On the problem of measure of sets of points on a line*

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The problem of the measure of sets of points on a line r is to determine for each set A of points of r a positive real number $\mu(A)$, to be said ***measure*** of A, such that:

1°) Two sets which coincide by means of a rigid motion have the same measure.

2°) The union of a family of a finite or countable number of sets, pairwise with no common points, has the sum of measures as measure.

3°) The measure of the set of all points in the interval $(0, 1)$ is 1. (*)

Let x be a point in r. Points in r which differ from x by any rational number, be it positive, negative or zero, do form a countable set A_x. If A_{x_1} and A_{x_2} are two such sets, they have no element in common or they do coincide.

Let us consider all sets A_x, viewed as elements of a set H. If P is any point of r, then it exists one and only one element in H whom P belongs.

Let us consider for each element α of H a point P_α in the interval $(0, \frac{1}{2})$ that belongs to α and let us denote by G_0 the set of all points P_α. Moreover, if ρ is any rational number, let us denote by G_ρ the set of points $P_\alpha - \rho$.

The sets G_ρ corresponding to different rational values of ρ are pairwise with no common points, and moreover they are a countable set and they have, by 1ª), the same measure.

The sets

$$G_0, \quad G_{\frac{1}{2}}, \quad G_{\frac{1}{3}}, \quad G_{\frac{1}{4}}, \dots$$

lie all in the interval $(0, 1)$, therefore their union must have measure $m \leq 1$.

But one also has

$$\begin{aligned} m &= \mu(G_0) + \sum_{n=2}^{\infty} \mu\left(G_{\frac{1}{n}}\right) \\ &= \lim_{n=\infty} n \cdot \mu(g_0), \end{aligned}$$

*Some terms are translated in their modern counterpart: for example *gruppo* translates as *set* and not as *group* in thjis context, etc.

(*) v. *Leçons* sur l'intègration etc. par H. Lebesgue p.103. Paris, Gauthier-Villars, 1904.

so that

$$\mu(G_0) = 0.$$

Therefore, the union of all G_ρ corresponding to different rational values of ρ also has measure zero. But this union is the set of all points in r hence it should have infinite measure. That is enough to conclude that: ***the problem of measure of sets of points on a line is impossible***.

One could object something about the consideration of set G_0. That can be be perfectly justified if one admits that the continuum can be well ordered. For people who don't want to accept this our result means: *the possibility of the problem of measure for sets of points on a line and the possibility to well order the continuum cannot coexist.*