Solving equations by completing the square

Here are two harder examples of solving an equation by completing the square.

1. Solve $x^2 - 6x + 4 = 0$ (messy).

Complete the square:

$$x^2 - 6x + \square = (x - \triangle)^2$$

$$x^2 - 6x + \boxed{9} = (x-3)^2$$

3 is half of 6 and $9 = 3^2$.

Solve:

$$x^2 - 6x + 4 = 0$$

$$x^2 - 6x = -4$$

$$+9 \quad +9$$

$$x^2 - 6x + 9 = 5$$

$$(x-3)^2 = 5$$

$$\sqrt{(x-3)^2} = \sqrt{5}$$

$$\swarrow \qquad \searrow$$

$$x - 3 = \sqrt{5} \qquad x - 3 = -\sqrt{5}$$

$$x = \sqrt{5} + 3 \qquad x = -\sqrt{5} + 3$$

Check one solution:

$$(\sqrt{5}+3)^2 - 6(\sqrt{5}+3) + 4 \stackrel{?}{=} 0$$

$$(\sqrt{5}+3)(\sqrt{5}+3)$$

$$5 + 6\sqrt{5} + 9 - 6\sqrt{5} - 18 \quad + 4 \stackrel{?}{=} 0$$

$$5 + 9 - 18 \quad + 4 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

2. Solve $x^2 - 7x = 3$ (really messy).

Complete the square:

$$x^2 - 7x + \square = (x - \triangle)^2$$

$$x^2 - 7x + \boxed{\tfrac{49}{4}} = (x - \tfrac{7}{2})^2$$

$\frac{7}{2}$ *is half of 7 and* $\frac{49}{4} = (\frac{7}{2})^2 = \frac{7}{2} \cdot \frac{7}{2}$.

Solve:

$$x^2 - 7x = 3$$

$$+\tfrac{49}{4} \quad +\tfrac{49}{4}$$

$$x^2 - 7x + \tfrac{49}{4} = \tfrac{12}{4} + \tfrac{49}{4}$$

$$x^2 - 7x + \tfrac{49}{4} = \tfrac{61}{4}$$

$$(x - \tfrac{7}{2})^2 = \tfrac{61}{4}$$

$$\sqrt{(x - \tfrac{7}{2})^2} = \sqrt{\tfrac{61}{4}}$$

$$\left|x - \tfrac{7}{2}\right| = \tfrac{\sqrt{61}}{2}$$

$$\swarrow \qquad \searrow$$

$$x - \tfrac{7}{2} = \tfrac{\sqrt{61}}{2} \qquad x - \tfrac{7}{2} = -\tfrac{\sqrt{61}}{2}$$

$$x = \tfrac{\sqrt{61}}{2} + \tfrac{7}{2} \qquad x = \tfrac{-\sqrt{61}}{2} + \tfrac{7}{2}$$

$$x = \tfrac{\sqrt{61}+7}{2} \qquad x = \tfrac{-\sqrt{61}+7}{2}$$

Checking the solution is a pain. Here are the details (but it's probably more helpful to do it yourself):

Check one solution:

$$(\tfrac{\sqrt{61}+7}{2})^2 - 7(\tfrac{\sqrt{61}+7}{2}) \stackrel{?}{=} 3$$

$$(\tfrac{\sqrt{61}+7}{2})(\tfrac{\sqrt{61}+7}{2}) - \tfrac{7\sqrt{61}+49}{2} \stackrel{?}{=} 3$$

$$\tfrac{(\sqrt{61}+7)(\sqrt{61}+7)}{4} - \tfrac{7\sqrt{61}+49}{2} \stackrel{?}{=} 3$$

$$\tfrac{61+14\sqrt{61}+49}{4} - \tfrac{7\sqrt{61}+49}{2} \stackrel{?}{=} 3 \quad \begin{matrix}\cdot 2 \\ \cdot 2\end{matrix}$$

$$\tfrac{61+14\sqrt{61}+49}{4} - \tfrac{14\sqrt{61}+98}{4} \stackrel{?}{=} 3$$

$$\tfrac{61+\cancel{14\sqrt{61}}+49-\cancel{14\sqrt{61}}-98}{4} \stackrel{?}{=} 3$$

$$\tfrac{12}{4} \stackrel{?}{=} 3$$

$$3 = 3 \checkmark$$