

Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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I. This problem review continuity for functions on real line.

We say a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

- (a) Show that $f(x) = x^2$ is continuous at $x = 2$.

Given an $\epsilon > 0$, when $|f(x) - 4| < \epsilon$, $|x^2 - 4| < \epsilon$ therefore let x be such that $|(x - 2)^2 - 4| < \epsilon$ then $(x - 2)^2 < \epsilon + 4$ or this is true whenever we choose $\delta < \sqrt{\epsilon + 4} + 2$ and $|x - 2| < \delta$

- (b) Suppose that f is continuous at a and $f(a) \neq 0$. Show that f is nonzero in some open interval containing a .

II. This problem review derivatives.

- (a) Let $f(x) = x^n$ for some positive integer n . Using the definition of the derivative, and the binomial theorem, show that f'^{n-1} .
- (b) Is the function

$$f(x) = \begin{cases} x, & x \leq 0, \\ -x^2, & x \geq 0 \end{cases}$$

differentiable at $x = 0$.

III. This problem reviews sup and inf.

For any subset $A \subset \mathbb{R}$, we say that M is an *upper bound* for A if $x \leq M$ for all $x \in A$. If a set A has a finite upper bound, we say it is *bounded above*. It is a theorem about the set \mathbb{R} that *for any set A subset \mathbb{R} that is bounded above, there exists a least (smallest) upper bound for A* . This least upper bound is called supremum of A , and denoted $\sup A$. By definition, the number $\sup A$ has two properties.

- (i) $x \leq \sup A$ for all $x \in A$ (i.e., $\sup A$ is an upper bound for M)
- (ii) for any M that is an upper bound for A , we have $\sup A \leq M$.

For sets that are not bounded above, we say that $\sup A = +\infty$. we often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set $\{f(x) : x \in A\}$, where f is a some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted $\inf A$. It satisfies the same properties as $\sup A$ with the inequalities reversed.

- (a) Find $\sup A$ and $\inf A$ for $A = (1, 2]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, and $A = \{0, 1, 2, 3, \dots\}$.
- (b) Find $\sup_{x \in (0,1)} (2 + x^2)^{-1}$
- (c) Assume that $\sup A < \infty$, and show that for every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$.
- (d) For any two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, and any set $A \subset \mathbb{R}$, show that $\sup_{x \in A} (f(x) + g(x)) \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$.

IV. Section 1.1, Exercise 5, 6, 13.

Exercise 1.1.5. Let $n \geq 1$, and let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be real numbers verify the identity

$$\left(\sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right), \quad (1.3)$$

and conclude *Cauchy-Schwarz inequality*

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

Exercise 1.1.6 Show that (\mathbb{R}^n, d_{l^2}) in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5)

Example 1.1.6 (Euclidean spaces). Let $n \geq 1$ be a natural number, and let \mathbb{R}^n be the space of n -tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}$$

We define the *Euclidean metric* (also called the l^2 metric) $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \end{aligned}$$

Exercise 1.1.13 Prove Proposition 1.1.19. **Proposition 1.1.19** (Convergence in a the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X . let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in X , and let x be a point in X . Then $(x^{(n)})_{n=m}^\infty$ convergent to x with respect to the discrete metric d_{disc} if and only if there exists $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.*

For this problem only, you do not need to give proofs. just write the answers.

For each set, identify the boundary, interior, and closure of A , and say whether A is open, closed, both or neither. We are working in \mathbb{R}^2 with the standard distance. Unless otherwise noted, the ambient space is \mathbb{R}^2 .

- (a) $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$
- (b) $A = \{(1/n, 2/n) : n = 1, 2, 3, \dots\}$ (Note: $(1/n, 2/n)$ is a vector in \mathbb{R}^2 , not an open interval in \mathbb{R}).
- (c) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$, in the relative topology with respect to $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$.

Let (X, d) be a metric space.

- (a) For a given point $x_0 \in X$, show the singleton set $\{x_0\}$ is closed.
- (b) Let $x_0 \in X$ and $r > 0$. Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open