

Math 5050 – Special Topics: Manifolds– Fall 2025

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Section 11: The Rank of a Smooth Map – July 2, 2025

Problems

11.1. Tangent vectors to a sphere

The unit sphere S^n in \mathbb{R}^{n+1} is defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i} \Big|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

All X_p are in the tangent plane and therefore perpendicular to the Normal of that plane. Let $f(x) = \sum_{i=1}^{n+1} (x^i)^2 - 1$, then the tangent to the unit sphere $S^n = f^{-1}(0)$ in \mathbb{R}^{n+1} at $(x^1, \dots, x^{n+1}) \in S^{n+1}$ we compute

$$\frac{\partial f}{\partial x^i} = 2x^i$$

at $p = (p^1, \dots, p^{n+1})$,

$$\frac{\partial f}{\partial x^i}(p) = 2p^i$$

The equation of a tangent space at a point p is

$$\sum_{k=1}^{n+1} \frac{\partial f}{\partial x^k}(p)(x^k - p^k) = 0$$

With the $n + 1$ -dimensional norm being

$$N = \langle 2p^1, \dots, 2p^{n+1} \rangle$$

any X in the tangent space will be

$$\begin{aligned} X \cdot N &= 0 \\ &= \langle a^1, \dots, a^{n+1} \rangle \cdot \langle 2p^1, \dots, 2p^{n+1} \rangle \\ &= \sum_{k=1}^{n+1} 2a^k p^{k+1} \end{aligned}$$

11.2. Tangent vectors to a plane curve

- (a) Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the standard coordinates of \mathbb{R}^2 and by \hat{x}, \hat{y} their restrictions to S^1 . Thus, $\hat{x} = i^*x$ and $\hat{y} = i^*y$. On the upper semicircle $U = \{(a, b) \in S^1 \mid b > 0\}$, \hat{x} is a local coordinate, so that $\frac{\partial}{\partial \hat{x}}$ is defined. Prove that for $p \in U$,

$$i_* \left(\frac{\partial}{\partial \hat{x}} \Big|_p \right) = \left(\frac{\partial}{\partial x} + \frac{\partial \hat{y}}{\partial \hat{x}} \frac{\partial}{\partial y} \right) \Big|_p$$

Thus, although $i_* : T_p S^1 \rightarrow T_p \mathbb{R}^2$ is injective, $\frac{\partial}{\partial \hat{x}} \Big|_p$ cannot be identified with $\frac{\partial}{\partial x} \Big|_p$ (Figure 11.9).

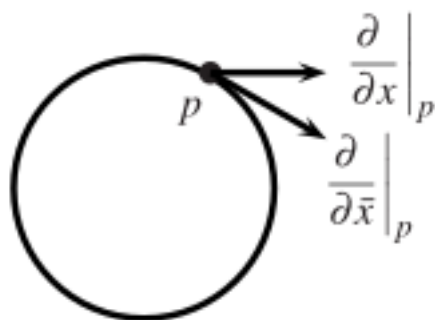


Fig. 11.9. Tangent vector $\partial/\partial \hat{x}|_p$ to a circle.

As a reminder, the pullback is defined as

$$\begin{aligned} i^* f &= f \circ i \\ \hat{x}(p) &= (i^* x)(p) = x(i(p)) = x(p) \end{aligned}$$

that is the x -coordinate of p

$$\begin{aligned} X &= \sum \frac{\partial}{\partial \hat{x}} \Big|_p \\ (i_* X)(g) &= X(g \circ i) \\ &= \sum \frac{\partial g(i(x))}{\partial \hat{x}} \Big|_p \\ &= \frac{\partial g}{\partial x}(p) + \frac{\partial g}{\partial y}(p) \end{aligned}$$

- (b) Generalize (a) to a smooth curve C in \mathbb{R}^2 , letting U be a chart in C on which \hat{x} , the restriction of x to C , is a local coordinate.

11.3. Critical points of a smooth map on a compact manifold

Show that a smooth map f from a compact manifold N to \mathbb{R}^m has critical point. (*Hint:* Let $\pi : \mathbb{R}^m \rightarrow \mathbb{R}$ be the projection to the first factor. Consider the composite map $\pi \circ f : N \rightarrow \mathbb{R}$. A second proof uses Corollary 11.6 and the connectedness of \mathbb{R}^m .)

11.4. Differential of an inclusion map

On the upper hemisphere of the unit sphere S^2 , we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \text{ and } v(a, b, c) = b.$$

So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \rightarrow \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on \mathbb{R}^3 . The differential $i_* : T_p S^2 \rightarrow T_p \mathbb{R}^3$ maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p \mathbb{R}^3$. Thus,

$$\begin{aligned} i_* \left(\frac{\partial}{\partial u} \Big|_p \right) &= \alpha^1 \frac{\partial}{\partial x} \Big|_p + \beta^1 \frac{\partial}{\partial y} \Big|_p + \gamma^1 \frac{\partial}{\partial z} \Big|_p, \\ i_* \left(\frac{\partial}{\partial v} \Big|_p \right) &= \alpha^2 \frac{\partial}{\partial x} \Big|_p + \beta^2 \frac{\partial}{\partial y} \Big|_p + \gamma^2 \frac{\partial}{\partial z} \Big|_p, \end{aligned}$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

11.5. One-to-one immersion of a compact manifold

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \rightarrow M$ is an embedding.

11.6. Multiplication map in $\mathrm{SL}(n, \mathbb{R})$

Let $f : \mathrm{GL}(n, \mathbb{R})$ be the determinant map $f(A) = \det A = \det[a_{ij}]$. For $A \in \mathrm{SL}(n, \mathbb{R})$, there is at least one (k, ℓ) such that the partial derivative $\partial f / \partial a_{k\ell}(A)$ is nonzero (Example 9.13). Use Lemma 9.10 and the implicit function theorem to prove that

- (a) there is a neighborhood of A in $\mathrm{SL}(n, \mathbb{R})$ in which $a_{ij}, (i, j) \neq (k, \ell)$, form a coordinate system, and $a_{k\ell}$ is a C^∞ function of the other entries $a_{ij}, (i, j) \neq (k, \ell)$;
- (b) the multiplication map

$$\hat{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \rightarrow \mathrm{SL}(n, \mathbb{R})$$

is C^∞ .