HOMOTOPY LIMITS AND COLIMITS

The goal of this talk is to develop a little bit of the theory of homotopy limits and colimits via derived functors in a homotopical category. If you've seen some model category theory, this might look a little familiar. And if you're curious, the standard model categories reference is [DS95].

OUTLINE

(1) Left deformations of homotopical categories [Rie14, Section 2.2]

   (a) introduce left deformations of homotopical categories [Rie14, def 2.2.1] and left deformations of functors [Rie14, def 2.2.4]
   (b) The goal is to state [Rie14, theorem 2.2.8], but you don't need to prove it if you don't want to give a proof in the talk.
   (c) You can say as much or as little about model categories as you like here, depending on how much you know and or like model categories.
   (d) After theorem 2.2.8, you should skip the rest of section 2.2

(2) Theory of homotopy limits and colimits

   (a) introduce the idea of homotopy colimits of shape $\mathcal{J}$ as left derived functors of the colimit functor colim: $\mathcal{C}^{\mathcal{J}} \to \mathcal{C}$.
   (b) Explain the difficulty that homotopy colimits are very rarely colimits in the homotopy category: this is explained in [Rie14, Remark 6.3.1]
   (c) Make explicit what we need to compute a homotopy colimit: a left deformation of the colim functor
   (d) Maybe say the words "the bar construction gives a left deformation of $\mathcal{C}^{\mathcal{J}}$" but you really shouldn't define the bar construction in general if you don't want to

(3) Homotopy colimits in the category of spaces

   (a) Explain that CW approximation (replacing a space X by a weakly equivalent CW complex X') is a left deformation on the homotopical category of CGWH spaces (this is cofibrant replacement, if you're a model category person). To get a functorial cofibrant replacement, we usually take the geometric realization of the singular simplicial set: $X \mapsto |\mathrm{Sing}(X)|$.
   (b) Say that you can get a left deformation on the category of diagrams in spaces by the bar construction, and give the bar construction in this context (see [Mal23, 7.3.23]

(4) Examples

   (a) Explain explicitly how to construct the homotopy pushout, homotopy cofiber (mapping cylinder), and homotopy sequential colimit (mapping telescope).
   (b) Make sure to include the example of the unreduced suspension as a homotopy pushout.
   (c) If you have time, explain the homotopy pullback and homotopy fiber. A good reference for this (with pictures!) is [Mal23, Section 1.5]

REFERENCES

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.