LOCALIZATION NAME: ____________________

These problems are not due and will not be graded.

Reading: [vK13, Sections 3 and 4] or [Bou79, Sections 1 and 2]. I also found these slides of Aras Ergus helpful [Erg19].

(1) Let $\mathcal{S}\mathrm{p}_{\mathbb{Q}}$ be the full subcategory of $\mathcal{S}\mathrm{p}$ on the $\mathbb{Q}$-local spectra (the rational spectra).

(a) Show that if R is a ring spectrum, any R-module is R-local.

(b) Show that any $\mathbb{Q}$-local spectrum is an $H\mathbb{Q}$-module in the stable homotopy category.

(c) Show that any map of $\mathbb{Q}$-local spectra is automatically a map of $H\mathbb{Q}$-modules in the stable homotopy category.

(d) Conclude that $\mathrm{ho}(\mathcal{S}\mathrm{p}_{\mathbb{Q}})$ is equivalent to the category of $H\mathbb{Q}$-modules in $\mathrm{ho}(\mathcal{S}\mathrm{p})$.

(2) Let $\widehat{\mathcal{S}\mathrm{p}}$ be your favorite symmetric monoidal category of spectra (e.g. symmetric or orthogonal spectra), and let $\widehat{\mathcal{S}\mathrm{p}}_E$ be the full subcategory of $\widehat{\mathcal{S}\mathrm{p}}$ on the E-local spectra.

(a) If $f\colon W \to X$ and $g\colon Y \to Z$ are E-equivalences, show that

$$L_E(W \wedge Y) \xrightarrow{L_E(f \wedge g)} L_E(X \wedge Z)$$

is a stable equivalence.

(b) Define $X \wedge^E Y := L_E(X \wedge Y)$. Show that $\wedge^E$ defines a symmetric monoidal structure on $\mathrm{ho}(\widehat{\mathcal{S}\mathrm{p}}_E)$ with unit $L_E(\mathbb{S})$.

(c) Conclude that L_E is a strong monoidal functor and the composite $\mathrm{ho}(\widehat{\mathcal{S}\mathrm{p}}) \xrightarrow{L_E} \mathrm{ho}(\widehat{\mathcal{S}\mathrm{p}}_E) \xrightarrow{\iota} \mathrm{ho}(\widehat{\mathcal{S}\mathrm{p}})$ is lax symmetric monoidal. Hence, $L_E(\mathbb{S})$ is always a commutative monoid in the stable homotopy category.

(3) The *Bousfield class* of a spectrum E is the set of E-acyclic spectra, denoted $\langle E \rangle$. The set of Bousfield classes of spectra forms a poset with $\langle E \rangle \geq \langle D \rangle$ if being E-acyclic implies being D-acyclic.

(a) Show that $\langle * \rangle$ is a maximum and $\langle \mathbb{S} \rangle$ is a minimum in this poset.

(b) Show that if $\langle E \rangle \geq \langle D \rangle$, then there is a natural map $L_E X \to L_D X$.

(c) Show that if $\langle E \rangle \geq \langle D \rangle$, then $L_D L_E X \simeq L_D X$.

REFERENCES

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[Erg19] Aras Ergus. The localization of spectra with respect to homology by A. K. Bousfield, eCHT Kan Seminar 2019. https://www.aergus.net/academic/documents/assorted/bousfield-localization.pdf, 2019.

[vK13] Paul van Koughnett. Spectra and localization. https://people.math.harvard.edu/~hirolee/pretalbot2013/notes/2013-02-07-Paul-VanKoughnett-Bousfield_Localization.pdf, 2013.

Credit for all problems to Bert Guillou.