Math 725 – Advanced Linear Algebra

Paul Carmody

Assignment #7 – Due 10/25/23

1. Let V be an n-dimensional vector space over the field F. What is the characteristic and minimal polynomials of the zero operator on V ? What is the characteristic and minimal polynomials of the identity operator on V?

$$\text{Let } T \equiv 0$$
$$\det(xI - T) = \det(xI) = x^n = 0$$

0 is the only eigenvalue. The characteristic polynomial is x^n and the minimum polynomial is x.

$$\begin{aligned} \text{Let } T(v) &= v, \text{ for all } v \in V \\ [T]_B^B &= I, \text{ for any basis } B \\ \det(xI - T) &= \det(xI - I) = (x-1)^n \\ \text{Notice } p(x) &= x - 1 \implies p(T) = T - I \equiv 0 \end{aligned}$$

1 is the only eigenvalue. The characteristic polynomial is $(x-1)^n$ and the minimum polynomial is $x - 1$.

2. Let $A, B \in \mathcal{M}_{n\times n}(F)$. Prove that AB and BA have the same eigenvalues.

If A, B are both diagonalizable then let A', B' be the 'diagonalized' versions of A, B, respectively. Then $AB = A'B' = B'A' = BA$, hence, commutative. Thus $\det(xI - AB) = \det(xI - BA)$.

$$AB_{ij} = \sum_{k=1}^{n} a_{kj} b_{jk}$$

3.a) Let $N \in \mathcal{M}_{2\times 2}(F)$ such that $N^2 = 0$. Show that either $N = 0$ or it is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

b) Suppose A is a 2×2 matrix with complex entries. Prove that A is similar to either $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$.

4.a) Let $A = \begin{pmatrix} 0 & 0 & 0 & \cdots & -a_0 \\ 1 & 0 & 0 & \cdots & -a_1 \\ 0 & 1 & 0 & \cdots & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$. What is the characteristic polynomial of A? [Hint: use induction]. Conclude that any monic polynomial is the characteristic polynomial of some matrix.

b) Show that the minimal polynomial and the characteristic polynomial of $\begin{pmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{pmatrix}$ are equal.

5. Prove that any square matrix A where $A^2 = A$ is diagonalizable.

6. Let T be a linear operator an a finite dimensional vector space V. Show that if every subspace of V is T-invariant, then T is a scalar multiple of the identity operator.

7. True or false?: if a triangular matrix is similar to a diagonal matrix then it is already a diagonal matrix.

Extra Questions

1. Let V be the vector space of all real valued continuous functions on the real line. Let T be the operator defined by $(Tf)(x) = \int_0^x f(t)\,dt$. Prove that T does not have an eigenvalue.

2. Let D be the differentiation operator on $\mathcal{P}^{(n)}$. Compute the characteristic and minimal polynomials of D.

3. Prove that a square matrix A and its transpose A^t have the same eigenvalues.

4. Let A be an $n \times n$ matrix with complex entries. Suppose λ is an eigenvalue of A and $v = (v_1, \ldots, v_n)$ an eigenvector of A. Let k be an index where $|v_k| \geq |v_i|$ for all $i = 1, \ldots, n$. Prove that $|\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{kj}|$. In other words, the eigenvalue λ lies in a disk in the complex plane with center at a_{kk} and radius $\sum_{j \neq k} |a_{jk}|$. Of course, most of the time we know neither the eigenvalues nor the associated eigenvectors. Therefore, we consider the region $R(A)$ in the complex plane that is the union of these disks for each row $k = 1, \ldots, n$. Moreover, since A and A^t have the same eigenvalues (see the previous exercise), we have n disks obtained from the columns of A. So also consider the region $C(A)$ in the complex plane that is the union of the disks obtained from each column. The region $G(A) = R(A) \cap C(A)$ is known as the Gersgorin region of A and contains all the eigenvalues of A.

5. We call a matrix $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ *strictly row dominant* if $|a_{kk}| > \sum_{j \neq k} |a_{kj}|$ for every $k = 1, \ldots, n$. Show that a strictly row dominant matrix is invertible. [Hint: do the above exercise first].