

Conjugate Bayesian analysis

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See [here](#) for a PDF version of this vignette.

Overview

This vignette introduces the idea of “conjugate prior” distributions for Bayesian inference for a continuous parameter. You should be familiar with [Bayesian inference for a binomial proportion](#).

Conjugate Priors for binomial proportion

Background

In [this example](#) we considered the following problem.

Suppose we sample 100 elephants from a population, and measure their DNA at a location (“locus”) in their genome, where there are two types (“alleles”). We label these alleles as “0” and “1”.

In my sample, I observe that 30 of the elephants have the 1 allele and 70 have the 0 allele. What can I say about the frequency, q , of the 1 allele in the population?

The example showed how to compute the posterior distribution for q using a *uniform* prior distribution. We saw that, conveniently, the posterior distribution for q was a Beta distribution.

Here we generalize this calculation to the case where the prior distribution on q is a Beta distribution. We will find that, in this case, the posterior distribution on q is again a Beta distribution. The property where the posterior distribution comes from the same family as the prior distribution is very convenient, and so has a special name: it is called “conjugacy”. We say, “the Beta distribution is the conjugate prior distribution for the binomial proportion.”

Details

As before, we use Bayes Theorem, which we can write in words as

$$\text{posterior} \propto \text{likelihood} \times \text{prior},$$

or in mathematical notation as

$$p(q | D) \propto p(D | q) p(q),$$

where D denotes the observed data. In this case, the likelihood $p(D | q)$ is given by

$$p(D | q) = q^{30}(1 - q)^{70}.$$

If our prior distribution on q is a Beta distribution, say, $\text{Beta}(a, b)$, then the prior density $p(q)$ is

$$p(q) \propto q^{a-1}(1-q)^{b-1} \quad (q \in [0, 1]).$$

Combining these two, we get

$$\begin{aligned} p(q | D) &\propto q^{30}(1-q)^{70}q^{a-1}(1-q)^{b-1} \\ &\propto q^{30+a-1}(1-q)^{70+b-1}. \end{aligned}$$

At this point, we again apply the trick of recognizing this density as the density of a Beta distribution — specifically, the Beta distribution with parameters $30 + a$, $70 + b$, written as $\text{Beta}(30 + a, 70 + b)$.

Generalization

There is nothing special about the 30 “1” alleles and 70 “0” alleles we observed. Suppose we observed n_1 of the 1 allele and n_0 of the 0 allele. Then the likelihood becomes

$$p(D | q) = q^{n_1}(1-q)^{n_0},$$

and you should be able to show — *this is an exercise* — that the posterior is

$$q | D \sim \text{Beta}(n_1 + a, n_0 + b).$$

Summary

When doing Bayesian inference for a binomial proportion, q , if the prior distribution is a Beta distribution, then the posterior distribution is also Beta. We say, “the Beta distribution is the conjugate prior for a binomial proportion.”

Exercise

Show that the Gamma distribution is the conjugate prior for a Poisson mean.

That is, suppose we have observations X that are Poisson distributed, $X \sim \text{Poisson}(\mu)$. Assume that your prior distribution on μ is a Gamma distribution with scale a and shape b . Show that the posterior distribution on μ is also a Gamma distribution.

Hint: You should take the following steps. 1. Write down the likelihood $p(X | \mu)$. (Look up the density function of the Poisson distribution if you cannot remember it.) 2. Write down the prior density for μ . (Look up the density of the Gamma distribution if you cannot remember it.) 3. Multiply them together to obtain the posterior density (up to a constant of proportionality), and notice that it has the same form as a Gamma distribution.