Chapter 4 Practice Problems

CAAM 335 • Matrix Analysis • Spring 2016

Here are some practice problems from the Chapter 4 material: column space, nullspace, and LU factorization.

Problem 1 Let A be the 4×6 matrix

$$A = \begin{bmatrix} 2 & 1 & 2 & 2 & 1 & 1 \\ -2 & 0 & -3 & -3 & 0 & 0 \\ 0 & -1 & -1 & 1 & 3 & 5 \\ 2 & 1 & 0 & 2 & 5 & 7 \end{bmatrix}.$$

i. Find an LU factorization of A.

Answer:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & -2 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

ii. What is A's rank?

Answer: 3.

iii. Find a basis for the column space of A.

Columns 1–3 of A are the pivot columns; they form one basis for its column space.

iv. Find a basis for the null space of A.

One basis is: $$\begin{bmatrix} 3/2 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ -2 \\ -3 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Problem 2 Let A be the following 3×3 matrix, with an unknown third column $\vec{a}_3$:

$$A = \begin{bmatrix} 1 & 0 & \\ 1 & 1 & \vec{a}_3 \\ 0 & 1 & \end{bmatrix}$$

i. What is the rank of A if the three columns ($\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, and $\vec{a}_3$) are linearly *independent*?

Answer: 3. (The columns of A are linearly independent, so they form a basis for the column space.)

ii. What is the rank of A if they are linearly *dependent*?

Answer: 2 (Because the first two columns are linearly independent, the column space has dimension at least 2. Since all three columns together are linearly dependent, the dimension is less than 3.)

iii. Find a vector $\vec{a}_3$ that makes $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$, and $\vec{a}_3$ linearly *independent*.

Infinitely many possible answers here: $\vec{a}_3$ can be anything not in the span of the first two columns. Some choices are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.