Due at the beginning of class on 22 April 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 6.1 and 6.2].

(1) Let $F \dashv G$ be an adjoint pair of functors between monoidal categories.

(a) Show that if F is strong symmetric monoidal, then G is lax symmetric monoidal.

SOLUTION: Let $F \dashv G$ be an adjunction between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$. We will only assume that F is oplax (we only ask for the "inverse" of the structure maps, we do not need the forward direction) i.e. that there are transformations

$$d\colon F(I) \to I \text{ and } n\colon F(A \otimes B) \to F(A) \otimes F(B)$$

which are compatible with the associativity, unit, and symmetry isomorphisms. We will show that G has an associated lax structure i.e. that there are transformations

$$e\colon I \to G(I) \text{ and } m\colon G(A) \otimes G(B) \to G(A \otimes B)$$

with similar compatibility.

The map e is the adjunct of d, meaning the image of d under the isomorphism $\mathcal{C}(F(I), I) \cong \mathcal{D}(I, G(I))$, but I find it hard to prove the axioms with this definition, and m is not just the adjunct of n. Let $\eta\colon \mathrm{id} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1$ be the unit and counit of the adjunction. Then, e is the composite

$$e\colon I \xrightarrow{\eta} G(F(I)) \xrightarrow{G(d)} G(I).$$

Similarly, m is the composite

$$m\colon G(A) \otimes G(B) \xrightarrow{\eta} G(F(G(A) \otimes G(B))) \xrightarrow{G(n)} G(F(G(A)) \otimes F(G(B))) \xrightarrow{G(\epsilon\otimes\epsilon)} G(A \otimes B).$$

It remains to show that these new transformations have the right compatibility.

To show compatibility with the symmetry, we need to show that if

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "n"] \arrow[d, "F(\sigma)"] & F(A) \otimes F(B) \arrow[d, "\sigma"] \\ F(B \otimes A) \arrow[r, "n"] & F(B) \otimes F(A) \end{tikzcd} \quad \text{commutes then} \quad \begin{tikzcd} G(A) \otimes G(B) \arrow[r, "m"] \arrow[d, "\sigma"] & G(A \otimes B) \arrow[d, "G(\sigma)"] \\ G(B) \otimes G(A) \arrow[r, "m"] & G(B \otimes A) \end{tikzcd}$$

commutes. Expanding the right diagram, we are tasked with showing that

$$\begin{tikzcd} G(A) \otimes G(B) \arrow[r, "\eta"] \arrow[d, "\sigma"] & G(F(G(A) \otimes G(B))) \arrow[r, "G(n)"] & G(F(G(A)) \otimes F(G(B))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(A \otimes B) \arrow[d, "G(\sigma)"] \\ G(B) \otimes G(A) \arrow[r, "\eta"] & G(F(G(B) \otimes G(A))) \arrow[r, "G(n)"] & G(F(G(B)) \otimes F(G(A))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(B \otimes A) \end{tikzcd}$$

commutes which we show by filling in the squares.

$$\begin{tikzcd}
G(A)\otimes G(B) \arrow[r, "\eta"] \arrow[d, "\sigma"] & G(F(G(A)\otimes G(B))) \arrow[r, "G(n)"] \arrow[d, "G(F(\sigma))"] & G(F(G(A))\otimes F(G(B))) \arrow[r, "G(\epsilon\otimes\epsilon)"] \arrow[d, "G(\sigma)"] & G(A\otimes B) \arrow[d, "G(\sigma)"] \\
G(B)\otimes G(A) \arrow[r, "\eta"] & G(F(G(B)\otimes G(A))) \arrow[r, "G(n)"] & G(F(G(B))\otimes F(G(A))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(B\otimes A)
\end{tikzcd}$$

The outside squares commute by the naturality of η and σ, and the middle square is the compatibility square for F plugged into G.

To show compatibility with the (left) unit λ, we need to show that if

$$\begin{tikzcd}
F(I\otimes A) \arrow[r, "n"] \arrow[d, "F(\lambda)"] & F(I)\otimes F(A) \arrow[d, "d\otimes 1"] \\
F(A) & I\otimes F(A) \arrow[l, "\lambda"]
\end{tikzcd}
\quad \text{commutes then} \quad
\begin{tikzcd}
I\otimes G(A) \arrow[r, "e\otimes 1"] \arrow[d, "\lambda"] & G(I)\otimes G(A) \arrow[d, "m"] \\
G(A) & G(I\otimes A) \arrow[l, "G(\lambda)"]
\end{tikzcd}$$

commutes. Expanding the right diagram, we are tasked to show that

$$\begin{tikzcd}
I\otimes G(A) \arrow[r, "\eta\otimes 1"] \arrow[dd, "\lambda"] & G(F(I))\otimes G(A) \arrow[r, "G(d)\otimes 1"] & G(I)\otimes G(A) \arrow[d, "\eta"] \\
 & & G(F(G(I)\otimes G(A))) \arrow[d, "G(n)"] \\
G(A) & G(I\otimes A) \arrow[l, "G(\lambda)"] & G(F(G(I))\otimes F(G(A))) \arrow[l, "G(\epsilon\otimes\epsilon)"]
\end{tikzcd}$$

commutes. This is an exercise in patience using naturality and the triangle axioms. See the diagram below:

$$\begin{tikzcd}
I\otimes G(A) \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\eta"] \arrow[ddddd, bend right=60, "\lambda"'] & GF(I)\otimes G(A) \arrow[r, "G(d)\otimes\mathrm{id}"] \arrow[d, "\eta"] & G(I)\otimes G(A) \arrow[d, "\eta"] \\
GF(I\otimes G(A)) \arrow[r, "GF(\eta\otimes\mathrm{id})"] \arrow[d, "G(n)"] \arrow[dddr, bend right=40, "GF(\lambda)"'] & GF(GF(I)\otimes G(A)) \arrow[r, "GF(G(d)\otimes\mathrm{id})"] \arrow[d, "G(n)"] & GF(G(I)\otimes G(A)) \arrow[d, "G(n)"] \\
G(F(I)\otimes FG(A)) \arrow[r, "GF(\eta\otimes\mathrm{id})"] \arrow[dr, "\mathrm{id}"] & GFG(F(I)\otimes FG(A)) \arrow[r, "GF(G(d)\otimes\mathrm{id})"] \arrow[d, "G(\varepsilon\otimes\mathrm{id})"] & G(FG(I)\otimes FG(A)) \arrow[d, "G(\varepsilon\otimes\mathrm{id})"] \\
 & G(F(I)\otimes FG(A)) \arrow[r, "G(d\otimes\mathrm{id})"] & G(I\otimes FG(A)) \arrow[d, "G(1\otimes\varepsilon)"] \arrow[dl, "G(\lambda)"] \\
 & GFG(A) \arrow[dr, "G(\varepsilon)"] & G(I\otimes A) \arrow[d, "G(\lambda)"] \\
 & G(A) \arrow[u, "\eta"] \arrow[r, "\mathrm{id}"] & G(A)
\end{tikzcd}$$

The compatibility with the associativity is similar although slightly less absurd looking.

(b) If G is lax symmetric monoidal, can you say anything about F? Give a proof or counterexample.

SOLUTION: Replacing F and G with their opposite functors $F\colon \mathcal{D}^{op} \to \mathcal{C}^{op}$ and $G\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ switches which functor is which adjoint and switches lax and oplax. Thus, using the argument above, if a right adjoint is lax symmetric monoidal, the left adjoint is oplax symmetric monoidal.

In fact, the procedures of obtaining one structure from the other are inverse; given an adjunction, the data of an oplax structure on the left adjoint is bijective to the data of a lax structure on the right adjoint.

(2) Every spectrum X is an $\mathbb{S}$-module. Describe the $\pi_*\mathbb{S}$-action on $\pi_* X$.

SOLUTION: Since π_* is lax monoidal, there are maps

$$\pi_*\mathbb{S} \otimes \pi_* X \to \pi_*(\mathbb{S} \wedge X) \cong \pi_* X$$

making $\pi_* X$ into a $\pi_*\mathbb{S}$-module. Given $\alpha \in \pi_n \mathbb{S}$ and $\xi \in \pi_m X$, we can describe $\alpha \cdot \xi$ as follows. Suppose that α is represented by the homotopy class of a map $a\colon S^{n+k} \to S^k$ and ξ is represented by the homotopy class of a map $x\colon S^{m+\ell} \to X_\ell$. To produce the class of a map in $\pi_{n+m} X$, consider the diagram:

$$\begin{tikzcd}
S^{n+m+k+\ell} \arrow[rrr, "a \cdot x"] \arrow[d, "\cong"] & & & X_{k+\ell} \\
S^{n+m} \wedge S^{k+\ell} \arrow[r, "\chi_{m,k}"] & S^{n+k} \wedge S^{m+\ell} \arrow[rr, "a \wedge f"] & & S^k \wedge X_\ell \arrow[u, "\sigma"]
\end{tikzcd}$$

The map $a \cdot x$ is the composite of the maps around the outside of the rectangle, where σ is an iterated structure map of the spectrum X. The shuffle permutation $\chi_{m,k}$ is a map of degree $(-1)^{mk}$ on $S^{n+m+k+\ell}$, so the total composite is

$$\alpha \cdot \chi = [\sigma \circ (a \wedge x) \circ (-1)^{mk}].$$

(3) Let R be a commutative ring spectrum.

(a) Show that the forgetful functor from the category of R-module spectra to symmetric (or orthogonal) spectra has a left adjoint.

SOLUTION: A symmetric ring spectrum is a monoid object in the category of symmetric spectra, and a module over it is a module object. Writing $\mathcal{S}\mathrm{p}$ for the category of symmetric spectra, the left adjoint of the forgetful functor $\mathcal{M}\mathrm{od}_R \to \mathcal{S}\mathrm{p}$ is the functor $\mathcal{S}\mathrm{p} \to \mathcal{M}\mathrm{od}_R \colon X \mapsto R \wedge X$. The module structure on $R \wedge X$ comes from the multiplication $R \wedge R \to R$.

Let X be a symmetric spectrum and M an R-module. To each map $X \to M$ of spectra, we may associate a map $R \wedge X \to R \wedge M \to M$ where the second map is the action map for M. This is in fact a map of R-module spectra, as witnessed by the square

$$\begin{tikzcd}
R \wedge R \wedge X \arrow[r] \arrow[d] & R \wedge R \wedge M \arrow[r] \arrow[d] & R \wedge M \arrow[d] \\
R \wedge X \arrow[r] & R \wedge M \arrow[r] & M
\end{tikzcd}$$

where the right square commutes because M is a module.

Further, to each map $R \wedge X \to M$ of R-module spectra, we may associate a map $X \xrightarrow{\cong} \mathbb{S} \wedge X \to R \wedge X \to M$ using the unit of R. These two associations are natural in X and M, and in fact, they are inverses of each other because of the unitality axioms. This establishes the desired adjunction. The argument for orthogonal spectra is identical.

(b) Let M be an R-module spectrum such that $\pi_* M$ is free as a graded $\pi_* R$-module. Show that M is stably equivalent to a wedge sum of shifts of copies of R.

SOLUTION: Suppose the generators of $\pi_* M$ as an R-module are x_i in degree d_i. These generators produce maps $\mathbb{S}[-d_i] \xrightarrow{x_i} M$ which, on homotopy, send the generator of $\pi_*\mathbb{S}[-d_i]$ to x_i. By the previous part, these maps correspond to R-module homomorphisms $R[-d_i] \cong R \wedge S[-d_i] \to M$. By construction, this sends the $\pi_* R$-module generator of $\pi_* R[-d_i]$ to x_i. We may wedge these maps together to obtain a map

$$\bigvee_i R[-d_i] \xrightarrow{\bigvee_i x_i} M$$

which is an isomorphism on homotopy groups by construction. Thus, M is stably equivalent to a wedge of shifts of R.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.