Due at the beginning of class on 9 April 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 6.1 and 6.2].

(1) Let $F \dashv G$ be an adjoint pair of functors between monoidal categories.

(a) Show that if F is strong symmetric monoidal, then G is lax symmetric monoidal.

SOLUTION: Let $F \dashv G$ be an adjunction between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$. We will only assume that F is oplax (we only ask for the "inverse" of the structure maps, we do not need the forward direction) i.e. that there are transformations

$$d\colon F(I) \to I \text{ and } n\colon F(A \otimes B) \to F(A) \otimes F(B)$$

which are compatible with the associativity, unit, and symmetry isomorphisms. We will show that G has an associated lax structure i.e. that there are transformations

$$e\colon I \to G(I) \text{ and } m\colon G(A) \otimes G(B) \to G(A \otimes B)$$

with similar compatibility.

The map e is the adjunct of d, meaning the image of d under the isomorphism $\mathcal{C}(F(I), I) \cong \mathcal{D}(I, G(I))$, but I find it hard to prove the axioms with this definition, and m is not just the adjunct of n. Let $\eta\colon \mathrm{id} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow 1$ be the unit and counit of the adjunction. Then, e is the composite

$$e\colon I \xrightarrow{\eta} G(F(I)) \xrightarrow{G(d)} G(I).$$

Similarly, m is the composite

$$m\colon G(A) \otimes G(B) \xrightarrow{\eta} G(F(G(A) \otimes G(B))) \xrightarrow{G(n)} G(F(G(A)) \otimes F(G(B))) \xrightarrow{G(\epsilon\otimes\epsilon)} G(A \otimes B).$$

It remains to show that these new transformations have the right compatibility.

To show compatibility with the symmetry, we need to show that if

$$\begin{tikzcd} F(A \otimes B) \arrow[r, "n"] \arrow[d, "F(\sigma)"] & F(A) \otimes F(B) \arrow[d, "\sigma"] \\ F(B \otimes A) \arrow[r, "n"] & F(B) \otimes F(A) \end{tikzcd} \quad \text{commutes then} \quad \begin{tikzcd} G(A) \otimes G(B) \arrow[r, "m"] \arrow[d, "\sigma"] & G(A \otimes B) \arrow[d, "G(\sigma)"] \\ G(B) \otimes G(A) \arrow[r, "m"] & G(B \otimes A) \end{tikzcd}$$

commutes. Expanding the right diagram, we are tasked with showing that

$$\begin{tikzcd} G(A) \otimes G(B) \arrow[r, "\eta"] \arrow[d, "\sigma"] & G(F(G(A) \otimes G(B))) \arrow[r, "G(n)"] & G(F(G(A)) \otimes F(G(B))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(A \otimes B) \arrow[d, "G(\sigma)"] \\ G(B) \otimes G(A) \arrow[r, "\eta"] & G(F(G(B) \otimes G(A))) \arrow[r, "G(n)"] & G(F(G(B)) \otimes F(G(A))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(B \otimes A) \end{tikzcd}$$

commutes which we show by filling in the squares.

$$
\begin{tikzcd}
G(A) \otimes G(B) \arrow[r, "\eta"] \arrow[d, "\sigma"] & G(F(G(A) \otimes G(B))) \arrow[r, "G(n)"] \arrow[d, "G(F(\sigma))"] & G(F(G(A)) \otimes F(G(B))) \arrow[r, "G(\epsilon\otimes\epsilon)"] \arrow[d, "G(\sigma)"] & G(A \otimes B) \arrow[d, "G(\sigma)"] \\
G(B) \otimes G(A) \arrow[r, "\eta"] & G(F(G(B) \otimes G(A))) \arrow[r, "G(n)"] & G(F(G(B)) \otimes F(G(A))) \arrow[r, "G(\epsilon\otimes\epsilon)"] & G(B \otimes A)
\end{tikzcd}
$$

The outside squares commute by the naturality of η and σ, and the middle square is the compatibility square for F plugged into G.

To show compatibility with the (left) unit λ, we need to show that if

$$
\begin{tikzcd}
F(I \otimes A) \arrow[r, "n"] \arrow[d, "F(\lambda)"] & F(I) \otimes F(A) \arrow[d, "d\otimes 1"] \\
F(A) & I \otimes F(A) \arrow[l, "\lambda"]
\end{tikzcd}
\quad \text{commutes then} \quad
\begin{tikzcd}
I \otimes G(A) \arrow[r, "e\otimes 1"] \arrow[d, "\lambda"] & G(I) \otimes G(A) \arrow[d, "m"] \\
G(A) & G(I \otimes A) \arrow[l, "G(\lambda)"]
\end{tikzcd}
$$

commutes. Expanding the right diagram, we are tasked to show that

$$
\begin{tikzcd}
I \otimes G(A) \arrow[r, "\eta\otimes 1"] \arrow[dd, "\lambda"] & G(F(I)) \otimes G(A) \arrow[r, "G(d)\otimes 1"] & G(I) \otimes G(A) \arrow[d, "\eta"] \\
 & & G(F(G(I) \otimes G(A))) \arrow[d, "G(n)"] \\
G(A) & G(I \otimes A) \arrow[l, "G(\lambda)"] & G(F(G(I)) \otimes F(G(A))) \arrow[l, "G(\epsilon\otimes\epsilon)"]
\end{tikzcd}
$$

commutes. This is an exercise in patience using naturality and the triangle axioms. See the diagram below:

$$
\begin{tikzcd}
I \otimes G(A) \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\eta"] \arrow[dddddr, bend right, "\lambda"'] & GF(I) \otimes G(A) \arrow[r, "G(d)\otimes\mathrm{id}"] \arrow[d, "\eta"] & G(I) \otimes G(A) \arrow[d, "\eta"] \\
GF(I \otimes G(A)) \arrow[r, "GF(\eta\otimes\mathrm{id})"] \arrow[d, "G(n)"] \arrow[ddddr, bend right, "GF(\lambda)"'] & GF(GF(I) \otimes G(A)) \arrow[r, "GF(G(d)\otimes\mathrm{id})"] \arrow[d, "G(n)"] & GF(G(I) \otimes G(A)) \arrow[d, "G(n)"] \\
G(F(I) \otimes FG(A)) \arrow[r, "GF(\eta\otimes\mathrm{id})"] \arrow[dr, "\mathrm{id}"] & GFG(F(I) \otimes FG(A)) \arrow[r, "GF(G(d)\otimes\mathrm{id})"] \arrow[d, "G(\varepsilon\otimes\mathrm{id})"] & G(FG(I) \otimes FG(A)) \arrow[d, "G(\varepsilon\otimes\mathrm{id})"] \\
 & G(F(I) \otimes FG(A)) \arrow[r, "G(d\otimes\mathrm{id})"] & G(I \otimes FG(A)) \arrow[d, "G(1\otimes\varepsilon)"] \arrow[dl, "G(\lambda)"] \\
 & GFG(A) \arrow[dr, "G(\varepsilon)"] & G(I \otimes A) \arrow[d, "G(\lambda)"] \\
 & G(A) \arrow[u, "\eta"] \arrow[r, "\mathrm{id}"] & G(A)
\end{tikzcd}
$$

The compatibility with the associativity is similar although slightly less absurd looking.

(b) If G is lax symmetric monoidal, can you say anything about F? Give a proof or counterexample.

SOLUTION: Replacing F and G with their opposite functors $F\colon \mathcal{D}^{op} \to \mathcal{C}^{op}$ and $G\colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ switches which functor is which adjoint and switches lax and oplax. Thus, using the argument above, if a right adjoint is lax symmetric monoidal, the left adjoint is oplax symmetric monoidal.

In fact, the procedures of obtaining one structure from the other are inverse; given an adjunction, the data of an oplax structure on the left adjoint is bijective to the data of a lax structure on the right adjoint.

(2) Every spectrum X is an $\mathbb{S}$-module. Describe the $\pi_*\mathbb{S}$-action on π_*X.

SOLUTION: Since π_* is lax monoidal, there are maps

$$\pi_*\mathbb{S} \otimes \pi_*X \to \pi_*(\mathbb{S} \wedge X) \cong \pi_*X$$

making π_*X into a $\pi_*\mathbb{S}$-module. Given $\alpha \in \pi_n\mathbb{S}$ and $\xi \in \pi_m X$, we can describe $\alpha \cdot \xi$ as follows.

Suppose that α is represented by the homotopy class of a map $a\colon S^{n+k} \to S^k$ and ξ is represented by the homotopy class of a map $x\colon S^{m+\ell} \to X_\ell$. To produce the class of a map in $\pi_{n+m}X$, consider the diagram:

$$\begin{tikzcd} S^{n+m+k+\ell} \arrow[rr, "a\cdot x"] \arrow[d, "\cong"] & & X_{k+\ell} \\ S^{n+m} \wedge S^{k+\ell} \arrow[r, "\chi_{m,k}"] & S^{n+k} \wedge S^{m+\ell} \arrow[r, "a\wedge f"] & S^k \wedge X_\ell \arrow[u, "\sigma"'] \end{tikzcd}$$

The map $a \cdot x$ is the composite of the maps around the outside of the rectangle, where σ is an iterated structure map of the spectrum X. The shuffle permutation $\chi_{m,k}$ is a map of degree $(-1)^{mk}$ on $S^{n+m+k+\ell}$, so the total composite is

$$\alpha \cdot \chi = [\sigma \circ (a \wedge x) \circ (-1)^{mk}].$$

(3) Recall the cobordism spectrum MO from [Mal23, Example 2.1.21]. For a construction of MO as a symmetric/orthogonal spectrum, see [Sch07, Example 1.16].

(a) Prove that there is a pullback square of vector bundles

$$\begin{tikzcd} \gamma_n \oplus \gamma_m \arrow[r] \arrow[d] & \gamma_{n+m} \arrow[d] \\ BO(n) \times BO(m) \arrow[r] & BO(n+m), \end{tikzcd}$$

where $\gamma_k \to BO(k)$ is the tautological bundle.

SOLUTION: This follows from the universal property of the space $BO(n+m)$: any real vector bundle of dimension $n+m$ is a pullback of the tautological $n+m$-bundle γ_{n+m}, but we give some details.

Fix a linear isomorphism $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$. Using this isomorphism, we can define a map $p\colon BO(n) \times BO(m) \to BO(n+m)$ by taking the cartesian product of an n-plane and an m-plane. Then claim that the pullback bundle $p^*(\gamma_{n+m})$ is isomorphic to $\gamma_n \oplus \gamma_m$. By inspecting the fibers of $\gamma_n \oplus \gamma_m$ and $p^*(\gamma_{n+m})$, we see that they contain the same data and are therefore isomorphic as bundles.

(b) Use the pullback square to produce maps $MO(n) \wedge MO(m) \to MO(n+m)$ for all $n, m \geq 0$.

SOLUTION: First, we prove a lemma. Let $E_1 \to X_1$ and $E_2 \to X_2$ be vector bundles. Claim that $\mathrm{Th}(E_1 \oplus E_2) \cong \mathrm{Th}(E_1) \wedge \mathrm{Th}(E_2)$. This can be seen because $\mathrm{Th}(E_1 \oplus E_2)$ is the one-point compactification of the space $E_1 \times E_2$, which can be identified with the one-point compactification of E_1 smashed with the one-point compactification of E_2.

Given this lemma, the map $\gamma_n \oplus \gamma_m \to \gamma_{n+m}$ from part (a) extends to a map on the Thom spaces

$$\mu_{n,m} \colon MO(n) \wedge MO(m) = Th(\gamma_n) \wedge Th(\gamma_m) \cong Th(\gamma_n \oplus \gamma_m) \to Th(\gamma_{n+m}) = MO(n+m)$$

(c) Show that these maps make MO into a commutative ring orthogonal spectrum.

SOLUTION: We define the unit maps of this spectrum MO by

$$\iota_0 = \mathrm{id} \colon S^0 \to MO(0) = S^0$$

and $\iota_1 \colon S^1 \to MO(1)$ as follows. The trivial rank 1 bundle over a point is by the universal property a pullback of the tautological bundle γ_1 over $BO(1)$. (Note that $BO(1)$ is the space of lines in $\mathbb{R}^\infty$ – that is, $BO(1) \cong \mathbb{R}P^\infty$, and γ_1 is the tautological line bundle over $\mathbb{R}P^\infty$.)

$$\begin{tikzcd} \varepsilon^1 \arrow[r] \arrow[d] & \gamma^1 \arrow[d] \\ * \arrow[r] & BO(1) \cong \mathbb{R}P^\infty \end{tikzcd}$$

Then applying the Thom space construction to the map $\varepsilon^1 \to \gamma^1$ gives a map $S^1 \to MO(1)$; this is the unit $\iota_1 \colon S^1 \to MO(1)$.

We want to show that, equipped with these unit maps and the multiplication maps from part (b), that MO becomes a commutative orthogonal ring spectrum.

First, let's check the unit conditions: note that the maps

$$\mu_{0,n} \colon MO(0) \wedge MO(n) \to MO(n)$$

are just identities, since $MO(0) \cong S^0$. Therefore, the diagrams

$$\begin{tikzcd} MO(n) \cong S^0 \wedge MO(n) \arrow[r, "\iota_0 \wedge \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & MO(0) \wedge MO(n) \arrow[d, "\mu_{0,n}"] \\ & MO(n) \end{tikzcd} \quad \begin{tikzcd} MO(0) \wedge MO(n) \arrow[d, "\mu_{n,0}"] & MO(n) \wedge S^0 \cong MO(n) \arrow[l, "\mathrm{id} \wedge \iota_0"'] \arrow[dl, "\mathrm{id}"] \\ MO(n) & \end{tikzcd}$$

both commute, and the spectrum is unital.

Next, we check associativity. Consider the diagrams

$$\begin{tikzcd} \gamma^n \oplus \gamma^m \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^{n+m} \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^{n+m+k} \arrow[d] \\ BO(n) \times BO(m) \times BO(k) \arrow[r] & BO(n+m) \times BO(k) \arrow[r] & BO(n+m+k) \end{tikzcd}$$

$$\begin{tikzcd} \gamma^n \oplus \gamma^m \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^n \oplus \gamma^{m+k} \arrow[r] \arrow[d] & \gamma^{n+m+k} \arrow[d] \\ BO(n) \times BO(m) \times BO(k) \arrow[r] & BO(n) \times BO(m+k) \arrow[r] & BO(n+m+k) \end{tikzcd}$$

In the above diagrams, all squares and outer rectangles are pullbacks, the right squares and the outer rectangles by the universal property of $BO(\ell)$, and the left squares by the pasting lemma. After applying the functor Th to the two top composites, we get either $\mu_{n+m,k} \circ \mu_{n,m} \circ \mathrm{id}$ or $\mu_{n+m+k} \circ \mathrm{id} \wedge \mu_{m,k}$. To show that these are equal, it suffices to show that the two composites along the bottom of the rectangles are the same, whence it follows that the composites along the top are the same because the rectangles are pullbacks.

So we want to show that the following diagram commutes:

$$\begin{tikzcd}
BO(n) \times BO(m) \times BO(k) \arrow[r] \arrow[d] & BO(n+m) \times BO(k) \arrow[d] \\
BO(n) \times BO(m+k) \arrow[r] & BO(n+m+k)
\end{tikzcd}$$

Indeed, appropriately choosing isomorphism $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$, one sees that both directions around this square send a triple of planes (U, V, W) in $\mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty$ to the same $(n+m+k)$-plane $U \oplus V \oplus W$ in $\mathbb{R}^\infty$. So this diagram commutes, and MO is associative.

To show centrality, let $\chi_{n,m} \in O(n+m)$ be the permutation matrix corresponding to the permutation which shuffles the first n elements of a vector $v \in \mathbb{R}^{n+m}$ to the end. Applying the functor B, we obtain maps $\chi_{n,m} \colon BO(n+m) \to BO(n+m)$. Assume that we have chosen isomorphisms $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ such that the following diagram commutes for all n and m:

$$\begin{tikzcd}
BO(n) \times BO(m) \arrow[r] \arrow[d, "\tau"] & BO(n+m) \arrow[d, "\chi_{n,m}"] \\
BO(m) \times BO(n) \arrow[r] & BO(n+m),
\end{tikzcd}$$

where τ is the symmetry isomorphism. Then we argue similarly to associativity. We have the following diagrams, where every square and every rectangle is a pullback:

$$\begin{tikzcd}
\varepsilon^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^{1+n} \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\
* \times BO(n) \arrow[r] & BO(1) \times BO(n) \arrow[r] & BO(1+n) \arrow[r, "\chi_{1,n}"] & BO(n+1)
\end{tikzcd}$$

$$\begin{tikzcd}
\varepsilon^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^n \oplus \varepsilon^1 \arrow[r] \arrow[d] & \gamma^n \oplus \gamma^1 \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\
* \times BO(n) \arrow[r, "\tau"] & BO(n) \times * \arrow[r] & BO(n) \times BO(1) \arrow[r] & BO(n+1)
\end{tikzcd}$$

As before, if the bottom rows compose to the same morphism, then applying the functor Th to the two rows yields the centrality diagram. Again, it's easy to see that the bottom rows are the same morphism, so MO satisfies the centrality condition.

Commutativity is similar.

(4) Let R be a commutative ring spectrum.

  (a) Show that the forgetful functor from the category of R-module spectra to symmetric (or orthogonal) spectra has a left adjoint.

  SOLUTION: A symmetric ring spectrum is a monoid object in the category of symmetric spectra, and a module over it is a module object. Writing $\mathcal{S}p$ for the category of symmetric spectra, the left adjoint of the forgetful functor $\mathcal{M}od_R \to \mathcal{S}p$ is the functor $\mathcal{S}p \to \mathcal{M}od_R \colon X \mapsto R \wedge X$. The module structure on $R \wedge X$ comes from the multiplication $R \wedge R \to R$.

  Let X be a symmetric spectrum and M an R-module. To each map $X \to M$ of spectra, we may associate a map $R \wedge X \to R \wedge M \to M$ where the second map is the action map for M. This is in fact a map of R-module spectra, as witnessed by the square

$$\begin{tikzcd} R \wedge R \wedge X \arrow[r] \arrow[d] & R \wedge R \wedge M \arrow[r] \arrow[d] & R \wedge M \arrow[d] \\ R \wedge X \arrow[r] & R \wedge M \arrow[r] & M \end{tikzcd}$$

where the right square commutes because M is a module.

Further, to each map $R \wedge X \to M$ of R-module spectra, we may associate a map $X \xrightarrow{\cong} \mathbb{S} \wedge X \to R \wedge X \to M$ using the unit of R. These two associations are natural in X and M, and in fact, they are inverses of each other because of the unitality axioms. This establishes the desired adjunction. The argument for orthogonal spectra is identical.

(b) Let M be an R-module spectrum such that $\pi_* M$ is free as a graded $\pi_* R$-module. Show that M is stably equivalent to a wedge sum of shifts of copies of R.

SOLUTION: Suppose the generators of $\pi_* M$ as an R-module are x_i in degree d_i. These generators produce maps $\mathbb{S}[-d_i] \xrightarrow{x_i} M$ which, on homotopy, send the generator of $\pi_* \mathbb{S}[-d_i]$ to x_i. By the previous part, these maps correspond to R-module homomorphisms $R[-d_i] \cong R \wedge \mathbb{S}[-d_i] \to M$. By construction, this sends the $\pi_* R$-module generator of $\pi_* R[-d_i]$ to x_i. We may wedge these maps together to obtain a map

$$\bigvee_i R[-d_i] \xrightarrow{\bigvee_i x_i} M$$

which is an isomorphism on homotopy groups by construction. Thus, M is stably equivalent to a wedge of shifts of R.

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.