Math 5301 – Numerical Analysis– Spring 2025
w/Professor Du

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Homework #4 – March 26, 2025

Assignment: Consider 1D Poisson Equation $-\Delta x = \sin(\pi x)$, over the region $(0, \pi/2)$ with boundary conditions $u(0) = 0$ and $u(\pi/2) = 1$. Using central difference scheme and a mesh of 128, obtain a linear system of $Au = f$ for the problem, then the solve the system using the following methods until a relative residual of 10^{-4} is reached. For all methods below, plot the analytical solution, the numerical solution and the error distribution. Use zero vectors as your initial guess.

(a) Jacobis's Method. Plot the rate of convergence and compareit with analysis.

(b) Steepest Descent Method. Compare the rate of convergence with that obtained in (a).

Analytical and Numerical Solutions to Poisson's Equation

Introduction:

We will begin by describing the equation and providing an analytical solution. Then solve this equation using Jacobi's Method and the Steepest Descent Method. Comparisons will be made as to accurancy and rate of convergence.

Poison's Equation and an Analytical Solution:

As described in the assignment we will focus our attenion on this Poisson equation and initial conditions.

$$-\Delta x = \sin(\pi x)$$
$$u(0) = 0 \text{ and } u(\pi/2) = 1.$$

When we solve this problem analtyically we get

$$\begin{aligned}
u'(x) &= \int -\sin(\pi x) dx \\
&= \frac{1}{\pi}\cos(\pi x) + C \\
u(x) &= \int \left(\frac{1}{\pi}\cos(\pi x) + C\right) dx \\
&= \frac{1}{\pi^2} sin(\pi x) + Cx + D \\
u(0) = 0 &= \frac{1}{\pi^2} sin(\pi 0) + Cx + D \\
D &= 0 \\
u(\pi/2) = 1 &= \frac{1}{\pi^2} sin(\pi^2/2) + C(\pi/2) \\
C &= \frac{2\left(1 - \frac{1}{\pi^2}\right) sin(\pi^2/2)}{\pi^2} = \frac{2\left(1 - \frac{1}{9.869604064}\right) 0.086022097}{9.869604064} = 1.998233797 \approx 2 \\
u(x) &\approx \frac{1}{\pi^2} sin(\pi x) + 2x
\end{aligned}$$

Centered Different Scheme

We will attempt to approximate the curve of the solution at particular points u_i by calculating a slope at a point by using the preceeding point u_{i-1} and succeeding point u_{i+1}.

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \approx -\sin(\pi x)$$

where $h = (\pi/2)/129 = \pi/258$ (we use $N + 1$ as we start with the left boundary 0). This can be reduced to

$$-u_{i+1} + 2u_i - u_{i-1} = h^2 \sin(\pi x).$$

This expands to a linear function over the a matrix A and vector $u = \{u_i\}$ reflecting the left hand side and the value to our function on the right with $f = \{f_i\}, f_i = \sin(\pi x_i)$ or

$$Au = h^2 f$$

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_i \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ f_i \end{pmatrix}$$

$$u = h^2 A^{-1} f$$

remembering the boundary conditions. Since A is tri-diagonal we can use several methods and compare the cost and efficiency. The MatLab code for the Central Difference Method is

```
clc; clear; close all;

h_values = [0.1, 0.01, 0.001];

figure; hold on;

function [x, dsc] = centered_difference_scheme(mesh)

    h = 1/mesh;
    x = 0:h:pi/2;
    N = length(x) - 2;

    % Construct finite difference matrix A
    A = (1/h^2) * (diag(-2*ones(N,1)) + diag(ones(N-1,1),1) + diag(ones(N-1,1),-1));
    b = sin(pi*x(2:end-1));

    % Solve the linear system A*u = b
    u = A \ reshape(b, [], 1);

    % Include boundary values u(0) = 0, u(pi/2) = 1)
    dsc = [0; u; 1];

end

[x, u_full]= centered_difference_scheme(128);
%analytic = (-1/pi^2)*sin(pi*x);
analytic = (-1/pi^2)*sin(pi*x)+2*x;
% Plot the solution
plot(x, u_full, '-b', 'DisplayName', sprintf('h = %.3f', 1/128))
plot(x, analytic, '-r', 'DisplayName', 'analytic')
xlabel('x');
ylabel('u(x)');
title('Poisons Equation on [0,pi/2]: Solution using 128');
legend;
```

Jacobi's Iteration Method

Since A is a tri-diagonal matrix it can be divided into three separate matrices that add up. Let D be zero everywhere except the diagonal where it will hold the values of A_{ii} (namely all 2s). Let L be the same except that it will hold values of the diagonla lower $A_{i-1,i}$ (all -1) and let U be the same except that it will hold the values of the diagonal above $A_{i,i+1}$ (also all -1). Then, applying this iteratively we get

$$u_i^{[k+1]} = \frac{1}{2}(u_{i-1}^{[k]} + u_{i+1}^{[k]} - h^2 f_i)$$

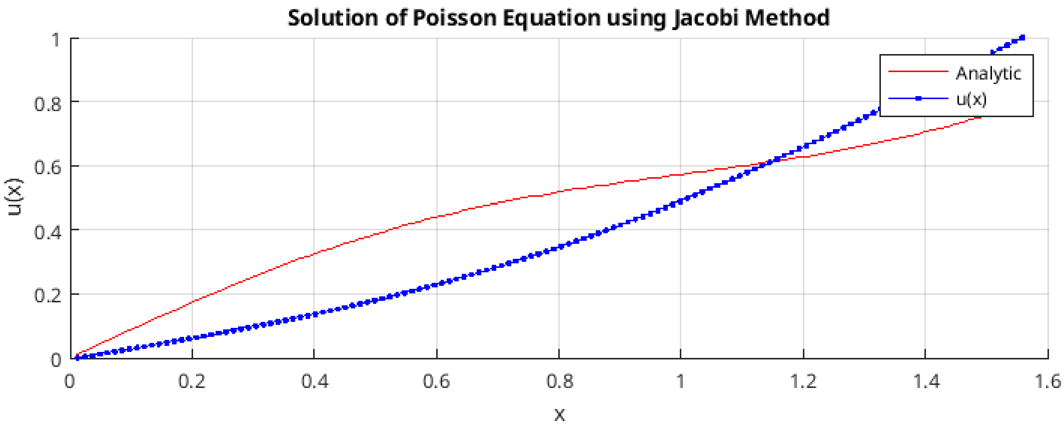
Here is the MatLab code used to generate the graphs that follow:

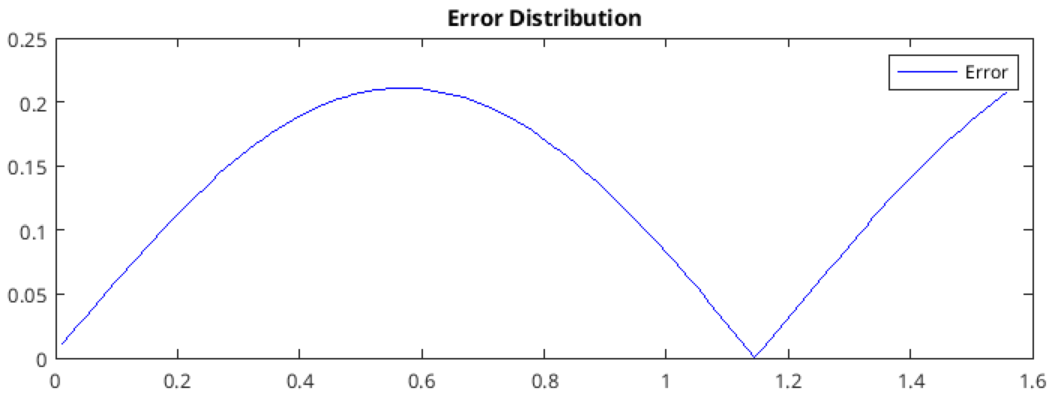
```
clc; clear; close all;

figure; hold on;

```

```
function x = jacobi_method(A, b, h, tol, max_iter)
    % Solves Ax = b using Jacobi's iterative method
    % Inputs:
    %   A        - Coefficient matrix (NxN)
    %   b        - Right-hand side vector (Nx1)
    %   tol      - Convergence tolerance
    %   max_iter - Maximum number of iterations
    % Output:
    %   x - Solution vector (Nx1)

    N = length(b);                  % Number of equations
    x = b;%zeros(N, 1);                 % Initial guess (zero vector)
    x_old = x;                      % Store previous iteration
    h2=h^2;
    i=2:N;

    for k=1:max_iter
        for i=2:N-1
            x(i) = 1/2*( x_old(i-1) + x_old(i+1) - h2*b(i) );
        end

        % Check for convergence
%        if norm(x - x_old, 2) < tol
        if norm(x - x_old, Inf) < tol
            fprintf('Converged in %d iterations.\n', k);
            return;
        end
        x_old = x;  % Update solution
    end

    fprintf('Max iterations reached without convergence.\n');
end
% Define problem parameters
N = 128;                        % Number of internal grid points
h = (pi/2) / (N+1);             % Grid spacing
x = linspace(h, pi/2-h, N)';  % Grid points
f = h^2* sin(pi * x);     % Right-hand side vector

% Construct tridiagonal matrix A
A = 2 * eye(N) - diag(ones(N-1,1),1) - diag(ones(N-1,1),-1);

% Modify last element of f to include boundary condition u(pi/2) = 1
f(1) = 0; f(N) = 1;

% Solve using Jacobi method
tol = 1e-4;  % Convergence tolerance
max_iter = 100000;  % Maximum iterations

u = jacobi_method(A, f, h, tol, max_iter);

grid on;
tiledlayout(2,1);
tile1=nexttile;
hold(tile1,'on');
%Analytic plot
analytic = 1/pi^2*sin(pi*x)+2/pi*(1-1/pi^2)*x;
%analytic = 1/pi^2*sin(pi*x)+2*x;
plot(tile1, x,analytic, '-r', 'DisplayName', 'Analytic');
legend;
% Plot solution
plot(tile1, x, u, 'b.-', 'DisplayName', 'u(x)');
xlabel('x'); ylabel('u(x)');
title('Solution of Poisson Equation using Jacobi Method');
legend;
```

```
grid on;

diff = abs(analytic-u);
tile2=nexttile;
plot(tile2,x,diff,'b-', 'DisplayName', 'Error');
title('Error Distribution');
legend;
```

From this we generate the following graphs.





Steepest Descent Method

This technique for approximating our solution to the Poisson equation is to make each iteration in the direction of greatest change. That is,

$$\nabla\phi(u_{k-1}) = Au_{k-1} - f \equiv -r_{k-1}$$

where $\phi : \mathbb{R}^m \to \mathbb{R}$ of the form

$$\phi(u) = \frac{1}{2}u^T Au - u^T f$$

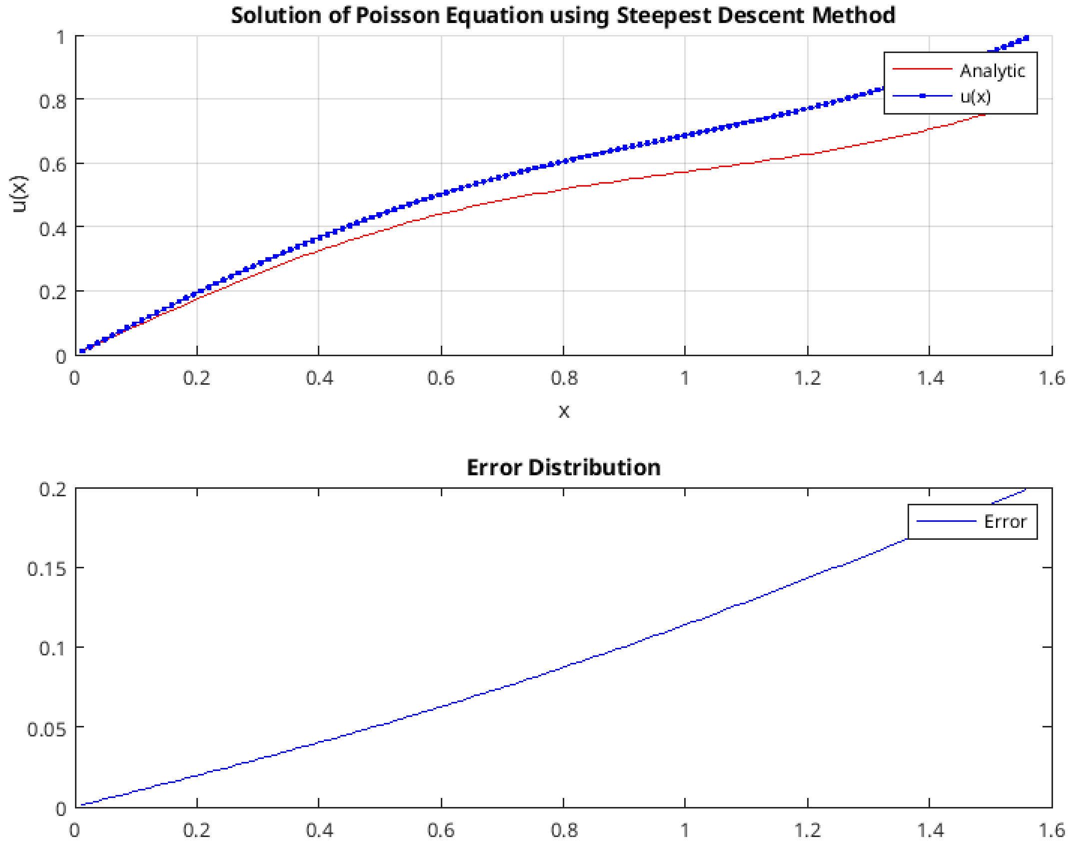
which is a quadratic function in u and can be mapped with local extrema either as a top, a bowl or a saddle point, all based on the eigenvalues of A (negative, positive, or neither, respectively). Thus, when A is SPD we can expect r_k to progress ever closer towards the extremum.

This is the source code

```
clc; clear; close all;

figure; hold on;

function u = steepest_descent(A, f, tol, max_iter)
    % Solves Ax = b using steepest descent method
    % Inputs:
    %   A        - Coefficient matrix (NxN)
    %   b        - Right-hand side vector (Nx1)
    %   tol      - Convergence tolerance
    %   max_iter - Maximum number of iterations
```

```
    % Output:
    %   x - Solution vector (Nx1)

    N = length(f);                % Number of equations
    u = zeros(N, 1);              % Initial guess (zero vector)
    u_old = u;

    for k=1:max_iter
        r_old = f - A*u_old;
        if norm(r_old) < tol
            fprintf('Converged in %d iterations.\n', k);
            return;
        end
        alpha_old = (r_old.' * r_old)/(r_old.' * A * r_old);
        u = u_old +  alpha_old*r_old;
        u_old = u;
    end

    fprintf('Max iterations reached without convergence.\n');
end
% Define problem parameters
N = 128;                    % Number of internal grid points
h = (pi/2) / (N+1);         % Grid spacing
x = linspace(h, pi/2-h, N)';  % Grid points
f = h^2* sin(pi * x);     % Right-hand side vector

% Construct tridiagonal matrix A
A = 2 * eye(N) - diag(ones(N-1,1),1) - diag(ones(N-1,1),-1);

% Modify last element of f to include boundary condition u(pi/2) = 1
f(1) = 0; f(N) = 1;

% Solve using steepest descent method
tol = 1e-4;  % Convergence tolerance
max_iter = 100000;  % Maximum iterations

u = steepest_descent(A, f, tol, max_iter);

grid on;
tiledlayout(2,1);
tile1=nexttile;
hold(tile1,'on');
%Analytic plot
analytic = 1/pi^2*sin(pi*x)+2/pi*(1-1/pi^2)*x;
%analytic = 1/pi^2*sin(pi*x)+2*x;
plot(tile1, x,analytic, '-r', 'DisplayName', 'Analytic');
legend;
% Plot solution
plot(tile1, x, u, 'b.-', 'DisplayName', 'u(x)');
xlabel('x'); ylabel('u(x)');
title('Solution of Poisson Equation using Steepest Descent Method');
legend;
grid on;

diff = abs(analytic-u);
tile2=nexttile;
plot(tile2,x,diff,'b-', 'DisplayName', 'Error');
title('Error Distribution');
legend;
```

From this we generate the following graphs.


Solution of Poisson Equation using Steepest Descent Method
Analytic
u(x)
u(x)
x
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Error Distribution
Error
0
0.05
0.1
0.15
0.2