Gibbs Sampling for a mixture of normals

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Pre-requisites

You should know about Gibbs sampling and mixture models, and be familiar with Bayesian inference for the normal mean and for the two class problem.

Overview

We consider using Gibbs sampling to perform inference for a normal mixture model,

$$X_1, \dots, X_n \sim f(\cdot)$$

where

$$f(\cdot) = \sum_{k=1}^{K} \pi_k N(\cdot; \mu_k, 1).$$

Here $\pi_1, \dots, \pi_K$ are non-negative and sum to 1, and $N(\cdot; \mu, \sigma^2)$ denotes the density of the $N(\mu, \sigma^2)$ distribution.

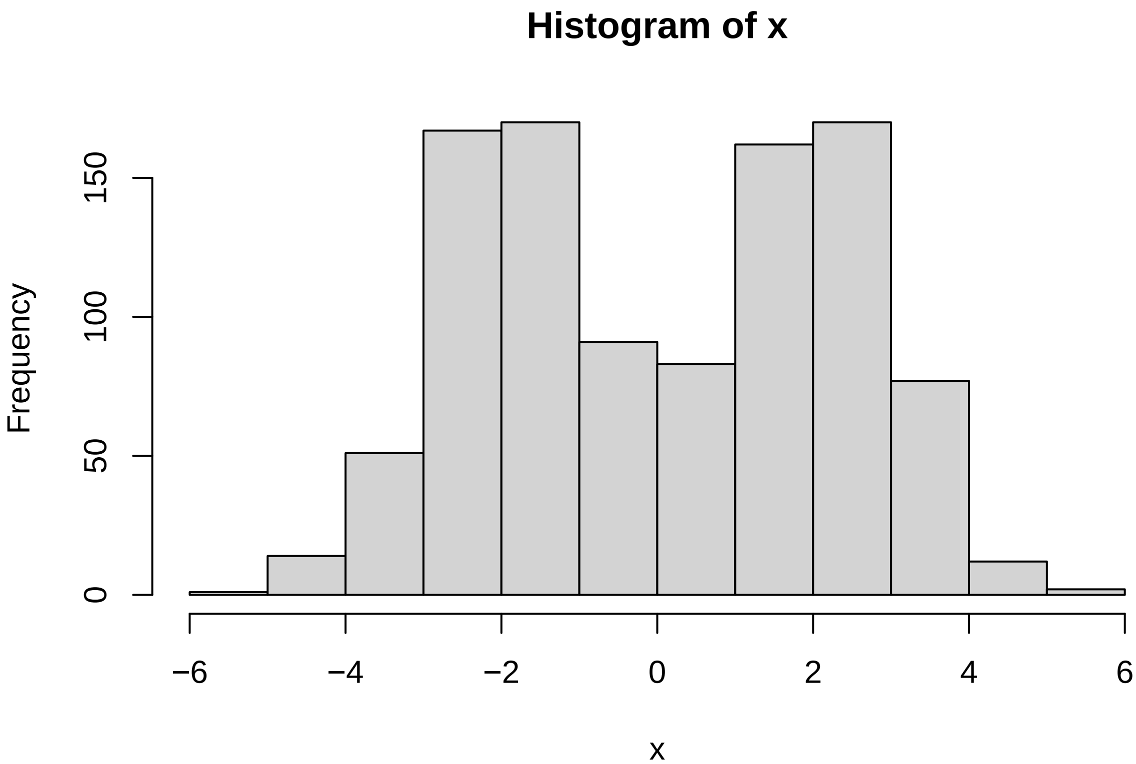
Recall the latent variable representation of this model:

$$\Pr(Z_j = k) = \pi_k$$

$$X_j | Z_j = k \sim N(\mu_k, 1)$$

To illustrate, let's simulate data from this model:

```r
set.seed(33)

# generate from mixture of normals
#' @param n number of samples
#' @param pi mixture proportions
#' @param mu mixture means
#' @param s mixture standard deviations
rmix = function(n,pi,mu,s){
  z = sample(1:length(pi),prob=pi,size=n,replace=TRUE)
  x = rnorm(n,mu[z],s[z])
  return(x)
}
x = rmix(n=1000,pi=c(0.5,0.5),mu=c(-2,2),s=c(1,1))
hist(x)
```



Gibbs sampler

Suppose we want to inference for the parameters μ, π. That is, we want to sample from $p(\mu, \pi|x)$. We can use a Gibbs sampler. However, to do this we have to augment the space to sample from $p(z, \mu, \pi|x)$, not only $p(\mu, \pi|x)$.

Here is the algorithm in outline:

- sample μ from $\mu|x, z, \pi$
- sample π from $\pi|x, z, \mu$
- sample z from $z|x, \pi, \mu$

The point here is that all of these conditionals are easy to sample from.

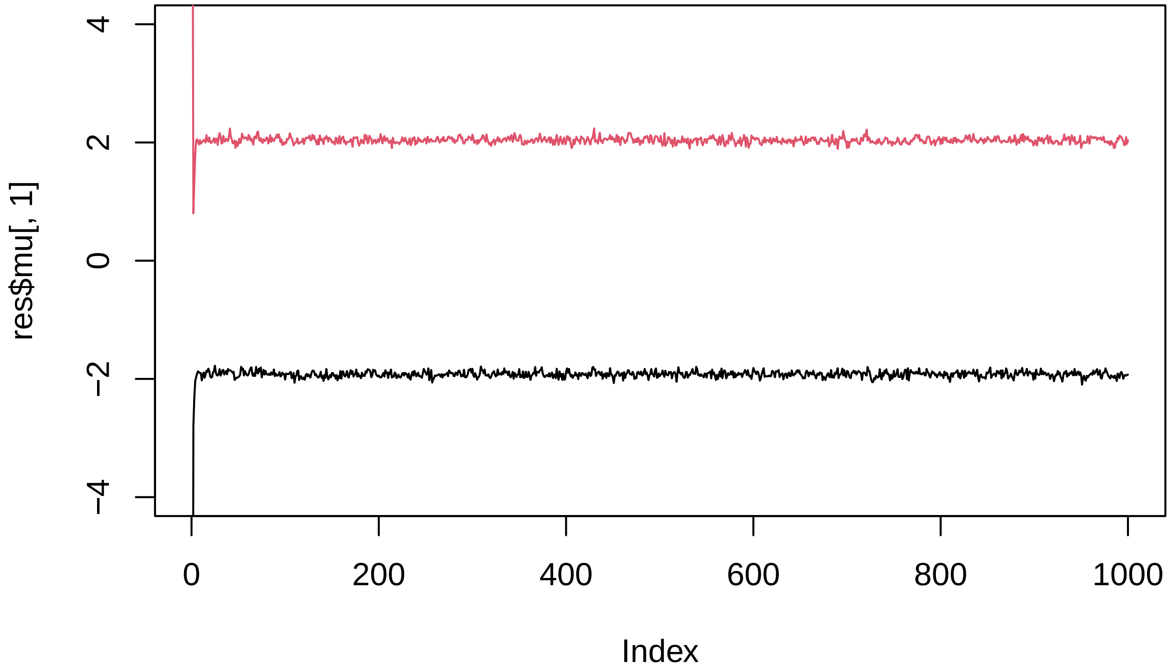
Code

```
normalize = function(x){return(x/sum(x))}

#' @param x an n vector of data
#' @param pi a k vector
#' @param mu a k vector
sample_z = function(x,pi,mu){
  dmat = outer(mu,x,"-") # k by n matrix, d_kj =(mu_k - x_j)
  p.z.given.x = as.vector(pi) * dnorm(dmat,0,1)
  p.z.given.x = apply(p.z.given.x,2,normalize) # normalize columns
  z = rep(0, length(x))
  for(i in 1:length(z)){
    z[i] = sample(1:length(pi), size=1,prob=p.z.given.x[,i],replace=TRUE)
  }
```

```r
  return(z)
}


#' @param z an n vector of cluster allocations (1...k)
#' @param k the number of clusters
sample_pi = function(z,k){
  counts = colSums(outer(z,1:k,FUN="=="))
  pi = gtools::rdirichlet(1,counts+1)
  return(pi)
}

#' @param x an n vector of data
#' @param z an n vector of cluster allocations
#' @param k the number o clusters
#' @param prior.mean the prior mean for mu
#' @param prior.prec the prior precision for mu
sample_mu = function(x, z, k, prior){
  df = data.frame(x=x,z=z)
  mu = rep(0,k)
  for(i in 1:k){
    sample.size = sum(z==i)
    sample.mean = ifelse(sample.size==0,0,mean(x[z==i]))

    post.prec = sample.size+prior$prec
    post.mean = (prior$mean * prior$prec + sample.mean * sample.size)/post.prec
    mu[i] = rnorm(1,post.mean,sqrt(1/post.prec))
  }
  return(mu)
}

gibbs = function(x,k,niter =1000,muprior = list(mean=0,prec=0.1)){
  pi = rep(1/k,k) # initialize
  mu = rnorm(k,0,10)
  z = sample_z(x,pi,mu)
  res = list(mu=matrix(nrow=niter, ncol=k), pi = matrix(nrow=niter,ncol=k), z = matrix(nrow=niter, nc
  res$mu[1,]=mu
  res$pi[1,]=pi
  res$z[1,]=z
  for(i in 2:niter){
      pi = sample_pi(z,k)
      mu = sample_mu(x,z,k,muprior)
      z = sample_z(x,pi,mu)
      res$mu[i,] = mu
      res$pi[i,] = pi
      res$z[i,] = z
  }
  return(res)
}
```

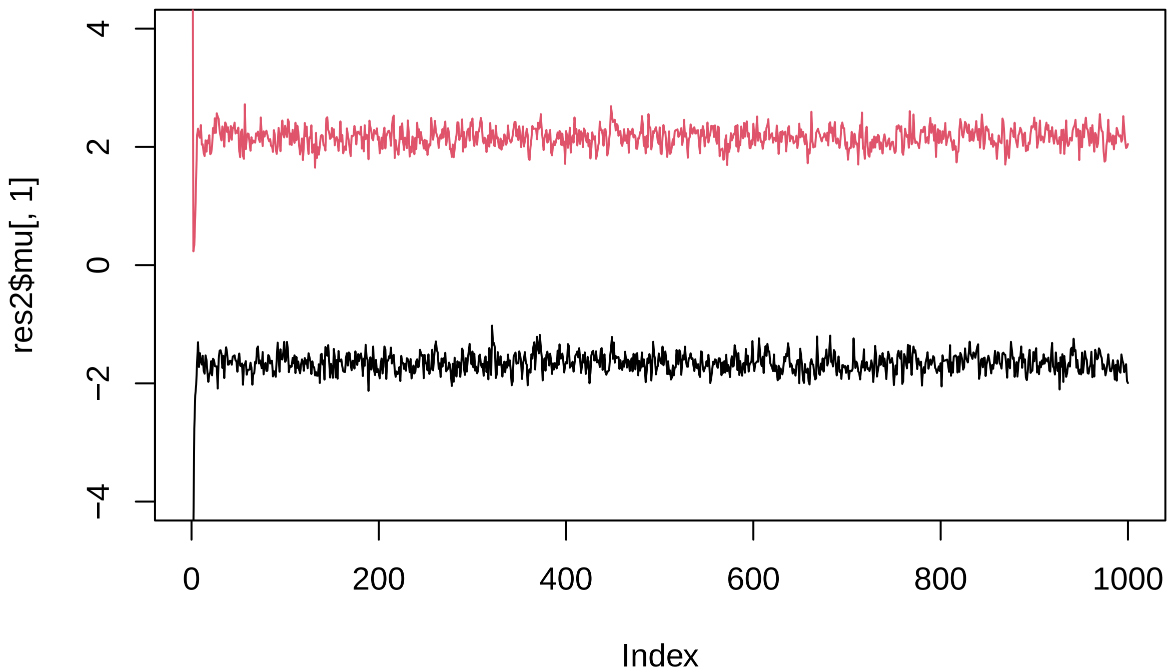
Try the Gibbs sampler on the data simulated above. We see it quickly moves to a part of the space where the mean parameters are near their true values (-2,2).

```r
res = gibbs(x,2)
plot(res$mu[,1],ylim=c(-4,4),type="l")
lines(res$mu[,2],col=2)
```



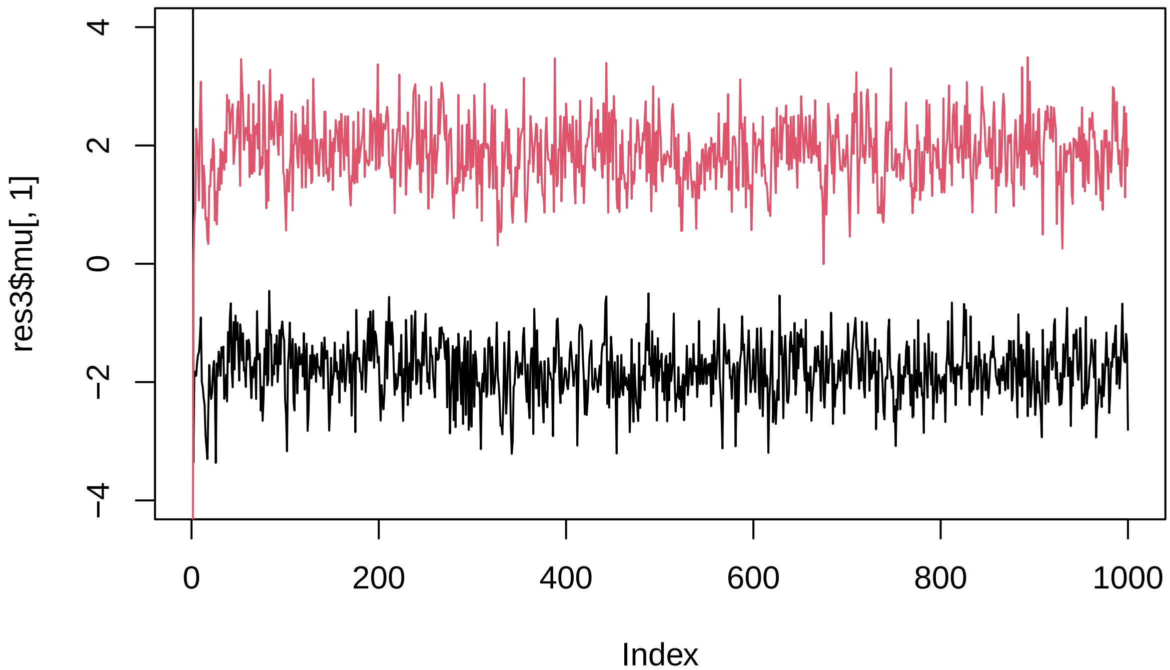
If we simulate data with fewer observations we should see more uncertainty

```r
x = rmix(100,c(0.5,0.5),c(-2,2),c(1,1))
res2 = gibbs(x,2)
plot(res2$mu[,1],ylim=c(-4,4),type="l")
lines(res2$mu[,2],col=2)
```



And fewer observations still…

```r
x = rmix(10,c(0.5,0.5),c(-2,2),c(1,1))
res3 = gibbs(x,2)
plot(res3$mu[,1],ylim=c(-4,4),type="l")
lines(res3$mu[,2],col=2)
```



And we can get credible intervals (CI) from these samples (discard the first few samples as "burn-in").

For example, to get 90% posterior CIs for the mean parameters:

```r
quantile(res3$mu[-(1:10),1],c(0.05,0.95))
```

```
##     5%    95%
## -2.645 -1.004
```

```r
quantile(res3$mu[-(1:10),2],c(0.05,0.95))
```

```
##    5%   95%
## 0.940 2.777
```