Integrals with Absolute Values

Overview: The first two sections talk about total distance, and how to find it with integrals. The last section walks through computing integrals with absolute values in them.

Idea: Imagine that one early morning you skate from UW to Green Lake and back. There's two ways you could think about your trip:

- You just traveled 6 miles and got some exercise.
- You ended up right back where you started. so you didn't actually get anywhere.

Or, to be technical, your *(total) distance traveled* was 6 miles, but your *displacement* was zero. You could also refer to your displacement as net change in position.

Using Integration: Suppose you know your velocity, $v(t)$, over a time interval $a \le t \le b$. To find displacement, or change in position, just integrate $v(t)$:

$$\text{displacement} = \int_a^b v(t)\,dt.$$

To compute total distance takes more work. In the skating example, although you traveled 6 miles, your displacement was zero, because at some point you turned around and headed home. The issue is that your direction changed. When you're dealing with total distance, you're not worried about which direction you're going, only your speed— the absolute value of velocity. That's why total distance is the integral of $|v(t)|$:

$$\text{total distance} = \int_a^b |v(t)|\,dt.$$

Computing Integrals with Absolute Values: In total distance calculations, you have to compute integrals that look like

$$\int_{-1}^{3} \left|x^3 - x\right|\,dx.$$

Almost always, you have to deal with absolute values *before* starting to integrate. Most of the time, this involves splitting up the integral. Here's the idea.

First, take a look at the function inside the absolute values; here it's $x^3 - x$. Find its zeros:

$$\begin{aligned} x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \\ x &= -1, 0, 1. \end{aligned}$$

Next, split up the integral at each of the zeros you found:

$$\int_{-1}^{3} \left|x^3 - x\right|\,dx = \int_{-1}^{0} \left|x^3 - x\right|\,dx + \int_{0}^{1} \left|x^3 - x\right|\,dx + \int_{1}^{3} \left|x^3 - x\right|\,dx.$$

If you think about it, $x^3 - x$ can't cross the x-axis between -1 and 0 (or any of the other three intervals)— it has to be either all positive or all negative. This also means you can move the absolute value signs outside the integrals (does it makes sense why?):

$$\int_{-1}^{3} |x^3 - x|\, dx = \left|\int_{-1}^{0} (x^3 - x)\, dx\right| + \left|\int_{0}^{1} (x^3 - x)\, dx\right| + \left|\int_{1}^{3} (x^3 - x)\, dx\right|.$$

Notice that you have to split up the integral *before* moving absolute value signs. Now you can do the three integrals in the normal way, take absolute values, and add them up.

More practice:

- Evaluate $\displaystyle\int_{-10}^{10} |x|\, dx$.

- Evaluate $\displaystyle\int_{1/2}^{3} \left|\frac{1}{x} - \frac{1}{x^2}\right| dx$.

- A ball attached to a spring has velocity $v(t) = \sin(\pi t)$ at time t seconds. What is the total distance it travels between $t = 0$ and $t = 3$?