ as $t \to \pm\infty$. Moreover,

$$\lim_{t\to\pm\infty} \alpha'(t) = \left(\lim_{t\to\pm\infty} \frac{4at}{(1+t^2)^2}, \lim_{t\to\pm\infty} \frac{6at^2 + 2at^4}{(1+t^2)^2}\right) = (0, 2a).$$

Exercise 3.4. a. We have that $\sin t, \cos t$, and $\tan(t/2)$ are differentiable for all $t \in (0, \pi)$. Moreover, $\tan(t/2)$ is positive for $t \in (0, \pi)$, and log is differentiable on the positive reals. Thus α is a differentiable parameterized curve with

$$\begin{aligned}\alpha'(t) &= \left(\cos t, -\sin t + \frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right) \\ &= \left(\cos t, -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) \\ &= \left(\cos t, -\sin t + \frac{1}{\sin t}\right) \\ &= (\cos t, \cos t\cot t)\end{aligned}$$

for all $t \in (0, \pi)$. Then $\alpha'(t) = 0$ if and only if $\cos t = 0$, that is, $t = \pi/2$. Hence α is regular except at $t = \pi/2$.

b. We have that for all $t \in (0, \pi)$ with $t \neq \pi/2$, the tangent line to α at t is given by $\beta : \mathbf{R} \to \mathbf{R}^2$ where

$$\begin{aligned}\beta(s) &= \alpha(t) + s\alpha'(t) \\ &= \left(\sin t, \cos t + \log \tan \frac{t}{2}\right) + s(\cos t, \cos t \cot t) \\ &= \left(\sin t + s\cos t, (1 + s\cot t)\cos t + \log \tan \frac{t}{2}\right).\end{aligned}$$

Then the tangent line intersects the y-axis when

$$\sin t + s\cos t = 0,$$

that is, when $s = -\tan t$ (since $t \neq \pi/2$, $\cos t \neq 0$). Then at time t, the length of the segment of the tangent line from the point of tangency to the y-axis is given by

$$\begin{aligned}\|\alpha(t) - \beta(-1)\| &= \left\|\left(\sin t, \cos t + \log \tan \frac{t}{2}\right) - \left(\sin t + s\cos t, (1 + s\cot t)\cos t + \log \tan \frac{t}{2}\right)\right\| \\ &= \|(\sin t, -s\cot t \cos t)\| \\ &= \|(\sin t, \cos t)\| \\ &= 1.\end{aligned}$$

Exercise 3.5. a. We have that for $t \in (-1, \infty)$, $1 + t^3 \neq 0$ and so

$$\begin{aligned}\alpha'(t) &= \left(\frac{(3a)(1+t^3) - (3at)(3t^2)}{(1+t^3)^2}, \frac{(6at)(1+t^3) - (3at^2)(3t^2)}{(1+t^3)^2}\right) \\ &= \left(\frac{3a - 6at^3}{(1+t^3)^2}, \frac{6at - 3at^4}{(1+t^3)^2}\right).\end{aligned}$$

In particular,

$$\alpha'(0) = (3a, 0),$$

and so at $t = 0$, α is tangent to the x-axis.

b. We have that

$$\lim_{t\to\infty} \frac{3at}{1+t^3} = 0$$

and

$$\lim_{t\to\infty} \frac{3at^2}{1+t^3} = 0.$$

Thus $\alpha(t) \to (0, 0)$ as $t \to \infty$. Moreover, $(1 + t^3)^2 = 1 + 2t^3 + t^6$ is a degree 6 polynomial in t, and so

$$\lim_{t\to\infty} \frac{3a - 6at^3}{(1+t^3)^2} = 0$$

and

$$\lim_{t\to\infty} \frac{6at - 3at^4}{(1+t^3)^2} = 0.$$

Hence $\alpha'(t) \to (0, 0)$ as $t \to \infty$.

c. For $t \in (-1, \infty)$, we have

$$\begin{aligned}\alpha_1(t) + \alpha_2(t) + a &= \frac{3at}{1+t^3} + \frac{3at^2}{1+t^3} + a \\ &= \frac{3at(1+t)}{1+t^3} + a \\ &= \frac{3at}{1-t+t^2} + a \\ &= a\frac{1+2t+t^2}{1-t+t^2}.\end{aligned}$$

Hence

$$\lim_{t\to -1^+}(\alpha_1(t) + \alpha_2(t) + a) = a \lim_{t\to -1^+} \frac{1+2t+t^2}{1-t+t^2} = 0.$$

Morover, we computed in part a. that

$$\alpha'(t) = \left(\frac{3a - 6at^3}{(1+t^3)^2}, \frac{6at - 3at^4}{(1+t^3)^2}\right).$$

Then assuming that $a \neq 0$, we have

$$\frac{\alpha_2'(t)}{\alpha_1'(t)} = \frac{6at - 3at^4}{3a - 6at^3} = \frac{6t - 3t^4}{3 - 6t^3}$$

and so

$$\lim_{t\to -1^+} \frac{\alpha_2'(t)}{\alpha_1'(t)} = \frac{-6-3}{3+6} = -1.$$

Thus as $t \to -1$, α and its tangent approach the line $x + y + a = 0$.

Exercise 3.6. a. Since $a > 0$ and $b < 0$, $ae^{bt} \to 0$ as $t \to \infty$ with $ae^{bt} > 0$ for all $t \in \mathbf{R}$. Then since $(\cos t, \sin t)$ traverses the unit circle counterclockwise, we have as $t \to \infty$ that α spirals counterclockwise around the origin with $\alpha(t) \to 0$ (since $\cos t, \sin t$ are bounded).

b. We compute for all $t \in \mathbf{R}$ that

$$\alpha'(t) = ae^{bt}(-\sin t + b\cos t, \cos t + b \sin t).$$

Then from $ae^{bt} \to 0$ as $t \to \infty$ (with the remaining terms being bounded), we have that $\alpha'(t) \to (0,0)$. Additionally, since $ae^{bt} > 0$ for all $t \in \mathbf{R}$ we have

$$\begin{aligned}|\alpha'(t)| &= ae^{bt}|(-\sin t + b\cos t, \cos t + b\sin t)| \\ &= ae^{bt}\sqrt{(-\sin t + b\cos t)^2 + (\cos t + b\sin t)^2} \\ &= ae^{bt}\sqrt{1+b^2}.\end{aligned}$$

Thus for any $t_0 \in \mathbf{R}$, the arc length of α along $[t_0, \infty)$ is

$$\begin{aligned}\int_{t_0}^{\infty} |\alpha'(t)|\,dt &= \int_{t_0}^{\infty} ae^{bt}\sqrt{1+b^2}\,dt \\ &= a\sqrt{1+b^2}\int_{t_0}^{\infty} e^{bt}\,dt \\ &= a\sqrt{1+b^2}\lim_{t\to\infty}\frac{e^{bt}-e^{bt_0}}{b} \\ &= -\frac{a\sqrt{1+b^2}e^{bt_0}}{b}.\end{aligned}$$

In particular, the arc length of α along $[t_0, \infty)$ is finite.

Exercise 3.7. a. For all nonzero $h \in \mathbf{R}$, the line determined by $\alpha(0+h) = (h^3, h^2)$ and $\alpha(0) = (0,0)$ is given by $x = hy$. This line has limit position $x = 0$ as $h \to 0$, so α has a weak tangent at $t = 0$. But for nonzero $h \in \mathbf{R}$, the line determined by $\alpha(0+h) = (h^3, h^2)$ and $\alpha(0-h) = (-h^3, h^2)$ is given by $y = h^2$ which has limit position $y = 0$. Hence α does not have a strong tangent at $t = 0$.

b. For all nonzero $h, k \in \mathbf{R}$, by the mean value theorem there exists t between h and k such that

$$\frac{\alpha'(t_0+h)-\alpha'(t_0+k)}{h-k} = \alpha'(t).$$

Hence by continuity of α',

$$\lim_{h,k\to 0}\frac{\alpha'(t_0+h)-\alpha'(t_0+k)}{h-k} = \alpha'(t_0).$$

Then since $\alpha'(t_0) \neq 0$, the line determined by $\alpha(t_0+h)$ and $\alpha(t_0+k)$ converges to the line through $\alpha(t_0)$ in the direction of $\alpha'(t_0)$ as $h, k \to 0$.

c. We have that

$$\lim_{h\to 0^+}\frac{\alpha(0+h)-\alpha(0)}{h} = \lim_{h\to 0^+}\frac{(h^2,h^2)-(0,0)}{h} = \lim_{h\to 0^+}(h,h) = (0,0)$$

and

$$\lim_{h\to 0^-}\frac{\alpha(0+h)-\alpha(0)}{h} = \lim_{h\to 0^-}\frac{(h^2,-h^2)-(0,0)}{h} = \lim_{h\to 0^-}(h,-h) = (0,0)$$

Then $\alpha'(0) = (0,0)$ and so

$$\alpha'(t) = \begin{cases}(2t, 2t) & \text{if } t \geq 0 \\ (2t, -2t) & \text{if } t \leq 0.\end{cases}$$

Clearly α' is continuous on $\mathbf{R}_{\geq 0}$ and $\mathbf{R}_{\leq 0}$ since

$$\lim_{t\to 0^+}\alpha'(t) = \alpha'(0) = \lim_{t\to 0^-}\alpha'(t),$$

and so α' is continuous on $\mathbf{R}$. Thus α is C^1. But for $h > 0$,

$$\frac{\alpha'(0+h)-\alpha'(0)}{h} = \frac{(2h,2h)-(0,0)}{h} = (2,2)$$

while for $h < 0$,

$$\frac{\alpha'(0+h)-\alpha'(0)}{h} = \frac{(2h,-2h)-(0,0)}{h} = (2,-2).$$

Then $\alpha''(0)$ does not exist and hence α is not C^2. The trace of α is the graph of $x = |y|$.

Exercise 3.8. Since $|\alpha'|$ is continuous on $[a,b]$, the integral $\int_a^b \|\alpha'(t)\|\,dt$ exists and so there exists $\delta > 0$ such that for $|P| < \delta$,

$$\left|\int_a^b \|\alpha'(t)\|\,dt - \sum_{i=1}^n (t_i - t_{i-1})\|\alpha'(t_{i-1})\|\right| < \varepsilon/2.$$

Since $\|\alpha'\|$ is continuous on the compact set $[a,b]$, it is uniformly continuous, and hence making $\delta > 0$ smaller if necessary yields

$$|\|\alpha'(s)\| - \|\alpha'(t)\|| < \frac{\varepsilon}{2(b-a)}$$

for $s, t \in [a,b]$ with $|s-t| < \delta$. Then for $|P| < \delta$ we have

$$\left|\|\alpha'(t_{i-1})\| - \left\|\frac{\alpha(t_i) - \alpha(t_{i-1})}{t_i - t_{i-1}}\right\|\right| < \frac{\varepsilon}{2(b-a)}$$

for all $i = 1, \ldots, n$ by the mean value theorem. Thus

$$|(t_i - t_{i-1})\|\alpha'(t_{i-1})\| - \|\alpha(t_i) - \alpha(t_{i-1})\|| < (t_i - t_{i-1})\frac{\varepsilon}{2(b-a)}$$

and so

$$\left|\sum_{i=1}^n (t_i - t_{i-1})\|\alpha'(t_{i-1})\| - l(\alpha, P)\right| < \sum_{i=1}^n (t_i - t_{i-1})\frac{\varepsilon}{2(b-a)} = \varepsilon/2.$$

Hence

$$\left|\int_a^b \|\alpha'(t)\|\,dt - l(\alpha, P)\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

proving the claim.

Exercise 3.9. a. Following Exercise 8, we define the arc length of α along $[a,b] \subseteq I$ by

$$\lim_{|P| \to \delta} l(\alpha, P)$$

if it exists, where the limit is over partitions P of $[a,b]$. Exercise 8 shows that if α is C^1, then this limit exists and

$$\lim_{|P| \to \delta} l(\alpha, P) = \int_a^b \|\alpha'(t)\|\,dt.$$

b. Clearly α is differentiable on $(0,1)$ and continuous at $t = 1$. At $t = 0$, we have

$$\lim_{t \to 0^+} \alpha(t) = \left(\lim_{t \to 0^+} t, \lim_{t \to 0^+} t \sin\left(\frac{\pi}{t}\right)\right) = (0,0) = \alpha(0)$$

since sin is bounded. Thus α is C^0. We assume that a "reasonable definition" of arc length of a curve α of class C^0 on an interval $[a,b]$ has the property that for any partition P of $[a,b]$, the arc length is at least $l(\alpha, P)$. For $n \in \mathbf{N}$, consider the partition

$$\frac{1}{n+1} < \frac{1}{n+\frac{1}{2}} < \frac{1}{n}$$

of the interval $[1/(n+1), 1/n]$. Then the arc length of α along $[1/(n+1), 1/n]$ is at least

$$\begin{aligned}
l(\alpha, P) &= \left\|\alpha\left(\frac{1}{n+\frac{1}{2}}\right) - \alpha\left(\frac{1}{n+1}\right)\right\| + \left\|\alpha\left(\frac{1}{n}\right) - \alpha\left(\frac{1}{n+\frac{1}{2}}\right)\right\| \\
&= \left\|\left(\frac{1}{n+\frac{1}{2}}, \frac{(-1)^n}{n+\frac{1}{2}}\right) - \left(\frac{1}{n+1}, 0\right)\right\| + \left\|\left(\frac{1}{n}, 0\right) - \left(\frac{1}{n+\frac{1}{2}}, \frac{(-1)^n}{n+\frac{1}{2}}\right)\right\| \\
&\geq \left|\frac{(-1)^n}{n+\frac{1}{2}}\right| + \left|\frac{(-1)^n}{n+\frac{1}{2}}\right| \\
&= \frac{2}{n+\frac{1}{2}}.
\end{aligned}$$

Hence the arc length of α on $[1/N, 1]$ for any $N \in \mathbf{N}$ is at least

$$\sum_{n=1}^{N} \frac{2}{n+\frac{1}{2}} \geq 2\sum_{n=1}^{N} \frac{1}{n+1}.$$

But $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges, so the arc length of α on $[1/N, 1]$ tends to infinity as $N \to \infty$.

Exercise 3.10. a. It is immediate from the bilinearity of the dot product that

$$(q-p) \cdot v = \left(\int_a^b \alpha'(t)\, dt\right) \cdot v = \int_a^b \alpha'(t) \cdot v\, dt.$$

Since $\|v\| = 1$, we have $\alpha'(t) \cdot v \leq \|\alpha'(t)\|$ for all $t \in I$ and hence

$$\int_a^b \alpha'(t) \cdot v\, dt \leq \int_a^b \|\alpha'(t)\|\, dt.$$

The right-hand side is the arc length of α on $[a, b]$.

b. From the inequality in part a., we find that

$$(q-p) \cdot \frac{q-p}{\|q-p\|} \leq \int_a^b \|\alpha'(t)\|\, dt.$$

The left-hand side is simply

$$\frac{\|q-p\|^2}{\|q-p\|} = \|q-p\| = \|\alpha(b) - \alpha(a)\|,$$

so

$$\|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\|\, dt.$$

4. The Vector Product in $\mathbf{R}^3$

Exercise 4.1. a. The change of basis from $\{(1,3),(4,2)\}$ to $\{(1,0),(0,1)\}$ has determinant

$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = (1)(2) - (4)(3) = -10$$

and so $\{(1,3),(4,2)\}$ is negative.

b. The change of basis from $\{(1,3,5),(2,3,7),(4,8,3)\}$ has determinant

$$\begin{aligned}\begin{vmatrix}1 & 2 & 4\\ 3 & 3 & 8\\ 5 & 7 & 3\end{vmatrix} &= 1\begin{vmatrix}3 & 8\\ 7 & 3\end{vmatrix} - 2\begin{vmatrix}3 & 8\\ 5 & 3\end{vmatrix} + 4\begin{vmatrix}3 & 3\\ 5 & 7\end{vmatrix}\\ &= 1((3)(3)-(8)(7)) - 2((3)(3)-(8)(5)) + 4((3)(7)-(3)(5))\\ &= -47+62+24\\ &= 39,\end{aligned}$$

so $\{(1,3,5),(2,3,7),(4,8,3)\}$ is positive.

Exercise 4.2. For any $(x,y,z),(x_0,y_0,z_0)\in P$, we have

$$ax+by+cz=d$$

and

$$ax_0+by_0+cz_0=d$$

so

$$(a,b,c)\cdot(x-x_0,y-y_0,z-z_0)=a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

Then $v=(a,b,c)$ is perpendicular to P. Moreover, for all $(x,y,z)\in P$, we have $(a,b,c)\cdot(x,y,z)=-d$ and so by the Cauchy-Schwarz inequality,

$$\|(a,b,c)\|\|(x,y,z)\|\geq|d|.$$

Hence

$$\|(x,y,z)\|\geq\frac{|d|}{\sqrt{a^2+b^2+c^2}}.$$

The distance $|d|/\sqrt{a^2+b^2+c^2}$ is achieved by

$$(x,y,z)=\left(-\frac{ad}{a^2+b^2+c^2},-\frac{bd}{a^2+b^2+c^2},-\frac{cd}{a^2+b^2+c^2}\right)\in P,$$

so $|d|/\sqrt{a^2+b^2+c^2}$ is the distance from the plane to the origin.

Exercise 4.3. The angle of intersection of two planes is given by the angle of their normal vectors. By Exercise 2, the normal vector of the plane $5x+3y+2z-4=0$ is $(5,3,2)$ and the normal vector of $3x+4y-7z=0$ is $(3,4,-7)$. If θ denotes the angle between $(5,3,2)$ and $(3,4,-7)$, then

$$\cos\theta=\frac{(5,3,2)\cdot(3,4,-7)}{\|(5,3,2)\|\|(3,4,-7)\|}=\frac{13}{\sqrt{38}\sqrt{74}}.$$

Hence

$$\theta=\arccos\left(\frac{13}{2\sqrt{703}}\right).$$

Exercise 4.4 (TODO)**.**

Exercise 4.5. We have that $(p-p_1)\wedge(p-p_2)\cdot(p-p_3)=0$ for $p=p_1,p_2,p_3$ and so it suffices to show that this equation defines a plane. Indeed, if $p=(x,y,z)$ then

$$\begin{aligned}(p-p_1)\wedge(p-p_2)\cdot(p-p_3) &= \begin{vmatrix} x-x_1 & x-x_2 & x-x_3 \\ y-y_1 & y-y_2 & y-y_3 \\ z-z_1 & z-z_2 & z-z_3 \end{vmatrix} \\ &= \begin{vmatrix} x-x_1 & x_1-x_2 & x_1-x_3 \\ y-y_1 & y_1-y_2 & y_1-y_3 \\ z-z_1 & z_1-z_2 & z_1-z_3 \end{vmatrix} \\ &= a(x-x_1)+b(y-y_1)+c(z-z_1)\end{aligned}$$

where

$$\begin{aligned} a &= \begin{vmatrix} y_1-y_2 & y_1-y_3 \\ z_1-z_2 & z_1-z_3 \end{vmatrix}, \\ b &= -\begin{vmatrix} x_1-x_2 & x_1-x_3 \\ z_1-z_2 & z_1-z_3 \end{vmatrix}, \\ c &= \begin{vmatrix} x_1-x_2 & x_1-x_3 \\ y_1-y_2 & y_1-y_3 \end{vmatrix}. \end{aligned}$$

Thus letting $d=-ax_1-by_1-cz_1$, we see that $(p-p_1)\wedge(p-p_2)\cdot(p-p_3)=0$ defines the plane given by $ax+by+cz+d=0$.

Exercise 4.6. The planes are defined by $v_i\cdot(x,y,z)=d_i$. Then if (x_0,y_0,z_0) lies in their intersection, (x,y,z) lies in the intersection if and only if $v_i\cdot(x-x_0,y-y_0,z-z_0)=0$ for $i=1,2$. But since the planes are nonparallel, we have that v_1 and v_2 are linearly independent and so (x,y,z) is in the intersection if and only if $(x-x_0,y-y_0,z-z_0)$ is parallel to $u=v_1\wedge v_2$. Hence if $u=(u_1,u_2,u_3)$, the intersection of the two planes is parameterized by

$$x-x_0=u_1t,\quad y-y_0=u_2t,\quad z-z_0=u_3t$$

for $t\in\mathbf{R}$.

Exercise 4.7. By Exercise 2, the normal vector to the plane given by $ax+by+cz+d=0$ is (a,b,c). Then the plane and the line are parallel if and only if $(a,b,c)\cdot(u_1,u_2,u_3)=0$, that is,

$$au_1+bu_2+cu_3=0.$$

Exercise 4.8. Since the lines are nonparallel, u and v are linearly independent. Then $u\wedge v\neq 0$ and $u\wedge v$ is perpendicular to both lines, so the distance ρ is given by the length of the projection of r onto $u\wedge v$, that is,

$$\frac{|(u\wedge v)\cdot r|}{\|u\wedge v\|}.$$

Exercise 4.9. The normal vector to the plane is $(3,4,7)$ and the line is in the direction of $(3,5,9)$. Then the angle of intersection $\theta\in[0,\pi/2]$ satisfies

$$\begin{aligned}\sin\theta &= \frac{|(3,4,7)\cdot(3,5,9)|}{\|(3,4,7)\|\|(3,5,9)\|} \\ &= \frac{92}{\sqrt{74}\sqrt{115}} \\ &= \frac{92}{\sqrt{8510}}\end{aligned}$$

and so

$$\theta = \arcsin\left(\frac{92}{\sqrt{8510}}\right).$$

Exercise 4.10.

Exercise 4.11.

Exercise 4.12.

Exercise 4.13.

Exercise 4.14.

5. The Local Theory of Curves Parameterized by Arc Length

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10.

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

Exercise 5.16.

Exercise 5.17.

Exercise 5.18.

6. The Local Canonical Form

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

7. Global Properties of Plane Curve

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Exercise 7.9.

Exercise 7.10.

Exercise 7.11.

Exercise 7.12.

Exercise 7.13.

Exercise 7.14.

Exercise 7.15.

CHAPTER 2

Regular Surfaces

2. Regular Surfaces; Inverse Images of Regular Values

Exercise 2.1.

Exercise 2.2.

Exercise 2.3.

Exercise 2.4.

Exercise 2.5.

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Exercise 2.9.

Exercise 2.10.

Exercise 2.11.

Exercise 2.12.

Exercise 2.13.

Exercise 2.14.

Exercise 2.15.

Exercise 2.16.

Exercise 2.17.

Exercise 2.18.

Exercise 2.19.

3. Change of Parameters; Differentiable Functions on Surface

Exercise 3.1.

Exercise 3.2.

Exercise 3.3.

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

Exercise 3.10.

Exercise 3.11.

Exercise 3.12.

Exercise 3.13.

Exercise 3.14.

Exercise 3.15.

Exercise 3.16.

4. The Tangent Plane; The Differential of a Map

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Exercise 4.9.

Exercise 4.10.

Exercise 4.11.

Exercise 4.12.

Exercise 4.13.

Exercise 4.14.

Exercise 4.15.

Exercise 4.16.

Exercise 4.17.

Exercise 4.18.

Exercise 4.19.

Exercise 4.20.

Exercise 4.21.

Exercise 4.22.

Exercise 4.23.

Exercise 4.24.

Exercise 4.25.

Exercise 4.26.

Exercise 4.27.

Exercise 4.28.

5. The First Fundamental Form; Area

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10.

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

6. Orientation of Surfaces

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

CHAPTER 3

The Geometry of the Gauss Map

2. The Definition of the Gauss Map and Its Fundamental Properties

Exercise 2.1.

Exercise 2.2.

Exercise 2.3.

Exercise 2.4.

Exercise 2.5.

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Exercise 2.9.

Exercise 2.10.

Exercise 2.11.

Exercise 2.12.

Exercise 2.13.

Exercise 2.14.

Exercise 2.15.

Exercise 2.16.

Exercise 2.17.

Exercise 2.18.

Exercise 2.19.

Exercise 2.20.

3. The Gauss Map in Local Coordinates

Exercise 3.1.

Exercise 3.2.

Exercise 3.3.

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

Exercise 3.10.

Exercise 3.11.

Exercise 3.12.

Exercise 3.13.

Exercise 3.14.

Exercise 3.15.

Exercise 3.16.

Exercise 3.17.

Exercise 3.18.

Exercise 3.19.

Exercise 3.20.

Exercise 3.21.

Exercise 3.22.

Exercise 3.23.

Exercise 3.24.

4. Vector Fields

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Exercise 4.9.

Exercise 4.10.

Exercise 4.11.

Exercise 4.12.

Exercise 4.13.

5. Ruled Surfaces and Minimal Surfaces

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10.

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

CHAPTER 4

The Intrinsic Geometry of Surfaces

2. Isometries; Conformal Maps

Exercise 2.1.

Exercise 2.2.

Exercise 2.3.

Exercise 2.4.

Exercise 2.5.

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Exercise 2.9.

Exercise 2.10.

Exercise 2.11.

Exercise 2.12.

Exercise 2.13.

Exercise 2.14.

Exercise 2.15.

Exercise 2.16.

Exercise 2.17.

Exercise 2.18.

Exercise 2.19.

Exercise 2.20.

3. The Gauss Theorem and the Equations of Compatibility

Exercise 3.1.

Exercise 3.2.

Exercise 3.3.

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

4. Parallel Transport. Geodesics.

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Exercise 4.9.

Exercise 4.10.

Exercise 4.11.

Exercise 4.12.

Exercise 4.13.

Exercise 4.14.

Exercise 4.15.

Exercise 4.16.

Exercise 4.17.

Exercise 4.18.

Exercise 4.19.

Exercise 4.20.

Exercise 4.21.

Exercise 4.22.

Exercise 4.23.

5. The Gauss-Bonnet Theorem and Its Applications

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

6. The Exponential Map. Geodesic Polar Coordinates

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Exercise 6.9.

Exercise 6.10.

Exercise 6.11.

Exercise 6.12.

Exercise 6.13.

7. Further Properties of Geodesics; Convex Neighborhoods

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

CHAPTER 5

Global Differential Geometry

2. The Rigidity of the Sphere

Exercise 2.1.

Exercise 2.2.

Exercise 2.3.

Exercise 2.4.

3. Complete Surfaces. Theorem of Hopf-Rinow

Exercise 3.1.

Exercise 3.2.

Exercise 3.3.

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

Exercise 3.10.

Exercise 3.11.

4. First and Second Variations of Arc Length; Bonnet's Theorem

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

5. Jacobi Fields and Conjugate Points

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Exercise 5.9.

6. Covering Spaces; The Theorems of Hadamard

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Exercise 6.9.

Exercise 6.10.

Exercise 6.11.

8. Surfaces of Zero Gaussian Curvature

9. Jacobi's Theorems

Exercise 9.1.

Exercise 9.2.

10. Abstract Surfaces; Further Generalizations

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Exercise 10.9.

Exercise 10.10.

11. Hilbert's Theorem

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.