Due at the beginning of class on 12 March 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Section 3.2] and [Wei94, Section 10.2]

(1) (a) A spectrum X is *rational* if each of its homotopy groups is an $\mathbb{Q}$-vector space. Prove that the full subcategory of $\mathrm{ho}(\mathcal{S}p)$ consisting of the rational spectra is a triangulated subcategory.

SOLUTION: Denote this subcategory by $\mathrm{Sp}_{\mathbb{Q}}$. Since the homotopy groups of a finite product/wedge of spectra are just the products of the homotopy groups of its summands, we know that $\mathrm{Sp}_{\mathbb{Q}}$ is additive. Moreover, since Σ only affects the homotopy groups of a spectrum by shifting their degrees, we know that Σ restricts to an equivalence $\mathrm{Sp}_{\mathbb{Q}} \to \mathrm{Sp}_{\mathbb{Q}}$.

We are left with checking that the cofiber of a map of rational spectra is rational. Let $f : X \to X$ be a map of rational spectra, and let $B = \mathrm{cof}(f)$. We have a long exact sequence in homotopy, from which we can extract a bunch of short exact sequences of the form:

$$0 \to \ker\delta_n \to \pi_n B \to \mathrm{im}\delta_n \to 0$$

where δ_n is the connecting homomorphism $\pi_n \to \pi_{n-1} A$. By exactness, we have

$$\ker\delta_n \cong \pi_n X/\mathrm{im}(\pi_n A \to \pi_n X)$$

Which is a quotient of a $\mathbb{Q}$-vector space by a $\mathbb{Q}$-vector subspace, and thus rational. Similarly $\mathrm{im}\delta_n \subset \pi_{n-1} A$ and is thus a rational vector space. By choosing bases for these, we obtain isomorphisms

$$\pi_n B \cong \ker\delta_n \oplus \mathrm{im}\delta_n$$

for all n, showing that $\pi_n B$ is rational for all n and thus $B = \mathrm{cof}(f)$ is rational. Therefore $\mathrm{Sp}_{\mathbb{Q}}$ is a triangulated category of $\mathrm{ho}(\mathrm{Sp})$

(b) Define a triangulated functor $H\colon \mathcal{D}(\mathbb{Q}) \to \mathrm{ho}(\mathcal{S}p)$ such that $\pi_n H(V_\bullet) = H_n(V_\bullet)$ for all $n \in \mathbb{Z}$. *Hint: any chain complex of $\mathbb{Q}$-vector spaces is quasi-isomorphic to its homology.*

SOLUTION: Given a chain complex $V_\bullet$, define $H(V_\bullet) = \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet))$. That is, H takes a chain complex to the wedge sum of shifts of the Eilenberg–MacLane spectra of its homology groups. Since quasi-isomorphisms induce isomorphisms on homology, this functor sends quasi-isos to weak equivalences. Therefore, it is a homotopical functor and descends to a functor $H\colon \mathcal{D}(\mathbb{Q}) \to \mathrm{ho}(\mathcal{S}p)$. Note that the image of this functor is contained in the full subcategory $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ spanned by the rational spectra.

To show that this functor is triangulated, we must show that it is additive, that it commutes with translation, and it preserves triangles.

To see that this functor is additive, first note that H applied to the zero chain complex is just a point. Then:

$$\begin{aligned}
H(V_\bullet \oplus W_\bullet) &= \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet \oplus W_\bullet)) \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet) \oplus H_n(W_\bullet)) && \text{homology distributes over products} \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n (H(H_n(V_\bullet)) \vee H(H_n(W_\bullet))) && \text{Eilenberg–MacLane functor is additive (HW 5)} \\
&\cong \bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet)) \vee \Sigma^n H(H_n(W_\bullet)) && \text{suspension commutes with coproducts} \\
&\cong \left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet))\right) \vee \left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(W_\bullet))\right) \\
&= H(V_\bullet) \vee H(W_\bullet)
\end{aligned}$$

So H is an additive functor.

To see that H commutes with translation, recall that translation in the derived category is shifting the chain complex by 1: $V_\bullet \mapsto V_{\bullet+1}$. Then

$$\begin{aligned}
H(V_{\bullet+1}) &= \bigvee_{k\in\mathbb{Z}} \Sigma^k H(H_k(V_{\bullet+1})) \\
&= \bigvee_{k\in\mathbb{Z}} \Sigma^k H(H_{k-1}(V_\bullet)) \\
&= \bigvee_{n\in\mathbb{Z}} \Sigma^{n+1} H(H_n(V_\bullet)) && \text{reindexing with } n = k-1 \\
&= \Sigma\left(\bigvee_{n\in\mathbb{Z}} \Sigma^n H(H_n(V_\bullet))\right) \\
&= \Sigma H(V_\bullet)
\end{aligned}$$

So H commutes with translation.

Finally, we must check that H preserves distinguished triangles. Consider a distinguished triangle in $\mathcal{D}(\mathbb{Q})$, which is quasi-isomorphic to one of the form

$$V_\bullet \xrightarrow{f_\bullet} W_\bullet \to \operatorname{cone}(f_\bullet) \to V_{\bullet+1}$$

We want to show that

$$H(V_\bullet) \xrightarrow{H(f_\bullet)} H(W_\bullet) \to H(\operatorname{cone}(f_\bullet))$$

is a cofiber sequence in $\operatorname{ho}(\mathcal{S}p)$. Consider the morphism g from the cofiber of $H(f_\bullet)$ to $H(\operatorname{cone}(f_\bullet))$:

$$\begin{tikzcd}
H(V_\bullet) \arrow[r, "H(f_\bullet)"] \arrow[d, equal] & H(W_\bullet) \arrow[r] \arrow[d, equal] & \operatorname{cof}(H(f_\bullet)) \arrow[d, dashed, "g"] \\
H(V_\bullet) \arrow[r, "H(f_\bullet)"] & H(W_\bullet) \arrow[r] & H(\operatorname{cone}(f_\bullet))
\end{tikzcd}$$

Taking homotopy, we get a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}\operatorname{cof}(H(f_\bullet)) \arrow[r] \arrow[d, "g_*"] & \pi_n H(V_\bullet) \arrow[r] \arrow[d, equal] & \pi_n H(W_\bullet) \arrow[r] \arrow[d, equal] & \pi_n \operatorname{cof}(H(f_\bullet)) \arrow[r] \arrow[d, "g_*"] & \pi_{n-1} H(V_\bullet) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \pi_{n+1} H(\operatorname{cone}(f_\bullet)) \arrow[r] & \pi_n H(V_\bullet) \arrow[r] & \pi_n H(W_\bullet) \arrow[r] & \pi_n H(\operatorname{cone}(f_\bullet)) \arrow[r] & \pi_{n-1} H(V_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

The top row is exact because it's the LES of a cofiber sequence. The bottom row is exact because it's isomorphic to the LES in homology for the distinguished triangle $V_\bullet \xrightarrow{f_\bullet} W_\bullet \to \mathrm{cone}(f_\bullet)$ in $\mathcal{D}(\mathbb{Q})$. Then the five lemma shows that $\mathrm{cof}(H(f_\bullet))$ is stably equivalent to $H(\mathrm{cone}(f_\bullet))$. Therefore, H sends triangles to triangles.

We have shown that H is additive, commutes with shift functors, and sends triangles to triangles. So H is a triangulated functor.

(2) Show that the class of stable equivalences is saturated in $\mathcal{S}p$.

SOLUTION: Recall that a homotopical category $\mathcal{C}$ is saturated if it has the following property: if $f \in \mathcal{C}$ becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then f is a weak equivalence.

So the question is asking us to show that if $f\colon X \to Y$ becomes an isomorphism in $\mathrm{ho}(\mathcal{S}p)$, then f is a stable equivalence.

Let $\mathrm{gr}\mathcal{A}b$ denote the category of graded abelian groups, and consider it as a homotopical category equipped with the minimal homotopical structure (ie only isomorphisms are weak equivalences). Then $\pi_*\colon \mathcal{S}p \to \mathrm{gr}\mathcal{A}b$ is a homotopical functor, by definition. Hence, it descends to a functor $\mathrm{ho}(\mathcal{S}p) \to \mathrm{gr}\mathcal{A}b$, and the triangle below commutes:

$$\begin{tikzcd} \mathcal{S}p \arrow[r, "\pi_*"] \arrow[d] & \mathrm{gr}\mathcal{A}b \\ \mathrm{ho}(\mathcal{S}p) \arrow[ur, "\mathrm{ho}(\pi_*)"'] & \end{tikzcd}$$

Suppose that f is an isomorphism in $\mathrm{ho}(\mathcal{S}p)$. Then $\mathrm{ho}(\pi_*)(f)$ is an isomorphism in $\mathrm{gr}\mathcal{A}b$, so f must have always been a stable equivalence.

(3) (a) By giving a counterexample, show that $\mathrm{ho}(\mathcal{S}p)$ is not an abelian category.

SOLUTION: First, we show that all monics are split. Given a monic $f\colon A \to B$, we may extend to an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and more powerfully to an exact sequence

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \to \ldots .$$

It follows that $-\Sigma f \circ h = 0$ and thus that $f \circ \Sigma^{-1} h = 0$ and thus that $\Sigma^{-1} h = 0$ and $h = 0$ since f is monic. It follows that $B \cong A \oplus C$ so that f is split.

Now consider the map $H\mathbb{Z}/p^2 \to H\mathbb{Z}/p$ arising from the quotient map $\mathbb{Z}/p^2 \to \mathbb{Z}/p$. This map must have nontrivial kernel K because there is a nontrivial kernel after taking π_*. The map $K \to H\mathbb{Z}/p^2$ splits so that there is a decomposition $H\mathbb{Z}/p^2 \cong K \oplus J$ for some J. There is no such nontrivial decomposition after taking π_*, so it much be that $\pi_* J = 0$ and thus that $J \cong 0$ and thus $H\mathbb{Z}/p^2 \cong K$. This is a contradiction because $H\mathbb{Z}/p^2 \to H\mathbb{Z}/p$ is not the zero map after taking π_*.

(b) Let X be a spectrum and $e\colon X \to X$ an *idempotent* map: $e \circ e = e$. Construct a spectrum X_e such that for all $n \in \mathbb{Z}$,

$$\pi_n(X_e) = \mathrm{im}(\pi_n X \xrightarrow{e_*} \pi_n X).$$

Thus, idempotent maps have "images" in $\mathrm{ho}(\mathcal{S}p)$, even though it is not an abelian category.

SOLUTION: First, recall the previously discussed facts that sequential homotopy colimits of spectra are computed level-wise and that homotopy groups commute with sequential colimits.

Let $X_e = \mathrm{hocolim}(X \xrightarrow{e} X \xrightarrow{e} \dots)$. It follow that

$$\begin{aligned}
\pi_n X_e &\cong \operatorname*{colim}_k \pi_{n+k}(X_e)_k \\
&\cong \operatorname*{colim}_k \pi_{n+k}\,\mathrm{hocolim}(X_k \xrightarrow{e_k} X_k \xrightarrow{e_k} \dots) \\
&\cong \operatorname*{colim}_k \mathrm{colim}(\pi_{n+k}X_k \xrightarrow{e_{k*}} \pi_{n+k}X_k \xrightarrow{e_{k*}} \dots) \\
&\cong \mathrm{colim}(\operatorname*{colim}_k \pi_{n+k}X_k \xrightarrow{e_*} \operatorname*{colim}_k \pi_{n+k}X_k \xrightarrow{e_*} \dots) \\
&\cong \mathrm{colim}(\pi_n X \xrightarrow{e_*} \pi_n X_k \xrightarrow{e_*} \dots) \cong \mathrm{im}(e_*)
\end{aligned}$$

where the last isomorphism is a standard fact about abelian groups.

(4) Let $\mathcal{C}$ be a triangulated category with shift functor Σ. Suppose that

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

is a commuting square in $\mathcal{C}$. Prove that there is a diagram as below, in which each row and each column is a triangle in $\mathcal{C}$, and the diagram commutes except for the bottom right square (marked with -1), which *anticommutes*: $fg = -gf$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-1"] & \Sigma X'' \arrow[d] \\
\Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

SOLUTION: This approach is essentially brute forcing the diagram with the octahedral axiom. All proofs that I have seen look like this. I am very interested in a more geometric argument involving some sort of diagram calculus for triangulated categories, if these is one. I think, in the same sense that an exact triangle $X \to Y \to Z \to \Sigma X$ is triangular, by gluing X to ΣX, I suspect this diagram is an exact is an exact torus (or maybe sphere, because of the -1, idk), and I suspect there is an elegant reason that it exists.

First, extend the four maps in the original square to a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[d, "i''"] & Y'' \arrow[d, "j''"] & ? & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

where the last row and column are just rotations of the first row and column.

Extend the diagonal $a = jf = f'i \colon X \to Y'$ to an exact triangle $X \xrightarrow{a} Y' \xrightarrow{b} W \xrightarrow{c} \Sigma X$. I am unsure if there is a good way to draw this in the diagram, so I will not.

Now, we use the octahedral axiom with the triangles $X \to X' \to X'' \to \Sigma X$, $X' \to Y' \to Z'$, and $X \to Y' \to W \to \Sigma X$ to get the octahedron

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "i"] & X \arrow[d, "a"] & & \\
X' \arrow[r, "f'"] \arrow[d, "i'"] & Y' \arrow[r, "g'"] \arrow[d, "b"] & Z' \arrow[r, "h'"] \arrow[d, equal] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[r, dotted, "u"] \arrow[d, "i''"] & W \arrow[r, dotted, "v"] \arrow[d, "c"] & Z' \arrow[r, dotted, "w"] & \Sigma X'' \\
\Sigma X \arrow[r, equal] & \Sigma X & &
\end{tikzcd}$$

such that $\Sigma i \circ c = h' \circ v \colon W \to \Sigma X'$. See [Hub] for how this is an octahedron. I think this is the clearest diagram because my simple mind likes squares.

Now we use a reflection of this argument with the triangles $X \to Y \to Z \to \Sigma X$, $Y \to Y' \to Y'' \to \Sigma Y$ and $X \to Y' \to W \to \Sigma X$ to get the octahedron

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, dotted, "r"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "a"] & Y' \arrow[r, "b"] \arrow[d, "j'"] & W \arrow[r, "c"] \arrow[d, dotted, "s"] & \Sigma X \\
& Y'' \arrow[r, equal] \arrow[d, "j''"] & Y'' \arrow[d, dotted, "t"] & \\
& \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z &
\end{tikzcd}$$

such that $\Sigma f \circ c = j'' \circ s \colon W \to \Sigma Y$.

This provides a map $v \circ r \colon Z \to Z'$ which we call k. Verify that $k \circ g = v \circ r \circ g = v \circ b \circ j = g' \circ j$. Extend k to a triangle $Z \to Z' \to Z'' \to \Sigma Z$ which fills in most of the remaining diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] \arrow[d, "k'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[d, "i''"] & Y'' \arrow[d, "j''"] & Z'' \arrow[d, "k''"] & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

Now, we need to fill the third row in such a way that the relevant squares (anti)commute. Note that we have additionally shown that TR3 is an irrelevant axiom.

Now apply the octahedral axiom to the triangles $Z \to W \to Y'' \to \Sigma Z$, $W \to Z' \to \Sigma X \to \Sigma W$, and

$Z \to Z' \to Z'' \to \Sigma Z$ to get the octahedron

$$\begin{tikzcd}
Z \arrow[r, equal] \arrow[d, "r"] & Z \arrow[d, "k"] & & \\
W \arrow[r, "v"] \arrow[d, "s"] & Z' \arrow[r, "w"] \arrow[d, "k'"] & \Sigma X'' \arrow[r, "-\Sigma u"] \arrow[d, equal] & \Sigma W \arrow[d, "\Sigma s"] \\
Y'' \arrow[r, dotted, "g''"] \arrow[d, "t"] & Z'' \arrow[r, dotted, "h''"] \arrow[d, "k''"] & \Sigma X'' \arrow[r, dotted, "-\Sigma f''"] & \Sigma Y'' \\
\Sigma Z \arrow[r, equal] & \Sigma Z & &
\end{tikzcd}$$

such that $\Sigma r \circ k'' = -\Sigma u \circ h''$ where we have given the dotted maps suggestive names such that they fit into the original diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"] & Y' \arrow[r, "g'"] \arrow[d, "j'"] & Z' \arrow[r, "h'"] \arrow[d, "k'"] & \Sigma X' \arrow[d, "\Sigma i'"] \\
X'' \arrow[r, "f''"] \arrow[d, "i''"] & Y'' \arrow[r, "g''"] \arrow[d, "j''"] & Z'' \arrow[r, "h''"] \arrow[d, "k''"] & \Sigma X'' \arrow[d, "\Sigma i''"] \\
\Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma g"] & \Sigma Z \arrow[r, "\Sigma h"] & \Sigma^2 X
\end{tikzcd}$$

Now, verify that the unchecked commuting squares follow from the octahedra. The anti-commutativity of the bottom right square follows from the fact that

$$\Sigma h \circ k'' = \Sigma c \circ \Sigma r \circ k'' = -\Sigma c \circ \Sigma u \circ h'' = -\Sigma i'' \circ h''.$$

The other squares are similar.

References

[Hub] Andrew Hubery. Notes on the Octahedral Axiom. "https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=2246900fb2f9694d965b6b6482f76d4d3c6b1206".

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.