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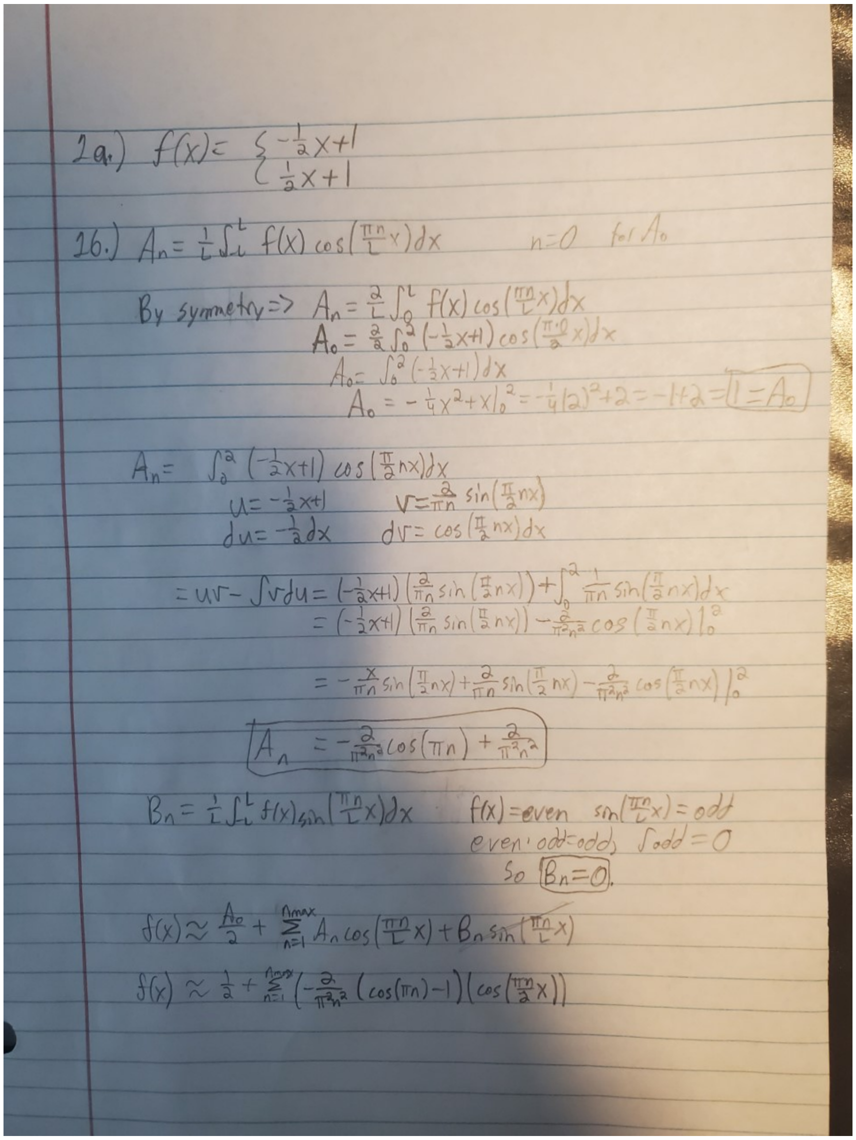
PHYS 331, HW07

1a&b.)



1a.) $f(x) = \begin{cases} -\frac{1}{2}x+1 \\ \frac{1}{2}x+1 \end{cases}$

1b.) $A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n}{L}x\right)dx$ $\quad n=0$ for A_0

By symmetry => $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n}{L}x\right)dx$

$A_0 = \frac{2}{2}\int_0^2 \left(-\frac{1}{2}x+1\right)\cos\left(\frac{\pi \cdot 0}{2}x\right)dx$

$A_0 = \int_0^2 \left(-\frac{1}{2}x+1\right)dx$

$A_0 = -\frac{1}{4}x^2+x\Big|_0^2 = -\frac{1}{4}(2)^2+2 = -1+2 = \boxed{1 = A_0}$

$A_n = \int_0^2 \left(-\frac{1}{2}x+1\right)\cos\left(\frac{\pi}{2}nx\right)dx$

$u = -\frac{1}{2}x+1 \qquad v = \frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)$

$du = -\frac{1}{2}dx \qquad dv = \cos\left(\frac{\pi}{2}nx\right)dx$

$= uv - \int v\,du = \left(-\frac{1}{2}x+1\right)\left(\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)\right) + \int_0^2 \frac{1}{\pi n}\sin\left(\frac{\pi}{2}nx\right)dx$

$= \left(-\frac{1}{2}x+1\right)\left(\frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right)\right) - \frac{2}{\pi^2 n^2}\cos\left(\frac{\pi}{2}nx\right)\Big|_0^2$

$= -\frac{x}{\pi n}\sin\left(\frac{\pi}{2}nx\right) + \frac{2}{\pi n}\sin\left(\frac{\pi}{2}nx\right) - \frac{2}{\pi^2 n^2}\cos\left(\frac{\pi}{2}nx\right)\Big|_0^2$

$\boxed{A_n = -\frac{2}{\pi^2 n^2}\cos(\pi n) + \frac{2}{\pi^2 n^2}}$

$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n}{L}x\right)dx$ $\quad f(x) = \text{even} \quad \sin\left(\frac{\pi n}{L}x\right) = \text{odd}$

even · odd = odd, $\int$ odd $= 0$

So $\boxed{B_n = 0}$.

$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{n_{max}} A_n\cos\left(\frac{\pi n}{L}x\right) + \cancel{B_n\sin\left(\frac{\pi n}{L}x\right)}$

$f(x) \approx \frac{1}{2} + \sum_{n=1}^{n_{max}}\left(-\frac{2}{\pi^2 n^2}\left(\cos(\pi n)-1\right)\left(\cos\left(\frac{\pi n}{2}x\right)\right)\right)$

1d.) The plot is now essentially a repeat of the original plot, repeated over the intervals -6 to -2 and 2 to 6. This is to be expected. The actual values of A0, An, and Bn have not changed, so it will obviously graph the fourier series over -2 to 2. Because fourier series work using phasors, it has a cyclical nature when extrapolated beyond the original intended interval. Because the new interval is an added period to the left and the right, the function is repeated once to the left and to the right.

2a.) The condition number is essentially a measure of how well conditioned a matrix is, with a perfectly conditioned matrix having a condition number of 1, and a larger condition number corresponding to a more ill-conditioned matrix. Additionally, we discussed that a matrix is ill-conditioned if its determinant is much smaller than its norm, hence matrices are ill-conditioned when R is small. As such, we expect to see high R values correspond to low condition number values, creating a negative slope. This is what we observed, as the graph is clustered along a negative trend in R as C increases. The matrices are randomized, hence giving this scattered, clustered appearance.

2b.) If a matrix is well conditioned, it should be fairly resistant to small perturbations, and as such should have a fairly small residual when used to solve for a vector. Oppositely, if it is ill-conditioned, changes in input vectors will create large changes in outputs, leading to large discrepancies and hence large residuals. The Euclidean norm sum the squares of elements of a matrix. As such, the norm of a matrix of large numbers will be large, and the norm of a matrix of small numbers will be small. Taking the norm of the residual is essentially mapping the "bigness" or "smallness" of the elements in it to a single scalar representation. A matrix that is well conditioned has a residual matrix with small values, and hence a small residual norm. A matrix that is ill conditioned has a residual matrix with large values, and hence a large residual norm. Because a small condition number signifies a matrix is well conditioned, a small condition number correlates to a small residual norm. Likewise, a large condition number correlates to a large residual norm. Thus we expect to see the norm of the residual increase with the condition number. This is what we observe, as the graph is clustered along a positive trend in |r| as C increases. The matrices are randomized, hence giving this scattered, clustered appearance.

3a.) $f(x) = e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n}{n!} \approx \frac{(-1)^0 (x)^0}{0!} + \frac{(-1)^1 (x)^1}{1!} + \frac{(-1)^2 (x)^2}{2!} + \frac{(-1)^3 (x)^3}{3!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$

3c.) Here is a table of the polyfit coefficients obtained:

	Second Order Polyfit	Third Order Polyfit	Fourth Order Polyfit
a_0	0.54209901	-0.17790243	0.04403354
a_1	-1.10386229	0.53676193	-0.17614109
a_2	0.99521795	-0.9971991	0.49906168
a_3		0.99628493	-0.99795346
a_4			1.00005169

The best fit coefficients alternate signs, between + - +, then - + - +, then + - + - +. Additionally, the fit coefficients are correlated with each other fairly closely when shifted up by one with each fitting. So a_2 in the second order fitting is approximately a_3 in the third order fitting and a_4 in the fourth order fitting. Likewise, a_1 in the second order fitting is approximately a_2 in the third order fitting and a_3 in the fourth order fitting, etc.

The plots of the second, third, and fourth order polyfits are all quite close to each other. The second order plot fits f(x) the least, the third order fits it the second most, and the fourth order fits it the closest. However, they all fit the function quite well despite being relatively low order.

The difference plots make the discrepancies between second, third, fourth order, and f(x) a great deal more apparent. The second order difference is much higher as its peaks are the greatest. The third order difference is lower by a good margin, and the fourth order difference is the lowest. The second order difference appears to be an odd second order polynomial function, while the third appears to be an even third order polynomial function. The fourth order is so close it appears to be a straight line, although very close inspection reveals it is an odd fourth order polynomial function. The fourth order is definitively the best approximation of the three approximations, which is exactly to be expected.