HW02 Linear Algebra

Problem 1 (3 points)

In Julia randomly generate a 5×5 symmetric positive matrix A and a 5×1 vector b. Use the Cholesky decomposition to find the solution to $Ax = b$. As long as it is clear what you are doing, you do not need to typeset the Julia output for this problem, but you should use words to describe the workflow. You must use the decomposition to obtain your answer, but you may use the backslash operator in place of forward and backward substitutions.

Note: the object from the Julia `cholesky` is a `struct`, the Julia term for a composite data type. Use the help feature `?` to learn how to get the relevant matrices from this.

Solution

In [1]:
```
#Generate a 4x4 random matrix
M = 10*rand(5,5)
#Generate a symmetric positive definite matrix
A = M*M'
```

Out[1]:
```
5×5 Matrix{Float64}:
 208.657  136.431  188.319  148.588  222.396
 136.431  136.32   187.144  146.654  150.032
 188.319  187.144  281.222  208.727  235.494
 148.588  146.654  208.727  180.082  160.869
 222.396  150.032  235.494  160.869  283.769
```

In [2]:
```
#Generate a random 4x1 vector
b = 10*rand(5,1)
```

Out[2]:
```
5×1 Matrix{Float64}:
 1.5482822721451917
 2.6261334036357775
 8.2334658253686
 8.718184409310895
 1.9808854106537555
```

In [3]:
```
#Grab the upper triangular matrix from the cholesky function.
#Thus, G'G = A => G'Gx = b
using LinearAlgebra
G = cholesky(A).U
```

Out[3]:
```
5×5 UpperTriangular{Float64, Matrix{Float64}}:
 14.445  9.44488  13.037    10.2865    15.3961
   ⋅     6.86399   9.32567   7.21151   0.672789
   ⋅      ⋅        4.92854   1.49539   5.78284
   ⋅      ⋅         ⋅        4.47522  -2.45848
   ⋅      ⋅         ⋅         ⋅        2.60607
```

In [4]:
```
#Letting Gx = y, G'y = b => y = G'\b
y = G'\b
```

Out[4]:
```
5×1 Matrix{Float64}:
  0.10718495214244461
  0.2351087547299727
  0.9421753308432127
  1.0080443848335305
 -1.073535039618823
```

In [5]:
```
#Gx = y => x = G\y
x = G\y
```

Out[5]:
```
5×1 Matrix{Float64}:
  0.38814062073738376
 -0.8411107906170889
  0.6748247873009939
 -0.0010485195143112968
 -0.41193583163493014
```

In [6]:
```
A*x
```

Out[6]:
```
5×1 Matrix{Float64}:
 1.5482822721451797
 2.6261334036357713
 8.233465825368596
 8.718184409310892
 1.9808854106537455
```

In [7]:
```
A*x - b
```

Out[7]:
```
5×1 Matrix{Float64}:
 -1.199040866595169e-14
 -6.217248937900877e-15
 -5.329070518200751e-15
 -3.552713678800501e-15
 -9.992007221626409e-15
```

The residual error is quite small, approximately on the order of 10^{-14}

Problem 1 (3 points)

(Analytical). If A can by decomposed into $A = G^T G$ with G upper triangular, and into $A = LDL^T$ with L unit lower triangular and D diagonal, show that $G^T = LD^{1/2}$ where $D^{1/2}$ is a diagonal matrix whose elements are the square roots of the elements of D.

Solution

First, show that $D = D^{\frac{1}{2}} D^{\frac{1}{2}}$

$$D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_{11}}^2 & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}}^2 \end{bmatrix} = D$$

Thus we can write $A = LDL^T$ as $A = LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T$

Upon examining the structure of L as a unit lower triangular matrix, we see that L^T is a unit upper triangular matrix

Now consider $D^{\frac{1}{2}}L^T$

$$\begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} 1 & l_{12} & \dots & l_{1n} \\ 0 & 1 & \dots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_{11}} & \sqrt{d_{11}}l_{12} & \dots & \sqrt{d_{11}}l_{1n} \\ 0 & \sqrt{d_{22}} & \dots & \sqrt{d_{22}}l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

Which is upper triangular in structure. Similarly, we also find that $LD^{\frac{1}{2}}$ is the transpose of $D^{\frac{1}{2}}L^T$ by $(AB)^T = B^TA^T$, and thus lower triangular in structure

Thus, if we let $F = D^{\frac{1}{2}}L^T$, we have that $A = F^TF$

However, we also have $A = G^TG$, and by Cholesky decomposition, we must have a unique G such that $A = G^TG$

Thus, since $G \neq F$ is a contradiction, we have $G = F$

Problem 2 (4 points)

The hyperbolic tangent is defined mathematically (but not numerically) as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

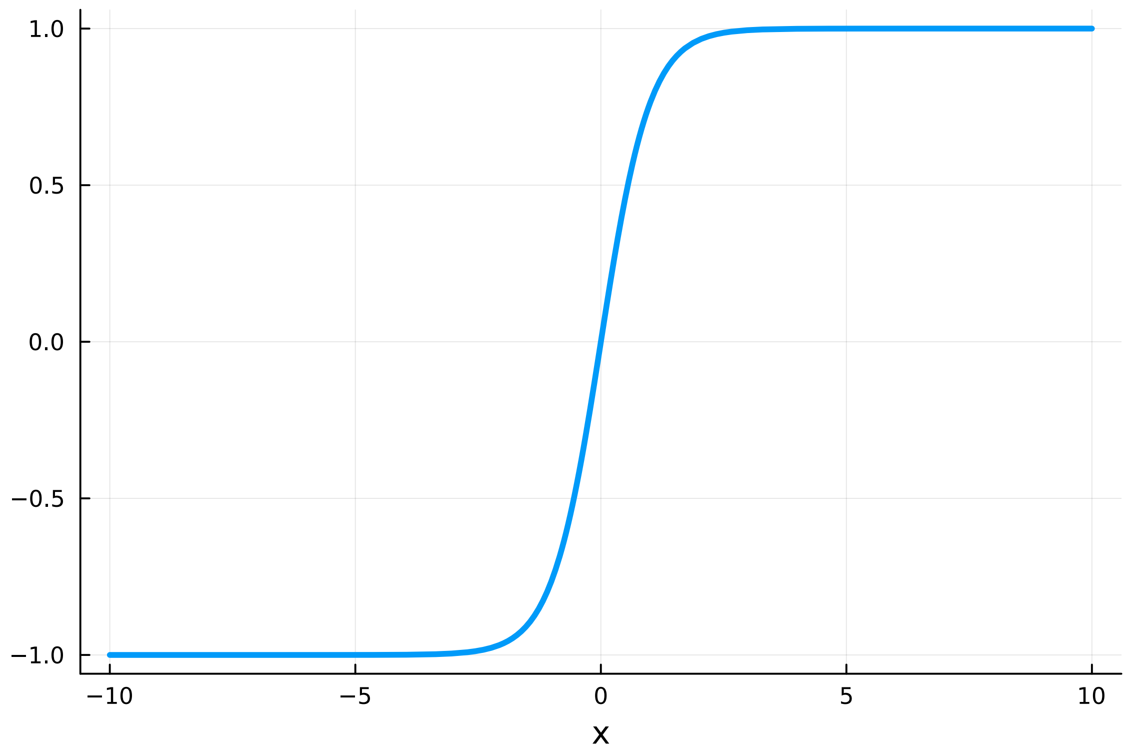
We can visualize this using the `Plots.jl` package in Julia:

In [8]:
```
import Pkg; Pkg.add("Plots")  # you should only need to do this once for the semest
```

```
  Resolving package versions...
 No Changes to `~/.julia/environments/v1.9/Project.toml`
 No Changes to `~/.julia/environments/v1.9/Manifest.toml`
```

In [9]:
```
using Plots
plot(tanh,-10,10,legend=:false,lw=3,xlabel="x")
```

Out[9]:



As you can see,

$$\lim_{x \to \pm\infty} \tanh(x) = \pm 1.$$

1. Use the build-in Julia function, `tanh` to compute `tanh(1000)` and `tanh(-1000)`.

2. Evaluate again at `x=1000` and `x=-1000` using the formal definition,

```
f(x) = (exp(x) - exp(-x))/(exp(x) + exp(-x))
```

3. Rewrite the mathematical expression so that you don't get `NaN` as answers. Typeset the work you did to get this expression. Hint: what happens when you evaluate `exp(1000)` ?

Solution

In [10]:
```
tanh(1000)
```
Out[10]:
```
1.0
```

In [11]:
```
tanh(-1000)
```
Out[11]:
```
-1.0
```

In [12]:
```
f(x) = (exp(x) - exp(-x))/(exp(x) + exp(-x))
```
Out[12]:
```
f (generic function with 1 method)
```

In [13]:
```
f(1000)
```
Out[13]:
```
NaN
```

In [14]:
```
f(-1000)
```
Out[14]:
```
NaN
```

$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \times \frac{e^{-x}}{e^{-x}}$

$f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}$

Or, $f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} \times \frac{e^{2x}}{e^{2x}}$

$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$

In [15]:
```
f(x) = (x > 0) ? (1 - exp(-2x)) / (1 + exp(-2x)) : (exp(2x) - 1) / (exp(2x) + 1)
```
Out[15]:
```
f (generic function with 1 method)
```

In [16]:
```
f(1000)
```
Out[16]:
```
1.0
```

In [17]:
```
f(-1000)
```
Out[17]:
```
-1.0
```