Solving IVPs with Piecewise Functions

This worksheet walks step by step through the process of solving IVPs with piecewise functions with several examples.

Table of Laplace Transforms:

f	$\mathcal{L}[f]$	f	$\mathcal{L}[f]$
1	$\frac{1}{s}$	$\cos bt$	$\frac{s}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$	$\sin bt$	$\frac{b}{s^2+b^2}$
t^n	$\frac{n!}{s^{n+1}}$	$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

Rules for Step Functions:

$$\boxed{\begin{aligned}\mathcal{L}\{u_c(t)f(t)\} &= e^{-cs}\mathcal{L}\{f(t+c)\}\\ \mathcal{L}^{-1}\{e^{-cs}F(s)\} &= u_c(t)\mathcal{L}^{-1}\{F(s)\}(t-c)^*.\end{aligned}}$$

* $\boldsymbol{\mathcal{L}^{-1}\{F(s)\}(t-c)}$ *means take the inverse Laplace transform of $F(s)$, and then plug in $t-c$ wherever you see t.*

Example 1: Solve the IVP $y'' + 2y' - 3y = F(t)$, with

$$F(t) = \begin{cases} -9t, & t < 2 \\ 9, & t \geq 2 \end{cases} \qquad\qquad y(0) = 1,\ y'(0) = 2.$$

Step 1: Rewrite $F(t)$ using step functions.

Step 2: Take the Laplace transform of both sides of the equation. Let $Y = \mathcal{L}\{y\}$.

Step 3: Solve for $Y(s)$.

Step 4: Let $Y_1(s) = \frac{(s+4)s^2-9}{s^2(s+3)(s-1)}$, $Y_2(s) = \frac{27s+9}{s^2(s+3)(s-1)}$. Find the partial fractions decomposition of Y_1 and Y_2.

Step 5: Take the inverse Laplace transform to find the solution $y(t)$.

Step 6: Write $y(t)$ as a piecewise function.

(Answers on back)

Answers:

Step 1 $F(t) = [1 - u_2(t)]\,(-9t) + 9u_2(t)$

Step 2 $(s^2 + 2s - 3)Y(s) - s - 4 = -\dfrac{9}{s^2} + e^{-2s}\left(\dfrac{9}{s^2} + \dfrac{27}{s}\right)$

Step 3 $Y(s) = \dfrac{(s+4)s^2 - 9}{s^2(s+3)(s-1)} + e^{-2s}\dfrac{27s+9}{s^2(s+3)(s-1)}$

Step 4 $Y_1(s) = \dfrac{2}{s} + \dfrac{3}{s^2} + \dfrac{-1}{s-1}$, $Y_2(s) = \dfrac{-11}{s} + \dfrac{-3}{s^2} + \dfrac{2}{s+3} + \dfrac{9}{s-1}$

Step 5 $y(t) = 2 + 3t - e^t + u_2(t)\left[-3t - 5 + 2e^{-3t+6} + 9e^{t-2}\right]$

Step 6 $y(t) = \begin{cases} 3t + 2 - e^t, & t < 2 \\ -3 - e^t + 2e^{-3t+6} + 9e^{t-2}, & t \geq 2 \end{cases}$ or $y(t) = \begin{cases} 3t + 2 - e^t, & t < 2 \\ -3 + (9e^{-2} - 1)e^t + 2e^{-3t+6}, & t \geq 2 \end{cases}$

Example 2: Solve the IVP $y'' - 5y' + 6y = F(t)$, with

$$F(t) = \begin{cases} 0, & t < 1 \\ 3e^{2t}, & 1 \le t < 3 \\ 0, & t > 3, \end{cases} \qquad y(0) = 2,\ y'(0) = 3.$$

Step 1: Rewrite $F(t)$ using step functions.

Step 2: Take the Laplace transform of both sides of the equation. Let $Y = \mathcal{L}\{y\}$.

Step 3: Solve for $Y(s)$.

Step 4: Find the partial fractions decomposition of each fraction appearing in your answer for $Y(s)$. *Hint:* factor out e^2 and e^6.

Step 5: Take the inverse Laplace transform to find the solution $y(t)$.

Step 6: Write $y(t)$ as a piecewise function.

(Answers on back)

Answers

Step 1 $F(t) = [u_1(t) - u_3(t)]3e^{2t}$.

Step 2 $(s^2 - 5s + 6)Y(s) - 2s + 7 = \frac{3e^2}{s-2}e^{-s} - \frac{3e^6}{s-2}e^{-3s}$.

Step 3 $Y(s) = \frac{2s-7}{(s-2)(s-3)} + e^2\frac{3}{(s-2)^2(s-3)}e^{-s} + (-e^6)\frac{3}{(s-2)^2(s-3)}e^{-3s}$.

Step 4 $Y_1(s) = \frac{3}{s-2} + \frac{-1}{s-3}$, $Y_2(s) = \left(\frac{-3}{s-2} + \frac{-3}{(s-2)^2} + \frac{3}{s-3}\right)e^2$, $Y_3(s) = \left(\frac{-3}{s-2} + \frac{-3}{(s-2)^2} + \frac{3}{s-3}\right)(-e^6)$.

Step 5 $y(t) = 3e^{2t} - e^{3t} + u_1(t)\left[-3te^{2t} + 3e^{-1}e^{3t}\right] + u_3(t)\left[-6e^{2t} + 3te^{2t} - 3e^{-3}e^{3t}\right]$.

Step 6 $y(t) = \begin{cases} 3e^{2t} - e^{3t}, & t < 1 \\ 3e^{2t} - 3te^{2t} + (-1 + 3e^{-1})e^{3t}, & 1 \le t < 3 \\ -3e^{2t} + (-1 + 3e^{-1} - 3e^{-3})e^{3t}, & t \ge 3. \end{cases}$