HW02 Linear Algebra

Problem 1 (3 points)

In Julia randomly generate a 5×5 symmetric positive matrix A and a 5×1 vector b. Use the Cholesky decomposition to find the solution to $Ax = b$. As long as it is clear what you are doing, you do not need to typeset the Julia output for this problem, but you should use words to describe the workflow. You must use the decomposition to obtain your answer, but you may use the backslash operator in place of forward and backward substitutions.

Note: the object from the Julia `cholesky` is a `struct`, the Julia term for a composite data type. Use the help feature `?` to learn how to get the relevant matrices from this.

Solution

In [1]:
```
#Generate a 4x4 random matrix
M = 10*rand(5,5)
#Generate a symmetric positive definite matrix
A = M*M'
```

Out[1]:
```
5×5 Matrix{Float64}:
 138.684  167.667  175.151  133.395  174.942
 167.667  325.602  219.086  250.246  210.584
 175.151  219.086  236.994  192.503  221.7
 133.395  250.246  192.503  245.845  148.64
 174.942  210.584  221.7    148.64   234.223
```

In [2]:
```
#Generate a random 4x1 vector
b = 10*rand(5,1)
```

Out[2]:
```
5×1 Matrix{Float64}:
 6.295150943484
 9.679646247049916
 8.877538919340871
 3.6553596552167544
 1.8197447698692992
```

In [3]:
```
#Grab the upper triangular matrix from the cholesky function.
#Thus, G'G = A => G'Gx = b
using LinearAlgebra
G = cholesky(A).U
```

Out[3]:
```
5×5 UpperTriangular{Float64, Matrix{Float64}}:
 11.7764  14.2376  14.873     11.3273   14.8553
   ⋅      11.0857   0.661185   8.02581  -0.0829284
   ⋅        ⋅       3.91785    4.77941   0.206967
   ⋅        ⋅        ⋅         5.50274  -3.6263
   ⋅        ⋅        ⋅          ⋅        0.585806
```

In [4]:
```
#Letting Gx = y, G'y = b => y = G'\b
y = G'\b
```

Out[4]:
```
5×1 Matrix{Float64}:
   0.5345564805740155
   0.18662375151328026
   0.20512894389378283
  -0.8864556705687682
 -15.98273933842438
```

In [5]:
```
#Gx = y => x = G\y
x = G\y
```

Out[5]:
```
5×1 Matrix{Float64}:
   8.126740308850325
  11.537282330521013
  23.623699835589917
 -18.14078178043559
 -27.28332284640167
```

In [6]:
```
A*x
```

Out[6]:
```
5×1 Matrix{Float64}:
 6.2951509434827555
 9.679646247048527
 8.877538919340399
 3.655359655216354
 1.8197447698679232
```

In [7]:
```
A*x - b
```

Out[7]:
```
5×1 Matrix{Float64}:
 -1.2443379659998755e-12
 -1.3891110484109959e-12
 -4.725109192804666e-13
 -4.005684672847565e-13
 -1.376010416720419e-12
```

The residual error is quite small, approximately on the order of 10^{-14}

Problem 1 (3 points)

(Analytical). If A can by decomposed into $A = G^T G$ with G upper triangular, and into $A = LDL^T$ with L unit lower triangular and D diagonal, show that $G^T = LD^{1/2}$ where $D^{1/2}$ is a diagonal matrix whose elements are the square roots of the elements of D.

Solution

First, show that $D = D^{\frac{1}{2}} D^{\frac{1}{2}}$

$$D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_{11}}^2 & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}}^2 \end{bmatrix} = D$$

Thus we can write $A = LDL^T$ as $A = LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T$

Now consider $D^{\frac{1}{2}}L^T$

$$\begin{bmatrix} \sqrt{d_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{d_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} 1 & l_{12} & \dots & l_{1n} \\ 0 & 1 & \dots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_{11}} & \sqrt{d_{11}}l_{12} & \dots & \sqrt{d_{11}}l_{1n} \\ 0 & \sqrt{d_{22}} & \dots & \sqrt{d_{22}}l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{d_{nn}} \end{bmatrix}$$

Which is upper triangular in structure.

Recall that $(AB)^T = B^T A^T$. Thus, $(D^{\frac{1}{2}}L^T)^T = (L^T)^T(D^{\frac{1}{2}})^T = LD^{\frac{1}{2}}$.

Thus, we have $A = (D^{\frac{1}{2}}L^T)^T(D^{\frac{1}{2}}L^T)$

Thus, if we let $F = D^{\frac{1}{2}}L^T$, we have that $A = F^T F$ where F is upper triangular.

However, we also have $A = G^T G$, and by Cholesky decomposition, we must have a *unique* G such that $A = G^T G$

Thus, since $G \neq F$ is a contradiction, we have $G = F$

$$G^T = F^T = LD^{\frac{1}{2}}$$

Problem 2 (4 points)

The hyperbolic tangent is defined mathematically (but not numerically) as

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

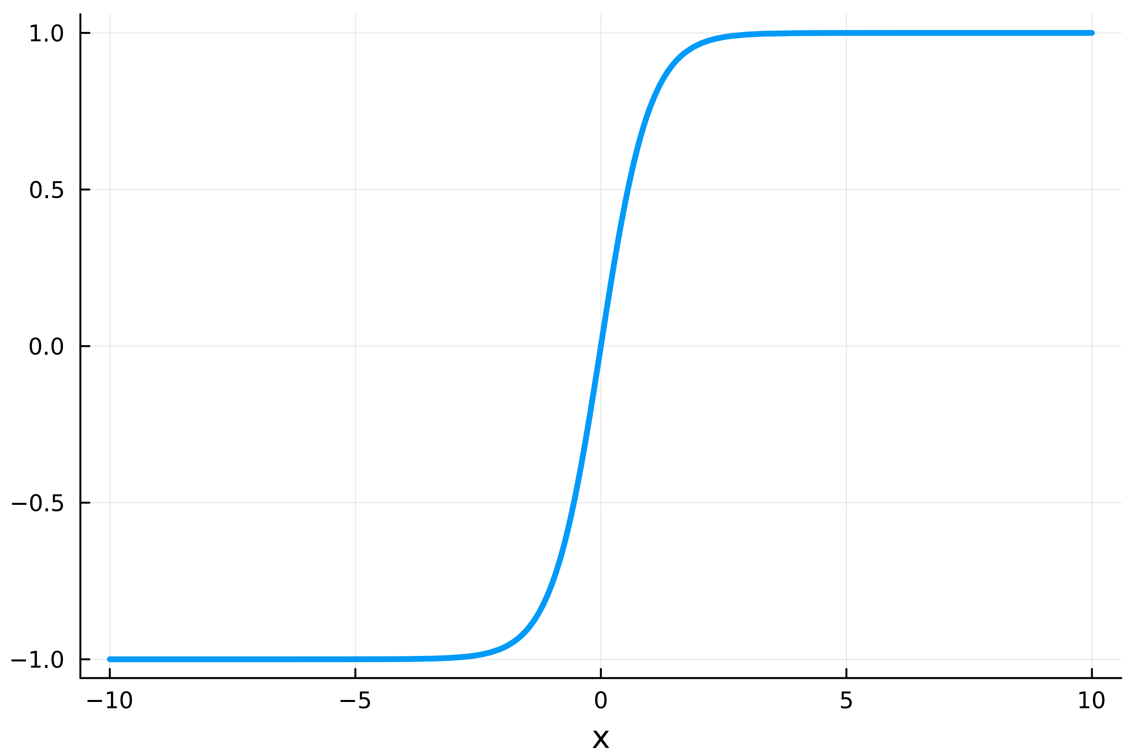
We can visualize this using the `Plots.jl` package in Julia:

In [8]:
```
import Pkg; Pkg.add("Plots")  # you should only need to do this once for the semest
```

```
  Resolving package versions...
 No Changes to `~/.julia/environments/v1.9/Project.toml`
 No Changes to `~/.julia/environments/v1.9/Manifest.toml`
```

In [9]:
```
using Plots
plot(tanh,-10,10,legend=:false,lw=3,xlabel="x")
```

Out[9]:



As you can see,

$$\lim_{x\to\pm\infty} \tanh(x) = \pm 1.$$

1. Use the build-in Julia function, `tanh` to compute `tanh(1000)` and `tanh(-1000)` .
2. Evaluate again at `x=1000` and `x=-1000` using the formal definition,

```
f(x) = (exp(x) - exp(-x))/(exp(x) + exp(-x))
```

3. Rewrite the mathematical expression so that you don't get `NaN` as answers. Typeset the work you did to get this expression. Hint: what happens when you evaluate `exp(1000)` ?

Solution

In [10]:
```
tanh(1000)
```

Out[10]: `1.0`

In [11]:
```
tanh(-1000)
```

Out[11]: `-1.0`

In [12]:
```
f(x) = (exp(x) - exp(-x))/(exp(x) + exp(-x))
```

Out[12]: `f (generic function with 1 method)`

In [13]:
```
f(1000)
```

Out[13]: `NaN`

In [14]:
```
f(-1000)
```

Out[14]: `NaN`

$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \times \frac{e^{-x}}{e^{-x}}$

$f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}$

Or, $f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} \times \frac{e^{2x}}{e^{2x}}$

$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$

In [15]:
```
f(x) = (x > 0) ? (1 - exp(-2x)) / (1 + exp(-2x)) : (exp(2x) - 1) / (exp(2x) + 1)
```

Out[15]: `f (generic function with 1 method)`

In [16]:
```
f(1000)
```

Out[16]: `1.0`

In [17]:
```
f(-1000)
```

Out[17]: `-1.0`