Homework 6

CAAM 335 • Matrix Analysis • Spring 2016

Due Date: March 11, 4pm

Submission Instructions: Homework submission will be on OWL-Space, as with Homework 1. You can take a look at the Homework 1 problems page for details on the process.

You are welcome to collaborate with other CAAM 335 students, use a calculator, consult the textbook, and get help from an instructor or TA. For this assignment, you ***may not use any other resources***, including MATLAB, Octave, or another program to do your matrix computations.

Problem 1 Find the QR decomposition of the following matrix A using the Gram-Schmidt procedure:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

In order to pin down a unique answer, make sure that R's diagonal entries are all positive. If you follow the recipe from class or the book, you should find this happens automatically.

Problem 2 Let $Q \in \mathbb{R}^{n\times k}$ be an orthogonal matrix, not necessarily square.

Which of the following is always equal to $I - QQ^T$?

(a) The projection matrix onto the column space of Q.

(b) The projection matrix onto the left nullspace of Q.

(c) The projection matrix onto the column space of QQ^T.

(d) The projection matrix onto the nullspace of Q^TQ.

(e) The zero matrix.

(f) None of the above.

Problem 3 (Solving Equations with QR) QR decomposition can be used to solve linear equations, as an alternative to Gaussian elimination (or LU). Suppose $A = QR$, where

$$Q = \frac{1}{3} \cdot \begin{bmatrix} 0 & 2 & -2 & 1 \\ -2 & 0 & 1 & 2 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

You can check that Q is orthogonal.

Without using Gaussian elimination, solve $A\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{bmatrix} 3 \\ -1 \\ -5 \\ -1 \end{bmatrix}.$$

Hint: write $A = QR$ and move Q to the other side of the equation.

Problem 4 (Least Squares and QR) Consider the 4×3 matrix B with the QR decomposition $B = QT$, where

$$Q = \frac{1}{3} \cdot \begin{bmatrix} 0 & 2 & -2 & 1 \\ -2 & 0 & 1 & 2 \\ 2 & -1 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\vec{b}$ be as in problem 3.

Since B has only 3 columns (in fact B is just the first 3 columns of problem 3's A), its column space isn't all of $\mathbb{R}^4$. So we don't expect to be able to solve $B\vec{x} = \vec{b}$, but we can try to minimize the least-squares difference $\|B\vec{x} - \vec{b}\|$.

Without solving the normal equations or calculating B directly, find the $\vec{x}$ minimizing $\|B\vec{x} - \vec{b}\|$.

Hint 1: If Q is a square orthogonal matrix and $\vec{y}$ is any vector, then $\vec{y}$ and $Q\vec{y}$ have the same norm.

Hint 2: It's easy to find the projection of a vector onto the column space of T.