FINAL EXAM NAME: SOLUTIONS

Due April 30th.

- *Your answers should be neatly written and logically organized.*
- *Do your best to solve these problems by yourself, but ask for help from others if you're stuck. Asking for help is usually a good move with research problems!*
- *The solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

ESSAY QUESTION

Using examples from class (or elsewhere), write 1-2 pages of TEX* to argue for or against the following claim:

> Spectra are primarily important in algebraic topology as a tool for studying stable homotopy groups of spaces.

Cite any resources you use.

PROBLEMS

Answer 3 out of the following 4 problems.

(1) Define a category $\mathcal{Z}p$ of "zpectra" whose objects are $\mathbb{Z}$-indexed sequences of spaces $\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ together with structure maps $\sigma_i \colon \Sigma X_i \to X_{i+1}$ for all $i \in \mathbb{Z}$. A *morphism of zpectra* $f \colon X \to Y$ is a sequence of continuous maps $f_i \colon X_i \to Y_i$ that commute with the structure maps of X and Y.

The *stable homotopy groups* of a zpectrum X are defined by $\pi_k X := \operatorname{colim}_{n\in\mathbb{Z}} \pi_{n+k} X_n$. A *stable equivalence* of zpectra is a map that induces isomorphisms on stable homotopy groups.

Prove that the homotopy category of the homotopical category of zpectra and stable equivalences is equivalent to $\mathrm{ho}(\mathcal{S}p)$.

SOLUTION: Define a functor $F \colon \mathcal{S}p \to \mathcal{Z}p$ by setting $F(X)_n = X_n$ for $n \geq 0$ and $F(X)_n = *$ for $n < 0$. F sends a morphism to the same thing in nonnegative degree, and a constant map $* \to *$ in negative degree.

Define a functor $G \colon \mathcal{Z}p \to \mathcal{S}p$ by truncating a zpectrum Y: $G(Y)_n = Y_n$ for $n \geq 0$. G truncates morphisms in the same way.

Note that both F and G are homotopical functors: if f is a stable equivalence of spectra, then $F(f)$ is a stable equivalence of zpectra. Likewise, if g is a stable equivalence in $\mathcal{Z}p$, then $G(g)$ is a stable equivalence in $\mathcal{S}p$. This follows from the fact that $\operatorname*{colim}_{n\in\mathbb{Z}} X_n = \operatorname*{colim}_{n\in\mathbb{N}} X_n$ for any $\mathbb{Z}$-indexed sequence of objects

$$\cdots \to X_{-2} \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

in a category. The fancy way to say this is "$\mathbb{N}$ is cofinal in $\mathbb{Z}$."

The functors F and G are adjoint to each other. The unit is the identity natural transformation $\eta \colon \mathrm{id}_{\mathcal{S}p} \to GF$: for any spectrum X, $GF(X) = X$. The counit is the natural transformation $\varepsilon \colon FG \to \mathrm{id}_{\mathcal{Z}p}$ defined by

$$(\varepsilon_X)_n \colon FG(X)_n \to X_n = \begin{cases} \mathrm{id}_{X_n} & (n \geq 0), \\ * \to X_n & (n < 0). \end{cases}$$

*Or 1-2 pages single spaced in a word processor, about 500-1000 words. I am not a stickler for essay length.

That is, ε_X is the identity in nonnegative degree and the inclusion of a basepoint in negative degree. By the discussion about colimits above, this means that ε is a natural stable equivalence.

So we have an adjunction $F \dashv G$ between homotopical functors, so by [HHR16, Paragraph following Definition B.7], it descends to an adjunction between the homotopy categories $F\colon \mathrm{ho}(\mathcal{S}p) \leftrightarrows \mathrm{ho}(\mathcal{Z}p)\colon G$. Furthermore, the unit of the adjunction is an isomorphism, and the counit of the adjunction is a stable equivalence (i.e. an isomorphism in $\mathrm{ho}(\mathcal{Z}p)$), so the adjunction is actually an equivalence of categories. (This is a general fact: if the unit and the counit of an adjunction are isomorphisms, then the adjunction is an equivalence of categories.)

(2) Let $X \xrightarrow{f} Y \to Z$ be a cofiber sequence such that f is zero in $\mathrm{ho}(\mathcal{S}p)$. Show that $Z \simeq Y \vee \Sigma X$.

SOLUTION: Let $g\colon Y \to Z$ be the map in the cofiber sequence above. We may extend the cofiber sequence:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X \xrightarrow{\Sigma f} \Sigma Y,$$

but since f is zero in $\mathrm{ho}(\mathcal{S}p)$, so is Σf. Note that precomposition with f or Σf is zero as well. Therefore, after applying the functor $[-, Y]$, we get a short exact sequence of abelian groups

$$0 \leftarrow [Y, Y] \xleftarrow{g^*} [Z, Y] \leftarrow [\Sigma X, Y] \leftarrow 0.$$

Since g^* is surjective, there is a (homotopy class of a) map $h\colon Z \to Y$ such that $g^*h = \mathrm{id}_Y$, that is $h \circ g = \mathrm{id}_Y$, in $\mathrm{ho}(\mathcal{S}p)$.

Now consider the long exact sequence in homotopy for the cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. Since f is zero on homotopy, this splits into a bunch of short exact sequences

$$0 \to \pi_n Y \xrightarrow{g_*} \pi_n Z \to \pi_n \Sigma X \to 0.$$

But g has a left inverse in $\mathrm{ho}(\mathcal{S}p)$, so g_* has a left inverse as a homomorphism of abelian groups. Therefore, this short exact sequence is split, and

$$\pi_n Z \cong \pi_n Y \oplus \pi_n \Sigma X.$$

Hence, $Z \simeq Y \vee \Sigma X$, and the map witnessing the stable equivalence is $Z \xrightarrow{(h,\partial)} Y \vee \Sigma X$.

(3) Let $\widehat{\mathcal{S}p}$ be any symmetric monoidal category of spectra. Given a spectrum X, define $T(X) := \bigvee_{n\geq 0} X^{\wedge n}$.

(a) Prove that $T(X)$ is an associative ring spectrum.

SOLUTION: This is all a very fancy and categorical way of something that's very simple if you think about it algebraically. If V is a vector space, let $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ be the tensor algebra of V. Elements are tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for some n, and the product is taking two lists of tensors and tensoring them together:

$$(v_1 \otimes \cdots \otimes v_n, u_1 \otimes \ldots \otimes u_m) \mapsto v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \ldots \otimes u_m$$

Keep that intuition in mind while we translate this into $\widehat{\mathcal{S}p}$.
Define a unit map $\iota\colon \mathbb{S} \to T(X)$ by inclusion into the coproduct (note that $X^{\wedge 0} = \mathbb{S}$):

$$\iota\colon \mathbb{S} \hookrightarrow \mathbb{S} \vee X \vee X^{\wedge 2} \vee X^{\wedge 3} \vee \ldots = T(X)$$

Define a multiplication map $\mu\colon T(X) \wedge T(X) \to T(X)$ as follows. First, we have

$$T(X) \wedge T(X) = \left(\bigvee_{n\geq 0} X^{\wedge n}\right) \wedge \left(\bigvee_{m\geq 0} X^{\wedge m}\right) \cong \left(\bigvee_{n,m\geq 0} X^{\wedge n} \wedge X^{\wedge m}\right)$$

Then to define a map out of a coproduct (i.e. a wedge sum), it suffices to define maps on each of the summands. In particular, we will define $\mu\colon T(X) \wedge T(X) \to T(X)$ by sending the copies $X^{\wedge n} \wedge X^{\wedge m}$ indexed by some pair (n, m) in $T(X) \wedge T(X)$ to the copy of $X^{\wedge(n+m)}$ indexed by $n + m$ in $T(X)$.

We now need to check that this product is associative and unital. To check the unit condition, we want to show that

$$T(X) \xrightarrow{\iota \wedge \mathrm{id}_{T(X)}} T(X) \wedge T(X) \xrightarrow{\mu} T(X)$$

is the identity on $T(X)$. Since this is a map out of the coproduct, it will instead suffice to show that on each summand of the domain, this composite is the canonical inclusion $X^{\wedge n} \hookrightarrow T(X)$. To that end, consider

$$X^{\wedge n} \xrightarrow{\iota \wedge \mathrm{id}} \mathbb{S} \wedge X^{\wedge n} \cong X^{\wedge n} \hookrightarrow T(X)$$

Because $\mathbb{S}$ is the unit for the smash product, this is the canonical inclusion of $X^{\wedge n}$ into $T(X)$. The fact that ι is also a right unit is similar.

To check that the product is associative, we again use the fact that we can check this on components. We want to show that for any summand $X^{\wedge n} \wedge X^{\wedge m} \wedge X^{\wedge k}$ of $T(X)^{\wedge 3}$, the two ways of performing iterated multiplications in $T(X)$ yield the same result. This boils down to commutativity of the diagram below, which follows from associativity of the smash product.

$$\begin{tikzcd}
X^{\wedge n} \wedge X^{\wedge m} \wedge X^{\wedge k} \arrow[r] \arrow[d] & X^{\wedge n} \wedge X^{\wedge(m+k)} \arrow[d] \\
X^{\wedge(n+m)} \wedge X^{\wedge k} \arrow[r] & X^{\wedge(n+m+k)}.
\end{tikzcd}$$

(b) Prove that the functor $T\colon \widehat{\mathcal{S}p} \to \mathrm{Mon}(\widehat{\mathcal{S}p})$ is left adjoint to the forgetful functor $U\colon \mathrm{Mon}(\widehat{\mathcal{S}p}) \to \widehat{\mathcal{S}p}$, where $\mathrm{Mon}(\widehat{\mathcal{S}p})$ is the category of monoids in $\widehat{\mathcal{S}p}$, i.e. associative ring spectra.

SOLUTION: To show that T is left adjoint to the forgetful functor, we write down unit and counit maps and check the triangle identities.

The unit map is $\eta\colon \mathrm{id}_{\mathcal{S}p} \to UT$ given by the inclusion of X into the $n = 1$ summand of the wedge product $T(X)$.

The counit map $\varepsilon\colon TU \to \mathrm{id}_{\mathrm{Mon}(\widehat{\mathcal{S}p})}$ is given on the summand of $T(UR) = T(R)$ indexed by n by

$$\mu_n\colon R^{\wedge n} \to R,$$

where μ_n is an iterated multiplication, or the unit map of R when $n = 0$. The collection of all of these maps together defines a map out of the coproduct $\bigvee_n R^{\wedge n} = T(R)$ into R. To show that this is a map of ring spectra, it suffices to show that the following diagrams commute for all i and j:

$$\begin{tikzcd}
R^{\wedge i} \wedge R^{\wedge j} \arrow[r, "\mu_i \wedge \mu_j"] \arrow[d] & R \wedge R \arrow[d, "\mu"] \\
R^{\wedge i+j} \arrow[r, "\mu_{i+j}"] & R,
\end{tikzcd}$$

because the terms in the left column are summands of $T(X) \wedge T(X)$ or $T(X)$, the left vertical maps is multiplication in $T(X)$, and the horizontal maps are the counit. Commutativity of the above diagram follows from the associativity of multiplication in R (or unitality when $i = 0$ or $j = 0$).

Now we verify the triangle identities.

$$\begin{tikzcd}
U \arrow[r, "\eta_U"] \arrow[dr, "\mathrm{id}"'] & UTU \arrow[d, "U\varepsilon"] \\
& U
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "\mathrm{id}"'] & TUT \arrow[d, "\varepsilon_T"] \\
& T
\end{tikzcd}$$

By definition, the composite

$$U(R) = R \xrightarrow{\eta_{U(R)}} UT(U(R)) = T(R) \xrightarrow{U\eta_R} U(R) = R$$

is the identity on R.

To verify the other triangle identity, claim that $\varepsilon_T \circ T\eta$, when restricted to the summand of the domain indexed by n, is just the inclusion $X^{\wedge n} \hookrightarrow T(X)$. This will suffice to show that this composite is the identity. We prove this claim by induction.

For $n = 0$, the restriction of $\varepsilon_{T(X)} \circ T\eta_X$ to the copy of $\mathbb{S}$ inside the domain $T(X)$ is

$$\begin{tikzcd}
X^{\wedge 0} = \mathbb{S} \arrow[r, "\mathrm{id}"] \arrow[dr, hook] & \mathbb{S} = T(X)^{\wedge 0} \arrow[d, hook] \\
& T(X),
\end{tikzcd}$$

which commutes. For $n = 1$, the restriction of $\varepsilon_{T(X)} \circ T\eta_X$ to the copy of X inside the domain $T(X)$ is

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\eta_X"'] & T(X) \arrow[d, "\mathrm{id}_{T(X)}"] \\
& T(X)
\end{tikzcd}$$

which again commutes.

For the inductive step, consider the diagram

$$\begin{tikzcd}
X^{\wedge(n+1)} \arrow[r, "\eta_X^{\wedge n}"] \arrow[d, "\cong"] \arrow[dr, phantom, "(a)"] & T(X)^{\wedge(n+1)} \arrow[r, hook] \arrow[d, "\cong"] \arrow[dr, phantom, "(b)"] & TT(X) \arrow[r, "\varepsilon_{T(X)}"] & T(X) \\
X \wedge X^n \arrow[r, "\eta_X \wedge \eta_X^{\wedge n}"] \arrow[d, "\cong"] \arrow[rr, bend right, "(c)", "\eta_X \wedge i_n"'] & T(X) \wedge T(X)^{\wedge n} \arrow[r, "\mathrm{id}_{T(X)} \wedge \mu_n"] \arrow[dr, phantom, "(d)"] & T(X) \wedge T(X) \arrow[ur, "\mu_2"'] & \\
X^{\wedge(n+1)} \arrow[rrruu, bend right, "i_{n+1}"] & & &
\end{tikzcd}$$

The smaller diagrams within this square commute:

(a) by the associativity of the smash product,
(b) by the definition of the counit,
(c) by the inductive hypothesis,
(d) by the fact that $\mathrm{id}_{T(X)}$ is a morphism of ring spectra; in this diagram, the way this appears is after restricted to summands in the domain of $\mathrm{id}_{T(X)}$. Note that $\eta_X = i_1$ if i_n is the inclusion of X^n into $T(X)$.

(4) Let X be a symmetric spectrum. Recall that the symmetric spectrum $\mathrm{sh}^1 X$ is the symmetric spectrum with $(\mathrm{sh}^1 X)_n = X_{1+n}$, where Σ_n acts on X_{1+n} as the subgroup of Σ_{1+n} consisting of those permutations of $\{1, \ldots, n+1\}$ leaving 1 fixed.

(a) Construct a symmetric spectrum $(\mathrm{sh}^{-1} X)$ with n-th space $(\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_{n-1}$.

(b) Show that sh^{-1} is a functor on symmetric spectra which is left adjoint to sh^1.

SOLUTION: See [Sch07, Example 3.17] for both parts (a) and (b) of this problem.

It's useful to note that for G a finite group and H a subgroup of G, and X a space with H-action, the space $G_+ \wedge_H X$ can be described as the equivalence classes of pairs (g, x) under the equivalence relation $(gh, x) \sim (g, hx)$, and any $(g, *)$ is identified with the basepoint.

For part (b), you can use the fact that the functor $G_+ \wedge_H (-)$ is left adjoint to the forgetful functor from G-spaces to H-spaces. This gives you a levelwise adjunction, and then you just have to check it commutes with the structure maps.

REFERENCES

[HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.