$$
\begin{tikzcd}
 & \mathrm{Alg} \arrow[dr, dash, bend left] & \\
\mathrm{Cat} \arrow[ur, dash, bend left] \arrow[dr, bend right] & & \mathrm{Geo} \arrow[ll, Rightarrow, bend right=40] \arrow[ll, Rightarrow] \arrow[ll, Rightarrow, bend left=40] \arrow[dl, bend left] \\
 & \mathrm{Phys} &
\end{tikzcd}
$$

MATH-PHYS-CAT

INTRODUCTION TO GAUGE THEORY

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PREFACE

Various of the studies of our research group, Math-Phys-Cat, involves the notion of *Gauge Theories*, and so, this is the result of me learning the basics of such theories.

These notes are the product of a seminar presented by me at the Federal University of Minas Gerais (UFMG), during the first semester of 2020 under the supervision of Professor Rodney Josué Biezuner and Yuri Ximenes Martins . The goal of this seminar was to learn the basics of gauge theory in a mathematical framework. That being said, we study mainly principal fiber bundles and the theory of their connections, and by the end we give a little glimpse into the world of Field Theories.

This material only assumes from the reader basic notions of differential geometry, such as smooth manifolds, metric tensors, differential forms and integration on manifolds. One thing to note is that thanks to the mathematical machinery we will develop, we can present several results — such as the Structural Equation and Bianchi's Identity— in a coordinate free manner, giving the results much more clarity and insight, which often times is lost in the chaos of working locally with coordinates and indices, thus this material could be of great interest to physicists as well.

The main text used in this seminar was the book "Gauge Theory and Variational Principles" by David Bleecker ([Ble05]), which gave us a direction to follow, although here we give place to the study of associated bundles.

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1

VECTOR BUNDLES

This chapter is dedicated to the study of vector bundles. Although it is very interesting, this is probably the most "skipable" chapter, since it develops notions which will not be required for the rest of text. The goal of what follows is to define algebra valued forms in a very general sense, which can in turn be used to study the notion of algebra valued gauge theories, a generalization of traditional gauge theory, where our forms take place in an lie algebra $\mathfrak{g}$. If you are interested, you can take a look at [MB19a] and [MB19b] .

1.1 VECTOR BUNDLES

In what follows, the theory of vector bundles will be developed. Essentially, a vector bundle is a collection of vector spaces indexed in a nice enough way (in our case in a smooth way). As is common in mathematics, we shall see that there is a strong relation between our geometry and some algebraic structure, namely vector bundles and projective modules. This relation takes the form of a functor, which will be studied in detail, so, without further a do, lets define vector bundles.

Definition 1.1. A vector bundle of rank k consists of two topological spaces E and M together with a continuous surjection $\pi : E \longrightarrow M$ such that:

(i) For each $p \in M$ the subset $E_p = \pi^{-1}(p)$ has the structure of k-dimensional real vector space.

(ii) For each $p \in M$ there exists an open neighborhood U of p, and a homeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

(iii) For each $p \in M$ the restriction of ϕ to E_p is a linear transformation to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$. $\square$

Remark 1.2. The homeomorphisms are called local trivializations of the bundle and the vector spaces E_p its fibers. The space E is called total space, M the base space and the surjection is called the projection. We write $E \to M$ to say that E is a vector bundle over M.

Definition 1.3. A smooth vector bundle is a vector bundle in which E and M are smooth manifolds, the projection is differentiable and the local trivializations are diffeomorphisms. $\square$

When M itself, the base space, is associated with a local trivialization we say that the fiber bundle admits a global trivialization, and the vector bundle is called trivial.

Exemplo 1.4. One of the most important examples of vector bundles is the tangent TM bundle of a manifold M. It is easy to check that it is indeed a vector bundle.

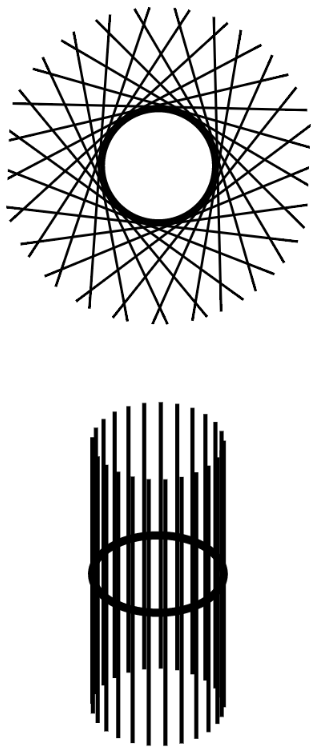

Figure 1.1: The tangent bundle of the circle S^1

We will soon see that the collection of vector bundles over a given space forms a category, but first we must define what would be the morphisms between them.

Definition 1.5. Let $\pi : E \to M$, $\pi' : E' \to M'$ be vector bundles, a morphism between them is a pair (F, f) of continuous (differentiable) functions $F : E \longrightarrow E'$, $f : M \longrightarrow M'$ such that:

(i) The following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

(ii) The restriction of F to a fiber E_p is a linear transformation to $E_{f(p)}$. $\square$

Remark 1.6. In the special case where $M = M'$, we define a morphism from E to E' over M to be a bundle morphism with $f = \mathrm{id}_M$. Thus we condition (i) of the definition above translates to

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

It is easy to see that the composition of bundle morphisms is a bundle morphism, also, since it is the composition of functions, it is associative. The identity function is a bundle morphism. Therefore, we may define the following:

Definition 1.7. Let M be a smooth manifold, we define $\mathbf{Vec}_M$ to be the category whose objects are smooth vector bundles over M and morphisms are bundle morphisms. □

In what follows, it will be shown that this category has some things in common with the category of finite dimensional real vector spaces $\mathbf{Vect}_{\mathbb{R}}$.

Proposition 1.8. *Let $M_p = \{p\}$ be a manifold with a single point, then $\mathbf{Vec}_{M_p} \cong \mathbf{Vect}_{\mathbb{R}}$.*

Proof: First we define two functors $F : \mathbf{Vec}_{M_p} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ and $G : \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vec}_{M_p}$ which will be done in steps as follows:

(i) For E a bundle over M_p, $E = \pi^{-1}(p)$, so it's a vector space. So we can set $F(E) = E$.

(ii) Since is consists M_p consists of a single point, a bundle morphism is just a linear transformation f, so we let $F(f) = f$.

(iii) If V is a vector space, we let $\pi : V \longrightarrow M_p$ be the constant map (in fact the only possible map). So we may define $G(V) = V \to M_p$.

(iv) A linear transformation g between vector spaces is a morphism of vector bundles, so $G(g) = g$.

Now it is easy to see that the compositions of these functors are the identity functors in their respective category. ■

This is a simple, but nice correspondence. Actually, there are less trivial similarities between these categories for arbitrary manifolds, for instance, it is possible to define the tensor product of two vector bundles, as well as their direct sum, as we shall see. Their construction is quite similar, so first we enunciate a helpful lemma about the construction of smooth vector bundles.

Lema 1.9. *Let M be a smooth manifold, and suppose that we are given*

- *for each $p \in M$ a vector space E_p of a fixed dimension k. We may assume that they are disjoint for each p. Let $E = \bigsqcup_{p \in M} E_p$ and let $\pi(x) = p$ if $x \in E_p$;*
- *an open cover $\{U_\alpha\}_{\alpha \in A}$ of M;*
- *for each $\alpha \in A$ a bijective map $\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$ whose restriction to E_p is linear;*
- *for each α, β such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ a smooth map $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL(k, \mathbb{R})$ such that*
$$\phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p)v).$$

Then E has a smooth structure such that π is differentible and E is a smooth vector bundle over M with the trivializations ϕ_α.

Proof: The proof will be given ahead, in the more general setting of G-bundles. ■

The utility of this lemma is that it captures the idea of making a new bundle out of sufficient data. This occurs quite frequently, the situation where we are given a bunch of vector spaces indexed by a manifold and a way to “glue" them, so it's worthwhile seeing the overview of the process. The proof of this lemma is quite straightforward, but a little tiresome. The important part is to observe that the transition functions $g_{\alpha\beta}$ make the compositions of the trivializations smooth, so we may compose them with the charts of the manifold M, getting an atlas for E. With this lemma at our disposal we may prove the following:

Proposition 1.10. *Given two smooth vector bundles E and E' over M of rank kWhen M itself, the base space, is associated with a local trivialization we say that the fiber bundles admits a global trivialization, and the vector bundle is called trivial. and l respectively, the following are also smooth vector bundles over M:*

(i) $E \otimes_M E'$, *where a fiber at p is the tensor product $E_p \otimes_{\mathbb{R}} E'_p$;*

(ii) $E \oplus E'$, *where a fiber at p is the direct sum $E_p \oplus E'_p$;*

(iii) $Hom_M(E, E')$, *where a fiber at p is the vector space $Hom(E_p, E'_p)$.*

Proof: We will use the construction lemma.
(i) We may assume that the local trivializations of E, E' have the same open domain (otherwise we take their intersections), it's clear that they cover M . Let ϕ^1, ϕ^2 be trivializations of E, E' over U. So we define a trivialization of $E \otimes_M E'$ over U being:

$$\phi^1 \otimes \phi^2 : \pi^{-1}(U) \longrightarrow U \times (\mathbb{R}^k \otimes \mathbb{R}^l) \cong U \times \mathbb{R}^{kl}$$

$$\text{for } x \in E_p \otimes_{\mathbb{R}} E'_p, \quad x \mapsto (p, [\phi^1_2 \otimes \phi^2_2](p)x)$$

Here, the lower index represents the coordinate of the trivialization, of course their first coordinates are equal. Observe that the tensor product of linear transformations $\phi^1_2 \otimes \phi^2_2$ is well defined because the restriction of trivializations to fibers is linear. With fixed bases, the matrix of $[\phi^1_2 \otimes \phi^2_2](p)$ is the Kronecker product of the original matrices, thus its entries are the product of different entries of the original matrices, which were differentiable functions, so this defined transformation varies smoothly with p, and all the requisites of the construction lemma are attained. The proofs of (ii) and (iii) are analogous. ■

Now we focus our attention on other concepts, such as smooth sections of a vector bundle, which will be defined shortly, but first, a preliminary on rings and modules.

Definition 1.11. A ring is a set R together with two binary operations called sum $+ : R \times R \longrightarrow R$, and multiplication $\cdot : R \times R \longrightarrow R$, such that for all $a, b, c, r \in R$;

(i) R is an abelian group under sum;

(ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(iii) $r \cdot (a + b) = r \cdot a + r \cdot b$;

(iv) $(a + b) \cdot r = a \cdot r + b \cdot r$.

Furthermore , if there exists $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ for all all $a \in R$, we say that R is a ring with identity, and if the multiplication is commutative, we say that the R is a commutative ring. □

From now on we will omit the $\cdot$ in multiplication so $a \cdot b = ab$. The only use of this symbol is to indicate the binary operation.

Definition 1.12. Let R and S be rings, a function $f : R \longrightarrow S$ is said to be a ring homomorphism if

$$f(a+b) = f(a) + f(b),\ f(ab) = f(a)f(b) \text{ for all } a, b \in R.$$

It is easy to see that the composition of ring homomorphisms is a homomorphism. □

Definition 1.13. The collection of rings together with ring homomorphisms forms a category, the category **Rng** of rings.

Definition 1.14. And let R be a ring. An (left) R-module is an abelian group A together with a function $R \times A \to A$, such that for all $r, s \in R$ $a, b \in A$:

(i) $r(a+b) = ra + rb$.

(ii) $(r+s)a = ra + sa$.

(iii) $r(sa) = (rs)a$.

Furthermore, if R has identity 1 and $1a = a$ for all $a \in A$, A is said to be an unitary R-module. □

Definition 1.15. Let A, B be R-modules, a function $f : A \longrightarrow B$ is said to be a R-homomorphism between modules if

$$f(a+b) = f(a) + f(b),\ f(ra) = rf(a) \text{ for all } r \in R,\ a, b \in A.$$

Again, the composition of R-homomorphisms is a R-homomorphism. □

Definition 1.16. Let R be a ring. Then, the collection of R-modules together with R-homomorphisms forms a category, the category $\mathbf{Mod}_R$ of R-modules. □

Definition 1.17. A submodule of an R-module A is a subgroup $B \leq A$ such that $rb \in B$ $\forall b \in B$. It's easy to see that it is also an R-module. For a subset $X \subset A$, we define the R-module generated by X as the intersection of all submodules containing X. If there is a finite subset $X \subset A$ such that the module generated by X is A, we say that A is finitely generated. If A is an unitary module, it is immediate that the submodule generated by X is of the form $\{\sum_{i=1}^{n} r_i x_i \mid n \in \mathbb{N}, x_i \in X\}$ □

Definition 1.18. We say that an unitary R-module A is free if there exists a generating set X of A, such that

$$\text{for all } n \in \mathbb{N}\ r_1 a_1 + \ldots r_n a_n = 0 \implies r_i = 0 \text{ for all } a_i \in A,\ r_i \in R.$$

In this case we say that X is a basis for A. Whenever a module is finitely generated and free, it admits a finite basis. □

As with other algebraic structures, given two R-modules A, B we may form a new module $A \oplus B$ by acting with R in each entry of the cartesian product $A \times B$. A quick check will show you that it is indeed a module.

Definition 1.19. An unitary R-module A is said to be projective if it there exists a free R-module B such that $B = A' \oplus C$, with A isomorphic to A'. □

There are in fact a handful of (equivalent) ways of defining a projective module, but given our interests, this is the most practical one. Now we go back to vector bundles. Below is the definition of a section, one can readily see that is a generalization of vector fields in smooth manifolds.

Definition 1.20. Let $E \to M$ be a smooth vector bundles. A smooth section is a differentiable function $s : M \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_M$, that is to say that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[dr, "\mathrm{id}_M"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

□

Since each fiber is a vector space, it's easy to see that the sum and multiplication by a real scalar are well defined, and that such operations are closed, that is, the sum of sections is a section and a section times a real number is a section. So the set of sections forms a real vector space. We denote by $\Gamma_M(E)$ the space of sections of a vector bundle E over M. As we shall see, $\Gamma_M(E)$ can also be regarded as a module over $C^\infty(M)$, the ring of smooth real valued functions in M.

Proposition 1.21. *Let $E \to M$ be a smooth vector bundle, then $\Gamma_M(E)$ is a $C^\infty(M)$-module with the operation of multiplication by scalar defined as $[fs](p) = f(p)s(p)$ for all $f \in C^\infty(M), s \in \Gamma_M(E)$.*

Proof: The composition $\pi \circ fs$ is clearly the identity in M. So all we need to check is differentiability. To see this, let $f \in C^\infty(M)$ be a smooth function, so we may define a function $\tilde{f} : E \longrightarrow E$, $\tilde{f}(x) = f(p)x$ if x is in the fiber of p, which is differentiable (just use the chart induced by the local trivializations to check). So, $fs = \tilde{f} \circ s$, so it's differentiable, and the result follows. ■

We saw sections that are defined in M, those are called global sections, in contrast with local sections defined below.

Definition 1.22. Let U be an open subset of M. A local section is a differentiable function $s : U \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_U$. □

Lema 1.23. *Let s be a smooth section defined on a closed subset A of M. Then, for any open subset U containing A, s extends to a global section $\tilde{s} : M \longrightarrow E$ with support supp $\tilde{s} = \overline{\{p \mid \tilde{s}(p) \neq 0 \in E_p\}}$ contained in U.*

Proof: The result follows from the extension lemma for smooth functions locally defined, which in turn follows from the existence of partitions of the unity. ■

Sections are a way of associating to each point of a manifold a vector in its fiber. That being said, we can try to smoothly give a basis to each fiber. This is the notion of frames!

Definition 1.24. A local smooth frame in an open subset U of a vector bundle $E \to M$ of rank k is a tuple $(\sigma_1, \ldots, \sigma_k)$ of k smooth sections such that for each $p \in M$ the set $\{\sigma_1(p), \ldots, \sigma_k(p)\}$ is a basis for E_p. A global frame is a local frame defined in all of M. □

Now we prove a theorem that makes a connection between local frames and local trivializations, actually, they are essentially the same thing.

Theorem 1.25. *Every smooth local frame for a smooth vector bundle is associated with a local trivialization and every local trivialization is associated with a smooth local frame.*

Proof: First, let ϕ be a local trivialization in U. Let $\{e_i\}$ be the canonical base in $\mathbb{R}^k$ and $\tilde{e}_i : U \longrightarrow U \times \mathbb{R}^k$ be the smooth section defined by $\tilde{e}_i(p) = (p, e_i)$, so we have the following diagram:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"'] \\
& U \arrow[ul, dotted, bend left, "\sigma_i"] \arrow[ur, bend right, "\tilde{e}_i"'] &
\end{tikzcd}$$

it is easy to see that $\sigma_i = \phi^{-1} \circ \tilde{e}_i$ is a smooth frame.

Now, let σ_i be a local frame in U. We define a map $\psi : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$ by

$$\psi(p, (v_1, \dots, v_k)) = \sum v_i \sigma_i(p)$$

siince σ_i is a local frame, this map is bijective. It remains to show that it is a diffeomorphism. It suffices to show that it is a local diffeomorphism. Given a point $p \in U$ there exists a local trivialization ϕ in $V \ni p$, we may assume that $V \subset U$. So it suffices to show that $\phi \circ \psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism (since ϕ is a diffeomorphism).

For each σ_i, the composite map $\phi \circ \sigma_i|_V : V \to V \times \mathbb{R}^k$ is of the form

$$\phi \circ \sigma_i|_V(p) = (p, (\sigma_i^1(p), \dots, \sigma_i^k(p)))$$

where σ_i^k is a smooth functions, so

$$\Phi \circ \Psi\left(p, (v_1, \dots, v_k)\right) = \left(p, \left(v_i\sigma_i^1(p), \dots, v_i\sigma_i^k(p)\right)\right)$$

and so it's smooth. It remains to show that its inverse is smooth. But this follows from the fact that the matrix $\sigma_i^j(p)$ has smooth entries and so, since matrix inversion is smooth, the inverse matrix has smooth entries, and with it we may express the inverse of the map. ∎

Remark 1.26. Note that the same proof is valid for topological vector bundles, just change the words *diffeomorphism* to *homeorphism* and *smooth* to *continuous*.

Corollary 1.27. *A vector bundle is trivial if and only if it admits a global frame.*

From here until the end of this section, we explore how vector bundles and $C^\infty(M)$ modules interact. First let's look at their morphisms.

Definition 1.28. Let $E, E' \to M$ be smooth vector bundles, and $F : E \longrightarrow E'$ a bundle morphism, we define the map $F : \Gamma_M(E) \longrightarrow \Gamma_M(E')$ as

$$F(\sigma) = F \circ \sigma.$$

Proposition 1.29. *Let $F : E \longrightarrow E'$, then $F : \Gamma_M(E) \longrightarrow \Gamma_M(E')$ is well defined and is a $C^\infty(M)$-homomorphism of modules.*

Proof: This follows from the fact that $\pi' \circ F = \pi$ and that the restriction of F to the fibers is linear (because F is a bundle morphism). ∎

So, with this last proposition we can see that the notion of space of section gives rise to a functor.

Proposition 1.30. *Let $\Gamma_M(-)$ be the rule that assigns to each $E \in \mathbf{Vec}_M$ its section space $\Gamma_M(E)$ and to each $F : E \longrightarrow E'$ bundle morphism its induced map $\Gamma_M(F) = F : \Gamma_M(E) \longrightarrow \Gamma_M(E')$, then $\Gamma_M(-)$ is a functor between the catgories $\mathbf{Vec}_M$ and $\mathbf{Mod}_{C^\infty(M)}$.*

Proof: It follows the proposition above. It's easy to see that the identity bundle morphism is taken to the identity module homomorphism and that the composition behaves well under this rule. ■

We write $\Gamma_M(-) : \mathbf{Vec}_M \longrightarrow \mathbf{Mod}_{C^\infty(M)}$ to express this functor. In the following proposition we prove that this functor has some nice properties, in particular that every $F : \Gamma_M(E) \longrightarrow \Gamma_M(E')$ module homomorphism is induced by a bundle map.

Lema 1.31. *Let $E, E' \to M$ be vector bundles, and let $\mathcal{F} : \Gamma_M(E) \longrightarrow \Gamma_M(E')$ be a module homomorphism. Then there exists a bundle morphism $F : E \longrightarrow E'$ such that $\Gamma_M(F) = \mathcal{F}$.*

Proof: For any $v \in E_p$, there exists a smooth section $\sigma_{(p,v)}$ such that $\sigma_{(p,v)}(p) = v$ by the extension Lemma 1.23, so, we may define a bundle morphism F by

$$F(v) = \mathcal{F}(\sigma_{(p,v)})(p)$$

this map is readily seen to be differentiable and a bundle morphism. Also, by its definition it induces the module homomorphism $\mathcal{F}$. ■

Lema 1.32. *Let $F, G : E \longrightarrow E'$ be bundle morphisms, if $\Gamma_M(F) = \Gamma_M(G)$, then $F = G$.*

Proof: From the extension lemma of sections (Lemma 1.23), for every $x \in E_p$ there exists a section σ such that $\sigma(p) = x$, so $F(\sigma(p)) = G(\sigma(p)) = F(x) = G(x)$ for every $x \in E$. ■

Theorem 1.33. *The functor $\Gamma_M(-) : \mathbf{Vec}_M \longrightarrow \mathbf{Mod}_{C^\infty(M)}$ is a fully faithful functor.*

Proof: Injectivity follows from Lemma 1.32, and surjectivity from Lemma 1.31. ■

We dedicate the end of this section to exploring this functor and seeing some of its consequences. The following proposition is a purely categorical one.

Proposition 1.34. *A fully faithful functor $F : C \longrightarrow D$ is essentially injective on objects i.e. $F(x) \cong F(y) \implies x \cong y$.*

Proof: The following morphisms that make these diagrams commute exist because the functor is full.

$$\begin{tikzcd} & F(y) \arrow[dr, "f^{-1}"] & \\ F(x) \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_{F(x)}"'] & & F(x) \end{tikzcd} \qquad \begin{tikzcd} & F(y) \arrow[dr, "Fh"] & \\ F(x) \arrow[ur, "Fg"] \arrow[rr, "\mathrm{id}_{F(x)}"'] & & F(x) \end{tikzcd}$$

So $F(g)F(h) = F(gh) = \mathrm{id}_{F(x)}$, so $gh = \mathrm{id}_x$. The rest of the proof is analogous. ■

Proposition 1.35. *Let E, E' be vector bundles over M, then they are isomorphic over M if and on if their section spaces are isomorphic as $C^\infty(M)$-modules.*

Proof: If they are isomorphic as bundles, the induced map is a module isomorphism. Reciprocally, the section functor is full and faithul, so by Proposition 1.34, their respective bundles must be isomorphic. ■

Proposition 1.36. *The following statements hold:*

(i) $\Gamma_M(E \otimes_M E') \cong \Gamma_M(E) \otimes_{C^\infty(M)} \Gamma_M(E')$;

(ii) $\Gamma_M(E \oplus E') \cong \Gamma_M(E) \oplus \Gamma_M(E')$;

(iii) $\mathrm{Hom}_M(E; E') \cong \mathrm{Hom}_{C^\infty(M)}\left(\Gamma_M(E); \Gamma_M(E')\right)$.

Furthermore, all of these isomorphisms are natural.

Proof: All of the results are proven analogously, for simplicity we prove (ii). All we need to do is define a natural isomorphism.

A section $\sigma : M \longrightarrow E \oplus E'$ is the direct sum is of the form $\sigma(p) = (v, v')$, where $v \in E_p$, $v' \in E'_p$, and of course this assignment must me differentiable in each coordinate, so each entry is a smooth $\sigma = (\sigma_1, \sigma_2)$, where σ_1 is a section of E and σ_2 is a section of E'. Thus, we may define the morphism $\sigma \mapsto \sigma_1 \oplus \sigma_2$. This is easily checked to be an isomorphism. ■

We conclude that the section functor preserves additional structure: the tensor product. Therefore it is a monoidal functor.

We can prove more! The proof will be omitted, but the following theorem holds.

Theorem 1.37 (Serre-Sawn). *Let M be a connected smooth manifold, and let P be a $C^\infty(M)$-module. Then P is isomorphic to some section module $\Gamma_M(E)$ if and only if P is finitely generated and projective.*

Proof: See [Nes06], Theorem 11.32 p.181. ■

This makes the section functor even more useful. To illustrate, we prove a very nifty fact.

Theorem 1.38. *Let M be a connected manifold. Then any vector bundle E over M is the direct summand of some trivial vector bundle.*

Proof: Let $\Gamma_M(E) = P$, by the preceding theorem P is projective, therefore is the direct summand of some $P \oplus Q$ finitely generated free module. Q is also a projective finitely generated module. Therefore there is a vector bundle E' such that $\Gamma_M(E') = Q$ (or isomorphic to Q), then $\Gamma_M(E \oplus E') \cong P \oplus Q$, which is free. It's easy to see that a bundle is trivial if and only if their section space is a free module (take a global frame), so we conclude that $E \oplus E'$ is trivial. ■

Now we prove one last result, again a purely categorical one, to finish this section by showing that the section functor defines an equivalence of categories.

Theorem 1.39. *Every fully faithful essentially surjective functor establishes an equivalence of categories.*

Proof: Let $F : C \longrightarrow D$ be a fully faithful essentially surjective functor. For $d \in D$ chose Gd (repair the suggestive notation!) such that $FGd \cong d$ and an isomorphism $\epsilon_d : FGd \cong d$. For each $l : d \longrightarrow d'$ it can be shown that there exists an unique morphism making the following diagram commute

$$\begin{tikzcd} FGd \arrow[r, "\epsilon_d"] \arrow[d, dotted] & d \arrow[d, "l"] \\ FGd' \arrow[r, "\epsilon_{d'}"'] & d' \end{tikzcd}$$

this unique morphism id defined to be Gl. This is setup so that each isomorphism ϵ_d becomes the components of a natural isomorphism $\epsilon : FG \Rightarrow \mathrm{id}_D$. It is readily checked that the assignment $l \mapsto Gl$ is functorial. Now it remains to define the natural isomorphism $\eta : 1_C \Rightarrow GF$. We may define the components of η by specifying morphisms $F\eta_c : Fc \longrightarrow FGFc$

because F is full and faithful. So we define $F\eta_c$ to be ϵ_{Fc}^{-1}, for any $f : c \longrightarrow c'$ we have that the outer square of

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "f"'] & FGFc \arrow[r, "\epsilon_{Fc}"] \arrow[d, "FGFf"'] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_c"'] & FGFc' \arrow[r, "\epsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

commutes. The right square commutes by naturality while the left square commutes because ϵ_{Fc} is an isomorphism. Therefore, the faithfulness of F implies that $\eta_{c'} \cdot f = GFf \cdot \eta_c$ and we conclude that η is a natural isomorphism. ■

Now, by theorems 1.38 and 1.39, we may conclude that if M is connected its vector bundle category is equivalent to the category of finitely generated projective $C^\infty(M)$-modules. For $\Gamma_M(E)$ to be a finitely generated projective $C^\infty(M)$-module it is not required that M be connected, so even if M is not connected, its vector bundle category is equivalent to a full subcategory of the category of finitely generated projective $C^\infty(M)$-modules (every functor is surjective in its image).

1.2 ALGEBRA BUNDLES

One might imagine if we can endow our fibers with an algebraic structure other than that of a vector space, so we explore the idea of R-algebra bundles. As one can guess, we will have that each fiber has the structure of an R-algebra. In what follows, R will always be a commutative ring with identity and module over R will be unitary and, furthermore, we require that ring homomorphisms preserve the identity. First we define an R-module bundle.

Definition 1.40. An R-module bundle consists of two smooth manifolds E and M together with a differentiable surjection $\pi : E \longrightarrow M$ and an R-module A with a differentiable structure such that:

(i) For each $p \in M$ the subset $E_p = \pi^{-1}(p)$ has the structure of an R-module.

(ii) For each $p \in M$ there exists an open neighborhood U of p, and a diffeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times A$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times A \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

(iii) For each $p \in M$ the restriction of ϕ to E_p is an R-homomorphism to $\{p\} \times A \cong A$. □

As said before, the definition is almost identical to that of a vector bundle. In fact, a vector bundle is just the case where $R = \mathbb{R}$. Before we move on, let's define what is an R-algebra.

Definition 1.41. An R-algebra is a R-module together with a bilinear multiplication and an identity with respect to it. Oftentimes the multiplication is associative, giving the module a ring structure. □

Definition 1.42. Let A, B be R-algebras, then a R-algebra morphism is a R-homomorphism $f : A \longrightarrow B$ that preserves the multiplication. □

Now we may define an R-algebra bundle. Repair that in the definition the only change is that the word “module" is replaced by “algebra".

Definition 1.43. An R-algebra bundle consists of two smooth manifolds E and M together with a differentiable surjection $\pi : E \longrightarrow M$ and an R-algebra A with a differentiable structure such that:

(i) For each $p \in M$ the subset $E_p = \pi^{-1}(p)$ has the structure of an R-algebra.

(ii) For each $p \in M$ there exists an open neighborhood U of p, and a diffeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times A$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times A \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

(iii) For each $p \in M$ the restriction of ϕ to E_p is an R-algebra homomorphism to $\{p\} \times A \cong A$. □

In what follows, we will replicate some things of the previous section, for instance, now we talk about sections in R-module bundles.

Definition 1.44. Let $E \to M$ be an R-module bundle, then a global section is a differentiable function $\sigma : M \longrightarrow E$ such that $\pi \circ \sigma = id_M$. □

This definition is the same as the one from vector bundles, and many more times throughout this text we will have this same definition for other types of bundle. That being said, from now on these definitions will no longer be explicitly said.

One thing to note is that until now we haven't talked about the relation between our algebraic and differentiable structures. To make sense of what follows we need an “agreement" between the two, so we will assume that the operations (multiplication in a module, summation, etc.) are differentiable. More precisely:

Definition 1.45. Let $E \to M$ be an R-module bundle (or R-algebra) with typical module A, then we say that the module(algebra) operations are differentiable if $+ : A \times A \longrightarrow A$ and $a \mapsto ra$ ($\cdot : A \times A \longrightarrow A$ for the algebra case) are smooth. □

Now we show that the section space $\Gamma_M(E)$ inherits the R-module structure from the fibers.

Proposition 1.46. *Let $E \to M$ be an R-module bundle, then $\Gamma_M(E)$ is an R-module with the oprations*

$$[\sigma + \beta](p) = \sigma(p) + \beta(p) \text{ and } [r\sigma](p) = r\sigma(p).$$

Proof: This from the fact that our typical fiber A has multiplication and addition compatible with the differentiable structure. ■

As you may have guessed,the same is true for R-algebra bundles, that is, their section space is a R-algebra.

Proposition 1.47. *Let $E \to M$ be an R-algebra bundle, then $\Gamma_M(E)$ is an R-module with the operations*

$$[\sigma + \beta](p) = \sigma(p) + \beta(p),\ [\sigma\beta](p) = \sigma(p)\beta(p) \text{ and } [r\sigma](p) = r\sigma(p).$$

Proof: Is the exact same proof given in section 1 and the previous proposition, this follows because the operations in our algebraic structure are differentiable. ■

One last remark to be made is regarding the case of $\mathbb{R}$-algebra bundles. Our typical fiber A will be an $\mathbb{R}$-vector space, which is also a smooth manifold. Thus, as a manifold, it has dimension $n \in \mathbb{N}$. What we'd like to show is that A is a finite dimensional vector space. Its tangent spaces have dimension n, now suppose that $\dim_{\mathbb{R}} A > n$. Thus we have at least $n+1$ LI vectors $x_1 \ldots x_{n+1}$. Since we require that our operations be differentiable, the curves $\alpha_i : \mathbb{R} \longrightarrow A$ defined by $\alpha_i(t) = p + tx_i$ for $p \in A$ are smooth, an easy calculation would show that the derivations $X_p^i : C^\infty(M) \longrightarrow \mathbb{R}$ defined by this curves are LI, which would implies $\dim_{\mathbb{R}} TM_p > n$.

1.3 E-VALUED DIFFERENTIAL FORMS

Now we will generalize the concept of a differential form. To do this, we will rely heavily on the theory developed in the last two sections. Also, it's worth mentioning a few things about the tensor product of real vector spaces. It's easy to check that for vector spaces V, W, Z, their tensor product is associative and commutative that is: $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$ and $V \otimes W \cong W \otimes V$. All these isomorphisms are natural isomorphisms! Therefore we can simply identify such spaces. This isomorphisms are easily extended to vector bundles, since we may use an isomorphism in each fiber, this map will be differentiable. So we say that the tensor product of vector bundles is associative and commutative i.e. $\mathbf{Vec}_M$ is a commutative monoidal category.

Now let us remember the definition of the wedge product, in what follows $\Lambda^k(\mathbb{R}^n)$ is the space of all k-linear functionals.

Definition 1.48. The wedge product is a linear transformation $\wedge^{k,l} : \Lambda^k(\mathbb{R}^n) \otimes_{\mathbb{R}} \Lambda^k(\mathbb{R}^n) \longrightarrow \Lambda^{k+l}(\mathbb{R}^n)$ defined by its action on the basis as

$$\wedge^{k,l}(e^i \otimes e^j) = \frac{1}{k!l!}\mathbf{Alt}(e^i \otimes e^j) = e^i \wedge e^j$$

where **Alt** is the alternate transformation. □

Remember that by the universal property of the tensor product we may regard the wedge product as a bilinear transformation (as is traditional), hence the notation in the second equality. Now we see how we can define the wedge product between certain vector bundles.

Definition 1.49. Let M be a smooth manifold and T^*M its cotangent bundle. We define T^*M^k as the tensor product

$$T^*M \otimes_M \cdots \otimes_M T^*M$$

where the product is repeated k times. In the case where k=0 we set $T^*M^0 = C^\infty(M)$. □

Proposition 1.50. *The map* $\wedge^{k,l} : T^*M^k \otimes_M T^*M^l \longrightarrow T^*M^{k+l}$ *defined by*

$$(p, \omega \otimes \eta) \mapsto (p, \omega \wedge \eta)$$

is a bundle morphism. For $l = 0$ *or* $k = 0$*, we take the map to be the multiplication by a smooth function.*

Proof: It is clear that it respects the composition law with the projection. Differentiability follows from that fact that in a local chart the coordinates of the new vector varies linearly with the coordinates of the original vectors, which were smooth. ■

Corollary 1.51. *The map $\wedge^{k,l} : T^*M^k \otimes_M T^*M^l \longrightarrow T^*M^{k+l}$ defines a unique bundle morphism $\tilde{\wedge}^{k,l}_M : \Gamma(T^*M^k \otimes_M T^*M^l) \longrightarrow \Gamma(T^*M^{k+l})$ which in turns defines a unique map $\wedge^{k,l}_M : \Gamma(T^*M^k) \otimes \Gamma(T^*M^l) \longrightarrow \Gamma(T^*M^{k+l})$.*

Proof: This follows from the fact that the map is a bundle morphism and from the isomorphisms from section 1. ■

Definition 1.52. Let $\Lambda^k(M) = \wedge^{k,0}(\Gamma(T^*M^k) \otimes C^\infty(M))$ and E be a vector bundle. Then an E-valued differential form is simply an element of $\Gamma(\Lambda^k(M) \otimes E) \cong \Lambda^k(M) \otimes_{C^\infty(M)} \Gamma(E)$. □

Remark 1.53. The important map is the map $\wedge^{k,l}_M : \Gamma(T^*M^k) \otimes \Gamma(T^*M^l) \longrightarrow \Gamma(T^*M^{k+l})$, but of course they are equivalent. Also, we can define a map $\wedge^{k,l}_M : \Lambda^k(M) \otimes \Lambda^l(M) \longrightarrow \Lambda^{k+l}(M)$ just by restricting the original map. From now on we will only be interested in $\Lambda^k(M)$.

Other way to define $\Lambda^k(M)$ is to think of the section module of the vector bundle $\Lambda^k T^*M$ generated by taking each fiber $\pi^{-1}(p)$ to be the space $\Lambda^k(TM_p)$, the space of k-linear alternating forms. Then, we have that the inclusion of modules induces a inclusion of bundles, and we can think of this bundle as a subbundle of T^*M^k. As said before, the tensor product inherits some natural isomorphisms from the tensor product of vector spaces. So given E a vector bundle, we may define natural isomorphisms between $[\Lambda^k(M) \otimes \Gamma(E)] \otimes [\Lambda^l(M) \otimes \Gamma(E)]$ to $[\Lambda^k(M) \otimes \Lambda^l(M)] \otimes [\Gamma(E) \otimes \Gamma(E)]$. Thus it's possible define a map

$$\wedge^{k,l}_E : \left[\Lambda^k(M) \otimes \Gamma(E)\right] \otimes \left[\Lambda^l(M) \otimes \Gamma(E)\right] \to \Lambda^{k+l}(M) \otimes [\Gamma(E) \otimes \Gamma(E)]$$

by taking the tensor product of the map $\wedge^{k,l}_M$ with the identity map in $\Gamma(E) \otimes \Gamma(E)$.

To further continue our extension, we prove a simple Lemma:

Lema 1.54. *Let E be an $\mathbb{R}$-algebra bundle with typical algebra A, then the multiplication $* : A \times A \longrightarrow A$ extends to a $C^\infty(M)$ multiplication in the sections modules.*

Proof: As seen in the previous section, we know that it extends to a differentiable multiplication $\tilde{*} : E \otimes_M E \longrightarrow E$. Is easy to check that it is a bundle morphism, so it corresponds to a multiplication $*_E : \Gamma(E) \otimes \Gamma(E) \longrightarrow \Gamma(E)$. ■

With this map in hand we maw define a map

$$*^{k+l}_E : \Lambda^{k+l}(M) \otimes [\Gamma(E) \otimes \Gamma(E)] \to \Lambda^{k+l}(M) \otimes \Gamma(E)$$

by taking the tensor product of $*_E$ and the identity in $\Lambda^{k+l}(M)$. Now we can compose $*^{k+l}_E$ with $\wedge k, l_E$ to get a map

$$\wedge^{k,l}_* : \left[\Lambda^k(M) \otimes \Gamma(E)\right] \otimes \left[\Lambda^l(M) \otimes \Gamma(E)\right] \to \Lambda^{k+l}(M) \otimes \Gamma(E).$$

Now we can proceed to the final form of this operation, but first one definition.

Definition 1.55. Let E be an $\mathbb{R}$-algebra bundle, we define the space of E-valued k-differential forms to be the (module) space

$$\Lambda^k(M; E) = \Gamma(\Lambda^k T^*M \otimes E).$$

By our standard isomorphisms, we can use the map $\wedge^{k+l}_*$ to get a map

$$\wedge^{k,l}_* : \Lambda^k(M; E) \otimes \Lambda^l(M; E) \to \Lambda^{k+l}(M; E)$$

this is the wedge product for E-valued differential forms.

Definition 1.56. We define $\Lambda^*(M;E)$ to be the direct sum over all non negatives integers

$$\Lambda^*(M;E) = \bigoplus_{i=0}^{\infty} \Lambda^i(M;E).$$

We can also define a map $\wedge_* : \Lambda^*(M;E) \otimes \Lambda^*(M;E) \to \Lambda^*(M;E)$ by taking the direct sum over $k+l$. More precisely, we define the nth entry of the product of $\Lambda = (\Lambda_1, \Lambda_2, \dots)$ with $\eta = (\eta_1, \eta_2, \dots)$ to be

$$\sum_{k+l=n} \wedge_*^{k,l}(\Lambda_k \otimes \eta_l).$$

□

Proposition 1.57. *$\Lambda^*(M;E)$ together with $\wedge_* : \Lambda^*(M;E) \otimes \Lambda^*(M;E) \to \Lambda^*(M;E)$ constitutes an $C^\infty(M)$-algebra.*

Proof: It suffices to check in each coordinate since operations are defined coordinate wise. Then bilinearity follows from the fact that , for each coordinate, it is the sum of bilinear products. ■

Definition 1.58. Let $(G,+)$ be a monoid, then an R-algebra A is said to be a G-graded R-algebra if it can be expressed as $A = \bigoplus_{i \in G} A^i$ such that the algebra multiplication respects the operation of the underlying monoid, that is

$$* : A^i \otimes A^j \longrightarrow A^{i+j}.$$

An $\mathbb{N}$-graded R-algebra A is said to be connected if $A^0 = R$. □

So, lastly, we prove the following:

Proposition 1.59. *$\Lambda^*(M;E)$ is a $\mathbb{N}$-graded $C^\infty(M)$-algebra. If E is trivial of rank one, then it is connected.*

Proof: This follows immediately from our construction. Let $\Lambda^i(M;E) \hookrightarrow \Lambda^*(M;E)$ be the natural inclusion, if $\Lambda \in \Lambda^k(M;E)$ and $\eta \in \Lambda^l(M;E)$ then the only non zero entry is in the $k+l$th one, as desired. If E is trivial and of rank 1, then a section can be identified with a smooth function via a global frame, thus $\Lambda^0(M,E) = \Gamma(\Lambda^0 T^*M \otimes E) = \Gamma(M \times \mathbb{R} \otimes E) = \Gamma(E) = C^\infty(M)$ ■

2

PRINCIPAL FIBER BUNDLES

This chapter is dedicated to the study of fiber bundles, more specifically, *Principal Fiber Bundles and G-Bundles*. Fiber bundles are important in a lot of areas of research, and in Gauge Theory it is central: one could say that Gauge Theory is the study of principal fiber bundles and its connection forms.

2.1 LIE GROUPS AND LIE ALGEBRAS

In this section we discuss basic concepts of the theory of Lie groups and Lie algebras. Lie groups are extremely useful objects and arise naturally in many areas of mathematics and physics. Due to the group structure, we obtain access to a lot of tools to study geometry, for instance, we can think of *differentiable group actions*. With Lie groups comes Lie algebras, since every Lie group has an associated Lie algebra, they are also quite useful and interesting in their own right. That being said, let's define what a Lie group is exactly.

Definition 2.1. Let G be a group with the structure of a differentiable manifold. G is said to be a Lie group if the multiplication

$$G \times G \longrightarrow G$$
$$(g, h) \mapsto gh$$

is differentiable and the inversion

$$G \longrightarrow G$$
$$g \mapsto g^{-1}$$

is differentiable as well. □

Given $g \in G$, we can think of the transformation "multiplication by g", that is $h \mapsto gh$ or $h \mapsto hg$. By the definition of a Lie group, these maps are smooth. To get an inverse, it suffices to multiply by g^{-1}, which is also smooth, therefore, multiplication is a diffeomorphism.

Definition 2.2. Given $g \in G$, we define the diffeomorphisms L_g and R_g to be the left and right multiplication by g. □

Notação 2.3. Let M be a smooth manifold, we denote by $\Gamma(TM)$ the space of smooth vector fields (sections) of its tangent bundle TM. □

Definition 2.4. A vector field $X \in \Gamma(TM)$ is said to be left invariant if for every $g \in G$ we have that

$$(L_g)_* X = X$$

explicitly

$$(dL_g)_h X_h = X_{gh}.$$

□

Proposition 2.5. *A left invariant field is completely defined by a single tangent vector.*

Proof: Let X be left invariant and take X_e, where e is the identity of the group. Then, for every $g \in G$, $X_g = (dL_g)_e(X_e)$. ■

Remark 2.6. Knowing that a field is determined by its value in some tangent space, we often write $X \in TG_e$ to represent the whole field.

Invariant fields are extremely important, as it is from them that we define the Lie algebra associated with a Lie group. The following theorem shows that the space of invariant fields has finite dimension as vector space.

Theorem 2.7. *If X is a left invariant field (not necessarily smooth), then X is smooth.*

Proof: See [Bie20], Proposition 13.23, p.281. ■

Corollary 2.8. *The space of left invariant fields has a finite dimension equal to n, where n is the dimension of G.*

Proof: By Proposition 2.5 and Theorem 2.7, the result follows. Just take a base of TG_e, then we can write any invariant field in TG_e and "transport it" using $d(L_g)$ with the appropriate g. ■

Now we discuss the notion of Lie algebras. Since we are only interested in real Lie algebras, we will define just this case.

Definition 2.9. A Lie algebra is a real vector space V equipped with a bilinear operation $[\,.\,,\,.\,] : V \times V \longrightarrow V$ satisfying

(i) Anticommutativity: $[X, Y] = -[Y, X]$.

(ii) The Jacobi's Identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. □

As usual, we can look at a vector field X as a derivation $X : C^\infty(M) \longrightarrow C^\infty(M)$, where $(Xf)(p) = X_p(f)$. So it makes sense to think about the composition of vector fields. We now define the Lie bracket.

Definition 2.10. Given $X, Y \in \Gamma(TM)$, we define the Lie bracket of X, Y as the vector field $[X, Y] \in \Gamma(TM)$ defined by

$$[X, Y] = XY - YX.$$

□

Proposition 2.11. *$\Gamma(TM)$ together with the Lie bracket is a Lie algebra.*

Proof: Bilinearity and anticommutativity follow immediately. The proof that the field $[X, Y]$ is in fact a derivation is also straightforward. For Jacobi's identity, just manipulate using bilinearity and anticommutativity:

$$\begin{aligned}&[[Y,Z],X]+[[Z,X],Y]=-[X,[Y,Z]]-[Y,[Z,X]]\\&=-[X,YZ-ZY]-[Y,ZX-XZ]\\&=-[X,YZ]+[X,ZY]-[Y,ZX]+[Y,XZ]\\&=-XYZ+YZX+XZY-ZYX-YZX+ZXY+YXZ-XZY\\&=-XYZ+ZXY+YXZ-ZYX\\&=-[[X,Y],Z].\end{aligned}$$

■

Now, one last proposition is enough before we can define the Lie algebra of a Lie group.

Proposition 2.12. *The Lie bracket of a left invariant field is a left invariant field.*

Proof:

$$\begin{aligned}\left(dL_g\right)_h[X,Y]_h f &= [X,Y]_h\left(f\circ L_g\right)\\&=XY_h\left(f\circ L_g\right)-YX_h\left(f\circ L_g\right)\\&=X\left(dL_g\right)_h Y_h(f)-Y\left(dL_g\right)_h X_h(f)\\&=XY_{gh}(f)-YX_{gh}(f)\\&=[X,Y]_{gh}f.\end{aligned}$$

■

Corollary 2.13. *Left invariant fields form a Lie algebra.*

Since invariant fields are completely defined by their values in the tangent space of any given point, we can define a Lie algebra in any tangent space.

Definition 2.14. Let G be a Lie group, the Lie algebra of G, denoted $\mathfrak{g}$, is the space TG_e together with the product $[\,.\,,\,.\,]: TG_e\times TG_e\longrightarrow TG_e$ defined by

$$[X_e,Y_e]=[X,Y]_e$$

which is well defined, as the fields extend in a unique way. □

The next step is to define the exponential map of a Lie algebra, so we state the following theorem:

Theorem 2.15. *A left invariant field is always complete i.e. there is a flow $\varphi:\mathbb{R}\times G\longrightarrow G$.*

Proof: [Lee03] Lemma 20.1, p519. ■

We now define what a one parameter subgroup of a Lie group is.

Definition 2.16. An one parameter subgroup of G is a smooth homomorphism $F:\mathbb{R}\longrightarrow G$. □

Theorem 2.17. *Let G be a Lie group. Then its one parameter subgroups are exactly the integral curves of left invariant fields starting at the identity $e\in G$.*

Proof: Let $X\in\mathfrak{g}$ and let $\{\varphi_t\}$ be the family of integral curves defined by the global flow of X. Since X is complete, these curves are defined on $\mathbb{R}$. Define $\gamma:\mathbb{R}\longrightarrow G$ as $\gamma(t)=\varphi_t(e)$, now we show that γ is an one parameter subgroups . Fix $s\in\mathbb{R}$ and let $\gamma_1(t)=\gamma(s+t)$ and $\gamma_2(t)=\gamma(s)\gamma(t)$. Then $\gamma_1'(t)=\gamma'(s+t)=X_{\gamma(s+t)}$ e $\gamma_2'(t)=(L_{\gamma(s)}\circ\gamma(t))'(t)=X_{\gamma(s)\gamma(t)}$

by the chain rule. Thus, they define integral curves starting at s of the same field X, and therefore are equal by uniqueness of said curves, then $\gamma_1 = \gamma_2$ for all $s \in \mathbb{R}$. Conversely, let $\gamma : \mathbb{R} \longrightarrow G$ be an one parameter subgroup, we define a family of diffeomorphisms $\psi_t : G \longrightarrow G$ where $\psi_t(g) = g\gamma(t)$, this is a family of diffeomorphisms that have the composition properties of a flow, thus $X_g = (\gamma_t(g))'(0)$ and by its definition its a left invariant field. ■

Definition 2.18. Using the notation from the previous demonstration, we define the exponential map $\exp : \mathfrak{g} \longrightarrow G$ as $\exp(X) = \gamma(1)$. In this case, γ is the curve that starts at the identity and is defined by the family of diffeomorphisms generated by X. As there is a bijection by Theorem 2.17, this is well defined. □

Remark 2.19. Note that $\exp(tA) = \gamma(t)$ (take the curve $\gamma'(t') = \gamma(tt')$ and evaluate the derivative, then it follows from the uniqueness of the integral curves). Also, note that $\varphi_t(g) = g\varphi_t(e) = g\exp(tA)$, again, this follows from the uniqueness of the flow.

Definition 2.20. A subgroup H of a Lie group G is a submanifold which is also a subgroup. Clearly H is also a Lie group, so we see that its one parameter subgroups are also subgroups of G. So $\exp : \mathfrak{h} \longrightarrow H$ is just the restriction of the exponential map of $\mathfrak{g}$. □

Proposition 2.21. *Let G e G' be Lie groups and $F : G \longrightarrow G'$ a differentiable homomorphism, then $dF_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is such that $dF_e([X_e, Y_e]) = [dF_e(X_e), dF_e(Y_e)]$ i.e. it is a morphism of Lie algebras.*

Proof: Observe that $F \circ L_g = F(g)F = L_{F(g)} \circ F$, then $d(F \circ L_g) = d(L_{F(g)} \circ F)$. Let X be invariant e and take its image $F_*(X)$. Let $g' = F(g)$, we have $(L_{g'})_*(F_*(X)) = d(L_{F(g)} \circ F)(X) = d(F \circ L_g)(X) = dF(X) = F_*(X)$. Thus we can think that the image of an invariant field is again an invariant field, since we know how to extend it. Let $f \in C^\infty(G')$, so $dF_p(XY)f = X_p(Y(f \circ F)) = dF_p(X_p)(dF(Y)f)$, therefore $F_*[X, Y] = [F_*X, F_*Y]$. ■

Remark 2.22. The proposition above is telling us that the rule which assigns to a Lie group its Lie algebra is a functor. This functor is very important and there are several important results regarding it. For instance, every finite dimensional Lie algebra is the Lie algebra of a Lie group, that is to say that this functor is essentially surjective. Under certain restrictions of the category of Lie groups, the Lie algebra functor becomes an equivalence of categories!

Definition 2.23. Let G be a Lie group, for $g \in G$ we define the differentiable isomorphism $Ad_g : G \longrightarrow G$ given by $h \mapsto ghg^{-1}$. So, by the previous Proposition, we have a smooth homomorphism $\mathcal{A}d : G \longrightarrow GL(\mathfrak{g})$ where $g \mapsto d(Ad_g)_e = \mathcal{A}d_g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Again by the last Proposition, we have an induced map $\mathfrak{Ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, namely $d(\mathcal{A}d)_0 : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$. □

Remark 2.24. Since $GL(V) \cong GL_n(\mathbb{R})$ for some n, we have an Lie group isomorphism between their Lie algebras, so $\mathfrak{gl}(V) \cong \mathfrak{gl}_n(\mathbb{R})$, which, as proven below (2.26), is just the space of all linear transformation from $T : \mathbb{R}^n \to \mathbb{R}^n$. Actually, this is stronger, for any vector space V there is natural isomorphism between $\mathfrak{gl}(V)$ and $\{T : V \longrightarrow V \mid T \text{ is linear.}\}$ (See [Lee03], Corollary 4.24, p.99.). This justifies the following Proposition.

Proposition 2.25. *For $X, Y \in \mathfrak{g}$, we have:*

$$[\mathfrak{Ad}(X)](Y) = \frac{\partial^2}{\partial t \partial s}(\exp(tX)\exp(sY)\exp(-tX)) = [X, Y]$$

Proof:

$$\begin{aligned}
[X, Y] = [X, Y]_e &= \frac{d}{dt}\varphi_{-t*}\left(Y_{\varphi,(e)}\right) \\
&= \frac{d}{dt}\varphi_{-t*}\left(\frac{d}{ds}\varphi_t(e)\exp(sY)\right) = \frac{d}{dt}\frac{d}{ds}\varphi_{-t}\left(\varphi_t(e)\exp(sY)\right) \\
&= \frac{\partial^2}{\partial t \partial s}(\exp(tX)\exp(sY)\exp(-tX)) \\
&= \frac{d}{dt}(d(\mathcal{A}d))(\exp(tX))(Y) = d(\mathcal{A}d)_e(X)(Y) = \mathfrak{Ad}(X)(Y).
\end{aligned}$$

In the first equality we are using the equivalence between the Lie derivative of a field Y in the direction of a field X with the Lie brackets of X with Y. φ is the flow of X and we used that $\varphi_t(g) = g\varphi_t(e) = g\exp(tX)$. ■

To conclude this section we will see some properties of a very important class of Lie groups, the matrix groups. In $\mathbb{R}^n$, $GL_n(\mathbb{R})$, is the space of all invertible $n \times n$ matrices, is easy to see that such space is naturally isomorphic to the space of linear automorphisms of $\mathbb{R}^n$.

Proposition 2.26. *The Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $GL_n(\mathbb{R})$ can be identified with $M_n(\mathbb{R})$, the space of $n \times n$ matrices with real entries.*

Proof: Since $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, it is an open set of $\mathbb{R}^{n^2} = M_n(\mathbb{R})$. Thus $T\left(GL_n(\mathbb{R})\right)_{id} = M_n(\mathbb{R})$.In this identification the matrix A is identified with the tangent vector

$$A = \sum_{k,l=1}^{n} A^{kl}\partial_{kl}|_{id}$$

where $\partial_{ij}|_{id}$ is a basis of the tangent space of the identity related to some local coordinate system. ■

Now let's see that the exponential map of the matrix group is an exponential series of matrices, hence the name!

Proposition 2.27. *Let* $\exp : \mathfrak{gl}_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ *be the exponential map, then*

$$\exp(A) = e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!}.$$

Proof: To show that the series converges, just observe that

$$\|\sum_{k=0}^{n}\frac{A^k}{k!}\| \leq \sum_{k=0}^{n}\frac{\|A^k\|}{k!} \leq \sum_{k=0}^{n}\frac{\|A\|^k}{k!} \leq \sum_{k=0}^{\infty}\frac{\|A\|^k}{k!} < \infty$$

Also, it's easy to see that $e^A e^{-A} = id$, so $e^A \in GL_n(\mathbb{R})$.

Now see that the curve $\gamma(t) = e^{tA}$ is such that

$$\left.\frac{d}{dt}e^{tA}\right|_{t=0} = A.$$

Thus, it is (the) an integral curve for A, therefore it is the exponential as defined beforehand. ■

Finally, we see that the Lie bracket in $\mathfrak{gl}_n(\mathbb{R})$ is just the commutator of the product of matrices.

Theorem 2.28. *Given* $A, B \in \mathfrak{gl}_n(\mathbb{R})$ *we have* $[A, B] = AB - BA$.

Proof: By propositions 2.25 and 2.27, we have:

$$[A, B] = \frac{\partial^2}{\partial s \partial t}(e^{tA}e^{sB}e^{-tA})\bigg|_{s,t=0} = AB - BA.$$

■

2.2 PRINCIPAL FIBER BUNDLES

A fiber bundle can be thought as the generalization of the product of two manifolds. The idea is that while the product $M \times N$ of two manifolds "behaves the same" everywhere, the bundle has local behavior, as if you would "twist" the product of manifolds independently in different places. Below follows the precise definition.

Definition 2.29. A differentiable fiber bundle consists of three manifolds E,M and F together with a differentiable surjection $\pi : E \longrightarrow M$ such that:

(i) For each $p \in M$, we have that $\pi^{-1}(p)$ is a manifold diffeomorphic to F.

(ii) For each $p \in M$ there exists an open neighborhood $U \ni p$ and a diffeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\
& M &
\end{tikzcd}$$

E is called the total space, M the base space, F the typical fiber and π the projection. □

Remark 2.30. The maps Φ are called local trivializations. If there exists a global trivialization *i.e.* a trivialization with E as its domain, we say that the bundle is trivial.

It follows from the definition that $E_p = \pi^{-1}(p)$ is diffeomorphic to F, using a local trivialization and identifying $\{p\} \times F$ with F, thus, it preserves the fibers. Also, it's not hard to see that π is a submersion, making E_p a submanifold of E, as all its values are regular.

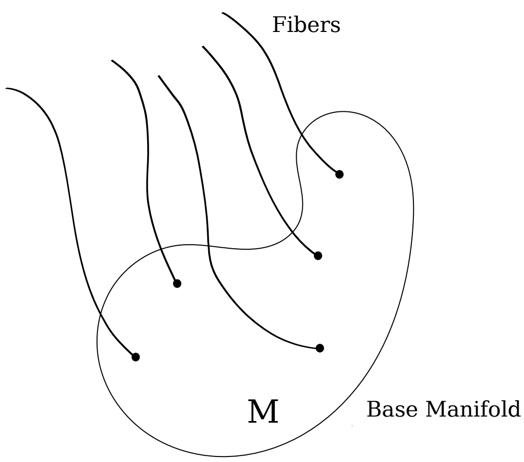

Figure 2.1: Intuitive depiction of a Fiber Bundle

This is the general case, but our interest leans towards the notion of *principal fiber bundles*. In this case, we'll have a Lie group "dictating" how the fibers twist when we move along them.

Definition 2.31. Let G be a Lie group. A principal fiber bundle is a fiber bundle $\pi : P \to M$ with typical fiber G together with a free differentiable right action

$$P \times G \longrightarrow P$$

$$(p, g) \mapsto pg$$

such that:

(i) For all $p \in M$, $\pi^{-1}(p) = \{xg \mid g \in G\}$ for some $x \in \pi^{-1}(p)$.

(ii) A local trivialization $\Phi = (\Phi^1, \Phi^2)$ is such that $\Phi(xg) = (\Phi^1(x), \Phi^2(x)g)$. □

Note that condition (i) is saying that the action preserves the fibers (and is transitive in the fibers) and (ii) is saying that the trivializations preserve the action. For g fixed, the action becomes a diffeomorphism, we denote it by R_g. Another nice feature is that the change of local trivializations is merely an element of G (that is to say that a principal bundle is a G-*bundle*). This is shown below:

Proposition 2.32. *Let Φ_α, Φ_β be local trivializations with domains U_α, U_β such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. Let $x \in E_p$, then $\Phi_\alpha|_{E_p} \circ (\Phi_\beta|_{E_p})^{-1}(p, g) = (p, g_{\alpha\beta}(p)g)$, where $g_{\alpha\beta}(p) \in G$. Moreover, the function $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G \;\; p \mapsto g_{\alpha\beta}(p)$ is smooth.*

Proof: Define $g_{\alpha\beta}(p) = \Phi^2_\alpha(x)(\Phi^2_\beta(x))^{-1}$ where $x \in E_p$. First to see that it is well defined take $y \in E_p$ then $y = xh$ for some $h \in G$, so

$$\begin{aligned}\Phi^2_\alpha(xh)(\Phi^2_\beta(xh))^{-1} &= \Phi^2_\alpha(x)h(\Phi^2_\beta(x)h)^{-1}\\ &= \Phi^2_\alpha(x)hh^{-1}(\Phi^2_\beta(x))^{-1}\\ &= \Phi^2_\alpha(x)(\Phi^2_\beta(x))^{-1}.\end{aligned}$$

Now to see that they are in fact the change of local trivializations take $(p, g) \in U_{\alpha\beta} \times G$, then $g = \Phi^2_\beta(x)$ for some x, so $\Phi^{-1}_\beta(p, g) = x$. Now just compute $\Phi_\alpha \circ (\Phi_\beta)^{-1}(p, g) = \Phi_\alpha(x) = (p, \Phi^2_\alpha(x)) = (p, \Phi^2_\alpha(x)(\Phi^2_\beta(x))^{-1}\Phi^2_\beta(x)) = (p, g_{\alpha\beta}(p)g)$. To check that it is differentiable just see that it is the composition of differentiable functions

$$\begin{tikzcd} U_{\alpha\beta} \times G \arrow[r, "\Phi_\alpha \circ (\Phi_\beta)^{-1}"] & U_{\alpha\beta} \times G \arrow[d, "pr_2"] \\ U_{\alpha\beta} \arrow[u, hook, "i"] \arrow[r, dotted, "g_{\alpha\beta}"'] & G \end{tikzcd}$$

where i is the inclusion in the identity e. ■

Now we take a look at the notion of sections. Sections are the generalization of vector fields, typically defined for tangent bundles.

Definition 2.33. A local section in a fiber bundle $P \to M$ is a smooth function $\sigma : U \longrightarrow P$ defined on an open set $U \subset M$ such that

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[dr, "id_U"'] & P \arrow[d, "\pi"] \\ & U \end{tikzcd}$$

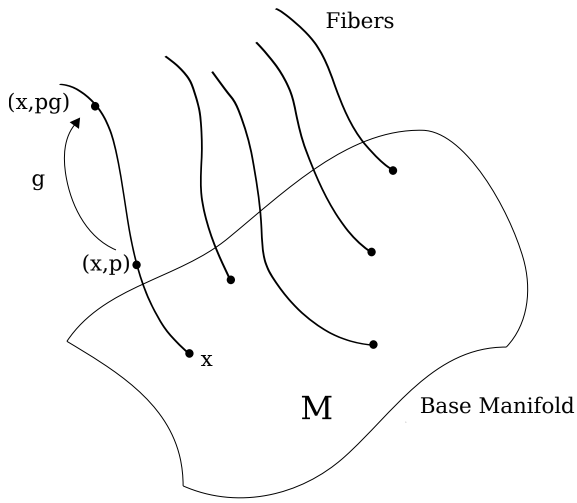

Figure 2.2: Intuitive depiction of a Principal Fiber Bundle

commutes. A section is said to be global if it is defined in all of M. □

There is a strong relation between local sections and local trivializations, actually, local sections in principal fiber bundles are essentially the same thing as local trivializations:

Theorem 2.34. *Let P be a principal fiber bundle. Then there exists a correspondence between sections defined in U and local trivializations of U.*

Proof: Let $\sigma : U \longrightarrow P$ be a local section. Define the trivialization $\Phi_\sigma : \pi^{-1}(U) \longrightarrow U \times G$ as $\Phi_\sigma(\sigma(p)g) = (p, g)$. It is easy to see that it is a bijection since the action is free and transitive on fibers. To show that it is differentiable, note that the inverse of such map is given by

$$U \times G \xrightarrow{\sigma \times id_G} \pi^{-1}(U) \times G \xrightarrow{A} \pi^{-1}(U)$$

where A is the action. Since for fixed $g \in G$ A is diffeomorphism, A is a submersion. Also $d(\pi)_{\sigma(p)} d(\sigma)_p = id$ so σ has constant maximal rank, now you can show with the chain rule that this map has maximal rank, thus by the inverse function theorem it is a diffeomorphism (it's already a bijection).

Conversely, given a trivialization $\Phi : \pi^{-1}(U) \longrightarrow U \times G$, define the section $\sigma(p) = \Phi^{-1}(p, e)$. This map is differentiable, because its the composition of the inclusion of U in $U \times \{e\}$ composed with Φ^{-1}. ■

Corollary 2.35. *A principal fiber bundle is trivial if and only if it admits a global section.*

2.3 G-BUNDLES AND ASSOCIATED BUNDLES

In this section we explore the notion of G bundles and associated bundles. Given a principal bundle with group G and a smooth action of G in some manifold F it is possible to construct a G-bundle, called the associated bundle. We'll see that every G-bundle can be constructed in such a manner.

Definition 2.36. A G-bundle is a fiber bundle $E \to M$ with typical fiber F together with a Lie group G that acts smoothly on F, with local trivializations satisfying the following property: Given local trivializations $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ and $\psi_\beta : \pi^{-1}(U_\beta) \longrightarrow U_\beta \times F$ with $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, there exists a smooth function $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ such that the transition function

$$\psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha\beta} \times F \longrightarrow U_{\alpha\beta} \times F$$

is given by

$$\psi_\alpha \circ \psi_\beta^{-1}(p, f) = (p, g_{\alpha\beta}(p) \cdot f).$$

In this case, we say that $g_{\alpha\beta}$ is the transition function. The action on F can be thought as a group homomorphism $\rho : G \longrightarrow \mathrm{Diff}(F)$ where $\rho(g)(x) = g \cdot x$, when the homomorphism ρ is injective we say that the action in effective. □

This definition may seem a little weird, but is actually quite natural. For most people, as it was for me, their first encounter with bundles was the tangent bundle of a manifold M, namely TM. TM is a vector bundle, so the trivializations induced by the derivative of charts are just an atlas for which the structural group is $GL_n(\mathbb{R})$! More generally, every (real) vector bundle is just a G-bundle where $G = GL_n(\mathbb{R})$.

Remark 2.37. Note that we may always assume that the action is effective, since we may take the quotient of the Lie group G by the kernel H of such action, getting an effective action of G/H.

Using the notion of G-bundles, we can define a principal bundle in a different manner, so we'll have to show that our definitions are equivalent.

Definition 2.38. A principal G-bundle (or principal fiber bundle) is G-bundle with typical fiber G and with action of G given by left multiplication. □

Theorem 2.39. *Definition 2.38 and 2.31 are equivalent.*

Proof: Our first definition does in fact give rise to a principal fiber bundle, as proven in Proposition 2.32. Now assume we have a G-bundle in the sense of 2.38, we have to define a differentiable right action that is free and transitive on fibers and show that the local trivializations preserve the fibers and the action. So we define for $p \in \pi^{-1}(x),\ g \in G$

$$p \cdot g = \psi^{-1}(x, hg)$$

for some local trivialization containing $\{x\} \times G$ such that $\psi(p) = (x, h)$. First we have to show that this action doesn't depend on the choice of trivialization. Let $\psi_\beta : \pi^{-1}(U_\beta) \longrightarrow U_\beta \times F$ and $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ be local trivilizations with $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$. Since P is a G-bundle, we have $\left(\psi_\alpha \circ \psi_\beta^{-1}\right)(x, g) = \left(x, g_{\alpha\beta}(x) g\right)$ so

$$\left(\psi_\alpha \circ \psi_\beta^{-1}\right)(x, h_\beta) = \left(x, g_{\alpha\beta}(x) h_\beta\right) = (x, h_\alpha)$$

thus, $h_\alpha = g_{\alpha\beta} h_\beta$ so

$$\begin{aligned} \psi_\alpha^{-1}(x, h_\alpha g) &= \psi_\alpha^{-1}\left(x, g_{\alpha\beta}(x)(h_\alpha g)\right) = \psi_\beta^{-1}\left(x, g_{\beta\alpha}(x)(h_\alpha g)\right) \\ &= \psi_\beta^{-1}\left(x, \left(g_{\beta\alpha}(x h_\alpha)\, g\right)\right) = \psi_\beta^{-1}(x, h_\beta g) \end{aligned}$$

Which implies that it doesn't depend on the choice of local trivialization. Now we must show that this function is indeed an action *i.e.* $(p \cdot g_1) \cdot g_2 = p \cdot (g_1 g_2)$. Since $\psi(p \cdot g_1) = (x, hg_1)$, by definition

$$\psi_\alpha\left((p \cdot g_1) \cdot g_2\right) = \left(x, (hg_1)\, g_2\right) = \left(x, h\,(g_1 g_2)\right) = \psi_\alpha\left(p \cdot g_1 g_2\right).$$

The transitivity of this action on fibers follows from the fact that the action of G in G (left multiplication) is transitive and, since left multiplication is free, the action is free as well.

The fact that this action preserves (satisfies condition (ii) of 2.31) fibers is clear as well, which concludes the proof. ■

Now we know that a principal bundle is a special case of a G-bundle, so we'll be able to justify the name "principal" by showing that every G-bundle is an associated bundle of some principal G-bundle. But first, we will construct associated fiber bundles explicitly.

Let G be a Lie group acting on a manifold F of dimension m. And let $\pi : P \to M$ be a principal fiber bundle. Then we define an action

$$(P \times F) \times G \to P \times F$$

by setting

$$(p, f) \cdot g = \left(p \cdot g, g^{-1} \cdot f\right).$$

Let $P \times_G F = (P \times F)/G$ be set of orbits of the action, $\tilde{\pi} : P \times F \longrightarrow P \times_G F$ be the canonical projection and $\pi'([p, f]) = \pi(p)$. We get the following commutative diagram

$$\begin{array}{ccc} P \times F & \xrightarrow{\tilde{\pi}} & P \times_G F \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi'} \\ P & \xrightarrow[\pi]{} & M \end{array}$$

What will be shown is that the vertical arrows define G-bundles while the horizontal ones define principal bundles.

Theorem 2.40. *Let $\pi : P \to M$ be a principal G-bundle, F a manifold, and let G act on F. Let π, $\tilde{\pi}$, π' and $P \times_G F$ be defined as above then:*

(i) $P \times_G F$ is a manifold such that $\pi' : P \times_G F \longrightarrow M$ and $\tilde{\pi} : P \times F \longrightarrow P \times_G F$ are smooth.

(ii) Vertical arrows are G-bundles and horizontal ones are principal G-bundles.

(iii) Every local trivialization $\Phi : \pi^{-1}(U) \longrightarrow U \times G$ of $\pi : P \longrightarrow M$ induces a local trivialization $\Phi_F : \pi'^{-1}(U) \longrightarrow M \times F$ of $P \times_G F$ such that the condition $\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)(x, g) = \left(x, g_{\alpha\beta}(x) g\right)$ implies

$$\left((\Phi_\alpha)_F \circ (\Phi_\beta)_F^{-1}\right)(x, f) = \left(x, g_{\alpha\beta}(x) \cdot f\right).$$

(iv) Every local trivialization $\Phi : \pi^{-1}(U) \longrightarrow U \times G$ of $\pi : P \longrightarrow M$ induces a local trivialization $\tilde{\Phi} : \tilde{\pi}^{-1}(\pi'^{-1}(U)) \longrightarrow \pi'^{-1}(U) \times G$ of $P \times F$ such that the condition $\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)(x, g) = \left(x, g_{\alpha\beta}(x) g\right)$ implies

$$(\tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1})([p, f], g) = \left([p, f], g_{\alpha\beta}(\pi[p, f]) g\right).$$

Proof: Observe that items (iii) and (iv) are giving the desired structure to our constructions, so we'll begin by them. First we must define the induced trivializations:

$$\Phi_F[p, f] = \left(\pi(p), \Phi^2(p) \cdot f\right)$$
$$\tilde{\Phi}(p, f) = \left([p, f], \Phi^2(p)\right)$$

Note that $\tilde{\pi}^{-1}(\pi'^{-1}(U)) = \pi^{-1}(U) \times F$, it makes sense, so in both cases p ranges through $\pi^{-1}(U)$ and f through F. Now we have to show that Φ_F is well defined, this follows from

$$\left(\pi(p \cdot g), \Phi^2(p \cdot g) \cdot \left(g^{-1} \cdot f\right)\right) = \left(\pi(p), \left(\Phi^2(p)g\right) \cdot \left(g^{-1} \cdot f\right)\right) = \left(\pi(p), \Phi^2(p) \cdot f\right)$$

observe that we are using the fact that the trivializations of P are G-equivariant. To show that they are bijections, we can construct inverses using local sections. We know that there exists exactly one local section $\sigma : U \longrightarrow P$ such that $\Phi^{-1}(x, g) = \sigma(x) \cdot g$. So, it's easy to see that $\Phi^2(\sigma(x)) = e$ and that $p = \sigma(\pi(p)) \cdot \Phi^2(p)$, so we can set

$$\Phi_F^{-1}(x, f) = [\sigma(x), f]$$

$$\tilde{\Phi}^{-1}([p, f], g) = \tilde{\sigma}[p, f] \cdot g$$

where $\tilde{\sigma}[p, f] = (p, f) \cdot \Phi^2(p)^{-1} = \left(\sigma(\pi(p)), \Phi^2(p) \cdot f\right)$ (which can easily be checked to be well defined). It's clear that those are the inverse functions of Φ_F and $\tilde{\Phi}$. Now let Φ_α, Φ_β be local trivializations such that $\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)(x, g) = \left(x, g_{\alpha\beta}(x)g\right)$ for a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$. Taking $g = e$, and applying Φ_α^{-1}, we get that

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot g_{\alpha\beta}(x)$$

and that

$$\Phi_\beta^2(p) = g_{\alpha\beta}(\pi(p))^{-1}\Phi_\alpha^2(p)$$

finally, we get

$$\begin{aligned} \left((\Phi_\alpha)_F \circ (\Phi_\beta)_F^{-1}\right)(x, f) &= (\Phi_\alpha)_F \left[\sigma_\beta(x), f\right] = (\Phi_\alpha)_F \left[\sigma_\alpha(x) \cdot g_{\alpha\beta}(x), f\right] \\ &= (\Phi_\alpha)_F \left[\sigma_\alpha(x), g_{\alpha\beta}(x) \cdot q\right] = \left(x, g_{\alpha\beta}(x) \cdot f\right) \end{aligned}$$

and

$$\begin{aligned} (\tilde{\Phi}_\alpha \circ \tilde{\Phi}_\beta^{-1})([p, f], g) &= \tilde{\Phi}_\alpha \left(\left(\sigma_\beta(\pi(p)), \Phi_\beta^2(p) \cdot f\right) \cdot g\right) \\ &= \tilde{\Phi}_\alpha \left(\left(\sigma_\alpha(\pi(p)) \cdot g_{\alpha\beta}(\pi(p)), \left(g_{\alpha\beta}(\pi(p))^{-1}\Phi_\alpha^2(p)\right) \cdot f\right) \cdot g\right) \\ &= \tilde{\Phi}_\alpha \left(\left(\sigma_\alpha(\pi(p)), \Phi_\alpha^2(p) \cdot f\right) \cdot g_{\alpha\beta}(\pi(p))g\right) \\ &= \left([p, f], g_{\alpha\beta}(\pi(p))g\right) \end{aligned}$$

which concludes the proofs of (iii) and (iv). Now all we need to do is equip $P \times_G F$ with the structure of a manifold, and we'll do this in a manner that makes the trivializations of (iii) and (iv) charts. If we do this, (ii) follows trivially from (iii) and (iv). Let's prove (i) by defining such a structure. We have bijections

$$\Phi_F : \pi'^{-1}(U) \to U \times F$$

$$\tilde{\Phi} : \pi^{-1}(U) \times F \to \pi'^{-1}(U) \times G$$

so we can compose them with charts of M, G and F to get bijections with some euclidian open set such that different charts compose to diffeomorphisms, thus there is a unique topology (see [Bie20], theorem 1.10 p.25.) making $P \times_G F$ into a smooth manifold. The maps $\tilde{\pi}$ and π' compose with charts to projections, thus they're smooth. ■

Definition 2.41. Let $P \to M$ be a principal G-bundle, and let G act on F smoothly. Then the bundle $P \times_G F \to M$ is called the associated bundle of P relative to the action of G on F. □

Definition 2.42. When $F = G$ and the action of G in G is given by $g \cdot h = ghg^{-1}$ we call the associated fiber bundle $P \times_G F = Ad_P(M)$ as the adjoint bundle of P.

Lema 2.43 (Construction of G-bundles). *Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open covering for a smooth manifold M and $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G\}_{\alpha,\beta \in \mathcal{A}}$ a family of smooth functions taking in values in a Lie group G satisfying the cocycle property:*

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma$$

Then, given a manifold F and an action of G on F, there exists a unique (up to isomorphisms) G-bundle $\pi : E \to M$ with typical fiber F such that the transition functions are given by the family $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G\}_{\alpha,\beta \in \mathcal{A}}$.

Proof: Let $E_A = \bigcup_{\alpha \in A} \{\alpha\} \times U_\alpha \times F$ be the manifold given by the disjoint union of the product manifolds $U_\alpha \times F$. Then we can define a differentiable surjection $\pi_A : E_A \longrightarrow M$ by

$$\pi_A(\alpha, x, f) = x.$$

now we define an equivalence relation on E_A by setting

$$(\alpha, x_\alpha, f_\alpha) \sim (\beta, x_\beta, f_\beta) \text{ if } x_\alpha = x_\beta \text{ and } g_{\alpha\beta}(x) \cdot f_\beta = f_\alpha.$$

The relation is symmetric since $g_{\alpha\alpha} = e$, it is reflexive because the cocycle property together with the identity property implies that $g_{\beta\alpha}^{-1} = g_{\alpha\beta}$, and finally, transitivity follows from the cocycle property. Now, we define $E = E_A / \sim$ and $p : E_A \longrightarrow E$ as the canonical projection. Since π_A is constant on the equivalence classes, we have an induced projection $\pi : E \longrightarrow M$. The projection p when restricted to $\{\alpha\} \times U_\alpha \times F \cong U_\alpha \times F$ is a bijection with $\pi^{-1}(U_\alpha)$, thus we may define the local trivializations Φ_α as $\Phi_\alpha = (p|_{\{\alpha\} \times U_\alpha \times F})^{-1}$. These maps are bijections which compose to smooth bijections in $\mathbb{R}^m$ with charts of M and F, thus we have an unique topology making E into a smooth manifold with charts given by the trivializations Φ_α compose with charts of $U_\alpha \times F$ (reducing the sets U_α if necessary). The transition functions are the original ones by construction, to see this, lets calculate for $x \in U_\alpha \cap U_\beta$

$$\begin{aligned} \Phi_\alpha \circ \Phi_\beta^{-1}(\beta, x, f) &= \Phi_\alpha \circ p(\beta, x, f) = \Phi_\alpha([\beta, x, f]) \\ &= \Phi_\alpha([\alpha, x, g_{\alpha\beta}(x) \cdot f]) = (\alpha, x, g_{\alpha\beta} \cdot f) \end{aligned}$$

Thus $\pi : E \to M$ is the desired G-bundle. ■

Lema 2.44. *Let E be a G-bundle, if the action of G in the typical fiber F is effective, then the transition functions satisfy the cocycle property.*

Proof: The commutative diagram

$$\begin{tikzcd} U_{\alpha\beta\gamma} \arrow[r, "\Phi_\alpha \circ \Phi_\beta^{-1}"] \arrow[rr, bend right, "\Phi_\alpha \circ \Phi_\gamma^{-1}"'] & U_{\alpha\beta\gamma} \arrow[r, "\Phi_\beta \circ \Phi_\gamma^{-1}"] & U_{\alpha\beta\gamma} \end{tikzcd}$$

implies that $g_{\alpha\beta}(x) g_{\beta\gamma}(x) \cdot f = g_{\alpha\gamma}(x) \cdot f$ for every $x \in U_{\alpha\beta\gamma}$, $f \in F$. Since the action is effective we get $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$, as desired. ■

Theorem 2.45. *Every G-bundle whose action is effective is isomorphic to some associated bundle of a principal G-bundle.*

Proof: Let $\pi : E \to M$ be a G-bundle with typical fiber F whose action is effective. Since the action of G is effective, the transition functions $\{g_{\alpha\beta}\}$ satisfy the cocycle condition, so by letting G act in G by left multiplication, by the preceding lemma, we get a G-bundle $P \to M$ whose transitions functions are $\{g_{\alpha\beta}\}$. By definition, such a bundle is a principal G-bundle. Since we have an action of G in F, there is an associated G-bundle $\pi' : P \times_G F \to M$ of P. We affirm that this bundle is isomorphic to E over M. To prove this, we must show that there exists a diffeomorphism $\mathfrak{F} : P \times_G F \longrightarrow E$ such that

$$\begin{tikzcd} P \times_G F \arrow[r, "\mathfrak{F}"] \arrow[dr, "\pi'"'] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes. Let $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ and $(\Phi_\alpha)_F : \pi'^{-1}(U_\alpha) \longrightarrow U \times F$ be local trivializations (they have the same domain by construction). For $p \in \pi'^{-1}(U_\alpha)$, define the map $\mathfrak{F} : P \times_G F \longrightarrow E$ by setting

$$\mathfrak{F}([p, f]) = [\psi_\alpha^{-1}(\Phi_\alpha)_F]([p, f]).$$

Now we need to see that it is independent of the choice of trivializations To see this, note that $\psi_\beta \circ \psi_\alpha^{-1} = L_{g_{\beta\alpha}}$ where $L_{g_{\beta\alpha}}(x, f) = (x, g_{\beta\alpha}(x) \cdot f)$ so, we get that $\psi_\beta = L_{g_{\alpha\beta}} \circ \psi_\alpha$. By item (iii) of theorem 2.40, the trivializations $(\Phi_\alpha)_F$ have the same property, and by construction the transition functions are the same, so $(\Phi_\beta)_F = L_{g_{\alpha\beta}} \circ (\Phi_\alpha)_F$, now we get that

$$\begin{aligned} [\psi_\beta^{-1}(\Phi_\beta)_F]([p, f]) &= [(L_{g_{\alpha\beta}} \circ \psi_\alpha)^{-1}(L_{g_{\alpha\beta}} \circ (\Phi_\alpha)_F)]([p, f]) \\ &= [\psi_\alpha^{-1} \circ L_{g_{\alpha\beta}}^{-1} \circ L_{g_{\alpha\beta}}(\Phi_\alpha)_F]([p, f]) = [\psi_\alpha^{-1}(\Phi_\alpha)_F]([p, f]) \end{aligned}$$

So the map is well defined. The map $\mathfrak{F}$ clearly makes the diagram commute, since it preserves the first entry of the trivializations, thus we get our result . ■

FINAL REMARK

The last section can be summarized as follows. Let M be a fixed base space gor the following categories and let $\mathbf{PBund}_G$ be the category of Principal G-bundles, $\mathbf{Bund}_G$ be the category of G-bundles and let $\mathrm{Hom}(BG, \mathbf{Diff})$ be the category of covariant functors (left smooth actions) from G, regarded as an one object grupoid, to the category of smooth manifolds. Then we showed that the functorial map

$$\times_G : \mathbf{PBund}_G \times \mathrm{Hom}(BG, \mathbf{Diff}) \longrightarrow \mathbf{Bund}_G$$

$$(P, F) \mapsto P \times_G F$$

is essentially surjective. This is a nice way of summarizing what was said.

3

GAUGES

As mentioned before, the Gauge Theory might be thought of as the study of principal fiber bundles and its *connections*. We already know what a principal fiber bundle is, now is time to dive into the notion of connections, and its interactions with other elements present in the theory of principal fiber bundles. The word gauge is used very loosely, but in this chapter we try to give it a definition. Also we talk about gauge potentials and fields.

3.1 CONNECTIONS ON PRINCIPAL BUNDLES

Generally speaking connections are a way to make sense of the interaction between tangent spaces of close points. That's why they're useful to "measure" several things, such as *curvature*. From now on, $P \longrightarrow M$ will always mean a principal fiber bundle with group G and projection π, also, n will denote the dimension of the base manifold M. Now we give three definitions of what is a connection and prove that they're equivalent.

Definition 3.1. A connection is a function that assigns to each $p \in P$ a vector space $H_p \subset TP_p$ such that:

(i) $TP_p = H_p \oplus V_p$, where $V_p = \{X \in TP_p \mid d(\pi)_p(X) = 0\}$.

(ii) $(R_g)_*(H_p) = H_{pg}$.

Furthermore, we require this assignment to be smooth with p, in the sense that there are n vector fields defined on a neighborhood U of p such that they span H_q for every $q \in U$. The spaces H_p are called horizontal spaces while V_p are called the vertical spaces. $\square$

The definition above is saying that a connection is in particular a vector subbundle H of TP such that $TP = H \oplus V$, where V is the vertical bundle, *i.e.* the bundle whose fiber at p is V_p.

Definition 3.2. Let $\mathfrak{g}$ be the Lie algebra of G. For $A \in \mathfrak{g}$, let A^* denote the field defined by $A^*_p = \frac{d}{dt}(p\exp(tA))|_{t=0}$. Then a connection is a $\mathfrak{g}$-valued 1-form ω on P such that:

(i) $\omega_p(A^*_p) = A$.

(ii) For $g \in G$, let $Ad_g : G \longrightarrow G$ be the adjunction map. And let $\mathcal{A}d_g = d(Ad_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$. We require that $\omega_{pg}(d(R_g)_p X_p) = \mathcal{A}d_{g^{-1}}\omega_p(X_p)$, that is to say that $(R_g)^*\omega = \mathcal{A}d_{g^{-1}}\omega$.

The form ω is called a connection 1-form. $\square$

Definition 3.3. A connection assigns to each local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$, a $\mathfrak{g}$-valued 1-form ω_α on U_α such that if Φ_β is local a trivialization with $U_{\alpha\beta} \neq \emptyset$ and with change of coordinates $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ we have that

$$\omega_\beta(Y) = L^{-1}_{g_{\alpha\beta}(x)*}\left(g_{\alpha\beta*}(Y)\right) + \mathcal{A}d_{g_{\alpha\beta}(x)^{-1}}\left(\omega_\alpha(Y)\right)$$

for all $Y \in TM_x$. □

Theorem 3.4. *The definitions 3.1 and 3.2 are equivalent.*

Proof: Let ω be a connection 1-form. Define $H_p = \{X \in TP_p \mid \omega_p(X) = 0\}$. This assignment is smooth because the form is smooth. Since it is a one form, ω_p is just linear transformation and since we require that $\omega_p(A^*) = A$, we know that it is surjective, thus $\dim H_p = n$. Now, for $p \in P$, let ϕ_p denote the differentiable function $g \mapsto pg$. It's easy to see that $A^*_p = d(\phi_p)_e A$. So we get that $d(\pi)_p A^*_p = d(\pi)_p d(\phi_p)_e A$, which by the chain rule must be zero, since $\pi \circ \phi_p$ is constant, thus we get than $A^*_p \in V_p$ for all $A \in \mathfrak{g}$. Since $\dim\mathfrak{g}=\dim V_p$, we get that $TP_p = H_p \oplus V_p$. Now note that $(R_g)_* H_p = H_{pg}$ since $\omega_{pg}(d(R_g)_p X_p) = \mathcal{A}d_{g^{-1}}\omega_p(X_p)) = 0$ for all $X_p \in H_p$.

Conversely, given a connection $p \mapsto H_p$ we define the one form $\omega(A^*_p + X_p) = A$. It is smooth because we have a local smooth frame for H_p. Condition (i) of definition 3.2 is already satisfied. Now let's check (ii), for elements of $X_p \in H_p$ the equality holds since $(R_g)_* H_p = H_{pg}$ which implies $((R_g)^*\omega)(X_{pg}) = 0 = \mathcal{A}d_{g^{-1}}\omega_p(X_p)$. So, take A^*_p, we have that

$$\begin{aligned}
\omega_{pg}\left(R_{g*}A^*_p\right) &= \omega_{pg}\left(\frac{d}{dt}[p(\exp tA)g]\right) = \omega_{pg}\left(\frac{d}{dt}\left[pgg^{-1}(\exp tA)g\right]\right) \\
&= \omega_{pg}\left(\frac{d}{dt}\left[pg\exp\left(t\mathcal{A}d_{g^{-1}}A\right)\right]\right) = \omega_{pg}\left(\left(\mathcal{A}d_{g^{-1}}A\right)^*_{pg}\right) \\
&= \mathcal{A}d_{g^{-1}}A = \mathcal{A}d_{g^{-1}}\left(\omega_p\left(A^*_p\right)\right).
\end{aligned}$$

Thus we conclude that (ii) holds. ■

Theorem 3.5. *Definitions 3.2 and 3.3 are equivalent.*

Proof: Let ω be a connection 1-form as in definition 3.2 and Φ_α a local trivialization and σ_α its associated local section as theorem 2.34. We define $\omega_\alpha = \sigma^*_\alpha\omega$, now we must show that those forms satisfy the condition in definition 3.3. Observe that

$$\begin{aligned}
\Phi_\alpha\left(\sigma_\alpha(x)\Phi^2_\alpha(p)\right) &= \left(x, \Phi^2_\alpha\left(\sigma_\alpha(x)\Phi^2_\alpha(p)\right)\right) \\
&= \left(x, \Phi^2_\alpha\left(\sigma_\alpha(x)\right)\Phi^2_\alpha(p)\right) = \left(x, e\Phi^2_\alpha(p)\right) \\
&= \left(x, \Phi^2_\alpha(p)\right) = \Phi_\alpha(p)
\end{aligned}$$

where $x = \pi(p)$. Therefore, we get that $p = \sigma_\alpha(x)\Phi^2_\alpha(p)$ and similarly $p = \sigma_\beta(x)\Phi^2_\beta(p)$, thus $\sigma_\beta(x) = \sigma_\alpha(x)\Phi^2_\alpha(p)(\Phi^2_\beta(p))^{-1} = \sigma_\alpha(x)g_{\alpha\beta}(x)$. For $Y_x \in TM_x$ and let $\gamma : \mathbb{R} \longrightarrow M$ be any curve such that $\gamma'(0) = Y$, then evaluating at $t = 0$,

$$\begin{aligned}
\omega_\beta(Y) = \sigma_{\beta*}(Y) = \frac{d}{dt}\sigma_\beta(\gamma(t)) &= \frac{d}{dt}\left[\sigma_\alpha(\gamma(t))g_{\alpha\beta}(\gamma(t))\right] \\
&= \frac{d}{dt}\left[\sigma_\alpha(x)g_{\alpha\beta}(\gamma(t))\right] + \frac{d}{dt}\left[\sigma_\alpha(\gamma(t))g_{\alpha\beta}(x)\right] \\
&= \frac{d}{dt}\left[\sigma_\beta(x)g_{\alpha\beta}(x)^{-1}g_{\alpha\beta}(\gamma(t))\right] + R_{g_{\alpha\beta}(x)*}\sigma_{\alpha*}(Y) \\
&= \left[L^{-1}_{g_{\alpha\beta}(x)*}g_{\alpha\beta*}(Y)\right]^*_{\sigma_\beta(x)} + R_{g_{\alpha\beta}(x)*}\sigma_{\alpha_*}(Y) \\
&= \omega_\beta\,(Y) = L^{-1}_{g_{\alpha\beta}(x)*}\left(g_{\alpha\beta*}\,(Y)\right) + \mathcal{A}d_{g_{\alpha\beta}(x)^{-1}}\left(\omega_\alpha\,(Y)\right)
\end{aligned}$$

where in the last equality we used properties (i) and (ii) of the connection form.

Conversely, let $\Phi_\alpha \mapsto \omega_\alpha$ be a connection as in definition 3.3 and σ_α the associated local section. For $x \in U_\alpha$, $p = \sigma_\alpha(x)$, $Y \in TM_x$ and $A \in \mathfrak{g}$, we define $\omega^\alpha_p : TP_p \longrightarrow \mathfrak{g}$ as $\omega^\alpha_p(\sigma_{\alpha^*}(Y) + A^*) = \omega_\alpha(Y)$. Now we extend this by a connection form on $\pi^{-1}(U_\alpha)$ by the formula $\omega^\alpha_{pg}(X) = \mathcal{A}d_{g^{-1}}\omega^\alpha_p\left(R_{g^{-1}*}X\right)$ for $X \in TP_{pg}$. By its definition ω^α is a connection 1-form on $\pi^{-1}(U_\alpha)$. Now it remains to show that those are well defined, that is if we have two intersecting local trivializations, then the forms coincide in the intersection. Let $U_{\alpha\beta}$ be the domain of two local trivializations. By condition (ii) of the definition 3.2, it suffices to show that the forms $\omega^\alpha = \omega^\beta$ in $\sigma_\alpha(U_{\alpha\beta})$, because it will imply that they coincide in $\pi^{-1}(U_{\alpha\beta})$. For A^* we have that $\omega^\alpha(A^*) = \omega^\beta(A^*) = A$ so it suffices to check for $Y \in TM_x$. First, $\omega^\beta\left(\sigma_{\beta*}Y\right) = \omega_\beta(Y)$

$$\begin{aligned}
\omega^\alpha\left(\sigma_{\beta*}Y\right) &= \omega^\alpha\left(\left[L^{-1}_{g_{\alpha\beta}(x)*}g_{\alpha\beta*}(Y)\right]^*_{\beta_\beta(x)} + R_{g_{\alpha\beta}(x)*}\sigma_{\alpha*}(Y)\right) \\
&= (L^{-1}_{g_{\alpha\beta}(x)*})g_{\alpha\beta*}(Y) + \mathcal{A}d_{g_{\alpha\beta}(x)^{-1}}\omega_\alpha(Y) = \omega_\beta(Y)
\end{aligned}$$

The result follows by the equality that must be satisfied in definition 3.3. ∎

Remark 3.6. It is possible to show that every principal fiber bundle admits a connection, a detailed discussion of this fact can be found in [KN63], at section 2.2.

In conclusion, we saw three separate ways of defining the same idea. Definition 3.1 is the most intuitive one, since it tells us a way of defining "directions" consistently throughout the tangent bundle of the principal bundle. However, having more than one definition will come in handy, especially when computing, since it is relatively easy to manipulate differential forms.

3.2 THE ALGEBRA OF $\mathfrak{g}$-VALUED FORMS

As we saw in the last section, a connection is just a $\mathfrak{g}$-valued differential form. In the case of ordinary differential forms we have a natural algebraic structure, which is very useful on a lot of occasions, one good example being that it is essential to define the de Rham cohomology of a manifold. In this section we will give the space of differential $\mathfrak{g}$-valued forms an algebraic structure and generalize the concept of exterior derivative.

Definition 3.7. Let $\mathfrak{g}$ be a Lie algebra and N a manifold. And let $\Lambda^k(N, \mathfrak{g})$ denote the set of $\mathfrak{g}$-valued forms, then we define the bilinear transformation $[\,.\,,\,.\,] : \Lambda^i(N, \mathfrak{g}) \times \Lambda^j(N, \mathfrak{g}) \longrightarrow$

$\Lambda^{i+j}(N,\mathfrak{g})$ as

$$[\omega,\eta]\left(X_1,\ldots,X_{i+j}\right)=\frac{1}{i!j!}\sum_{\sigma\in S_{i+j}}(-1)^{\sigma}\left[\omega\left(X_{\sigma(1)},\ldots,X_{\sigma(i)}\right),\eta\left(X_{\sigma(i+1)},\ldots,X_{\sigma(i+j)}\right)\right]$$

where S_{i+j} is the $i+j$th symmetric group. □

The above definition looks like the conjunction of the wedge product on differential forms with the Lie algebra product, and that is exactly right! Let $\Lambda^k(N,\mathbb{R})$ be set of differential forms, for $\omega\in\Lambda^k(N,\mathbb{R})$ and $A\in\mathfrak{g}$, we denote (suggestively) by $\omega\otimes A$ the $\mathfrak{g}$-valued form defined by $\omega\otimes A(X_1,\ldots,X_k)=\omega(X_1,\ldots,X_k)A$. Given a basis $E_1,\ldots,E_m$ we define the *structure constants* $c^{\gamma}_{\alpha\beta}$ using $\left[E_\alpha,E_\beta\right]=\sum c^{\gamma}_{\alpha\beta}E_\gamma$. It's not hard to see that given $\omega\in\Lambda^i(N,\mathfrak{g})$ and $\eta\in\Lambda^j(N,\mathfrak{g})$ there are unique forms $\omega^\alpha\in\Lambda^i(N,\mathbb{R}),\eta^\beta\in\Lambda^j(N,\mathbb{R})$ such that $\omega=\sum\omega^\alpha\otimes E_\alpha$ and $\eta=\sum\eta^\beta\otimes E_\beta$, then we get that

$$[\omega,\eta]=\sum_{\alpha,\beta}\left(\omega^\alpha\wedge\eta^\beta\right)\otimes\left[E_\alpha,E_\beta\right]=\sum_{\alpha,\beta,\gamma}c^{\gamma}_{\alpha\beta}\left(\omega^\alpha\wedge\eta^\beta\right)\otimes E_\gamma. \tag{3.1}$$

This equation will be quite useful. As stated the notation of the tensor product is intentional, we can make this identification due to the fact that, for vector spaces V,W, the space of $\Lambda^k(V,W)$ of linear forms corresponds naturally to the space $\Lambda^k(V,\mathbb{R})\otimes W$, that is to say that there is a natural isomorphism between them. This correspondence translates to a natural isomorphism in the *tensor product of vector bundles*! But for now what we have will suffice.

Proposition 3.8. *For $\omega\in\Lambda^i(N,\mathfrak{g}),\eta\in\Lambda^j(N,\mathfrak{g})$ and $\rho\in\Lambda^k(N,\mathfrak{g})$ we have that*

(i) $[\omega,\eta]=-(-1)^{ij}[\eta,\omega]$

(ii) $(-1)^{ik}[[\omega,\eta],\rho]+(-1)^{kj}[[\rho,\omega],\eta]+(-1)^{ji}[[\eta,\rho],\omega]=0$

Proof: By equation 3.1, (i) follows from the fact that $\omega^\alpha\wedge\eta^\beta=(-1)^{ij}\eta^\beta\wedge\omega^\alpha$ and $[E_\alpha,E_\beta]=-1[E_\beta,E_\alpha]$. For (ii)

$$\begin{aligned}&\textstyle\sum_{\alpha,\beta,\gamma}\Big\{(-1)^{ik}\omega^\alpha\wedge\eta^\beta\wedge\rho^\gamma\otimes\left[\left[E_\alpha,E_\beta\right],E_\gamma\right]\\&\quad+(-1)^{kj}\rho^\gamma\wedge\omega^\alpha\wedge\eta^\beta\otimes\left[\left[E_\gamma,E_\alpha\right],E_\beta\right]\\&\quad+(-1)^{ji}\eta^\beta\wedge\rho^\gamma\wedge\omega^\alpha\otimes\left[\left[E_\beta,E_\gamma\right],E_\alpha\right]\Big\}\\&=\textstyle\sum_{\alpha,\beta,\gamma}(-1)^{ik}\omega^\alpha\wedge\eta^\beta\wedge\rho^\gamma\otimes\Big\{\left[\left[E_\alpha,E_\beta\right],E_\gamma\right]+\left[\left[E_\gamma,E_\alpha\right],E_\beta\right]\\&\quad+\left[\left[E_\beta,E_\gamma\right],E_\alpha\right]\Big\}=0\end{aligned}$$

where in the last equality we use the Jacobi's identity in $\mathfrak{g}$. ■

Definition 3.9. Given a form $\omega\in\Lambda^k(N,\mathfrak{g})$ we can write it as $\omega=\sum\omega^\alpha\otimes E_\alpha$, we define the exterior derivative $d\omega$ of ω as $d\omega=\sum d\omega^\alpha\otimes E_\alpha$. □

Proposition 3.10. *For $\omega\in\Lambda^i(N,\mathfrak{g})$ and $\eta\in\Lambda^j(N,\mathfrak{g})$, we have* $d[\omega,\eta]=[d\omega,\eta]+(-1)^i[\omega,d\eta]$.

Proof: Using equation 3.1, we can see that this follows from $d(\omega^\alpha\wedge\eta^\beta)=d\omega^\alpha\wedge\eta^\beta+(-1)^i\omega^\alpha\wedge d\eta^\beta$, which is a property of the ordinary exterior derivative. ■

3.3 CURVATURE

Now we discuss the notion of curvature and prove some useful equations relating to it. In what follows P will be a principal fiber bundle over M and Lie group G and ω will be a connection 1-form. Let H^p the horizontals spaces associated with the connection form, note that for every $X \in TP_p$, $X_p = X^H + X^V$ where $X_p^H \in H_p$, $X_p^V \in V_p$, furthermore, for a smooth vector field X, the vector field X^H defined point wise as $(X^H)_p = X_p^H$ is also smooth. This happens because the horizontal spaces are assigned smoothly as discussed before.

Definition 3.11. Let $\varphi \in \Lambda^k(P, \mathfrak{g})$, then we define the horizontal form φ^H associated with φ by

$$\varphi^H(X_1, \dots, X_k) = \varphi\left(X_1^H, \dots, X_k^H\right)$$

as discussed above, this form is smooth. □

Definition 3.12. The *exterior covariant derivative* of a form $\varphi \in \Lambda^k(P, \mathfrak{g})$, written $d^\omega\varphi$, is defined as

$$d^\omega \varphi = (d\varphi)^H.$$

Note that this definition depends on the connection form. □

Now we may finally define the curvature of a connection.

Definition 3.13. The curvature of a connection one form ω is the 2-form $\Omega^\omega = d^\omega\omega \in \Lambda^2(P, \mathfrak{g})$. □

Now, we'll prove a very useful identity for the curvature, the *Structural Equation*. Namely, it states that $\Omega^\omega = d\omega + \frac{1}{2}[\omega, \omega]$, but first, we prove a few lemmas to help us.

Lema 3.14. *Given a vector field X on M, there is a unique vector field $\tilde{X}$, called the horizontal lift of X, on P such that $\omega(\tilde{X}) = 0$ and $\pi_*(\tilde{X}_p) = X_{\pi(p)}$ for all $p \in P$. Moreover, $(R_g)_*\tilde{X} = \tilde{X}$ for all $g \in G$.*

Proof: Existence and uniqueness follow from the fact that $d\pi_p : H_p \longrightarrow TM_{\pi(p)}$ is an isomorphism. Now observe that

$$\pi_*((R_g)_*\tilde{X}_p) = (\pi \circ R_g)_*(\tilde{X}_p) = \pi_*(\tilde{X}_p) = X_{\pi(p)}$$

since the action preserve the fibers, as π_* is injective, the result follows. ■

Lema 3.15. *Let $A, B \in \mathfrak{g}$, and let A^* be the fundamental field as defined in 3.2, then $[A, B]^* = [A^*, B^*]$ as vector fields in P.*

Proof: Defining $\varphi_t(p) = p \cdot \exp(tA)$ we see that it is the family of diffeomorphisms generated by the flow of of A^*, using the relation between the Lie derivative and the Lie bracket we compute, evaluating all derivatives at $t = 0$

$$\begin{aligned}
[A^*, B^*]_p &= \frac{d}{dt}\varphi_{t*}^{-1}\left(B^*_{\varphi_t(p)}\right) = \frac{d}{dt}\frac{d}{ds}\varphi_t(p)\exp(sB)\exp(tA)^{-1} \\
&= \frac{d}{dt}\frac{d}{ds}p\exp(tA)\exp(sB)\exp(tA)^{-1} = \frac{d}{dt}\frac{d}{ds}p\exp\left(s\mathcal{A}d_{\exp tA}B\right) \\
&= \frac{d}{ds}p\exp\left(s\frac{d}{dt}\left[\mathcal{A}d_{\exp tA}B\right]\right) = \frac{d}{ds}p\exp(s[A, B]) = [A, B]^*_p
\end{aligned}$$

which is the desired result. ■

Lema 3.16. *Let X be a vector field of M and $A \in \mathfrak{g}$, then $[\tilde{X}, A] = 0$.*

Proof: Let φ_t be as in the last proposition. Then, by lemma 3.14, we have that $\varphi_{t*}^{-1}(\tilde{X}) = \tilde{X}$, using the correspondence of Lie brackets and derivatives we get

$$[A^*, \tilde{X}] = \frac{d}{dt}\varphi_{t*}^{-1}(\tilde{X}) = 0. \quad ■$$

With these lemmas in hand, we may state and prove the following:

Theorem 3.17 (Cartan's Structural Equation). *Given a connection form ω, its curvature is given by the equation*

$$\Omega^{\omega} = d\omega + \frac{1}{2}[\omega, \omega].$$

Proof: First, we observe that $\frac{1}{2}[\omega,\omega](Y,Z) = \frac{1}{2}(\omega(Y,Z) - \omega(Z,Y)) = \omega(Y,Z)$, thus it suffices to that $d\omega\left(Y^H, Z^H\right) = d\omega(Y,Z) + [\omega(Y), \omega(Z)]$ holds. Since every tangent space splits in the direct sum of its horizontal and vertical space, the proof is divided in three cases.

Case 1: $Y, Z \in H_p$. By definition $\omega(Y) = \omega(Z) = 0$, and $Y^H = Y$, $Z^H = Z$, so the equation holds.

Case 2: $Y, Z \in V_p$, we can suppose that $Y = A^*_p$ and $Z = B^*_p$ for $A, B \in \mathfrak{g}$ then $d\omega(Y,Z) = A^*\left[\omega\left(B^*\right)\right] - B^*\left[\omega\left(A^*\right)\right] - \omega([(A^*, B^*])$, this follows from the property $d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$ of ordinary differential forms (see [Bie20], proposition 9.40). Note that the only non vanishing term is $-\omega([A^*, B^*])$, since $\omega(A^*)$ is constant, so by Lemma 3.15 we get $-\omega([A^*, B^*]) = -\omega([A,B]^*)$ and the right side of the equation equals zero. Now, since $Y^H = Z^H = 0$, the equation holds.

Case 3: $Y \in V_p$, $Z \in H_p$. Since we Z is horizontal by lemma 3.14 we may assume that $Z = \tilde{X}_p$, where X is the horizontal lift of some vector field in M and that $Y = A^*_p$ for $A \in \mathfrak{g}$, thus $d\omega(Y,Z) = A^*[\omega(\tilde{X})] - \tilde{X}\left[\omega\left(A^*\right)\right] - \omega([(A^*, \tilde{X}]) = 0$, since $\omega(A^*) = A$ which is constant, $\omega(\tilde{X}) = 0$ and $[A^*, \tilde{X}] = 0$ by lemma 3.16. So both sides of the equation vanish.

This concludes the proof, since $TP_p = H_p \oplus V_p$. ■

Theorem 3.18 (Bianchis's Identity). *If ω is a connection form with curvature Ω^{ω}, then $d^{\omega}\Omega = 0$. Moreover, $d\Omega^{\omega} = [\Omega^{\omega}, \omega]$.*

Proof: We know from the Structural equation that $d\Omega^{\omega} = d(d\omega + \frac{1}{2}[\omega,\omega]) = d^2\omega + \frac{1}{2}[d\omega, \omega] - \frac{1}{2}[\omega, d\omega] = [d\omega, \omega] = [d\omega + 1/2[\omega,\omega], \omega] = [\Omega^{\omega}, \omega]$. Now, the result follows from the fact that ω is zero in horizontal vectors. ■

Proposition 3.19. *For all $g \in G$, $R^*_g\Omega^{\omega} = \mathcal{A}d_{g^{-1}}\Omega^{\omega}$.*

Proof: Is easy to see that pullbacks preserve the product of $\mathfrak{g}$-valued algebras (by its definition). Then

$$\begin{aligned}
&R^*_g\Omega^{\omega} = R^*_g\left(d\omega + \frac{1}{2}[\omega,\omega]\right) = dR^*_g\omega + \frac{1}{2}\left[R^*_g\omega, R^*_g\omega\right] = \\
&d\mathcal{A}d_{g^{-1}}\omega + \frac{1}{2}\left[\mathcal{A}d_{g^{-1}}\omega, \mathcal{A}d^{g}_{g^{-1}}\omega\right] = \mathcal{A}d_{g^{-1}}\left(d\omega + \frac{1}{2}[\omega,\omega]\right) \\
&= \mathcal{A}d_{g^{-1}}\Omega^{\omega}
\end{aligned}$$

where we used the property (ii) of a connection one from. ■

Definition 3.20. For every local trivialization Φ we have an associated section $\sigma_u : U \longrightarrow P$. The $\mathfrak{g}$-valued one form in U of our third definition is given by $\omega_u = \sigma^*_u\omega \in \Lambda^1(U, \mathfrak{g})$. The *field strength* of the ω_u is defined as $\Omega_u = \sigma^*_u\Omega^{\omega}$. □

The field strength is of much importance to physics, so it'd be desirable to have a way of computing it only in terms of the form ω_u. So we prove the following:

Proposition 3.21. *The field strength is given by $\Omega_u = d\omega_u + \frac{1}{2}[\omega_u, \omega_u]$.*

Proof:

$$\Omega_u = \sigma_u^* (\Omega^\omega) = \sigma_u^* \left(d\omega + \frac{1}{2}[\omega, \omega] \right) = d\left(\sigma_u^* \omega\right) +$$
$$\frac{1}{2}\left[\sigma_u^* \omega, \sigma_u^* \omega\right] = d\omega_u + \frac{1}{2}\left[\omega_u, \omega_u\right].$$

■

Now it will be shown that the change of local field strengths is quite easy (when compared by the rule given by our third definition).

Proposition 3.22. *Let Φ_α and Φ_β be local trivializations with change of coordinates $g_{\alpha\beta}$: $U_{\alpha\beta}$longrightarrow$G$. Then, in $U_{\alpha\beta}$, $\Omega_\beta = \mathcal{A}d_{g_{\alpha\beta}^{-1}}\Omega_\alpha$.*

Proof: Let σ_α and σ_β be the forms associated with the local trivializations, then for $Y \in TM_x$, from the proof of theorem 3.5

$$\sigma_{\beta *}(Y) = \left[L^{-1}_{g_{\alpha\beta}(x)*} g_{\alpha\beta *}(Y)\right]^*_{\sigma_\beta(x)} + R_{g_{\alpha\beta}(x)*}\sigma_{\alpha *}(Y).$$

For any Z, W, $\Omega^\omega(W, Z) = 0$ if either of the two vectors is vertical. Therefore

$$\begin{aligned}
\Omega_\beta(X, Y) &= \Omega^\omega\left(\sigma_{\beta *}(X), \sigma_{\beta *}(Y)\right) \\
&= \Omega^\omega\left(R_{g_{\alpha\beta}(x)*}\sigma_{\alpha *}(X), R_{g_{\alpha\beta}(x)*}\sigma_{\alpha *}(Y)\right) \\
&= \mathcal{A}d_{g_{\alpha\beta}(x)^{-1}}\Omega_\alpha(X, Y).
\end{aligned}$$

In this we used the linearity of the form together with the fact that $\sigma_{\beta *}(Y) = [L^{-1}_{g_{\alpha\beta}(x)*} g_{\alpha\beta *}(Y)]^*_{\sigma_\beta(x)}$ is a vertical vector. ■

Theorem 3.23 (Local Bianchi's Identity). *Given a local trivialization in U, we have*

$$d\Omega_u = [\Omega_u, \omega_u].$$

Proof: It follows immediately from 3.18, Bianchi's Identity, since the brackets are preserved by pullbacks. ■

WHAT IS A GAUGE?

The goal of all of this, as stated in the title, is to study *Gauge Theories*, but until now,we haven't talked about what a *gauge* is! Actually, we already know what is is, but with another name, so we dedicate this small section to rename stuff and to familiarize ourselves with some terms.

Let $P \to M$ be a principal fiber bundle. A (local) **gauge** (or choice of gauge) is a local trivialization, equivalently a local section is a gauge. Gauges are important to physicists because they almost always work with local coordinates, never really worrying about the global picture.

A connection 1-form ω is called a **gauge potential**, and, for a given local gauge $\sigma_u : U \longrightarrow P$, the 1-form $\omega_u = \sigma_u^* \omega$ is called a **local gauge potential**. The curvature of a connection is regarded as the strength (in a physical sense) of a field, hence the name **field strength**. As we saw, it is defined as $d^\omega \omega = \Omega^\omega$. Again, we have its local relative, namely $\Omega_u = \sigma_u^* \Omega^\omega$, this is the **local field strength**.

To conclude this little discussion we talk about the physical motivation of gauge theories, even though our interest lies in mathematics. Time and time again in physics, the universe was seen to behave in a *geometrical manner*, that is, physical quantities were oftentimes described by geometrical objects, such as curvature. The most famous example of such things being Einstein's General Relativity, where gravity is but curvature and objects travel through geodesics in space-time. With the theory of principal fiber bundles it is possible to "geometrize" other physical aspects of reality. For instance, electromagnetism can be formulated in terms of principal bundles (more specifically the trivial bundle with group $U(1)$), and quantities, such as the intensity of an electric field, are nothing more than the curvature of a connection. Using this theory physicists and mathematicians were able to come up with the celebrated *Standard Model* for particles.

Now that we have some more jargon and a little motivation, we may continue to develop the theory.

3.4 PARTICLE FIELDS

A particle field can be regarded as a section of a vector bundle associated to some principal fiber bundle, or equivalently, as a vector-valued function on P with certain transformation properties. These properties arise from the action of a Lie group in some manifold (often times vector spaces), which can be seen as a fiber of some associated bundle to the principal fiber bundle.

Definition 3.24. Let G be a Lie group acting smoothly on some manifold F by the left. As we are accustomed, let $L_g : F \longrightarrow F$ be the diffeomorphism induced by fixing $g \in G$. If F is a vector space and $L_g \in GL(F)$ for all $g \in G$ we say that we have a *representation* of G in F. Suppose that we have a representation of G in two vector spaces F and V, then a morphism between representations is a linear transformation $T : F \longrightarrow V$ such that

$$\begin{array}{ccc} F & \xrightarrow{L_g} & F \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ V & \xrightarrow[L'_g]{} & V \end{array}$$

commutes for all $g \in G$ (*i.e.* a natural transformation). If T is an isomorphism we say that the representations are equivalent. $\square$

Definition 3.25. Let P be a principal G-bundle and F a manifold being acted by G from the left. Then we define the set $C^\infty_G(P, F)$ to be

$$C^\infty_G(P, F) = \{\tau : P \longrightarrow F \mid \tau(pg) = g^{-1}\tau(p)\ \forall\ g \in G\}.$$

In the case that F is a vector space, and the action of G a representation, an element of $C^\infty_G(P, F)$ is called a *particle field*. $\square$

Now, for the sake of completeness, we briefly define what an associated bundle of a principal fiber is and show that, in particular, particle fields are just sections of an associated vector bundle.

Let G be a vector bundle acting on a manifold F of dimension n. And let $P \to M$ be a principal fiber bundle. Then we can define an action $(P \times F) \times G \to P \times F$ by setting $(p, f) \cdot g = \left(p \cdot g, g^{-1} \cdot f\right)$. Let $P \times_G F = (P \times F)/G$ be set of orbits of the action, $\tilde{\pi} : P \times F \longrightarrow$

$P \times_G F$ be the canonical projection and $\pi'([p, f]) = \pi(p)$. Then it can be shown that

$$\begin{tikzcd} P \times F \arrow[r, "\tilde{\pi}"] \arrow[d, "pr_1"'] & P \times_G F \arrow[d, "\pi'"] \\ P \arrow[r, "\pi"'] & M \end{tikzcd}$$

commutes and that $\pi' : P \times_G F \to M$ is a G-bundle with typical fiber F (see [Bie20] section 22.3 p.420). This is the associated fiber bundle of P of the action of G in the fiber F. If F is a vector space and the action of G is a representation, the associated fiber bundle is a vector bundle, called the associated vector bundle of said representation. Now we show that $C_G^\infty(P, F)$ is equivalent to the space of the section of the bundle $P \times_G F$.

Theorem 3.26. *There exists a natural bijection between $C_G^\infty(P, F)$ and $\Gamma(P \times_G F)$.*

Proof: Let $\tau \in C_G^\infty(P, F)$, we define the section $s_\tau : M \longrightarrow P \times_G F$ by setting $s_\tau(x) = [p, \tau(p)]$ where p is any element of $\pi^{-1}(x)$. To see that it is well defined just note that for any $q \in \pi^{-1}(x)$, $q = pg$ for some g, so $(q, \tau(q)) = (pg, g^{-1}\tau(p)) = (p, \tau(p)) \cdot g$, so they are in the same orbit.

Now let $s : M \longrightarrow P \times_G F$ be a section, then $s(x) = [p, f] = \{(p, f) \cdot g \mid g \in G\}$ where $p \in \pi^{-1}(x)$. Define $\tau(q) = g^{-1}f$ where $pg = q$. Is easy to see that this doesn't depend on the choice of generating element (p, f). These two maps are clearly the inverses of one another. ■

Corollary 3.27. *A particle field can be thought as a section in the vector bundle $P \times_G V$.*

Proof: The bijection holds, of course, but since V is a vector space, we see that $C_G^\infty(P, V)$ has the natural structure of a vector space, and the bijection above can be checked to be a linear isomorphism. ■

Now we continue with our discussion, in what follows V will be a vector space.

Definition 3.28. Let $\Lambda_{eq}^k(P, V)$ be the space of V-valued forms satisfying the following properties:

(i) $\varphi\left(R_{g*}X_1, \ldots, R_{g*}X_k\right) = g^{-1} \cdot \varphi\left(X_1, \ldots, X_k\right)$, that is $R_g^*\varphi = g^{-1} \cdot \varphi$.

(ii) If X_i is vertical for some i, then $\varphi\left(X_1, \ldots, X_k\right) = 0$.

Forms satisfying (i) are called equivariant forms while forms satisfying (ii) are called horizontal forms. □

Proposition 3.29. *If $V = \mathfrak{g}$ and the representation is the adjoint representation $g \mapsto \mathcal{A}d_g$, then $\Omega^\omega \in \Lambda_{eq}^2(P, \mathfrak{g})$ for any connection.*

Proof: Condition (i) follows from the definition of the curvature of a connection and (ii) from Proposition 3.19. ■

Definition 3.30. For a connection ω, let $d^\omega : \Lambda_{eq}^k(P, V) \longrightarrow \Lambda_{eq}^{k+1}(P, V)$ be the linear map defined as $d^\omega\varphi = (d\varphi)^H$ (defined as in 3.11). Note that

$$R_g^* d^\omega \varphi = R_g^*(d\varphi)^H = \left(R_g^* d\varphi\right)^H = \left(dR_g^*\varphi\right)^H = \left(dg^{-1} \cdot \varphi\right)^H = g^{-1} \cdot (d\varphi)^H = g^{-1} \cdot d^\omega \varphi$$

so the map is well defined. □

Definition 3.31. If $G \to GL(V)$ is a representation, then by proposition 2.21, we get a map $\mathfrak{g} \to \mathfrak{gl}(V)$ which, is given by

$$A \cdot v = \frac{d}{dt}(\exp tA) \cdot v\bigg|_{t=0}$$

for $v \in V$ and $A \in \mathfrak{g}$. For $\varphi \in \Lambda^k(P, V)$ and $\rho \in \Lambda^j(P, \mathfrak{g})$ we define $\rho \dot{\wedge} \varphi \in \Lambda^{j+k}(P, V)$ by

$$(\rho \dot{\wedge} \varphi)\left(X_1, \dots, X_{j+k}\right) = \frac{1}{j!k!} \sum_{\sigma} (-1)^{\sigma} \rho\left(X_{\sigma(1)}, \dots, X_{\sigma(j)}\right) \cdot \varphi\left(X_{\sigma(j+1)}, \dots, X_{\sigma(j+k)}\right)$$

where σ represents a permutation. □

Proposition 3.32. *For* $\tau \in \Lambda^k_{eq}(P, V)$*, we have* $d^{\omega}\tau = d\tau + \omega \dot{\wedge} \tau$.

Proof: We have to show that

$$d\tau\left(X_1^H, \dots, X_{k+1}^H\right) = d\tau\left(X_1, \dots, X_{k+1}\right) + \frac{1}{k!} \sum_{\sigma} (-1)^{\sigma} \omega\left(X_{\sigma(1)}\right) \cdot \tau\left(X_{\sigma(2)}, \dots, X_{\sigma(k+1)}\right)$$

holds. Similar to the demonstration of the structural equation we look at some cases. If $(X_1, \dots, X_{k+1})$ are horizontal the equations hold since $X_i^H = X_i$ and $\omega(X_i) = 0$. If two or more vectors are vertical, we have to show that $d\tau(X_1, \dots, X_{k+1}) = 0$ since τ will vanish. Extending the vertical fields to fundamental ones and using the same strategy as in case 2 of the demonstration of theorem 3.17, we get:

$$\begin{aligned} d\tau\left(X_1, \dots, X_{k+1}\right) &= \sum_{i=1}^{k+1} (-1)^{i+1} X_i\left[\tau\left(X_1, \dots, \hat{X}_i, \dots, X_{k+1}\right)\right] \\ &+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \tau\left(\left[X_i, X_j\right], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}\right) \end{aligned}$$

thus the first sum is zero since at least one vector is always vertical and the same is true for the second sum by lemma 3.15. The remaining case is that of which only one vector is vertical, assume without loss of generality that this vector is X_1. So we can extend X_1 to a fundamental field and by lemma 3.14 (extend the projection by π of X_1) we can extend X_i to invariant fields over R_{g*}. So, by lemma 3.16, we have that $[X_1, X_j] = 0$ for all $j \neq 1$, thus we get that $d\tau(X_1, \dots, X_{k+1}) = X_1\left[\tau\left(X_2, \dots, X_{k+1}\right)\right]$. So, for our equation to hold, we need that $X_1\left[\tau\left(X_2, \dots, X_{k+1}\right)\right] + \omega\left(X_1\right) \cdot \tau\left(X_2, \dots, X_{k+1}\right) = 0$. By letting $X_1 = A^*$ and $g_t = \exp tA$ we get

$$\begin{aligned} X_1\left[\tau\left(X_2, \dots, X_{k+1}\right)\right] &= \frac{d}{dt}\left[\tau\left(R_{g_t*}X_2, \dots, R_{g_t*}X_{k+1}\right)\right] \\ &= \frac{d}{dt}\left[g_t^{-1} \cdot \tau\left(X_2, \dots, X_{k+1}\right)\right] \\ &= -A \cdot \tau\left(X_2, \dots, X_{k+1}\right) \\ &= -\omega\left(X_1\right) \cdot \tau\left(X_2, \dots, X_{k+1}\right) \end{aligned}$$

and our equation holds once more. ■

Corollary 3.33. *Given the adjoint representation* $\mathfrak{g} \to GL(\mathfrak{g})$*, we have* $d^{\omega}\tau = d\tau + [\omega, \tau]$.

Proof: The lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $A \cdot B = [A, B]$, so we get that $\omega \dot{\wedge} \tau = [\omega, \tau]$. ■

3.5 GAUGE TRANSFORMATIONS

We now talk about gauge transformations, and as one might expect, it has to do with change of coordinates (gauges).

Definition 3.34. A morphism of principal fiber bundles $P \to M$ and $P' \to M'$ with same group G is a pair of smooth functions $F : P \longrightarrow P'$, $f : M \longrightarrow M'$ with $F(pg) = F(p)g$ such that the following commutes

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "\pi"'] & P' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

A gauge transformation is a bundle isomorphism such that $P \to M = P' \to M'$ and $f = \mathrm{id}_M$. $GA(P)$ is the set of gauge transformations of the bundle. □

Remark 3.35. One thing to note is that any morphism of the principal bundle over M (that is, a morphism with $f = \mathrm{id}_M$) is an isomorphism! This follows from the fact that it must respect the action of G which is free and transitive on fibers. In other words, the category $\mathbf{PBund}_G$ of principal bundles with structural group G and base space M is a grupoid.

Theorem 3.36. *Let $C^\infty_G(P, G)$ be as in definition 3.25 where G acts on itself by $g \cdot h = ghg^{-1}$, then there is natural isomorphism between $C^\infty_G(P, G)$ and $GA(P)$.*

Proof: If $\tau \in C^\infty_G(P, G)$, then define $f : P \to P$ by $f(p) = p\tau(p)$. since $f(pg) = pg\tau(pg) = pgg^{-1}\tau(p)g = p\tau(p)g = f(p)g$, it follows that $f \in GA(P)$. Conversely, if $f \in GA(P)$, define $\tau : P \to G$ by the relation $f(p) = p\tau(p)$. Note that $pg\tau(pg) = f(pg) = f(p)g = p\tau(p)g$, whence $\tau(pg) = g^{-1}\tau(p)g$, and it follows that $\tau \in C^\infty_G(P, G)$. Finally, if $f, f' \in GA(P)$ with $f(p) = p\tau(p)$ and $f'(p) = p\tau'(p)$, then $\left(f \circ f'\right)(p) = p\left(\tau(p)\tau'(p)\right)$ ■

Proposition 3.37. *If $f \in GA(P)$ and ω is a connection 1-form, then $f^*\omega$ is a connection 1-form.*

Proof: To prove this, we need to check conditions (i) and (ii) of definition 3.2. For (i), we have

$$(f^*\omega)\left(A^*_p\right) = \omega(f_*A^*_p) = \omega\left(\frac{d}{dt} f(p \exp tA)\bigg|_{t=0}\right) = \omega\left(\frac{d}{dt} f(p) \exp tA\bigg|_{t=0}\right) = \omega\left(A^*_{f(p)}\right) = A.$$

Now we need to check (ii), since $f(p)g = f(pg)$, we have that $R_g \circ f = f \circ R_g$, thus we get that

$$R^*_g f^*\omega = \left(f \circ R_g\right)^* \omega = \left(R_g \circ f\right)^* \omega = f^* R^*_g \omega = f^* \mathcal{A}d_{g^{-1}}\omega = \mathcal{A}d_{g^{-1}} f^*\omega.$$

■

Proposition 3.38. *If f is a gauge transformation and $G \to GL(V)$ is a representation, then the pullback of f is an linear automorphism of $\Lambda^k_{eq}(P, V)$*

Proof: Since f is a diffeomorphism, its pullback its already injective, so it suffices to show that the map is well defined, that is, if $\varphi \in \Lambda^k_{eq}(P, V)$, then $f^*\varphi \in \Lambda^k_{eq}(P, V)$. We have to show that conditions (i) and (ii) of definition 3.28 hold. For (i) we have shown that $R^*_g f^* = f^* R^*_g$, so for $\varphi \in \Lambda^k_{eq}(P, V)$, we have

$$R^*_g f^*\varphi = f^* R^*_g \varphi = f^*(g^{-1}\varphi) = g^{-1} f^*\varphi.$$

For (ii), let A^* be a fundamental field, then, by the previous demonstration, we have that $f_*A^* = A^*$, so

$$\left(f^*\varphi\right)\left(A^*\right) = \varphi\left(f_*A^*\right) = \varphi\left(A^*\right) = 0.$$

Thus, we conclude that $f^*\varphi \in \Lambda^k_{eq}(P, V)$. ∎

Definition 3.39. Let $P \to M$ be a principal G-bundle. Then we denote by $\mathrm{Conn}(P)$ the set of connection 1-forms. As observed in remark 3.6, this set is not empty. □

Theorem 3.40. *Let $\omega \in \mathrm{Conn}(P)$, and $\Lambda^1_{eq}(P, \mathfrak{g})$ be the set of horizontal equivariant forms for the adjoint action, then the map $\Lambda^1_{eq}(P, \mathfrak{g}) \to \mathrm{Conn}(P)$ defined by $\varphi \mapsto \varphi + \omega$ is a bijection.*

Proof: If $\varphi \in \Lambda^1_{eq}(P, \mathfrak{g})$, then $(\varphi + \omega)(A^*) = \varphi(A^*) + \omega(A^*) = 0 + A = A$, also, we have that

$$R^*_g(\varphi + \omega) = R^*_g\varphi + R^*_g\omega = \mathcal{A}d_{g^{-1}}\varphi + \mathcal{A}d_{g^{-1}}\omega = \mathcal{A}d_{g^{-1}}(\varphi + \omega)$$

using the properties of each form, so the map is well defined. Now, if $\eta \in \mathrm{Conn}(P)$, it's not hard to see $\eta - \omega \in \Lambda^1_{eq}(P, \mathfrak{g})$. These maps are clearly the inverses of one another, thus we proved the desired result. ∎

The theorem above allows us to see the set of connection forms and an affine subspace of the vector space of 1-forms.

Proposition 3.41. *Let $C^\infty_G(P, \mathfrak{g})$ with the adjoint action, then, if $\tau, \tau' \in C^\infty_G(P, \mathfrak{g})$, the map $[\tau, \tau'] : P \longrightarrow \mathfrak{g}$ defined by $[\tau, \tau'](p) = [\tau(p), \tau'(p)]$ is also in $C^\infty_G(P, \mathfrak{g})$.*

Proof: The proof it's just a matter of computation, for any p and g, we have

$$\begin{aligned}\left[\tau, \tau'\right](pg) &= \left[\tau(pg), \tau'(pg)\right] = \left[\mathcal{A}d_{g^{-1}}\tau(p)\,,\, \mathcal{A}d_{g^{-1}}\tau'(p)\right] \\ &= \mathcal{A}d_{g^{-1}}\left[\tau(p), \tau'(p)\right] = \mathcal{A}d_{g^{-1}}\left[\tau, \tau'\right](p)\end{aligned}$$

which is the desired result. ∎

Corollary 3.42. *$C^\infty_G(P, \mathfrak{g})$ has a natural Lie algebra structure.*

Proof: Just use the product defined in the proposition above. ∎

Definition 3.43. Let $G \to GL(\mathfrak{g})$ be the adjoint representation, the we call the space $C^\infty_G(P, \mathfrak{g})$ the Gauge Algebra of the bundle. As seen above, the name is justified. □

Since in $C^\infty_G(P, G)$, G is a group, we have a natural group structure in it, namely for $\tau, \sigma \in C^\infty_G(P, G)$, we define the multiplication point wise . So $(\tau\sigma)(pg) = \tau(pg)\sigma(pg) = (g^{-1}\tau(p)g)(g^{-1}\sigma(p)g) = g^{-1}\tau(p)\sigma(p)g$, thus $\tau\sigma \in C^\infty_G(P, G)$. The identity is clearly the constant function to the identity.

Proposition 3.44. *The map $\mathrm{Exp} : C^\infty_G(P, \mathfrak{g}) \longrightarrow C^\infty_G(P, G)$ defined by $\mathrm{Exp}(\sigma)(p) = \exp(\sigma(p))$ is such that for $r \in \mathbb{R}$, $t \mapsto \mathrm{Exp}(t\sigma)$ is group homomorphism. Also this assignment satisfies*

$$\frac{d}{dt}\mathrm{Exp}(t\sigma)(p) = \sigma(p).$$

Moreover, for $\sigma, \tau \in C^\infty_G(P, \mathfrak{g})$, we have that

$$[\sigma, \tau](p) = \frac{\partial^2}{\partial t \partial s} \mathrm{Exp}(t\sigma)_p\, \mathrm{Exp}(s\tau)_p\, \mathrm{Exp}(t\sigma)_p^{-1}\bigg|_{s,t=0}.$$

Proof: For $t, s \in \mathbb{R}$ and $p \in P$, we have $\mathrm{Exp}((t+s)\sigma)(p) = \exp((t+s)\sigma(p)) = \exp(t\sigma(p))\exp(s\sigma(p)) = [\mathrm{Exp}(t\sigma)\mathrm{Exp}(s\sigma)](p)$. The remaining proofs are similar and consist of using results already proven, but point wise, results like 2.25. ∎

Definition 3.45. We define a map the map $\exp : C^\infty_G(P, \mathfrak{g}) \longrightarrow GA(P)$ by $\exp(H)(p) = p\exp(H(p))$, this is just the map from theorem 3.40 composed with Exp. □

Lema 3.46. *Let $f \in GA(P)$ and $\tau \in C^\infty_G(P, G)$ be its corresponding function i.e. $f(p) = p\tau(p)$, then for $X \in TP_p$ we have*

$$f_*(X) = \left(L^{-1}_{\tau(p)*}\tau_*(X)\right)^*_{f(p)} + R_{\tau(p)*}(X).$$

Proof: Let $\gamma : \mathbb{R} \longrightarrow P$ be such that $\gamma'(0) = X$, then

$$\begin{aligned} f_*(X) &= \frac{d}{dt}f(\gamma(t)) = \frac{d}{dt}\gamma(t)\tau(\gamma(t)) = \frac{d}{dt}p\tau(\gamma(t)) + \frac{d}{dt}\gamma(t)\tau(p) \\ &= \frac{d}{dt}p\tau(p)\tau(p)^{-1}\tau(\gamma(t)) + \frac{d}{dt}R_{\tau(p)}(\gamma(t)) \\ &= \frac{d}{dt}f(p)\tau(p)^{-1}\tau(\gamma(t)) + R_{\tau(p)*}\left(\gamma'(0)\right) \\ &= \left(L^{-1}_{\tau(p)*}\tau_*(X)\right)^*_{f(p)} + R_{\tau(p)*}(X). \quad \blacksquare \end{aligned}$$

Proposition 3.47. *For $\omega \in \mathrm{Conn}(P)$, $f \in GA(P)$ and its corresponding function $\tau \in C^\infty_G(P, G)$, we have*

$$\left(f^*\omega\right)_p = L_{\tau(p)*^{-1}}\tau_{p*} + \mathcal{A}d_{\tau(p)^{-1}}\omega_p$$

Proof: This follows from the lemma above, apply ω to both sides of the equation and use the definition of the pullback. ■

Lema 3.48. *For $\varphi \in \Lambda^k_{eq}(P, V)$, $f \in GA(P)$ and its corresponding function $\tau \in C^\infty_G(P, G)$, we have*

$$f^*\varphi = \tau^{-1} \cdot \varphi.$$

Proof: Thus the result also follows from Lemma 3.47. Just recall that fundamental vector fields are vertical, and so φ vanishes on it, the result follows by condition (i) of the definition for $\varphi \in \Lambda^k_{eq}(P, V)$. ■

Theorem 3.49. *If ω is a connection and $\sigma \in C^\infty_G(P, \mathfrak{g})$, then*

$$\frac{d}{dt}(\exp t\sigma)^*\omega\bigg|_{t=0} = d\sigma + [\omega, \sigma] = d^\omega\sigma \in \Lambda^1_{eq}(P, \mathfrak{g}).$$

Proof: We have that $\exp(t\sigma) \in GA(P)$ for every t. Using proposition 3.47, we have

$$\frac{d}{dt}(\exp t\sigma)^*\omega = \sigma_* - [\sigma, \omega] = d\sigma + [\omega, \sigma] = d^\omega\sigma$$

Here, we are using corollary 3.33 and its proof. The first part of the equation in proposition 3.47 becomes σ_* and the rest is $[\omega, \sigma]$. ■

Theorem 3.50. *Let $\varphi \in \Lambda^k_{eq}(P, V)$ and $\sigma \in C^\infty_G(P, \mathfrak{g})$. Then*

$$\frac{d}{dt}(\exp t\sigma)^*\varphi\bigg|_{t=0} = -\sigma \cdot \varphi.$$

Proof: This follows from Lemma 3.48, taking f to be $\exp(t\sigma)$. ■

4

SOME FIELD THEORY

In this final chapter we put the theory we developed so far into use in the study of *Field Theories*. Field theory is the study in physics where the notion of a particle is that of a field (a function taking values of some vector space) instead of a "punctual thing". Principal fiber bundles are important because they allow us to treat formally the notion of *symmetries of a system*, that is, to find solutions about physical quantities of a given system which are invariant in some way by the action of a Lie group, the Group of Symmetries of a system.

4.1 LAGRANGIANS

In what follows, we will discuss the notions of Langrangians and some specific associated functions to it. With the Lagrangian it will be possible to define the *principle of least* action for particle fields in a bundle with fixed curvature form. Particle fields satisfying this principle represent *accessible physical states*, also we'll also give an explicit method of calculation of such states via a differential equation, namely Lagrange's equation.

Definition 4.1. Let $P \to M$ be a principal fiber bundle and $G \to GL(V)$ be a representation, a 1-jet is a tuple (p, v, θ) where $p \in P$, $v \in V$ and $\theta : TP_p \longrightarrow V$ is linear. The set of 1-jets from P to V is denoted $J(P, V)$. □

Theorem 4.2. *The set of 1-jets can be made into a smooth manifold in a natural manner.*

Proof: We just give the general idea. Fixing a basis in V we have an isomorphism with $\mathbb{R}^m$ for some m. Now, given a chart (φ, U) on P, we have a local frame in TP, thus we may represent the transformations θ with this frame and basis the V, therefore obtaining a chart for $J(P, V)$. ■

Definition 4.3. A Lagrangian is a smooth function $L : J(P, V) \longrightarrow \mathbb{R}$ such that for all $(p, v, \theta) \in J(P, V)$, $g \in G$, we have $L(pg, g^{-1}v, g^{-1} \cdot \theta \circ R_{g^{-1}*}) = L(p, v, \theta)$. □

Theorem 4.4. *Let* $L : J(P, V) \longrightarrow \mathbb{R}$ *be a Lagrangian, then there exists a function* $\mathcal{L}_0 : C^\infty_G(P, V) \longrightarrow C^\infty(M)$ *such that* $\mathcal{L}_0(\psi)(x) = L(p, \psi(p), d\psi_p)$ *for* $\psi \in C^\infty_G(P, V)$ *and* $p \in \pi^{-1}(x)$.

Proof: We just need to show that the function is well defined, since smoothness will follow from the differentiability of L. Observe that $\psi \circ R_g = g^{-1}\psi$, $d\psi_{pg} \circ R_{g*} = g^{-1} d\psi_p$, thus $d\psi_{pg} = g^{-1} \cdot d\psi_p$, so

$$L\left(pg, \psi(pg), d\psi_{pg}\right) = L\left(pg, g^{-1} \cdot \psi(p), g^{-1} \cdot d\psi_p \circ R_{g^{-1}*}\right) = L\left(p, \psi(p), d\psi_p\right).$$

■

Definition 4.5. A Lagrangian is called G-invariant if $L(p, gv, g\theta) = L(p, v, \theta)$. Most of the traditional Lagrangians satisfy this property. □

As we saw, all Lagrangians have an associated function $\mathcal{L}_0$. The function $\mathcal{L}_0$ is called gauge invariant if $\mathcal{L}_0(\psi) = \mathcal{L}_0\left(f^{-1*}\psi\right)$ for all $f \in GA(P)$. Unfortunately, L being G-invariant does not imply that $\mathcal{L}_0$ is gauge invariant, but there is a way to remedy this. We will consider a closely related function, but with some tweaks.

Theorem 4.6. *Let $L : J(P, V) \longrightarrow \mathbb{R}$ be a G-invariant Lagrangian, we define the function $\mathcal{L} : C^\infty_G(P, V) \times \mathrm{Conn}(P) \longrightarrow C^\infty(M)$ by $\mathcal{L}(\psi, \omega)(x) = L\left(p, \psi(p), d^\omega\psi_p\right)$ for $p \in \pi^{-1}(x)$. The function $\mathcal{L}$ is well defined and it is gauge invariant.*

Proof: First, it needs to be shown that the map is well defined. From definition 3.30 and the fact that $\Lambda^0_{eq}(P, V) = C^\infty_G(P, V)$, we know that $D^\omega\psi \in \Lambda^1_{eq}(P, V)$, so

$$d^\omega\psi_{pg} \circ R_{g*} = g^{-1} \cdot d^\omega\psi_p \text{ or } d^\omega\psi_{pg} = g^{-1} \cdot d^\omega\psi_p \circ R_{g^{-1}*}$$

so we have

$$\begin{aligned} L\left(pg, \psi(pg), d^\omega\psi_{pg}\right) &= L\left(pg, g^{-1}\psi(p), g^{-1} \cdot d^\omega\psi_p \circ R_{g^{-1}*}\right) \\ &= L\left(p, \psi(p), d^\omega\psi_p\right). \end{aligned}$$

Now we have to check the gauge invariance :

$$\begin{aligned} \mathcal{L}\left(f^*\psi, f^*\omega\right)(x) &= L\left(p, \left(f^*\psi\right)(p), d\left(f^*\psi\right)_p + f^*\omega_p \cdot \left(f^*\psi\right)(p)\right) \\ &= L\left(p, \tau(p)^{-1} \cdot \psi(p), f^*\left(d\psi_p + \omega_p \cdot \psi(p)\right)\right) \\ &= L\left(p, \tau(p)^{-1} \cdot \psi(p), f^*\left(d^\omega\psi_p\right)\right) \\ &= L\left(p, \tau(p)^{-1} \cdot \psi(p), \tau(p)^{-1} \cdot d^\omega\psi_p\right) \\ &= L\left(p, \psi(p), d^\omega\psi_p\right) = \mathcal{L}(\psi, \omega)(x) \end{aligned}$$

where in the first equality we used 3.32 and in the fourth equality we used lemma 3.48. ■

Remark 4.7. Observe that we didn't use the fact that L is G-invariant to prove that $\mathcal{L}$ is well defined.

Definition 4.8. Let $L : J(P, V) \longrightarrow \mathbb{R}$ be a Lagrangian and $\mathcal{L} : C^\infty_G(P, V) \times \mathrm{Conn}(P) \longrightarrow C^\infty(M)$ as in theorem 4.6, for $\omega \in \mathrm{Conn}(P)$ and $\psi \in C^\infty_G(P, V)$, we call the function $\mathcal{L}(\psi, \omega) : M \longrightarrow C^\infty(M)$ the action density of the pair (ψ, ω). □

4.2 LAGRANGE'S EQUATION

We are ready to define the principle of least action. The action will be defined as an integral over the base space M, so we will assume that M is orientable and has a metric g and a volume form μ associated with it. However, we will not assume that M is compact, thus an integral over M may not exist, so we will work locally with open sets whose closure is compact. For simplicity, we will denote the action density $\mathcal{L}(\psi, \omega)$ by $\mathcal{L}^\omega(\psi)$.

Definition 4.9. Let $U \subset M$ be an open set with compact closure. For $\psi \in C^\infty_G(P, V)$ we define the action of ψ over U as $\int_U \mathcal{L}^\omega(\psi)\mu$, where μ is a volume form associated with the metric g. The projected support of a particle field $\psi \in C^\infty_G(P, V)$ is the closure of the set $\{p \in P \mid \psi(p) \neq 0\}$. □

Definition 4.10. We say that $\psi \in C^\infty_G(P,V)$ satisfies the principle of least action if for any open set $U \subset M$ with compact closure and $\sigma \in C^\infty_G(P,V)$ with projected support contained in U we have

$$\frac{d}{dt}\int_U \mathcal{L}^\omega(\psi + t\sigma)\mu\bigg|_{t=0} = 0.$$

Another way to say that ψ satisfies the principle of least action is to say that ψ is stationary. □

The main goal of the following is to prove that ψ satisfies the principle of least action if and only if it satisfies some specific differential equations, namely Lagrange's equations. To do this, we'll need some tools, which will be developed below.

Since $\pi_* : H_p \longrightarrow TM_x$ is an isomorphism, the metric g_x in TM_x induces a metric $\tilde{g}_p$ in H_p via $\tilde{g}_p(X,Y) = g_x(\pi_* X, \pi_* Y)$. The same is true for the volume element μ_x, getting a volume element $\tilde{\mu}_p$ in H_p, thus we may define a Hodge star operator in H_p with respect to this volume element as follows: First, let $\star$ be the Hodge star operator in M with respect to g, the pullback of $\pi^* : \Lambda^k(TM_x) \longrightarrow \Lambda^k(H_p)$ is a linear isomorphism as well, so we define $\tilde{\star}_p : \Lambda^k(H_p) \to \Lambda^{n-k}(H_p)$ by setting $\tilde{\star}_p(\pi^*\varphi) = \pi^*(\star_x\varphi)$, this map is clearly linear and well defined.

Definition 4.11. We define the linear transformation $\tilde{\star} : \Lambda^k_{eq}(P,V) \longrightarrow \Lambda^{n-k}_{eq}(P,V)$ by setting $(\tilde{\star}\varphi)|_{H_p} = \tilde{\star}_p(\varphi|_{H_p})$ and $(\tilde{\star}\varphi)|_{V_p} = 0$. Since $TP_p = H_p \oplus V_p$, this definition makes sense. □

Proposition 4.12. *If $\varphi \in \Lambda^k_{eq}(P,V)$ and $\sigma : U \longrightarrow P$ is a local section, then the pullback of the section and the star operator commute, that is, $\sigma^*(\tilde{\star}\varphi) = \star(\sigma^*\varphi)$.*

Proof: Let $\sigma(x) = p$, since π_* vanishes on vertical vectors, we know that $\pi_*\left((\sigma_*\pi_* X)^H\right) = \pi_*(\sigma_*\pi_* X) = \pi_*(X)$ since $\pi\circ\sigma = id_U$, thus if $X \in H_p$, $(\sigma_*\pi_* X)^H = X$ by the injectivity of π_* in H_p. This implies that for any $\psi \in \Lambda^i_{eq}(P,V)$, we get $\pi^*\sigma^*\psi|_{H_p} = \psi|_{H_p}$, so it suffices to show that $\pi^*(\sigma^*(\tilde{\star}\varphi)) = \pi^*(\star(\sigma^*\varphi))$ on H_p.

$$\pi^*\left(\sigma^*(\tilde{\star}\varphi)\right)|_{H_p} = (\tilde{\star}\varphi)|_{H_p} = \tilde{\star}_p\left(\varphi|_{H_p}\right) = \pi^*\left(\star\sigma^*(\varphi)\right)\Big|_{H_p} = \tilde{\star}_p\left(\left(\pi^*\sigma^*\varphi\right)|_{H_p}\right)$$

■

Important Construction

In the discussion to come, the vector V will be equipped with a norm h, and we will require that the representation $G \to GL(V)$ be orthogonal, *i.e.* $h(x,y) = h(gx,gy)$. Since $\tilde{g}_p$ is a metric on H_p we can define a metric $(\tilde{g}_p h)$ in the space of V-valued forms. This is done in the following manner: fixing basis $e_1,\dots,e_n$ in H_p and $f_1,\dots,f_n$ in V we define for $\psi,\varphi \in \Lambda^k(H_p,V)$, $\psi = \sum\psi^\alpha f_\alpha$ and $\varphi = \sum\varphi^\beta f_\beta$,

$$\langle\psi,\varphi\rangle_{Hp} = \frac{1}{k!}\sum h_{\alpha\beta}\tilde{g}_p^{i_1j_1j}\tilde{g}_p^{i_2j_2}\cdots\tilde{g}_p^{i_kj_k}\psi^\alpha_{i_1\cdots i_k}\varphi^\beta_{j_1\cdots j_k} \tag{4.1}$$

where $\tilde{g}_p^{ij}$ is the (i,j) entry of the the inverse matrix of the matrix $\tilde{g}_{p_{ij}} = \tilde{g}_p(e_i,e_j)$, $h_{\alpha\beta} = h(f_\alpha,f_\beta)$, and $\psi^\alpha_{i_1\cdots i_k}$ are the components of the form $\psi^\alpha \in \Lambda^k(H_p,\mathbb{R})$ with respect to the chosen basis. It's not hard to shown that this definition doesn't depend on the choice of basis. Observe that we can define a function $\langle\cdot,\cdot\rangle_M : \Lambda^k(M,V)\times\Lambda^k(M,V) \longrightarrow C^\infty(M)$ by setting $\langle\psi,\varphi\rangle_M(x) = \langle\psi_x,\varphi_x\rangle_{Mx}$, where $\langle\cdot,\cdot\rangle_{Mx}$ is defined similarly as $\langle\cdot,\cdot\rangle_{Hp}$. For ordinary differential forms, we have a norm by setting $h_{\alpha\beta} = 1$, in this case we just write $g(\psi,\varphi)$.

When it's clear, we will omit the subscript indicating the base manifold of the forms (P or M). Now will define a similar function but for forms in P and not H_p.

Throughout the rest of this text, these inner products will be of great importance, so be careful which one is which because it is indeed very confusing! To remember which one we are using just observe where the forms are taking values, so you know how the inner product is construct: we use the metric em M to induce in a metric in regular (taking values in $\mathbb{R}$) forms in P and if the forms are taking values in other spaces, we need a metric in them to combine the two and get yet another metric for forms taking values there!

Definition 4.13. For $\psi, \varphi \in \Lambda^k_{eq}(P, V)$, we define $\langle \psi_p, \varphi_p \rangle_p = \langle \psi|_{H_p}, \varphi|_{H_p} \rangle_{Hp}$. □

Proposition 4.14. *The function* $\langle \cdot, \cdot \rangle_P : \Lambda^k_{eq}(P, V) \times \Lambda^k_{eq}(P, V) \longrightarrow C^\infty(M)$ *given by*

$$\langle \psi, \varphi \rangle(x) = \langle \psi_p, \varphi_p \rangle_p \text{ for some } p \in \pi^{-1}(x)$$

is well defined.

Proof: We have $\psi_{pg} = g^{-1} \cdot \psi_p \circ R_{g^{-1}*}$ by the definition of $\Lambda^k_{eq}(P, V)$, thus

$$\langle \psi_{pg}, \varphi_{pg} \rangle_{pg} = \langle g^{-1} \cdot \psi_p \circ R_{g^{-1}*},\ g^{-1} \cdot \varphi_p \circ R_{g^{-1}*} \rangle_{pg} = \langle \psi_p, \varphi_p \rangle_p,$$

since $R_{g^{-1}*} : H_{pg} \to H_p$ is an isometry, and $G \to GL(V)$ is orthogonal relative to h. ■

Proposition 4.15. *Let* $\psi, \varphi \in \Lambda^k_{eq}(P, V)$ *and let* $\sigma : U \to P$ *be a local section, then*

$$\langle \sigma^* \psi, \sigma^* \varphi \rangle_M = \langle \psi, \varphi \rangle_P.$$

Proof: We know that $\pi_* : H_p \longrightarrow TM_x$ is an isometry and that $\pi^* \sigma^* \psi|_{H_p} = \psi|_{H_p}$, so

$$\begin{aligned}\langle \sigma^* \psi, \sigma^* \varphi \rangle_x &= \langle \pi^* \sigma^* \psi \Big|_{H_p}, \pi^* \sigma^* \varphi \Big|_{H_p} \rangle_{Hp} \\ &= \langle \psi \Big|_{H_p}, \beta \Big|_{H_p} \rangle_{Hp} = \langle \psi_p, \varphi_p \rangle_{Pp} \\ &= \langle \psi, \varphi \rangle_P(x).\end{aligned}$$

■

Definition 4.16. The covariant codifferential is a function $\delta^\omega : \Lambda^k_{eq}(P, V) \longrightarrow \Lambda^{k-1}_{eq}(P, V)$, defined by $\delta^\omega(\varphi) = -(-1)^g(-1)^{n(k+1)} \tilde{\star} d^\omega(\tilde{\star}\varphi)$ where $(-1)^g$ is the sign of the determinant of $g(\partial_i, \partial_j)$. □

Theorem 4.17. *Let* $U \subset M$ *be an open set with compact closure,* $\psi \in \Lambda^k_{eq}(P, V)$ *with projected support contained in* U*. Then for any* $\varphi \in \Lambda^{k+1}_{eq}(P, V)$

$$\int_U \langle d^\omega \psi, \varphi \rangle \mu = \int_U \langle \psi, \delta^\omega \varphi \rangle \mu.$$

Proof: We first assume that there is a local section $\sigma : U \to P$ so, by proposition 4.15

$$\int_U \langle d^\omega \psi, \varphi \rangle_P \mu = \int_U \langle \sigma^*(d^\omega \psi), \sigma^* \varphi \rangle_M \mu$$

using proposition 3.32 we get that

$$\sigma^*\left(d^\omega \psi\right) = \sigma^*(d\psi + \omega \dot{\wedge} \psi) = d\left(\sigma^* \psi\right) + \left(\sigma^* \omega\right) \dot{\wedge} \sigma^*(\psi)$$

thus,

$$\int_U \langle d^\omega \psi, \varphi\rangle_P \mu = \int_U \langle d\left(\sigma^*\psi\right), \sigma^*\varphi\rangle_M \mu + \int_U \langle \sigma^*\omega \dot{\wedge} \sigma^*\psi, \sigma^*\varphi\rangle_M \mu.$$

The first term is reduced to $\int_U \langle \sigma^*\psi, \delta\left(\sigma^*\varphi\right)\rangle_M \mu$ with $\delta = -(-1)^g(-1)^{kn} \star d\star$. For the second term, let $v_1, \ldots, v_m$ be a basis for V and $e_1, \ldots, e_f$ a basis for $\mathfrak{g}$ so that $\sigma^*\psi = \psi^i v_i$ and $\sigma^*\varphi = \varphi^j v_j$ thus $\langle \sigma^*\omega \wedge \sigma^*\psi, \sigma^*\varphi\rangle_M = \langle \left(\omega^l \wedge \psi^i\right)\left(e_l \cdot v_i\right), \varphi^j v_j\rangle_M = g\left(\omega^l \wedge \psi^i, \varphi^j\right) h\left(e_l \cdot v_i, v_j\right)$. At $t = 0$, we have

$$0 = \frac{d}{dt} h\left(\left(\exp te_l\right) \cdot v_i, \left(\exp te_l\right) \cdot v_j\right) = h\left(e_l \cdot v_i, v_j\right) + h\left(v_i, e_l \cdot v_j\right).$$

Also, $g\left(\omega^l \wedge \psi^i, \varphi^j\right)\mu = \left(\omega^l \wedge \psi^i\right) \wedge \star\varphi^j = (-1)^l \psi^i \wedge \left(\omega^l \wedge \star\varphi^j\right) = (-1)^l(-1)^g(-1)^{k(n-k)}\psi^i \wedge \star\left(\star\left(\omega^l \wedge \star\varphi^j\right)\right) = (-1)^g(-1)^{nk} g\left(\psi^i, \star\left(\omega^p \wedge \star\varphi^j\right)\right)\mu$. So we get $g\left(\omega^l \wedge \psi^i, \varphi^j\right) = (-1)^g(-1)^{nk} g\left(\psi^i, \star\left(\omega^l \wedge \star\varphi^j\right)\right)$ and $h\left(e_l \cdot v_i, v_j\right) = -h\left(v_i, e_l \cdot v_j\right)$, using these two we may conclude that

$$\begin{aligned}\langle \sigma^*\omega \dot{\wedge} \sigma^*\psi, \sigma^*\varphi\rangle_M &= -(1)^h(-1)^{nk} g\left(\psi^i, \star\left(\omega^l \wedge \star\varphi^j\right)\right) h\left(v_i, e_l \cdot v_j\right) \\ &= -(-1)^g(-1)^{nk}\langle \sigma^*\psi, \star\left(\sigma^*(\omega) \dot{\wedge} \star \sigma^*(\varphi)\right)\rangle_M\end{aligned}$$

Finally,

$$\begin{aligned}\int_U \langle d^\omega\psi, \varphi\rangle_P \mu &= -(-1)^g(-1)^{nk}\int_U \langle (\sigma^*\psi, \star d \star \sigma^*\varphi + \star\left(\sigma^*(\omega) \dot{\wedge} \star \sigma^*(\varphi)\right)\rangle_M \mu \\ &= \int_U \langle \sigma^*\psi, \sigma^*\left(\delta^\omega\varphi\right)\rangle_M \mu = \int_U \langle \psi, \delta^\omega\varphi\rangle_P \mu\end{aligned}.$$

For the case where we don't have a local section defined in U, we may use a partition of the unity. Since the projected support K of ψ in contained in U, there are $U_1, \ldots, U_N$ open subsets of U together with N local sections $\sigma_i : U_i \longrightarrow P$ that cover K, also there are N smooth functions $\rho_i : M \longrightarrow \mathbb{R}$ such that $\mathrm{supp}(\rho_i) \subset U_i$ and $\sum p_i(x) = 1$ for all $x \in K$. Define $\tilde{f}_i\psi \in \Lambda^k_{eq}(P, V)$ where $\tilde{f}_i = \pi^* f_i$, by $(\tilde{f}_i\psi)_p = \tilde{f}_i(p)\psi_p, p \in P$. since $\tilde{f}_i\psi$ has projected support in $U_i \subset U$, our previous result yields

$$\int_U \langle d^\omega(\tilde{f}_i\psi), \varphi\rangle_P \mu = \int_U \langle \tilde{f}_i\psi, \delta^\omega\varphi\rangle_P \mu$$

but, since $\sum \tilde{f}_i\psi = \psi$, we get the result by summing over i. ■

Definition 4.18. Let $L : J(P, V) \longrightarrow \mathbb{R}$ be a Lagrangian and $\Lambda^1_{eq}(TP_p, V)$ be the set of linear transformations that vanish on vertical vectors. Then for $(p, v, \theta) \in J(P, V)$ we define the linear transformation $\nabla L(p, v, \theta) \in \Lambda^1_{eq}(TP_p, V)$ by the equation

$$\langle \nabla L(p, v, \theta), \varphi\rangle_p = \frac{d}{dt} L(p, v, \theta + t\varphi)\Big|_{t=0}.$$

By equation 4.1 we can see that this in fact defines a linear transformation. For $\psi \in C^\infty_G(P, V)$, we define a V-valued 1-form $\partial L/\partial\left(d^\omega\psi\right)$ on P by $\partial L/\partial\left(d^\omega\psi\right)_p = \nabla L\left(p, \psi(p), d^\omega\psi_p\right)$ □

Proposition 4.19. $\partial L/\partial\left(d^\omega\psi\right) \in \Lambda^1_{eq}(P,V)$

Proof: Since $\partial L/\partial\left(d^\omega\psi\right)$ vanishes on vertical vectors, all we need to show is that $R_g^*\partial L/\partial\left(d^\omega\psi\right) = g^{-1}\cdot\partial L/\partial\left(d^\omega\psi\right)$

$$\begin{aligned}
\langle g\cdot\frac{\partial L}{\partial\left(d^\omega\psi\right)_{pg}}\circ R_{g*},\varphi_p\rangle_p &= \langle\frac{\partial L}{\partial\left(d^\omega\psi\right)_{pg}}, g^{-1}\cdot\varphi_p\circ R_{g^{-1}*}\rangle_{pg}\\
&= \langle\frac{\partial L}{\partial\left(d^\omega\psi\right)_{pg}},\varphi_{pg}\rangle_{pg} = \frac{d}{dt}L\left(pg,\psi(pg),d^\omega\psi_{pg}+t\varphi_{pg}\right)\Big|_{t=0}\\
&= \frac{d}{dt}L\left(pg,g^{-1}\psi(p),g^{-1}\cdot\left(d^\omega\psi_p+t\varphi_p\right)\circ R_{g^{-1}*}\right)\Big|_{t=0}\\
&= \frac{d}{dt}L\left(p,\psi(p),d^\omega\psi_p+t\varphi_p\right)\Big|_{t=0}\\
&= \langle\frac{\partial L}{\partial\left(d^\omega\psi\right)_p},\varphi_p\rangle_p.
\end{aligned}$$

In the first equality we are using the fact that the representation is orthogonal and that R_{g*} is an isometry. We get that $g\cdot\partial L/\partial\left(d^\omega\psi\right)_{pg}\circ R_{g*} = \partial L/\partial\left(d^\omega\psi\right)_p$. ∎

Definition 4.20. For $(p,v,\theta)\in J(P,V)$ we define the vector $\nabla' L(p,v,\theta)\in V$ by the equation

$$h\left(\nabla' L(p,v,\theta),w\right) = \frac{d}{dt}L(p,v+tw,\theta)\Big|_{t=0}.$$

If $\psi\in C^\infty_G(P,V)$, we define a V-valued function $\partial L/\partial\psi$ on P by $\partial L/\partial\psi(p) = \nabla' L\left(p,\psi(p),d^\omega\psi(p)\right)$. By the same arguments of the last proposition, $\partial L/\partial\psi\in C^\infty_G(P,V) = \Lambda^0_{eq}(P,V)$. □

Theorem 4.21. *Let $U\subset M$ be an open subset with compact closure, and let $\tau\in C^\infty_G(P,V)$ with projected support contained in U, then*

$$\frac{d}{dt}\int_U \mathcal{L}^\omega(\psi+t\tau)\mu\Big|_{t=0} = \int_U h\left(\delta^\omega\frac{\partial L}{\partial\left(d^\omega\psi\right)}+\frac{\partial L}{\partial\psi},\tau\right)\mu$$

Proof: Evaluating the derivatives at $t=0$, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}^\omega(\psi+t\tau)(\pi(p)) &= \frac{d}{dt}L\left(p,\psi(p)+t\tau(p),d^\omega\psi_p+td^\omega\tau_p\right)\\
&= h\left(\frac{\partial L}{\partial\psi(p)},\tau(p)\right)+\langle\frac{\partial L}{\partial\left(d^\omega\psi\right)_p},d^\omega\tau_p\rangle_P
\end{aligned}$$

now, by integrating both sides over U and applying theorem 4.17 we get the result by observing that $\langle\cdot,\cdot\rangle_P = h$ in $C^\infty_G(P,V)$. ∎

Theorem 4.22 (Lagrange's Equation). *A particle field $\psi\in C^\infty_G(P,V)$ satisfies the principle of least action if and only if Lagrange's equation,*

$$\delta^\omega\frac{\partial L}{\partial\left(d^\omega\psi\right)}+\frac{\partial L}{\partial\psi} = 0$$

holds.

Proof: If the principle of least action holds, by theorem 4.21 we see that that the equation must hold, otherwise we might find a τ such that the right side integral doesn't vanish. Explicitly, if the equation doesn't hold, there exists a $\tilde\tau\in C^\infty_G(P,V)$ such that

$h(\delta^\omega[\partial L/\partial\,(d^\omega\psi)] + \partial L/\partial\psi, \tilde{\tau}) > 0$ at $\pi(p)$, thus for an open neighborhood U of $\pi(p)$ with compact closure of M the inequality still holds. It's not hard to see that there exists a smooth non-negative function $f : P \longrightarrow \mathbb{R}$ constant on fibers with $f(p) > 0$ and with projected support contained in U, thus we have that $\tau = f\tilde{\tau}$ has projected support contained in U and is such that the right side of theorem 4.21 is positive, so we get a contradiction. The converse follows trivially, since the ride hand side will be 0. ■

AN EXAMPLE IN ELECTRODYNAMICS

Here we give a little application of the theory developed so far, an example in electrodynamics. Our base manifold M will be a Lorentzian manifold, we have an principal fiber bundle $\pi : P \to M$ whose structural group is $G = U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. The Lie algebra $\mathfrak{u}(1)$ of $U(1)$ is simply $\mathfrak{u}(1) = \{i\theta \mid \theta \in \mathbb{R}\}$, that is, pure imaginary numbers, since matrix multiplication is commutative for complex numbers, our Lie bracket is trivial. Let ω be a fixed connection of such a bundle.

Now that we have the first ingredient, we need a vector space V, which will give rise to our particle fields, we let $V = \mathbb{C}$, regarded as a 2 dimensional real vector space. We need a representation of $U(1)$ in $\mathbb{C}$, which will set to be complex multiplication, *i.e.* $e^{i\theta} \cdot z = e^{i\theta} z$, so the induced action of the Lie algebra $\mathfrak{u}(1)$ in V is

$$\frac{d}{dt} e^{t(i\theta)} \cdot z|_{t=0} = i\theta z$$

so it is just complex multiplication again! Let h be a metric in $\mathbb{C}$ given by $h(z, w) = \frac{1}{2}(\bar{z}w + \bar{w}z)$. As before, the metric g in M induces a metric on H_p which in turn induces a metric on horizontal forms with $\langle\cdot,\cdot\rangle_P$, so we may set our lagrangian as

$$L(p, z, \theta) = \langle\theta^H, \theta^H\rangle_{P_p} - \frac{1}{2}m^2\bar{z}z.$$

For $T \in \Lambda^1_{eq}(P, V)$ we have that

$$\langle\nabla L(p, z, \theta), T\rangle = \frac{d}{dt} L(p, z, \theta + T) = \langle\theta^H, T\rangle$$

so that $\nabla L(p, z, \theta) = \theta^H$, similarly, $\nabla' L(p, z, \theta) = -m^2 z$.

Now we must look at particle fields, which re remind that bust be smooth and equivariant with the action of $U(1)$, so $C^\infty_G = \{\psi : P \longrightarrow \mathbb{C} \mid \psi(p \cdot e^{i\theta}) = e^{-i\theta}\psi(p)\}$. With the same calculations we get that

$$\frac{\partial L}{\partial d^\omega\psi} = d^\omega\psi \qquad \text{and} \qquad \frac{\partial L}{\partial\psi} = -m^2\psi$$

so Lagrange's equation yields

$$\delta^\omega\psi - m^2\psi = 0.$$

The equation above is called the "first-quantized " wave function of a spin-0 charged particle of mass m under the influence of an electromagnetic potential ω. Note that the equation is quite elegant, but it is not just elegant, it is quite general, since we didn't give any restrictions on M other than it is a Lorentzian manifold, so it's ready to use even in curved space-times!

4.3 INHOMOGENEOUS FIELDS

In this section we explore the notion of the current 1-form.

We will assume the the Lie algebra $\mathfrak{g}$ of our Lie group G is equipped with a metric k such that the adjoint representation is orthogonal, so by proposition 4.14, we have a function

$$\langle\cdot,\cdot\rangle_{\mathfrak{g}} : \Lambda^j_{eq}(P,\mathfrak{g}) \times \Lambda^j_{eq}(P,\mathfrak{g}) \to C^\infty(M).$$

Definition 4.23. Let $\omega \in \mathrm{Conn}(P)$, $\psi \in C^\infty_G(P,V)$ and $p \in P$, then we define the current $J_p \in \Lambda^1(TP_p,\mathfrak{g})$ by the equation

$$\langle J_p,\tau\rangle_{\mathfrak{g}_p} = \langle \frac{\partial L}{\partial\left(d^\omega\psi\right)},\tau\cdot\psi\rangle_{P_p}$$

where $\tau \in \Lambda^1_{eq}(P,\mathfrak{g})$ and τ acts on ψ as in 3.31. Again, this is well defined since $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is non degenerated. The current depends on the connection and the a particle field, somtimes we denote the current of the pair (ω,ψ) as $J^\omega(\psi)$. □

Proposition 4.24. *Given a basis $e_1,\ldots,e_f$ for $\mathfrak{g}$ and setting $k^{\alpha\beta}$ to be the inverse of the matrix defined by $k_{\alpha\beta} = k(e_\alpha,e_\beta)$, we get that*

$$J(X) = k^{\alpha\beta} h\left(\partial L/\partial\left(d^\omega\psi\right)(X), e_\alpha\cdot\psi\right) e_\beta.$$

Proof: It is just a matter of computation:

$$\begin{aligned}\langle k^{\alpha\beta} h\left(\partial L/\partial\left(d^\omega\psi\right), e_\alpha\cdot\psi\right) e_\beta, \tau^\gamma e_\gamma\rangle_{\mathfrak{g}_p} &= k^{\alpha\beta}k_{\beta\gamma}\langle h\left(\partial L/\partial\left(d^\omega\psi\right), e_\alpha\cdot\psi\right), \tau^\gamma\rangle_p \\ = \langle \partial L/\partial\left(d^\omega\psi\right), \tau\cdot\psi\rangle_{P_p} &= \langle J_p, \tau^\gamma e_\gamma\rangle_{\mathfrak{g}_p}\end{aligned}$$

Thus, by non-degeneracy, the result follows. ■

Theorem 4.25. $J \in \Lambda^1_{eq}(P,\mathfrak{g})$.

Proof: All we have to show is that $R^*_g J = \mathcal{A}d_{g^{-1}} J$ since it already vanishes on vertical vectors by definition. For $\tau \in \Lambda^1_{eq}(P,\mathfrak{g})$, we have that

$$\begin{aligned}\langle \mathcal{A}d_{g^{-1}} J_p \circ R_{g^{-1}*}, \tau_{pg}\rangle_{\mathfrak{g}_p} &= \langle J_p, \mathcal{A}d_g \tau_{pg}\cdot R_{g^*}\rangle_{\mathfrak{g}_p} \\ &= \langle J_p,\tau_p\rangle_{\mathfrak{g}_p} = \langle \partial L/\partial\left(d^\omega\psi\right)_p, \tau_p\cdot\psi_p\rangle_{P_p}.\end{aligned}$$

Also $\partial L/\partial\left(d^\omega\psi\right) \in \Lambda^1_{eq}(P,V)$, and, since $\tau\cdot\psi \in \Lambda^1_{eq}(P,V)$, $\left(R^*_g\tau\cdot\psi\right)_p = \mathcal{A}d_{g^{-1}}\tau_p\cdot\psi_{pg} = g^{-1}\cdot(\tau\cdot\psi)_p$, now since our representations are orthogonal, we have

$$\langle(\partial L/\partial\left(d^\omega\psi\right))_p, \tau_p\cdot\psi_p\rangle_{P_p} = \langle \partial L/\partial\left(d^\omega\psi\right)_{pg}, \tau_{pg}\cdot\psi_{pg}\rangle_{P_{pg}} = \langle J_{pg},\tau_{pg}\rangle_{\mathfrak{g}_{pg}}$$

so, $\mathcal{A}d_{g^{-1}} J_p \circ R_{g^{-1*}} = J_{pg}$, which is the desired result. ■

Lema 4.26. *Let $J^\omega(\psi)$ be the current associated with the pair (ω,ψ) and $\tau \in \Lambda^1_{eq}(P,\mathfrak{g})$, then*

$$\frac{d}{dt}\mathcal{L}(\psi,\omega+t\tau)\Big|_{t=0} = \langle J^\omega(\psi),\tau\rangle_{\mathfrak{g}}.$$

Proof: Let $p \in \pi^{-1}(x)$, so

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(\psi, \omega + t\tau)(x) &= \frac{d}{dt}L\left(p, \psi(p), d\psi_p + \omega_p \cdot \psi(p) + t\tau_p \cdot \psi(p)\right)\\ &= \langle \nabla L\left(p, \psi(p), d^\omega\psi_p\right), \tau_p \cdot \psi(p)\rangle_{P_p}\\ &= \langle \frac{\partial L}{\partial\left(d^\omega\psi\right)}, \tau \cdot \psi\rangle_{P_p}\\ &= \langle (J, \tau)\rangle_{\mathfrak{g}_p}(x)\end{aligned}$$

where in the first equality we used proposition 3.32. ■

Theorem 4.27 (Conservation of Charge)**.** *Let* $L : J(P, V) \longrightarrow \mathbb{R}$ *be a* G*-invariant Lagrangian and* $\omega \in \mathrm{Conn}(P)$ *be a fixed connection. If* $\psi \in C^\infty_G(P, V)$ *satisfies the principle of least action, then*

$$\delta^\omega(J^\omega(\psi)) = 0.$$

The above equation is called the generalized continuity equation.

Proof: Since L is G invariant we have that $\mathcal{L}$ is gauge invariant, thus for any $F \in C^\infty_G(P, \mathfrak{g})$ we have $\mathcal{L}\left((\exp tF)^*\psi, (\exp tF)^*\omega\right) = \mathcal{L}(\psi, \omega)$ (we are using the notation of 3.45). So using theorems 3.49 and 3.50, we have

$$\frac{d}{dt}\mathcal{L}(\psi - tF \cdot \psi, \omega)\Big|_{t=0} + \frac{d}{dt}\mathcal{L}\left(\psi, \omega + td^\omega F\right)\Big|_{t=0} = 0$$

Now, if $U \subset M$ is open with compact closure, and F has projected support in U, we can integrate the above expression over U and use 4.26 to get

$$\frac{d}{dt}\int_U \mathcal{L}^\omega(\psi - tF \cdot \psi)\mu + \int_U \langle (J^\omega(\psi), d^\omega F\rangle_{\mathfrak{g}}\mu = 0$$

The first term vanishes since ψ is stationary, now using theorem 4.17, we get

$$\int_U k\left(\delta^\omega\left(J^\omega(\psi)\right), F\right)\mu = 0$$

for all $F \in C^\infty_G(P, \mathfrak{g})$ with projected support in U, so which implies the result by the same argument used in the proof of Lagrange's equation. ■

Theorem 4.28. *Let* $k^{\alpha\beta}$ *be as in 4.24, then for any* $\psi \in C^\infty_G(P, \mathfrak{g})$ *we have*

$$k^{\alpha\beta}h\left(\delta^\omega\left[\partial L/\partial\left(d^\omega\psi\right)\right] + \partial L/\partial\psi, e_\alpha \cdot \psi\right)e_\beta = \delta^\omega J^\omega(\psi)$$

Proof: Using theorem 4.21, we see that the first term of the second equation of our last proof becomes

$$\frac{d}{dt}\int_U \mathcal{L}^\omega(\psi - tF \cdot \psi)\mu = -\int_U h\left(\delta^\omega\frac{\partial L}{\partial\left(d^\omega\psi\right)} + \frac{\partial L}{\partial\psi}, F^\alpha e_\alpha \cdot \psi\right)\mu$$

now using proposition 4.24, we can express the integrands as

$$k\left(k^{\alpha\beta}h\left(\delta^\omega[\partial L/\ \partial\left(d^\omega\psi\right)] + \partial L/\partial\psi, e_\alpha \cdot \psi\right)e_\beta, F\right)$$

now, noting what the second terms become in the last proof, we see that it is just a matter of combining the integrals and noting that it always equals zero, so the integrand must be zero and the result follows. ■

Definition 4.29. Let k be a metric in $\mathfrak{g}$ such that the adjoint representation in orthogonal, then we define the function $\mathfrak{D} : \text{Conn}(P) \longrightarrow C^\infty(M)$ by $\mathfrak{D}(\omega) = -\frac{1}{2}\langle \Omega^\omega, \Omega^\omega \rangle_{\mathfrak{g}}$. This function is called the self-action density of ω, by proposition 3.29 we know that it is well defined. □

We have a similar notion of what it means to a pair (ω, ψ) to be stationary, as defined below.

Definition 4.30. We say that a pair (ω, ψ), $\omega \in \text{Conn}(P)$, $\psi \in C^\infty_G(P, V)$ is stationary if for all open sets $U \subset M$ with compact closure, and $\sigma \in C^\infty_G(P, V), \tau \in \Lambda^1_{eq}(P, \mathfrak{g})$, both with projected support contained in U the following holds

$$\frac{d}{dt}\left(\int_U (\mathcal{L} + \mathfrak{D})(\psi + t\sigma, \omega + t\tau)\mu\right)\Big|_{t=0} = 0.$$

Above, we we let $\mathfrak{D}(\omega, \psi) = \mathfrak{D}(\omega)$. □

Now, as before, we have a characterization of stationary pairs in terms of an equation.

Theorem 4.31 (Inhomogeneous Field Equations). *A pair is stationary with respect to $(\mathcal{L} + \mathfrak{D})$ if and only the following equations hold*

(i) *Lagrange's equation:*

$$\delta^\omega \frac{\partial L}{\partial\, (d^\omega \psi)} + \frac{\partial L}{\partial \psi} = 0.$$

(ii) *Inhomogeneous field equation:*

$$\delta^\omega \Omega^\omega = J^\omega(\psi).$$

Proof: By the Structural Equation (3.17), we have that $\Omega^{\omega+t\tau} = d(\omega + t\tau) + \frac{1}{2}[\omega + t\tau, \omega + t\tau]$ so that

$$\frac{d}{dt}\Omega^{\omega+t\tau}\Big|_{t=0} = d\tau + [\omega, \tau] = d^\omega \tau$$

where in the last equality we used corollary 3.33. Now, evaluating the derivatives at $t = 0$, we compute

$$\begin{aligned}
&\frac{d}{dt}\int_U (\mathcal{L} + \mathfrak{D})(\psi + t\sigma, \omega + t\tau)\mu \\
&\qquad = \frac{d}{dt}\int_U \mathcal{L}(\psi + t\sigma, \omega + t\tau)\mu - \frac{1}{2}\frac{d}{dt}\int_U \langle \Omega^{\omega+t\tau}, \Omega^{\omega+t\tau}\rangle_{\mathfrak{g}}\mu \\
&\qquad = \frac{d}{dt}\int_U \mathcal{L}(\psi + t\sigma, \omega)\mu + \frac{d}{dt}\int_U \mathcal{L}(\psi, \omega + t\tau)\mu - \int_U \langle \Omega^\omega, d^\omega\tau\rangle_{\mathfrak{g}}\mu \\
&\qquad = \int_U h\left(\delta^\omega \frac{\partial L}{\partial\,(d^\omega\psi)} + \frac{\partial L}{\partial\psi}, \sigma\right)\mu + \int_U \langle J^\omega(\psi), \tau\rangle_{\mathfrak{g}}\mu - \int_U \langle \delta^\omega\,(\Omega^\omega)\,, \tau\rangle_{\mathfrak{g}}\mu
\end{aligned}$$

So by the same arguments we used before, if this derivative equals zero, we must have that the integrands equal zero, so the equations hold. Conversely, if the equations hold, we have that the actions equals zero. ■

4.4 THE YANG-MILLS EQUATION

Let $P \to M$ be a principal G-bundle. One important thing to notice is that in general, especially in physics, we are worried about things taking values according to the base manifold manifold M, and not P. For instance, when we talked about lagrangians and inhomogeneous fields, we developed tools for associating to a connection ω and a particle field ψ and function in $C^\infty(M)$, *so we could integrate over* M. In this section we explore another way of doing this, using associated bundles, then we'll take a look at the famous *Yang-Mills equation*.

Remark 4.32. Given a principal G-bundle, all we need to construct an associated bundle is an action of G is some manifold. If said action is a representation of G in some vector space V *i.e.* a homomorphism $G \to GL(V)$, is easy to see that $P \times_G V$ has the natural structure of a vector bundle.

Now we repeat a definition, just so it's clear what we're doing.

Definition 4.33. Let V be a vector space with a representation $G \to GL(V)$. And let $\Lambda^k(P, V)$ be the space of V valued one k-forms. We say that a form $\varphi \in \Lambda^k(P, V)$ is G equivariant if

$$\varphi\left(R_{g*}X_1, \dots, R_{g*}X_k\right) = g^{-1} \cdot \varphi\left(X_1, \dots, X_k\right)$$

and horizontal if

$$\varphi\left(X_1, \dots, X_k\right) = 0 \text{ if } X_i \text{ is vertical for some } i.$$

The space of space of equivariant horizontal k-forms is denoted by $\bar{\Lambda}^k(P, V)$. □

Definition 4.34. Let $E \to M$ be a vector bundle, a E-valued k-form is simply an element of $\Lambda^k(M) \otimes_{C^\infty(M)} \Gamma(E)$. Given a local frame e_i for E we may express a form ψ as

$$\psi = \sum \psi^i \otimes e_i$$

with $\psi^i \in \Lambda^k(M)$. We denote the space of E-valued k-forms by $\Lambda^k(M, E)$ □

Theorem 4.35. *Let $P \to M$ a principal G-bundle and $G \to GL(V)$ a representation, then there exists a natural linear isomorphism between $\bar{\Lambda}^k(P, V)$ and $\Lambda^k(M, P \times_G V)$.*

Proof: For $\varphi \in \bar{\Lambda}^k(P, V)$, we define $\tilde{\varphi} \in \Lambda^k(M, P \times_G V)$ to be

$$\tilde{\varphi}_{\pi(p)}\left(\pi_* X_1, \dots, \pi_* X_k\right) = \left[p, \varphi_p\left(X_1, \dots, X_k\right)\right].$$

First we need to show that this is well defined. Let Y be such that $\pi_* X_1 = \pi_* Y$, then $X_1 - Y \in \mathrm{Ker}(\pi_*) = V_p$, thus $Y = X_1 + V$ for some $V \in V_p$, so

$$\left[p, \varphi_p\left(Y, \dots, X_k\right)\right] = \left[p, \varphi_p\left(X_1, \dots, X_k\right)\right] + \left[p, \varphi_p\left(V, \dots, X_k\right)\right].$$

Since the form is horizontal we get an equality. Now let $\pi(p) = \pi(p') = x$, then $p' = p \cdot g$ for some g, and $R_{g*}X_i$ are such that $\pi_* R_{g*} X_i = \pi_* X_i$, thus

$$\begin{aligned}(p', \varphi_{p'}(R_{g*}X_1, \dots, R_{g*}X_k)) &= (p \cdot g, g^{-1} \cdot \varphi_p(X_1, \dots, X_k)) \\ &= g \cdot (p, \varphi_p(X_1, \dots, X_k)).\end{aligned}$$

Now it remains to show that this map is in fact an isomorphism. It is clear that it is linear, now we need to show bijectivity. If we have an $\psi \in \Lambda^k(M, P \times_G V)$ we set

$$\bar{\psi}_p(X_1, \dots, X_k) = (p, v) \in \psi_{\pi(p)}(\pi_* X_1, \dots, \pi_* X_k)$$

$\bar{\psi}$ is well defined because the action is free in P. It's clear that it is vertical, now $\bar{\psi}_{pg}(R_{g*}X_1, \dots, R_{g*}X_k))$ is the unique element $(pg, u) \in \psi_{\pi(p)}(\pi_* X_1, \dots, \pi_* X_k)$, so necessarily, $u = g^{-1} \cdot v$, so $\bar{\psi}$ is equivariant. Now is easy to see that the maps $\varphi \mapsto \tilde{\varphi}$ and $\psi \mapsto \bar{\psi}$ are the inverses of one another ■

Corollary 4.36. *For a connection 1-form ω, its curvature Ω^ω corresponds naturally with some 2-form of $\Lambda^2(M, Ad_P(M))$.*

Now we focus on the Yang-Mills equation, first we have to define the Yang-Mills action. To do so we will need to assume that our base manifold M has a volume form μ (is orientable), that our Lie algebra $\mathfrak{g}$ is equipped with a metric $k : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ such that the adjoint representation is orthogonal (which can always be done if G is compact). That being said, we have a Hodge star operator $\star : \Lambda(M) \longrightarrow \Lambda(M)$ associated with said volume form μ. Furthermore, for simplicity, we will assume that M is compact.

Definition 4.37. Let $\varphi, \psi \in \Lambda^k(M)$, then we define the inner product $\langle \varphi, \psi \rangle \in C^\infty(M)$ of the two forms as the smooth function

$$\langle \varphi, \psi \rangle = \frac{1}{k!} \sum_{\mu_1 \dots \mu_k} \varphi_{\mu_1 \dots \mu_k} \psi^{\mu_1 \dots \mu_k}$$

where $\varphi_{\mu_1 \dots \mu_k} = \varphi\left(\partial_{\mu_1}, \dots, \partial_{\mu_k}\right)$ for some local chart. Here we are using a metric g in M to raise the indices! □

Definition 4.38. Let $E \to M$ be a metric vector *i.e.* a vector bundle with a smooth map $\langle \cdot, \cdot \rangle_E : E \otimes E \longrightarrow C^\infty(M)$. We define the inner product of E-valued differential k-forms $\varphi = \sum \varphi^i \otimes e_i$, $\psi = \sum \psi^i \otimes e_i$ as

$$\langle \varphi, \psi \rangle_E = \sum \left\langle \varphi^i, \psi^j \right\rangle \langle e_i, e_j \rangle_E .$$

This definition does not depend on the choice of frame, thus it defines the inner product globally. □

Lets look again at the adjoint action on $\mathfrak{g}$, since we required it to be orthogonal we respect to a metric k, we may think of the induced metric k on $Ad_P(M)$, indeed, take a local trivialization $\Phi : \pi'^{-1}(U) \longrightarrow U \times \mathfrak{g}$, then we set for $u \otimes v \in \pi'^{-1}(U) \otimes \pi'^{-1}(U)$, as $k(\Phi(u), \Phi(v))$, this well defined because given another trivialization $\tilde{\Phi}$, we have that $\tilde{\Phi}(u) = g \cdot \Phi(U)$, and since the representation is orthogonal, we have an equality. We denote such metric by $\langle \cdot, \cdot \rangle_{Ad_P(M)}$.

Definition 4.39 (Yang-Mills Lagrangian). The Yang-Mills lagrangian is a function defined as

$$\begin{aligned} \mathcal{L}_{YM} &: \mathrm{Conn}(P) \longrightarrow C^\infty(M) \\ \omega &\mapsto -\frac{1}{2} \langle \Omega^\omega, \Omega^\omega \rangle_{Ad_P(M)}. \end{aligned}$$

Where we are using the fact that the curvature of a connection corresponds naturally with some $Ad_P(M)$-valued form. □

Definition 4.40 (Yang-Mills Action). The Yang-Mills action is function defined by

$$\begin{aligned} \mathcal{S}_{YM} &: \mathrm{Conn}(P) \longrightarrow \mathbb{R} \\ \omega &\mapsto \int_M \mathcal{L}_{YM}(\omega) \mu. \end{aligned}$$

We say that a connection ω satisfies the principle of least action (or is stationary) if for all $\sigma \in \bar{\Lambda}^1(P, \mathfrak{g})$ we have that

$$\left. \frac{d}{dt} \mathcal{S}_{YM}(\omega + t\sigma) \right|_{t=0} = 0$$

The action is well defined because M is compact. If that happens, we say that ω is a Yang-Mills connection. □

Now, as we did before, we need a characterization of stationary connections, but we already solved such a problem when we talked about inhomogeneous fields! Even though the constructions are different, we can check that the definition of the Yang-Mills lagrangian coincides with definition 4.29, that is $\mathfrak{D} = \mathcal{L}_{YM}$. Thus, we already have a characterization, namely the field equation.

Theorem 4.41 (Yang-Mills Equation)**.** *A connection ω satisfies the principle of least action if and only if it satisfies the Yang-Mills equation*

$$\delta^{\omega}\Omega^{\omega} = 0$$

where δ^{ω} is the codifferential defined as in 4.16.

Proof: Using the notation of theorem 4.31, take V to be the trivial vector space equipped with the trivial action, thus we get that $\mathcal{L} + \mathfrak{D} = \mathfrak{D}$ and that $J^{\omega} = 0$, thus, since $\mathfrak{D} = \mathcal{L}_{YM}$, the result follows from item (ii). ∎

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