Math 307

THE COMPLEX EXPONENTIAL FUNCTION

(These notes assume you are already familiar with the basic properties of complex numbers.)

We make the following *definition*

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1}$$

This formula is called Euler's Formula. In order to justify this use of the exponential notation appearing in (1), we will first verify the following form of the Law of Exponents:

$$e^{i\theta_1 + i\theta_2} = e^{i\theta_1}\, e^{i\theta_2} \tag{2}$$

To prove this we first expand the right-hand side of (1) by first multiplying out the product: $e^{i\theta_1}\, e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$. Next we apply to this the trigonometric identities:

$$\begin{aligned} \cos\theta_1\, \cos\theta_2 - \sin\theta_1\, \sin\theta_2 &= \cos(\theta_1 + \theta_2) \\ \sin\theta_1\, \cos\theta_2 + \cos\theta_1\, \sin\theta_2 &= \sin(\theta_1 + \theta_2). \end{aligned}$$

When all this is done the result is

$$e^{i\theta_1}\, e^{i\theta_2} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2).$$

The right hand side of the last equation is exactly what we would get if we wrote out (1) with θ replaced by $\theta_1 + \theta_2$. We have therefore proved (2).

To justify the use of $e = 2.718\ldots$, the base of the natural logarithm, in (1), we will differentiate (1) with respect to θ: We should get $ie^{i\theta}$. Treating i like any other constant, we find $\frac{d}{d\theta} e^{i\theta} = \frac{d}{d\theta}\big(\cos\theta + i\sin\theta\big) = -\sin\theta + i\cos\theta$. But $-\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$. Thus, as expected,

$$\frac{d}{d\theta}\, e^{i\theta} = i\, e^{i\theta} \tag{3}$$

If one does not define $e^{i\theta}$ by (1), then one must find some other mean to define e^z and then to derive (1) directly as a consequence. Often the definition of e^z is made using power series with complex numbers z but this requrires a considerable amount of preliminary work with power series. For a very brief discussion of this approach, see page 154 in the text.

Some examples: $e^{i\pi/2} = i,\ e^{\pi i} = -1,$ and $e^{2\pi i} = +1$.

Recall that the relation between the rectangular coordinates (x, y) and the polar coordinates (r, θ) of a point is

$$\begin{aligned} x &= r\cos\theta, & y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2}, & \theta &= \arctan\frac{y}{x} \end{aligned} \tag{4}$$

where arctan (also called tan^{-1}) is one of the "branches" of the inverse tangent function. (The quadrant which holds the point (x, y) determines the correct branch of $\tan^{-1}$.) If $z = 0$ then $r = 0$ and θ can be anything. Making use of Euler's formula, we can express polar representation in the following manner:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}, \tag{5}$$

where $r = |z| = \sqrt{x^2 + y^2}$ and θ is given by (4). The angle θ is also called an *argument* of z and we write $\theta = \arg(z)$.

As noted, there is an ambiguity in (4) about the inverse tangent formula for θ which can (and *must*) be resolved by looking at the signs of x and y in order to determine in which quadrant $e^{i\theta}$ lies. For example, if $x = 0$, then the formula for θ in (4) makes no sense; but $x = 0$ simply means that $z = 0 + iy$ lies on the imaginary axis so θ must be $\pi/2$ or $3\pi/2$ depending on whether y is positive or negative. Again, if $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$. Therefore $-4 + 4i = 4\sqrt{2}e^{3\pi i/4}$.

Note also that, due to the periodicity of $\sin\theta$ and $\cos\theta$, if $z = re^{i\theta}$, then we also have $z = re^{i(\theta + 2k\pi)}$, $k = 0, \pm1, \pm2, \ldots$. Thus, in our last example, $-4 + 4i = 4\sqrt{2}e^{11\pi i/4} = 4\sqrt{2}e^{-5\pi i/4}$ etc.

Here is another example: the complex number $2 + 8i$ may also be written as $\sqrt{68}e^{i\theta}$, where $\theta = \tan^{-1}(4) \approx 1.33$ rad. See Figure 1.

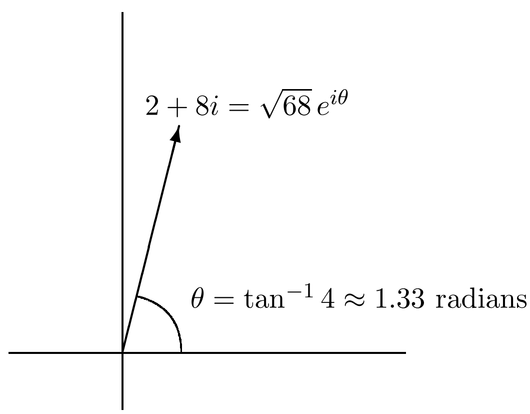

Figure 1.

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they are for rectangular coordinates. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 = z_2 \text{ if and only if } r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2k\pi,\ k = 0, \pm1, \pm2, \ldots$$

Despite this the polar representation is very useful when it comes to multplication and division:

$$\text{if } z_1 = r_1 e^{i\theta_1} \text{ and } z_2 = r_2 e^{i\theta_2}, \text{ then } z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}; \tag{6}$$

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \qquad (z_2 \neq 0). \tag{7}$$

This follows from the Law of Exponents in equation (2) and the rules: $|z_1 z_2| = |z_1||z_2| = r_1 r_2$; $|z_2^{-1}| = 1/r_2$; $\arg(1/z_2) = \arg(\overline{z_2}) = -\theta_2$. For example, let

$$\begin{aligned} z_1 &= 2 + i = \sqrt{5}e^{i\theta_1}, \quad \theta_1 = \tan^{-1}(\tfrac{1}{2}) \approx .464 \\ z_2 &= -2 + 4i = \sqrt{20}e^{i\theta_2}, \quad \theta_2 = \tan^{-1}(\tfrac{4}{-2}) = \pi + \mathrm{Tan}^{-1}(-2) \approx 2.034\ldots \end{aligned}$$

(Note: Tan^{-1} is the *principal* inverse tangent. It is the quantity computed on most scientific calculators.) Then $z_3 = z_1 z_2$ where:

$$z_3 = \sqrt{5}\sqrt{20}e^{i\theta_3} = 10e^{i\theta_3},$$

$$\theta_3 \approx .464 + 2.034 = 2.498\ldots$$

This gives $z_3 \approx 10(\cos(2.498) + \sin(2.498) \approx -7.995 + i6.001$. (The exact value is $z_3 = -8 + 6i$.) We leave it to the reader to find z_1/z_2 in this example using (7). (The exact value is $-i/2$ using the algebraic method.)

Applying (6) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2}e^{3\pi i/4}$, gives

$$(4 + 4i)^2 = \left(4\sqrt{2}e^{3\pi i/4}\right)^2 = 32e^{3\pi i/2} = -32i.$$

Indeed for any positive (or negative) integer it is quite straightforward to show that

$$\text{If } z = re^{i\theta} \neq 0, \text{ then } z^n = r^n e^{in\theta}.$$

This formula makes it quite easy to solve equations such as $z^3 = 1$. Write the unknown z as $re^{i\theta}$. Then for the equation $z^3 = 1$, we have $r^3 e^{3i\theta} = 1 = e^{0i}$. Hence, $r^3 = 1$ and $r = 1$, because r is supposed to be a positive real number, *and* $3\theta = 0 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. It follows that $\theta = 2k\pi/3$, $k = 0, \pm 1, \ldots$. There are only three *distinct* numbers of the form $e^{2k\pi i/3}$, namely: $1 = e^0$, $e^{2\pi i/3}$, and $e^{4\pi i/3}$.

The following figure illustrates the distinct solutions to another equation: $z^3 = 8i$. The solutions (called the *cube roots* of $8i = 8e^{\pi i/2}$) are: $z_1 = 2e^{i\pi/6}$, $z_2 = 2e^{5i\pi/6}$, and $z_3 = 2e^{9i\pi/6} = -2i$.

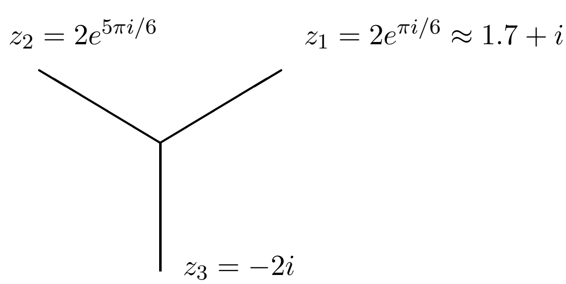

Figure 2. The three cube roots of $= 8i$.

Note that these roots are equally spaced on a circle with radius 2 and center 0. The n distinct $n^{\underline{\text{th}}}$ roots of any complex number $w \neq 0$ are equally spaced on a circle of radius $|w|^{1/n}$ centered at 0. One need only locate one of them on the circle. To get the other $n - 1$ roots, one rotatates the first one $n - 1$ times, each time through an angle $2\pi/n$, marking the points as one proceeds. Each distinct point corresponds to a distinct root of the equation $z^n = w$. After n rotations one goes right round the circle and arrives at the initial point.

From the fact that $\left(e^{i\theta}\right)^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Expanding on the left and equating real and imaginary parts, leads to trigonometric identities which can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example with $n = 2$ one gets:

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + i\,2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta.$$

Hence $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$. For $n = 3$, let us set $C = \cos\theta$ and $S = \sin\theta$. Then $(C + iS)^3 = C^3 + 3iC^2S - 3CS^2 - iS^3$, so

$$\cos 3\theta = \mathrm{Re}\ \{(C + iS)^3\} = C^3 - 3CS^2 = 4\cos^3\theta - 3\cos\theta.$$

(because $S^2 = 1 - C^2$)). One can derive a similar identity for $\sin 3\theta$.

1 The exponential of any complex number.

The definition of e^{x+iy} is given by the formula

$$e^{x+iy} = e^x e^{iy} \tag{8}$$

Each term on the right-hand side of (8) already has a well defined meaning. It is left as an exercise to show that

$$\frac{d}{dt}\,e^{(a+bi)t} = (a + bi)e^{(a+bi)t} \tag{9}$$

for any complex constant $a + bi$.

Exercises

1. Let $z_1 = 3i$ and $z_2 = 2 - 2i$.

(a) Plot the points $z_1 + z_2$, $z_1 - z_2$, and $\overline{z_2}$.

(b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.

(c) Express z_1 and z_2 in polar form.

2. Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot $z_1 z_2$, and z_1/z_2.

3. **(a)** Find and plot *all* complex numbers which satisfy $z^3 = -8$.

(b) Find all complex numbers $z = re^{i\theta}$, which satisfy $z^2 = \sqrt{2}e^{i\pi/4}$.

4. Verify (9). Note that $e^{(a+bi)t} = e^{at} \cdot e^{ibt}$. Use the product differentiation rule on this. You can differentiate e^{ibt} by means of (3) and the chain rule. You will still have some algebra to do to get the form on the right of (9).

5. Find an identity for $\sin 3\theta$ using $n = 3$ in De Moivre's formula. Write your identity in a way that involves only $\sin\theta$ and $\sin^3\theta$ if possible.

6. This problem explains the first real use of complex numbers. A cubic equation can be tranformed into the form:

$$x^3 = 3px + 2q,$$

where p and q are constants by replacing x with $ax+b$ and multiplying the cubic by a constant. The graph of the right side is a straight line which must cross the graph of x^3 and therefore there must be a (real) solution to the cubic. Cordano found a formula:

$$x = \left(q + \sqrt{q^2 - p^3}\right)^{1/3} + \left(q - \sqrt{q^2 - p^3}\right)^{1/3}.$$

Try finding the solution to $x^3 = 6x + 6$ using this formula ($p = 2$ and $q = 3$).

Here is where complex numbers arise: To solve $x^3 = 15x + 4$, $p = 5$ and $q = 2$, so we obtain:

$$x = (2 + 11i)^{1/3} + (2 - 11i)^{1/3}.$$

Even though this looks like a complex number, it actually is a real number: the second term is the complex conjugate of the first term. Check that $(2+i)^3 = 2+11i$, and thus the solution is $x = 4$.