Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Definitions

1. ***Diffeomorphism***: If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then f is said to be a ***diffeomorphism***. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be ***diffeomorphic*** to each other.
2. ***Tangent Space*** at a point p. The set of all vectors rooted at p, written as $T_p(\mathbb{R}^n)$.
3. ***Derivations***: any operation that supports the Liebniz Rule ($D(fg) = (Df)g + fDg$).
4. ***Derivation Space***. $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p. This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ defined as
$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p . \end{aligned}$$
5. ***Germ***: equivalence class of functions whose derivatives around a point are the same.
6. Vector Field vs Vector Space.
   - ***A Vector Field*** a function that assigns a vector to every point in the subset U.
   $$\begin{aligned} f : (U \subset \mathbb{R}^m) &\to T_p(\mathbb{R}^n) \\ X &\mapsto X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p . \end{aligned}$$
   consider a^i as coefficent functions. We say that X is C^∞ on U if $a^i \in C^\infty$, $\forall i = 1, \dots, n$.
   - ***A Vector Space*** is any abstraciton that is closed under addition and scalar multiplication.
7. Left R-***Module***: An Abelian group R with a scalar multiplication map:
$$\mu : R \times A \to A$$
usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a
   (i) (associative) $(rs)a = r(sa)$.
   (ii) (identity) $1a = a$ (1 is a multiplicative identity).
   (iii) (distributivity) $(r + s)a = ra + sa$ and $r(a + b) = ra + rb$.

   If R is a field then R-module is precisely a vector space over R.

   A K-***Algebra over a field*** K is also a ring A that is also a vector space over K such that the ring multiplication satisfies homogeniety (scalar distributes over vector multipliation to only one of the operators).
8. Exterior Algebras
9. ***Dual Basis and Dual Space***. The ***Dual Basis*** is a set of functions $\alpha^i : V \to \mathbb{R}$
$$\begin{aligned} \alpha^i &: V \to \mathbb{R} \\ \alpha^i(e_j) &= \delta^i_j \end{aligned}$$
the ***Dual Space*** $V^\vee$ is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called ***Functionals (Analysis)/1-Covectors (Differential Geometry)***.
10. ***Multi-Linear Functions*** Let V be a vector space and V^k be k-tuples of vectors in V. A K***-linear map or*** k***-tensor*** $f : V^k \to \mathbb{R}$ such that each i^{th} component is linear. The vector space of all k-tensors on V is denoted $L_k(V)$.
11. Permuting Mult-linear Functions
12. ***Tensor Product*** is an operator on $v \in V$ and $u \in U$ where
$$\begin{aligned} v \otimes u &: V \times U \to V \oplus U \\ (v \otimes u)_{i \cdot j} &= v_i \cdot u_j,\ \forall i = 1, \dots, \dim(V),\ j = 1, \dots, \dim(U) \end{aligned}$$

13. Wedge Product

14. Differential k-Forms

15. the Exterior Derivative

16. Tensor Product.

    Given two vector spaces V, W with bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

    α_i, β_j are all scalars. The real issue is the behavior of unit basis vectors v_i, w_j and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

NOTE: "In a typical school, there would be graduate level courses on Smooth Manifods and anoter on Remannian Manifolds."