FINAL EXAM NAME: ______________________

Due by 11:59pm on May 6th.

- *Your answers should be neatly written and logically organized.*
- *Do your best to solve these problems by yourself, but ask for help from others if you're stuck. Asking for help is usually a good move with research problems!*
- *The solutions you turn in should be your own.*
- *You may use any resource you find online (or elsewhere), but you must cite any resource you use.*

ESSAY QUESTION

Using examples from class (or elsewhere), write 1-2 pages of TEX* to argue for or against the following claim:

> Spectra are more fundamental to algebraic topology than spaces.

Cite any resources you use.

PROBLEMS

Answer 3 out of the following 5 problems.

(1) Define a category $\mathcal{Z}p$ of "zpectra" whose objects are $\mathbb{Z}$-indexed sequences of spaces $\{X_n\}_{n\in\mathbb{Z}}$ together with structure maps $\sigma_i \colon \Sigma X_i \to X_{i+1}$ for all $i \in \mathbb{Z}$. A *morphism of zpectra* $f \colon X \to Y$ is a sequence of continuous maps $f_i \colon X_i \to Y_i$ that commute with the structure maps of X and Y.

The *ztable homotopy groups* of a zpectrum X are defined by $\pi_k X := \operatorname{colim}_{n\in\mathbb{Z}} \pi_{n+k} X_n$. A *ztable equivalence* of zpectra is a map that induces isomorphisms on stable homotopy groups.

Prove that the homotopy category of zpectra and ztable equivalences is equivalent to $\mathrm{ho}(\mathcal{S}p)$.

(2) Let $X \xrightarrow{f} Y \to Z$ be a cofiber sequence such that f is zero in $\mathrm{ho}(\mathcal{S}p)$. Show that $Z \simeq Y \vee \Sigma X$.

(3) Let $\widehat{\mathcal{S}p}$ be any symmetric monoidal category of spectra. Given a spectrum X, define $T(X) := \bigvee_{n\geq 0} X^{\wedge n}$.

  (a) Prove that $T(X)$ is an associative ring spectrum.

  (b) Prove that the functor $T \colon \widehat{\mathcal{S}p} \to \mathrm{Mon}(\widehat{\mathcal{S}p})$ is left adjoint to the forgetful functor $U \colon \mathrm{Mon}(\widehat{\mathcal{S}p}) \to \widehat{\mathcal{S}p}$, where $\mathrm{Mon}(\widehat{\mathcal{S}p})$ is the category of monoids in $\widehat{\mathcal{S}p}$, i.e. associative ring spectra.

(4) Let X be a symmetric spectrum. Recall that the symmetric spectrum $\mathrm{sh}^1 X$ is the symmetric spectrum with $(\mathrm{sh}^1 X)_n = X_{1+n}$, where Σ_n acts on X_{1+n} as the subgroup of Σ_{1+n} consisting of those permutations of $\{1, \dots, n+1\}$ leaving 1 fixed.

  (a) Construct a symmetric spectrum $(\mathrm{sh}^{-1} X)$ with n-th space $(\Sigma_n)_+ \wedge_{\Sigma_{n-1}} X_{n-1}$.

  (b) Show that sh^{-1} is a functor on symmetric spectra which is left adjoint to sh^1.

(5) If R is a commutative ring spectrum and $x \in \pi_n R$, let $R[x^{-1}]$ be the homotopy colimit of

$$R \xrightarrow{x} \Sigma^{-n} R \xrightarrow{x} \Sigma^{-2n} R \xrightarrow{x} \cdots,$$

where by abuse of notation we write $x \colon \Sigma^n R \to R$ for any suspension/desuspension of the map

$$\Sigma^n R = \mathbb{S}^n \wedge R \xrightarrow{x \wedge \mathrm{id}_R} R \wedge R \xrightarrow{\mu} R.$$

Show that $\pi_*(R[x^{-1}]) = (\pi_* R)[x^{-1}]$.

*Or 1-2 pages single spaced in a word processor, about 500-1000 words. I am not a stickler for essay length.

References

[HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[Sch07] Stefan Schwede. An untitled book project about symmetric spectra. http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf, 2007.