# Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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## Problems

9.1. **Regular values**

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Fine all values $c \in \mathbb{R}$ for whcih the level $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

9.2. **Solution set of one equation.**

Let $x, y, z, w$ be the standard coordinates on $\mathbb{R}^4$. Is teh solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in $\mathbb{R}^4$ a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology).

9.3. **Solution set of two equations.**

Is teh solution set of thesysemt of equations

$$x^3 + y^3 + z^3 = 1,\, z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

9.4. **Regular submanifolds**

Suppose that a subset $S$ of $\mathbb{R}^2$ hat eh property that locally on $S$ one of the coordinates is $C^\infty$ function of the other coordinate. Show that $S$ is qa regular submanifold of $\mathbb{R}^2$. (Note that the unit circle defined by $x^2 + y^2 = 1$ has this property. AT every point of the circle, there is a neighborhood in whic h$y$ is a $C^\infty$ function of $x$ or $x$ is a $C^\infty$ function of $y$.)

9.5. **Graph of a smooth function**

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

$$\Gamma(f) = \left\{ \, (\, x, y, f(x, y)\, ) \in \mathbb{R}^3 \, \right\}$$

is a regular submanifold of $\mathbb{R}^3$.

9.6. **Euler's formula** A polynomial $F(x_0, \ldots, x_n) \in \mathbb{R}[x_0, \ldots, x_n]$ is *homogenous of degree $k$* if it is a linear combination of monomials $x_9^{i_o} \cdots x_n^{i_n}$ of degree $\sum\limits_{j=0}^{n} i + j = k$. Let $F(x_0, \ldots, x_n)$ be a homogenous polynomial of degree $k$. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_o, \ldots, tx_n) = t^k F(x_0, \ldots, x_n).$$

Show that

$$\sum +i = 0^n x_i \frac{\partial F}{\partial x_i} = kF.$$

9.7. *Smooth projectve hypersurface*

On the projective space $\mathbb{R}P^n$ a hmogenous polynomial $F(x_0, \ldots, x_n)$ of degree $k$ is not a function , since its value at a point $[a_0, \ldots, a_n]$ is not unique. However, the zer set in $\mathbb{R}P^n$ of a homogenous polynomial $F(x_0, \ldots, x_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0, \, \forall t \in R^\times : \mathbb{R} - \{0\}$$

The zero set of finitely many homogenous polynomials in $\mathbb{R}P^q$ is called a *real projective variety.* A projectvie variety defined bya single homogeneous polynomial of degree $k$ is called a *hypersurface* of degree $k$. Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smoot if $\partial F/\partial x_0, \partial F/\partial x_1$ and $\partial F/\partial x_2$ are simultaneously zero on $Z(F)$. (*Hint:* The standard coordinates on $U_0$ which is homeomorpic to $\mathbb{R}^2$, are $x = x_1/x_0, y = x_2/x_0$ (see Subsection 7.7). In $U_0, F(x_0, x_1, x_2) = x_0^t F(1, x_1/x_0, x_2/x_0) = x_0^k F(1, x, y)$. Defein $f(x, y) = F(1, x, y)$. Then $f$ and $F$ ahve the same zero set in $U_0$.)

9.8. **Product of regular submanifolds**

If $S_1$ is a regular submanifold of the manifold $M_i$ for $i = 1, 2$, prove that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.