Math 5301 – Numerical Analysis– Spring 2025
w/Professor Du

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Homework #3 – February 27, 2025

Question 1. (20 points)

Consider the function $f(x) = \cos(x)$ with the domain of $[-1, 1]$:

(a) Approximate $f(x)$ with the 4^{th} order polynomial $P_4(x)$ obtained from Taylor Expansion. Estimate the error bound for $|f(x) - P_4(x)|$ and compare the actual largest error with the error bound.

$$\begin{aligned}
P_4(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \\
|f(x) - P_4(x)|_{\max} &\le \left| \frac{f^{(5)}(\zeta(x))}{5!} \right| \le \left| \frac{-\cos x}{120} \right|_{\max[-1,1]} = \frac{1}{120} \\
&= 0.008333333 && \text{error bound} \\
|f(1) - P_4(1)| &= |0.54869 - 0.54166| \\
&= 0.00703 && \text{actual error}
\end{aligned}$$

(b) Construct the 3^{rd} order polynomial $P_3(x)$ that is the closest for $P_4(x)$, compute the error bound for $|P_4(x) - P_3(x)|$.

$$\begin{aligned}
P_3(x) &= 1 - \frac{x^2}{2} \\
|P_3(x) - P_4(x)| &= \frac{x^4}{24} \\
|P_3(1) - P_4(1)| &= \frac{1}{24} = 0.041667
\end{aligned}$$

(c) Based on (a) and (b), compute the error bound for $|f(x) - P_4(x)|$. Compare the real error with this error bound.

Question 2. (20 points)

Consider 1D Poisson's equation $-u_{xx} = 1 + x$ with boundary conditions $u(0) = 0$ and $u(1) = 0$:

(a) Find the solution analytically over the doman $[0, 1]$.

$$\int u_x x dx = -\int 1 + x dx$$

$$u_x + C = -(x + \frac{1}{2}x^2)$$

$$\int u_x + C dx = -\int (x + \frac{1}{2}x^2) dx$$

$$u(x) + Cx + D = -\left(\frac{x^2}{x} + \frac{x^3}{6}\right)$$

$$u(x) = -\frac{1}{6}x^3 - \frac{1}{2}x^2 - Cx - D$$

$$u(0) = D = 0$$

$$U(1) = 0$$

$$= -\left(\frac{1}{2} + \frac{1}{6} + C\right)$$

$$C = -\frac{1}{3}$$

$$u(x) = -\frac{1}{6}x^3 - \frac{1}{2}x^2 - \frac{1}{3}x$$

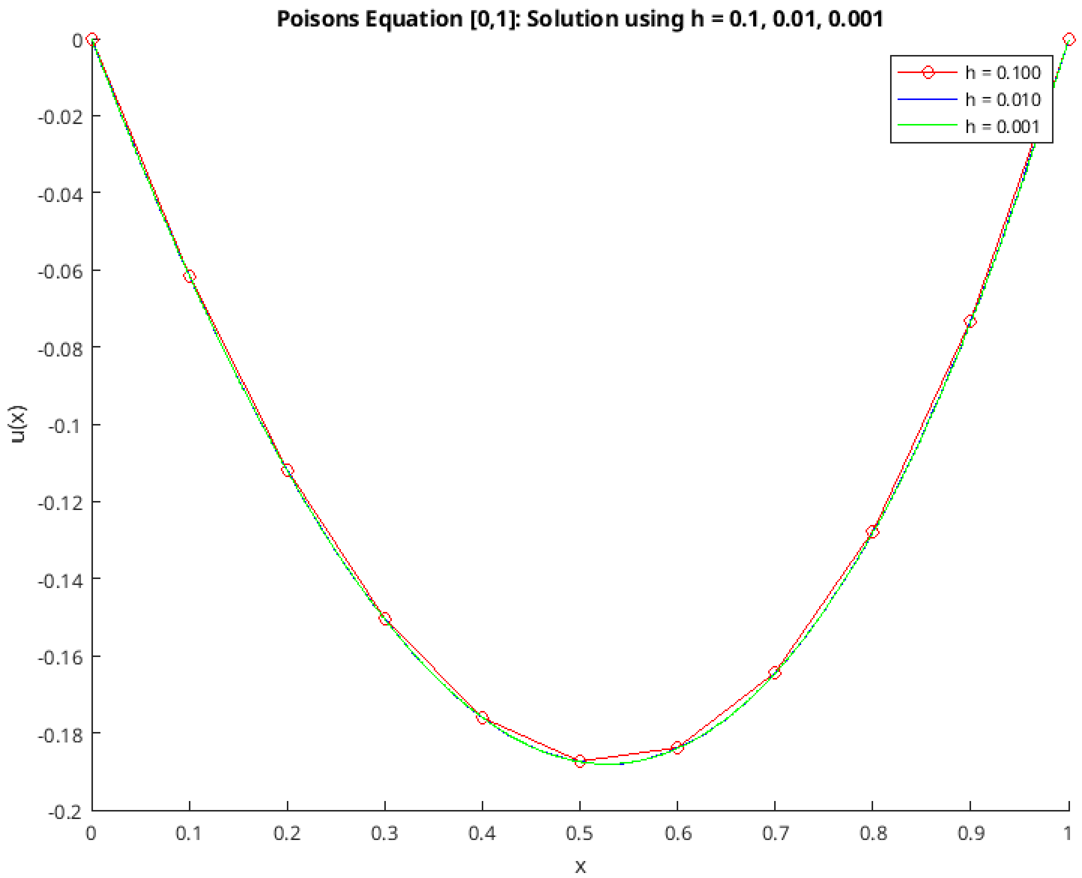
(b) Solve the equation numerically by centered difference discretization, with $h = 0.1, 0.01$, and 0.001.

The Source Code:

```
clc; clear; close all;

h_values = [0.1, 0.01, 0.001];

figure; hold on;

for h = h_values

    x = 0:h:1;
    N = length(x) - 2;

    % Construct finite difference matrix A
    A = (1/h^2) * (diag(-2*ones(N,1)) + diag(ones(N-1,1),1) + diag(ones(N-1,1),-1));
    b = 1 + x(2:end-1);

    % Solve the linear system A*u = b
    u = A \ reshape(b, [], 1);

    % Include boundary values (assuming Dirichlet B. C. u(0) = u(1) = 0)
     u_full = [0; u; 0];

    % Plot the solution
     if h==0.1
         plot_color = '-or';
     elseif h==0.01
         plot_color = '-b';
     else
         plot_color = '-g';
    end
     plot(x, u_full, plot_color, 'DisplayName', sprintf('h = %.3f', h))
     xlabel('x');
```

```
    ylabel('u(x)');
    title(sprintf('Poisons Equation [0,1]: Solution using h = 0.1, 0.01, 0.001'));
    legend;
end
```

The output



(c) Plot the error distributions over space, as well as 2-norms of the error vs. h. Analyze the order of accuracy of your discretization.

$$\| A \|_2 = \max_{||x||_2=1} \| Ax \|_2 = \sqrt{\left(\sum_{i=1}^{N} A_{ii}^2\right)}$$

```
clc; clear; close all;

h_values = [0.1, 0.01, 0.001];

figure; hold on;

for h = h_values

    x = 0:h:1;
    N = length(x) - 2;

    A = (1/h^2) * (diag(-2*ones(N,1)) + diag(ones(N-1,1),1) + diag(ones(N-1,1),-1))
    ;
    b = 1 + x(2:end-1);

    u = A \ reshape(b, [], 1);

```

```
        u_full = [0; u; 0];

        if h==0.1
            plot_color = '-r';
        elseif h==0.01
            plot_color = '-b';
        else
            plot_color = '-og';
        end
        e=reshape(eig(A), [], 1);
        e_full = [0; e ;0];
        plot(x, e_full, plot_color, 'DisplayName', sprintf('h = %.3f', h))
        xlabel('x');
        ylabel('u(x)');
        title(sprintf('Poisons Equation [0,1]: Error Distribution h = 0.1, 0.01, 0.001'));
        legend;
    end
```