Introduction to Hamiltonian Mechanics

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1 Basic symplectic linear algebra

1.1 Linear functionals and the dual space

Definition 1.1 (Dual spaces)**.** A linear functional on a vector space V is just a linear map $\phi : V \to \mathbb{R}$ The set of all linear functionals on V forms a vector space, called the dual space of V. We write

$$V^* := \mathrm{Hom}(V, \mathbb{R}).$$

The operations are given point wise, for instance $(\phi + \psi)(v) = \phi(v) + \psi(v)$.

Example 1.2. Let $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the dot product. Fix $v \in \mathbb{R}^n$. Define

$$\phi_v : \mathbb{R}^n \longrightarrow \mathbb{R}$$

by setting

$$\phi_v(u) = v \cdot u$$

Since the dot product is linear in each entry the map ϕ_v is a linear functional.

Example 1.3. A very important example of the above construction is the **differential of a function on a point**. Consider a smooth function $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$. For a point $p \in \mathbb{R}^n$, we have that the gradient vector of f at p is given by:

$$\nabla f(p) = \left(\frac{\partial f}{x_1}(p), \ldots, \frac{\partial f}{x_n}(p) \right) \in \mathbb{R}^n$$

Then the differential of f at the point p is the linear functional

$$df_p : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$df_p(v) = (\nabla f(p)) \cdot v$$

We now provide basis on the dual space of V.

Definition 1.4 (Dual basis)**.** Let V be a vector space and let $\{v_1, \ldots, v_n\}$ be a basis for V. For $1 \leq i \leq n$ we define the linear functional $\phi_i : V \to \mathbb{R}$ by

$$\phi_i(v_j) = \delta_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Since a linear map is defined by it's action on a basis this completely determines ϕ_i.

Remark 1.5. The numbers δ_{ij} is called the Kronecker delta. It's a very useful that makes writing a lot easier.

Proposition 1.6. *The elements ϕ_i defined above indeed form a basis for V^*.*

Proof: We know that the dimension of V^* is the same as the dimension of V and that the dimension of V is n. So if we show that $\{\phi_i\}$ is a linearly independent set we'll have shown that it is a basis.

Suppose that $\sum \lambda_i \phi_i = 0$, that is $\sum \lambda_i \phi_i = 0$ is the 0 linear functional. Then for any v_j a basis element we have that:

$$\left(\sum \lambda_i \phi_i \right) (v_j) = \sum \lambda_i \phi_i(v_j) = \lambda_j = 0$$

With this we show that λ_j is 0 for every j, and so we have a linearly independent set. ■

Now let's look at the case of $\mathbb{R}^n$. In $\mathbb{R}^n$ we have a canonical choice of basis $\{e_1, \ldots, e_n\}$ given by $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$ and so on. In this case we use another notation for the dual basis vectors. Instead of ϕ_i we write dx^i. So for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then

$$dx^i(v_1, \ldots, v_n) = v_i.$$

Example 1.7. In the notation just discussed above we conclude that

$$df_p = \sum \frac{\partial f}{\partial x_i}(p) dx^i.$$

1.2 Bilinear forms

Definition 1.8. A bilinear form is a function

$$B : V \times V \longrightarrow \mathbb{R}$$

such that B is linear in each entry.

Example 1.9. The dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear map on $\mathbb{R}^n$.

Example 1.10. Let $\phi, \psi \in V^*$. We define the bilinear form

$$\phi \otimes \psi : V \times V \longrightarrow \mathbb{R}$$

by

$$(\phi \otimes \psi)(v, w) = \phi(v)\psi(w).$$

This is called the **tensor product** of ϕ and ψ.

Suppose that $v_1, \dots, v_n$ is a basis for V and let $B : V \times V \to \mathbb{R}$ be a bilinear form on V.

Notice that by bilinearity if $v = \sum \lambda_i v_i$ and $w = \sum_j \mu_j v_j$ we can write

$$B(v, w) = B(\sum_i \lambda_i v_i, \sum_j \mu_j v_j) = \sum_i \sum_j \lambda_i \mu_j B(v_i, v_j).$$

so that the bilinear form is completely determined by the values $B(v_i, v_j)$.

With that we mind, we define the $n \times n$ matrix M_B by

$$(M_B)_{ij} = B(v_i, v_j).$$

Under the chosen basis, $v = (\lambda_1, \dots, \lambda_n)$ and $w = (\mu_1, \dots, \mu_n)$ so one checks that

$$B(v, w) = v^t M_B w$$

where v^t is the transpose of the vector v.

Notice, that for any matrix M, for a choice of basis for V, we can define a bilinear form

$$B_M : V \times V \to \mathbb{R}$$

by

$$B_M(v, w) = v^t M w.$$

It is immediate to check that the $B_M(v_i, v_j) = M_{ij}$, so that the matrix associated to to B_M is M. This whole discussion can be summarized in the following:

Proposition 1.11. *Let $\{v_1, \dots, v_n\}$ be a basis for V. Then for any bilinear form on B, there is a unique matrix M such that*

$$B(v, w) = v^t M w$$

where we see v and w as tuples via the chosen basis.

1.3 Non degeneracy

Given a bilinear form
$$B : V \times V \longrightarrow \mathbb{R}$$
we can consider the map
$$B^\sharp : V \longrightarrow V^*$$
defined by
$$B^\sharp(v)(w) = B(v, w)$$
so that $B^\sharp(v)$ is a linear functional on V. This map is readily checked to be linear. With this we can make the following definition:

Definition 1.12. Say that a bilinear form
$$B : V \times V \longrightarrow \mathbb{R}$$
is non-degenerate if
$$B^\sharp : V \longrightarrow V^*$$
is an isomorphism.

Example 1.13. Let B be a bilinear form on V. Chose a basis on V so that
$$B(v, w) = v^t M w$$
for some matrix M. Then B is non degenerate if and only if M is invertible.

Notice that if B is a non-degenerate bilinear form then for every linear functional $\phi \in V^*$ there is a unique vector $v \in V$ representing it via B. That is, for any $\phi \in V^*$ there is a unique $v \in V$ such that $B^\sharp(v) = \phi$.

Example 1.14. For the dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ we saw that the the gradient $\nabla f(p)$ of a function f on a point p is the vector representing the differential of f at p.

1.4 Skew symmetric bilinear forms

Definition 1.15. A skew-symmetric bilinear form on V is a bilinear form
$$\omega : V \times V \longrightarrow \mathbb{R}$$
such that $\omega(v, w) = -\omega(w, v)$.

Example 1.16. The determinant
$$det : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$
is a skew-symmetric bilinear form.

Example 1.17. Let $\phi, \psi \in V^*$. We define the bilinear form
$$\phi \wedge \psi : V \times V \longrightarrow \mathbb{R}$$
by setting
$$(\phi \wedge \psi)(v, w) = \phi(v)\psi(w) - \phi(w)\psi(v).$$
One immediately checks that the form obtained is skew-symmetric. This is called the **wedge product** of ϕ and ψ.

Example 1.18. Let B be a bilinear form on V. Chose a basis on V so that

$$B(v, w) = v^t M w.$$

Then, we have that B is skew-symmetric if and only if B is a skew-symmetric matrix, that is, $M^t = -M$.

1.5 The symplectic form

Definition 1.19. A symplectic form on V is a skew-symmetric non-degenerate bilinear form

$$\omega : V \times V \longrightarrow \mathbb{R}.$$

Example 1.20. Let $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and consider the bilinear form

$$B_M : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$B(v, w) = v^t M w.$$

Since $M^t = -M$, this a skew-symmetric bilinear form.

We note that not every space admits a symplectic form:

Theorem 1.21. *Suppose that ω is a symplectic form on V, then the dimension of V is even.*

Proof: Choose a basis for V so that the ω is represented by a matrix M. Then we know that M is skew-symmetric and invertible. A general linear algebra fact is that

$$-det(M) = det(-M) = det(M^t) = (-1)^{\dim V} \det(M).$$

Since M is invertible, $det(M) \neq 0$, and this forces $\dim V$ to be even, otherwise we would have $det(M) = -det(M)$. ■

Definition 1.22 (The Canonical Symplectic Form)**.** Denote vectors on $\mathbb{R}^{2n}$ by tuples $(q_1, \dots, q_n, p_1, \dots p_n)$. Let dp^i and dq^i denote the corresponding functionals of the dual basis. Then we define

$$\omega = \sum_i dq^i \wedge dp^i.$$

The above symplectic form is very easy to understand, in fact, it has a very nice matrix representation under the standard basis:

Proposition 1.23. *Under the canonical basis of $\mathbb{R}^{2n}$, the canonical symplectic form is represented by the block matrix*

$$\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where I_n denotes the $n \times n$ identity matrix.

Proof: We just need to check the action on the canonical basis of $\mathbb{R}^{2n}$. Denote such basis by $\{e_1, \dots, e_n, f_1, \dots, f_n\}$. Then

$$\omega(e_i, e_j) = \omega(f_l, f_k) = 0 \text{ and } \omega(e_i, f_j) = \delta_{ij}.$$

These follow from the definition of ω. For instance, let's compute $\omega(e_i, f_j)$:

$$\omega(e_i, f_j) = \sum_k dq^k \wedge dp^k(e_i, f_j) = \sum_k dq^k(e_i)dp^k(f_j) - dq^k(f_j)dp^k(e_i) = \sum_k dq^k(e_i)dp^k(f_i)$$

where is the last equality we are using the fact that $dq^l(f_g) = 0$ for any l, g. Now note that $dq^k(e_i)dp^k(f_j) \neq 0$ if and only if $k = i = j$. Thus we get

$$\sum_k dq^k(e_i)dp^k(f_i) = \delta_{ij}.$$

With this in hand, just note that the matrix $(\omega)_{ij} = \omega(e_i, f_j)$ is indeed our matrix J. ■

With this we conclude that

$$\omega(v, u) = v^t \Omega u.$$

The matrix Ω is called the **symplectic matrix**.

2 Hamiltonian mechanics

2.1 Vector Fields

Before we define what is a vector space we need to know what it means for a function of several variables to be smooth.

Let $U \subset \mathbb{R}^n$ be an open set. Then a function

$$f : U \longrightarrow \mathbb{R}^m$$

is of the form

$$f = (f_1, \ldots, f_m)$$

where each f_i is a function

$$f_i : U \longrightarrow \mathbb{R}.$$

With this characterization we say that f is **smooth** if each f_i has partials derivatives of every order. Smooth functions are often times referred to as C^∞ functions.

Definition 2.1. A smooth vector field on $U \subset \mathbb{R}^n$ is a smooth function

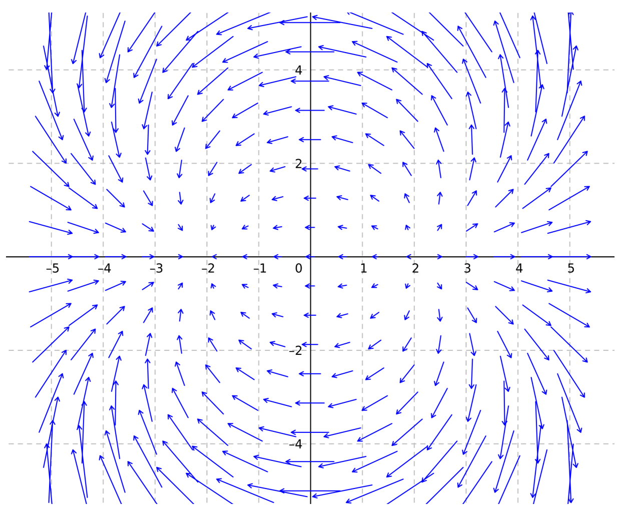
$$X : U \longrightarrow \mathbb{R}^n$$

You should think of a vector field as a function assigning a vector for each point of U. With this is mind, the value vector $X(p)$ is often written as X_p to indicate that it is a vector over a point p.

Example 2.2. Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$X(x, y) = (x^2 - y^2 - 4, 2xy)$$

The way to visualize a vector field is to imagine that at the point (x, y) you are a placing the vector $f(x, y)$. This is illustrated in the picture below:



2.2 Integral curves of vector fields

Suppose that you have a river. In it, the water flows along the current, and if you were to drop a leaf in the river that leaf would follow the current. That's the intuition behind the integral curve: the path points will follow if the vector field represents the flow of the river. The precise definition of an integral curve is the following:

Definition 2.3 (Integral curves)**.** Let $X : U \to \mathbb{R}^n$ be a vector field, then an integral curve for X is any smooth curve $\gamma : (-\epsilon, \epsilon) \to U$, $\epsilon > 0$, such that

$$\gamma'(t) = X(\gamma(t)) \text{ for all } t \in (-\epsilon, \epsilon).$$

The point $\gamma(0) = x_0$ is called the initial point of the curve.

Proposition 2.4. *Suppose that $\gamma : (-\epsilon, \epsilon) \to U$ is an integral curve for X. Then the curve*

$$\gamma_s(t) = \gamma(t + s)$$

is also an integral curve for X whenever it's defined.

Proof: This is just a matter of computation:

$$\dot{\gamma}_s(t) = \dot{\gamma}(t+s)\frac{d(s+t)}{dt} = \dot{\gamma}(t+s) = X(\gamma(t+s)) = X(\gamma_s(t)).$$

This shows that our curve is in fact an integral curve. ■

Let $\gamma : I \to U$ be an integral curve for X on an open interval I. An **extension** for γ is an integral curve $\tilde{\gamma} : \tilde{I} \to U$ such that $I \subset \tilde{I}$ and $\tilde{\gamma}(t) = \gamma(t)$ for all $t \in I$. This allows us to make the following definition:

Definition 2.5. Let $X : U \to \mathbb{R}^n$ be a smooth vector field. Then an integral curve $\gamma : I \to \mathbb{R}^n$ for X is said to be **maximal** if it cannot be extended to an integral curve on larger interval. An integral curve $\gamma : I \to \mathbb{R}^n$ is said to be **globally** defined if $I = \mathbb{R}$.

We are not going to prove the following theorem, but it is extremely useful.

Theorem 2.6 (Fundamental theorem of integral curves)**.** *Let $X : U \to \mathbb{R}^n$ be a smooth vector field on U and let $x_0 \in U$. Then, there exists a unique maximal integral curve $\gamma : I \to U$ such that $\gamma(0) = x_0$.*

An immediate corollary is the following:

Corollary 2.7. *Let $X : U \to \mathbb{R}^n$ be a smooth vector field on U and let $x_0 \in U$. Then, there exists a unique maximal integral curve $\gamma : I \to U$ such that $\gamma(t_0) = x_0$ where t_0 is any real number.*

Proof: By Proposition 2.4, if γ is an integral curve with $\gamma(0) = x_0$, then γ_{-t_0} is an integral curve with

$$\gamma_{-t_0}(t_0) = \gamma(t_0 - t_0) = \gamma(0) = x_0.$$

Thus, if γ is a maximal integral curve with $\gamma(0) = x_0$, then γ_{-t_0} is a maximal integral curve with $\gamma_{-t_0}(t_0) = x_0$. ■

This tells us that we can start "measuring the time" whenever we want. This is important for physics, since it would make no sense for the outcome of an experiment to depend on when it was performed.

2.3 Hamiltonian dynamics

In this section we see how to associate to any smooth function on $\mathbb{R}^n$ a smooth vector field. The integral curves of this vector field are going to correspond the physical trajectories of particles such that their energies is given by the initial smooth function.

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. We should think of this as a function $H(q, p)$ where q is the position of the particle and p its momentum. For a point $x \in \mathbb{R}^{2n}$, we have that

$$dH_x : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

is a linear functional.

$\mathbb{R}^{2n}$ comes equipped with the canonical symplectic form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ which gives an isomorphism

$$\omega^\sharp : \mathbb{R}^{2n} \longrightarrow (\mathbb{R}^{2n})^*$$

which has an inverse

$$(\omega^\sharp)^{-1} : (\mathbb{R}^{2n})^* \longrightarrow \mathbb{R}^{2n}.$$

Now for $x \in \mathbb{R}^{2n}$ we can define the vector

$$X_H(x) = (\omega^\sharp)^{-1}(dH_x).$$

This defines a vector a field

$$X_H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$$
$$x \mapsto X_H(x)$$

Definition 2.8. For a smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ we call the vector field $X_H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the **Hamiltonian vector field** associated to H.

The goal here is to study the dynamics of Hamiltonian vector fields. The dynamics given by a Hamiltonian vector field is often times referred to as **Hamiltonian dynamics**. It turns out that when H has a specific (and familiar) face, then dynamics of its vector field it's going to correspond to Newton's 2^{nd} law (Theorem 2.9). Below we discuss to what the integral curves of the vector field X_H looks like for a generic function H and how we can interpret them physically.

Let $x = (v, u)$ be a vector in $\mathbb{R}^{2n}$. Then we know that

$$\omega^\sharp(x) = (x^t\Omega) \cdot (-)$$

where Ω is the symplectic matrix. Note that

$$x^t\Omega = (v, u)^t\Omega = (v, u)^t \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = (-u, v).$$

Thus, if a linear functional ϕ is of the form $(-u, v) \cdot (-)$, then $(\omega^\sharp)^{-1}(\phi) = (u, v)$.

$dH_x : \mathbb{R}^{2n} \to \mathbb{R}$ is of the form $\nabla H(x) \cdot (-)$. $\nabla H(x)$ is given by

$$\nabla H(x) = \left(\frac{\partial H}{\partial q_1}(x), \dots, \frac{\partial H}{\partial q_n}(x), \frac{\partial H}{\partial p_1}(x), \dots, \frac{\partial H}{\partial p_n}(x)\right)$$

So that

$$(\omega^\sharp)^{-1}(dH_x) = \left(\frac{\partial H}{\partial p_1}(x), \dots, \frac{\partial H}{\partial p_n}(x), -\frac{\partial H}{\partial q_1}(x), \dots, -\frac{\partial H}{\partial q_n}(x)\right).$$

With this we conclude that $x(t) = (q(t), p(t))$ is a integral curve for X_H if and only if the equations

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}(x(t)) \quad \text{and} \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}(x(t)) \tag{1}$$

are satisfied for all $0 \leq i \leq n$.

The equations appearing in Equation 1 are known as **Hamilton Equations**, and they are the foundations of Hamiltonian mechanics. Up to now, we don't know how these equations have anything to do with classical mechanics, after all we are just computing integral curves for a somewhat arbitrary class of vector fields. Well, it turns out that when H is the energy of some classical system of particles, the integral curves of X_H correspond to the trajectories of such a system.

Theorem 2.9. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be given by*

$$H(q,p) = \frac{||p||^2}{2m} + V(q)$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function and m is a positive real number. Then $x(t) = (q(t), p(t))$ is an integral curve for X_H if and only if

$$m\ddot{q}(t) = -\nabla V(q(t)).$$

Proof: To see this, suppose that $(q(t), p(t))$ is a solution to Hamilton's equations. Then

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}(x(t)) = p_i(t)/m$$

by just computing $\frac{\partial H}{\partial p_i}$ directly. We can now see that

$$\ddot{q}_i(t) = \dot{p}_i(t)/m = -\frac{1}{m}\frac{\partial H}{\partial q_i}(x(t)) = -\frac{1}{m}\frac{\partial V}{\partial q_i}$$

where in the second equality we are using Hamilton's equations. With this we see that

$$m\ddot{q}(t) = -\nabla V(q(t)).$$

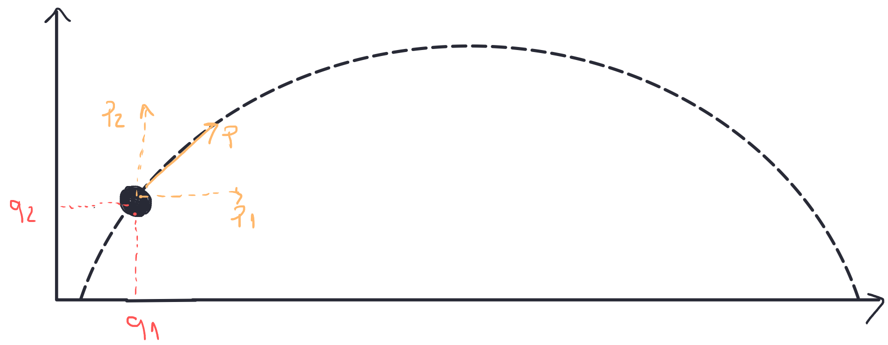
To show the other direction is analogous. Just assume you have a solution to Newton's 2^{nd} Law and show that Hamilton's equations are satisfied. ■

With this Theorem, we can now interpret our setting in a physical way. We started with $\mathbb{R}^{2n}$, and have been writing pairs (q, p) for its elements. In Hamiltonian mechanics, the space $\mathbb{R}^{2n}$ is called the **phase space** of the mechanical system. We should interpret it as the space of possible positions and momenta. That is, the coordinate q should be thought as the position of the system in space and p the momentum of the system. The function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is called the Hamiltonian of the system and it describes the dynamics of the system. That is, the physical trajectories that the system can take through the phase space are the integral curves of the vector field X_H.

In our theorem, $H(q,p) = \frac{||p||^2}{2m} + V(q)$. This tells that we have a system with position q and mass m being exerted a force given by the potential V, and the expression becomes exactly the energy of system, so that the Hamiltonian should be thought as the total energy of the system. If a system has a Hamiltonian H, an **initial state** is simply a point $x_0 = (q_0, p_0) \in \mathbb{R}^{2n}$ in the phase space. Then, by Theorem 2.6 and Corollary 2.7 we know that there's a unique maximal integral curve $x(t)$ of X_H such that $x(t_0) = x_0 = (q_0, p_0)$. Thus, if a system has Hamiltonian H and we know its position and momentum at an instant of time t_0, we know how it evolves in time. That's why we say that H gives the **time evolution** of the system.

2.4 An example

In this example we will describe the motion of a projectile, a cannonball! We assume that the motion of the cannonball occurs in the x-y plane, so the possible pairs of initial position and velocity are $\{(q_1, q_2, p_1, p_2)\} = \mathbb{R}^4$. In this case, we have the standard symplectic form in $\mathbb{R}^n$, which is the constant form $\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$.



The energy of this system is the sum of kinetic and potential energy. In this case, the potential energy is gravitational potential, which is proportional to the height, in our case q_2. Thus,

$$H = \frac{p_1^2 + p_2^2}{2m} + mgq_2$$

where m is the mass of the ball, and g is the acceleration due to gravity.

The solutions (dynamics) of this system are the integral curves of X_H. Therefore, the first step is to determine this vector field. Since the symplectic form is canonical, this vector field is

$$X_H = \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, -\frac{\partial H}{\partial q_1}, -\frac{\partial H}{\partial q_2}\right)$$

and, therefore, its integral curves are curves $x(t) = (q(t), p(t))$ such that

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}.$$

In our case,

$$X_H = \left(\frac{p_1}{m}, \frac{p_2}{m}, 0, -mg\right)$$

so we have the system of equations

$$\dot{q}_1(t) = \frac{p_1}{m}, \quad \dot{q}_2(t) = \frac{p_2}{m}, \quad \dot{p}_1(t) = 0, \quad \dot{p}_2(t) = -mg.$$

For (q_1, p_1), the solution is very simple. We have that p_1 is a constant equal to p_1^0, and therefore, $q_1(t) = p_1^0 t/m + q_1^0$, where q_1^0 is the initial position of q_1, and p_1^0 is the initial momentum. For $q_2(t)$, we obtain the following equation

$$\dot{q}_2(t) = \frac{-mgt + p_2^0}{m}$$

whose solution is

$$q_2(t) = -\frac{gt^2}{2} + \frac{p_2^0 t}{m} + q_2^0$$

thus obtaining the equations of motion for the launch of a cannonball.

2.5 Poisson brackets and constants of movement

Definition 2.10. Let $f, g : \mathbb{R}^{2n} \to \mathbb{R}$ be two smooth functions, then the **Poisson bracket** of f and g is the smooth function $\{f, g\} : \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$\{f, g\}(x) = \omega(X_f(x), X_g(x))$$

where ω is the canonical symplectic form.

Remark 2.11. Since the symplectic form is skew-symmetric, $\{f, g\} = -\{g, f\}$. This implies that for any function f, $\{f, f\} = 0$.

This whole definition may seem a bit weird, especially when physicists like to say that the time evolution of our quantities are given by the Poisson bracket. Why this is a a nice object from the point of view of Physics is made clear by Theorem 2.12 below.

Theorem 2.12. *Let H be a Hamiltonian in $\mathbb{R}^{2n}$. Then the dynamics of a quantity $f \in C^\infty(\mathbb{R}^{2n})$ is given by its Poisson bracket with the Hamiltonian. More precisely*

$$\frac{df(x(t))}{dt} = \{f, H\}(x(t))$$

for any integral curve $x(t)$.

Proof: Let $x(t)$ be an integral curve for H. By the chain rule

$$\frac{df(x(t))}{dt} = \nabla f(x(t)) \cdot \dot{x}(t) = \nabla f(x(t)) \cdot X_H(x(t)).$$

By definition $\omega(X_f(x(t)), -)$ represents the linear functional $\nabla f(x(t)) \cdot (-)$ so that

$$\nabla f(x(t)) \cdot X_H(x(t)) = \omega(X_f(x(t)), X_H(x(t))) = \{f, H\}(x(t)).$$

With this we conclude that

$$\frac{df(x(t))}{dt} = \{f, H\}(x(t))$$

which finishes the proof ■

Definition 2.13. A **constant of movement** (or conserved quantity) of the dynamics given by a Hamiltonian H is any smooth function $f : \mathbb{R}^{2n} \to \mathbb{R}$ such that $f(\gamma(t))$ is constant for any integral curve γ of X_H.

Corollary 2.14. *A quantity f is conserved if and only if $\{f, H\} = 0$.*

Corollary 2.15. *The energy of a Hamiltonian system is conserved.*

Proof: Just observe that by Remark 2.11, $\{H, H\} = 0$. ■

2.6 The Liouville–Arnold Theorem

In this final section we talk about a celebrated result in Hamiltonian mechanics, the so called Liouville–Arnold Theorem. Essentially, the theorem tells us that if we have enough constants of movement, then we know what the dynamics of the Hamiltonian system should look like. Before we state the theorem, we need an important definition, that of an **integrable Hamiltonian system**.

Definition 2.16. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be the Hamiltonian. Then we say that the system defined by H is integrable if the following two conditions are satisfied:

1. There exists (n - 1) constants of movement $f_1, \ldots, f_{n-1}$ such that $\{f_i, f_j\} = 0$ for all i and j;
2. $\{dH_x, d(f_1)_x, \ldots, d(f_{n-1})\}$ forms a linearly independent set for all x in some dense subset of $D \subset \mathbb{R}^{2n}$.

Theorem 2.17. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ define an integrable dynamical system. Let $L_c := H^{-1}(c)$ be the preimage (level set) of some regular value $c \in \mathbb{R}$, then the following holds:*

1. *L_c is a hyper surface of $\mathbb{R}^{2n}$ which is invariant under the Hamiltonian flow induced by of X_H.*
2. *If L_c is furthermore compact and connected, then it is diffeomorphic to the n-torus $T^n := (S^1)^n$*
3. *There exist local coordinates $(\theta_1, \cdots, \theta_n, \omega_1, \cdots, \omega_n)$ on L_c such that the ω_i are constant on the level set while $\dot{\theta}_i := \{H, \theta_i\} = \omega_i$. These coordinates are called. action-angle coordinate*