Homework 5

CAAM 335 • Matrix Analysis • Spring 2016

Due Date: February 26, 4pm

Submission Instructions: Homework submission will be on OWL-Space, as with Homework 1. You can take a look at the Homework 1 problems page for details on the process.

You are welcome to collaborate with other CAAM 335 students, use a calculator, consult the textbook, and get help from an instructor or TA. For this assignment, you ***may not use any other resources***, including MATLAB, Octave, or another program to do your matrix computations.

Problem 1 We're given the following data:

x_i	y_i
0	1
1	3
2	6
3	12

i. Set up a least-squares problem to find a linear model $y = ax + b$ that best fits the data (in terms of least-squares error). What are a and b? Round to one digit after the decimal place.

ii. Set up a linear least-squares problem to find an exponential model $y = ae^{bx}$ that minimizes the least-squares log error $\sum_{i=0}^{3} \ln(y(x_i) - y_i)^2$. What are a and b?

Problem 2 For each of the following statements, decide if it is always true, always false, or neither (it could be true or false).

i. If P is a projection, $\det(I + P) \neq 0$.

ii. If P is a projection, $\det(I - P) \neq 0$.

iii. If $P \in \mathbb{R}^{n\times n}$ is a projection and $\vec{v} \in \mathbb{R}^n$, then the projection $P\vec{v}$ and residual $(I - P)\vec{v}$ are linearly independent (i.e. $c_1 P\vec{v} + c_2(I - P)\vec{v} = 0$ implies $c_1 = c_2 = 0$.)

iv. If P is a projection, $\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{R}(I - P)$.

v. If P and Q are projections, then so is $P + Q$.

vi. If P and Q are projections, then so is PQ.

vii. Let $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{n\times \ell}$ be matrices whose column spaces are orthogonal. Let $C = \begin{bmatrix} A & B \end{bmatrix}$ *(i.e., the $n \times (k + \ell)$ matrix obtained by combining the columns of A and B)*. Then

$$A(A^TA)^{-1}A^T + B(B^TB)^{-1}B^T = C(C^TC)^{-1}C^T.$$

viii. If P is a nontrivial projection ($P \neq 0$ and $P \neq I$) and A is a permutation matrix (every row and column has exactly one 1 entry and the rest are zeros), then PA is also a projection.

Problem 3 Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

i. Find the projection matrix P onto the column space of A. What is its first row? *(Be careful! A's columns are not linearly independent.)*

ii. What is the rank of $I - P$?

Problem 4 Consider the matrix and vector

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}.$$

One solution to the normal equations $A^T A\vec{x} = A^T\vec{b}$ is

$$\vec{x} = \vec{x}_0 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Because it solves the normal equations, this $\vec{x}$ minimizes $\|A\vec{x} - \vec{b}\|$. However, because $\dim \mathcal{N}(A) > 0$ this $\vec{x}$ is just one of infinitely many possible solutions. Which one should we choose?

One criterion would be to make $\vec{x}$ as small as possible in norm. It turns out that out of all vectors $\vec{x}$ solving the normal equations, there is one with smallest norm. Find this vector.

Hint: use the Fundamental Theorem of Linear Algebra and projections.

Problem 5 (Uniqueness of QR Decomposition) Suppose A is square and invertible, and we have two QR decompositions of it: $A = Q_1R_1$ and $A = Q_2R_2$ (where Q_1, Q_2 are orthogonal matrices and R_1, R_2 are upper triangular.)

Can we say that these decompositions must be the same; i.e., $Q_1 = Q_2$ and $R_1 = R_2$?

(a) Yes, the QR decomposition is unique.

(b) No, Q_1 and Q_2 can be any bases for the column space of A.

(c) No, but Q_2 equals Q_1 multiplied by some diagonal matrix.

(d) No, but they must be the same ($Q_1 = Q_2$ and $R_1 = R_2$) if R_1 and R_2 have the same trace.