Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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I. *Exercise 7.2.2.* Let A be a subset of $\mathbb{R}^n$, and let B be a subset of $\mathbb{R}^m$. Note that the Cartesian product $\{(a,b) : a \in A, b \in B\}$ is then a subset of $\mathbb{R}_{n+m}$. Show that $m^*_{n+m}(A \times B) \leq m^*_n(A)m^*_m(B)$. (It is in fact true that $m^*_{n+m}(A \times B) = m^*_n(A)m^*_m(B)$. but is substantially harder to prove).

In Exercise 7.2.3-7.2.5, we assume that $\mathbb{R}^n$ is Euclidean space, and we have a notion of measurable set in $\mathbb{R}^n$ (which may or may not coincide with the notion of Lebesgue Measurable set) and a notion of measure (which may or may not coincide with Lebesque measure) which obeys axioms (i)-(xiii).

Since $m^*(\Omega)$ is defined as [1]

$$m^*(\Omega) = \inf\left\{ \sum_{j\in J} \text{vol}(C_j) \, : \, (C_j)_{j\in J} \text{ covers } \Omega; \, J \text{ at most countable} \right\}.$$

where $(C_j)_{j\in J}$ is a covering for Ω. Then there exists boxes $(\alpha_k)_{k\in K}$ and $(\beta_l)_{l\in L}$ which are coverings for A and B, respectively. And, clearly,

$$m^*(A) \leq \sum_{k\in K} \text{vol}(\alpha_k) \text{ and } m^*(B) \leq \sum_{l\in L} \text{vol}(\beta_l)$$

define a covering J such that

$$\delta_{k,l} = \alpha_k \times \beta_l$$

$\delta_{k,l}$ is countable as it is a union of two countable sets and it is a covering over $A \times B$. And, since each α_k and β_l is a box, then

$$\begin{aligned}
m^*(\delta_{k,l}) &= m^*(\alpha_k)m^*(\beta_l), \, \forall k \in K, \, l \in L \\
m^*(A \times B) &\leq \sum_{k\in K,\, l\in L} m^*(\delta_{k,l}) \\
\sum_{k\in K,\, l\in L} m^*(\delta_{k,l}) &= \sum_{k\in K,\, l\in L} m^*(\alpha_k)m^*(\beta_l) \\
&\leq \sum_{k\in K} m^*(\alpha_k) \sum_{l\in L} m^*(\beta_l) && \text{Cauchy-Schwarz} \\
&\leq m^*(A)m^*(B)
\end{aligned}$$

[1] I've substitute B with C from the definition in the text.

II. Section 7.4, problems 1,4 (only parts (e) and (f)).

Exercise 7.4.1. If A is an open interval in $\mathbb{R}$, show that $m^*(A) = m^*(A \cap (0,\infty)) + m^*(A\backslash(0,\infty))$.

From Lemma 7.4.8 (*countable additivity)*, we can see that the two sets are disjoint, i.e.,

$$(A \cap (0,\infty)) \cap (A\backslash(0,\infty)) = \emptyset$$
$$\text{and } A = (A \cap (0,\infty)) \sqcup (A\backslash(0,\infty))$$
$$m^*(A) = m^*(A \cap (0,\infty)) + m^*(A\backslash(0,\infty))$$

Exercise 7.4.4. Prove Lemma 7.4.4. (Hints: for (c) first prove that

(e) *Every open box, and every closed box, is measurable.*

Let C be an open box, $C = \prod_{i=1}^{n}(a_i, b_i)$. Define two half-spaces $A_i = \{\,(x_1,\ldots,x_n) \in \mathbb{R}^n \,|\, x_i > a_i)\,\}$ and $B_i = \{\,(x_1,\ldots,x_n) \in \mathbb{R}^n \,|\, x_i < b_i)\,\}$ for each $i = 1,\ldots,n$. Given any $x \in C$ where $x = (x_1,\ldots,x_n)$ then $x_i \in (A_i \cap B_i)$, therefore $C = \bigcap_{i=1}^{n}(A_i \cap B_i)$. Each half-space A_i and B_i is measurable and C is the intersection of finitely many measurable sets and is therefore measurable.

(f) *any set E of outer measure zero (i.e., $m^*(E) = 0$) is measurable.*

Let $T \subseteq \mathbb{R}^n$ be any measurable subset. Let E be a set with $m^*(E) = 0$. Then,

$$\begin{aligned} T &= (T \cap E)\bigcup(T\backslash E) \\ m^*(T) &= m^*\Big((T \cap E)\bigcup(T\backslash E)\Big) \\ &= m^*\,(\,(T \cap E)\,) + m^*\,(\,(T\backslash E)\,) \end{aligned}$$

Given any $x \in T \cap E$ clearly $x \in E$ thus $m^*(T \cap E) = 0$. Therefore $m^*(T) = m^*\,(\,(T\backslash E)\,)$. Also, $x \in T\backslash E$ indicates that $x \in T$ thus $m^*(T) = m^*\,(\,(T\backslash E)\,)$. This is true for all open sets $T \in \mathbb{R}$ therefore E is measurable.

III. Let C be a parameterized curve in $\mathbb{R}^2$. In other words, C is the image for a function $\phi : [a, b] \to \mathbb{R}^2$. Show that if ϕ is continuously differentiable in $[a, b]$, then C has outer measure 0.

Hint: Partition $[a, b]$ into N equal subintervals, and use the Mean Value Inequality to show that the image of each subinterval is bounded in terms of N, i.e., fits inside an open rectangle of side length that can be explicitly bounded in terms of N. Add up the total 2-dimensional volume of the covering obtained in this way, and show that it can be made arbitrarily small by taking N large.

Warning: If ϕ is only continuous, then the result fails. One can construct a continuous ϕ such that

$$\phi([a, b]) = [0, 1] \times [0, 1].$$

Divide $[a, b]$ into n equal subintervals. Each subinterval, $[a_i, b_i]$ has lenght $1/n$ and will have an intermediate value $\zeta_i \in [a_i, b_i]$ such that $\phi'(\zeta_i)/n = \phi(b_i) - \phi(a_i)$. The volume of each subinterval can be represented by $\phi'(\zeta_i)/n$. ϕ is bounded, and if it is continuously differentiable, ϕ' is bounded, too. Thus, $\lim_{n\to\infty} \phi'(\zeta_i)/n = 0$ and $\lim_{n\to\infty} \sum_{i=1}^{n} \phi'(\zeta_i)/n = 0$. m_2^* is a measure of area and therefore $m^*(C) = \int_a^b \phi(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} \phi'(\zeta_i)/n = 0$.

IV. skip

V. Suppose $A_i \in \mathcal{M}, A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$.

(a) if $m(A_1) < \infty$, show that

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

Since each A_i is strictly contained in A_{i+1} we can say that

$$\begin{aligned} A_{i-1} \cap A_i &= A_{i-1} \\ \text{or } A_1 \cap A_2 &= A_1 \\ \therefore A_1 \cap A_i &= A_1, \ \forall i = 1, \ldots \text{ and } \lim_{n\to\infty} m(A_n) = m(A_1) \\ A_1 &= \bigcap_{n=1}^{\infty} A_n = A_1 \\ m\left(\bigcap_{n=1}^{\infty} A_n\right) &= m(A_1) \end{aligned}$$

(b) Show by example that if $m(A_1) = \infty$, the above conclusion may be wrong.

Let A_1 be an open box with $A_1 = \prod_{i=1}^{n}(a_i, b_i)$ where all $a_i, b_i < \infty$ except $b_1 = \infty$. Clearly, A_1 is a half-space and is therefore measureable. Let $A_2 \supset A_1$ be the same as A_1 except that $b_2 = \infty$. In general,let $A_i \supset A_{i-1}$ and be the same as A_{i-1} except for $b_i = \infty$. We can clearly see that

$$A_1 \subset A_2 \subset \cdots A_{n-1} \subseteq A_n \subseteq \cdots$$

but we cannot know $\lim_{n\to\infty} m(A_n)$.

VI. Let $\Omega \subset \mathbb{R}^n$ be measurable. $f : \Omega \to \mathbb{R}$ is a function. If f^2 is measurable, and the set

$$A = \{x \in \Omega \,|\, f(x) > 0\}$$

is also measurable. Show that f is measurable.

Let $h = f^2$ which is measurable. Therefore. given open set $V \in \mathbb{R}^n$ there exists a measurable set $U \in \mathbb{R}^n$ such that $h(U) = V$. Let $g = f|_A$ and choose $V = g(A)$ and U such that $h(U) = V$. Therefore, $g(f(U)) = V$ or $f|_A(f(U)) = V$ or $f_A(A) = V$. A is measurable and is mapped to an open set. Therefore, f is measurable.