Stable Phenomena in Spaces Talk Outline

The goal of this talk is to describe spaces as a homotopical category, and to introduce many of the ideas that originally led to spectra. A good reference for this is [Sto22, Chapter 2].

Outline

(1) What do we mean by spaces?

  (a) define categories $\mathcal{T}op$ and $\mathcal{T}op_*$ of CGWH spaces and pointed CGWH spaces
  (b) explain that $\mathcal{T}op$ is symmetric monoidal for cartesian product, $\mathcal{T}op_*$ is closed symmetric monoidal for smash
  (c) we make these into homotopical categories by taking the weak equivalences to be the homotopy equivalences [Rie14, Digression 2.1.5]
  (d) define the function spaces $F(X, Y)$ in $\mathcal{T}op_*$ and the adjunction $X \wedge (-) \dashv F(X, -)$, and note that this adjunction extends to one on function spaces $F(X \wedge Y, Z) \cong F(Y, F(X, Z))$
  (e) note that $\pi_0 F(X, Y) = [X, Y]$
  (f) explain that if we didn't use CGWH spaces, this symmetric monoidal stuff would fail (see Riehl Lemma 6.1.3)

(2) Suspension and loopspaces

  (a) define the suspension $\Sigma X = S^1 \wedge X$ and the loopspace $\Omega X = F(S^1, X)$. This is a special case of the adjunction above
  (b) give the example that $\Sigma S^n \cong S^{n+1}$
  (c) ΣX is the homotopy pushout of $* \leftarrow X \rightarrow *$ and ΩX is the homotopy pullback of the same
  (d) note that $\Sigma \dashv \Omega$, and that this adjunction works both on the level of function spaces $F(-, -)$ and homotopy classes of maps $[-, -]$
  (e) $\pi_n F(X, Y) = [\Sigma^n X, Y]$
  (f) Define H-group. ΩX is an H-group. The consequence of this is that $[Y, \Omega X]$ becomes a group
  (g) mention that ΣX is a co-H-group

(3) Freudenthal Suspension Theorem

  (a) recall the definition of n-connected space
  (b) define the map $\pi_k X \rightarrow \pi_{k+1} \Sigma X$
  (c) state the Freudenthal suspension theorem (but do not prove it) (pick your favorite formulation)
  (d) state [Sto22, corollary 2.23]
  (e) define stable homotopy groups $\pi_*^s(X)$
  (f) (if you have time) define the space QX and show that it's homotopy groups are the same as the homotopy groups of X

(4) Blakers–Massey theorem

  (a) state the Blakers–Massey theorem in the form of a homotopy pullback/pushout square

(b) mention that this is homotopy excision (you don't have to prove it)

(c) state [Sto22, Corollary 2.29] and prove it from the theorem

(5) Optional Extra Topics, if you have time:

(a) Hurewicz as a corollary of the above Freudenthal or Blakers–Massey

(b) Stable classes of maps between spaces $\{X, Y\}$

(c) Localization of spaces at the stable equivalences / Spanier–Whitehead Category

(d) Puppe sequences, cofiber seq gives LES in homology, fiber seq gives LES in homotopy

References

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

[Sto22] Bruno Stonek. Introduction to stable homotopy theory. https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf, July 2022.