Random problems with R

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Abstract

R (Version 3.5.1 patched) has two issues with its random sampling functionality. First, it uses a version of the Mersenne Twister known to have a seeding problem, which was corrected by the authors of the Mersenne Twister in 2002. Updated C source code is available at `http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/MT2002/CODES/mt19937ar.c`. Second, R generates random integers between 1 and m by multiplying random floats by m, taking the floor, and adding 1 to the result. Well-known quantization effects in this approach result in a non-uniform distribution on $\{1, \ldots, m\}$. The difference, which depends on m, can be substantial. Because the `sample` function in R relies on generating random integers, random sampling in R is biased. There is an easy fix: construct random integers directly from random bits, rather than multiplying a random float by m. That is the strategy taken in Python's `numpy.random.randint()` function, and recommended by the authors of the Mersenne Twister algorithm, among others. Example source code in Python is available at `https://github.com/statlab/cryptorandom/blob/master/cryptorandom/cryptorandom.py` (see functions `getrandbits()` and `randbelow_from_randbits()`).

A textbook way to generate a random integer on $\{1, \ldots, m\}$ is to start with $X \sim U[0, 1)$ and define $Y \equiv 1 + \lfloor mX \rfloor$. If X is truly uniform on $[0, 1)$, Y is then uniform on $\{1, \ldots, m\}$. But if X has a discrete distribution derived by scaling a pseudorandom w-bit integer (typically $w = 32$) or floating-point number, the resulting distribution is, in general, not uniformly distributed on $\{1, \ldots, m\}$ even if the underlying pseudorandom number generator (PRNG) is perfect. Theorem 1 illustrates the problem.

Theorem 1 (Knuth [1997])**.** *Suppose X is uniformly distributed on w-bit binary fractions, and let $Y_m \equiv 1 + \lfloor mX \rfloor$. Let $p_+(m) = \max_{1 \le k \le m} \Pr\{Y_m = k\}$ and $p_-(m) = \min_{1 \le k \le m} \Pr\{Y_m = k\}$. There exists $m < 2^w$ such that, to first order, $p_+(m)/p_-(m) = 1 + m2^{-w+1}$.*

A better way to generate random elements of $\{1, \ldots, m\}$ is to use pseudorandom bits directly, avoiding floating-point representation, multiplication, and the floor operator. Integers between 0 and $m-1$ can be represented with $\mu(m) \equiv \lceil \log_2(m-1) \rceil$ bits. To generate a pseudorandom integer between 1 and m, first generate $\mu(m)$ pseudorandom bits (for instance, by taking the most significant $\mu(m)$ bits from the PRNG output, if $w \ge \mu(m)$, or by concatenating successive outputs of the PRNG and taking the first $\mu(m)$ bits of the result, if $w < \mu(m)$). Cast the result as a binary integer M. If $M > m - 1$, discard it and draw another $\mu(m)$ bits; otherwise, return $M+1$.[1] Unless $m = 2^{\mu(m)}$, this procedure is expected to discard some random draws—up to almost half the draws if $m = 2^p + 1$ for some integer p. But if the input bits are IID Bernoulli(1/2), the output will be uniformly distributed on $\{1, \ldots, m\}$. This is how the Python function `numpy.random.randint()` (Version 1.14) generates

[1] See Knuth [1997, p.114]. This is also the approach recommended by the authors of the Mersenne Twister. See `http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/efaq.html`, last accessed 18 September 2018.

pseudorandom integers.[2]

The algorithm that R (Version 3.5.1 patched) [R Core Team, 2018] uses to generate random integers in `R_unif_index()` (in `RNG.c`) has the issue pointed out in Theorem 1 in a more complicated form, because R uses a pseudorandom float at an intermediate step, rather than multiplying a binary fraction by m. The way the float is constructed depends on m. Because `sample` relies on random integers, it inherits the problem.

When m is small, R uses `unif_rand` to generate pseudorandom floating-point numbers X on $[0, 1)$ starting from a 32-bit random integer generated from the (obsolete version of the) Mersenne Twister algorithm [Matsumoto and Nishimura, 1998]. The range of `unif_rand` contains (at most) 2^{32} values, which are approximately equi-spaced (but for the vagaries of converting a binary number into a floating-point number [Goldberg, 1991], which R does using floating-point multiplication by 2.3283064365386963e-10).

When $m > 2^{31}$, `R_unif_index()` calls `ru` instead of `unif_rand`.[3] `ru` combines two floating-point numbers, R_1 and R_2, each generated from a 32-bit integer, to produce the floating-point number X, as follows: the first float is multiplied by $U = 2^{25}$, added to the second float, and the result is divided by U:

$$X = \frac{\lfloor U R_1 \rfloor + R_2}{U}.$$

The relevant code is in `RNG.c`.

The cardinality of the range of `ru` is certainly not larger than 2^{64}. The range of

[2]However, Python's built-in `random.choice()` (Versions 2.7 through 3.6) does something else biased: it finds the closest integer to mX, where X is a binary fraction between 0 and 1.

[3]A different function, `sample2`, is called when $m > 10^7$ and $k < m/2$. `sample2` uses the same method to generate pseudorandom integers.

`ru` is unevenly spaced on $[0, 1)$ because of how floating-point representation works. The inhomogeneity can make the probability that $X \in [x, x + \delta) \subset [0, 1)$ vary widely with x.

For the way `R_unif_index()` generates random integers, the non-uniformity of the probabilities of $\{1, \ldots, m\}$ is largest when m is just below 2^{31}. The upper bound on the ratio of selection probabilities approaches 2 as m approaches 2^{31}, about 2 billion. For m close to 1 million, the upper bound is about 1.0004.

We recommend that the R developers replace the obsolete (1997) version of the Mersenne Twister code with the current (2002) version, and replace the algorithm in `R_unif_index()` with the algorithm based on generating a random bit string large enough to represent m and discarding integers that are larger than m. The resulting code would be simpler and more accurate. Other routines that generate random integers using the multiply-and-floor method `(int) unif_rand() * n`, for instance, `walker_ProbSampleReplace()` in `random.c`, should also be updated to use an unbiased integer generator (e.g., to call the new version of `R_unif_index()`).

References

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