Unforced Oscillations

Here's a quick review of the main facts about unforced oscillations (those with no external driving force applied).

Unforced, undamped vibration

The basic equation for a mass on a spring is $mu'' + \gamma u' + ku = F(t)$, where m is the mass, γ the damping factor, k the spring constant, and $F(t)$ the external driving force. First, we'll look at the case where $\gamma = 0$ (no damping) and $F(t) = 0$ (no driving force).

So our equation of motion becomes

$$mu'' + ku = 0$$

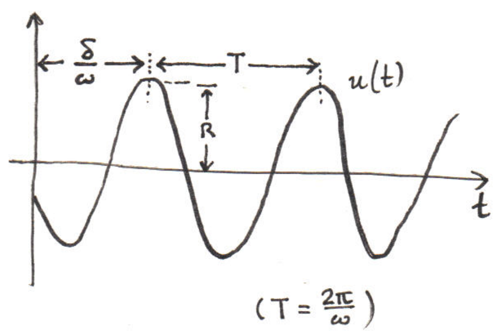
To solve it, we find that the characteristic equation is $mr^2 + k = 0$, with roots $r = \pm i\sqrt{k/m}$, so the general solution is

$$u(t) = c\cos(\omega t) + d\sin(\omega t),$$

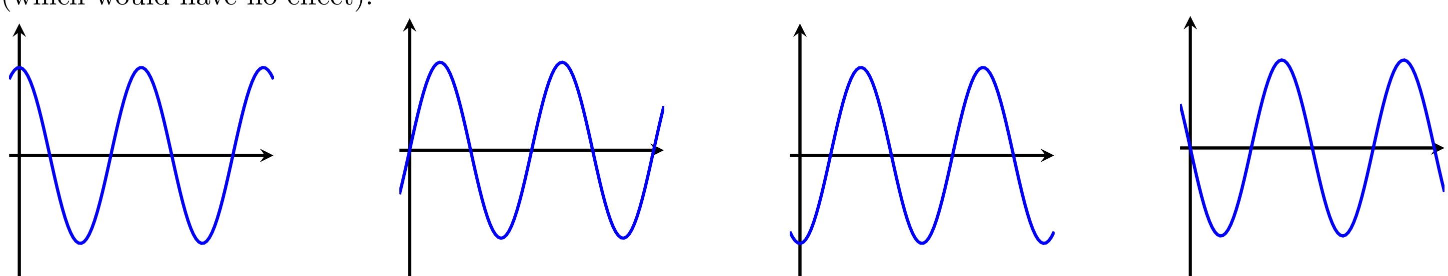
where $\omega = \sqrt{k/m}$ (ω is the *frequency*). The way the general solution is written, it's inconvenient to tell what the properties of the oscillation are (how far the mass moves up and down, or when the mass will reach its highest and lowest points). As in class, we can try to combine the sine and cosine terms into one cosine term with a shift δ:

$$u(t) = R\cos(\omega t - \delta).$$

R is called the *amplitude*, and δ the *phase* (note δ is an angle). The picture below shows how R and δ appear in the graph of the solution $u(t)$. The *period* of the oscillation is $T = 2\pi/\omega$, and is the distance between two peaks, or two troughs (low points):



The picture shows that the amplitude represents the height of the oscillation, while the phase is a little trickier: it represents how far horizontally the wave is shifted. A phase of zero represents a cosine wave (where the first peak occurs right on the y-axis). A phase of $\pi/2$ would respresent shifting right by 1/4 of a full period, and a phase of π would represent a shift by 1/2 a full period, and a phase of 2π would represent shifting right by a full period (which would have no effect).

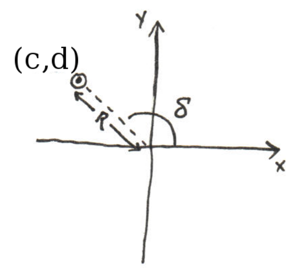


$\delta = 0$ $\delta = \frac{\pi}{2}$ $\delta = \pi$ $\delta = \frac{3\pi}{2}$

Using the trigonometric identity $\cos(a-b) = (\cos a)(\cos b) + (\sin a)(\sin b)$, we can show that $c\cos(\omega t) + d\sin(\omega t)$ will equal $R\cos(\omega t - \delta)$ if

$$R = \sqrt{c^2 + d^2} \qquad\qquad \tan\delta = \frac{d}{c}.$$

If we think of (c, d) as x and y, then R is the distance between (x, y) and the origin, and δ is the angle between the ray through (x, y) and the positive x-axis:



To get δ from $\tan\delta$, we can use the arctangent. Unfortunately, it isn't quite as simple as $\delta = \tan^{-1}(d/c)$. The problem is that $\tan\alpha$ and $\tan(\alpha+\pi)$ are always the same. So, for any x, there are two angles whose tangents are equal to x, and arctangent can only pick one of them. So, after taking the arctangent of d/c, we may have to add π to make sure δ is in the correct quadrant. You can figure out whether to add π by plotting the point (c, d) in the xy-plane (like in the picture above), or remember the rule that you add π when c is negative.

This works as long as c is nonzero. If $c = 0$, then $\delta = \pi/2$ or $\delta = 3\pi/2$, and you can again plot the point (c, d) in the xy-plane to find δ. Or if you'd like to memorize the rule, $\delta = \pi/2$ if $d > 0$ and $\delta = 3\pi/2$ is $d < 0$.

If this process looks familiar to you, it might be because it's the same process as converting from standard Cartesian (x, y) coordinates to polar coordinates (r, θ), where (c, d) are x and y, and (R, δ) are r and θ.

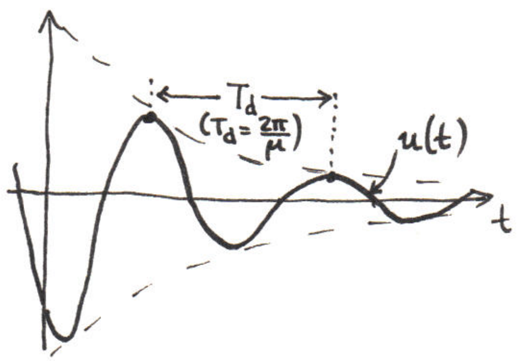
Unforced, damped vibration

With damping included (which may be due to air resistance, or energy loss in the spring), the equation of motion is $mu'' + \gamma u' + ku = 0$. There are three scenarios, corresponding to the three kinds of homogeneous equations we've studied:

more damping ↓			
	$\gamma^2 - 4km < 0$ (complex roots)	solution: $u(t) = ce^{\lambda t}\cos(\mu t) + de^{\lambda t}\sin(\mu t)$ where the roots are $\lambda \pm \mu i$	*vibrating*
	$\gamma^2 - 4km = 0$ (repeated roots)	solution: $u(t) = cte^{rt} + de^{rt}$	*critical damping*
	$\gamma^2 - 4km > 0$ (distinct real roots)	solution: $u(t) = ce^{r_1 t} + de^{r_2 t}$	*overdamping*

Notice that only when $\gamma^2 - 4km < 0$ does the system oscillate: when the damping γ becomes large enough that $\gamma^2 \geq 4km$, we have exponential decay.

The most interesting case is when $\gamma^2 < 4km$, and the solution oscillates, but the oscillations decay:



Since the solution isn't strictly periodic (it decays instead of repeating exactly), we don't talk about it having a frequency or period. But it does have a *quasi-frequency* (μ) and *quasi-period* (T_d), which are the frequency and period the solution would have if it weren't decaying. If the general solution is $u(t) = ce^{\lambda t) \cos(\mu t) + de^{\lambda t}\sin(\mu t)}$, then μ is the quasi-frequency, and $T_d = 2\pi/\mu$.