Applications of 1st Order ODEs

1. Water containing 2 lb/gal of magnesium is entering a storage tank at a rate of 6 gal/sec. The magnesium mixes with the water and drains at a rate of 4 gal/sec. The storage tank originally contains 25 gallons of water, and 75 lb of magnesium.

   Find the quantity of magnesium present in the tank after t seconds. Find the limit of the *concentration* of magnesium as $t \to \infty$.

   Let's say $Q(t)$ is the quantity of magnesium (in lb) in the tank after t seconds.

   Volume of tank: $V(t) = 25\ gal + (6\frac{gal}{s} - 4\frac{gal}{s})t = 25 + 2t$.

   Rate in: $2\frac{lb}{gal} \cdot 6\frac{gal}{s} = 12\frac{lb}{s}$.

   Rate out: $\frac{Q}{V(t)}\frac{lb}{gal} \cdot 4\frac{gal}{s} = \frac{4Q}{25+2t}\frac{lb}{s}$.

   Differential equation: $Q' = 12 - \frac{4Q}{25+2t}$ *(this is linear, but not separable, so use integrating factors to solve)*

   Initial condition: $Q(0) = 75$.

   Integrating factor equation: $\mu' = \frac{4}{25+2t}\mu$.

   Integrating factor: $\mu(t) = (25 + 2t)^2$.

   Quantity of magnesium after t seconds: $Q(t) = 2(25 + 2t) + \frac{C}{(25+2t)^2}$. *Plugging in the initial condition $Q(0) = 75$ and solving for C gives $C = 25^3 = 15625$.*

   Concentration of magnesium after t seconds: $\frac{Q(t)}{V(t)} = 2 + \frac{15625}{(25+2t)^3}$.

   Limit of the concentration as $t \to \infty$: $\lim_{t\to\infty} \frac{Q(t)}{V(t)} = 2\frac{lb}{gal}$. *(note this is equal to the concentration of magnesium entering the tank— can you see why?)*

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2. *(2.3, #22a–b, modified) A ball with mass 0.1 kg is thrown upward with initial velocity 20m/s from a building 30 m high. There is a force due to air resistance of $v^2/4$, where the velocity v is measured in m/s. Use $g = 10\ m/s^2$ for gravity.*

   Find the velocity of the ball, $v(t)$, after t seconds have passed. Find the maximum height above the ground that the ball reaches.

   (Answers) Part 1: Finding the velocity function

   Let's just worry about the case when the ball is traveling upward (this should really have been mentioned in the question; I'm sorry.) We can choose either upwards or downwards to be the positive direction; here let's choose upwards.

   Then the force from air resistance is $-v^2/4$, and the force from gravity is $-mg = -(0.1)(10) = -1$. Using Newton's second law ($F = ma$, or $F = m\frac{dv}{dt}$), our differential equation is $mv' = -1 - \frac{v^2}{4}$. Dividing both sides by $m = 0.1$, we get $v' = -10(1 + \frac{v^2}{4})$.

   This is a separable equation:

$$\frac{dv}{dt} = -10\left(1 + \frac{v^2}{4}\right)$$

$$\frac{1}{1 + \frac{v^2}{4}}\,dv = -10\,dt$$

$$\int \frac{1}{1 + \frac{v^2}{4}}\,dv = \int -10\,dt$$

To solve the integral on the left, you can use the fact that the derivative of $\arctan x$ *is* $\frac{1}{1+x^2}$ *and substitute* $u = v/2$*:*

$$\begin{aligned}\int \frac{1}{1+u^2}\,2\,du &= \int -10\,dt \\ 2\arctan u &= -10t + C_1 \\ u &= \tan\left(-5t + C_2\right) \\ v(t) = 2u &= 2\tan\left(-5t + C_2\right).\end{aligned}$$

(C_1 *and* C_2 *are arbitrary constants). The value for* C_2 *comes from the initial condition that the velocity is 20 at time* $t = 0$*, so* $v(0) = 20$*:*

$$\begin{aligned}20 = v(0) &= 2\tan\left(C_2\right) \\ \arctan(10) &= C_2\end{aligned}$$

So the solution is $\boxed{v(t) = 2\tan\left(-5t + \arctan(10)\right)}$.

Part 2: Finding maximum height

There are two obstacles to overcome here: first, we don't have the equation for the height of the object (just its velocity), and second, we need to know the time the object reaches its maximum height.

Since velocity is zero when the ball reaches maximum height, we can do the second part by solving $v(t) = 0$*:*

$$0 = 2\tan\left(-5t + \arctan(10)\right)$$

The values of x *for which* $\tan x = 0$ *are when* x *is a multiple of* π*:* $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$*. So we need to find when* $-5t + \arctan(10)$ *is a multiple of* π*. We don't know* $\arctan(10)$ *without using a calculator, but it must be positive since* 10 *is positive, and it is less than* $\pi/2$*, since* $-\pi/2 < \arctan y < \pi/2$ *for any* y*. So the first time that* $-5t + \arctan(10)$ *will be a multiple of* π *occurs when* $-5t + \arctan(10) = 0$*, or* $\boxed{t = \arctan(10)/5}$.

Now we need to find how far the ball moved in the first $t = \arctan(10)/5$ *seconds before it reached maximum height. Since velocity is the derivative of position, we can find how far the ball has moved in that time by integrating* $v(t)$ *from* $t = 0$ *to* $t = \arctan(10)/5$*:*

$$\int_0^{\arctan(10)/5} 2\tan\left(-5t + \arctan(10)\right)\,dt$$

Referring to an integral table, the integral of $\tan(x)$ *is* $\ln|\sec x|$ *(on a test, you wouldn't be expected to have this memorized). Making the substitution* $u = -5t + \arctan(10)$*, we find that the distance the ball travels is*

$$-\frac{2}{5}\left(\ln(\sec 0) - \ln(\sec(\arctan(10))\right) = \frac{2}{5}\ln(\sec(\arctan(10)),$$

(which isn't very far, because the air resistance is so large). So the maximum height the ball reaches is 30 feet (its starting height), plus $\frac{2}{5}\ln(\sec(\arctan(10))$*.*

3. *An oil droplet of mass m falls through an electric field. There are three forces acting on it: gravity pulls it downward, the electric field exerts a constant force F downward, and there is a force due to air resistance of μv in the opposite direction of the droplet's velocity.*

   Find the velocity of the droplet at time t, if the initial velocity is v_0.

   Answers:

   Let's say $v(t)$ is the velocity of the droplet at time t.

   Again, we can choose which direction (upwards or downwards) we want to consider as positive. Again, let's say the upward direction is positive. Then the forces acting on the droplet are: the force from the electric field, $-F$, gravity, $-mg$, and the air resistance force, which is $-\mu v$. The negative sign is there on the air resistance force because it acts in the direction opposite to velocity: if v is negative (the droplet is moving downward), the air resistance force should be positive (pushing the particle upward).

   So using Newton's second law, the differential equation for $v(t)$ is $mv' = -mg - F - \mu v$. This equation is both separable and linear, so we can solve it using either method.

   Using separation of variables (this is one way to do it; you might, for instance, have put the negative sign on the left-hand side instead, which is just fine)

$$\begin{aligned} m\frac{dv}{dt} &= -mg - F - \mu v \\ \frac{m}{mg + F + \mu v}\, dv &= -1\, dt \\ \frac{m}{\mu} \ln|mg + F + \mu v| &= -t + C_1 \\ \ln|mg + F + \mu v| &= -\frac{\mu}{m} t + C_2 \\ mg + F + \mu v &= e^{-\mu t/m + C_2} \\ mg + F + \mu v &= C_3 e^{-\mu t/m} \\ v &= \frac{1}{\mu}\left(C_3 e^{-\mu t/m} - F - mg\right) \end{aligned}$$

   (Here C_1, C_2, C_3 are arbitrary constants)

   ***Or**, we can use integrating factors, since the differential equation is linear. Since μ is already taken, let's use $f(t)$ for the integrating factor. First, we divide both sides of the DE by m and move the term with v to the left-hand side: $\frac{dv}{dt} + \frac{\mu}{m} v = -g - \frac{F}{m}$. Then the equation for the integrating factor is $f' = \frac{\mu}{m} f$, and one solution is $f(t) = e^{\mu t/m}$. This we multiply both sides of the original equation by, and then we integrate:*

$$\begin{aligned} e^{\mu t/m}\frac{dv}{dt} + \frac{\mu}{m} e^{\mu t/m} v &= \left(-g - \frac{F}{m}\right) e^{\mu t/m} \\ \int \left(e^{\mu t/m}\frac{dv}{dt} + \frac{\mu}{m} e^{\mu t/m} v\right) dt &= \int \left(-g - \frac{F}{m}\right) e^{\mu t/m}\, dt \\ e^{\mu t/m} v &= \frac{m}{\mu}\left(-g - \frac{F}{m}\right) e^{\mu t/m} + C \\ v &= \frac{m}{\mu}\left(-g - \frac{F}{m}\right) + C e^{-\mu t/m} \\ v &= \frac{1}{\mu}\left(-mg - F + C\mu e^{-\mu t/m}\right) \end{aligned}$$

Notice this is the same as what we got before, if $C_3 = C\mu$ (which is OK, because, since C is an arbitrary constant, so is $C\mu$).

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4. *The population of bacteria on a counter grows at a rate proportional to the current population. At time $t = 0$ minutes, there are 100 bacteria, while at $t = 1$ minute, there are 400 bacteria.*

   (a) *What is the population at $t = 3$?*
   Answers:
   Let's say $P(t)$ is the population at time t.
   Differential equation setup: $\frac{dP}{dt} = kP$, where k is a (unknown to start with) constant. The k is there because we're told the population's rate of change, $\frac{dP}{dt}$ is proportional to the current population P.
   This equation is both linear and separable, but I think it's easier to solve as a separable equation:

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{1}{P}\,dP &= k\,dt \\
\int \frac{1}{P}\,dP &= \int k\,dt \\
\ln|P| &= kt + C_1 \\
P &= e^{kt+C_1} \\
P &= C_2 e^{kt}.
\end{aligned}$$

   From the initial condition $P(0) = 100$, we can conclude by plugging $t = 0$ into the equation we just found for P that $C_2 = 100$, so $P(t) = 100e^{kt}$.
   Then, we need to find k. The information that the population grows to 300 after 1 minute should tell us something about its growth rate k, and we can set $P(1) = 300$ and solve to find k:

$$\begin{aligned}
300 = P(1) &= 100e^{k} \\
3 &= e^{k} \\
k &= \ln 3.
\end{aligned}$$

   Now we've found all the unknown constants in the formula for population: $P(t) = 100e^{(\ln 3)t}$ (We can rewrite using the exponent rules as $P(t) = 100(e^(\ln 3))^t = 100 \cdot 3^t$).
   Plugging in $t = 3$, we get $P(3) = 2700$.

   (b) *Starting at $t = 3$ minutes, the bacteria begin to die at a rate of 50/minute. How many bacteria are there at $t = 5$?*
   Answer:
   The bacteria dying changes the DE for $P(t)$, so we have to start all over with the equation for population, because our old equation didn't take the bacteria dying into account.
   The old equation was $\frac{dP}{dt} = kP$, and we found that k (which had to do with the rate the bacteria grew) was $\ln 3$. Now, the rate at which the population changes is

$$\begin{aligned}
\frac{dP}{dt} &= \textit{(bacteria growth rate)} - \textit{(bacteria death rate)} \\
\frac{dP}{dt} &= (\ln 3)P - 50.
\end{aligned}$$

Now we have to solve this new equation to find P. It's also both linear and separable. If we solve it as a separable equation, we get

$$\begin{aligned}
\frac{dP}{(\ln 3)P - 50} &= dt \\
\frac{1}{\ln 3}\ln((\ln 3)P - 50) &= t + C_1 \\
\ln((\ln 3)P - 50) &= (\ln 3)t + C_2 \\
(\ln 3)P - 50 &= C_3 e^{(\ln 3)t} \\
P &= \frac{50 + C_3 e^{(\ln 3)t}}{\ln 3}.P \qquad\qquad = \frac{50 + C_3 \cdot 3^t}{\ln 3}.
\end{aligned}$$

When the bacteria started dying, at $t = 3$, the population was 2700, so we can plug $t = 3$ into the equation for $P(t)$, set $P(3) = 2700$, and solve for C_3:

$$\begin{aligned}
2700 = P(3) &= \frac{50 + C_3 \cdot 27}{\ln 3} \\
C_3 &= \frac{(2700 \ln 3) - 50}{27}.
\end{aligned}$$

Finally, we plug in $t = 5$ to find the population after 5 minutes:

$$P(5) = \frac{50 + \frac{(2700)\ln 3) - 50}{27} \cdot 3^5}{\ln 3} \approx 23936.$$