HOMOTOPY LIMITS AND COLIMITS   NAME: ____________________

Due at the beginning of class on 11 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

(1) Let $\mathcal{C}$ be a homotopical category, and let $\mathcal{I}$ be any category. The category $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ of functors from $\mathcal{I}$ to $\mathcal{C}$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal{C}$ and a category $\mathcal{I}$, show that the limit functor

$$\lim\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}, \qquad F \mapsto \lim F$$

is *not* a homotopical functor.

(2) Let $\mathcal{C}$ be a homotopical category. Prove that for any discrete (the only morphisms are identities) category $\mathcal{I}$ with finitely many morphisms, $\mathrm{ho}(\mathcal{C})^{\mathcal{I}}$ is equivalent to $\mathrm{ho}(\mathcal{C}^{\mathcal{I}})$, where $\mathcal{C}^{\mathcal{I}}$ has weak equivalences defined pointwise. Prove that finite products in $\mathrm{ho}(\mathcal{C})$ are homotopy products and finite coproducts in $\mathrm{ho}(\mathcal{C})$ are homotopy coproducts.

(3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

(a) Prove that the data of the coequalizer of two parallel morphisms $A \underset{g}{\overset{f}{\rightrightarrows}} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla\colon A \amalg A \to A$ is the fold map.

(b) Use part (a) to describe the homotopy coequalizer of two maps in the category $\mathcal{T}op$ of (unpointed) topological spaces[1].

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

[1]To be precise, we assume all spaces are compactly generated and weakly Hausdorff.