THE SMASH PRODUCT NAME: SOLUTIONS

Due at the beginning of class on 19 March 2024

- *Your answers should be neatly written and logically organized.*
- *You may collaborate on solving the problems, but the solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

Reading: [Mal23, Sections 4.1 and 4.3] and the introduction to [Lew91].

(1) Define spectrum structure maps for a "fake smash product" $\wedge_{\mathrm{fake}} \colon \mathcal{S}p \times \mathcal{S}p \to \mathcal{S}p$ with n-th space

$$(X \wedge_{\mathrm{fake}} Y)_n = X_n \wedge Y_n.$$

Argue that, with your choice of structure maps, $(\mathcal{S}p, \wedge_{\mathrm{fake}}, \mathbb{S})$ is *not* a symmetric monoidal category.

SOLUTION: The structure maps are irrelevant here; this doesn't even have a well-defined unit so it can't be a monoidal category.

If $\mathbb{S}$ is to be the unit for this fake smash product, then we should have $X \wedge_{\mathrm{fake}} \mathbb{S} \cong X$, and in particular $(X \wedge_{\mathrm{fake}} \mathbb{S})_n \cong X_n$. But by the definition above,

$$(X \wedge_{\mathrm{fake}} \mathbb{S})_n = X_n \wedge S^n = \Sigma^n X_n \ncong X_n.$$

In fact, this can't even define a smash product on the homotopy category, because $\Sigma^n X_n$ is not stably equivalent to X_n.

(2) (a) Let $\mathcal{C}$ be a category that is both additive and symmetric monoidal with product $\otimes$ and unit I. Assume that $\otimes$ preserves coproducts in each variable separately. Prove that the product induces bilinear maps

$$\mathcal{C}(A, B) \times \mathcal{C}(X, Y) \to \mathcal{C}(A \otimes X, B \otimes Y)$$

SOLUTION: The bilinear pairing is obtained from the fact that $\otimes$ is a functor:

$$\otimes \colon \mathcal{C}(A, B) \times \mathcal{C}(X, Y) \cong (\mathcal{C} \times \mathcal{C})((A, X), (Y, B)) \to \mathcal{C}(A \otimes X, B \otimes Y).$$

The problem is complete once we show this preserves addition in each argument separately, and it suffices to show this for just the left argument. It is standard that coproducts in additive categories are biproducts and that the sum of $f, g \colon A \to B$ is given by

$$f + g \colon A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\nabla} B.$$

Further, coproduct preserving functors are the same as biproduct preserving functors are the same as additive functors. See [bip] for a discussion of these facts.

Then for $h \colon X \to Y$ it follows that,

$$(f + g) \otimes h = ((f + g) \otimes 1) \circ (1 \otimes h).$$

Since ∇ is part of the structure of the coproduct, it follows that $\nabla_A \otimes 1_Y = \nabla_{A \otimes Y}$. Similarly, Δ is part of the structure of the biproduct, so $\Delta \otimes 1 = \Delta$. Similarly again, $(f \oplus g) \otimes 1 = f \otimes 1 \oplus g \otimes 1$ (after composing with the relevant isomorphisms). Altogether, this verifies that

$$(f + g) \otimes 1 = (\nabla \circ (f \oplus g) \circ \Delta) \otimes h = (\nabla \otimes 1) \circ ((f \oplus g) \otimes 1) \circ (\Delta \otimes 1) = \nabla \circ (f \otimes 1 \oplus g \otimes 1) \circ \Delta = f \otimes 1 + g \otimes 1$$

(This follows from that coproduct preserving functors are additive, but I figured I'd show more details.) Thus,

$$\begin{aligned}(f+g)\otimes h &= ((f+g)\otimes 1)\circ(1\otimes h) = (f\otimes 1 + g\otimes 1)\circ(1\otimes h)\\ &= (f\otimes 1)\circ(1\otimes h) + (g\otimes 1)\circ(1\otimes h) = f\otimes h + g\otimes h.\end{aligned}$$

This completes the proof.

(b) Show that there is no natural transformation in the stable homotopy category $\delta\colon X \to X\wedge X$ that agrees with the diagonal on suspension spectra

$$\Sigma^\infty K \xrightarrow{\Sigma^\infty\Delta} \Sigma^\infty(K\wedge K) \cong \Sigma^\infty K \wedge \Sigma^\infty K.$$

Hint: argue by contradiction and apply δ *to* $\mathbb{S}\xrightarrow{2}\mathbb{S}$.

SOLUTION: By naturality, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\delta} & \mathbb{S}\wedge\mathbb{S} \\ \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2\wedge 2} \\ \mathbb{S} & \xrightarrow{\delta} & \mathbb{S}\wedge\mathbb{S} \end{array}$$

Since δ agrees with the map on suspension spectra and $\mathbb{S} = \Sigma^\infty S^0$, $\delta\colon \mathbb{S}\to\mathbb{S}\wedge\mathbb{S}$ is an isomorphism. By the previous part, $2\wedge 2 = 4$. But, this means that a degree 2 and a degree 4 map are equal, and this is false on π_0.

(3) Assume that we are given a closed symmetric monoidal category of spectra $(\widehat{\mathcal{S}p}, \wedge, \mathbb{S})$ such that $\wedge$ preserves colimits and weak equivalences in each variable separately, and for any pointed space K, $\Sigma^\infty K\wedge X \cong K\wedge X$. The right adjoint to $X\wedge -$ is the function spectrum $F(X,-)$.

(a) Prove that for any integer $n\in\mathbb{Z}$, the smash product $\mathbb{S}^n\wedge X$ is stably equivalent to $\Sigma^n X$.

SOLUTION: Let $n\geq 0$. Then $\mathbb{S}^n\wedge X = \Sigma^\infty S^n\wedge X \cong S^n\wedge X$ (note the difference between blackboard bold $\mathbb{S}$ and regular S – one is a spectrum, the other is a space). Then $S^n\wedge X$ has, by definition, k-th space $(S^n\wedge X)_k = S^n\wedge X_k = \Sigma^n X_k$, with structure maps suspensions of the structure maps of X. Hence, $S^n\wedge X\cong\Sigma^n X$ when $n\geq 0$.

For $n>0$, now consider $\mathbb{S}^{-n}\wedge X$. By definition, $\mathbb{S}^{-n}$ is the spectrum $F_n S^0 \simeq \Sigma^{-n}\Sigma^\infty S^0 = \Sigma^{-n}\mathbb{S}$. Then

$$\Sigma^n(\mathbb{S}^{-n}\wedge X)\cong(\Sigma^n\mathbb{S}^{-n})\wedge X\cong\mathbb{S}\wedge X\cong X$$

Applying Σ^{-n} to both sides of the above isomorphism yields

$$\mathbb{S}^{-n}\wedge X\cong\Sigma^{-n}X.$$

(b) Prove that for any integer $n\in\mathbb{Z}$, the function spectrum $F(\mathbb{S}^n, X)$ is stably equivalent to $\Sigma^{-n}X$.

SOLUTION: By part (a), $\mathbb{S}^n\wedge -$ is isomorphic to Σ^n. The right adjoint to $\mathbb{S}^n\wedge -$ is $F(\mathbb{S}^n,-)$, and the right adjoint to Σ^n is $\Sigma^{-n}=\Omega^n$. Since the functors $\mathbb{S}^n\wedge -$ and Σ^n are isomorphic, their right adjoints are isomorphic as well.

(4) Let X be an $H\mathbb{F}_p$-module spectrum. What does this imply about the homotopy groups of X? Can you say anything about the homology groups?

SOLUTION: If X is an $H\mathbb{F}_p$-module map, there is an action $\mu\colon H\mathbb{F}_p\wedge X\to X$. The homotopy groups functor from spectra to graded abelian groups is lax symmetric monoidal. Thus, there is an action map

$$\pi_*(H\mathbb{F}_p)\otimes\pi_*(X)\to\pi_*(H\mathbb{F}_p\wedge X)\to\pi_*(X).$$

Since $\pi_*(H\mathbb{F}_p) = \mathbb{F}_p$ (concentrated in degree 0), this is an action map $\mathbb{F}_p \otimes \pi_*(X) \to \pi_*(X)$. Equivalently, $\pi_*(X)$ is p-torsion.

If E is the spectra representing any cohomology theory (in particular $E = HA$), then $E_*(X) = \pi_*(X \wedge E)$, and $X \wedge E$ is an $H\mathbb{F}_p$-module with action map $\mu \wedge E$. Thus, $E_*(X)$ is also p-torsion.

References

[bip] nLab: biproduct. "https://ncatlab.org/nlab/show/biproduct".

[Lew91] L. Gaunce Lewis, Jr. Is there a convenient category of spectra? *J. Pure Appl. Algebra*, 73(3):233–246, 1991.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.