Homework 7

CAAM 335 • Matrix Analysis • Spring 2016

Due Date: March 25, 4pm

Submission Instructions: Homework submission will be on OWL-Space, as with Homework 1. You can take a look at the Homework 1 problems page for details on the process.

You are welcome to collaborate with other CAAM 335 students, use a calculator, consult the textbook, and get help from an instructor or TA. For this assignment, you ***may not use any other resources***, including MATLAB, Octave, or another program to do your matrix computations.

Problem 1 Find the Laplace transforms of the following functions:

i. $t^2 e^{-4t}$

ii. $\sin(t)\cos(t)$

Your answers should be one of the following:

(a) $\dfrac{1}{s^2+4}$

(b) $\dfrac{s^2}{s+4}$

(c) $\dfrac{1}{(s+4)^3}$

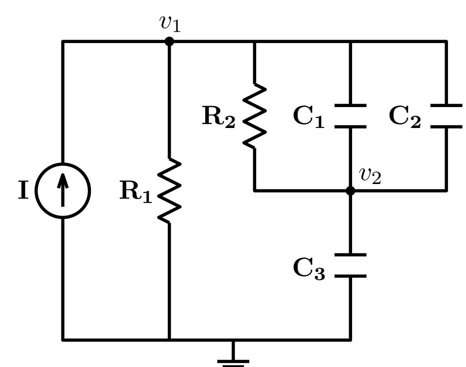
(d) $\dfrac{s^2}{(s+4)^2}$

(e) s^2+4

(f) $\cos(s+4)$

(g) $\dfrac{2s}{(s^2+4)^2}$

Problem 2 Consider the following resistor-capacitor network with a current source:



Use $C_1 = C_2 = 0.5$mF, $C_3 = 1$mF, $R_1 = 200\Omega$, $R_2 = 500\Omega$, and $I = 3$mA.

Write down a differential equation for the voltages $\vec{v} = (v_1, v_2)^T$ in the form

$$\frac{d\vec{v}}{dt} + B\vec{v} = \vec{a}.$$

What are B and $\vec{a}$?

Problem 3 In class, we studied three time-stepping methods (forward Euler, backward Euler, and the trapezoidal rule) for first-order ordinary differential equations:

$$x' = f(x, t).$$

Here is another one, known as *Heun's Method*. [1] The rule for moving from the approximation x_i at time t_i to an approximate solution x_{i+1} at the next time step $t_{i+1} = t_i + \Delta t$ consists of two parts:

$$\begin{aligned} y &= x_i + \Delta t \cdot f(x_i, t_i), \\ x_{i+1} &= x_i + \Delta t \cdot \frac{1}{2}\big[f(x_i, t_i) + f(y, t_{i+1})\big] \end{aligned}$$

The first equation is a normal forward Euler step. In the second equation, the derivatives at the beginning point and the forward Euler point are *averaged* for a better approximation of the correct step, producing the new value x_{i+1}.

i. Apply Heun's method to the vector linear equation $\vec{v}' = B\vec{v} + \vec{a}(t)$, where $\vec{v}(t)$, $\vec{a}(t) \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$.

   Now solve for $\vec{x}_{i+1}$ in terms of $\vec{x}_i$. Your answer should have the form

   $$\begin{aligned} \vec{x}_{i+1} = &\big(c_0 I + c_1(\Delta t \cdot B) + c_2(\Delta t \cdot B)^2\big)\,\vec{x}_i \\ &+ \Delta t\big(c_3\vec{a}(t_i) + c_4\vec{a}(t_{i+1})\big) + (\Delta t)^2\big(c_5 B\vec{a}(t_i) + c_6 B\vec{a}(t_{i+1})\big). \end{aligned}$$

   for constants $c_0, \ldots, c_6$, some of which may be zero. Find these constants.

ii. What is the order of accuracy of Heun's method? One way to find this is by comparing your answer for part (i) to the Taylor expansion of $\vec{x}(t_{i+1})$ centered at $t = t_i$.

Problem 4 Suppose A is the 3×3 matrix having the following eigenvectors and eigenvalues:

$$\begin{aligned} \lambda_1 &= -1 & \vec{x}_1 &= \begin{bmatrix}-2 & 1 & 2\end{bmatrix}^T, \\ \lambda_2 &= 0 & \vec{x}_2 &= \begin{bmatrix}2 & 2 & 1\end{bmatrix}^T, \\ \lambda_3 &= 2 & \vec{x}_3 &= \begin{bmatrix}1 & -2 & 2\end{bmatrix}^T. \end{aligned}$$

i. What is A? As a check on your work, you should find $A(1, 3) = 8/9$ and $A(2, 2) = 7/9$. *Hint: you don't need to use brute force to invert the eigenvalue matrix.*

ii. Find the rank of A.

Problem 5 A *Householder reflector* is a fancy name for a matrix that reflects vectors across a hyperplane. These matrices have the form

$$H = I - 2\frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}},$$

where $\vec{v}$ is a vector orthogonal to the hyperplane.

If $\vec{v} \in \mathbb{R}^4$, list all the eigenvalues of H. If an eigenvalue has k linearly independent eigenvectors, list it k times. The answer doesn't depend on what $\vec{v}$ is!

For instance, if 2 and 4 were the eigenvalues, and both had two linearly independent eigenvectors, your answer would be "2, 2, 4, 4."

[1] Warning: the book's formula for Heun's method has a typo.

Problem 6 True or false? In each of these questions, A is an $n \times n$ matrix.

i. If A has n linearly independent eigenvectors, and all its eigenvalues are either 0 or 1, then A must be a projection.

ii. If $A = VDV^{-1}$, where D is some diagonal matrix, and V is invertible, then D must contain A's eigenvalues and V must contain A's eigenvectors.

iii. As long as A has n linearly independent eigenvectors, its eigenvectors must be unique (up to scaling).

iv. If A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, and $B \in \mathbb{R}^{n \times n}$ is another matrix that also has $\vec{v}_1, \ldots, \vec{v}_n$ as eigenvectors, then $AB = BA$.