These problems are not due and will not be graded.

Reading: [vK13, Sections 3 and 4] or [Bou79, Sections 1 and 2]. I also found these slides of Aras Ergus helpful [Erg19].

(1) Let $\mathcal{S}p_{\mathbb{Q}}$ be the full subcategory of $\mathcal{S}p$ on the $\mathbb{Q}$-local spectra (the rational spectra).

(a) Show that if R is a ring spectrum, any R-module is R-local.

(b) Show that any $\mathbb{Q}$-local spectrum is an $H\mathbb{Q}$-module in the stable homotopy category.

(c) Show that any map of $\mathbb{Q}$-local spectra is automatically a map of $H\mathbb{Q}$-modules in the stable homotopy category.

(d) Conclude that $\mathrm{ho}(\mathcal{S}p_{\mathbb{Q}})$ is equivalent to the category of $H\mathbb{Q}$-modules in $\mathrm{ho}(\mathcal{S}p)$.

SOLUTION:

(a) Suppose M is an R-module spectrum. To show that M is R-local, it suffices to show that for any R-acyclic module A we have

$$[A, M] = 0$$

To show this, let $f : A \to M$ and consider the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i \wedge id_A"'] & M \arrow[d, "i \wedge id_M"'] \arrow[dr, equal] & \\ R \wedge A \arrow[r, "id_R \wedge f"'] & R \wedge M \arrow[r, "\mu"'] & M \end{tikzcd}$$

where i is the unit map $i \colon S \to R$. This diagram commutes. Along the top, we have the map f, so the same map is the composition along the bottom. However, since A is R-acyclic, we know that $R \wedge A \simeq *$, so f factors through $*$, and thus must be trivial. This is it.

(b) Let $L_{H\mathbb{Q}}$ be the localization with respect to $H\mathbb{Q}$. In fact, we know that this is a smashing localization, so $L_{H\mathbb{Q}}X = H\mathbb{Q} \wedge X$ for a given spectrum X.

If a spectrum Z is $\mathbb{Q}$-local, in particular this implies that the localization of Z is equivalent to Z, and there is a map $\eta_Z \colon Z \to L_{H\mathbb{Q}}Z = H\mathbb{Q} \wedge Z$ realizing this. In the homotopy category, weak equivalences are inverted, so we may take our module action map $\alpha : H\mathbb{Q} \wedge Z \to Z$ to be the inverse of the above weak equivalence.

It remains to check that the proper diagrams commute. However, this is probably obvious.

(c) Starting with a map $f \colon Z_1 \to Z_2$ of $\mathbb{Q}$-local spectra, consider the image of f under the natural transformation $\eta \colon id \Rightarrow L_{H\mathbb{Q}}$

$$\begin{tikzcd} Z_1 \arrow[r, "f"] \arrow[d, "\eta_{Z_1}"'] & Z_2 \arrow[d, "\eta_{Z_2}"] \\ H\mathbb{Q} \wedge Z_1 \arrow[r, "1 \wedge f"] & H\mathbb{Q} \wedge Z_2 \end{tikzcd}$$

This diagram commutes (up to homotopy, by naturality of η). Since the action is defined through taking a homotopy inverse of η, f is a $H\mathbb{Q}$-module homomorphism.

(d) This is an immediate consequence of parts (b) and (c).

(2) Let $\widehat{\mathcal{S}p}$ be your favorite symmetric monoidal category of spectra (e.g. symmetric or orthogonal spectra), and let $\widehat{\mathcal{S}p}_E$ be the full subcategory of $\widehat{\mathcal{S}p}$ on the E-local spectra.

(a) If $f: W \to X$ and $g: Y \to Z$ are E-equivalences, show that

$$L_E(W \wedge Y) \xrightarrow{L_E(f \wedge g)} L_E(X \wedge Z)$$

is a stable equivalence.

(b) Define $X \wedge^E Y := L_E(X \wedge Y)$. Show that $\wedge^E$ defines a symmetric monoidal structure on $\mathrm{ho}(\widehat{\mathcal{S}p}_E)$ with unit $L_E(\mathbb{S})$.

(c) Conclude that L_E is a strong monoidal functor and the composite $\mathrm{ho}(\widehat{\mathcal{S}p}) \xrightarrow{L_E} \mathrm{ho}(\widehat{\mathcal{S}p}_E) \xrightarrow{\iota} \mathrm{ho}(\widehat{\mathcal{S}p})$ is lax symmetric monoidal. Hence, $L_E(\mathbb{S})$ is always a commutative monoid in the stable homotopy category.

SOLUTION:

(a) By the E-Whitehead Theorem, it suffices to show that $L_E(f \wedge g)$ is an E-equivalence. This will follow if we can show that $f \wedge g$ is an E-equivalence, since L_E preserves E-homology.

Recall that a map is an E-equivalence iff the fiber is E-acyclic, and observe that $f \wedge g = (1_Y \wedge g) \circ (f \wedge 1_Y)$. Then to show that $f \wedge g$ is an E-equivalence, it suffices to show that $f \wedge 1_Y$ and $1_Y \wedge g$ are, i.e., that their fibers are E-acyclic.

Since smashing with a fixed spectrum is a left adjoint, it preserves fibers, and thus $\mathrm{fib}(f \wedge 1_Y) \simeq \mathrm{fib}(f) \wedge Y$. Now since $\mathrm{fib}(f)$ is E-acyclic, we have $E \wedge \mathrm{fib}(f) \simeq *$, and it follows that $\mathrm{fib}(f \wedge 1_Y)$ is E-acyclic as well. Therefore $f \wedge 1_Y$ is an E-equivalence, and the proof that $1_Y \wedge g$ is an E-equivalence is essentially identical.

(b) Recall that $X \to L_E X$ is an E-equivalence, for any spectrum X. If X is E-local, then this is a stable equivalence.

For the unit, apply part (a) with $f: \mathbb{S} \to L_E\mathbb{S}$ and $g = \mathrm{id}_X$. This gives a stable equivalence

$$L_E(X) = L_E(\mathbb{S} \wedge X) \simeq L_E(L_E\mathbb{S} \wedge X) = L_E\mathbb{S} \wedge^E X$$

Composing with $X \simeq L_E(X)$ yields a stable equivalence $X \simeq L_E\mathbb{S} \wedge^E X$. Similarly, $X \simeq X \wedge^E L_E\mathbb{S}$.

For the associativity, apply part (a) with $f: X \wedge Y \to L_E(X \wedge Y)$ and $g = \mathrm{id}_Z$. This gives a stable equivalence

$$L_E(X \wedge Y \wedge Z) \to L_E(L_E(X \wedge Y) \wedge Z) = (X \wedge^E Y) \wedge^E Z$$

Similarly, $X \wedge^E (Y \wedge^E Z)$ is stably equivalent to $L_E(X \wedge Y \wedge Z)$, and therefore stably equivalent to $(X \wedge^E Y) \wedge^E Z$.

I won't check the coherence axioms here.

(c) The maps that define the strong monoidal structure on L_E are $\mathrm{id}: L_E\mathbb{S} \to L_E\mathbb{S}$ and

$$L_E X \wedge^E L_E Y \to L_E(X \wedge Y)$$

coming from the inverse (in $\mathrm{ho}(\widehat{\mathcal{S}p}_E)$) of the stable equivalence deduced from part (a) using $f: X \to L_E(X)$ and $g: Y \to L_E(Y)$.

It is a tedious but straightforward exercise to check the right diagrams commute.

(3) The *Bousfield class* of a spectrum E is the set of E-acyclic spectra, denoted $\langle E \rangle$. The set of Bousfield classes of spectra forms a poset with $\langle E \rangle \geq \langle D \rangle$ if being E-acyclic implies being D-acyclic.

(a) Show that $\langle * \rangle$ is a maximum and $\langle \mathbb{S} \rangle$ is a minimum in this poset.

(b) Show that if $\langle E \rangle \geq \langle D \rangle$, then there is a natural map $L_E X \to L_D X$.

(c) Show that if $\langle E \rangle \geq \langle D \rangle$, then $L_D L_E X \simeq L_D X$.

SOLUTION:

(a) Every spectrum X is $*$-acyclic, because $* \wedge X \simeq *$. Therefore, being E-acyclic implies being $*$-acyclic for any E, so $\langle * \rangle$ is a minimum in this poset.

On the other hand, a spectrum X is $\mathbb{S}$-acyclic if and only if $\mathbb{S} \wedge X \simeq *$. So only $*$ is $\mathbb{S}$-acyclic. So X being $\mathbb{S}$-acyclic implies that X is E-acyclic for any E. So $\langle \mathbb{S} \rangle$ is a maximum.

(b) If $\langle E \rangle \geq \langle D \rangle$, then any E-acyclic spectrum is D-acyclic. Claim that $L_D X$ is E-local, so it admits a map from the initial E-localization $L_E X$ of X. To see that $L_D X$ is E-local, we must show that for any E-acylic A, $[A, L_D X] = 0$. But if A is E-acyclic, then A is also D-acyclic, so $[A, L_D X] = 0$ because $L_D X$ is D-local. Hence, $L_D X$ is E-local. Therefore, there is a map $L_E X \to L_D X$ from the initial E-localization of X.

(c) The map $X \to L_E X$ is an E-equivalence, which means that the fiber F of this map is E-acyclic. Since $\langle E \rangle \geq \langle D \rangle$, this means the fiber is D-acyclic, which is equivalent to the map $X \to L_E X$ being a D-equivalence. Then applying L_D to this map yields $L_D X \to L_D L_E X$, which is a D-equivalence between D-local spectra. By the D-Whitehead theorem, this is a stable equivalence.

REFERENCES

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