HOMOTOPICAL CATEGORIES NAME: SOLUTIONS

**Due at the beginning of class on 4 February 2025**

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

**Reading:** Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

(0) Introduce yourself on Zulip. Include:

(a) Your name and pronouns

(b) Two pictures of yourself: a serious one and a silly one

(c) Your university

(d) Your mathematical interests

(e) A nonmathematical interest – be specific!

- "I like reading fantasy books" :(
- "I just finished reading book 5 of the Stormlight Archive by Brandon Sanderson" :)

(1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms $f$ and $g$, if any two of $f$, $g$, and $gf$ are in $\mathcal{W}$, then so is the third.

(a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.

SOLUTION: Let $\mathcal{C}$ be a homotopical category with weak equivalences the subcategory $\mathcal{W}$. We will show that the $\mathcal{W}$ has the two-out-of-three property by using its two-out-of-six property. Consider the morphisms $f$, $g$, and $gf$ in the following diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

Suppose $f, g \in \mathcal{W}$. Then $gf \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, phantom] \arrow[dr, "\mathrm{id}\cdot f"'] \arrow[ddrr, phantom] \arrow[drr, "g\cdot \mathrm{id}\cdot f"] & \bullet \arrow[d, "\mathrm{id}"] \arrow[dr, "g\cdot \mathrm{id}"] & \\ & \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

Suppose $g, gf \in \mathcal{W}$, then $f \in \mathbb{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g\cdot f"'] \arrow[drr, "\mathrm{id}\cdot g\cdot f"] & \bullet \arrow[d, "g"] \arrow[dr, "\mathrm{id}\cdot g"] & \\ & \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}$$

Finally, suppose $f, gf \in \mathcal{W}$, then $g \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"] \arrow[dr, "f\cdot\mathrm{id}"'] \arrow[drr, "g\cdot f\cdot\mathrm{id}"] & \bullet \arrow[d, "f"] \arrow[dr, "g\cdot f"] & \\
& \bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}$$

(b) Is the two-out-of-three property equivalent to the two-out-of-six property?

SOLUTION: No. It is weaker. For a counterexample, consider the category:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Let $\mathcal{W} = \{gf, hg, \mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, \mathrm{id}_D\}$. $\mathcal{W}$ clearly satisfies the two-out-of-six property. It does not satisfy the two-out-of-three property: $gf$ and $hg$ are in $\mathcal{W}$, while $hgf$ is not.

(2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism $f$ in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if $f$ becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

(a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.

SOLUTION: Suppose $f, g, h$ are composable in $\mathcal{C}$ and $fg, gh \in \mathcal{W}$. Then, $fg$ and $gh$ are isomorphisms in $\mathrm{ho}\,\mathcal{C}$. Isomorphisms satisfy 2-out-of-6, so $f, g, h$ are isomorphisms in $\mathrm{ho}\,\mathcal{C}$. Thus, $f, g, h \in \mathcal{W}$. This proves the claim. [Rie14, Lemma 2.1.10]

(b) Give an example of a homotopical category that is *not* saturated.

SOLUTION: Consider the following category, which represents a morphism of retracts.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{id}"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r] & B
\end{tikzcd}$$

and let the weak equivalence be given by $f$ and the identity morphisms. This satisfies 2-out-of-6 since the only way to obtain $f$ via a composite of two morphisms is with $f$ and an identity, and the only way to obtain an identity via a composite of two morphisms either uses identities or is oen of the horizontal composites, but the maps $X \to A$ and $Y \to B$ cannot be applied before another morphism to either give $f$ or an identity.

The weak equivalences are not closed under retract, since $g$ is not a weak equivalence, but isomorphisms are closed under retract, so $g$ must be an isomorphism in the homotopy category. Thus, this construction is a homotopical category which is not saturated.

This example is nice because it is clear that it cannot have a model structure, as the weak equivalences in a model category are closed under retract by definition.

# REFERENCES


[HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.