

Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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I. This problem review continuity for functions on real line.

We say a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

- (a) Show that $f(x) = x^2$ is continuous at $x = 2$.

Given an $\epsilon > 0$, when $|f(x) - 4| < \epsilon$, $|x^2 - 4| < \epsilon$. Let $\delta < \sqrt{\epsilon + 4}$
If

$$(2 + \delta)^2 - 4 < \epsilon$$

$$(2 + \delta)^2 < \epsilon + 4$$

$$2 + \delta < \sqrt{\epsilon + 4}$$

$$\delta < \sqrt{\epsilon + 4} - 2$$

- (b) Suppose that f is continuous at a and $f(a) \neq 0$. Show that f is nonzero in some open interval containing a .

Since f is continuous at a and $f(a) \neq 0$ then for every $\epsilon > 0$ such that when $|f(x) - f(a)| < \epsilon$. Without loss of generality, assume $f(a) > 0$. Choose $\epsilon < f(a)$ then $0 < f(a) - \epsilon < f(x) < f(a) + \epsilon$. Therefore, $f(x) \neq 0$

II. This problem review derivatives.

- (a) Let $f(x) = x^n$ for some positive integer n . Using the definition of the derivative, and the binomial theorem, show that $f^{(n-1)}$.
- (b) Is the function

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x \geq 0 \end{cases}$$

differentiable at $x = 0$.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$f(x)$ is both continuous and differentiable at $x = 0$.

III. This problem reviews sup and inf.

For any subset $A \subset \mathbb{R}$, we say that M is an *upper bound* for A if $x \leq M$ for all $x \in A$. If a set A has a finite upper bound, we say it is *bounded above*. It is a theorem about the set \mathbb{R} that *for any set $A \subset \mathbb{R}$ that is bounded above, there exists a least (smallest) upper bound for A* . This least upper bound is called supremum of A , and denoted $\sup A$. By definition, the number $\sup A$ has two properties.

- (i) $x \leq \sup A$ for all $x \in A$ (i.e., $\sup A$ is an upper bound for M).
- (ii) for any M that is an upper bound for A , we have $\sup A \leq M$.

For sets that are not bounded above, we say that $\sup A = +\infty$. we often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set $\{f(x) : x \in A\}$, where f is a some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted $\inf A$. It satisfies the same properties as $\sup A$ with the inequalities reversed.

- (a) Find $\sup A$ and $\inf A$ for $A = (1, 2]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, and $A = \{0, 1, 2, 3, \dots\}$.

- $A = (1, 2]$, $\sup A = 2$ and $\inf A = 1$
- $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $\sup A = 1$, and $\inf A = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$.
- $A = \{0, 1, 2, 3, \dots\}$. $\sup A = \lim_{n \rightarrow \infty} = \infty$, and $\inf A = 0$

- (b) Find $\sup_{x \in (0,1)} (1+x^2)^{-1}$

Let $f(x) = (1+x^2)^{-1}$. On the interval $(0, 1)$ we can see that it is strictly decreasing, that is $a < b \implies f(a) > f(b)$. Thus, $\sup_{x \in (0,1)} f(x) = f(0) = (1+0^2)^{-1} = 1$.

- (c) Assume that $\sup A < \infty$, and show that for every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$.

Given any $\epsilon > 0$ let $x > \sup A - \epsilon$. If $x \notin A$ then x is an upper bound of A , i.e., $x \in M$ and $x < \sup A$, but that violates proper (ii). Hence, $x \in A$.

- (d) For any two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$, and any set $A \subset \mathbb{R}$, show that $\sup_{x \in A} (f(x) + g(x)) \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$.

$$\begin{aligned} f(x) &\leq \sup_{x \in A} f(x) \text{ and } g(x) \leq \sup_{x \in A} g(x), \forall x \in x \in A \\ \therefore f(x) + g(x) &\leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x), \forall x \in A \\ \text{and } \sup_{x \in A} (f(x) + g(x)) &\leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x) \end{aligned}$$

IV. Section 1.1, Exercise 5, 6, 13.

Exercise 1.1.5. Let $n \geq 1$, and let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be real numbers verify the identity

$$\left(\sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right), \quad (1.3)$$

and conclude *Cauchy-Schwarz inequality*

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2}$$

Let's start by expanding the center term

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \sum_{i=1}^n \sum_{j=1}^n ((a_i b_j)^2 + (a_j b_i)^2 - 2(a_i b_j a_j b_i)) \\ &= \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2 + \sum_{i=1}^n b_i^2 \sum_{j=1}^n a_j^2 - 2 \sum_{i=1}^n a_i b_i \sum_{j=1}^n a_j b_j \\ &= 2 \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) - 2 \left(\sum_{i=1}^n a_i b_i \right)^2 \end{aligned}$$

Equation 1.3 then becomes

$$\begin{aligned} \left(\sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \\ \left(\sum_{i=1}^n a_i b_i \right)^2 + \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) - \left(\sum_{i=1}^n a_i b_i \right)^2 &= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \end{aligned}$$

which is true. Since

$$\begin{aligned} \left(\sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \\ \left(\sum_{i=1}^n a_i b_i \right)^2 &= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 \\ \therefore \left| \sum_{i=1}^n a_i b_i \right| &\leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2} \end{aligned}$$

Let's start by taking the square of the distance from $a + b$ to zero using the ℓ^2 .

$$\begin{aligned} d_{\ell^2}(a + b, 0)^2 &= \sum_{i=1}^n (a_i + b_i)^2 \\ &= \sum_{i=1}^n (a_i^2 + b_i^2 + 2a_i b_i) \\ &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \end{aligned}$$

apply Cauchy-Schwarz and factor.

$$\begin{aligned}
 d_{\ell^2}(a+b, 0)^2 &\leq d_{\ell^2}(a, 0) + d_{\ell^2}(b, 0) + 2 \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{j=1}^n b_j^2 \right)^{1/2} \\
 &\leq d_{\ell^2}(a, 0) + d_{\ell^2}(b, 0) + 2 (d_{\ell^2}(a, 0) \cdot d_{\ell^2}(b, 0))^{1/2} \\
 &\leq \left(d_{\ell^2}(a, 0)^{1/2} + d_{\ell^2}(b, 0)^{1/2} \right)^2
 \end{aligned}$$

Expand the ℓ^2 metrics and take the square root of both sides and

$$\left(\sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2}$$

Exercise 1.1.6 Show that (\mathbb{R}^n, d_{l^2}) in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5)

Example 1.1.6 (Euclidean spaces). Let $n \geq 1$ be a natural number, and let \mathbb{R}^n be the space of n -tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}$$

We define the *Euclidean metric* (also called the l^2 metric) $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \end{aligned}$$

We must prove that d_{l^2} is symmetric, positive definite and that the triangle inequality holds.

Symmetric: show that $d_{l^2}(x, y) = d_{l^2}(y, x)$.

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^n (y_i - x_i)^2 \right)^{1/2} \\ &= d_{l^2}((y_1, \dots, y_n), (x_1, \dots, x_n)) \end{aligned}$$

Positive Definite: show that $d_{l^2}(x, y) \geq 0$ and $d_{l^2}(x, y) = 0 \rightarrow x = y$.

The square root is taken as a positive value. $d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) = 0$ implies that each $x_i - y_i = 0$ therefore $x = y$.

Triangle Inequality: show that $d_{l^2}(x, z) \leq d_{l^2}(x, y) + d_{l^2}(y, z)$

Exercise 1.1.5 proves the triangle inequality replacing $a_i = x_i$ and $b_i = y_i$.

Exercise 1.1.13 Prove Proposition 1.1.19.

Proposition 1.1.19 (Convergence in the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X . Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X , and let x be a point in X . Then $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc} if and only if there exists $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.*

Remember that:

$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

(\Rightarrow) assume that $x^{(n)} \rightarrow x$ under d_{disc} . Then, for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d_{\text{disc}}(x^{(n)}, x) < \epsilon$.

Clearly, $d_{\text{disc}}(x^{(n)}, x)$ can be equal to either 1 or 0. Thus, $d_{\text{disc}}(x^{(n)}, x) = 0$ or $d_{\text{disc}}(x^{(n)}, x) = 1$ and hence true for all $n > N$.

(\Leftarrow) assume that $\exists N > m$ such that when $n > N$, $x^{(n)} = x$. Given any $\epsilon > 0$ and $n > N$ we can see that $d_{\text{disc}}(x^{(n)}, x) = 0 < \epsilon$. Therefore $x^{(n)} \rightarrow x$.

V. For this problem only, you do not need to give proofs. Just write the answers.

For each set, identify the boundary, interior, and closure of A , and say whether A is open, closed, both or neither. We are working in \mathbb{R}^2 with standard distance. Unless otherwise noted, the ambient space is \mathbb{R}^2 .

- (a) $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$.

Boundary: $\partial A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$

Interior: $A^\circ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$

Closure: $\overline{A} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1\}$

- (b) $A = \{(1/n, 2/n) : n = 1, 2, 3, \dots\}$ (Note: $(1/n, 2/n)$ is a vector in \mathbb{R}^2 , not an open interval in \mathbb{R} .)

Boundary: $\partial A = A$

Interior: $A^\circ = A$

Closure: $\overline{A} = A$

- (c) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$, in the relative topology with respect to $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$.

Boundary: $\partial_Y A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, d(x, 0) = 1\}$ the right semi-circle combined with the y-axis from 1 to -1.

Interior: $A^\circ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) < 1\}$

Closure: $\overline{A} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, d(x, 0) \leq 1\}$

VI. Let (X, d) be a metric space.

- (a) For a given point $x_0 \in X$, show the singleton set $\{x_0\}$ is closed.
 (b) Let $x_0 \in X$ and $r > 0$. Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open.