

# Introduction to Spectral Methods

## Numerical Solution of Poisson's Equation

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# Outline

Poisson's Equation

One dimension

Two Dimensions

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### One dimension

### Two Dimensions

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When  $f = 0$ , we have Laplace's equation as a special case.

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- ▶ And much more ...

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One dimension

Two Dimensions

# Poisson's Equation in one dimension

$$\frac{d^2 u}{dx^2} = f(x)$$

Subject to  $u(a) = g_1$  and  $u(b) = g_2$ .

It is not always possible to integrate twice to find the solution.

We must translate this continuous problem into a discrete one.

For the space coordinate  $x \in [a, b]$ , we choose  $n$  nodes  $\{x_i\}_{i=1}^n$  on which we would like to know the value of  $u(x_i) = u_i$ .

$$\text{Input } u_1, u_n, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \vec{f} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}; \text{ output } \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$



The discrete solution must solve the continuous problem.

Is there a way to express  $u(x)$  in terms of  $u_i$ ?

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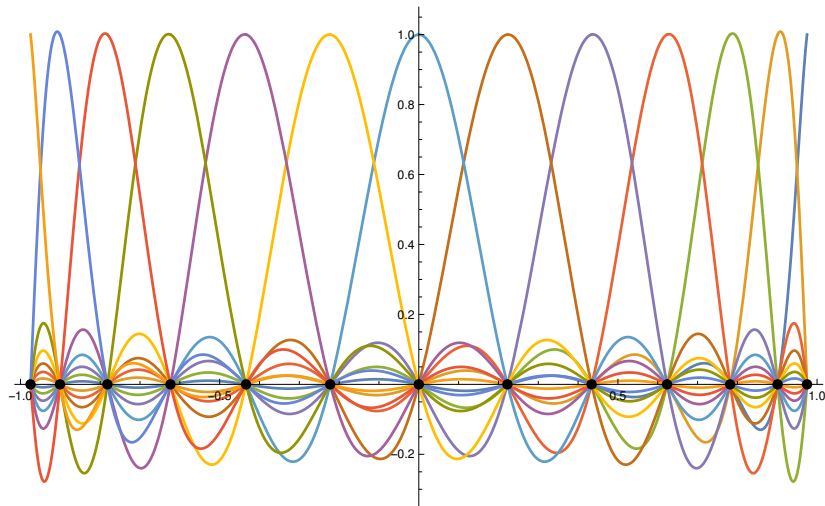
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$$u(x_i) = \sum_j u_j \delta_j(x_i) = u_i$$

$$\delta_j(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This is how the cardinal functions look like.



Now differentiate the expansion in cardinal functions.

$$u'(x_i) = \sum_j u_j \delta'_j(x_i)$$

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$$\frac{d}{dx} \vec{u} = D \vec{u}$$

$$\frac{d^2}{dx^2} \vec{u} = D^2 \vec{u} = \vec{f}$$

Solve the system of linear equations.

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$$\begin{bmatrix} D_{11}^2 & D_{1j}^2 & D_{1n}^2 \\ D_{i1}^2 & D_{ij}^2 & D_{in}^2 \\ D_{n1}^2 & D_{nj}^2 & D_{nn}^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_j \\ u_n \end{bmatrix} = \begin{bmatrix} D_{11}^2 \\ D_{i1}^2 \\ D_{n1}^2 \end{bmatrix} u_1 + \begin{bmatrix} D_{1j}^2 \\ D_{ij}^2 \\ D_{nj}^2 \end{bmatrix} [u_j] + \begin{bmatrix} D_{1n}^2 \\ D_{in}^2 \\ D_{nn}^2 \end{bmatrix} u_n = \begin{bmatrix} f_1 \\ f_i \\ f_n \end{bmatrix}$$

With  $i$  and  $j$  ranging from 2 to  $n - 1$ .

We have  $n - 2$  unknowns, but still  $n$  equations.

Discard the first and last rows.

$\nabla^2 u = f$  will not hold true at the boundary, but we will ensure the boundary conditions.

$$\begin{aligned} [D_{ij}^2] [u_j] &= [f_i] - u_1 [D_{i1}^2] - u_n [D_{in}^2] \\ [u_j] &= [D_{ij}^2]^{-1} [f_i - u_1 D_{i1}^2 - u_n D_{in}^2] \end{aligned}$$

With  $i$  and  $j$  ranging from 2 to  $n - 1$ .

## Cardinal functions are obtained from a mother function.

Usually we chose an orthogonal polynomial. It all depends on the domain of your solution.

Chebysheb	$\Psi_n(x) = T_n(x)$	$x \in [-1, 1]$
Legendre	$\Psi_n(x) = P_n(x)$	$x \in [-1, 1]$
Hermite	$\Psi_n(x) = H_n(x)e^{-x^2/2}$	$x \in (-\infty, \infty)$
Laguerre	$\Psi_n(x) = xL_{n-1}^{(1)}(x)e^{-x/2}$	$x \in [0, \infty)$

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Let  $\Psi_n(x)$  be a function with  $n$  roots on  $\{x_i\}_{i=1}^n$ , i. e.  $\Psi_n(x_i) = 0$ .  
By expanding  $\Psi_n(x)$  around its roots we can find a expression for  $\delta_j(x)$ .

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$$\Psi_n(x) = 0 + \Psi'_n(x_j)(x - x_j) + \frac{\Psi''_n(x_j)}{2}(x - x_j)^2 + O((x - x_j)^3)$$

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$$\delta_j(x) = \frac{\Psi_n(x)}{(x - x_j)\Psi'_n(x_j)} = 1 + \frac{\Psi''_n(x_j)}{2\Psi'_n(x_j)}(x - x_j) + O((x - x_j)^2)$$

Differentiate to obtain the elements of  $D$ .

$$D_{ij} = \delta'_j(x_i) = \frac{1}{x_i - x_j} \frac{\Psi'_n(x_i)}{\Psi'_n(x_j)}, \quad i \neq j$$

The diagonal is a bit tricky.

$$D_{jj} = \delta'_j(x_j) = \frac{\Psi''_n(x_j)}{2\Psi'_n(x_j)}$$



# MATLAB code

Solve Poisson's equation

$$\frac{d^2 u}{dx^2} = 1000 \cos(5\pi x) e^{-x^2}$$

with boundary conditions  $u(-1) = 2$ ,  $u(1) = -1$ .

# MATLAB code

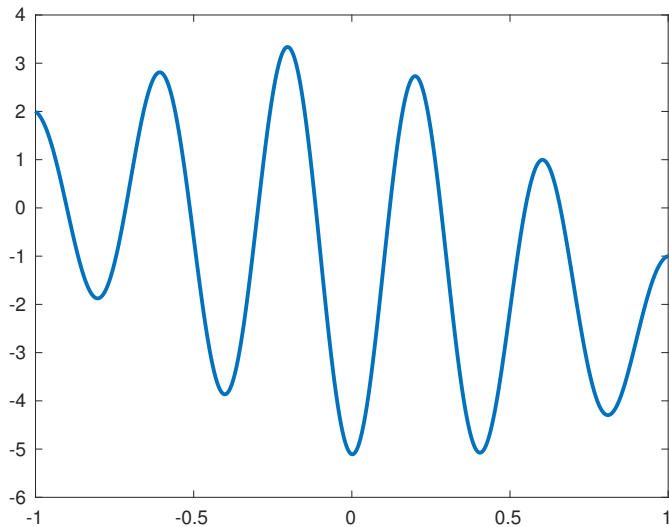
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```
1 [D,x]=chebD(n); D2=D*D; % Differentiation matrix and nodes
2 f=1000*cos(5*pi*x).*exp(-x.^2); % Source term
3 u=zeros(n,1); u([1,n])=[-1,2]; % Impose boundary conditions
4 % Solve and plot
5 u(2:n-1)=D2(2:n-1,2:n-1)\(f(2:n-1)-D2(2:n-1,[1,n])*u([1,n]));
6 plot(x,u);
```

Here is our solution.



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Two Dimensions

# Poisson's Equation in two dimensions

Our domain of solution will be the rectangle  $[a, b] \times [c, d]$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Subject to boundary conditions

$$u(x, y) = \begin{cases} g_1(y), & x = a \\ g_2(y), & x = b \\ h_1(x), & y = c \\ h_2(x), & y = d \end{cases}$$

In 2D we use matrices for  $u$  and  $f$ .

$$u(x_i, y_j) = u_{ij}$$

$$f(x_i, y_j) = f_{ij}$$

$$u(x, y) = \sum_{k,l} u_{kl} \delta_k(x) \delta_l(y)$$

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$$\nabla^2 u = D^2 U + U(D^2)^T$$

Now solve the system of linear equations.

$$D^2 U + U(D^2)^T = F$$

Boundary conditions are the tricky part.

$$U = \tilde{U} + U_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & u_{ij} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{1j} & u_{1n} \\ u_{i1} & 0 & u_{in} \\ u_{n1} & u_{nj} & u_{nn} \end{bmatrix}$$

$$D^2 \tilde{U} + \tilde{U}(D^2)^T = F - D^2 U_B - U_B(D^2)^T = \tilde{F}$$

Discard first and last rows and columns.

$\nabla^2 u = f$  will not hold true at the boundary, but we will ensure the boundary conditions.

$$(D_{ij}^2)\tilde{U}_{ij} + \tilde{U}_{ij}(D_{ij}^2)^T = \tilde{F}_{ij}$$

With  $i$  and  $j$  ranging from 2 to  $n - 1$ .

Now we just have to solve  $(n - 2)^2 \times (n - 2)^2$  system of linear equations.

## Matrix inversion does not help.

A matrix equation of the form  $AX + XB = C$  is known as a Sylvester equation.

There's a built-in MATLAB function that solves them.

```
X=sylvester(A,B,C)
```

This algorithm uses the Schur decomposition to solve a block-triangular matrix very efficiently.

## Two dimensional example

Solve Laplace's equation on  $(x, y) \in [-1, 1] \times [-1, 1]$

$$\nabla^2 u = 0$$

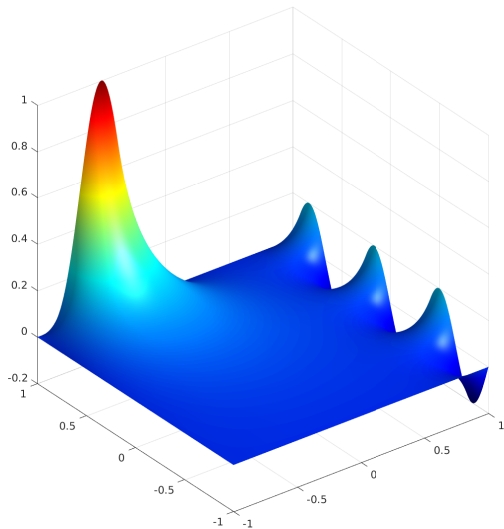
with boundary conditions

$$u(x, y) = \begin{cases} \sin^4(\pi x), & y = 1 \text{ and } -1 < x < 0, \\ \frac{1}{5} \sin(3\pi y), & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

# MATLAB Code

```
1  [D,x]=chebD(n); D2=D*D; y=x';
2  [xx, yy]=ndgrid(x);
3
4  % Boundary conditions
5  g=[0.2*sin(3*pi*y); 0*y];
6  h=[(x<0).*sin(pi*x).^4, 0*x];
7  uu=zeros(n);
8  uu([1 n],:)=g;
9  uu(:,[1 n])=h;
10
11 % Solve Laplace's equation
12 F=zeros(n);
13 RHS=F-D2(:,[1 n])*g-h*D2(:,[1 n])';
14 uu(2:n-1, 2:n-1)=sylvester(D2(2:n-1, 2:n-1), ...
15 D2(2:n-1, 2:n-1)', RHS(2:n-1, 2:n-1));
16
17 surf1(xx,yy,uu,'light'); colormap(jet(256));
18 shading interp; axis square;
```

Here is our solution.



# References



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