Stable Phenomena in Spaces   Name: ______________________

Due at the beginning of class on 18 February 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Sto22, Chapter 2].

(1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion $\mathbb{Z}$-module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.

(2) A pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. Let $f\colon X \to Y$ be a pointed map of well-based spaces.

  (a) Let $\mathrm{cof}(f)$ be the homotopy cofiber of f. Prove that the homotopy cofiber of $Y \to \mathrm{cof}(f)$ is homotopy equivalent to ΣX.

  (b) Prove the dual statement: if $\mathrm{fib}(f)$ is the homotopy fiber of f, then the homotopy fiber of $\mathrm{fib}(f) \to X$ is homotopy equivalent to ΩY.

(3) Let $f\colon X \to Y$ be a map between simply connected spaces such that $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for $i \le n$. We will show that f is an n-connected map.

  (a) Let C be the homotopy cofiber of f, and let F be the homotopy fiber of $Y \to C$. Use the Hurewicz theorem to show that C is n-connected and $F \to Y$ is an n-connected map.

  (b) Use the Blakers–Massey theorem to show that $X \to F$ is at least 2-connected.

  (c) Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n-connected.

(4) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of spaces. Prove that $\Omega \operatorname{hocolim}_i X_i \simeq \operatorname{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

References

[Sto22] Bruno Stonek. Introduction to stable homotopy theory. https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf, July 2022.