

Due at the beginning of class on 4 February 2025

- Your answers should be neatly written and logically organized.
- You may collaborate on solving the problems, but the solutions you turn in should be your own.
- You may use any resource you find online (or elsewhere), but you must cite any resource you use.

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

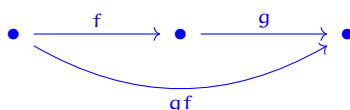
(0) Introduce yourself on Zulip. Include:

- Your name and pronouns
- Two pictures of yourself: a serious one and a silly one
- Your university
- Your mathematical interests
- A nonmathematical interest – be specific!
 - “I like reading fantasy books” :(
 - “I just finished reading book 5 of the Stormlight Archive by Brandon Sanderson” :)

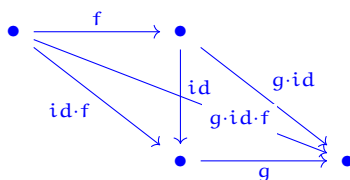
(1) A class of morphisms \mathcal{W} in a category \mathcal{C} satisfies the *two-out-of-three property* if given any two composable morphisms f and g , if any two of f , g , and gf are in \mathcal{W} , then so is the third.

(a) Prove that the class of weak equivalences \mathcal{W} in a homotopical category \mathcal{C} obeys the two-out-of-three property.

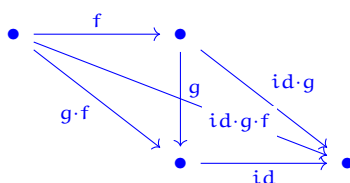
SOLUTION: Let \mathcal{C} be a homotopical category with weak equivalences the subcategory \mathcal{W} . We will show that the \mathcal{W} has the two-out-of-three property by using its two-out-of-six property. Consider the morphisms f , g , and gf in the following diagram:



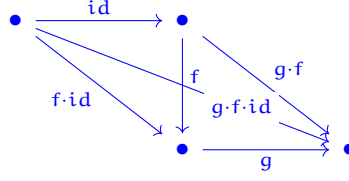
Suppose $f, g \in \mathcal{W}$. Then $gf \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Suppose $g, gf \in \mathcal{W}$, then $f \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Finally, suppose $f, gf \in \mathcal{W}$, then $g \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



- (b) Is the two-out-of-three property equivalent to the two-out-of-six property?

SOLUTION: No. It is weaker. For a counterexample, consider the category:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Let $\mathcal{W} = \{gf, hg, id_A, id_B, id_C, id_D\}$. \mathcal{W} clearly satisfies the two-out-of-six property. It does not satisfy the two-out-of-three property: gf and hg are in \mathcal{W} , while hgf is not.

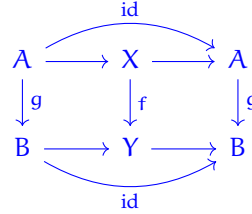
- (2) Let \mathcal{C} be any category equipped with a collection of morphisms \mathcal{W} . We say that \mathcal{W} is *saturated* if every morphism f in \mathcal{C} which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in \mathcal{W} . We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $ho(\mathcal{C})$, then $f \in \mathcal{W}$).

- (a) Prove that if \mathcal{W} is saturated, then \mathcal{W} has the two-out-of-six property.

SOLUTION: Suppose f, g, h are composable in \mathcal{C} and $fg, gh \in \mathcal{W}$. Then, fg and gh are isomorphisms in $ho \mathcal{C}$. Isomorphisms satisfy 2-out-of-6, so f, g, h are isomorphisms in $ho \mathcal{C}$. Thus, $f, g, h \in \mathcal{W}$. This proves the claim. [Rie14, Lemma 2.1.10]

- (b) Give an example of a homotopical category that is *not* saturated.

SOLUTION: Consider the following category, which represents a morphism of retracts.



and let the weak equivalence be given by f and the identity morphisms. This satisfies 2-out-of-6 since the only way to obtain f via a composite of two morphisms is with f and an identity, and the only way to obtain an identity via a composite of two morphisms either uses identities or is one of the horizontal composites, but the maps $X \rightarrow A$ and $Y \rightarrow B$ cannot be applied before another morphism to either give f or an identity.

The weak equivalences are not closed under retract, since g is not a weak equivalence, but isomorphisms are closed under retract, so g must be an isomorphism in the homotopy category. Thus, this construction is a homotopical category which is not saturated.

This example is nice because it is clear that it cannot have a model structure, as the weak equivalences in a model category are closed under retract by definition.

REFERENCES

- [HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

- [Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.