# Solutions – Practice Midterm 1

1. Solve the initial value problem
$$\frac{1}{t}y' + y - 3 = 0, \qquad y(0) = 2$$

Equation is linear → so we can use integrating factors (NOTE: it's also separable)

$$\frac{1}{t}y' + y - 3 = 0$$
$$y' + ty = 3t$$

Integrating factor:
$$\mu(t) = e^{(\int t\,dt)} = e^{t^2/2}$$

$$e^{t^2/2}y' + te^{t^2/2}y = 3te^{t^2/2}$$
$$(e^{t^2/2}y)' = 3te^{t^2/2}$$
$$e^{t^2/2}y = \int 3te^{t^2/2}\,dt \quad \left(\text{substitute: } u = t^2/2,\ du = t\,dt\right)$$
$$e^{t^2/2}y = \int 3e^u\,du$$
$$e^{t^2/2}y = 3e^u + C$$
$$e^{t^2/2}y = 3e^{t^2/2} + C$$

$$\rightarrow y(t) = 3 + Ce^{-t^2/2}$$

Solve for C:
$$2 = y(0) = 3 + Ce^0$$
$$\Rightarrow C = -1$$
$$\boxed{y(t) = 3 - e^{-t^2/2}}$$

2. Determine explicitly all the solutions to the differential equation
$$(1+t^2)y' + y = 1.$$

Like #1, this equation is both linear and separable. Let's solve it by separating variables.
(if you use integrating factors, the integrating factor is $e^{\tan^{-1}t}$)

$$(1+t^2)y' + y = 1$$
$$(1+t^2)\frac{dy}{dt} = 1 - y$$
$$\frac{dy}{1-y} = \frac{dt}{1+t^2}$$
$$\int \frac{1}{1-y}\,dy = \int \frac{1}{1+t^2}\,dt$$
$$-\ln|1-y| = \tan^{-1}t + C$$
$$\ln|1-y| = -\tan^{-1}t + C$$
$$|1-y| = e^{-\tan^{-1}t + C}$$
$$1 - y = ce^{-\tan^{-1}t}$$
$$\boxed{y = 1 - ce^{-\tan^{-1}t}}$$

3. Let $P(t)$ be the population of fish in Green Lake at time $t$. Suppose that fish are harvested at a constant rate $E$ from the total population, so that the population is given by the differential equation:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P - EP$$

where $K$ and $r$ are positive constants, and $E \geq 0$ is a nonnegative constant.

(a) Assume that $E < r$. Determine all the equilibrium solutions to this equation and classify them as stable, unstable, or semistable.

(b) Sketch the direction field without solving the differential equation.

(c) How does your answer to part (a) change if instead we assume $E > r$?

(d) Solve the differential equation if $K = 1$, $r = 4$, and $E = 2$, with the initial condition $P(0) = 100$.

(a) Solve $\frac{dP}{dt} = 0$:

factor: $\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P - EP$

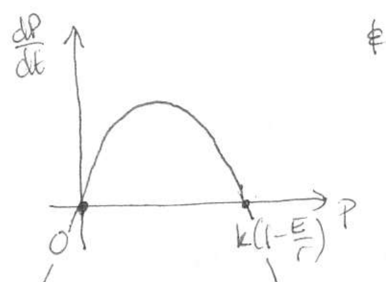
$= P\left(r - \frac{rP}{K} - E\right)$

So $\frac{dP}{dt} = 0$ when $\begin{cases} P = 0 \\ \text{or} \\ r - \frac{rP}{K} - E = 0 \rightarrow P = \frac{K}{r}(r-E) = \left(1-\frac{E}{r}\right)K > 0 \end{cases}$

equilibrium solutions: $\boxed{P = 0,\ P = K(1-E/r)}$

Classify equilibrium sol'ns:

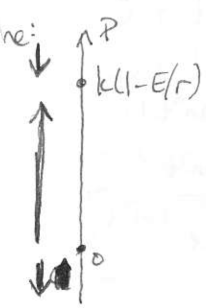
Graph $\frac{dP}{dt}$ versus $P$: $\frac{dP}{dt}$ is quadratic function of $P$ so its graph is a parabola & faces downward because coefficient of $P^2$ is negative.




From the graph:

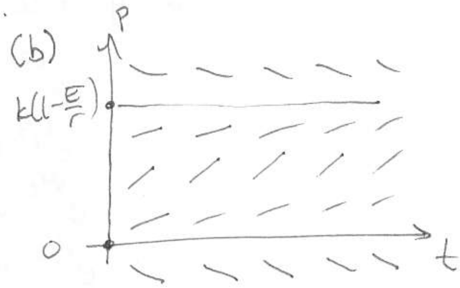
| | dP/dt |
|---|---|
| $P<0$ | $-$ |
| $0<P<K(1-\frac{E}{r})$ | $+$ |
| $P>K(1-\frac{E}{r})$ | $-$ |

Phase line:



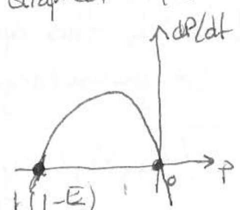

So $\boxed{\begin{array}{l} P=0 \text{ unstable} \\ P=K(1-\frac{E}{r}) \text{ stable} \end{array}}$

(b)

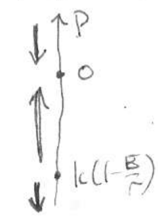



(c) $K(1-\frac{E}{r})$ is now negative if $E>r$.

Graph of $dP/dt$ vs. $P$:




Phase line:




$\boxed{\begin{array}{l} P=0 \text{ stable} \\ P=K(1-\frac{E}{r}) \text{ unstable} \end{array}}$ ← also $K(1-\frac{E}{r})$ is negative so not a very good population.

(d) $\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P - EP$ $\quad r=4,\ E=2,\ K=1$

$\frac{dP}{dt} = -4P^2 + 2P$

3(d) $\frac{dP}{dt} = 4\left(1-\frac{P}{1}\right)P - 2P$

$P' = -4P^2+2P$

This equation is separable; it's also a Bernoulli equation, so you can use Bernoulli substitution:

Using Separation of Variables

$$\frac{P'}{-4P^2+2P} = 1$$

$$\int \frac{dP}{-4P^2+2P} = \int dt$$

~~Integrand on left is a rational~~

Integrand on left is a rational function, so use partial fractions:

$$\frac{1}{-4P^2+2P} = \frac{1}{2P(1-2P)}$$ step 1: factor denominator

$$\frac{1}{-4P^2+2P} = \frac{A}{2P} + \frac{B}{(1-2P)}$$ step 2: split up

$$1 = A(1-2P) + B(2P)$$ step 3: clear denominators

$$1 + 0P = A + (2B-2A)P$$ step 4: collect like terms

$$\begin{cases} 1 = A \\ 0 = 2B - 2A \end{cases}$$ step 5: equate coefficients & solve

$$\rightarrow A = 1,\ B = 1$$

$$\frac{1}{-4P^2+2P} = \frac{1}{2P} + \frac{1}{1-2P}$$

Continue w/ integration

$$\int\left(\frac{1}{2P} + \frac{1}{1-2P}\right) dP = \int dt$$

$$\frac{1}{2}\ln|P| - \frac{1}{2}\ln|1-2P| = t + C_1$$

$$\ln|P| - \ln(1-2P) = 2t + C_2$$

$$\ln\left|\frac{2P}{1-2P}\right| = 2t + C_2$$

$$\frac{2P}{1-2P} = C_3 e^{2t}$$

$$\frac{1-2P}{2P} = \frac{1}{C_3 e^{2t}}$$

$$\frac{1}{2P} - 1 = \frac{1}{C_3 e^{2t}}$$

$$\frac{1}{2P} = 1 + \frac{1}{C_3}e^{-2t} \rightarrow P = \frac{1}{2 + \frac{2}{C_3}e^{-2t}} = \boxed{\frac{1}{2+C_4e^{-2t}}}$$

Plug in Initial condition:

$$P(0) = \frac{1}{2+C_4} = 100$$

$$\rightarrow C_4 = -199/200$$

Using Bernoulli

$$v = P^{1-2} = P^{-1}$$

$$\rightarrow P = v^{-1}$$

substitute $P = v^{-1}$ into DE:

$$(v^{-1})' = -4v^{-2} + 2v^{-1}$$

$$-v^{-2}v' = -4v^{-2} + 2v^{-1}$$

$$v' = +4 - 2v$$

This is linear and separable. Solving it either way

$$\rightarrow v(t) = 2 + ce^{-2t}$$

$$\rightarrow P(t) = \frac{1}{v(t)} = \frac{1}{2+ce^{-2t}}$$

4. Initially, a tank contains 6 gal of water containing 1 lb of salt. There is water flowing into the tank through two pipes: Water containing salt is entering the tank through the first pipe at rate of 2 gal/min. Several measurements indicate that the amount of salt contained in one gallon of the incoming water is $e^{-3t/2}$ lb at time $t$. One gallon of fresh water per minute is entering the tank through the second pipe. Finally, the well-stirred mixture is draining the tank at a rate of 3 gal/min.

Determine the amount of salt at any time $t \geq 0$.

$S(t)$ = amount of salt in tank (lb)

$t$ = time (min)

$\boxed{\text{Set up}}$ $\frac{dS}{dt} = (\text{rate in}) - (\text{rate out})$

Rate in:

First pipe:

$\left(2 \frac{\text{gal}}{\text{min}}\right) \cdot \left(e^{-3t/2} \frac{\text{lb}}{\text{gal}}\right) = 2e^{-3t/2} \frac{\text{lb}}{\text{min}}$

Second pipe brings in no salt (so no contribution to rate in)

Rate out:

$\left(3 \frac{\text{gal}}{\text{min}}\right) \cdot \left(\frac{S(t)\ \text{lb}}{6\ \text{gal}}\right) = \frac{S}{2}$

concentration of salt leaving tank.

volume of tank is constant (rate in = rate out)

$$\boxed{\frac{dS}{dt} = 2e^{-3t/2} - \frac{1}{2}S}$$

$\boxed{\text{Solve}}$ Equation is linear, not separable → use integrating factors

$S' = 2e^{-3t/2} - \frac{1}{2}S$

$S' + \frac{1}{2}S = 2e^{-3t/2}$

Integrating factor is $\mu(t) = e^{\frac{1}{2}t}$

$e^{\frac{1}{2}t}S' + \frac{1}{2}e^{\frac{1}{2}t}S = 2e^{-t}$

$(e^{t/2}S)' = 2e^{-t}$

$e^{t/2}S = \int 2e^{-t}$

$e^{t/2}S = -2e^{-t} + C$

$S = -2e^{-3t/2} + Ce^{-t/2}$

→ Apply initial condition

$S(0) = 1$ lb

$1 = S(0) = -2 + C$

→ $C = 3$

→ $\boxed{S(t) = -2e^{-3t/2} + 3e^{-t/2}}$

5. A mass is attached to a spring. Its velocity $v$ is given by the initial value problem

$$\frac{dv}{dt} = -2(x-3), \quad v = 4 \text{ when } x = 4$$

where $x$ is the position of the mass. Eliminate $t$ from the differential equation so it only involves $v$ and $x$, and solve it.

What is the domain of $v(x)$?

$\boxed{\text{Eliminate } t:}$ $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$ (chain rule) $= v\frac{dv}{dx}$ since $v = \frac{dx}{dt}$.

Substitute $\frac{dv}{dt} = v\frac{dv}{dx}$: $\boxed{v\frac{dv}{dx} = -2(x-3)}$

$\boxed{\text{Solve}}$ Equation is separable:

$$\int v\,dv = \int -2(x-3)\,dx$$

$$\frac{v^2}{2} = -(x-3)^2 + C$$

$$v^2 = C - 2(x-3)^2$$

$$v(x) = \sqrt{C - 2(x-3)^2}$$

Plug in initial condition:

$$4 = v(4) = \sqrt{C - 2(4-3)^2}$$

$$4 = \sqrt{C-2}$$

$$C = 18$$

$$\boxed{\begin{aligned} v(x) &= \sqrt{18 - 2(x-3)^2} \\ \text{or} \quad v(x) &= \sqrt{-2x^2 + 12x} \end{aligned}}$$ (expanding out $(x-3)^2$)

$\boxed{\text{Domain}}$: $-2x^2+12x$ needs to be positive

$$-2x^2 + 12x = x(-2x+12)$$

So either $x>0$, $-2x+12>0$ or $x<0$, $-2x+12<0$

$\to x>0$, $x<6$ $\qquad$ $\to$ (impossible)

$\boxed{\text{Domain: } 0<x<6}$