Integrating $e^x \cos x$ and Related Integrals

Today in class we ran up against $\int e^x \cos x \, dx$, and one of these integrals also appeared on the list of practice integrals for the first quiz. Solving these integrals requires a bit of a trick: we integrate by parts twice and then solve for the unknown integral. The main thing to remember here is to keep track of the original integral, and watch carefully which part of the integral is u and which is v. Also, watch those $+$ and $-$ signs. First, a reminder of integration by parts:

$$\textit{Integration by parts:} \quad \int u \, dv = uv - \int v \, du$$

Now, for our integral:

$$\begin{aligned}
&\int e^x \cos x \, dx \qquad \boxed{\begin{array}{ll} u = e^x & du = e^x \, dx \\ v = \sin x & dv = \cos x \, dx \end{array}} \\
&= e^x \sin x - \int e^x \sin x \, dx \\
&\qquad \boxed{\begin{array}{ll} u = e^x & du = e^x \, dx \\ v = -\cos x & dv = \sin x \, dx \end{array}} \\
&= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx \right] \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.
\end{aligned}$$

Now the integral we're looking for, $\int e^x \cos x \, dx$, is on both sides of the equation. Let's put both integrals on one side of the equation and solve:

$$\begin{aligned}
2 \int e^x \cos x \, dx &= e^x \sin x + e^x \cos x \\
\int e^x \cos x \, dx &= \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x.
\end{aligned}$$

That's it! A similar process works for integrals like $\int e^x \sin x \, dx$ or $\int e^{20x} \cos 300x$ (or whatever numbers you can imagine).