SEQUENTIAL SPECTRA NAME: ____________________

Due at the beginning of class on 4 March 2025

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 2.1, 2.2, and 2.3].

(1) Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ be a sequence of spectra. Show that the homotopy colimit [Mal23, Definition 2.3.17] of this sequence commutes with stable homotopy groups, in the sense that

$$\pi_k\left(\operatorname{hocolim}_n X_n\right) = \operatorname*{colim}_n \pi_k(X_n).$$

(2) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

(a) Prove that $H(\mathbb{Z}/p)$ is the cofiber of the map obtained by applying the functor H to $p\colon \mathbb{Z} \to \mathbb{Z}$.

(b) The *rationalization* $\mathbb{S}_{\mathbb{Q}}$ of the sphere spectrum $\mathbb{S}$ is the homotopy colimit of the diagram

$$\mathbb{S} \xrightarrow{1} \mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{3} \mathbb{S} \xrightarrow{4} \mathbb{S} \xrightarrow{5} \mathbb{S} \to \cdots$$

Prove that $\mathbb{S}_{\mathbb{Q}}$ is stably equivalent to $H\mathbb{Q}$.

(3) Let $\mathrm{ev}_0\colon \mathcal{S}p \to \mathcal{T}op_*$ be the functor that evaluates a spectrum at its zeroth space: $\mathrm{ev}_0 X = X_0$.

(a) Prove that Σ^∞ is left adjoint to ev_0.

(b) For any spectrum X, let $\Omega^\infty X = \mathrm{ev}_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Use [Rie14, Exercise 2.2.15] to prove that there is an adjunction

$$\Sigma^\infty\colon \operatorname{ho}(\mathcal{T}op_*) \leftrightarrows \operatorname{ho}(\mathcal{S}p)\colon \Omega^\infty.$$

REFERENCES

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.