Math 5050 – Special Topics: Differential Geometry– Fall 2025
w/Professor Berchenko-Kogan

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Section 17: Differential Forms– August 20, 2025

17.1. A 1-form on $\mathbb{R}^2 - \{(0,0)\}$

Denote the standard coordinates on $\mathbb{R}^2$ by x, y, and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \text{ and } Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector files on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

17.2. Transition formula for 1-form

Suppose $(U, x^1, \dots, X^n)$ and $(V, y^1, \dots, y^n)$ are two charts on M with nonempty overlap $U \cap V$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum a_j dx^j = \sum b_i dy^i.$$

Finda formula for a_j in terms of b_i.

17.3. Pullback of a 1-form on S^1

Multiplication in the unit circle S^1, viewed as a subset of the complex plane, is given by

$$e^{it} \cdot e^{i(t+u)},\ t, u \in \mathbb{R}$$

In terms of real imaginary parts,

$$(\cos t + i \sin t)(x + iy) = ((\cos t)x - (\sin t)y) + i((\sin t)x + (\cos t)y).$$

Hence, if $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$, then the left multiplication $\ell_g : S^1 \to S^1$ is given by

$$\ell_g(x, y) = ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y).$$

Let $\omega = -ydx + xdy$ be the 1-form found in Example 17.15. Prove that $\ell_g^*\omega = \omega$ for all $g \in S^1$.

17.4. Liouville form on the cotrangent bundle

(a) Leet $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart on a manifold M, and let

$$(\pi^{-1}U, \tilde{\phi}) = (\pi^{-1}U, \tilde{x}^1, \dots, \tilde{x}^n, c_1, \dots, c_n)$$

be the induced chart of the cotangent bundle T^*M. Find a formula for Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\tilde{x}^1, \dots, \tilde{x}^n, c_1, \dots, c_n$.

(b) Prove that the Liouville form λ on $T * M$ is C^∞. (*Hint*: Use (a) and Proposition 17.6)

17.5. Pullback of a sum and a product

Prove Proposition 17.11 by verifying both sides of each equality on a trangent vector X_p at a point p.

17.6. Construction of the cotangent bundle

let M be a manifold of dimension n. Mimicking the construction of the tangent bundle in Section 12, write out a detailed proof that $\pi : T^*M \to M$ is a $C\infty$ vector bundle of rank n