STABLE HOMOTOPY CATEGORY TALK OUTLINE

Read [Mal23, Section 3.2]. If you know model categories, I think that [Dug22, Section 1.2] is a nice explanation of the triangulated structure on any stable model category. You can translate the arguments there to our category of spectra, if you are so inclined.

The goal of this talk is to convince everyone that $\mathrm{ho}(\mathcal{S}\mathrm{p})$ is a triangulated category. You probably won't have time to prove it carefully, so you should instead focus on carefully defining a triangulated category and explaining how things in $\mathrm{ho}(\mathcal{S}\mathrm{p})$ fit into the triangulated category structure.

OUTLINE

(1) Recall that we defined the stable homotopy category $\mathrm{ho}(\mathcal{S}\mathrm{p})$ as the homotopy category of the homotopical category of spectra with stable equivalences.

  (a) Explicitly, this means that objects are spectra, and morphisms are zig-zags of morphisms in $\mathrm{ho}(\mathcal{S}\mathrm{p})$ where the stable equivalences are allowed to go backwards.

  (b) There are other, equivalent definitions $\mathrm{ho}(\mathcal{S}\mathrm{p})$ given in [Mal23, Section 3.1]. You should feel free to pick any one of these that you find convenient and claim without proof that it's the same. Feel no obligation to prove that they're the same, but you can sketch the proof if think you'll have time. It may be worth looking through the rest of the talk and seeing what you'll need to figure out the most convenient description of $\mathrm{ho}(\mathcal{S}\mathrm{p})$.

(2) $\mathrm{ho}(\mathcal{S}\mathrm{p})$ as an additive category

  (a) Define an additive category. I find [Lur17, Definition 1.1.2.1] particularly clear, but there are many equivalent definitions.

  (b) Explain the difference between an additive and an abelian category – for example, abelian categories have kernels and images of maps, but additive ones do not necessarily.

  (c) Give a few examples of additive categories – $\mathcal{A}\mathrm{b}$, $\mathcal{M}\mathrm{od}(R)$, etc.

  (d) Sketch the proof that $\mathrm{ho}(\mathcal{S}\mathrm{p})$ is an additive category. You can cite things that we've seen before. For example, a homework problem showed that $X \wedge Y \simeq X \times Y$ and that both are coproduct/product in $\mathrm{ho}(\mathcal{S}\mathrm{p})$.

  (e) Include some consequences of this: the abelian group structure on the stable maps from X to Y is unique, and [Mal23, Corollary 3.2.15], and [Mal23, Proposition 3.2.19]. Feel free to include some other things you find interesting too.

(3) $\mathrm{ho}(\mathcal{S}\mathrm{p})$ as a triangulated category

  (a) The next goal is to show that there's more structure on $\mathrm{ho}(\mathcal{S}\mathrm{p})$ than just an additive category.

  (b) Define a triangulated category. You can find this definition in [Lur17, Definition 1.1.2.5] or [Knu17].

  (c) Define the shift in $\mathrm{ho}(\mathcal{S}\mathrm{p})$ as the suspension $\Sigma \simeq \mathrm{sh}^1$. This is an equivalence by [Mal23, Proposition 3.2.1].

  (d) Define the triangles $\mathrm{ho}(\mathcal{S}\mathrm{p})$ as the fiber/cofiber sequences (which are the same).

  (e) Sketch the proof that $\mathrm{ho}(\mathcal{S}\mathrm{p})$ is triangulated. You probably won't have time for all of this, so pick a few axioms and convince us that they're true. The proofs can be found in [Dug22, Proposition 1.2.11], which refers to other propositions in the same book. You'll probably have to translate them to our context.

(4) Optional extra stuff to cover if you have time or interest

    (a) Every stable model category is a triangulated category. You'll have to explain what it means to have a stable model category, but this is a neat fact!
    (b) Other examples of triangulated categories: derived category of an abelian group, or a stable module category.
    (c) Stuff in [Mal23, Section 3.2.2], as you have interest.

References

[Dug22] Daniel Dugger. Stable categories and spectra via model categories. In *Stable categories and structured ring spectra*, pages 75–150. Cambridge: Cambridge University Press, 2022.

[Knu17] Ben Knudsen. Configuration Spaces in Algebraic Topology: Lecture 25. https://scholar.harvard.edu/files/knudsen/files/lecture_25.pdf, 2017.

[Lur17] Lurie. Higher Algebra. https://www.math.ias.edu/~lurie/papers/HA.pdf, 2017.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.