Homework 9

CAAM 335 • Matrix Analysis • Spring 2016

Due Date: April 15, 4pm

You are welcome to collaborate with other CAAM 335 students, use a calculator, consult the textbook, and get help from an instructor or TA. For this assignment, you ***may not use any other resources***, including MATLAB, Octave, or another program to do your matrix computations.

Problem 1 True or false? In the following questions, A and B are $n \times n$ complex matrices.

i. If A, B are Hermitian, AB must also be.

ii. If A is Hermitian and B is unitary, then BAB^{-1} must be Hermitian.

iii. If A, B are unitary, AB must be unitary also.

iv. $\det S = \det(S^*)$.

v. If λ is an eigenvalue of S, then $\bar{\lambda}$ is always an eigenvalue of S^*.

vi. If $u(x, y)$ has continuous partial derivatives, there is always another function $v(x, y)$ such that $f(x + yi) = u(x, y) + iv(x, y)$ is complex differentiable.

vii. If A is Hermitian, then $\|e^{iA}\vec{v}\| = \|\vec{v}\|$ for all $\vec{v} \in \mathbb{C}^n$.

Problem 2 Complex analysis can be used to solve 2D ideal fluid flow problems. An "ideal fluid" is one that is incompressible (can't be squished) and irrotational (an object in the fluid will not rotate).

The *velocity field* of the flow is a vector field that at each point (x, y) tells us the direction and magnitude of the fluid's velocity at that point. This velocity is a vector in $\mathbb{R}^2$ we call (u, v), or more completely $(u(x, y), v(x, y))$, since it depends on the point (x, y).

Now suppose we make a complex function out of the velocity field, $f(x + yi) = u(x, y) + iv(x, y)$. It turns out that the fluid flow is ideal if, and only if, $\bar{f}$ is complex differentiable. Note the complex conjugate on f.

Furthermore, suppose we have a function g that is an antiderivative of $\bar{f}$; that is, $g'(z) = \bar{f}(z)$. Write out g in terms of real and imaginary parts:

$$g(x + yi) = \phi(x, y) + i\psi(x, y).$$

Then ϕ and ψ have special meanings too. ϕ is called the *velocity potential*, and you can check that the velocity field (u, v) is the gradient of ϕ: $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$.

ψ is called the *stream function*. It turns out the fluid is flowing along the contour lines of ψ, that is, the curves where $\psi(x, y)$ is constant. These lines are called *streamlines*.

i. Consider the following stream functions, defining three different flow patterns:

$$\psi_1(x,y) = 1 - 2xy$$
$$\psi_2(x,y) = x - y$$
$$\psi_3(x,y) = y - \frac{y}{x^2+y^2}$$

For each ψ, find a corresponding velocity potential ϕ so that $g = \phi + i\psi$ is complex differentiable, among the following choices. You can use the Cauchy–Riemann equations, or try to find a formula for $g(z)$ directly.

(a) $x + y$
(b) $2x^2 + y$
(c) $x\left(1 + \frac{1}{x^2+y^2}\right)$
(d) $x + \ln(x^2+y^2) - \frac{1}{x^2+y^2}\left(\ln(x^2+y^2)\right)^2$
(e) $3 - x - y$
(f) $xy\ln|x^2-y^2|$
(g) $\frac{x^2-y^2}{2}$
(h) $y^2 - x^2$
(i) $-x - \frac{2xy}{(x^2+y^2)^2}$
(j) none of the above.

ii. What is the velocity field for ψ_1, written as a complex function $f(z)$?

(a) $z - \frac{1}{3}z^3$
(b) $\bar{z} - \frac{1}{3}\bar{z}^3$
(c) $2z + (z^2 - \bar{z}^2)$
(d) $\bar{z} - \frac{z^2-\bar{z}^2}{2}$
(e) $-2\bar{z}$
(f) $2z$
(g) $z^2 - \bar{z}^2$
(h) none of the above.

iii. Each of the three flow patterns can be interpreted as going around different obstacles O_1, O_2, O_3. For this question, a flow *going around* an obstacle O just means that streamlines starting outside O never go inside O, and vice-versa. Which of the following shapes are possible obstacles for each of the three flows? (there is only one correct choice for each flow.)

(a) the hyperbola $x^2 - y^2 = 1$
(b) the vertical line $x = -3$
(c) the unit circle $|z| = 1$
(d) the diagonal line $y = x - 3$
(e) the corner consisting of the positive y-axis and positive x-axis, joined at the origin
(f) the intersecting line segments $y = x$ and $y = -x$
(g) none of the above are possible

Problem 3 Calculate the contour integral

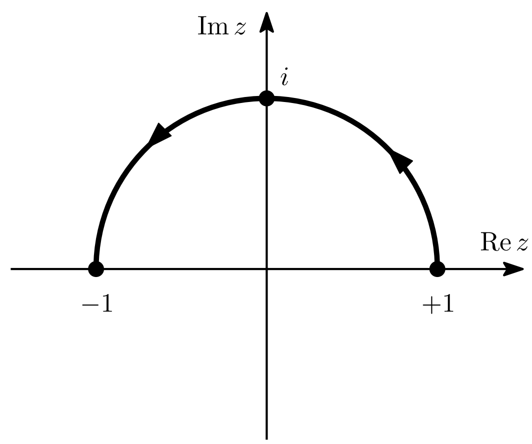
$$\int_{C_1} \frac{1}{z}\,dz$$

over the line segment C_1 from $1 - i$ to $1 + i$.

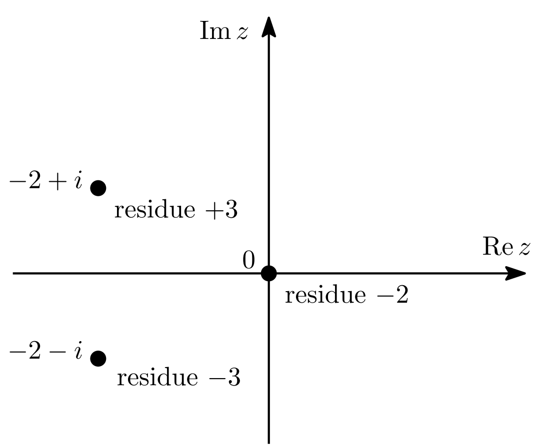
Problem 4 Calculate the contour integral

$$\int_{C_2} \ln z \, dz$$

over the upper half of the unit circle $|z| = 1$, $\operatorname{Im} z \geq 0$, traversed counterclockwise as pictured below:



Problem 5 The Laplace transform of a function $f(t)$ has residues at the following points:



What is $f(t)$?