Partial Fractions

Partial fraction decomposition is a way of rewriting a *rational function* (a rational function is just one polynomial divided by another) into a sum of smaller, manageable fractions. Here are two examples of partial fraction decompositions:

$$\frac{2s^2-3s+3}{s(s+1)(s-3)} = \frac{-1}{s} + \frac{2}{s+1} + \frac{1}{s-3}$$

$$\frac{16s-24}{(s^2+1)(s^2-4s+5)} = \frac{s-5}{s^2+1} + \frac{(s-2)+3}{(s-2)^2+1}$$

In this class, partial fractions is a crucial part of solving problems with the Laplace transform, as we will soon see.

The Process

Step 1 *(We will never have to do this step in this class unless we get to delta functions.)* If the polynomial in the denominator of the fraction has a smaller degree than the polynomial in the numerator, divide the numerator by the denominator, leave the remainder on top of the fraction, and move the quotient out of the fraction.

Example A: $\dfrac{s^3+2s^2-3s+1}{s^2-1}$. The numerator has degree 3 (because of the s^3), while the denominator has degree 2 (s^2). Dividing s^3+2s^2-3s+1 by s^2-1, the quotient is $s+2$, and the remainder is $-2s+1$:

$$\frac{s^3+2s^2-3s+1}{s^2-1} = s+2+\frac{-2s+1}{s^2-1}.$$

Now the numerator has degree 1, less than the degree of the denominator, so we can move on.

Step 2 *(Usually this step is already done too)* Factor denominator completely and group like terms. After this step, the denominator should have only linear factors and irreducible (unfactorable) quadratic factors:

Example B: $\dfrac{3s+1}{s^5-16s}$. Factoring the denominator, we get something that looks like

$$s^5-16s = s(s^4-16) = s(s^2-4)(s^2+4) = s(s-2)(s+2)(s^2+4)$$

The term s^2+4 can't be factored any further. If you're unsure whether a quadratic factor can be factored into a linear factor, try plugging it in the quadratic formula: if it has real roots, it can be factored; if it complex roots, it is irreducible. In this case, the roots of $s^2+4=0$ are $s=\pm 2i$, so it is irreducible.

Example C: $\dfrac{12s^2+s}{(s^2-4s+4)(s^2-2s+10)}$

$$\frac{12s^2+s}{(s^2-4s+4)(s^2-2s+10)} = \frac{12s^2+s}{(s-2)^2(s^2-2s+10)}.$$

The factor $s^2-2s+10$ is irreducible.

Step 3 Complete the square for any irreducible quadratic factors (write them in the form $(s+a)^2+b^2$).

Example C (continued): In this example, $s^2-2s+10$ is irreducible. To complete the square, we take the coefficient of s, and divide it by 2, getting -1, which is our a. Now we solve for b:

$$s^2-2s+10 = (s-1)^2+b^2 \quad \Rightarrow \quad s^2-2s+10 = s^2-2s+1+b^2 \quad \Rightarrow \quad b=3.$$

Step 4 Write out the partial fraction template. Each factor in the denominator gets its own fraction. In the numerators, we will put unknown coefficients (A, B, $\dots$) like the method of undetermined coefficients. In the next step, we will solve for those coefficients. Here is a quick example:

$$\frac{s}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}.$$

Now let's see how to write out each fraction.

Factor		Fraction
$s+a$	$\to$	$\dfrac{A}{s+a}$
$(s+a)^2+b^2$	$\to$	$\dfrac{A(s+a)+bB}{(s+a)^2+b^2}$

Example D: $\dfrac{4s^2-1}{(s-3)(s+2)((s-3)^2+2^2)} = \dfrac{A}{s-3} + \dfrac{B}{s+2} + \dfrac{C(s-3)+2D}{(s-3)^2+2^2}.$

Example E: $\dfrac{1}{s(s^2+3^2)} = \dfrac{A}{s} + \dfrac{Bs+3C}{s^2+3^2}.$

If any factors appear more than once in the denominator (like $(s-2)^3$ or s^4 for instance), they get special treatment: if the factor appears with power n, its fraction is repeated n times, with the exponents from 1 to n:

Factor		Fraction
$(s+a)^n$	$\to$	$\dfrac{A}{s+a} + \dfrac{B}{(s+a)^2} + \dfrac{C}{(s+a)^3} + \cdots + \dfrac{Z}{(s+a)^n}$
$\left[(s+a)^2+b^2\right]^n$	$\to$	$\dfrac{A(s+a)+bB}{(s+a)^2+b^2} + \dfrac{C(s+a)+bD}{[(s+a)^2+b^2]^2} + \cdots + \dfrac{Y(s+a)+bZ}{[(s+a)^2+b^2]^n}$

Example F: $\dfrac{3s}{s(s-3)^3} = \dfrac{A}{s} + \dfrac{B}{s-3} + \dfrac{C}{(s-3)^2} + \dfrac{D}{(s-3)^3}.$

Example G: $\dfrac{3s}{(s-1)\left[(s-2)^2+2^2\right]^2} = \dfrac{A}{s-1} + \dfrac{B(s-2)+2C}{(s-2)^2+2^2} + \dfrac{D(s-2)+2E}{[(s-2)^2+2^2]^2}.$

Step 5 Find the unknown coefficients. One way to do this is to clear denominators and solve for the coefficients, but if there are no repeated factors, we can take a shortcut. Let's look at an example that illustrates the procedure:

Example H:

$$\frac{3s+5}{(s-1)(s+1)\left((s-1)^2+2^2\right)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C(s-1)+2D}{(s-1)^2+2^2}.$$

To find A, cover up $(s-1)$ in the denominator on the left hand side, then plug $s=1$ into what remains. The result is A:

$$\text{Cover: } \frac{3s+5}{\cancel{(s-1)}(s+1)\left((s-1)^2+2^2\right)}$$

$$\text{Plug in: } \frac{3(1)+5}{(1+1)\left((1-1)^2+2^2\right)} = 1$$

So $A=1$. To find B, we would cover up $(s+1)$ in the denominator, and plug in $s=-1$.

Here's the general rule for linear factors (those that look like $s + a$):

Linear factor rule: To find the coefficient for a factor $s + a$, cover up $s + a$ in the denominator and plug in $s = -a$ into what remains.

For irreducible quadratic factors, it's a bit more complicated. Here's the rule:

Irreducible quadratic factor rule: To find the coefficients for a factor $(s + a)^2 + b^2$, cover it up in the denominator and plug in $s = -a + bi$ into what remains. Suppose you get the complex number $x + yi$ when you do this. Then the fraction for this factor is

$$\frac{\frac{y}{b}(s + a) + b\frac{x}{b}}{(s + a)^2 + b^2}$$

In other words, the unknown coefficients are $\frac{y}{b}$ and $\frac{x}{b}$. We'll see later why we write it in this funny way.

Practice

1. $\dfrac{(s+1)(s+2)}{s(s^2+2s-3)}$.

2. $\dfrac{2s-3}{(s^2+1)(s^2-4s+5)}$.