

# Conjugate Bayesian analysis

**Matthew Stephens**

University of Chicago

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See [here](#) for a PDF version of this vignette.

## Overview

This vignette introduces the idea of “conjugate prior” distributions for Bayesian inference for a continuous parameter. You should be familiar with [Bayesian inference for a binomial proportion](#).

## Conjugate Priors for binomial proportion

### Background

In [this example](#) we considered the following problem.

Suppose we sample 100 elephants from a population, and measure their DNA at a location (“locus”) in their genome, where there are two types (“alleles”). We label these alleles as “0” and “1”.

In my sample, I observe that 30 of the elephants have the 1 allele and 70 have the 0 allele. What can I say about the frequency,  $q$ , of the 1 allele in the population?

The example showed how to compute the posterior distribution for  $q$  using a *uniform* prior distribution. We saw that, conveniently, the posterior distribution for  $q$  was a Beta distribution.

Here we generalize this calculation to the case where the prior distribution on  $q$  is a Beta distribution. We will find that, in this case, the posterior distribution on  $q$  is again a Beta distribution. The property where the posterior distribution comes from the same family as the prior distribution is very convenient, and so has a special name: it is called “conjugacy”. We say, “the Beta distribution is the conjugate prior distribution for the binomial proportion.”

### Details

As before, we use Bayes Theorem, which we can write in words as

$$\text{posterior} \propto \text{likelihood} \times \text{prior},$$

or in mathematical notation as

$$p(q \mid D) \propto p(D \mid q) p(q),$$

where  $D$  denotes the observed data. In this case, the likelihood  $p(D \mid q)$  is given by

$$p(D \mid q) = q^{30}(1 - q)^{70}.$$

If our prior distribution on  $q$  is a Beta distribution, say,  $\text{Beta}(a, b)$ , then the prior density  $p(q)$  is

$$p(q) \propto q^{a-1}(1-q)^{b-1} \quad (q \in [0, 1]).$$

Combining these two, we get

$$\begin{aligned} p(q \mid D) &\propto q^{30}(1-q)^{70} q^{a-1}(1-q)^{b-1} \\ &\propto q^{30+a-1}(1-q)^{70+b-1}. \end{aligned}$$

At this point, we again apply the trick of recognizing this density as the density of a Beta distribution — specifically, the Beta distribution with parameters  $30 + a$ ,  $70 + b$ , written as  $\text{Beta}(30 + a, 70 + b)$ .

### Generalization

There is nothing special about the 30 “1” alleles and 70 “0” alleles we observed. Suppose we observed  $n_1$  of the 1 allele and  $n_0$  of the 0 allele. Then the likelihood becomes

$$p(D \mid q) = q^{n_1}(1-q)^{n_0},$$

and you should be able to show — *this is an exercise* — that the posterior is

$$q \mid D \sim \text{Beta}(n_1 + a, n_0 + b).$$

### Summary

When doing Bayesian inference for a binomial proportion,  $q$ , if the prior distribution is a Beta distribution, then the posterior distribution is also Beta. We say, “the Beta distribution is the conjugate prior for a binomial proportion.”

### Exercise

Show that the Gamma distribution is the conjugate prior for a Poisson mean.

That is, suppose we have observations  $X$  that are Poisson distributed,  $X \sim \text{Poisson}(\mu)$ . Assume that your prior distribution on  $\mu$  is a Gamma distribution with scale  $a$  and shape  $b$ . Show that the posterior distribution on  $\mu$  is also a Gamma distribution.

*Hint:* You should take the following steps. 1. Write down the likelihood  $p(X \mid \mu)$ . (Look up the density function of the Poisson distribution if you cannot remember it.) 2. Write down the prior density for  $\mu$ . (Look up the density of the Gamma distribution if you cannot remember it.) 3. Multiply them together to obtain the posterior density (up to a constant of proportionality), and notice that it has the same form as a Gamma distribution.