Generalized Cohomology Talk Outline

For the Brown Representability Theorem and generalized cohomology theories, see [Hat02, Sections 4.3 and 4.E]. For K-theory, [Hat17, Chapter 2] is a good reference, or [May99, Chapter 24]. I also like [Zak17, Section 12], because this is where I learned this stuff.

Outline

1. Generalized Cohomology Theories

   (a) Define a generalized cohomology theory following [Hat02, Page 448]: functors $h^n : \mathcal{T}op_* \to \mathcal{A}b$ with natural isomorphisms $h^n \to h^{n+1}\Sigma$ satisfying homotopy, exactness, and additivity axioms. (Hatcher uses pointed CW complexes, but we can just use all pointed spaces. The difference between pointed and unpointed stuff is in [Hat02, Theorem 4.59], if you're curious.)

   (b) Construct the cofibration sequence $A \to X \to Q \to \Sigma A \to \Sigma X \to \ldots$ (where Q is the cofiber of $A \to X$) using a problem from the second problem set. This is known as a *Puppe Sequence*. Show that this yields a long exact sequence in cohomology for any generalized cohomology theory, using the exactness axiom.

   (c) Give the example of reduced cohomology with coefficients in an abelian group A: $h^i = \widetilde{H}^i(-; A)$.

   (d) Another important example of a generalized cohomology theory is K-theory, which we will go into some detail on.

2. K-theory

   (a) Let X be a pointed space. We define $K^0(X)$ to be the free abelian group on isomorphism classes of complex vector bundles over X, modulo the relation that $[E] + [E'] = [E \oplus E']$.

   (b) An example: when $X = *$ is a point, then $K^0(*) \cong \mathbb{Z}$. Vector bundles on a point are just vector spaces, and the only information retained by the isomorphism classes is dimension.

   (c) Another (nontrivial) example: $K^0(S^1) \cong \mathbb{Z}$.

   (d) It's not obvious, but it's a fact that $K^0(X)$ depends on the space X only up to homotopy. If Y is another space such that $Y \simeq X$, then $K^0(Y) \cong K^0(X)$.

   (e) If we want to define a generalized cohomology theory, we have to satisfy the wedge axiom. Note that the wedge axiom implies that $h^i(*) = 0$ for all i, including $i = 0$. See [Hat02, Item (1) at bottom of page 449]. But $K^0(*) = \mathbb{Z}$.

   (f) To fix this, we define *reduced* K-*theory* as $\widetilde{K}^0(X) = \ker(\dim)$, where $\dim : K^0(X) \to \mathbb{Z}$ is the homomorphism given by $[E] \mapsto \dim(E)$. That is, it takes the dimension of a vector bundle. Then $\widetilde{K}^0(*) = 0$.

   (g) This group $\widetilde{K}^0(X)$ has an interpretation in terms of vector bundles too: it is the quotient of $K^0(X)$ by the relation that $[E] = [E']$ if there are trivial vector bundles ε^i and ε^j of dimension i and j such that $E \oplus \varepsilon^i \cong E' \oplus \varepsilon^j$, for some $i, j \in \mathbb{N}$. Elements of this group are called *stable equivalence classes of complex vector bundles on* X.

   (h) If we define $\widetilde{K}^i(X) = \widetilde{K}^0(\Sigma X)$, then this defines a generalized cohomology theory.

3. Generalized Cohomology Theories and Infinite Loopspaces

   (a) An *infinite loopspace* is a space X such that there are spaces X_i for $i \in \mathbb{N}$ and homotopy equivalences $X \simeq \Omega^i X_i$ (we say that $X = X_0$).

(b) Mention that we also call such a sequence of spaces an Ω-*spectrum*.

(c) An example of an infinite loopspace is an Eilenberg–MacLane space. Recall the definition of an Eilenberg–MacLane space $K(G, n)$. It is characterized by the property that $\pi_i K(A, n) = 0$ for $i \neq n$ and $\pi_n K(A, n) = A$. This implies that $\Omega K(A, n+1) \simeq K(A, n)$, so each $K(A, n)$ is an infinite loopspace.

(d) Prove that if you have an infinite loopspace X, then $h^i = [-, X_n]$ is a generalized cohomology theory. This is [Hat02, Theorem 4.58], but taking the definition of a generalized cohomology theory from [Hat02, Page 448] makes the proof even easier.

4. Brown Representability Theorem

(a) State the Brown Representability Theorem [Hat02, Theorem 4E.1]. Do not prove it.

(b) State the theorem that $\widetilde{H}^n(X; A) = [X, K(A, n)]$, but do not prove it.

(c) K-theory is also represented by an infinite loopspace. In this case, the infinite loopspace is the loopspace of BU, the classifying space of the infinite unitary group U.

(d) The infinite unitary group is $U = \operatorname{colim}_n U_n$. This is the union of all of the unitary groups U_n, including $U_n \hookrightarrow U_{n+1}$ by adding a row and a column to an $n \times n$ matrix and putting a 1 in the $(n+1, n+1)$ entry and zeros elsewhere.

(e) This infinite unitary group is, surprisingly, an infinite loopspace. This is a theorem known as *Bott Periodicity*: $U \simeq \Omega^2 U$. The spaces in between are $\Omega U \simeq \mathbb{Z} \times BU$. So we have a sequence of spaces

$$\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$$

References

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[Hat17] Allen Hatcher. Vector bundles and k-theory. `https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html`, 2017.

[May99] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

[Zak17] Inna Zakharevich. Math 6530: K-theory and characteristic classes lecture notes. `https://pi.math.cornell.edu/~zakh/6530/`, 2017.