First-Order Linear Equations and Examples

In class we did some examples of solving first-order linear equations using integrating factors. Here are some more examples and a formula to speed things up.

Example 1: Solve the equation $xy' + y = \frac{1}{x}$.

The left-hand side is the derivative of $xy(x)$ with respect to x (where the two terms come from using the product rule, since y is a function of x). Since $\frac{d}{dx}(xy) = xy' + y$, the integral of the left-hand side is $\int (xy' + y)\, dx = xy$. So

$$\begin{aligned} xy' + y &= \frac{1}{x} \\ \int (xy' + y)\, dx &= \int \frac{1}{x}\, dx \\ xy &= (\ln|x|) + C \\ y &= \frac{(\ln|x|) + C}{x}. \end{aligned}$$

Again, there are lots of solutions to a differential equation, and for this equation we get one solution for any value of C.

Example 2: Solve the equation $xy' + 3y = \frac{1}{x}$.

As it is now, the left-hand side isn't the derivative of anything nice like the last example, because 3 (the coefficient of y) isn't the derivative of x (the coefficient of y'). First off, to make things easier later, we'll divide both sides by x so y' is by itself:

$$y' + \frac{3}{x}y = \frac{1}{x^2}.$$

Then we'll try to multiply both sides by the right function (the integrating factor) so that the left-hand side is the derivative of something nice. If $\mu(x)$ is the integrating factor, multiplying both sides by $\mu(x)$ gives

$$\mu(x)y' + \frac{3}{x}\mu(x)y = \frac{\mu(x)}{x^2}.$$

We want $\frac{3}{x}\mu(x)$ to be the derivative of $\mu(x)$, so μ has to solve the differential equation

$$\mu'(x) = \frac{3}{x}\mu(x).$$

So we have a new differential equation to solve for μ. It's separable, because the right-hand side is a function of x times a function of μ. Let's solve it:

$$\begin{aligned}
\frac{d\mu}{dx} &= \frac{3}{x}\mu(x) \\
\frac{d\mu}{\mu} &= \frac{3}{x}\,dx \\
\int \frac{d\mu}{\mu} &= \int \frac{3}{x}\,dx \\
\ln|\mu| &= 3\ln|x| + C \\
|\mu| &= e^{3\ln|x|+C} \\
|\mu| &= e^C\,|x|^3 \\
|\mu| &= K\,|x|^3 \\
\mu &= Kx^{3\dagger}
\end{aligned}$$

As usual, there are many solutions, and all are possible choices for μ, but we just need one, so let's choose $K = 1$, so $\mu = x^3$.

Returning to the original ODE, we now know we need to multiply both sides by $\mu = x^3$:

$$\begin{aligned}
x^3\left(y' + \frac{3}{x}y\right) &= x^3\left(\frac{1}{x^2}\right) \\
x^3y' + 3x^2y &= x
\end{aligned}$$

By the product rule, the left-hand side is the derivative of x^3y (with respect to x), so integrating both sides with respect to x, we get

$$\begin{aligned}
\int (x^3y' + 3x^2y)\,dx &= \int x\,dx \\
x^3y &= \frac{x^2}{2} + C \\
y &= \frac{1}{2x} + \frac{C}{x^3}.
\end{aligned}$$

Example 3: Solve the equation $xy' + 4xy = x^2e^{-4x}$ and find the solution with $y(1) = 3e^{-4}$.

Step 1: Like before, let's start by dividing both sides by x to get y' by itself:

$$y' + 4y = xe^{-4x}.$$

Step 2: Multiply both sides by the (unknown at this point) integrating factor $\mu(x)$:

$$\mu(x)y' + 4\mu(x)y = \mu(x)xe^{-4x}.$$

Step 3: Find $\mu(x)$.

†We can drop the absolute values here, but the reasons are slightly technical and not too interesting. In general, you can usually drop the absolute values that come from integrating $1/x$ when solving differential equations, just like we did here.

We want $4\mu(x)$ to be the derivative of $\mu(x)$, so we get the differential equation

$$\mu' = 4\mu$$

and solving:

$$\begin{aligned}
\frac{d\mu}{dx} &= 4\mu \\
\frac{d\mu}{\mu} &= 4\,dx \\
\int \frac{d\mu}{\mu} &= \int 4\,dx \\
\ln|\mu| &= 4x + C \\
|\mu| &= e^C e^{4x} \\
|\mu| &= Ke^{4x} \\
\mu &= Ke^{4x}.
\end{aligned}$$

and choosing $K = 1$ again, we get $\mu = e^{4x}$.

Step 4: Returning to the original ODE, integrate and solve for y:

Plugging in $\mu = e^{4x}$, we get

$$\begin{aligned}
e^{4x}y' + 4e^{4x}y &= x \\
\int (e^{4x}y' + 4e^{4x}y)\,dx &= \int x\,dx \\
e^{4x}y &= \frac{x^2}{2} + C \\
y &= \frac{x^2e^{-4x}}{2} + Ce^{-4x}.
\end{aligned}$$

Final step: In this problem, we were asked to pinpoint the solution with $y(0) = 3$. So far, we've found that $y = \frac{x^2e^{-4x}}{2} + Ce^{-4x}$ is a solution for any value of C. So what we want to do now is to find the value of C so that $y(0) = 3$. Plugging $x = 0$ into the equation, we get

$$\begin{aligned}
y(0) &= \frac{1^2e^{-4\cdot 1}}{2} + Ce^{-4\cdot 1} \\
&= \frac{1}{2}e^{-4} + Ce^{-4} \\
&= \left(C + \frac{1}{2}\right)e^{-4}.
\end{aligned}$$

Since $y(0) = 3e^{-4}$, we get $3e^{-4} = \left(C + \frac{1}{2}\right)e^{-4}$, so $3 = C + \frac{1}{2}$, or $C = \frac{5}{2}$. The final solution, then, is

$$y(x) = \frac{x^2e^{-4x}}{2} + \frac{5}{2}e^{-4x}.$$

Note: The condition $y(1) = 3$ is called an *initial condition*, and a differential equation together with initial condition(s) is called an *initial value problem*. Soon we'll discuss initial value problems further in class.