Symmetric & Orthogonal Spectra — Name: Solutions

Due at the beginning of class on 15 April 2024

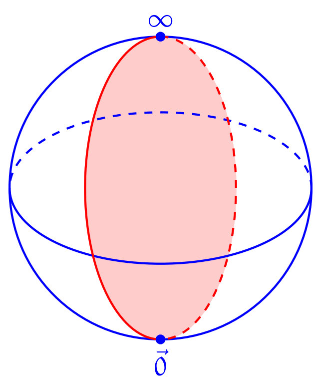
- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 6.1 and 6.2] and [Sch23, Section 1].

(1) Let X be a symmetric spectrum such that Σ_n acts trivially on X_n for all n.

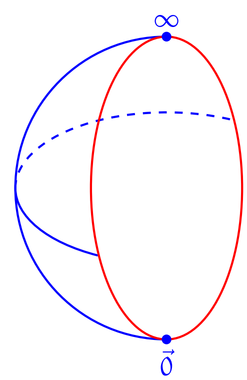
(a) Prove that the orbit space $(S^n)_{\Sigma_n}$ is contractible for all $n \geq 2$, with Σ_n-action on S^n by permutation of coordinates, viewing S^n as the one-point compactification of $\mathbb{R}^n$.

Solution: Let's first prove this for $n = 2$. Let τ be the generator of Σ_2; τ acts on $\mathbb{R}^2$ by reflecting over the line $y = x$. In the sphere S^2, this is reflecting across a great circle through the north pole ∞ and south pole $(0, 0)$. Quotienting by this reflection action smushes one hemisphere into the other, like deflating a beach ball, and $(S^2)_{\Sigma_2}$ is homeomorphic to a disk, which is contractible.



$\Sigma_2 \circlearrowright S^2$

τ reflects across the red plane



$(S^2)_{\Sigma_2}$

west is identified with east, leaving a disk

Now let's consider $(S^n)_{\Sigma_n}$, which is the quotient of S^n by the equivalence relation $x \sim \sigma x$ for all $\sigma \in \Sigma_n$. Let $H \subseteq \Sigma_n$ be a subgroup. We can take this quotient in stages, first identifying x with hx for all $h \in H$ and then identifying with σx for all $\sigma \in \Sigma_n$. In other words, $(S^n)_{\Sigma_n}$ is a further quotient of $(S^n)_H$, for any subgroup H.

Now let $\Sigma_2 \subseteq \Sigma_n$ be the subgroup generated by the simple transposition $\tau = (1\ 2)$. Then

$$(S^n)_{\Sigma_2} \simeq (S^2 \wedge S^{n-2})_{\Sigma_2} \simeq (S^2)_{\Sigma_2} \wedge S^{n-2} \simeq * \wedge S^{n-2} \simeq *,$$

so $(S^n)_{\Sigma_2}$ is contractible. Hence, $(S^n)_{\Sigma_n}$ is contractible as the quotient of a contractible space.

(b) Show that the (naive) homotopy groups of X are trivial.

Solution: Recall that if X is a symmetric spectrum, the iterated bonding maps

$$\xi_{i,j} \colon X_i \wedge S^j \to X_{i+j}$$

are equivariant for the $\Sigma_i \times \Sigma_j$-actions on source and target. Since the symmetric group actions on Σ_n are trivial, these iterated bonding maps factor through the orbit space

$$\begin{tikzcd} X_i \wedge S^j \arrow[r, "\xi_{i,j}"] \arrow[d] & X_{i+j} \\ (X_i)_{\Sigma_i} \wedge (S^j)_{\Sigma_j} \arrow[ur, dashed] & \end{tikzcd}$$

But $(X_i)_{\Sigma_i} \wedge (S^j)_{\Sigma_j} \simeq X_i \wedge * \simeq *$, so the map $\xi_{i,j}$ is nullhomotopic. Hence, in the colimit

$$\pi_k X = \operatorname*{colim}_n \pi_{k+n} X_n = \operatorname{colim}\left(\cdots \to \pi_{k+n_i}(X_n \wedge S^i) \xrightarrow{\xi_{n,i}} \pi_{k+n+i} X_{n+i} \to \cdots \right)$$

many (or even most) of the maps are zero, so the colimit is zero as well.

(2) A *symmetric ring spectrum* is a symmetric spectrum R together with $\Sigma_n \times \Sigma_m$-equivariant multiplication maps $\mu_{n,m} \colon R_n \wedge R_m \to R_{n+m}$ and unit maps $\iota_0 \colon S^0 \to R_0$ and $\iota_1 \colon S^1 \to R_1$ satisfying associativity, unit, multiplicativity, and centrality conditions (see [Sch07, Definition 1.3]).

Show that a symmetric ring spectrum in the sense above is a monoid in the monoidal category of symmetric spectra, with smash product as in [Mal23, Definition 6.2.1].

SOLUTION: This is [Sch07, Theorem 5.25], but we write a full proof here.

First, note by [Sch07, Remark 1.6(iii)], the bonding maps of R must be built out of ι_1 and μ as follows:

$$\begin{tikzcd} R_n \wedge S^1 \arrow[r, "1\wedge\iota_1"] \arrow[rr, bend right, "\xi_n"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \end{tikzcd}$$

and similarly, the iterated bonding maps $R_n \wedge S^k \xrightarrow{\xi_{n,k}} R_{n+k}$ are given by

$$\begin{tikzcd} R_n \wedge S^k \arrow[r, "1\wedge\iota_1^{\wedge n}"] \arrow[rrr, bend right, "\xi_{n,k}"'] & R_n \wedge R_1^{\wedge k} \arrow[r, "\mu_k"] & R_n \wedge R_k \arrow[r, "\mu_{n,k}"] & R_{n+k}, \end{tikzcd} \tag{1}$$

where $\mu_k \colon R_1^{\wedge k} \to R_k$ is the unique way to multiply k copies of R_1 into R_k using repeated $\mu_{i,j}$ (unique by associativity).

We want to construct monoid maps $S \to R$ and $R \wedge R \to R$ from the maps given by [Sch07, Definition 1.3].

To construct $S \to R$, note that $\iota_0 \colon S^0 \to R_0$ corresponds under the adjunction $\Sigma^\infty \dashv \mathrm{ev}_0$ to a map $\iota \colon S \to R$. This is the unit map we desire, but we can put it into slightly more useful terms. At level zero, this map is ι_0. At level n, this map is determined by the bonding maps in S and R (since it's a map out of a suspension spectrum):

$$\begin{tikzcd} S^0 \wedge S^n \arrow[r, "\iota_0\wedge\mathrm{id}"] \arrow[d, "\cong"] & R_0 \wedge S^n \arrow[d, "\xi_{0,n}"] \\ S^n \arrow[r, "\iota_n"] & R_n \end{tikzcd}$$

From this diagram and (1), we see that at level 1 the unit map $\mathbb{S} \to R$ is $\iota_1 \colon S^1 \to R_1$, and at level n the unit map $S^n \to R_n$ is

$$S^n \xrightarrow{\iota_1^{\wedge n}} R_1^{\wedge n} \xrightarrow{\mu_n} R_n.$$

To construct the product $\mu \colon R \wedge R \to R$, we first construct maps

$$\mu'_{i,j} \colon (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (R_i \wedge R_j) \to R_n$$

from the maps $\mu_{i,j} \colon R_i \wedge R_j \to R_n$ using the restriction/induction adjunction between pointed $(\Sigma_i \times \Sigma_j)$-spaces and pointed Σ_n-spaces. The collection $\{\mu'_{i,j}\}_{i+j=n}$ defines a map

$$\bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (R_i \wedge R_j) \to R_n,$$

and to get a map $R \wedge R \to R$, we must check that this map descends to the quotient $(R \wedge R)_n$. This amounts to checking the commutativity of the following diagram:

$$\begin{tikzcd}
(\Sigma_{i+j+k})_+ \wedge_{\Sigma_i \times \Sigma_j \times \Sigma_k} (R_i \wedge R_j \wedge S^k) \arrow[rr, "\sigma_{j,k} \wedge 1 \wedge \gamma"] \arrow[d, "1 \wedge 1 \wedge \xi_{j,k}"] & & (\Sigma_{i+j+k})_+ \wedge_{\Sigma_i \times \Sigma_j \times \Sigma_k} (R_i \wedge S^k \wedge R_j) \arrow[d, "1 \wedge \xi_{i,k} \wedge 1"] \\
(\Sigma_{i+j+k})_+ \wedge_{\Sigma_i \times \Sigma_{j+k}} (R_i \wedge R_{j+k}) \arrow[dr, "\mu'_{i,j+k}"'] & & (\Sigma_{i+j+k})_+ \wedge_{\Sigma_{i+k} \times \Sigma_j} (R_{i+k} \wedge R_j) \arrow[dl, "\mu'_{i+j,k}"] \\
& R_{i+j+k}. &
\end{tikzcd}$$

Commutativity of this diagram is a consequence of commutativity of the centrality diagram. Thus, we obtain maps

$$\mu_n \colon (R \wedge R)_n \to R_n.$$

To see that this is a spectrum map, we should check that it commutes with the bonding maps of R. However, that follows from our understanding of the bonding maps in terms of $\mu_{i,j}$ and ι.

Thus, we have a spectrum map $\mu \colon R \wedge R \to R$.

We must show that the multiplication on R is associative and unital.

Let's start with unital. Since we are working in a symmetric monoidal category, it suffices to check only one of the unit diagrams commutes. Let's check this one:

$$\begin{tikzcd}
R \wedge R \arrow[d, "\mu"'] & R \wedge \mathbb{S} \arrow[l, "\mathrm{id} \wedge \iota"'] \arrow[dl, "\rho"] \\
R &
\end{tikzcd}$$

where ρ is the right unitor. Recall that ρ is determined by the iterated bonding maps

$$\rho_{i,j} = \xi_{i,j} \colon R_i \wedge S^j \to R_{i+j},$$

and the iterated bonding maps are themselves are exactly $\mu \circ (1 \wedge \iota)$ by (1). Hence, the diagram commutes.

Finally, to show associativity, consider the triple smash product

$$(R \wedge R \wedge R)_n = \left(\bigvee_{i+j+k=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j \times \Sigma_k} (R_i \wedge R_j \wedge R_k) \right) / \sim .$$

The maps $(\mu\circ(1\wedge\mu))_n$ and $(\mu\circ(\mu\wedge 1))_n$ are determined on the factors $R_i\wedge R_j\wedge R_k$ by $(\mu_{i,j+k}\circ(1\wedge\mu_{j,k})$ and $(\mu_{i+j,k}\circ(\mu_{i,j}\wedge 1))$ respectively. By the associativity property of the maps $\mu_{a,b}$, these are the same. Hence, $\mu\circ(1\wedge\mu)$ and $\mu\circ(\mu\wedge 1)$ are the same map.

(3) Cobordism of manifolds is captured by the spectrum MO. Read about this spectrum in [Mal23, Example 2.1.20] and [Sch07, Example 2.8].

(a) Prove that there is a pullback square of vector bundles

$$\begin{tikzcd} \gamma_n\oplus\gamma_m \arrow[r] \arrow[d] & \gamma_{n+m} \arrow[d] \\ BO(n)\times BO(m) \arrow[r] & BO(n+m), \end{tikzcd}$$

where $\gamma_k\to BO(k)$ is the tautological bundle.

SOLUTION: This follows from the universal property of the space $BO(n+m)$: any real vector bundle of dimension $n+m$ is a pullback of the tautological $n+m$-bundle γ_{n+m}, but we give some details.

Fix a linear isomorphism $\mathbb{R}^\infty\times\mathbb{R}^\infty\to\mathbb{R}^\infty$. Using this isomorphism, we can define a map $p\colon BO(n)\times BO(m)\to BO(n+m)$ by taking the cartesian product of an n-plane and an m-plane. Then claim that the pullback bundle $p^*(\gamma_{n+m})$ is isomorphic to $\gamma_n\oplus\gamma_m$. By inspecting the fibers of $\gamma_n\oplus\gamma_m$ and $p^*(\gamma_{n+m})$, we see that they contain the same data and are therefore isomorphic as bundles.

(b) Use the pullback square to produce multiplication maps $\mu_{n,m}\colon MO(n)\wedge MO(m)\to MO(n+m)$ for all $n,m\geq 0$.

SOLUTION: First, we prove a lemma. Let $E_1\to X_1$ and $E_2\to X_2$ be vector bundles. Claim that $\mathrm{Th}(E_1\oplus E_2)\cong\mathrm{Th}(E_1)\wedge\mathrm{Th}(E_2)$. This can be seen because $\mathrm{Th}(E_1\oplus E_2)$ is the one-point compactification of the space $E_1\times E_2$, which can be identified with the one-point compactification of E_1 smashed with the one-point compactification of E_2.

Given this lemma, the map $\gamma_n\oplus\gamma_m\to\gamma_{n+m}$ from part (a) extends to a map on the Thom spaces

$$\mu_{n,m}\colon MO(n)\wedge MO(m)=\mathrm{Th}(\gamma_n)\wedge\mathrm{Th}(\gamma_m)\cong\mathrm{Th}(\gamma_n\oplus\gamma_m)\to\mathrm{Th}(\gamma_{n+m})=MO(n+m)$$

(c) Define unit maps $\iota_0\colon S^0\to MO(0)$ and $\iota_1\colon S^1\to MO(1)$.

SOLUTION: We define the unit maps of this spectrum MO by

$$\iota_0=\mathrm{id}\colon S^0\to MO(0)=S^0$$

and $\iota_1\colon S^1\to MO(1)$ as follows. The trivial rank 1 bundle over a point is by the universal property a pullback of the tautological bundle γ_1 over $BO(1)$. (Note that $BO(1)$ is the space of lines in $\mathbb{R}^\infty$ – that is, $BO(1)\cong\mathbb{R}P^\infty$, and γ_1 is the tautological line bundle over $\mathbb{R}P^\infty$.)

$$\begin{tikzcd} \varepsilon^1 \arrow[r] \arrow[d] & \gamma^1 \arrow[d] \\ * \arrow[r] & BO(1)\cong\mathbb{R}P^\infty \end{tikzcd}$$

Then applying the Thom space construction to the map $\varepsilon^1\to\gamma^1$ gives a map $S^1\to MO(1)$; this is the unit $\iota_1\colon S^1\to MO(1)$.

(d) Show that these maps make MO into a commutative ring orthogonal spectrum.

SOLUTION: We want to show that, equipped with these unit maps and the multiplication maps from part (b), that MO becomes a commutative orthogonal ring spectrum.

First, let's check the unit conditions: note that the maps

$$\mu_{0,n}\colon MO(0) \wedge MO(n) \to MO(n)$$

are just identities, since $MO(0) \cong S^0$. Therefore, the diagrams

$$\begin{tikzcd} MO(n) \cong S^0 \wedge MO(n) \arrow[r, "\iota_0 \wedge \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & MO(0) \wedge MO(n) \arrow[d, "\mu_{0,n}"] \\ & MO(n) \end{tikzcd} \qquad \begin{tikzcd} MO(0) \wedge MO(n) \arrow[d, "\mu_{n,0}"] & MO(n) \wedge S^0 \cong MO(n) \arrow[l, "\mathrm{id} \wedge \iota_0"'] \arrow[dl, "\mathrm{id}"] \\ MO(n) \end{tikzcd}$$

both commute, and the spectrum is unital.

Next, we check associativity. Consider the diagrams

$$\begin{tikzcd} \gamma^n \oplus \gamma^m \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^{n+m} \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^{n+m+k} \arrow[d] \\ BO(n) \times BO(m) \times BO(k) \arrow[r] & BO(n+m) \times BO(k) \arrow[r] & BO(n+m+k) \end{tikzcd}$$

$$\begin{tikzcd} \gamma^n \oplus \gamma^m \oplus \gamma^k \arrow[r] \arrow[d] & \gamma^{n} \oplus \gamma^{m+k} \arrow[r] \arrow[d] & \gamma^{n+m+k} \arrow[d] \\ BO(n) \times BO(m) \times BO(k) \arrow[r] & BO(n) \times BO(m+k) \arrow[r] & BO(n+m+k) \end{tikzcd}$$

In the above diagrams, all squares and outer rectangles are pullbacks, the right squares and the outer rectangles by the universal property of $BO(\ell)$, and the left squares by the pasting lemma. After applying the functor Th to the two top composites, we get either $\mu_{n+m,k} \circ \mu_{n,m} \circ \mathrm{id}$ or $\mu_{n+m+k} \circ \mathrm{id} \wedge \mu_{m,k}$. To show that these are equal, it suffices to show that the two composites along the bottom of the rectangles are the same, whence it follows that the composites along the top are the same because the rectangles are pullbacks.

So we want to show that the following diagram commutes:

$$\begin{tikzcd} BO(n) \times BO(m) \times BO(k) \arrow[r] \arrow[d] & BO(n+m) \times BO(k) \arrow[d] \\ BO(n) \times BO(m+k) \arrow[r] & BO(n+m+k) \end{tikzcd}$$

Indeed, appropriately choosing isomorphism $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$, one sees that both directions around this square send a triple of planes (U, V, W) in $\mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty$ to the same $(n+m+k)$-plane $U \oplus V \oplus W$ in $\mathbb{R}^\infty$. So this diagram commutes, and MO is associative.

To show centrality, let $\chi_{n,m} \in O(n+m)$ be the permutation matrix corresponding to the permutation which shuffles the first n elements of a vector $v \in \mathbb{R}^{n+m}$ to the end. Applying the functor B, we obtain maps $\chi_{n,m}\colon BO(n+m) \to BO(n+m)$. Assume that we have chosen isomorphisms $\mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ such that the following diagram commutes for all n and m:

$$\begin{tikzcd} BO(n) \times BO(m) \arrow[r] \arrow[d, "\tau"] & BO(n+m) \arrow[d, "\chi_{n,m}"] \\ BO(m) \times BO(n) \arrow[r] & BO(n+m), \end{tikzcd}$$

where τ is the symmetry isomorphism. Then we argue similarly to associativity. We have the following diagrams, where every square and every rectangle is a pullback:

$$\begin{tikzcd}
\varepsilon^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^{1+n} \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\
* \times BO(n) \arrow[r] & BO(1) \times BO(n) \arrow[r] & BO(1+n) \arrow[r, "\chi_{1,n}"] & BO(n+1)
\end{tikzcd}$$

$$\begin{tikzcd}
\varepsilon^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^n \oplus \varepsilon^1 \arrow[r] \arrow[d] & \gamma^n \oplus \gamma^1 \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\
* \times BO(n) \arrow[r, "\tau"] & BO(n) \times * \arrow[r] & BO(n) \times BO(1) \arrow[r] & BO(n+1)
\end{tikzcd}$$

As before, if the bottom rows compose to the same morphism, then applying the functor Th to the two rows yields the centrality diagram. Again, it's easy to see that the bottom rows are the same morphism, so MO satisfies the centrality condition.

Commutativity is similar.

References

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.

[Sch23] Stefan Schwede. Lectures on equivariant stable homotopy theory. http://www.math.uni-bonn.de/people/schwede/equivariant.pdf, 2023.