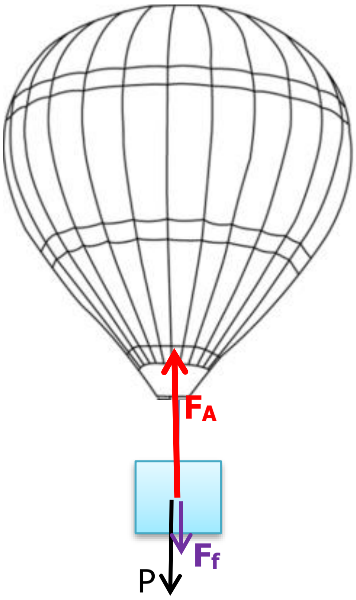
Weather balloon speed.

Equation.



List of forces:

F_A Archimedes thrust.
P : balloon and payload weight.
F_f : friction force in the air.

$\mathbf{F_A = \rho_{air}\, V_{balloon}\, g}$

$\mathbf{P = m_{total}\, g}$

$\mathbf{F_f = ½\, C_x\, \rho_{air}\, S\, v^2}$ = drag force.

$\mathbf{C_x}$ is the air penetration (or drag) coefficient
S the effective area of the object (largest section perpendicular to the movement).
v is the vertical speed of the balloon

Newton's equation: sum of the forces (with respective sign) **= m a.**

$$F_A - P - F_A = m_{\ total}\ a = m_{\ total}\frac{dv(t)}{dt}$$

$$\rho_{\text{air}}\ \mathrm{V}_{balloon}\ \mathrm{g} - \ \mathrm{m}_{\text{total}}\ \mathrm{g} - \frac{1}{2}C_x\rho_{\text{air}}S\ v^2(t) = m_{\ total}\frac{dv(t)}{dt}$$

That could be written :

$$\frac{dv(t)}{dt} + Av^2(t) - B = 0$$

With :

$$A = \frac{1}{2\mathrm{m}_{\text{total}}}C_x\rho_{\text{air}}S \qquad B = g\left(\frac{\rho_{\text{air}}\ \mathrm{V}_{balloon}}{\mathrm{m}_{\text{total}}} - 1\right)$$

Limit values.

Maximum boardable weight at take-off (acceleration must be positive):

$$\frac{\rho_{air}\ V_{balloon}\ g}{m_{total}} - \ g = B = \frac{dv(t)}{dt} \geq 0 \ \rightarrow \ m_{total} \leq \rho_{air}\ V_{balloon}$$

B>0 is needed to allow the take-off of the balloon.

Limit speed: speed for which the acceleration is zero.

$$\rho_{air}\ V_{ballon}\ g - \ m_{total}\ g - \frac{1}{2} C_x \rho_{air} S\ v^2(t) = 0$$

$$\rightarrow v_{lim} = \sqrt{\frac{2g}{C_x \rho_{air} S}\left(\rho_{air}\ V_{balloon}\ - \ m_{total}\right)} = \sqrt{\frac{B}{A}}$$

If we take the numerical value for a typical balloon (1.5 m in diameter)

ρ_{air} = 1.22 kg/m^3, $V_{balloon}$=1.77m^3, m_{total} = 1.4 kg, S= 1.77 m^2, C_x=0.45 g=9.81 m/s^2

we obtain: A = 0.347 m^{-1} et B=5.321 m/s^2

$$v_{lim} = \sqrt{13.6/0.53} = 3.9\ m/s$$

Maximum mass to allow take-off: m_{total} = 2.16 kg

Analytical solution.

The speed equation is:

$$\frac{dv(t)}{dt} + Av(t)^2 - B = 0 \qquad with\ A > 0\ and\ B > 0$$

It is a first order nonlinear differential equation. We can solve it by separation of variables :

$$\frac{dv}{B - Av^2} = dt$$

Integrating both sides gives :

$$\int \frac{dv}{B - Av^2} = \int dt$$

Using Sympy to integrate and doing some clean-up gives:

$$\frac{\ln\left(v(t) + \sqrt{\frac{B}{A}}\right) - \ln\left(v(t) - \sqrt{\frac{B}{A}}\right)}{m} = t + k \qquad with\ m = 2\sqrt{AB}$$

Taking exponential of both sides leads to:

$$e^{\ln\left(v(t)+\sqrt{\frac{B}{A}}\right)-\ln\left(v(t)-\sqrt{\frac{B}{A}}\right)} = e^{m(t+k)}$$

$$\frac{v(t) + \sqrt{\frac{B}{A}}}{v(t) - \sqrt{\frac{B}{A}}} = Ke^{mt}$$

If speed=0 at t=0 then K=-1

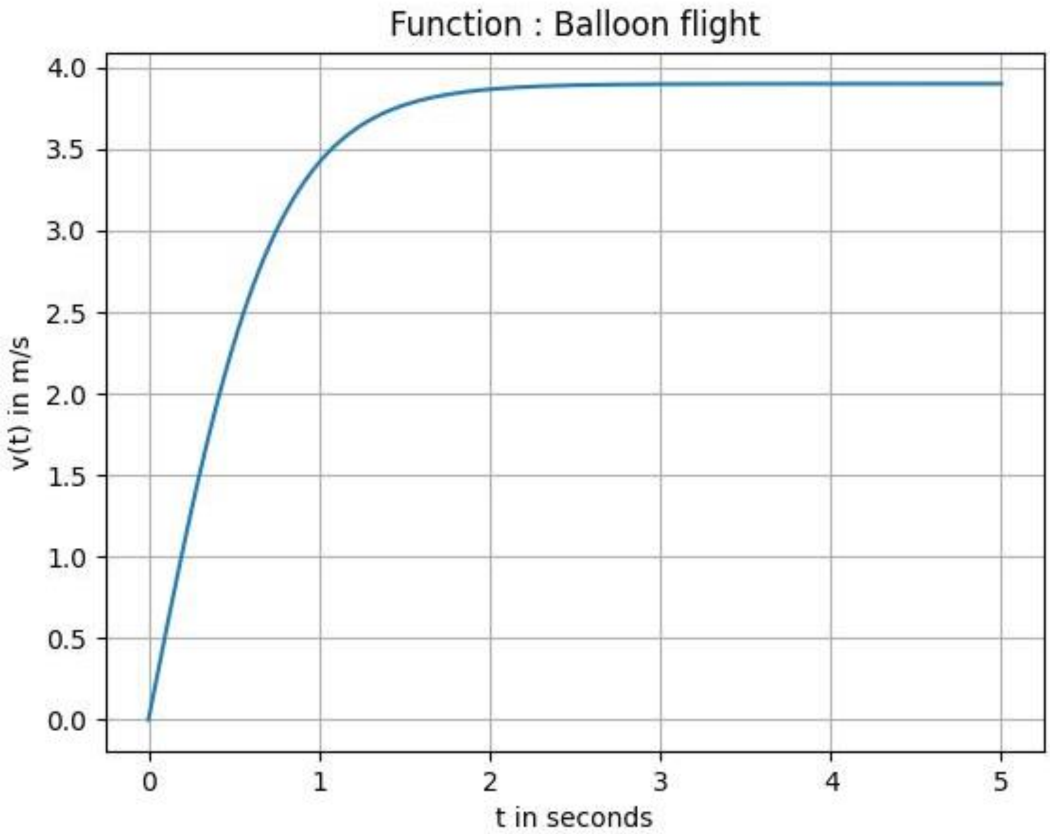
$$v(t) + \sqrt{\frac{B}{A}} = -\left(v(t) - \sqrt{\frac{B}{A}}\right)e^{mt}$$

That is :

$$v(t) = \sqrt{\frac{B}{A}}\left(\frac{e^{mt} - 1}{e^{mt} + 1}\right)$$

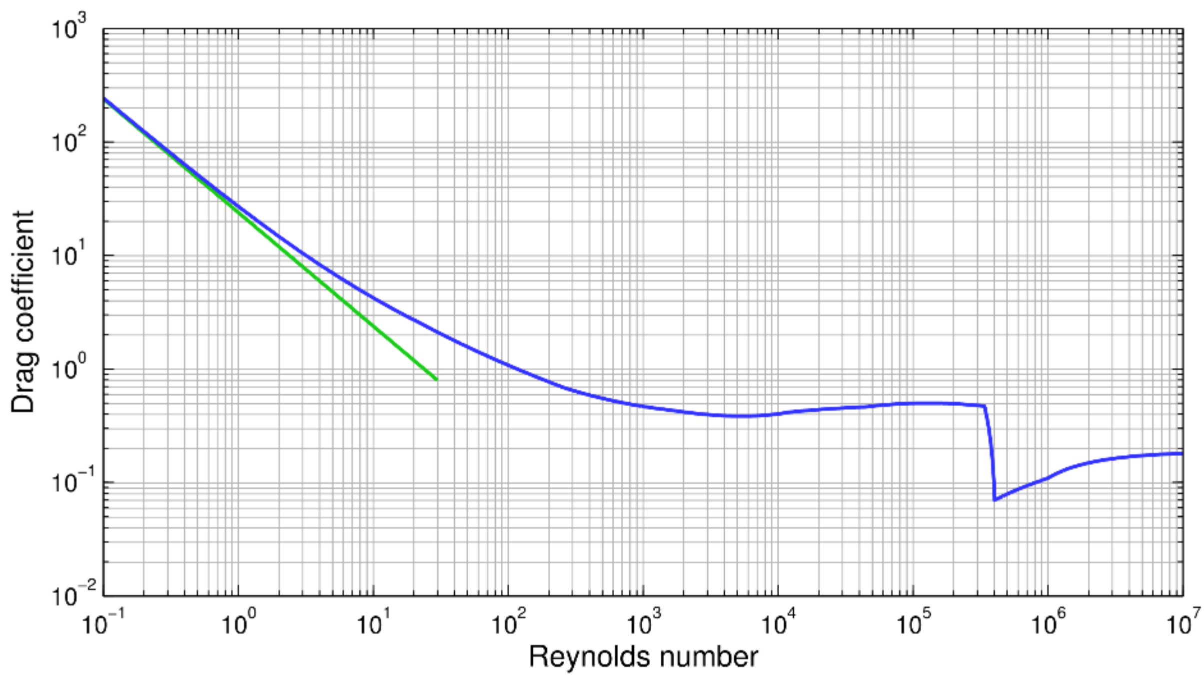
With the previous numerical values, the equation is: $v(t) = 3.9\left(\frac{e^{2.7176t}-1}{e^{2.7176t}+1}\right)$

Plotting the result.



Cx variability.

Cx depends on several parameters, among others the Reynolds number which is a function of the speed, the viscosity and a characteristic dimension. We used a approximative value of 0.45, corresponding more or less on the flight conditions.



(Diagram from Wikipedia.org)