Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Section 4 problems:

Within the section: 4.3 (p.37): **(A basis for 3-covectors).** Let x^1, x^2, x^3, x^4 be the coordinates in $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

$$\Phi = \{ \, dx_p^i \wedge dx_p^j \wedge dx_p^k \ : i < j < k \le 4 \, \}$$
$$\{ \, dx_p^1 \wedge dx_p^2 \wedge dx_p^3,$$
$$dx_p^1 \wedge dx_p^2 \wedge dx_p^4,$$
$$dx_p^1 \wedge dx_p^3 \wedge dx_p^4,$$
$$dx_p^2 \wedge dx_p^3 \wedge dx_p^4 \, \}$$
$$|\,\Phi\,| = \binom{4}{3} = 4$$

Within the section: 4.4 (p.38), **Wedge product of a 2-form with a 1-form.** Let ω be a 2-form and τ be a 1-form on on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z

$$(\omega \wedge \tau)(X, Y, Z) = (\omega \otimes \tau)(X, Y, Z) - (\tau \otimes \omega)(X, Y, Z)$$
$$= \omega(X)\tau(Y, Z) - \tau(X, Y)\omega(Z)$$
$$(\omega \wedge \tau)(X, Y, Z) = \frac{1}{1!2!} A(\omega \otimes \tau)(X, Y, Z))$$
$$= \frac{1}{2}\,(\,\omega(X,Y)\tau(Z) + \omega(Y,Z)\tau(X) + \omega(Z,X)\tau(Y) - \omega(Z,Y)\tau(X) - \omega(Y,X)\tau(Z) - \omega(X,Z)\tau(Y)\,)$$
$$= \omega(X,Y)\tau(Z) + \omega(Y,Z)\tau(X) + \omega(Z,X)\tau(Y)$$

Within the section: 4.9 (p.40) **A closed 1-form on the punctured plane.** Define a 1-form on ω on $\mathbb{R}^2 - \{0\}$ by
$$\omega = \frac{1}{x^2 + y^2}(-ydx - xdy).$$

Show that ω is closed.

$$d\omega = \frac{\partial \omega}{\partial x}dx + \frac{\partial \omega}{\partial y}dy$$
$$= \left(\frac{-2x}{(x^2+y^2)^2}(-ydx - xdy) + \frac{1}{x^2+y^2}(-yd^2x - dydx) \right) dx +$$
$$\left(\frac{-2y}{(x^2+y^2)^2}(-ydx - xdy) + \frac{1}{x^2+y^2}(-dxdy - xd^2y) \right) dyx`x$$

End of the section: 1 through 6.

4.1 **A 1-form on $\mathbb{R}^3$.**

Let ω be the 1-form $zdx - dz$ and let X be the vector $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Computer $\omega(X)$ and $d(\omega)$.

$$\omega(X) = (zdx - dz)\,(\,y\partial/\partial x + x\partial/\partial y\,)$$
$$= (zdx - dz)\,(\,y\partial/\partial x\,) + (zdx - dz)\,(\,x\partial/\partial y\,)$$
$$= zy\frac{\partial}{\partial x}dx - y\frac{\partial}{\partial x}dz + zx\frac{\partial}{\partial y}dx - x\frac{\partial}{\partial y}dz \qquad\qquad \text{recall } \frac{\partial}{\partial x^i}dx^j = \delta_i^j$$
$$= zy$$

$$d(\omega) = d(zdx - dz) = d(zdx) - d^2z = dz \wedge dx + z \wedge d^2x = dz \wedge dx$$

4.2 A 2-form on $\mathbb{R}^3$ At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\mathbf{a}, \mathbf{b}) = \omega_p\left(\left(\begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array}\right), \left(\begin{array}{c} b^1 \\ b^2 \\ b^3 \end{array}\right)\right) = p^3 \det\left(\begin{array}{cc} a^1 & b^1 \\ a^2 & b^2 \end{array}\right),$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternaing bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

$$\omega(p) = c_{xy}(p)(dx \wedge dy) + c_{yz}(p)(dy \wedge dz) + c_x z(p)(dx \wedge dz)$$

$$c_{xy}(p) = \omega_p(e_x, e_y) = p^3 \left(\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{array}\right) = p^3 \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 0\right)$$

$$c_{yz}(p) = \omega_p(e_y, e_z) = p^3 \left(\begin{array}{cc} 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{array}\right) = 0$$

$$c_{xz}(p) = \omega_p(e_x, e_z) = p^3 \left(\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & 0 \end{array}\right) = 0$$

notice that $(dx \wedge dy)(a, b) = dx(a)dy(b) - dy(a)dx(b) = a^1 b^2 - a^2 b^1 = \det\left(\begin{array}{cc} a^1 & b^1 \\ a^2 & b^2 \end{array}\right)$ Thus.

$$\omega = p^3\, dx \wedge dy$$

4.3 Exterior Calculus.

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in of dr and $d\theta$.

$$dx = \cos\theta dr - r\sin\theta d\theta$$

$$dy = \sin\theta dr + r\cos\theta d\theta$$

$$\begin{aligned} dx \wedge dy &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\ &= (\cos\theta dr) \wedge (\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\ &= (\cos\theta dr) \wedge (\sin\theta dr) + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + (r\sin\theta d\theta) \wedge (r\cos\theta d\theta) \\ &= 0 + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + 0 \\ &= (\cos\theta dr) \wedge (r\cos\theta d\theta) + (\sin\theta dr) \wedge (r\sin\theta d\theta) \\ &= (r\cos^2\theta)(dr \wedge d\theta) + (r\sin^2\theta)(dr \wedge d\theta) \\ &= r(dr \wedge d\theta) \end{aligned}$$

4.4 Exterior Calculus.

Suppose the standard coordiantes on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

$$dx = \sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta$$

$$dy = \sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta$$

$$dz = \cos\theta\, d\rho - \rho\sin\phi\, d\phi$$

We will attempt to cancel out any terms which have a $dx^i \wedge d^\wedge$ by simplifying dx, dy, and dz in the following manner

$$\begin{aligned} dx \wedge dy \wedge dz &= (x_1\, d\rho + x_2\, d\phi + x_3\, d\theta) \wedge (y_1\, d\rho + y_2\, d\phi + y_3\, d\theta) \wedge (z_1\, d\rho + z_2\, d\phi + z_3\, d\theta) \\ &= (x_1\, d\rho \wedge y_2\, d\phi \wedge z_3\, d\theta) + (x_1\, d\rho \wedge y_3\, d\theta \wedge z_2\, d\phi) \\ &\quad + (x_2\, d\phi \wedge y_1\, d\rho \wedge z_3\, d\theta) + (x_2\, d\phi \wedge y_3\, d\theta \wedge z_2\, d\phi) \\ &\quad + (x_3\, d\theta \wedge y_1\, d\rho \wedge z_2\, d\phi) + (x_3\, d\theta \wedge y_2\, d\phi \wedge z_1\, d\rho) \\ &= (x_1 y_2 z_3 + x_1 y_3 z_2 + x_2 y_1 z_3 + x_2 y_3 z_2 + x_3 y_1 z_2 + x_3 y_2 z_1)(d\rho \wedge d\phi \wedge d\theta) \\ &= \left|\begin{array}{ccc} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{array}\right|(d\rho \wedge d\phi \wedge d\theta) \end{aligned}$$

Solving for the determinant by expanding the bottom row

$$\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & \end{vmatrix} = \rho^2 \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \sin\phi\cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix}$$

$$= \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix}$$

$$= \rho^2\sin\phi \left(\cos\phi \begin{vmatrix} \cos\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta & \cos\theta \end{vmatrix} + \sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\theta \end{vmatrix} \right)$$

$$= \rho^2\sin\phi \left(\cos^2\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \sin^2\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \right)$$

$$= \rho^2\sin\phi \left(\cos^2\phi + \sin^2\phi \right)$$

$$= \rho^2\sin\phi$$

That is

$$dx \wedge dy \wedge dx = (\rho^2\sin\phi)\, dr \wedge d\phi \wedge d\theta$$

4.5 Wedge Product. Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\alpha = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$$
$$\beta = b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

The resulting expression $\alpha \wedge \beta \in \Omega^3(\mathbb{R}^3)$. The $\dim(\Omega^3(\mathbb{R}^3)) = 1$. Thus, there will be one term of the form $dx^1 \wedge dx^2 \wedge dx^3$. Further by distributing the terms of α across the terms of β and ignoring any terms where any two elements are equal, i.e., $dx^i \wedge dx^i = 0$. We will then have

$$\alpha \wedge \beta = a_1\, dx^1 \wedge (b_1\, dx^2 \wedge dx^3) + a_2\, dx^2 \wedge (b_2\, dx^3 \wedge dx^1) + a_3\, dx^3 (b_3\, dx^1 \wedge dx^2)$$
$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3$$

4.6 Wedge product and cross product

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under the correspodence, the wedge product of 1-covectors correponds to the cross product of vectors $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha \wedge \beta} = v_\alpha \times v_\beta$.

First the cross product

$$v_\alpha \times v_\beta = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)$$

then the tensor product and ignoring any $a^i \wedge a^i$ terms when we expand them

$$\alpha \wedge \beta = (a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3) \wedge (b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3)$$
$$= a_1\alpha^1 \wedge b_2\alpha^2 + a_1\alpha_1 \wedge b_3\alpha_3 + a_2\alpha^2 \wedge b_1\alpha^1 + a_2\alpha^2 \wedge b_3\alpha^3 + a_3\alpha^3 \wedge b_1\alpha^1 + a_3\alpha^3 \wedge b_2\alpha^2$$
$$= \underbrace{(a_1 b_2 - a_2 b_1)}_{k^{\text{th}} \text{ of } v_\alpha \times v_\beta}\alpha^1 \wedge \alpha^2 + \underbrace{(a_2 b_3 - a_2 b_3)}_{i^{\text{th}} \text{ of } v_\alpha \times v_\beta}\alpha^2 \wedge \alpha^3 + \underbrace{(a_1 b_3 - a_3 b_1)}_{j^{\text{th}} \text{ of } v_\alpha \times v_\beta}\alpha^1 \wedge \alpha^3$$