Cover-up method with irreducible quadratic factors

If we are given a rational function $f(s) = \frac{p(s)}{q(s)}$, and $q(s)$ factors into linear factors $(s-a_1), \ldots, (s-a_n)$ and irreducible quadratic factors $(s^2+b_1 s+c_1), \ldots, (s^2+b_m s+c_m)$, we can write $f(s)$ in the form

$$f(s) = \frac{p(s)}{q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n} + \frac{B_1 s + C_1}{s^2+b_1 s + c_1} + \cdots + \frac{B_n s + C_n}{s^2 + b_n s + c_n},$$

where the A's, B's, and C's are constants. (This is assuming all the factors are *distinct*, and the degree of $p(s)$ is smaller than the degree of $q(s)$.)

One way to find the values of the A's, B's, and C's is to clear denominators, multiplying both sides by the denominator $q(s)$, and equate the coefficients of the polynomial on both sides.

Another way which is often faster is the cover-up method. For the linear factors $(s-a_i)$, to find A_i we "cover up" the factor $(s-a_i)$ in the denominator of $f(s)$ and plug $s=a_i$ into what's left: in other words $A_i = (s-a_i)f(s)\big|_{s=a_i}$.

For irreducible quadratic factors, we can still use the cover-up method with a little more work. First, we complete the square on each quadratic factor and modify the numerator in the partial fraction expansion slightly, so the partial fraction expansion of $f(s)$ looks like

$$f(s) = \frac{p(s)}{q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n} + \frac{B_1(s-\alpha_1)+C_1}{(s-\alpha_1)^2+\beta_1^2} + \cdots + \frac{B_n(s-\alpha_n)+C_n}{(s-\alpha_n)^2+\beta_n^2}.$$

where we choose the β's to be nonnegative. Then, to find the constants B_i and C_i, we cover up the corresponding factor $((s-\alpha_i)^2+\beta_i^2)$ in the denominator of $f(s)$ and plug in $s = \alpha + \beta i$. The result will be a new complex number $\gamma + \delta i$, and the values of the constants are $B_i = \delta/\beta$, $C_i = \gamma$.

Notice that if we now wanted to take the inverse Laplace transform of the fraction $\frac{B_i(s-\alpha_i)+C_i}{(s-\alpha_i)^2+\beta_i^2}$, we would get $B_i e^{\alpha_i t}\cos(\beta_i t) + \frac{C_i}{\beta_i} e^{\alpha_i} \sin(\beta_i t)$.

Or, in terms of γ and δ, that would be $\frac{1}{\beta_i} e^{\alpha_i t}\left(\delta\cos(\beta_i t) + \gamma \sin(\beta_i t)\right)$.