Functional Analysis– Spring 2024

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6. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.

   - ($\Rightarrow$) $T^{-1} : \mathcal{R}(T) \to X$ is bounded. Given any Cauchy sequence $(x_n) \in X$ we know that $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Further, we know that $\|Tx_n - Tx_m\| \leq \|T\| \|x_n - x_m\|$ which implies that $\|Tx_n - Tx_m\| \to 0$ as $n, m \to \infty$. Let x be such that $Tx_n \to x$ as $n \to \infty$. Clearly, $x \in X$ because X is complete and $Tx \in \mathcal{R}(T)$ and $\|Tx_n - Tx\| \leq \|T\| \|x_n - x\| \to 0$ as $n \to \infty$. Thus, $\mathcal{R}(T)$ is closed.
   - ($\Leftarrow$) $\mathcal{R}(T)$ is closed. Given any sequence $(y_n) \in \mathcal{R}(T)$ we know that it converges. Let y be such that $y_n \to y$ as $n \to \infty$. Since T is injective then $T^{-1} : \mathcal{R}(T) \to X$ is a function. Let $x_i = T^{-1}(y_i)$ for all $i \in \mathbb{N}$ and $x = T^{-1}y$. $\|y_n - y\| \to 0$ thus $\|T^{-1}(y_n - y)\| = \|T^{-1}(y_n) - T^{-1}(y)\| = \|x_n - x\| \leq \|T^{-1}\| \|y_n - y\| \to 0$ as $n \to \infty$.

$$\begin{aligned} \|T^{-1}(y_n - y)\| &= \|T^{-1}(y_n) - T^{-1}(y)\| \\ &= \|x_n - x\| \\ &\leq \|T^{-1}\| \|y_n - y\| \\ &\leq \|T^{-1}\| \|T\| \|x_n - x\| \to 0 \text{ as } n \to \infty \end{aligned}$$

   T is bounded.

7. Let $T : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, show that there are positive real numbers a and b such that $a \|x\| \leq \|Tx\| \leq b \|x\|$ for all $x \in X$.

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8. Let X and Y be normed spaces and let $T : X \to Y$ be a closed linear operator.

   (a) Show that the image A of a compact subset $C \subset X$ is closed in Y.

   Let $(x_n) \in C$. Since C is compact, let α be the ordered set of integers such that $(x_i)_{i\in\alpha}$ converges and let $x_{\alpha_i} \to x$ as $i \to \infty$. Then $T(x_{\alpha_i}) \in A$ for all $i \in \mathbb{N}$. Since T is a closed linear operator and C is compact (hence closed) the set $\mathcal{G}(T) = \{(x, y) \,|\, x \in C, y \in A\}$ must also be closed. Therefore $((x_{\alpha_i}, Tx_{\alpha_i})) \in \mathcal{G}(T)$ as $i \to \infty$ so must $(x, Tx) \in \mathcal{G}(T)$ which means that $Tx \in A$. Hence A is closed in Y.

   (b) Show that the inverse image B of a compact subset $K \subset Y$ is closed in X. (Cf. Def. 2.5-1)

   Let $(y_n) \in K$ and let $\alpha \subset \mathbb{N}$ be an ordered set of indices such that (y_{α_n}) converges and let $y = (y_{\alpha_n})$. Then $\|(y_{\alpha_n}) - y\| \to 0$ as $n \to \infty$. Thus, $\left\|T^{-1}y_{\alpha_n} - T^{-1}y\right\| = \left\|T^{-1}(y_{\alpha_n} - y)\right\| \leq \left\|T^{-1}\right\| \|y_{\alpha_n} - y\| \to 0$ as $n \to \infty$. Sine T^{-1} is closed $T^{-1}y \in B$ and B is closed.

9. If $T : X \to Y$ is a closed linear opearator, where X and Y are normed spaces and Y is compact, show that T is bounded.

   Let (x_n) be a sequence in X then since Y is compact (Tx_n) converges and let $y = Tx_n$ as $n \to \infty$ and let x be such that $Tx = y$. Thus $\|Tx_n - y\| = \|Tx_n - Tx\| \leq M \|x_n - x\| \to 0$. Thus, T is bounded.

10. Let X and Y be normed spaces and X compact. If $T : X \to Y$ is a bijective closed linear operator, show that T^{-1} is bounded.

    Let $A \subset X$ be closed and bounded. T is bijective implies that $T^{-1}TA = A$ thus $(T^{-1})^{-1}(C) = T(C) \subset Y$ which is compact, that implies that T^{-1} is continuous and hence, bounded.

p. 246 #2, 3, 4

2. Give a simpler proof of Lemma 4.6-7 for the case that X is a Hilbert space.
3. If a normed space X is reflexive, show that X' is reflexive.
4. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive. (*Hint.* It can be shown that a closed subspace fo a reflexive Banach space is reflexive. Use this fact, without proving it.)

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4. Show that weak convergence in footnote 6 implies weak* convergence. Shwo that the converse holds if X is reflecive.

7. Let $T_n \in B(X, Y)$, where X is a Banach space. If (T_n) is strongly operator convergent, show that $(\|T_n\|)$ is bounded.