

Math 5102 – Linear Algebra – Fall 2024
w/Professor Pendera

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Homework #2 – NONE

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9. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generates $M_{2 \times 2}(F)$.

$$\text{Let } v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Given any $A \in M_{2 \times 2}(F)$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in F$$

It is easy to see that

$$A = av_1 + bv_2 + cv_3 + dv_4.$$

Hence any element of $M_{2 \times 2}(F)$ can be described as a linear combination of $\{v_1, v_2, v_3, v_4\}$.

10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then $\text{span}\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Given any symmetric matrix B

$$\begin{aligned} B &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \\ &= aM_1 + bM_2 + cM_3 \end{aligned}$$

thus $\text{span}\{M_1, M_2, M_3\}$ is the set symmetric matrices. Moreover, $\text{span}\{M_1, M_2, M_3\}$ cannot span the set of $M_{2 \times 2}$ matrices as the dimensions are different.

12. Show that a subset W of a vector space V is a subspace of V if and only if $\text{span}(W) = W$.

Show that $\text{span}(W) \subseteq W$

Let $w \in \text{span}(W)$ then there exist a_i such that $w = \sum_{i=1}^n a_i w_i$ for some $w_i \in W$. These represent a linear combination of elements of W .

Show that $\text{span}(W) \supseteq W$

13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\text{span}(S_1) = V$, deduce that $\text{span}(S_2) = V$.

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6. In $M_{m \times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the i^{th} row and j^{th} column. Prove that $\{E^{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.
7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $M_{2 \times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.
11. Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space V over the field \mathbb{Z}_2 . How many vectors are there in $\text{span}(S)$? Justify your answer.
19. Prove that if $\{A_1, A_2, \dots, A_k\}$ is a linearly independent subset of $M_{m \times n}(F)$, then $\{A_1^t, A_2^t, \dots, A_k^t\}$ is also linearly independent.
20. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.