Math 5050 – Special Topics: Differential Equations– Fall 2025
w/Professor Berchenko-Kogan

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Section 16: Lie Algebras – August 13, 2025

Notes:

Examples

Problems

16.1. Skew-Hermitian Matrices

A complex matrix $X \in \mathbb{C}^{n\times n}$ is said to be ***skew-Hermitian*** if its conugat transpose $\hat{X}^T$ is equal $-X$. Let V be the vector space of $n \times n$ skew-Hermitian matrices. Show that $\dim V = n^2$.

16.2. Lie algebra of a unitary group

Show that the tangent space at the identity I of the unitary group $U(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

16.3. Lie algebra of a symplectic group

Refer to Problem 15.15 for the definition and notation concerning the symp[lectic group $\mathrm{Sp}(n)$. Show that the tangent space at teh identity I of the symplectic group $\mathrm{Sp}(n) \subset \mathrm{GL}(n, \mathbb{H})$ is the vector space of all $n \times n$ quaternionic matrices X such that $\hat{X}^T = -X$.

16.4. Lie algebra of a complex symplectic group

(a) Show that the tangent space at the identity I of $\mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{GL}(wn, \mathbb{C})$ is the vector space of all $2n \times 2n$ complex matrices X such that JX is symmetric.

(b) Calculate the dimension of $\mathrm{Sp}(2n, \mathbb{C})$.

16.5. left-invariant vector fields on $\mathbb{R}^n$

Find the left-invariant vector fields on $\mathbb{R}^n$.

16.6. Left-invariant vector fileds on a circle

Find the left-invariant vector fields on S^1.

16.7. Integral curves of a left-invariant vector field

Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A. Show that $c(t) = e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I. Find the integral curve of $\tilde{A}$ starting at $g \in \mathrm{GL}(n, \mathbb{R})$.

16.8. Parallelizable manifolds

A manifold whose tangent bundle is trivial is said to be ***parallelizable***. If M is a manifold of dimension n, show that parallelizability is equivalent to the existence of a smooth frame $X_1, \ldots, X_n$ on M.

16.9. Parallelizability of a Lie group

Show that every Lie group is parallelizable.

16.10. The pushforward of left-invariant vector fields

Let $F : H \to G$ be a Lie group homomorphism and let X and Y be left-invariant vector fields on H. Prove that $F_x[X, Y] = f_*X, F_*Y]$.

16.11. The adjoint representation

Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$

(a) For each $a \in G$, the differential at the identity of the conjugation map $c_a := \ell_a \circ r_{a^{-1}} : G \to G$ is a linear isomorphism $c_{a*} : \mathfrak{g} \to \mathfrak{g}$. Hence, $c_{a*} \in \mathrm{GL}(\mathfrak{g})$. Show that the map $\mathrm{Ad} : G\, \mathrm{GL}(\mathfrak{g})$ define by $\mathrm{Ad}(a) = c_{a*}$ is a group homomorphism. It is called the ***adjoint representation*** of the Lie group G.

(b) Show that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is C^∞.

16.12. A Lie algebra structure on $\mathbb{R}^3$

The Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $O(n)$ is the Lie algebra fo $n \times n$ skew-symmetric real matrices, with Lie bracket $[A, b] = AB - BA$. When $n = 3$, there is a vector space isomorphism $\varphi : \mathfrak{o}(3) \to \mathbb{R}^3$,

$$\varphi(A) = \varphi \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \\ a_3 \end{pmatrix} = a.$$

Prove that $\varphi([a, b]) = \varphi(A) \times \varphi(B)$. Thus $\mathbb{R}^3$ with the cross product is a Lie algebra.