3 (a) *Solve* $y'' - 3y = t\cos t$

Step 1 The first step is to solve the homogeneous equation: $y'' - 3y = 0$. The characteristic equation is $r^2 - 3 = 0$, with roots $r = \pm\sqrt{3}$. We have two distinct real roots, so we use the general solution formula $y_c(t) = ce^{r_1 t} + de^{r_2 t}$, getting the general solution $y_c(t) = ce^{\sqrt{3}t} + de^{-\sqrt{3}t}$. (The "$c$" in y_c reminds us that this is the solution for the homogeneous equation— the "c" stands for *complementary solution* in case you were wondering)

Step 2 Now let's write down what our trial solution $Y(t)$ will look like. Let's write the nonhomogeneous term as $g(t)$ (this is the part that doesn't involve y, which is $t\cos t$). We then take the derivative over and over, crossing out duplicate terms as we go, until we run out of terms or reach zero. If two terms are the same except for their coefficients, we count those as duplicates too.

$$\begin{aligned} g &= t\cos t \\ g' &= -t\sin t + \cos t \\ g'' &= \cancel{-t\cos t} - \sin t - \cancel{\sin t} \\ g''' &= \cancel{-\cos t} \end{aligned}$$

Now we take all the unique terms ($\cos t$, $\sin t$, $t\cos t$, $t\sin t$) and make a linear combination of them using unknown coefficients A, B, $\ldots$:

$$Y(t) = A\cos t + B\sin t + Ct\cos t + Dt\sin t$$

Step 3 Plugging in the trial solution into the equation $y'' - 3y = t\cos t$, let's solve for A, B, C, and D:

$$\begin{aligned} Y(t) &= A\cos t + B\sin t + Ct\cos t + Dt\sin t \\ Y'(t) &= -A\sin t + B\cos t + C(\cos t - t\sin t) + D(\sin t + t\cos t) \\ &= (B + C)\cos t + (D - A)\sin t + Dt\cos t + (-C)t\sin t \\ Y''(t) &= -(B + C)\sin t + (D - A)\cos t + D(\cos t - t\sin t) + (-C)(\sin t + t\cos t) \\ &= (2D - A)\cos t + (-B - 2C)\sin t + (-C)t\cos t + (-D)t\sin t. \end{aligned}$$

So,

$$\begin{aligned} t\cos t = Y'' - 3Y &= (2D - A)\cos t + (-B - 2C)\sin t + (-C)t\cos t + (-D)t\sin t \\ &\quad - 3\big(A\cos t + B\sin t + Ct\cos t + Dt\sin t\big) \\ &= (2D - 4A)\cos t + (-4B - 2C)\sin t + (-4C)t\cos t + (-4D)t\sin t. \end{aligned}$$

Next we match up the coefficients on the right-hand side with the coefficients on the left-hand side.

$$\begin{aligned} 2D - 4A &= 0 \\ -4B - 2C &= 0 \\ -4C &= 1 \\ -4D &= 0 \end{aligned}$$

Solving for the unknown coefficients, we get $C = -\frac{1}{4}$, $D = 0$ from the last two equations, then $A = 0$, $B = \frac{1}{8}$ from the first two equations. So $Y(t) = \frac{1}{8}\sin t - \frac{1}{4}t\cos t$ is the particular solution.

Step 4 The last step is to write down the general solution for the nonhomogeneous equation. Now that we have a particular solution $Y(t)$ and the general solution for the homogeneous equation $y_c(t)$, we can just add them together:

$$y(t) = Y(t) + y_c(t) = \frac{1}{8}\sin t - \frac{1}{4}t\cos t + ce^{\sqrt{3}t} + de^{-\sqrt{3}t}.$$