Math 307B Midterm 2 — Answers

1. Find the *trial solutions* only:

(a) $y'' - 4y' + 5y = \sin t - e^{2t} + te^{2t}\cos t - 2e^{2t}\sin t$
The solution of the homogeneous equation is $y(t) = Ae^{2t}\cos 2t + Be^{2t}\sin t$.
Based on the nonhomogeneous term (the right-hand side of the equation), the trial solution would be $Y(t) = A\sin t + B\cos t + Ce^{2t} + Dte^{2t}\cos t + Ete^{2t}\sin t + Fe^{2t}\cos t + Ge^{2t}\sin t$, *but because* F *and* G *overlap with the solution of the homogeneous equation, we multiply the* D, E, F, *and* G *terms by* t:
Answer: $Y(t) = A\sin t + B\cos t + Ce^{2t} + Dt^2e^{2t}\cos t + Et^2e^{2t}\sin t + Fte^{2t}\cos t + Gte^{2t}\sin t$

(b) $y'' + y' = t^5 - 2$
The solution of the homogeneous equation is $y(t) = ce^{-t} + d$.
Based on the nonhomogeneous term (the right-hand side of the equation), the trial solution would be $At^5 + Bt^4 + Ct^3 + Dt^2 + Et + F$, *but since* F *overlaps with the homogeneous equation's solution, we have to multiply everything by* t:
Answer: $Y(t) = At^6 + Bt^5 + Ct^4 + Dt^3 + Et^2 + Ft$.

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2. Solve the initial value problem

$$\begin{aligned} y'' - 4y' + 8y &= 0 \\ y(\pi) &= 1 \\ y'(\pi) &= -2 \end{aligned}$$

Roots of characteristic equation: $r = 2 \pm 2i$.

Homogeneous equation solution: $y = ce^{2t}\cos(2t) + de^{2t}\sin 2t$.

Final answer: $y = \frac{1}{e^{2\pi}}e^{2t}\cos(2t) - \frac{2}{e^{2\pi}}e^{2t}\sin 2t$.

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3. A 2kg mass is attached to a spring, stretching it 2m. There is a damping force of 24 N when the object is traveling at 2 m/s, and an external force of $13\cos t$. The object is 3m below its equilibrium position at $t = 0$, and has an initial velocity of 1m/s upward.

(You can use $g = 10$ m/s^2 for the acceleration due to gravity instead of 9.8 m/s^2.)

(a) Is the system underdamped, critically damped, or overdamped?
Differential equation: $2u'' + 12u' + 10u = 13\cos t$.
Characteristic equation has two real roots ($\gamma^2 - 4mk = 64 > 0$), *so the system is overdamped.*

(b) Find the position of the mass at time t.
Homogeneous equation solution: $y = ce^{-t} + de^{-5t}$.
Particular solution: $y = \frac{1}{2}\cos t + \frac{3}{4}\sin t$.
General solution: $y = \frac{1}{2}\cos t + \frac{3}{4}\sin t + ce^{-t} + de^{-5t}$.
Initial conditions: $y(0) = 3$, $y'(0) = -1$ *(positive direction downwards) or* $y(0) = -3$, $y'(0) = 1$ *(positive direction upwards).*
Answer after plugging in the initial conditions and solving for c, d: $y(t) = \frac{1}{2}\cos t + \frac{3}{4}\sin t + \frac{43}{16}e^{-t} - \frac{3}{16}e^{-5t}$ *(positive direction downwards) or* $y(t) = \frac{1}{2}\cos t + \frac{3}{4}\sin t - \frac{69}{16}e^{-t} + \frac{13}{16}e^{-5t}$ *(positive direction upwards).*

4. An object with unknown mass m kg is attached to a spring with spring constant 4 N/m. The damping coefficient is $\gamma = \frac{1}{2}$ N·s/m, and there is an external force of $F(t) = -\cos 2t$.

(a) find the amplitude of the *steady-state* solution;

(b) find the phase of the driving force and the phase of the steady-state solution.

(Your answers will involve m).

Differential equation: $mu'' + \frac{1}{2}u' + 4u = -\cos 2t$.

Steady-state solution: $A\cos 2t + B\sin 2t$, *where* $A = \frac{4m-4}{(4m-4)^2+1}$, $B = -\frac{1}{(4m-4)^2+1}$.

Amplitude: $R = \sqrt{A^2+B^2} = \frac{1}{\sqrt{(4m-4)^2+1}}$

Phase of steady-state solution: If $m < 1$, *then* $A < 0$, *so phase is* $\delta = \left(\tan^{-1}\frac{B}{A}\right) + \pi = \tan^{-1}\left(\frac{-1}{4m-4}\right) + \pi$. *If* $m > 1$, *then* $A > 0$, *so phase is the same, but without adding* π: $\delta = \tan^{-1}\left(\frac{-1}{4m-4}\right)$.

Phase of driving force: $\delta = \tan^{-1}\frac{0}{-1} + \pi = \pi$ *(add* π *because the angle is in the left half-plane).*

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5. Like the previous problem, you have an object with unknown mass m kg attached to a spring with spring constant 4 N/m. The damping coefficient is $\gamma = \frac{1}{2}$ N·s/m, and there is an external force of $F(t) = -\cos\omega t + 2\sin\omega t$.

You find that the amplitude of the steady-state solution is largest when $\omega = 4$. Find all possible values for m.

The amplitude of the steady-state solution is largest when ω *is at the resonant frequency,* $\omega = \omega_{res} = \sqrt{\frac{k}{m}\left(1 - \frac{\gamma^2}{2km}\right)}$. *Setting* $\omega_{res} = 4$, *plugging in* γ *and* k *from the problem and solving for* m *gives* $m = \frac{2}{3}$.

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6. Given that $y_1(t) = t^3$ is one solution of the ODE $t^2y'' - 7ty' + 15y = 0$, solve the initial value problem

$$\begin{aligned} t^2y'' - 7ty' + 15y &= 0 \\ y(1) &= 1 \\ y'(1) &= 1. \end{aligned}$$

Using the method of reduction of order, substitute $y = v(t)y_1(t) = t^3v(t)$.

Substituting $y = t^3v$ *into the ODE and simplifying, we get this equation for* $v(t)$: $t^5v'' - t^4v' = 0$.

Using separation of variables, we get $\int \frac{v''}{v'}\,dt = \frac{1}{t}\,dt$, *so* $\ln v' = (\ln t) + c$, *and* $v' = Ct$. *Integrating gives* $v = Ct^2 + D$.

General solution: $y(t) = Ct^5 + Dt^3$.

Answer after plugging in the initial conditions: $C = -1$, $D = 2$; *so* $y(t) = 2t^3 - t^5$.