Partial Fractions Worksheet

When solving differential equations using the Laplace transform, we often have to use partial fraction decomposition, which is a way of rewriting rational functions (functions with one polynomial divided by another) as a sum of simpler functions.

The goal of this worksheet is to get some practice doing partial fraction decomposition, using a handy technique known as the "cover-up" method. You may have already seen this in Math 125 or another calculus class. In each of the following problems, find the partial fraction decomposition:

1. $\dfrac{(s+1)(s+2)}{s(s^2+2s-3)}$.

Factor denominator: $\dfrac{(s+1)(s+2)}{s(s-1)(s+3)}$

$$\frac{(s+1)(s+2)}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$$

Find coefficients:

A Plug $s=0$ into $\dfrac{(s+1)(s+2)}{s(s-1)(s+3)}$ $\rightarrow A = \dfrac{(1)(2)}{(-1)(3)} = -\dfrac{2}{3}$

B Plug $s=1$ into $\dfrac{(s+1)(s+2)}{s(s-1)(s+3)}$ $\rightarrow B = \dfrac{(2)(3)}{(1)(4)} = \dfrac{3}{2}$

C Plug $s=-3$ into $\dfrac{(s+1)(s+2)}{s(s-1)(s+3)}$ $\rightarrow C = \dfrac{(-2)(-1)}{(-3)(-4)} = -\dfrac{1}{6}$

answer

$$\frac{-\frac{2}{3}}{s} + \frac{\frac{3}{2}}{s-1} + \frac{-\frac{1}{6}}{s+3}$$

2. $\dfrac{2s-3}{(s^2+1)(s^2-4s+5)}$.

Complete squares: $\dfrac{2s-3}{(s^2+1)((s-2)^2+1)}$

$$\frac{2s-3}{(s^2+1)((s-2)^2+1)} = \frac{As+B}{s^2+1} + \frac{C(s-2)+D}{((s-2)^2+1)}$$

Find coefficients:

A, B $s^2+1 = (s-0)^2+1^2$ so plug in $s = 0+1i$:

$$\frac{2s-3}{(s^2+1)((s-2)^2+1)} = \frac{2i-3}{((i-2)^2+1)} = -\frac{5}{8} + \frac{1}{8}i$$

do some calculations

$A = \dfrac{\frac{1}{8}}{1} = \dfrac{1}{8}$, $B = -\dfrac{5}{8}$

C, D $(s-2)^2+1$ so plug in $s = 2+1i$:

$$\frac{2s-3}{(s^2+1)((s-2)^2+1)} = \frac{2(2+i)-3}{((2+i)^2+1)} = \frac{3}{8} + \frac{1}{8}i$$

do some calculations

$C = \dfrac{\frac{1}{8}}{1} = \dfrac{1}{8}$, $D = \dfrac{3}{8}$

answer

$$\frac{\frac{1}{8}s - \frac{5}{8}}{s^2+1} + \frac{\frac{1}{8}(s-2) + \frac{3}{8}}{(s-2)^2+1}$$

(continued on back)

3. $\dfrac{8}{(s-1)(s-3)^3}$.

$$\frac{8}{(s-1)(s-3)^3} = \frac{A}{s-1} + \frac{B}{(s-3)} + \frac{C}{(s-3)^2} + \frac{D}{(s-3)^3}$$

Find coefficients:

A: plug in $s=1$ into $\dfrac{8}{(s-1)(s-3)^3} \longrightarrow A = \dfrac{8}{(1-3)^3} = -1$

B, C: can't use cover up method

D: plug in $s=3$ into $\dfrac{8}{(s-1)(s-3)} \longrightarrow D = \dfrac{8}{(3-1)} = 4$

So far:

$$\frac{8}{(s-1)(s-3)^3} = \frac{-1}{s-1} + \frac{B}{s-3} + \frac{C}{(s-3)^2} + \frac{4}{(s-3)^3}$$

Have two unknowns (B, C) → plug in two values for s into both sides (any two except s=1,3 work)

$s=2$: $\dfrac{8}{(1)(-1)^3} = \dfrac{-1}{2-1} + \dfrac{B}{2-3} + \dfrac{C}{(2-3)^2} + \dfrac{4}{(2-3)^3}$

$$-8 = -1 - B + C - 4$$

$$\rightarrow \boxed{-3 = -B + C} \leftarrow \quad (1)$$

$s=4$: $\dfrac{8}{(3)(1)^3} = \dfrac{-1}{4-1} + \dfrac{B}{4-3} + \dfrac{C}{(4-3)^2} + \dfrac{4}{(4-3)^3}$

$$\frac{8}{3} = -\frac{1}{3} + B + C + 4$$

$$\rightarrow \boxed{-1 = B + C} \leftarrow \quad (2)$$

Combining these two equations (1) & (2):

$$\begin{array}{rl} & -3 = -B + C \quad (1) \\ + & -1 = B + C \quad (2) \\ \hline & -4 = 2C \end{array} \rightarrow C = -2$$

Plug C into either (1) or (2) → $B = 1$

answer

$$\frac{8}{(s-1)(s-3)^3} = \frac{-1}{s-1} + \frac{1}{s-3} + \frac{-2}{(s-3)^2} + \frac{4}{(s-3)^3}$$