Implicit Solutions to ODEs

. . . and intervals where they are defined

When solving a separable equation (like problem 21, §2.2), we almost always get an implicit formula for the solution, like $y^3 + 3y^2 = x^4 - x^2 + k$, instead of an explicit formula for $y(x)$, like $y(x) = \sqrt{x+1}$. This brings up some delicate points when we describe where a solution is valid.

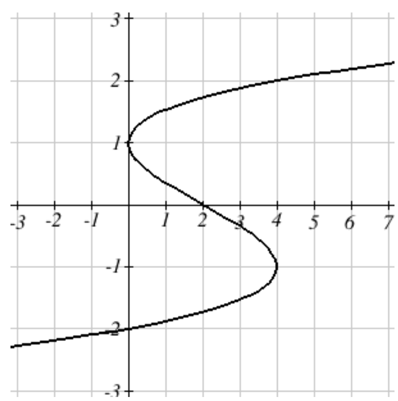
In this handout, we'll look at an example that's similar to our homework problem. Suppose we are faced with this initial value problem:

$$y' = \frac{1}{3y^2 - 3}$$
$$y(\tfrac{5}{8}) = \tfrac{1}{2}$$

and are asked the same question as in the homework problem:

Determine the interval on which the solution is defined
(or the interval where the solution is valid, which is the same thing)

This is a separable equation (do you see why?), and its general solution, you can check, is $y^3 - 3y = x + c$. Plugging in the initial condition $x = \frac{5}{8}$, $y = \frac{1}{2}$, we get $c = -2$. We can't easily solve for y in terms of x, so we'll have to leave the equation in implicit form. To get a handle on what's going on, let's look at a graph of the equation $y^3 - 3y = x - 2$:



Remember from the first day of class that a solution to this initial value problem is supposed to be a function $y(x)$. Notice that this not the graph of a function, because there are multiple values of y for the same x value. What's going on is that $y^3 - 3y = x - 2$ really defines *three* functions of x: the part from $y = -1$ to $y = 1$, the part with $y \geq 1$, and the part with $y \leq -1$. Our solution is going to be one of these three functions, and it has to contain the initial condition, $x = \frac{5}{8}$, $y = \frac{1}{2}$. So, it must be the part from $y = -1$ to $y = 1$.

Now, what's the domain of this function? From the graph, it appears that the domain is $x = 0$ to $x = 4$. We can verify this by plugging in $y = -1$ and $y = 1$ into the equation $y^3 - 3y = x - 2$ and solving for x. *Note:* for the homework problem in the book, solving for x is trickier, and you'probably have to guess and check, or use a calculator or computer.