# Quiz 2 solutions

(a) $\boxed{\text{* Set up differential equation}}$

Mixing problem setup: $S(t)$ = amount of soil in lake after $t$ seconds (in grams)

$$\frac{dS}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\begin{aligned}
\text{rate in} &= (\text{concentration in}) \cdot (100\ L/s) \\
&= (20\ g/L) \cdot (100\ L/s) \\
&= 2000\ g/s \\
\text{rate out} &= (\text{concentration of soil in lake})(100\ L/s) \\
&= \left(\frac{S(t)}{1000}\ \frac{g}{L}\right) \cdot (100\ L/s) \\
&= S(t)/10\ g/s.
\end{aligned}$$

$$\rightarrow \boxed{\frac{dS}{dt} = 2000 - \frac{S}{10}}$$

$\boxed{\text{* Solve differential equation:}}$

It's separable; so we can separate variables, and it's linear too so we can also use an integrating factor.

Option 1: Separate variables:

$$\frac{dS}{dt} = 2000 - \frac{S}{10}$$

$$\frac{dS}{2000 - \frac{S}{10}} = dt$$

$$\int \frac{1}{2000 - \frac{S}{10}}\, dS = \int 1\, dt$$

$$-10 \ln\left|2000 - \frac{S}{10}\right| = t + K_1$$

$$\ln\left|2000 - \frac{S}{10}\right| = -\frac{t}{10} + K_2$$

$$2000 - \frac{S}{10} = K_3 e^{-t/10}$$

$$S = 20000 - K_4 e^{-t/10}$$

If $S(0) = 0$ (initial condition) then $K_4 = 20000$

$$S(t) = 20000\left(1 - e^{-t/10}\right)$$

Option 2: Integrating Factors

First, rewrite: $S' = 2000 - \frac{S}{10}$

$$S' + \frac{1}{10}S = 2000$$

$$\boxed{\text{Integrating factor: } \mu(t) = e^{\int \frac{1}{10}dt} = e^{t/10}}$$

$$e^{\frac{t}{10}}S' + \frac{1}{10}e^{\frac{t}{10}}S = 2000e^{\frac{t}{10}}$$

$$\left(e^{\frac{t}{10}}S\right)' = 2000e^{\frac{t}{10}}$$

Integrate: $e^{\frac{t}{10}}S = \int 2000e^{\frac{t}{10}}\,dt$

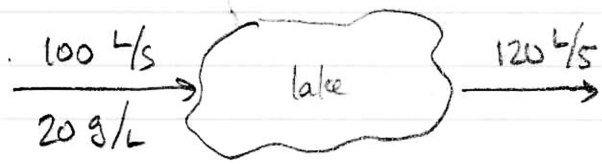
$$e^{\frac{t}{10}}S = 2000 \cdot 10e^{\frac{t}{10}} + K$$

$$S(t) = 20000 + Ke^{-t/10}$$

If $S(0)=0$ (initial condition), then $K = -20000$

$$\rightarrow \boxed{S(t) = 20000\left(1 - e^{-t/10}\right)}$$

(b) $\boxed{\text{What happens when the rate of flow out becomes } 120 \text{ L/s?}}$




Mixing problem setup: $\dfrac{dS}{dt} = (\text{rate in}) - (\text{rate out})$

Rate in isn't affected: $\text{rate in} = \left(20\ \frac{g}{L}\right)\left(100\ \frac{L}{s}\right) = 2000\ \frac{g}{s}.$

Rate out changes: $\text{rate out} = \left(\underset{\text{in lake}}{\text{concentration of soil}}\right)\left(120\ \frac{L}{s}\right)$

$$= \left(\frac{\text{mass of soil in lake}}{\text{volume of lake}}\right)\left(120\ \frac{L}{s}\right)$$

<u>Volume of lake</u>:

Water enters at 100 L/s and leaves at 120 L/s, so the volume of the lake is changing at a rate of $\left(100\frac{L}{s} - 120\frac{L}{s}\right) = -20$ L/s.

Every $t$ seconds, 20 L is lost, so the volume of the lake is $1000 - 20t$.

$$\text{rate out} = \frac{c}{1000-20t} \cdot \left(120\ \frac{L}{s}\right)$$

Final equation: $\dfrac{dS}{dt} = 2000 - \dfrac{120 \cdot c}{1000 - 20t}$