Intermediate Microeconomics: Advanced Notes

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Contents

1 Mathematics of Optimization

1.1 Unconstrained Optimization

- First order conditions: $\frac{\delta f}{\delta x} = 0, \frac{\delta f}{\delta y} = 0$
- Second order conditions: $\frac{\delta^2 f}{\delta x^2} < 0, \frac{\delta^2 f}{\delta y^2} < 0$
- Additionally, $(\frac{\delta^2 f}{\delta x^2} \times \frac{\delta^2 f}{\delta y^2}) - (\frac{\delta^2 f}{\delta x \delta y})^2 > 0$

Example: $G(T, C) = 50 + 10T + 16C - T^2 - 2TC - 2C^2$

$\frac{\delta G}{\delta T} = 10 - 2T - 2C$

$\frac{\delta G}{\delta C} = 16 - 2T - 4C$

$$\begin{bmatrix} -2 & -2 & -10 \\ -2 & -4 & -16 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 & -10 \\ 0 & -2 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 & -4 \\ 0 & -2 & -6 \end{bmatrix}$$

$\Rightarrow T = 2, C = 3$

$\frac{\delta^2 G}{\delta T^2} = -2 < 0$

$\frac{\delta^2 G}{\delta C^2} = -4 < 0$

$(\frac{\delta^2 G}{\delta T^2} \times \frac{\delta^2 G}{\delta C^2}) - (\frac{\delta^2 G}{\delta T \delta C})^2 = (-2 \times -4) - (-2)^2 = 8 - 4 = 4 > 0$

$\Rightarrow G(2, 3)$ is a maximum $\Rightarrow G(2, 3) = 50 + 20 + 48 - 4 - 12 - 18 = 84$

1.2 Constrained Optimization

To find the critical points of $f(x)$ subject to some constraints $g(x) = c$ we have

$$\begin{bmatrix} \frac{\delta f}{\delta x} & = \lambda \frac{\delta g}{\delta x} \\ \frac{\delta f}{\delta y} & = \lambda \frac{\delta g}{\delta y} \\ g(x) & = c \end{bmatrix}$$

Example: $T = 19.25 - 6\ln(R) - 4\ln(W)$, constraint is $R + W = 5$

$$\begin{bmatrix} 19.25 - \frac{6}{R} & = \lambda \\ 19.25 - \frac{4}{W} & = \lambda \\ R + W & = 5 \end{bmatrix}$$

$19.25 - \frac{6}{R} = 19.25 - \frac{4}{W} \Rightarrow 6W = 4R$

$W = 5 - R \Rightarrow 6(5-R) = 4R \Rightarrow 30 - 6R = 4R \Rightarrow 10R = 30 \Rightarrow R = 3, W = 2$

$T(3,2) = 19.25 - 6\ln(3) - 4\ln(2) \approx 9.89$

Example: $I = \frac{P^2}{250} + \frac{PA}{100} + \frac{A^2}{1000}$, constraint is $P + A = 110$

$$\begin{bmatrix} \frac{P}{125} + \frac{A}{100} & = \lambda \\ \frac{P}{100} + \frac{A}{500} & = \lambda \\ P + A & = 110 \end{bmatrix}$$

$\frac{P}{125} + \frac{A}{100} = \frac{P}{100} + \frac{A}{500}$

$\frac{P}{125} - \frac{P}{100} = \frac{-4A}{500}$

$\frac{-P}{500} = \frac{-4A}{500}$

$P = 4A \Rightarrow 5A = 110 \Rightarrow A = 22 \Rightarrow P = 88$

$I(88, 22) = 50.82$