Functional Analysis
– Spring 2024

Paul Carmody Assignment #2– March 1, 2024

p. 81 #7. If dim $Y < \infty$ in Riesz's lemma 2.5-4, show that one can even choose $\theta = 1$.

- p. 101 #3, 5, 6, 7, 8, 9.
- 3. If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that ||x|| < 1 we have the strict inequality ||Tx|| < ||T||.
- 5. Show tyhat the operator $T: \ell^{\infty} \to \ell^{\infty}$ defined by $y = (\eta_i) = Tx, \eta_j = \xi/j, x = (\xi_j)$, is linear and bounded.
- 6. (Range) Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T: X \to Y$ need not be closed in Y. Hint. Use T in Prob 5.
- 7. (Inverse operator) Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$||Tx|| \ge b ||x|| \text{ for all } x \in X$$

show that then $T^{-1}: Y - tox$ exists and is bounded.

- 8. Show that the inverse $T^{-1}: \mathcal{R}(T) \to X$ of a bounded linear operator $T: X \to Y$ need not be bounded. *Hint*. Use T in Prob. 5.
- 9. Let $T: C[0,1] \to C[0,1]$ be defined by

$$y(t) = \int_0^1 x(\tau)d\tau.$$

Find $\mathcal{R}(T)$ and $T^{-1}:\mathcal{R}(T)\to C[0,1]$. Is T^{-1} linear and bounded?

- p. 109 #2, 3, 4.
- 2. Show that the functionals defined on C[a, b] by

$$f_1(x) = \int_a^b x(t)y_0(t)dt \qquad (y_o \in C[a, b])$$

$$f_2(x) = \alpha x(a) + \beta x(b) \qquad (\alpha, \beta \text{ fixed})$$

are linear and bounded.

3. Find the norm of the linear functional f defined on C[-1,1] by

$$f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt.$$

4. Show that

$$f_1(x) = \max_{t \in J} x(t)$$

$$J = [a, b]$$

$$f(2) = \min_{t \in J} x(t)$$

define functionals on C[a, b]. Are they linear? Bounded?