HOMOTOPY LIMITS AND COLIMITS   NAME: ______________________

Due at the beginning of class on 30 January 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

(1) Let $\mathcal{C}$ be a homotopical category, and let $\mathcal{J}$ be any category. The category $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ of functors from $\mathcal{J}$ to $\mathcal{C}$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal{C}$ and a category $\mathcal{J}$, show that the limit functor

$$\lim\colon \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}, \qquad F \mapsto \lim F$$

is *not* a homotopical functor.

(2) Prove that if $\mathcal{J}$ is a discrete category (i.e. the only morphisms are identities), then $\mathrm{ho}(\mathcal{C})^{\mathcal{J}}$ is equivalent to $\mathrm{ho}(\mathcal{C}^{\mathcal{J}})$, where $\mathcal{C}^{\mathcal{J}}$ has weak equivalences defined pointwise. Conclude that products in $\mathrm{ho}(\mathcal{C})$ are homotopy products and coproducts in $\mathrm{ho}(\mathcal{C})$ are homotopy coproducts.

(3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

(a) Prove that the data of the coequalizer of two parallel morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla\colon A \amalg A \to A$ is the fold map.

(b) Use part (a) to describe the homotopy coequalizer of two maps in the category **Top** of (unpointed) topological spaces[1].

(4) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f\colon X \to Y$ be a pointed map of well-based spaces.

(a) Let $\mathrm{cof}(f)$ be the homotopy cofiber of f. Prove that the homotopy cofiber of $Y \to \mathrm{cof}(f)$ is homotopy equivalent to ΣX.

(b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

REFERENCES

[Mal23] Cary Malkiewich. *Spectra and Stable Homotopy Theory*. October 2023.

[MP12] J. P. May and K. Ponto. *More concise algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

[1] To be precise, we assume all spaces are compactly generated and weakly Hausdorff.