# Functional Analysis– Spring 2024

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p. 200 #2,3,4,5,6,10.

2. Let $H$ be a Hilbert space and $T : H \to H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*$$

$$\begin{aligned}
\langle Tx, y \rangle &= \langle x, T^* y \rangle \\
\left\langle T^{-1} Tx, y \right\rangle &= \left\langle T^{-1} x, T^* y \right\rangle \\
\langle x, y \rangle &= \left\langle T^{-1} x, T^* y \right\rangle \\
\left\langle x, (T^*)^{-1} y \right\rangle &= \left\langle T^{-1} x, (T^*)^{-1} T^* y \right\rangle \\
&= \left\langle T^{-1} x, y \right\rangle \\
&= \left\langle x, (T^{-1})^* y \right\rangle \\
(T^*)^{-1} &= (T^{-1})^*
\end{aligned}$$

3. If $(T_n)$ is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$, show that $T_n^* \to T^*$.

$$\begin{aligned}
\|T_n - T\|^2 \geq \|T_n\|^2 - \|T\|^2 = \|T_n^*\|^2 - \|T^*\|^2 \geq \|T_n^* - T^*\|^2 \\
\text{similarly, } \|T_n^* - T^*\|^2 \geq \|T_n^*\|^2 - \|T^*\|^2 = \|T_n\|^2 - \|T\|^2 \geq \|T_n - T\|^2 \\
\text{hence } \|T_n - T\|^2 = \|T_n^* - T^*\|^2
\end{aligned}$$

We know that given any $N > 0$ then for all $n > N$ when $\|T_n - T\| < \epsilon$ implies that $\|T_n^* - T^*\| < \epsilon$. Therefore, $T_n^* \to T^*$.

4. Let $H_1$ and $H_2$ be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1^\perp \subset T^*(M_2^\perp)$.

Let $x \in M_1$ and $z \in M_2^\perp$ and $x \notin \mathcal{N}(T)$.Then, $\langle Tx, z \rangle = 0$ implies $\langle x, T^* z \rangle = 0$ and either $T^* z \in \mathcal{N}(T^*)$ or $T^* z \perp x$. $x$ is arbitrary, therefore $T^* z \perp \text{span}(M_1)$ or $T^* z \in M_1^\perp$. Thus, $T^* z \in \mathcal{N}(T^*) \cup M_1^\perp$. $z$ is arbitrary so $T^*(M_2^\perp) = \mathcal{N}(T^*) \cup M_1^\perp$, hence, $M_1^\perp \subset T^*(M_2^\perp)$.

5. Let $M_1$ and $M_2$ in Prob. 4 be closed subspaces. Show that $T(M_1) \subset M_2$ if and only if $M_1^\perp \supset T^*(M_2^\perp)$.

6. If $M_1 = \mathcal{N}(T) = \{x \,|\, Tx = 0\}$ in Prob. 4, show that
   (a) $T^*(H_2) \subset M_1^\perp$
   (b) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$
   (c) $M_1 = [T^*(H_2)]^*$

10. **(Right shift operator)** Let $(e_n)$ be a total orthonormal sequence in a separable Hilbert space $H$ and define the *right shift operator* to be the linear operator $T : H \to H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of $T$.

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4. Show that for any bounded linear operator $T$ on $H$, the operators

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_3 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2 \ T^* = T_1 = iT_2.$$

Show uniquness, that is, $T_1 + iT_2 = S_1 + iS_2$ implies $S_i = T_i$ and $S_2 = T_2$; here, $S_1$ and $S_2$ are self-adjoint by assumptioin.

5. On $\mathbb{C}^2$ (cf. 3.1-4) let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find $T^*$. Show that we have $T^*T = TT^* = 2I$. Find $T_1$ and $T_2$ as defined in prob. 4.