STABLE PHENOMENA IN SPACES NAME: SOLUTIONS

**Due at the beginning of class on 6 February 2024**

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

**Reading:** [Sto22, Chapter 2].

(1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where $M$ is a torsion $\mathbb{Z}$-module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.

SOLUTION: Recall that the stable homotopy groups of $S^0$ are defined as follows:

$$\pi_i^s(S^0) = \operatorname{colim}_{n\to\infty}\pi_{i+n}(\Sigma^n S^0) = \operatorname{colim}_{n\to\infty}\pi_{i+n}(S^n)$$

Since $i > 0$, the fact that $\pi_n(S^n) \cong \mathbb{Z}$ is not relevant since it only impacts $\pi_0^s(S^0)$. Thus, we need only show that the colimit of each sequence containing $\pi_{4j-1}(S^{2j})$ is finite abelian. For $i > 0$, every sequence in the colimit $\pi_i^s(S^0)$ will eventually stabilize by the Freudenthal Suspension Theorem. Suppose this stabilization occurs at $\pi_{4j-1}(S^{2j})$. This would imply:

$$\pi_{4j-1}(S^{2j}) \cong \pi_{4j}(S^{2j+1})$$

But $\pi_{4j}(S^{2j+1})$ is finite abelian by Serre's Theorem, so stabilization cannot occur at $\pi_{4j-1}(S^{2j})$. Hence, $\pi_i^s(S^0)$ must be finite abelian for $i > 0$.

(2) Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. A *monoid* in $\mathcal{C}$ is an object $M$ together with morphisms $m\colon M \otimes M \to M$ and $i\colon 1 \to M$ such that the following diagrams commute:

$$\begin{tikzcd} M\otimes M\otimes M \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"] & M\otimes M \arrow[d, "m"] \\ M\otimes M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} 1\otimes M \arrow[r, "i\otimes id"] \arrow[dr, "\cong"'] & M\otimes M \arrow[d, "m"] & M\otimes 1 \arrow[l, "id\otimes i"'] \arrow[dl, "\cong"] \\ & M & \end{tikzcd}$$

A *morphism of monoids* $f\colon M \to N$ is one that commutes with the structure morphisms:

$$\begin{tikzcd} M\otimes M \arrow[r, "f\otimes f"] \arrow[d, "m"] & N\otimes N \arrow[d, "m"] \\ M \arrow[r, "f"] & N \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "i"] \arrow[dr, "i"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

$(M, m, i)$ is a *commutative monoid* if $m = m \circ s$, where $s\colon M \otimes M \to M \otimes M$ is the symmetry in $\mathcal{C}$.

(a) Let $M$ and $C$ be objects in $\mathcal{C}$. Prove that if $M$ is a monoid and $C$ is a comonoid, then $\mathcal{C}(C, M)$ is a monoid in the ordinary sense: a set with an associative and unital operation.

SOLUTION: Let $\mu, \eta$ be the multiplication/unit of $M$ and $\delta, \epsilon$ be the comultiplication/counit of $C$. The monoid operation on $\mathcal{C}(C, M)$ has multiplication

$$f * g\colon C \xrightarrow{\delta} C \otimes C \xrightarrow{f\otimes g} M \otimes M \xrightarrow{\mu} M$$

and unit

$$e\colon C \xrightarrow{\epsilon} 1 \xrightarrow{\eta} M.$$

Left unitality follows from

$$\mu(e \otimes f)\delta = \mu((\eta\epsilon) \otimes f)\delta = \mu(\eta \otimes \mathrm{id})(\mathrm{id} \otimes f)(\epsilon \otimes \mathrm{id})\delta = \mathrm{id} \otimes f = f$$

where the unitors have been omitted, and a similar argument shows right unitality. Associativity follows from

$$\begin{aligned}(f * g) * h = \mu((f * g) \otimes h)\delta &= \mu((\mu(f \otimes g)\delta) \otimes h)\delta \\ &= \mu(\mu \otimes \mathrm{id})(f \otimes g \otimes h)(\delta \otimes \mathrm{id})\delta \\ &= \mu(\mathrm{id} \otimes \mu)(f \otimes g \otimes h)(\mathrm{id} \otimes \delta)\delta \\ &= \mu(f \otimes (\mu(g \otimes h)\delta))\delta = \mu(f \otimes (g * h))\delta = f * (g * h)\end{aligned}$$

where the associators have been omitted.

(b) Let $M$ in $\mathcal{C}$ be a monoid in two different ways: $(M, m, i)$ and $(M, n, j)$. Further assume that $m$ and $n$ are morphisms of monoids. Prove that $M$ is a commutative monoid and the two structures are the same.

SOLUTION: The monoid structure on $M \otimes M$ induced by $m, i$ has multiplication

$$(m \otimes m)(\mathrm{id} \otimes s \otimes \mathrm{id})\colon (M \otimes M) \otimes (M \otimes M) \xrightarrow{\mathrm{id}\otimes s\otimes \mathrm{id}} M \otimes M \otimes M \otimes M \xrightarrow{m \otimes m} M \otimes M$$

and unit

$$i \otimes i\colon 1 \xrightarrow{\sim} 1 \otimes 1 \xrightarrow{i \otimes i} M \otimes M$$

where unitors and associators are supressed and $s$ is the symmetry isomorpism of $\mathcal{C}$. The use of the symmetry is needed to capture the idea of a 'component-wise' multiplication $(a, b)(c, d) = (ac, bd)$.

That $n\colon M \otimes M \to M$ is a morphism of monoids in this structure means that

$$\begin{tikzcd} (M \otimes M) \otimes (M \otimes M) \arrow[r, "(m\otimes m)(\mathrm{id}\otimes s\otimes \mathrm{id})"] \arrow[d, "n\otimes n"] & M \otimes M \arrow[d, "n"] \\ M \otimes M \arrow[r, "m"] & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & 1 \arrow[dl, "i\otimes i"'] \arrow[dr, "i"] & \\ M \otimes M \arrow[rr, "n"] & & M \end{tikzcd}$$

are commutative diagrams. There are similar diagrams witnessing that $m$ is a monoid morphism with the structures induced by $n$.

First, we show $i = j$.

$$\begin{aligned} i = m(i \otimes i) &= m((n(j \otimes i)) \otimes (n(i \otimes j))) = m(n \otimes n)(j \otimes i \otimes j \otimes i) \\ &= n(m \otimes m)(\mathrm{id} \otimes s \otimes \mathrm{id})(j \otimes i \otimes i \otimes j) = n(m \otimes m)(j \otimes i \otimes i \otimes j) \\ &= n((m(j \otimes i)) \otimes (m(j \otimes i))) = n(j \otimes j) = j \end{aligned}$$

Next, we show $m = n$.

$$\begin{aligned} m = m(\mathrm{id} \otimes \mathrm{id}) &= m((n(\mathrm{id} \otimes i)) \otimes (n(i \otimes \mathrm{id}))) = m(n \otimes n)(\mathrm{id} \otimes i \otimes i \otimes \mathrm{id}) \\ &= n(m \otimes m)(\mathrm{id} \otimes s \otimes \mathrm{id})(\mathrm{id} \otimes i \otimes i \otimes \mathrm{id}) = n(m \otimes m)(\mathrm{id} \otimes i \otimes i \otimes \mathrm{id}) \\ &= n((m(\mathrm{id} \otimes i)) \otimes (m(i \otimes \mathrm{id}))) = n(\mathrm{id} \otimes \mathrm{id}) = n \end{aligned}$$

Finally, we show that $m = n$ is commutative.

$$\begin{aligned} m = m(\mathrm{id}\otimes \mathrm{id}) &= m((m(i\otimes \mathrm{id}))\otimes(m(\mathrm{id}\otimes i))) = m(m\otimes m)(i\otimes \mathrm{id}\otimes \mathrm{id}\otimes i) \\ &= m(m\otimes m)(\mathrm{id}\otimes s\otimes \mathrm{id})(i\otimes \mathrm{id}\otimes \mathrm{id}\otimes i) = m(m\otimes m)(i\otimes \mathrm{id}\otimes \mathrm{id}\otimes i)s \\ &= m((m(i\otimes \mathrm{id}))\otimes(m(\mathrm{id}\otimes i)))s = m(\mathrm{id}\otimes \mathrm{id})s = ms \end{aligned}$$

This completes the proof.

(c) For any spaces X and Y, prove that $[X, \Omega^2 Y]$ and $[\Sigma X, \Omega Y]$ are abelian groups.

SOLUTION: Recall that $\Sigma X$ is an H-cogroup and $\Omega Y$ is an H-group. This yields that $\Sigma X$ is a cogroup in the homotopy category of spaces with the (derived) cocartesian monoidal structure and $\Omega Y$ is a group in the homotopy category of spaces with the (derived) cartesian monoidal structure. Further, $\Sigma X$ (and in fact any object) is a comonoid in the cartesian monoidal structure using the diagonal $\Sigma X \to \Sigma X \times \Sigma X$, and $\Omega Y$ (and in fact any object) is a monoid in the cocartesian structure using the fold map $\Omega Y \vee \Omega Y \to \Omega Y$.

This puts two group structures on $[\Sigma X, \Omega Y]$ using part a. The inverse comes with pre/post-composing with the (co)inverse of $\Sigma X / \Omega Y$. The fact that these operations distribute over each other (i.e. are homomorphisms for eachh other) will follow from interactions of $\times$ and $\vee$. Consequently, by the previous part, they are the same and this operation is abelian.

Since $[X, \Omega^2 Y] \cong [\Sigma X, \Omega Y]$, this makes $[X, \Omega^2 Y]$ into an abelian group as well. Alternatively, we may notice that $\Omega^2 Y \cong \Omega\Omega Y$ is an H-group in two (seemingly) distinct ways, using concatenation in the first or second $\Omega$. The fact that these structures are homomorphisms for each other follows from the functoriality of $\Omega$ and the naturality of the group structure. Therefore, these are also the same operation and this operation is abelian. In fact, this is the same operation induced by the structure on $[\Sigma X, \Omega Y]$.

(3) Let $f\colon X \to Y$ be a map between simply connected spaces such that $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$. We will show that $f$ is an $n$-connected map.

(a) Let $C$ be the homotopy cofiber of $f$, and let $F$ be the homotopy fiber of $Y \to C$. Use the Hurewicz theorem to show that $C$ is $n$-connected and $F \to Y$ is an $n$-connected map.

SOLUTION: The cofiber $C$ of $f$ is the homotopy pushout of $* \leftarrow X \xrightarrow{f} Y$, giving a Mayer-Vietoris exact sequence

$$\dots \to H_i(X) \xrightarrow{f_*} H_i(Y) \to H_i(C) \xrightarrow{\partial} H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \to \dots$$

The map $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $i \leq n$. By exactness, $H_i(C)$ is then $0$ for $i \leq n$. Furthermore, since $X$ and $Y$ are 1-connected, $C$ is 1-connected as well. Assuming $n > 1$, the Hurewicz Theorem can be used to assert that $\pi_2(C) \cong H_2(C) = 0$. $C$ is then 2-connected and we can iterate the argument to get $\pi_3(C) \cong H_3(C)$. This may proceed until we reach $\pi_n(C) \cong H_n(C) = 0$, and hence $C$ is $n$-connected. To show that $F \to Y$ is an $n$-connected map, consider the long exact sequence in homotopy induced by the fiber sequence $F \to Y \to C$:

$$\dots \to \pi_{i+1}(Y) \to \pi_{i+1}(C) \xrightarrow{\partial} \pi_i(F) \to \pi_i(Y) \to \pi_i(C) \to \dots$$

We have $\pi_i(C) = 0$ for $i \leq n$, so $\pi_i(F) \to \pi_i(Y)$ is an isomorphism in that range, showing that the map $F \to Y$ is $n$-connected.

(b) Use the Blakers–Massey theorem to show that $X \to F$ is at least 2-connected.

SOLUTION:

We have the following diagram:

$$\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[dr, dashed] \arrow[ddr, bend right, "g"'] & & \\
& F \arrow[r, "p"] \arrow[d] & Y \arrow[d, "q"] \\
& * \arrow[r] & C
\end{tikzcd}$$

where the outer square is a homotopy pushout and the inner square is a homotopy pullback. Thus, the Blakers-Massey theorem can be applied to the induced map $X \to F$.

Since $X$ is 1-connected, the map $g$ is at least 2-connected. Furthermore, since $Y$ is also 1-connected, the map $f$ is at least 1-connected. Hence, by the Blakers-Massey theorem, the map $X \to F$ is at least (2+1-1=2)-connected.

(c) Show that $f$ is at least 2-connected. Iterate your argument from part (b) to show that $f$ is $n$-connected.

SOLUTION: We can write $f$ as a composite:

$$f\colon X \underset{2}{\to} \mathrm{fib}(g) \underset{n}{\to} Y;$$

the connectivity is drawn under the arrows. $f$ is the composite of a 2-connected map and an $n$-connected map, so $f$ must be 2-connected.

Now we can repeat the argument of the previous part: by Blakers–Massey, $f$ is 2-connected and $X \to *$ is 2-connected, so $X \to \mathrm{fib}(g)$ must be at least $(2+2-1=3)$-connected. Then $f$ is the composite of a 3-connected map and an $n$-connected map, so it must be 3-connected. Rinse and repeat to conclude $f$ is $n$-connected.

(4) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of spaces. Prove that $\Omega \operatorname{hocolim}_i X_i \simeq \operatorname{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

SOLUTION: This relies on the fact that $S^1$ is compact, and that we are computing the homotopy colimit. This is a particularly thorny problem, and if you really get into the weeds you'll find yourself questioning what compactness even means. (Compact objects in **Top** vs compact spaces – they're different! And $S^1$ is only the latter.)

Recall that to compute $\operatorname{hocolim}_i X_i$, we cofibrantly replace the sequence $X_i$ with a sequence $Y_i$ and the $\operatorname{hocolim}_i X_i \simeq \operatorname{colim}_i Y_i$. The standard choice is to pick $Y_i$ to be the partial mapping telescopes, and the maps in the sequence of partial mapping telescopes are closed inclusions.

Since $\Omega = F(S^1, -)$ preserves homotopy, the $\Omega Y_i$ are equivalent to the $\Omega X_i$. Further, the maps between the $\Omega Y_i$'s are also closed inclusions ($\Omega Y_i$ can be identified with the functions into $Y_{i+1}$ that happen to land in $Y_i$). Therefore, $\operatorname{hocolim}_i \Omega X_i \simeq \operatorname{colim}_i \Omega Y_i$. Thus, it suffices to show that $\Omega \operatorname{colim}_i Y_i \simeq \operatorname{colim}_i \Omega Y_i$.

Let $Y = \operatorname{colim}_i Y_i = \bigcup_i Y_i$. There is a map $\operatorname{colim}_i \Omega Y_i \to \Omega Y$ induced by the universal property of the colimit. Concretely, it takes a function (or an equivalence class of functions) into a partial mapping telescope and interprets it as a function into the mapping telescope $Y$.

It is a standard exercise in point set topology that this is a homeomorphism. More generally, if $K$ is compact,

$$\operatorname*{colim}_i F(K, Y_i) \to F(K, Y)$$

is a homeomorphism. We give two arguments:

(a) To produce the inverse, recognize that the $Y_i \setminus X_i$'s form an open cover of Y, so their preimages cover K. Since K is compact, given any map $K \to Y$, only finitely many of these preimages are needed to cover K, and therefore K lands in only finitely many of the partial mapping telescopes. If we take $Y_i$ to be the large,st such partial mapping telescope, we get a map $K \to Y_i$. It is straight-forward to show that this is inverse to the canonical map.

(b) It is not too hard to see that this map is an injection, and that it is a surjection if and only if the natural map $K \to Y$ factors through $Y_i$ for some i.

So it remains to be seen that $f: K \to Y$ factors through one of the $X_i$. We are still assuming that these $Y_i \to Y_{i+1}$ are closed inclusions. It is now important that these spaces are CGWH. Assume for the sake of contradiction that $f: S^1 \to Y$ does not factor through any of the $Y_i$. Then there is $y_0 \in f(K)$ such that $y_0 \notin Y_0$. However, by properties of the colimit, there is $a_0 \in \mathbb{N}$ such that $y_0 \in Y_{a_0}$. Choose $y_1 \in f(S^1)$ such that $y_1 \notin X_{a_0}$. There exists $a_1 \in \mathbb{N}$ such that $y_1 \in Y_{a_1}$. Choose $y_2 \in f(K)$ such that $y_2 \notin Y_{a_1}$. Continue inductively to find a sequence of points $y_0, y_1, y_2, \ldots$ such that $y_i \in f(K) \setminus Y_{a_{i-1}}$. Consider the set $Z = \{y_0, y_1, y_2, \ldots\} \subseteq f(S^1)$. A subset $K \subseteq Z$ is closed if and only if $K \cap Y_i$ is closed for all i. By construction, $K \cap Y_i$ is finite and therefore closed by weak Hausdorff (in fact, $T_1$ would be enough). So any subset of Z is closed in Y, including Z itself.

Now consider $f^{-1}(Z) \subseteq S^1$. This is the preimage of a closed set, so closed itself. Therefore, $f^{-1}(Z)$ is compact, as a closed subset of a compact space. But by the above, it is also a discrete topological space since every subset is closed. But any discrete compact space is finite. This is a contradiction, since $f^{-1}(Z)$ is at least countable.

# References

[Sto22] Bruno Stonek. Introduction to stable homotopy theory. `https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf`, July 2022.