CELLULAR SPECTRA TALK OUTLINE

Read [Mal23, Section 2.1.6]. The goal of this talk is to introduce cellular spectra and cellular approximation of spectra, and use these to construct Postnikov towers of spectra and homology of spectra. You should also at least state Whitehead's theorem for spectra and the Hurewicz theorem for spectra.

OUTLINE

(1) Cellular spectra

  (a) Define cellular spectra [Mal23, Definition 2.6.2]

  (b) Give some examples: Eilenberg–MacLane spectra, the Thom spectrum MO, suspension spectra.

  (c) Recall the definition of what Malkiewich calls the *free spectrum* $F_n K \simeq \Sigma^n \Sigma^\infty (K_+)$.

  (d) Explain the process of attaching stable cells to construct a cellular spectrum. You can take [Mal23, Proposition 2.6.11] as the definition of a relative cellular spectrum. Then a cellular spectrum is just a relative cellular spectrum $* \to X$.

  (e) State and prove [Mal23, Theorem 2.6.12] and [Mal23, Corollary 2.6.13]. Make note that Q is a left deformation of $\mathcal{S}p$ with respect to the stable equivalences.

  (f) Recall the definition of a homotopy of spectra from [Mal23, Definition 2.3.11], and then state the Whitehead theorem for spectra [Mal23, Corollary 2.6.2]. You can sketch the proof if you want, but skip it if you think it'll take too long.

(2) Postnikov Towers of Spectra

  (a) Define n-*connected*, *connective* spectra [Mal23, Definition 2.6.27], and bounded below spectra and give examples.

  (b) Explain how to construct relative homotopy groups of spectra [Mal23, Proposition 2.6.25, Remark 2.6.26] (I prefer to think about this as the LES for the cofiber sequence $A \xrightarrow{f} X \to Cf$), and how to use these to kill elements in the homotopy of a spectrum. You don't have to prove much here; it works basically the same as for spaces.

  (c) Define the n-th Postnikov stage $P_n X$ of a spectrum X and the relative cellular spectrum $X \to P_n X$. Explain the construction. [Mal23, Definition 2.6.30].

  (d) Describe the Postnikov tower, [Mal23, Definition 2.6.34], and the fibers of maps $P_{n+1}X \to P_n X$.

  (e) Describe the Whitehead tower of X. Explain the notation $X\langle n\rangle$ means the n-*connective cover of* X: the spectrum whose homotopy is the same as X in degrees $\geq n$. This is also sometimes denoted $\tau_{\geq n} X$.

(3) Homology of Spectra

  (a) Define homology of a spectrum with coefficients in an abelian group A. Since we haven't yet defined the smash product, you should use take the first bullet point in **??**Definition 2.6.36]malkiewich2023 as your definition.

  (b) Give examples: homology of a suspension spectrum. The homology of a suspension spectrum is the same as homology of the space by the suspension isomorphism in homology – we say homology is a *stable invariant*.

  (c) In particular, give homology of the sphere spectrum.

(d) Give a counterexample to the claim that $H_*(\Omega^\infty X; A) = H_*(X; A)$ [Mal23, Warning 2.6.39].

(e) Define the Hurewicz map of spectra [Mal23, Definition 2.6.41]. Remember not to use the smash product version!

(f) State and prove the Hurewicz theorem for spectra [Mal23, Theorem 2.6.42]: you can reduce to the case where X is a cellular spectrum with only i-cells for $i \geq k$. Then an analysis of the dimensions of cells in the levels of X and the classical Hurewicz give the result.

(g) Give the corollary [Mal23, Corollary 2.6.43].

(4) (Optional) Extra things if you have the time or inclination to include them

(a) Introduce the Spanier–Whitehead Category of finite spectra and explain that it is equivalent to the category whose objects are pairs (K, n) of a finite CW-complex K and an integer n. [Mal23, Definition 4.2.12]. I also like these notes [Knu17].

(b) You can give a preview of what we'll learn in two weeks: the homology of a spectrum X with coefficients in A is represented by the spectrum HA in the sense that $H_n(X; A) = \pi_n(X \wedge A)$. This is the mysterious smash product that we haven't introduced yet. This gives us a general way to define homology of a spectrum with coefficients in another spectrum: $H_n(X; Y) = \pi_n(X \wedge Y)$. We'll come back to this in a few weeks.

(c) Use the Pontryagin–Thom isomorphism to compute homology of a Thom spectrum [Mal23, Example 2.6.38, Exercise 7 in Chapter 2].

REFERENCES

[Knu17] Ben Knudsen. Configuration Spaces in Algebraic Topology: Lecture 25. https://scholar.harvard.edu/files/knudsen/files/lecture_25.pdf, 2017.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.