Gravity and Gravitation

Gravity:

The force of attraction between the earth and any object is called gravity.
Example: *The force of attraction between earth and table.*

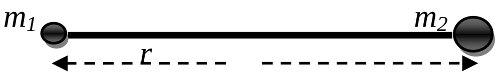
Gravitation:

The force of attraction between any two objects in the universe is called gravitation.
Example: *The force of attraction between earth and sun.*

Newton's Law of Gravitation:

Statement:

"Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them."



Explanation:
Let us consider m_1 and m_2 be the masses of the particles and F is the force of attraction between them. The distance between the centres of the masses is r. Then according to the Newton's law of gravitation, we can write,

$$F \propto \frac{m_1 m_2}{r^2}$$
$$F = G\frac{m_1 m_2}{r^2}.$$

Where, G is the universal constant of gravitation.

Gravitational field:

The space around a body within which the gravitational force of attraction is perceived is called gravitational field.
The gravitational field or intensity at a point is the force experienced by a unit mass at that point. If the gravitational field at a point is E, the force acting on a mass m is F,

$$\boldsymbol{F} = m\boldsymbol{E}$$
$$\boldsymbol{E} = \frac{\boldsymbol{F}}{m} \dots\dots\dots\dots\dots\dots\dots\dots.(1)$$

Also the gravitational field is defined as the negative gradient of gravitational potential

$$\boldsymbol{E} = -\frac{d\boldsymbol{V}}{dx}$$

Gravitational Potential:

Gravitational potential at a point is defined as the amount of work done is moving a unit mass from the point to infinity against the gravitational force of attraction.

If we denoting the gravitational potential at a point distant r from a body of mass M by V, we can write

$$V = -\frac{G}{r}.M$$

It is maximum and zero at infinity. At all other points it is less than zero, i.e., a negative value. Gravitational potential at a point is a scaler quantity.

Gravitational Potential Energy:

The gravitational potential energy of a body at a point in a gravitational field is equal to the product of the mass of the body and the gravitational potential at that point. Therefore, we can write

$$Gravitational\ Potential\ Energy = -\frac{G}{r}M.m.$$

Here, m is the mass of the body.

Gravitational Potential due to Point Mass:

Let a mass M be situated at O (fig.-1) and let a unit mass be situated at P. Then, the force of attraction on the unit mass due to M is clearly

$$F = G.\frac{M \times 1}{x^2}$$
$$= \frac{M}{x^2}.G$$

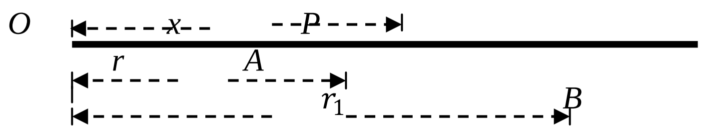


Fig.-1

where x is the distance of P from O, the force being directed towards O. Therefore, work done when the unit mass moves through a small distance dx, towards A, is equal to $M.G.\frac{dx}{x^2}$.
Therefore, work done when it moves from B to A

$$= \int_B^A F\,.dx$$

$$= \int_B^A \frac{M}{x^2}.G.dx$$
$$= G.M\int_B^A \frac{1}{x^2}.dx$$
$$= -G.M\,\left[\frac{1}{x}\right]_B^A$$
$$= -G.M\left[\frac{1}{r} - \frac{1}{r_1}\right]$$
$$= G.M\left[\frac{1}{r_1} - \frac{1}{r}\right]$$

where r and r_1 *are the distances of A and B from O. This, obviously, is the potential difference beween the points A and B.*
If B be at infinity, i.e., if $r_1 = \infty$, *we have*
Potential difference between A and $\infty = G.M\left[\frac{1}{\infty} - \frac{1}{r}\right]$

$$= -\frac{M}{r} \, . G.$$

But potential difference between A and ∞ is equal to the potential at A, because the gravitational force at ∞, due to M, is zero and therefore, the work done in moving a mass about, there is also zero. In other words, the potential at infinity is zero. Therefore, the gravitational potential at a due to the mass M,

$$= -\boldsymbol{\frac{GM}{r}}.$$

If we denoting the gravitational potential at a point distant r from a body of mass M by V, we can write

$$V = -\frac{G}{r} . M.$$

It is maximum and zero at infinity. At all other points it is less than zero,i.e., a negative value.

Gravitational Potential due to a Spherical Shell

(a) At a point outside the shell:

Let us consider a uniform thin spherical shell of mass M and radius R. We have to calculate the potential due to this shell at a point P. Let the centre of the shall be O and OP= r. Let ρ be the density per unit area of the shell.

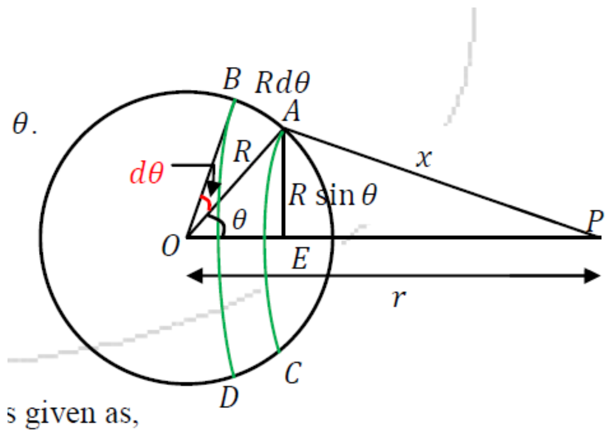
Let us draw a radius making an angle θ. Keeping the fixed ∠AOP=θ fixed; let us rotate this radius about OP. The point A traces a circle on the surface of the shell. Let us now consider another radius OB at an angle θ+ dθ and draw a similar circle about OP. The part of the spherical shell between these two circles can be treated as a ring. Therefore,

The radius of the ring is given as , AE=OA. Sinθ=R. Sinθ.

The width of the ring is given as AB= OB. Sindθ=R.dθ.
So that, its circumference= 2πRsinθ.
The area of the ring is hence given as,

$$= \text{its circumference} \times \text{its width}$$
$$= 2\pi R \sin\theta \times R.d\theta$$
$$= 2\pi R^2 \sin\theta d\theta.$$

Mass of the element or ring, $m = (2\pi R^2 \sin\theta d\theta)\rho$



Let us consider, $AP = x$*, every point of the element is at a distance x from P and therefore, the potential at P due to this small element is given by*

$$dV = -\frac{Gm}{x}$$

$$dV = -\frac{2G\pi R^2 \rho sin\theta d\theta}{x} \dots\dots\dots\dots\dots\dots.(1)$$

Now, in $\Delta\, OAP$, $x^2 = R^2 + r^2 - 2rR\cos\theta$

Differentiating the above expression, we have

$$2x.dx = 0 + 0 + 2rR\sin\theta\, d\theta. \qquad \textit{[R and r being constants]}$$

Hence , $x = \frac{rR\sin\theta\, d\theta}{dx}$

Substituting this value of x in expression (1) above, we have

$$dV = -\frac{2\pi R^2 \sin\theta \, d\theta \, \rho}{Rr \sin\theta \, d\theta} \, G \, dx$$

$$= -\frac{2\pi G R\rho \, dx}{r}$$

Integrating this between the limits, $x = AP = (r - R)$ *and* $x = BP = (r + R)$, *we get V, the potential due to the whole shell at the point P.*

Thus,

$$V = \int_{(r-R)}^{(r+R)} -\frac{2\pi R\rho G \, dx}{r}$$

$$= -\frac{2\pi R\rho G}{r} \int_{(r-R)}^{(r+R)} dx.$$

$$= -\frac{2\pi R\rho G}{r} [x]_{(r-R)}^{(r+R)}$$

$$= -\frac{2\pi R\rho G}{r} . 2R$$

$$= -\frac{4\pi R^2 \rho . G}{r}$$

Now, $4\pi R^2$ *is the surface area of the whole shell and therefore,* $4\pi R^2\rho$ *is equal to its mass M. We thus have*

$$\boldsymbol{V = -\frac{M}{r} G.}$$

Thus for a point outside the shell,the shell behaves as if the whole of its mass is concentrated at the centre of the shell.

Further $V \propto \frac{1}{r}$

Gravitational Field:

$$E = -\frac{dV}{dr}$$

(a) ***At a point inside the shell:***

Let us consider a uniform thin spherical shell of mass M and radius R. We have to calculate the potential due to this shell at a point P. Let the centre of the shall be O and OP= r. Let ρ be the density per unit area of the shell.

Let us draw a radius making an angle θ. Keeping the fixed ∠AOP=θ fixed; let us rotate this radius about OP. The point A traces a circle on the surface of the shell. Let us now consider another radius OB at an angle θ+ dθ and draw a similar circle about OP. The part of the spherical shell between these two circles can be treated as a ring. Therefore,

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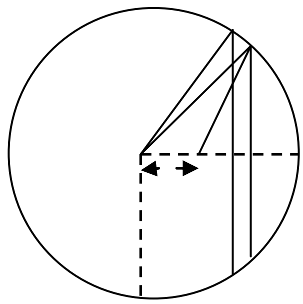

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Differentiating the above expression, we have

$$2x.dx = 0 + 0 + 2rR \sin\theta\, d\theta. \qquad \textit{[R and r being constants]}$$

Hence , $x = \frac{rR \sin\theta d\theta}{dx}$

Substituting this value of x in expression (1) above, we have

$$dV = -\frac{2\pi R^2 \sin\theta\, d\theta\, \rho}{Rr \sin\theta\, d\theta}\, G\, dx$$

$$= -\frac{2\pi GR\rho\, dx}{r}$$

For the whole shell integrate between the limits $(R-r)$ *and* $(R+r)$*. In this case, the limits of x are (R-r) and (R+r). So that, we have*

Thus,

$$V = \int_{(R-r)}^{(R+r)} -\frac{2\pi GR\rho}{r}\, dx.$$

$$= -\frac{2\pi GR\rho}{r}[x]_{(R-r)}^{(R+r)}$$

$$= -\frac{2\pi GR\rho}{r}.2r$$

$$= -4\pi GR\rho.$$

Or, multiplying and dividing by a, we have $V = -\frac{4\pi R^2\rho G}{R}$.

Now, $4\pi R^2\rho = M$*, the mass of the shell.*

Hence $\boldsymbol{V = -\frac{M}{R}G}, \dots\dots\dots\dots\dots\dots\dots.(\mathbf{2})$

Expression (2) shows that ***V*** *is independent of* ***r.*** *It is the same at all points inside the shell. This potential is equal to the potential on the surface of the shell and is constant.*

Gravitational Field:

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(c) At a point on the surface of the shell:

Let us consider a uniform thin spherical shell of mass M and radius R. We have to calculate the potential due to this shell at a point P. Let the centre of the shall be O and OP= r. Let ρ be the density per unit area of the shell.

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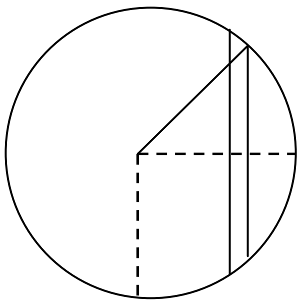

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Hence , $x = \frac{rR\sin\theta\, d\theta}{dx}$

Substituting this value of x in expression (1) above, we have

$$dV = -\frac{2\pi R^2 \sin\theta\, d\theta\, \rho}{Rr\sin\theta\, d\theta}\, G\, dx$$

$$= -\frac{2\pi GR\rho\ dx}{r}$$

In the above case, if we imagine the point P to lie at N, i.e., on the surface of the shell itself, we obtain the potential there by integrating the expression for dV between the limits $x = 0$ and $x = 2R$. So that in this case,

$$V = \int_0^{2R} -\frac{2\pi GR\rho}{r}.dx$$

$$= -\frac{2\pi GR\rho}{r}[x]_0^{2R}$$

$$= -\frac{4\pi R^2\rho.G}{r}$$

$$= -\frac{MG}{r}$$

$$= -\frac{MG}{r}$$

Thus,

$$\boldsymbol{V = -\frac{M}{r}G.}$$

Again, therefore, the whole mass of the shell behaves as though it were concentrated at its centre.

Escape Velocity:

Escape velocity is defined as the velocity with which a body has to be projected vertically upwards from the earth's surface so that it escapes the earth's gravitational field altogether.

If v is the escape velocity, then the initial kinetic energy of projection $\frac{1}{2}mv^2$ must be equal to the work done in moving the body from the surface of the earth to infinity.

Therefore, we can write

$$\frac{1}{2}mv^2 = \frac{GMm}{r}$$

$$v = \sqrt{2gr}$$

$$v = 11.2\ km/sec.$$

Kepler's Laws of Motion:

The three laws of planetary motion are:

(1) <u>Shape of the orbit:</u>

Every planet moves in an elliptical orbit with the sun being at one of its foci.

(2) <u>Velocity in the orbit [Law of areas]:</u>

The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time.

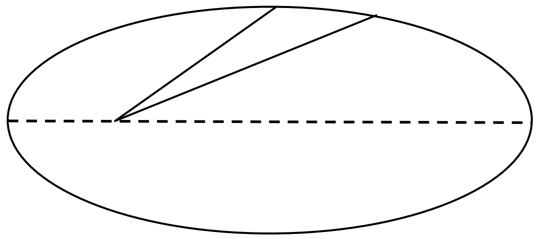


Fig.-1

(3) <u>Time periods of Planets:</u>

The square of the time period of revolution of a planet around the sun is proportional to the cube of the semi-major axis of the orbit.Thus, if a_1 and a_2 are the semi-major axis of the plantes and T_1 and T_2 are the time periods, respectively, then

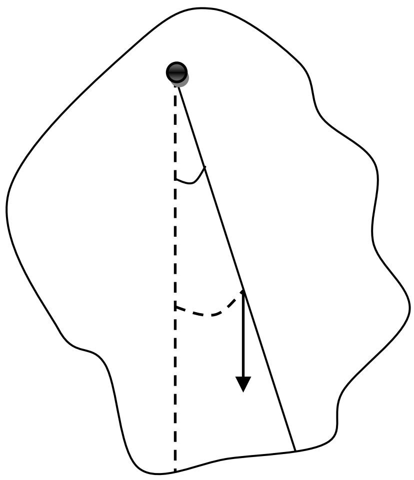
$$\frac{T_1^2}{a_1^3} = \frac{T_2^2}{a_2^3} = Constant$$

$$or\ \frac{T^2}{a^3} = Constant$$

$$\therefore T^2 \propto a^3.$$

Compound Pendulum:

A compound pendulum is a rigid mass capable of oscillating about a horizontal axis passing through any point of the mass. This point is called the point of suspension. In Fig.-1, G is the centre of gravity of the body and S is the point of suspension. At any instant of time, when the mass has been displaced, the force acting vertically downwards=Mg. At this position, the line SG makes an angle θ with the vertical and the restoring moment of this force about the point S=Mgl sinθ. This is the only moment which produces angular acceleration in the pendulum.



Let the moment of inertia of the pendulum about an axis passing through S and perpendicular to its length be I. If the angular acceleration at this instant is $\frac{d^2\theta}{dt^2}$*, then*

$$I\frac{d^2\theta}{dt^2} = -Mglsin\theta$$

[-ve sign shows that the force is directed towards the mean position].

$$\therefore\ I\frac{d^2\theta}{dt^2} + Mglsin\theta = 0$$

For small angular displacements, sinθ=θ

$$\therefore\ I\frac{d^2\theta}{dt^2} + Mgl\theta = 0$$

The moment of inertia of the pendulum about an axis passing through S and perpendicular to its plane =$MK^2 + Ml^2$*. Where K is the radius of gyration.*

$$\therefore\ M(K^2 + l^2)\frac{d^2\theta}{dt^2} + Mgl\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{lg}{K^2 + l^2}\right)\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

This equation is similar to the equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Comparing equation (1) and (2),

$$\omega^2 = \left(\frac{lg}{K^2 + l^2}\right)$$

Here ω is the angular frequency.

$\therefore$ *Time period* $T = \frac{2\pi}{\omega}$

$$T = 2\pi\sqrt{\frac{K^2 + l^2}{lg}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or $T = 2\pi\sqrt{\frac{(K^2/l)+l}{g}}$

Here $(K^2/l) + l$ *is called the equivalent length of the simple pendulum.*