SYMMETRIC & ORTHOGONAL SPECTRA NAME: ____________________

Due at the beginning of class on 15 April 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 6.1 and 6.2] and [Sch23, Section 1].

(1) Let X be a symmetric spectrum such that Σ_n acts trivially on X_n for all n.

  (a) Prove that the orbit space $(S^n)_{\Sigma_n}$ is contractible for all $n \geq 2$, with Σ_n-action on S^n by permutation of coordinates, viewing S^n as the one-point compactification of $\mathbb{R}^n$.
  (b) Show that the (naive) homotopy groups of X are trivial.

(2) A *symmetric ring spectrum* is a symmetric spectrum R together with $\Sigma_n \times \Sigma_m$-equivariant multiplication maps $\mu_{n,m}\colon R_n \wedge R_m \to R_{n+m}$ and unit maps $\iota_0\colon S^0 \to R_0$ and $\iota_1\colon S^1 \to R_1$ satisfying associativity, unit, multiplicativity, and centrality conditions (see [Sch07, Definition 1.3]).

  Show that a symmetric ring spectrum in the sense above is a monoid in the monoidal category of symmetric spectra, with smash product as in [Mal23, Definition 6.2.1].

(3) Cobordism of manifolds is captured by the spectrum MO. Read about this spectrum in [Mal23, Example 2.1.20] and [Sch07, Example 2.8].

  (a) Prove that there is a pullback square of vector bundles

$$\begin{tikzcd} \gamma_n \oplus \gamma_m \arrow[r] \arrow[d] & \gamma_{n+m} \arrow[d] \\ BO(n) \times BO(m) \arrow[r] & BO(n+m), \end{tikzcd}$$

  where $\gamma_k \to BO(k)$ is the tautological bundle.
  (b) Use the pullback square to produce unit maps $\mu_{n,m}\colon MO(n) \wedge MO(m) \to MO(n+m)$ for all $n, m \geq 0$.
  (c) Define unit maps $\iota_0\colon S^0 \to MO(0)$ and $\iota_1\colon S^1 \to MO(1)$.
  (d) Show that these maps make MO into a commutative ring orthogonal spectrum.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.

[Sch23] Stefan Schwede. Lectures on equivariant stable homotopy theory. http://www.math.uni-bonn.de/people/schwede/equivariant.pdf, 2023.