

Math 5050 – Special Topics: Manifolds– Spring 2025

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Notes – January – May, 2025

Definitions

1. **Diffeomorphism:** If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then f is said to be a **diffeomorphism**. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be **diffeomorphic** to each other.

2. **Tangent Space** at a point p . The set of all vectors rooted at p , written as $T_p(\mathbb{R}^n)$.

Let $p = (x^1, \dots, x^n)$. The directional derivative for each component would be described as

$$\text{notice } \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta_j^i, \forall p.$$

that is perpendicular and form a orthogonal basis. Thus, a Tangent Vector is also called a "Derivation".

3. **Derivations:** any operation that supports the Liebniz Rule $D(fg) = (Df)g + fDg$.
4. **Derivation Space.** $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p . This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \rightarrow \mathcal{D}(\mathbb{R}^n)$ defined as

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\rightarrow \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v = \sum v^i \frac{\partial}{\partial x^i} \Big|_p. \end{aligned}$$

5. **Germ:** equivalence class of functions whose derivatives around a point are the same.
6. Vector Field vs Vector Space.

- **A Vector Field** a function that assigns a vector to every point in the subset U .

$$\begin{aligned} f : (U \subset \mathbb{R}^m) &\rightarrow T_p(\mathbb{R}^n) \\ X &\mapsto X_p = \sum a^i(p) \frac{\partial}{\partial x^i} \Big|_p. \end{aligned}$$

consider a^i as coefficient functions. We say that X is C^∞ on U if $a^i \in C^\infty, \forall i = 1, \dots, n$.

- **A Vector Space** is any abstraciton that is closed under addition and scalar multiplication.

7. **Dual Basis and Dual Space.** The **Dual Basis** is a set of functions $\alpha^i : V \rightarrow \mathbb{R}$

$$\begin{aligned} \alpha^i : V &\rightarrow \mathbb{R} \\ \alpha^i(e_j) &= \delta_j^i \end{aligned}$$

the **Dual Space** V^\vee is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called **Functionals (Analysis)/1-Covectors (Differential Geometry)**.

8. **Multi-Linear Functions and Vector Space of k -tensors** $L_k(V)$ Let V be a vector space and V^k be k -tuples of vectors in V . A **K -linear map or k -tensor** $f : V^k \rightarrow \mathbb{R}$ such that each i^{th} component is linear. The vector space of all k -tensors on V is denoted $L_k(V)$.

Permuting Mult-linear Functions. Given any permutation $\sigma \in S_k$

$$f(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

e.g., $f(x, y, z) = xyz \rightarrow f(z, x, y) = zxy$. FYI: if x, y, z are from non-commutative rings (i.e., matrices) then we must be aware of the $\text{sgn}(\sigma)$.

9. **Left R -Module:** An Abelian group R with a scalar multiplication map:

$$\mu : R \times A \rightarrow A$$

usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a

- (i) (associative) $(rs)a = r(sa)$.

- (ii) (identity) $1a = a$ (1 is a multiplicative identity).
- (iii) (distributivity) $(r + s)a = ra + sa$ and $r(a + b) = ra + rb$.

If R is a field then R -module is precisely a vector space over R .

A **K -Algebra over a field K** is also a ring A that is also a vector space over K such that the ring multiplication satisfies homogeneity (scalar distributes over vector multiplication to only one of the operators).

A **graded Algebra** is an algebra A over a field K if it can be written as the direct sum

$$A = \bigoplus_{i=0}^{\infty} A^i$$

of vector spaces over K such that the multiplication map sends $A^k \times A^l \rightarrow A^{k+l}$

10. The set of all C^∞ -vector fields on U , denoted by $\mathfrak{X}(U)$, is not only a vector space over \mathbb{R} , but also a *module* over the $C^\infty(U)$ ring.

$$\mathfrak{X}(U) = \{ X : V \rightarrow V \mid X \in C^\infty(U) \} \text{ where } V = (\mathbb{R} \text{ or } \mathbb{C})^n$$

11. **Derivation:** A **derivation** on an algebra A is a K -multilinear function $D : A \rightarrow A$ such that

$$D(ab) = (Da)b + aDb, \forall a, b \in A$$

known as the **Liebniz Rule**.

The set of all derivations on A forms a vector space, $\text{Der}(C^\infty(U))$. Thus a $C^\infty(U)$ vector field gives rise to a derivation of the algebra $C^\infty(U)$. Thus the mapping

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\rightarrow \text{Der}(C^\infty(U)) \\ X &\mapsto (f \mapsto Xf) \end{aligned}$$

this map is an isomorphism of vector spaces.

12. **Exterior Algebras $\Lambda(V)$.** The exterior algebra $\Lambda(V)$ is obtained by imposing an **anti-commutative** relation:

$$v \otimes w + w \otimes v = 0, \forall v, w \in V$$

this means that the quotient algebra is:

$$\Lambda(V) = T(V) / \langle v \otimes w + w \otimes v \rangle.$$

Where $T(V)$ is the **tensor algebra**

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

13. **Symmetric Algebras $S(V)$.** The symmetric algebra $S(V)$ is obtained by imposing an **commutative** relation:

$$v \otimes w - w \otimes v = 0, \forall v, w \in V$$

this means that the quotient algebra is:

$$S(V) = T(V) / \langle v \otimes w - w \otimes v \rangle.$$

14. **Tensor Product** The tensor product between two 1-covectors, $f, g : V \rightarrow \mathbb{R}$ is the 2-covector $f \otimes g$.

$$(f \otimes g)(u, v) = f(u)g(v)$$

. In general, the tensor product of a k -covector $p : V^k \rightarrow \mathbb{R}$ with a l -covector $q : V^l \rightarrow \mathbb{R}$ is the $(k + l)$ -covector $p \otimes q : V^{k+l} \rightarrow \mathbb{R}$.

$$(p \otimes q)(u, v) = p(u)q(v), \forall u \in V^k, v \in V^l$$

15. **Tensor Product(?)** is an operator on $v \in V$ and $u \in U$ where

$$\begin{aligned} v \otimes u &: V \times U \rightarrow V \oplus U \\ (v \otimes u)_{i,j} &= v_i \cdot u_j, \forall i = 1, \dots, \dim(V), j = 1, \dots, \dim(U) \end{aligned}$$

Given two vector spaces V, W with bases v_1, \dots, v_n and w_1, \dots, w_m then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

α_i, β_j are all scalars. The real issue is the behavior of unit basis vectors v_i, w_j and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

16. Wedge Product

Between two covectors Let $f, g \in L_1(V)$ then for all $u, v \in V$

$$(f \wedge g)(u, v) = (f \otimes g)(u, v) - (g \otimes f)(u, v) = f(u)g(v) - f(v)g(u)$$

Between multiple 1-covectors.

$$\begin{aligned} (\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \dots, v_k) &= \det[\alpha^i(v_j)] \\ &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)}) \end{aligned}$$

Between k -covector and l -covector. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A_{k+l}(V)$$

or explicitly

$$(f \wedge g)(v_1, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

Anticommutative. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$(f \wedge g) = (-1)^{kl} g \wedge f$$

17. Differential k-Forms

1-forms, covectors

$$\begin{aligned} (dx^i) \left(\frac{\partial}{\partial x^j} \Big|_p \right) &= \frac{\partial}{\partial x^j} \Big|_p x^i = \delta_j^i \\ (df)_p(X_p) &= X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i} \Big|_p = \sum \frac{\partial f}{\partial x^i} dx^i \end{aligned}$$

18. $\Omega^k(U)$, Vector space of C^∞ k -forms on U .

$\Omega^0 = A_0(T_p(\mathbb{R}^n)) = C^\infty(U)$, e.g., $f \in \Omega^0$ then $f : V \rightarrow \mathbb{R}$ is a functional/covector/1-tensor.

Elements of 1-form $\Omega^1 = A_1(T_p(\mathbb{R}^n))$. For example, when $n = 3$

$$f dx + g dy + h dz, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

Elements of 2-form $\Omega^2 = A_2(T_p(\mathbb{R}^n))$. For example, when $n = 3$ ¹

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

if $n = 4$, that is coordinates for u, v, w, x . Each form is derived from these bases

0-form $\Omega^0(\mathbb{R}^4) \in \mathbb{R}$

1-forms $\Omega^1(\mathbb{R}^4)$ summing du, dv, dw, dx ,

2-forms $\Omega^2(\mathbb{R}^4)$ summing $du \wedge dv, du \wedge dw, du \wedge dx, dv \wedge dw, dv \wedge dx, dw \wedge dx$,

3-forms $\Omega^3(\mathbb{R}^4)$ summing $du \wedge dv \wedge dw, du \wedge dv \wedge dx, du \wedge dw \wedge dx, dv \wedge dw \wedge dx$

4-form $\Omega^4(\mathbb{R}^4)$ $du \wedge dv \wedge dw \wedge dx$.

Also, $U \subseteq \mathbb{R}^n$ then $k < n$. k -forms for $k > n$ are zero. Further $|\Omega^k(\mathbb{R}^n)| = \binom{n}{k}$ and $|\bigcup_k \Omega^k(\mathbb{R}^n)| = 2^n$ and think of $\Omega^*(U) = \bigcup_k \Omega^k(\mathbb{R}^n)$

Direct Sum. $\Omega^*(U) = \bigoplus_k \Omega^k(U)$ is an anti-commutative graded algebra over \mathbb{R} .

Since one can multiply C^∞ k -forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k -forms is both a vector space over \mathbb{R} and a module over $C^\infty(U)$ and $\Omega^*(U)$ is also a module over C^∞ of C^∞ functions.

19. Wedge Product of k -form.

Recall: $dx^i \wedge dx^i = 0$ for all $i = 1, \dots, n$. Therefore, \wedge only makes sense to be defined on *disjoint indice-lists*, that is, $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_l\}$ such that $I \cap J = \emptyset$. Then,

$$\begin{aligned} \wedge : \Omega^k(U) \times \Omega^l(U) &\rightarrow \Omega^{k+l}(U) \\ (\omega, \tau) &\mapsto (\omega \wedge \tau) = \sum_{I, J} a_I b_J dx^I \wedge dx^J. \end{aligned}$$

where $\omega = \sum_I a_I dx^I, \tau = \sum_J b_J dx^J$.

¹NOTE the cyclic order of the indices x, y, z . Switching any one of these will flip the sign.

20. **the Exterior Derivative.** If $k \geq 1$ and if $\omega = \sum_I a_I dx^I \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$ and

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_J \frac{\partial a_I}{\partial x_J} dx^J \right) \wedge dx^I$$

Example: Let $\omega \in \Omega^1(\mathbb{R}^2)$ and $\omega = f dx + g dy$, $f, g \in C^\infty(\mathbb{R}^2)$.

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\ &= (g_x - f_y) dx \wedge dy \end{aligned}$$

Definition: Let $\bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K . An **anti-derivation** of the graded algebra A is a K -linear map $D : A \rightarrow A$ such that $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k aDb$$

Proposition 4.7: Three Criterion for an Exterior Derivation

i) The **exterior derivation** $d : \Omega^*(U) \rightarrow \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

ii) $d^2 = 0$.

iii) If $f \in \mathbb{C}^\infty$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

NOTE: “In a typical school, there would be graduate level courses on Smooth Manifolds and another on Riemannian Manifolds.”

Q: What is the difference between $\mathfrak{X}(U)$ and $C^\infty(U)$?

The difference between $\mathfrak{X}(U)$ and $C^\infty(U)$ lies in the types of objects they contain:

1. **$C^\infty(U)$: The Space of Smooth Functions** - $C^\infty(U)$ consists of all smooth (infinitely differentiable) real-valued functions defined on an open subset U of a manifold M . - Elements of $C^\infty(U)$ are scalar functions $f : U \rightarrow \mathbb{R}$. - These functions can be added and multiplied pointwise, forming an algebra over \mathbb{R} .

2. **$\mathfrak{X}(U)$: The Space of Smooth Vector Fields** - $\mathfrak{X}(U)$ consists of all smooth vector fields on U . - A vector field X assigns to each point $p \in U$ a tangent vector $X_p \in T_p M$, smoothly varying with p . - Vector fields act as derivations on smooth functions, meaning they satisfy the Leibniz rule:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(U).$$

- The space $\mathfrak{X}(U)$ forms a module over $C^\infty(U)$, meaning smooth functions can scale vector fields: if $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then fX is also a vector field.

Key Differences

Feature	$C^\infty(U)$	$\mathfrak{X}(U)$
Elements	Smooth scalar functions $f : U \rightarrow \mathbb{R}$	Smooth vector fields $X : U \rightarrow TM$
Algebraic Structure	Commutative algebra (pointwise multiplication)	Module over $C^\infty(U)$, noncommutative under Lie bracket
Operations	Addition, multiplication	Addition, scalar multiplication by $C^\infty(U)$, Lie bracket $[X, Y]$

In summary, $C^\infty(U)$ consists of smooth functions, while $\mathfrak{X}(U)$ consists of smooth vector fields, which act as differential operators on $C^\infty(U)$.

Compare and contrast.

Set	Dim	index	basis	Delta
$L_1(U)$	n	$i = 1, \dots, n$	α^i	$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
$L_k(U)$	n^k	$I, J \in \underbrace{\{i_1, \dots, i_k\}}_{k \text{ times}}, i_k \in [1, \dots, n]$	$\alpha^I = \alpha^{i_1} \otimes \alpha^{i_2} \otimes \dots \otimes \alpha^{i_k}$	
$A_k(U)$	$\binom{n}{k}$	$I, J \in \underbrace{\{i_1, \dots, i_k\}}_{k \text{ times}}, i_1 < i_2 < \dots < i_k \in [1, n]$	$\alpha^I = \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}$	$\delta_I^J = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$

Supersets

Symbol	Name (set of)	Definition	Example
$\Omega^0(U)$	0-forms	{ scalar fields }	$f : V \rightarrow \mathbb{R} \quad f(x, y, z)$
$\Omega^1(U)$	1-forms	{ 1-forms, vector fields }	$d\omega(v) = A(v)dx + B(v)dy + C(v)dz$ $A, B, C : V \rightarrow \mathbb{R}$
$\Omega^k(U)$	k -forms	{ k -forms }	$\dots + dx^1 \wedge \dots \wedge dx^k + \dots$
$\Omega^*(U)$	sum of k -forms	{ $x = \sum y \mid y \in \oplus_k \Omega^k(U)$ }	$A dx + B dx \wedge dy + C dx \wedge dy \wedge dz, \quad A, B, C : V \rightarrow \mathbb{R}$
$\mathfrak{X}(U)$	vector fields on U	{ $X \rightarrow \exists f : U \rightarrow U$ }	
$C^\infty(U)$	smooth functions on U		
$X_p = T_p(U)$	a vector field at p	{ $v \in U \mid v = p + x$ for some $x \in U$ }	

Map of $\Omega^k(\mathbb{R}^3)$

$$\begin{array}{ccccccc}
 \Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \\
 \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
 C^\infty(U) & \xrightarrow{\text{grad}} & \mathfrak{X}(U) & \xrightarrow{\text{curl}} & \mathfrak{X}(U) & \xrightarrow{\text{div}} & C^\infty(U).
 \end{array}$$

Shorthand

$$\begin{aligned}
 \sum_{i,j} a_i b_j &= \sum_i a_i \sum_j b_j \\
 \sum_{i,j} a_i b_j &= \sum_i a_i \sum_j b_j \\
 \sum_I a_I &= \sum_{n=1}^k a_{i_n} \\
 \sum_{I,J} a_I b_J &= \sum_{n=1}^k a_{i_n} \sum_{m=1}^k b_{j_m} \\
 \delta_i^j &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\
 \delta_I^J &= \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots \delta_{i_k}^{j_k} = \begin{cases} 1 & i_n = j_n, \forall n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}
 \end{aligned}$$

Definition 0.0.1 (Exact and Closed k -forms). A k -form ω on U is **closed** if $d\omega = 0$; it is **exact** if there is a $(k-1)$ -form τ such that $\omega = d\tau$ on U . Since $d(d\tau) = 0$, every exact form is closed.

Definition 0.0.2 (de Rham Cohomology). .

The k^{th} -**cohomology** of U is defined as the quotient vector space

$$H^k(U) = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}$$

That is, each element is a vector space forming an equivalence class of k -forms.

Examples of de Rham Cohomology

De Rham cohomology provides a way to study the topology of smooth manifolds using differential forms. Below are some key examples illustrating how to compute and interpret de Rham cohomology groups.

****Example 1: Euclidean Space \mathbb{R}^n ****

For $M = \mathbb{R}^n$, we claim that the de Rham cohomology is:

$$H_{\text{dR}}^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

****Computation****

1. **** $H_{\text{dR}}^0(\mathbb{R}^n)$ ****

- The 0-forms are just smooth functions f .
- A function is closed if $df = 0$, meaning f is constant.
- Every constant function is not only closed but also exact since $f = d(fx)$.
- The space of closed 0-forms is \mathbb{R} (constant functions), and there are no exact forms to quotient out.
- So, $H_{\text{dR}}^0(\mathbb{R}^n) = \mathbb{R}$.

2. **** $H_{\text{dR}}^k(\mathbb{R}^n)$ for $k > 0$ ****

- Any closed k -form ω is locally exact due to ****Poincaré's lemma****.
- That is, every closed form is of the form $\omega = d\eta$, meaning it contributes nothing to cohomology.
- Thus, $H_{\text{dR}}^k(\mathbb{R}^n) = 0$ for $k > 0$.
This result reflects the fact that \mathbb{R}^n is ****contractible****, so it has trivial topology.

****Example 2: The Circle S^1 ****

For $M = S^1$, we find:

$$H_{\text{dR}}^k(S^1) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$

****Computation****

1. **** $H_{\text{dR}}^0(S^1) = \mathbb{R}$ ****

- Smooth functions f that satisfy $df = 0$ are constant.
- Thus, $H_{\text{dR}}^0(S^1) = \mathbb{R}$.

2. **** $H_{\text{dR}}^1(S^1) = \mathbb{R}$ ****

- Consider the 1-form $\omega = d\theta$, where θ is the angular coordinate.
- $d\omega = 0$, so ω is closed.
- Is ω exact? If $\omega = d\eta$ for some η , then $d\eta = d\theta$, but no globally defined function η exists on S^1 satisfying this.
- So ω represents a ****nontrivial cohomology class****, giving $H_{\text{dR}}^1(S^1) = \mathbb{R}$.

3. **** $H_{\text{dR}}^k(S^1) = 0$ for $k \geq 2$ ****

- There are no nontrivial 2-forms on a 1-dimensional manifold.

****Interpretation****

- The nontrivial $H_{\text{dR}}^1(S^1)$ reflects the existence of a **loop** in S^1 .
- This cohomology detects the ability to define a **non-exact closed form**, related to the winding number.

Example 3: The 2-Sphere S^2 For $M = S^2$:

$$H_{\text{dR}}^k(S^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ 0, & k = 1. \end{cases}$$

Computation

1. $H_{\text{dR}}^0(S^2) = \mathbb{R}$
 - As always, closed 0-forms are constant functions, so $H_{\text{dR}}^0(S^2) = \mathbb{R}$
2. $H_{\text{dR}}^1(S^2) = 0$
 - Any closed 1-form is exact by a higher-dimensional **Poincaré lemma**, so $H_{\text{dR}}^1(S^2) = 0$.
3. $H_{\text{dR}}^2(S^2) = \mathbb{R}$
 - The standard volume form $\omega = \sin \theta \, d\theta \wedge d\phi$ is closed.
 - It is not exact, because there is no 1-form η such that $d\eta = \omega$ (this follows from **Stokes' theorem**).
 - So ω represents a generator of $H_{\text{dR}}^2(S^2)$.

Interpretation

- $H_{\text{dR}}^1(S^2) = 0$ reflects that there are no **nontrivial loops** (all loops contract).
- $H_{\text{dR}}^2(S^2) = \mathbb{R}$ corresponds to the existence of a volume form, a global topological feature.

Example 4: The Torus $T^2 = S^1 \times S^1$

For T^2 , the de Rham cohomology groups are:

$$H_{\text{dR}}^k(T^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ \mathbb{R} \oplus \mathbb{R}, & k = 1, \\ 0, & k > 2. \end{cases}$$

Computation

1. $H_{\text{dR}}^0(T^2) = \mathbb{R}$ (constant functions).
2. $H_{\text{dR}}^1(T^2) = \mathbb{R} \oplus \mathbb{R}$
 - The torus has two independent 1-forms: $d\theta_1$ and $d\theta_2$, corresponding to the two loops in T^2 .
3. $H_{\text{dR}}^2(T^2) = \mathbb{R}$
4. The volume form $d\theta_1 \wedge d\theta_2$ represents a nontrivial 2-class.

Interpretation

- The rank of $H_{\text{dR}}^1(T^2)$ reflects the **two independent loops** in the torus.
- The nontrivial $H_{\text{dR}}^2(T^2)$ corresponds to the existence of a **volume form**.

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Summary Table

Manifold M	$H^0_{dR}(M)$	$H^2_{dR}(M)$	$H^1_{dR}(M)$
\mathbb{R}^n	\mathbb{R}	0	0
S^1	\mathbb{R}	\mathbb{R}	0
S^2	\mathbb{R}	0	\mathbb{R}
T^2	\mathbb{R}	$\mathbb{R} \oplus \mathbb{R}$	\mathbb{R}

§8 Tangent Space.

Definition 8.1.3 (Tangent vector). A *tangent vector* at a point p is a derivation at p .

Remark 8.1.4. In general, x^i, y^j are coordinates for manifolds N, M and r^i are coordinates associated with charts. Thus $p = (x^1, \dots, x^n)$ and $\phi(p) = (r^1, \dots, r^k)$ where $k \leq n$.

Definition 8.1.5 (Differential of a Map).