Review

Appendix A

Outline

A.1 Algebra Essentials

OBJECTIVES
1 Work with Sets (p. A1)
2 Graph Inequalities (p. A4)
3 Find Distance on the Real Number Line (p. A5)
4 Evaluate Algebraic Expressions (p. A6)
5 Determine the Domain of a Variable (p. A7)
6 Use the Laws of Exponents (p. A7)
7 Evaluate Square Roots (p. A9)
8 Use a Calculator to Evaluate Exponents (p. A10)

1 Work with Sets

A **set** is a well-defined collection of distinct objects. The objects of a set are called its **elements.** By **well-defined**, we mean that there is a rule that enables us to determine whether a given object is an element of the set. If a set has no elements, it is called the **empty set**, or **null set**, and is denoted by the symbol $\varnothing$.

For example, the set of **digits** consists of the collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. If we use the symbol D to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation, the braces { } are used to enclose the objects, or **elements**, in the set. This method of denoting a set is called the **roster method**. A second way to denote a set is to use **set-builder notation**, where the set D of digits is written as

$$D = \{ x \mid x \text{ is a digit} \}$$

Read as "D is the set of all x such that x is a digit."

EXAMPLE 1 Using Set-builder Notation and the Roster Method

(a) $E = \{x|x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$

(b) $O = \{x|x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$

Because the elements of a set are distinct, we never repeat elements. For example, we would never write $\{1, 2, 3, 2\}$; the correct listing is $\{1, 2, 3\}$. Because a set is a collection, the order in which the elements are listed is immaterial. $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, and so on, all represent the same set.

If every element of a set A is also an element of a set B, then A is a **subset** of B, which is denoted $A \subseteq B$. If two sets A and B have the same elements, then A **equals** B, which is denoted $A = B$.

For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ and $\{1, 2, 3\} = \{2, 3, 1\}$.

DEFINITION Intersection and Union of Two Sets

If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE 2

Finding the Intersection and Union of Sets

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

Solution

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$

(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$

(c) $$\begin{aligned} B \cap (A \cup C) &= \{3, 5, 7\} \cap [\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\}] \\ &= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\} \end{aligned}$$

Now Work PROBLEM 15

Usually, in working with sets, we designate a **universal set** U, the set consisting of all the elements that we wish to consider. Once a universal set has been designated, we can consider elements of the universal set not found in a given set.

DEFINITION Complement of a Set

If A is a set, the **complement** of A, denoted $\overline{A}$, is the set consisting of all the elements in the universal set that are not in A.*

EXAMPLE 3

Finding the Complement of a Set

If the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $A = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{2, 4, 6, 8\}$.

It follows from the definition of complement that $A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$. Do you see why?

Now Work PROBLEM 19

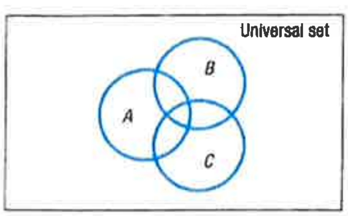


Figure 1 Venn diagram

It is often helpful to draw pictures of sets. Such pictures, called **Venn diagrams**, represent sets as circles enclosed in a rectangle, which represents the universal set. Such diagrams often help us to visualize various relationships among sets. See Figure 1.

If we know that $A \subseteq B$, we might use the Venn diagram in Figure 2(a). If we know that A and B have no elements in common—that is, if $A \cap B = \varnothing$—we might use the Venn diagram in Figure 2(b). The sets A and B in Figure 2(b) are said to be **disjoint**.

*Some texts use the notation A' or A^c for the complement of A.

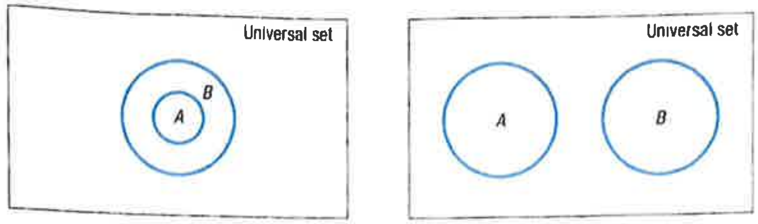


Figure 2

(a) $A \subseteq B$ subset

(b) $A \cap B = \varnothing$ disjoint sets

Figures 3(a), 3(b), and 3(c) use Venn diagrams to illustrate intersection, union, and complement, respectively.

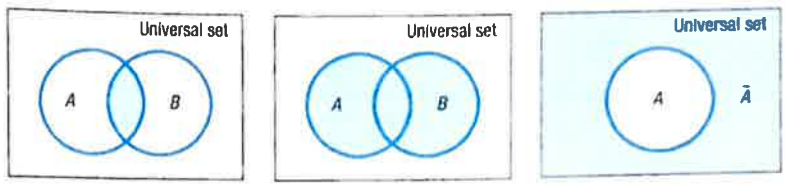


Figure 3

(a) $A \cap B$ intersection

(b) $A \cup B$ union

(c) $\bar{A}$ complement

Real Numbers

Real numbers are represented by symbols such as

$$25, \quad 0, \quad -3, \quad \frac{1}{2}, \quad -\frac{5}{4}, \quad 0.125, \quad \sqrt{2}, \quad \pi, \quad \sqrt[3]{-2}, \quad 0.666\ldots$$

The set of **counting numbers**, or **natural numbers**, contains the numbers in the set $\{1, 2, 3, 4, \ldots\}$. (The three dots, called an **ellipsis**, indicate that the pattern continues indefinitely.) The set of **integers** contains the numbers in the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. A **rational number** is a number that can be expressed as a *quotient* $\frac{a}{b}$ of two integers, where the integer b cannot be 0. Examples of rational numbers are $\frac{3}{4}, \frac{5}{2}, \frac{0}{4}$, and $-\frac{2}{3}$. Since $\frac{a}{1} = a$ for any integer a, every integer is also a rational number. Real numbers that are not rational are called **irrational**. Examples of irrational numbers are $\sqrt{2}$ and π (the Greek letter pi), which equals the constant ratio of the circumference to the diameter of a circle. See Figure 4.

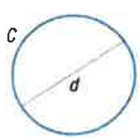


Figure 4 $\pi = \frac{C}{d}$

Real numbers can be represented as **decimals**. Rational real numbers have decimal representations that either **terminate** or are nonterminating with **repeating** blocks of digits. For example, $\frac{3}{4} = 0.75$, which terminates; and $\frac{2}{3} = 0.666\ldots$, in which the digit 6 repeats indefinitely. Irrational real numbers have decimal representations that neither repeat nor terminate. For example, $\sqrt{2} = 1.414213\ldots$ and $\pi = 3.14159\ldots$. In practice, the decimal representation of an irrational number is given as an approximation. We use the symbol $\approx$ (read as "approximately equal to") to write $\sqrt{2} \approx 1.4142$ and $\pi \approx 3.1416$.

Two frequently used properties of real numbers are given next.

Suppose that a, b, and c are real numbers.

Distributive Property

$$a \cdot (b + c) = ab + ac$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$

In Words
If a product equals 0, then one or both of the factors is 0.

Zero-Product Property

If $ab = 0$, then $a = 0$, $b = 0$, or both equal 0.

The Distributive Property can be used to remove parentheses:

$$2(x + 3) = 2x + 2 \cdot 3 = 2x + 6$$

The Zero-Product Property will be used to solve equations (Section A.6). For example, if $2x = 0$, then $2 = 0$ or $x = 0$. Since $2 \neq 0$, it follows that $x = 0$.

The Real Number Line

Real numbers can be represented by points on a line called the **real number line**. There is a one-to-one correspondence between real numbers and points on a line. That is, every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

Pick a point on a line somewhere in the center, and label it O. This point, called the **origin**, corresponds to the real number 0. See Figure 5. The point 1 unit to the right of O corresponds to the number 1. The distance between 0 and 1 determines the **scale** of the number line. For example, the point associated with the number 2 is twice as far from O as 1. Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of the origin correspond to the real numbers -1, -2, and so on. Figure 5 also shows the points associated with the rational numbers $-\frac{1}{2}$ and $\frac{1}{2}$ and with the irrational numbers $\sqrt{2}$ and π.

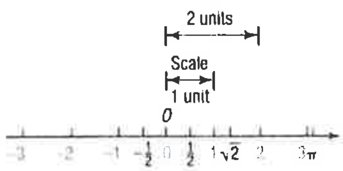


Figure 5 Real number line

DEFINITION Coordinate; Real Number Line

The real number associated with a point P is called the **coordinate** of P, and the line whose points have been assigned coordinates is called the **real number line**.

Now Work PROBLEM 23

The real number line consists of three classes of real numbers, as shown in Figure 6.

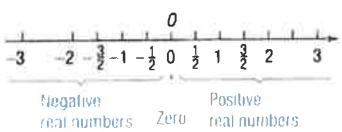


Figure 6

- The **negative real numbers** are the coordinates of points to the left of the origin O.
- The real number **zero** is the coordinate of the origin O.
- The **positive real numbers** are the coordinates of points to the right of the origin O.

2 Graph Inequalities

An important property of the real number line follows from the fact that, given two numbers a and b, either a is to the left of b, or a is at the same location as b, or a is to the right of b. See Figure 7.

If a is to the left of b, then "a is less than b," which is written $a < b$. If a is to the right of b, then "a is greater than b," which is written $a > b$. If a is at the same location as b, then $a = b$. If a is either less than or equal to b, then $a \leq b$. Similarly, $a \geq b$ means that a is either greater than or equal to b. Collectively, the symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols**.

Note that $a < b$ and $b > a$ mean the same thing. It does not matter whether we write $2 < 3$ or $3 > 2$.

Furthermore, if $a < b$ or if $b > a$, then the difference $b - a$ is positive. Do you see why?

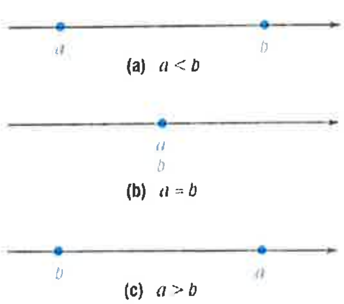


Figure 7

An **inequality** is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the **sides** of the inequality. Inequalities of the form $a < b$ or $b > a$ are called **strict inequalities**, whereas inequalities of the form $a \le b$ or $b \ge a$ are called **nonstrict inequalities**.

Based on the discussion so far, we conclude that

- $a > 0$ is equivalent to a is positive
- $a < 0$ is equivalent to a is negative

We sometimes read $a > 0$ by saying that "a is positive." If $a \ge 0$, then either $a > 0$ or $a = 0$, and we may read this as "a is nonnegative."

Now Work PROBLEMS 27 AND 37

EXAMPLE 4

Graphing Inequalities

(a) On the real number line, graph all numbers x for which $x > 4$.
(b) On the real number line, graph all numbers x for which $x \le 5$.

Solution

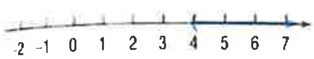


Figure 8 $x > 4$

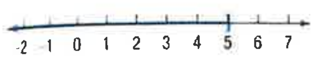


Figure 9 $x \le 5$

(a) See Figure 8. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.
(b) See Figure 9. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph.

Now Work PROBLEM 43

3 Find Distance on the Real Number Line

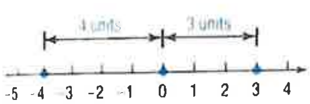


Figure 10

The *absolute value* of a number a is the distance from 0 to a on the number line. For example, -4 is 4 units from 0, and 3 is 3 units from 0. See Figure 10. That is, the absolute value of -4 is 4, and the absolute value of 3 is 3.

A more formal definition of absolute value is given next.

DEFINITION Absolute Value

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by the rules

$$|a| = a \quad \text{if } a \ge 0 \qquad \text{and} \qquad |a| = -a \quad \text{if } a < 0$$

For example, because $-4 < 0$, the second rule must be used to get

$$|-4| = -(-4) = 4$$

EXAMPLE 5

Computing Absolute Value

(a) $|8| = 8$ (b) $|0| = 0$ (c) $|-15| = -(-15) = 15$

Look again at Figure 10. The distance from -4 to 3 is 7 units. This distance is the difference $3 - (-4)$, obtained by subtracting the smaller coordinate from the larger. However, since $|3 - (-4)| = |7| = 7$ and $|-4 - 3| = |-7| = 7$, we can use absolute value to calculate the distance between two points without being concerned about which is smaller.

DEFINITION Distance Between Two Points

If P and Q are two points on the real number line with coordinates a and b, respectively, the **distance between P and Q**, denoted by $d(P, Q)$, is

$$d(P, Q) = |b - a|$$

Since $|b - a| = |a - b|$, it follows that $d(P, Q) = d(Q, P)$.

EXAMPLE 6 Finding Distance on a Number Line

Let P, Q, and R be points on the real number line with coordinates -5, 7, and -3, respectively. Find the distance

(a) between P and Q (b) between Q and R

Solution See Figure 11.

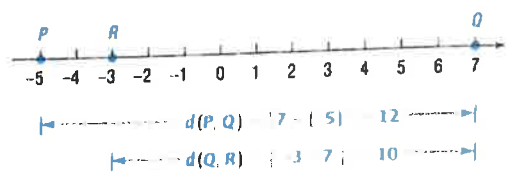


Figure 11

(a) $d(P, Q) = |7 - (-5)| = |12| = 12$

(b) $d(Q, R) = |-3 - 7| = |-10| = 10$

Now Work PROBLEM 49

4 Evaluate Algebraic Expressions

Remember, in algebra we use letters such as x, y, a, b, and c to represent numbers. If a letter used is to represent *any* number from a given set of numbers, it is called a **variable**. A **constant** is either a fixed number, such as 5 or $\sqrt{3}$, or a letter that represents a fixed (possibly unspecified) number.

Constants and variables are combined using the operations of addition, subtraction, multiplication, and division to form *algebraic expressions*. Examples of algebraic expressions include

$$x + 3 \qquad \frac{3}{1 - t} \qquad 7x - 2y$$

To evaluate an algebraic expression, substitute a numerical value for each variable.

EXAMPLE 7 Evaluating an Algebraic Expression

Evaluate each expression if $x = 3$ and $y = -1$.

(a) $x + 3y$ (b) $5xy$ (c) $\dfrac{3y}{2 - 2x}$ (d) $|-4x + y|$

Solution (a) Substitute 3 for x and -1 for y in the expression $x + 3y$.

$$x + 3y \underset{\uparrow}{=} 3 + 3(-1) = 3 + (-3) = 0$$
$$x = 3, y = -1$$

(b) If $x = 3$ and $y = -1$, then

$$5xy = 5 \cdot 3 \cdot (-1) = -15$$

(c) If $x = 3$ and $y = -1$, then

$$\frac{3y}{2 - 2x} = \frac{3(-1)}{2 - 2 \cdot 3} = \frac{-3}{2 - 6} = \frac{-3}{-4} = \frac{3}{4}$$

(d) If $x = 3$ and $y = -1$, then

$$|-4x + y| = |-4 \cdot 3 + (-1)| = |-12 + (-1)| = |-13| = 13$$

Now Work PROBLEMS 51 AND 59

5 Determine the Domain of a Variable

In working with expressions or formulas involving variables, the variables may be allowed to take on values from only a certain set of numbers. For example, in the formula for the area A of a circle of radius r, $A = \pi r^2$, the variable r is necessarily restricted to the positive real numbers. In the expression $\frac{1}{x}$, the variable x cannot take on the value 0, since division by 0 is not defined.

DEFINITION Domain of a Variable

The set of values that a variable may assume is called the **domain of the variable.**

EXAMPLE 8 Finding the Domain of a Variable

The domain of the variable x in the expression

$$\frac{5}{x - 2}$$

is $\{x \mid x \neq 2\}$ since, if $x = 2$, the denominator becomes 0, which is not defined.

EXAMPLE 9 Circumference of a Circle

In the formula for the circumference C of a circle of radius r,

$$C = 2\pi r$$

the domain of the variable r, representing the radius of the circle, is the set of positive real numbers, $\{r \mid r > 0\}$. The domain of the variable C, representing the circumference of the circle, is also the set of positive real numbers, $\{C \mid C > 0\}$.

In describing the domain of a variable, we may use either set notation or words, whichever is more convenient.

Now Work PROBLEM 69

6 Use the Laws of Exponents

Integer exponents provide a shorthand notation for representing repeated multiplications of a real number. For example,

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

DEFINITION a^n

If a is a real number and n is a positive integer, then the symbol a^n represents the product of n factors of a. That is,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \qquad (1)$$

WARNING Be careful with minus signs and exponents.

$$-2^4 = -1 \cdot 2^4 = -16$$

whereas

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

In the definition it is understood that $a^1 = a$. Furthermore, $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, and so on. In the expression a^n, a is called the **base** and n is called the **exponent**, or **power**. We read a^n as "a raised to the power n" or as "a to the nth power." We usually read a^2 as "a squared" and a^3 as "a cubed."

In working with exponents, the operation of *raising to a power* is performed before any other operation. As examples,

$$4 \cdot 3^2 = 4 \cdot 9 = 36 \qquad 2^2 + 3^2 = 4 + 9 = 13$$

$$-2^4 = -16 \qquad 5 \cdot 3^2 + 2 \cdot 4 = 5 \cdot 9 + 2 \cdot 4 = 45 + 8 = 53$$

Parentheses are used to indicate operations to be performed first. For example,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16 \qquad (2 + 3)^2 = 5^2 = 25$$

DEFINITION a^0

If $a \neq 0$, then

$$a^0 = 1$$

DEFINITION a^{-n}

If $a \neq 0$ and if n is a positive integer, then

$$a^{-n} = \frac{1}{a^n}$$

Whenever you encounter a negative exponent, think "reciprocal."

EXAMPLE 10 Evaluating Expressions Containing Negative Exponents

(a) $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ (b) $x^{-4} = \dfrac{1}{x^4}$ (c) $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{\left(\dfrac{1}{5}\right)^2} = \dfrac{1}{\dfrac{1}{25}} = 25$

Now Work PROBLEMS 87 AND 107

The following properties, called the **Laws of Exponents**, can be proved using the preceding definitions. In the list, a and b are real numbers, and m and n are integers.

THEOREM Laws of Exponents

$$a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} \text{ if } a \neq 0 \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \text{ if } b \neq 0$$

EXAMPLE 11

Using the Laws of Exponents

Write each expression so that all exponents are positive.

(a) $\dfrac{x^5y^{-2}}{x^3y} \quad x \neq 0, \quad y \neq 0$ (b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2} \quad x \neq 0, \quad y \neq 0$

Solution

(a) $\dfrac{x^5y^{-2}}{x^3y} = \dfrac{x^5}{x^3} \cdot \dfrac{y^{-2}}{y} = x^{5-3} \cdot y^{-2-1} = x^2y^{-3} = x^2 \cdot \dfrac{1}{y^3} = \dfrac{x^2}{y^3}$

(b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2} = \dfrac{(x^{-3})^{-2}}{(3y^{-1})^{-2}} = \dfrac{x^6}{3^{-2}(y^{-1})^{-2}} = \dfrac{x^6}{\frac{1}{9}y^2} = \dfrac{9x^6}{y^2}$

NOTE Always write the final answer using positive exponents.

Now Work PROBLEMS 89 AND 99

7 Evaluate Square Roots

In Words
$\sqrt{36}$ means "what is the nonnegative number whose square is 36?"

A real number is squared when it is raised to the power 2. The inverse of squaring is finding a **square root**. For example, since $6^2 = 36$ and $(-6)^2 = 36$, the numbers 6 and -6 are square roots of 36.

The symbol $\sqrt{\ }$, called a **radical sign**, is used to denote the **principal**, or nonnegative, square root. For example, $\sqrt{36} = 6$.

DEFINITION Principal Square Root

If a is a nonnegative real number, the nonnegative number b for which $b^2 = a$ is the **principal square root** of a, and is denoted by $b = \sqrt{a}$.

The following comments are noteworthy:

- Negative numbers do not have square roots (in the real number system), because the square of any real number is *nonnegative*. For example, $\sqrt{-4}$ is not a real number, because there is no real number whose square is -4.
- The principal square root of 0 is 0, since $0^2 = 0$. That is, $\sqrt{0} = 0$.
- The principal square root of a positive number is positive.
- If $c \geq 0$, then $(\sqrt{c})^2 = c$. For example, $(\sqrt{2})^2 = 2$ and $(\sqrt{3})^2 = 3$.

EXAMPLE 12

Evaluating Square Roots

(a) $\sqrt{64} = 8$ (b) $\sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$ (c) $(\sqrt{1.4})^2 = 1.4$

Examples 12(a) and (b) are examples of square roots of perfect squares, since $64 = 8^2$ and $\dfrac{1}{16} = \left(\dfrac{1}{4}\right)^2$.

Consider the expression $\sqrt{a^2}$. Since $a^2 \geq 0$, the principal square root of a^2 is defined whether $a > 0$ or $a < 0$. However, since the principal square root is nonnegative, we need an absolute value to ensure the nonnegative result. That is,

$$\sqrt{a^2} = |a| \quad a \text{ any real number} \tag{2}$$

EXAMPLE 13 Using Equation (2)

(a) $\sqrt{(2.3)^2} = |2.3| = 2.3$

(b) $\sqrt{(-2.3)^2} = |-2.3| = 2.3$

(c) $\sqrt{x^2} = |x|$

Now Work PROBLEM 95

Calculators and Graphing Utilities

Calculators are incapable of displaying decimals that contain a large number of digits. For example, some calculators are capable of displaying only eight digits. When a number requires more than eight digits, the calculator either truncates or rounds. To see how your calculator handles decimals, divide 2 by 3. How many digits do you see? Is the last digit a 6 or a 7? If it is a 6, your calculator truncates; if it is a 7, your calculator rounds.

There are different kinds of calculators. An **arithmetic** calculator can only add, subtract, multiply, and divide numbers; therefore, this type is not adequate for this course. **Scientific** calculators have all the capabilities of arithmetic calculators and also contain **function keys** labeled ln, log, sin, cos, tan, x^y, inv, and so on. **Graphing** calculators have all the capabilities of scientific calculators and contain a screen on which graphs can be displayed. We use the term **graphing utility** to refer generically to all graphing calculators and computer software packages, and use the symbol whenever a graphing utility needs to be used. In this text the use of a graphing utility is optional.

8 Use a Calculator to Evaluate Exponents

Your calculator has either a caret key, ^, or an x^y key, that is used for computations involving exponents.

EXAMPLE 14 Exponents on a Graphing Calculator

Evaluate: $(2.3)^5$

Solution Figure 12 shows the result using a TI-84 Plus C graphing calculator.

Figure 12

Now Work PROBLEM 125

A.1 Assess Your Understanding

Concepts and Vocabulary

1. A(n) __variable__ is a letter used in algebra to represent any number from a given set of numbers.

2. On the real number line, the real number zero is the coordinate of the __origin__.

3. An inequality of the form $a > b$ is called a(n) __strict__ inequality.

4. In the expression 2^4, the number 2 is called the __base__ and 4 is called the __exponent or power__.

5. ***Multiple Choice*** If a is a nonnegative real number, then which inequality statement best describes a?

   (a) $a < 0$ **(b)** $a > 0$

   (c) $a \leq 0$ **(d)** $a \geq 0$ d

6. ***Multiple Choice*** Let a and b be nonzero real numbers and m and n be integers. Which of the following is not a law of exponents?

   (a) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ **(b)** $(a^m)^n = a^{m+n}$

   (c) $\frac{a^m}{a^n} = a^{m-n}$ **(d)** $(ab)^n = a^n b^n$ b

7. ***Multiple Choice*** The set of values that a variable may assume is called the ________ of the variable.

   (a) domain **(b)** range **(c)** coordinate **(d)** origin a

8. ***True or False*** The distance between two distinct points on the real number line is always greater than zero. True

9. ***True or False*** The absolute value of a real number is always greater than zero. False

10. ***True or False*** The inverse of squaring is finding a square root. True

Skill Building

In Problems 11–22, use U = universal set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 4, 5, 9\}$, $B = \{2, 4, 6, 7, 8\}$, *and* $C = \{1, 3, 4, 6\}$ *to find each set.*

11. $A \cup B$ **12.** $A \cup C$ **13.** $A \cap B$ $\{4\}$ **14.** $A \cap C$ $\{1, 3, 4\}$

15. $(A \cup B) \cap C$ $\{1, 3, 4, 6\}$ **16.** $(A \cap B) \cup C$ $\{1, 3, 4, 6\}$ **17.** $\overline{A}$ $\{0, 2, 6, 7, 8\}$ **18.** $\overline{C}$ $\{0, 2, 5, 7, 8, 9\}$

19. $\overline{A \cap B}$ **20.** $\overline{B \cup C}$ $\{0, 5, 9\}$ **21.** $A \cup \overline{B}$ **22.** $\overline{B} \cap \overline{C}$ $\{0, 5, 9\}$

23. On the real number line, label the points with coordinates $0, 1, -1, \frac{5}{2}, -2.5, \frac{3}{4}$, and 0.25.

24. Repeat Problem 23 for the coordinates $0, -2, 2, -1.5, \frac{3}{2}, \frac{1}{3}$, and $\frac{2}{3}$.

In Problems 25–34, replace the question mark by <, >, or =, whichever is correct.

25. $\frac{1}{2}\ ?\ 0$ $>$ **26.** $5\ ?\ 6$ $<$ **27.** $-1\ ?\ -2$ $>$ **28.** $-3\ ?\ -\frac{5}{2}$ $<$ **29.** $\pi\ ?\ 3.14$ $>$

30. $\sqrt{2}\ ?\ 1.41$ $>$ **31.** $\frac{1}{2}\ ?\ 0.5$ $=$ **32.** $\frac{1}{3}\ ?\ 0.33$ $>$ **33.** $\frac{2}{3}\ ?\ 0.67$ $<$ **34.** $\frac{1}{4}\ ?\ 0.25$ $=$

In Problems 35–40, write each statement as an inequality.

35. x is positive $x > 0$ **36.** z is negative $z < 0$ **37.** x is less than 2 $x < 2$

38. y is greater than -5 $y > -5$ **39.** x is less than or equal to 1 $x \le 1$ **40.** x is greater than or equal to 2 $x \ge 2$

In Problems 41–44, graph the numbers x on the real number line.

41. $x \ge -2$ **42.** $x < 4$ **43.** $x > -1$ **44.** $x \le 7$

In Problems 45–50, use the given real number line to compute each distance.

45. $d(C, D)$ 1   **46.** $d(C, A)$ 3   **47.** $d(D, E)$ 2   **48.** $d(C, E)$ 3   **49.** $d(A, E)$ 6   **50.** $d(D, B)$ 2

In Problems 51–58, evaluate each expression if $x = -2$ *and* $y = 3$.

51. $x + 2y$ 4   **52.** $3x + y$ -3 **53.** $5xy + 2$ -28 **54.** $-2x + xy$ -2

55. $\dfrac{2x}{x - y}$ $\dfrac{4}{5}$ **56.** $\dfrac{x + y}{x - y}$ $-\dfrac{1}{5}$ **57.** $\dfrac{3x + 2y}{2 + y}$ 0   **58.** $\dfrac{2x - 3}{y}$ $-\dfrac{7}{3}$

In Problems 59–68, find the value of each expression if $x = 3$ *and* $y = -2$.

59. $|x + y|$ 1   **60.** $|x - y|$ 5   **61.** $|x| + |y|$ 5   **62.** $|x| - |y|$ 1   **63.** $\dfrac{|x|}{x}$ 1

64. $\dfrac{|y|}{y}$ -1 **65.** $|4x - 5y|$ 22   **66.** $|3x + 2y|$ 5   **67.** $||4x| - |5y||$ 2   **68.** $3|x| + 2|y|$ 13

In Problems 69–76, determine which of the values (a) through (d), if any, must be excluded from the domain of the variable in each expression.

(a) $x = 3$ (b) $x = 1$ (c) $x = 0$ (d) $x = -1$

69. $\dfrac{x^2 - 1}{x}$ $x = 0$ **70.** $\dfrac{x^2 + 1}{x}$ $x = 0$ **71.** $\dfrac{x}{x^2 - 9}$ $x = 3$ **72.** $\dfrac{x}{x^2 + 9}$ None

73. $\dfrac{x^2}{x^2 + 1}$ None   **74.** $\dfrac{x^3}{x^2 - 1}$ $x = 1, x = -1$ **75.** $\dfrac{x^2 + 5x - 10}{x^3 - x}$ $x = 0, x = 1, x = -1$ **76.** $\dfrac{-9x^2 - x + 1}{x^3 + x}$ $x = 0$

In Problems 77–80, determine the domain of the variable x in each expression.

77. $\dfrac{4}{x - 5}$ $\{x \mid x \ne 5\}$ **78.** $\dfrac{-6}{x + 4}$ $\{x \mid x \ne -4\}$ **79.** $\dfrac{x}{x + 4}$ $\{x \mid x \ne -4\}$ **80.** $\dfrac{x - 2}{x - 6}$ $\{x \mid x \ne 6\}$

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

In Problems 81–84, use the formula $C = \frac{5}{9}(F - 32)$ for converting degrees Fahrenheit into degrees Celsius to find the Celsius measure of each Fahrenheit temperature.

81. $F = 32°$ $0°C$ **82.** $F = 212°$ $100°C$ **83.** $F = 77°$ $25°C$ **84.** $F = -4°$ $-20°C$

In Problems 85–96, simplify each expression.

85. $(-4)^2$ 16 **86.** -4^2 -16 **87.** 4^{-2} $\frac{1}{16}$ **88.** -4^{-2} $-\frac{1}{16}$ **89.** $3^{-6} \cdot 3^4$ $\frac{1}{9}$ **90.** $4^{-2} \cdot 4^3$ 4

91. $(3^{-2})^{-1}$ 9 **92.** $(2^{-1})^{-3}$ 8 **93.** $\sqrt{25}$ 5 **94.** $\sqrt{36}$ 6 **95.** $\sqrt{(-4)^2}$ 4 **96.** $\sqrt{(-3)^2}$ 3

In Problems 97–106, simplify each expression. Express the answer so that all exponents are positive. Whenever an exponent is 0 or negative, assume that the base is not 0.

97. $(8x^3)^2$ $64x^6$ **98.** $(-4x^2)^{-1}$ $-\frac{1}{4x^2}$ **99.** $(x^2y^{-1})^2$ $\frac{x^4}{y^2}$ **100.** $(x^{-1}y)^3$ $\frac{y^3}{x^3}$ **101.** $\frac{x^2y^3}{xy^4}$ $\frac{x}{y}$

102. $\frac{x^{-2}y}{xy^2}$ $\frac{1}{x^3y}$ **103.** $\frac{(-2)^3x^4(yz)^2}{3^2xy^3z}$ **104.** $\frac{4x^{-2}(yz)^{-1}}{2^3x^4y}$ $\frac{1}{2x^6y^2z}$ **105.** $\left(\frac{3x^{-1}}{4y^{-1}}\right)^{-2}$ $\frac{16x^2}{9y^2}$ **106.** $\left(\frac{5x^{-2}}{6y^{-2}}\right)^{-3}$ $\frac{216x^6}{125y^6}$

In Problems 107–118, find the value of each expression if $x = 2$ and $y = -1$.

107. $2xy^{-1}$ -4 **108.** $-3x^{-1}y$ $\frac{3}{2}$ **109.** $x^2 + y^2$ 5 **110.** x^2y^2 4

111. $(xy)^2$ 4 **112.** $(x + y)^2$ 1 **113.** $\sqrt{x^2}$ 2 **114.** $(\sqrt{x})^2$ 2

115. $\sqrt{x^2 + y^2}$ $\sqrt{5}$ **116.** $\sqrt{x^2} + \sqrt{y^2}$ 3 **117.** x^y $\frac{1}{2}$ **118.** y^x 1

119. Find the value of the expression $2x^3 - 3x^2 + 5x - 4$ if $x = 2$. What is the value if $x = 1$? $10; 0$

120. Find the value of the expression $4x^3 + 3x^2 - x + 2$ if $x = 1$. What is the value if $x = 2$? $8; 44$

121. What is the value of $\frac{(666)^4}{(222)^4}$? 81 **122.** What is the value of $(0.1)^3(20)^3$? 8

In Problems 123–130, use a calculator to evaluate each expression. Round your answer to three decimal places.

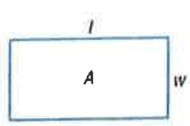
123. $(8.2)^6$ $304{,}006.671$ **124.** $(3.7)^5$ 693.440 **125.** $(6.1)^{-3}$ 0.004 **126.** $(2.2)^{-5}$ 0.019

127. $(-2.8)^6$ 481.890 **128.** $-(2.8)^6$ -481.890 **129.** $(-8.11)^{-4}$ 0.000 **130.** $-(8.11)^{-4}$ -0.000

Applications and Extensions

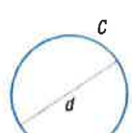
In Problems 131–140, express each statement as an equation involving the indicated variables.

131. Area of a Rectangle The area A of a rectangle is the product of its length l and its width w. $A = lw$

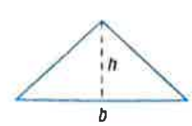


132. Perimeter of a Rectangle The perimeter P of a rectangle is twice the sum of its length l and its width w. $P = 2(l + w)$

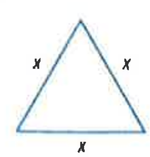
133. Circumference of a Circle The circumference C of a circle is the product of π and its diameter d. $C = \pi d$



134. Area of a Triangle The area A of a triangle is one-half the product of its base b and its height h. $A = \frac{1}{2}bh$



135. Area of an Equilateral Triangle The area A of an equilateral triangle is $\frac{\sqrt{3}}{4}$ times the square of the length x of one side. $A = \frac{\sqrt{3}}{4}x^2$



62. **The Gibb's Hill Lighthouse, Southampton, Bermuda,** in operation since 1846, stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that the light itself can be seen on the horizon about 26 miles distant. Verify the accuracy of this information. The brochure further states that ships 40 miles away can see the light and that planes flying at 10,000 feet can see it 120 miles away. Verify the accuracy of these statements. What assumption did the brochure make about the height of the ship?

A.3 Polynomials

OBJECTIVES
1 Recognize Monomials (p. A22)
2 Recognize Polynomials (p. A23)
3 Know Formulas for Special Products (p. A24)
4 Divide Polynomials Using Long Division (p. A25)
5 Factor Polynomials (p. A27)
6 Complete the Square (p. A29)

We have described algebra as a generalization of arithmetic in which letters are used to represent real numbers. From now on, we shall use the letters at the end of the alphabet, such as x, y, and z, to represent variables and use the letters at the beginning of the alphabet, such as a, b, and c, to represent constants. In the expressions $3x + 5$ and $ax + b$, it is understood that x is a variable and that a and b are constants, even though the constants a and b are unspecified. As you will find out, the context usually makes the intended meaning clear.

1 Recognize Monomials

DEFINITION Monomial

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is called the **degree** of the monomial.

NOTE The nonnegative integers are the whole numbers 0, 1, 2, 3, ...

EXAMPLE 1 Examples of Monomials

Monomial	Coefficient	Degree	
$6x^2$	6	2	
$-\sqrt{2}x^3$	$-\sqrt{2}$	3	
3	3	0	Since $3 = 3 \cdot 1 = 3x^0$, $x \neq 0$
$-5x$	-5	1	Since $-5x = -5x^1$
x^4	1	4	Since $x^4 = 1 \cdot x^4$

EXAMPLE 2 **Examples of Expressions that are Not Monomials**

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$, and $\frac{1}{2}$ is not a nonnegative integer.
(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3, and -3 is not a nonnegative integer.

Now Work PROBLEM 15

2 Recognize Polynomials

Two monomials with the same variable raised to the same power are called **like terms**. For example, $2x^4$ and $-5x^4$ are like terms. In contrast, the monomials $2x^3$ and $2x^5$ are not like terms.

We can add or subtract like terms using the Distributive Property. For example,

$$2x^2 + 5x^2 = (2+5)x^2 = 7x^2 \quad \text{and} \quad 8x^3 - 5x^3 = (8-5)x^3 = 3x^3$$

The sum or difference of two monomials having different degrees is called a **binomial**. The sum or difference of three monomials with three different degrees is called a **trinomial**. For example,

- $x^2 - 2$ is a binomial.
- $x^3 - 3x + 5$ is a trinomial.
- $2x^2 + 5x^2 + 2 = 7x^2 + 2$ is a binomial.

DEFINITION Polynomial

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (1)$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is called the **leading coefficient**, $a_n x^n$ is called the **leading term**, and n is the **degree** of the polynomial.

In Words
A polynomial is a sum of monomials.

The monomials that make up a polynomial are called its **terms**. If all of the coefficients are 0, the polynomial is called the **zero polynomial**, which has no degree.

Polynomials are usually written in **standard form**, beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

EXAMPLE 3 **Examples of Polynomials**

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 - 6x + 2$	$-8, 4, -6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x - 5$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 - 2x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

*The notation a_n is read as "a sub n." The number n is called a **subscript** and should not be confused with an exponent. We use subscripts to distinguish one constant from another when a large or undetermined number of constants are required.

Although we have been using x to represent the variable, letters such as y and z are also commonly used.

- $3x^4 - x^2 + 2$ is a polynomial (in x) of degree 4.
- $9y^3 - 2y^2 + y - 3$ is a polynomial (in y) of degree 3.
- $z^5 + \pi$ is a polynomial (in z) of degree 5.

Algebraic expressions such as

$$\frac{1}{x} \quad \text{and} \quad \frac{x^2 + 1}{x + 5}$$

are not polynomials. The first is not a polynomial because $\frac{1}{x} = x^{-1}$ has an exponent that is not a nonnegative integer. The second expression is not a polynomial because the quotient cannot be simplified to a sum of monomials.

Now Work PROBLEM 25

3 Know Formulas for Special Products

Certain products, which we call **special products**, occur frequently in algebra. We can calculate them easily using the **FOIL** (*F*irst, *O*uter, *I*nner, *L*ast) method of multiplying two binomials.

$$\begin{aligned}(ax + b)(cx + d) &= \overbrace{ax \cdot cx}^{\text{First}} + \overbrace{ax \cdot d}^{\text{Outer}} + \overbrace{b \cdot cx}^{\text{Inner}} + \overbrace{b \cdot d}^{\text{Last}} \\ &= acx^2 + adx + bcx + bd \\ &= acx^2 + (ad + bc)x + bd\end{aligned}$$

EXAMPLE 4 Using FOIL

(a) $(x - 3)(x + 3) = \underset{F}{x^2} + \underset{O}{3x} - \underset{I}{3x} - \underset{L}{9} = x^2 - 9$

(b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x + 3)(x + 1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$

Notice the factors in part (a). The first binomial is a difference and the second one is a sum. Now notice that the outer product O and the inner product I are additive inverses; their sum is zero. So the product is a difference of two squares.

Now Work PROBLEM 45

Some products have been given special names because of their form. The special products in equations (2), (3a), and (3b) are based on Examples 4(a), (b), and (c).

Difference of Two Squares

$$\boxed{(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}}$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \quad \text{(3a)}$$
$$(x - a)^2 = x^2 - 2ax + a^2 \quad \text{(3b)}$$

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \quad \text{(4a)}$$
$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \quad \text{(4b)}$$

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \quad \text{(5)}$$

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \quad \text{(6)}$$

Now Work PROBLEMS 49, 53, AND 57

4 Divide Polynomials Using Long Division

The procedure for dividing two polynomials is similar to the procedure for dividing two integers.

EXAMPLE 5 **Dividing Two Integers**

Divide 842 by 15.

Solution

$$\begin{array}{rrl} & 56 & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow & 15\overline{)842} & \leftarrow \text{Dividend} \\ & \underline{75} & \leftarrow 5 \cdot 15 \\ & 92 & \leftarrow \text{Subtract and bring down the 2.} \\ & \underline{90} & \leftarrow 6 \cdot 15 \\ & 2 & \leftarrow \text{Subtract; the remainder is 2.} \end{array}$$

So, $\frac{842}{15} = 56 + \frac{2}{15}$.

In the division problem detailed in Example 5, the number 15 is called the **divisor**, the number 842 is called the **dividend**, the number 56 is called the **quotient**, and the number 2 is called the **remainder**.

To check the answer obtained in a division problem, multiply the quotient by the divisor and add the remainder. The answer should be the dividend.

$$\text{Quotient} \cdot \text{Divisor} + \text{Remainder} = \text{Dividend}$$

For example, we can check the results obtained in Example 5 as follows:

$$56 \cdot 15 + 2 = 840 + 2 = 842$$

NOTE Remember, a polynomial is in standard form when its terms are written in descending powers of x. ■

To divide two polynomials, we first write each polynomial in standard form. The process then follows a pattern similar to that of Example 5. The next example illustrates the procedure.

EXAMPLE 6

Dividing Two Polynomials

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Solution Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

Step 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{rl} & 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \big) \!\!\!\! & \overline{3x^3 + 4x^2 + x + 7} \end{array}$$

Step 2: Multiply $3x$ by $x^2 + 1$, and enter the result below the dividend.

$$\begin{array}{rl} & 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \big) \!\!\!\! & \overline{3x^3 + 4x^2 + x + 7} \\ & \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \end{array} \quad \leftarrow 3x \cdot (x^2 + 1) = 3x^3 + 3x$$

Align the $3x$ term under the x to make the next step easier.

Step 3: Subtract and bring down the remaining terms.

$$\begin{array}{rl} & 3x \phantom{{}+4x^2+x+7} \\ x^2+1 \big) \!\!\!\! & \overline{3x^3 + 4x^2 + x + 7} \\ & \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \\ & \phantom{3x^3+{}} 4x^2 - 2x + 7 \end{array} \quad \begin{array}{l} \\ \\ \leftarrow \text{Subtract (change the signs and add)} \\ \leftarrow \text{Bring down the } 4x^2 \text{ and the } 7 \end{array}$$

Step 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$\begin{array}{rl} & 3x + 4 \phantom{{}+x+7} \\ x^2+1 \big) \!\!\!\! & \overline{3x^3 + 4x^2 + x + 7} \\ & \underline{3x^3 \phantom{{}+4x^2} + 3x \phantom{{}+7}} \\ & \phantom{3x^3+{}} 4x^2 - 2x + 7 \\ & \phantom{3x^3+{}} \underline{4x^2 \phantom{{}-2x} + 4} \\ & -2x + 3 \end{array} \quad \begin{array}{l} \\ \\ \\ \leftarrow \text{Divide } 4x^2 \text{ by } x^2 \text{ to get } 4. \\ \leftarrow \text{Multiply } (x^2 + 1) \text{ by } 4\text{; subtract.} \\ \\ \end{array}$$

COMMENT When the degree of the divisor is greater than the degree of the dividend, the process ends. ■

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

✓ **Check:** Quotient · Divisor + Remainder

$$\begin{aligned} &= (3x + 4)(x^2 + 1) + (-2x + 3) \\ &= 3x^3 + 3x + 4x^2 + 4 - 2x + 3 \\ &= 3x^3 + 4x^2 + x + 7 = \text{Dividend} \end{aligned}$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

The next example combines the steps involved in long division.

EXAMPLE 7 **Dividing Two Polynomials**

Find the quotient and the remainder when

$$x^4 - 3x^3 + 2x - 5 \quad \text{is divided by} \quad x^2 - x + 1$$

Solution In setting up this division problem, it is necessary to leave a space for the missing x^2 term in the dividend.

$$\begin{array}{lrrrrrrl}
 & & x^2 & -2x & -3 & & & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow & x^2 - x + 1 \,\big) & x^4 & -3x^3 & & +2x & -5 & \leftarrow \text{Dividend} \\
\text{Subtract} \rightarrow & & x^4 & -x^3 & +x^2 & & & \\
 & & & -2x^3 & -x^2 & +2x & -5 & \\
\text{Subtract} \rightarrow & & & -2x^3 & +2x^2 & -2x & & \\
 & & & & -3x^2 & +4x & -5 & \\
\text{Subtract} \rightarrow & & & & -3x^2 & +3x & -3 & \\
 & & & & & x & -2 & \leftarrow \text{Remainder}
\end{array}$$

✓ **Check:** Quotient · Divisor + Remainder

$$\begin{aligned}
&= (x^2 - 2x - 3)(x^2 - x + 1) + x - 2 \\
&= x^4 - x^3 + x^2 - 2x^3 + 2x^2 - 2x - 3x^2 + 3x - 3 + x - 2 \\
&= x^4 - 3x^3 + 2x - 5 = \text{Dividend}
\end{aligned}$$

As a result,

$$\frac{x^4 - 3x^3 + 2x - 5}{x^2 - x + 1} = x^2 - 2x - 3 + \frac{x - 2}{x^2 - x + 1}$$

The process of dividing two polynomials leads to the following result:

THEOREM

Let Q be a polynomial of positive degree, and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

Now Work PROBLEM 65

5 Factor Polynomials

Consider the following product:

$$(2x + 3)(x - 4) = 2x^2 - 5x - 12$$

The two polynomials on the left side are called **factors** of the polynomial on the right side. Expressing a given polynomial as a product of other polynomials—that is, finding the factors of a polynomial—is called **factoring**.

We restrict our discussion here to factoring polynomials in one variable into products of polynomials in one variable, where all coefficients are integers. We call this **factoring over the integers**.

Any polynomial can be written as the product of 1 times itself or as -1 times its additive inverse. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is **prime**. When a polynomial has been written as a product consisting only of prime factors, it is **factored completely**. Examples of prime polynomials (over the integers) are

$$2, \quad 3, \quad 5, \quad x, \quad x + 1, \quad x - 1, \quad 3x + 4, \quad x^2 + 4$$

COMMENT Over the real numbers, $3x + 4$ factors into $3\left(x + \frac{4}{3}\right)$. It is the fraction $\frac{4}{3}$ that causes $3x + 4$ to be prime over the integers. ■

The first factor to look for in a factoring problem is a common monomial factor present in each term of the polynomial. If one is present, use the Distributive Property to factor it out. Continue factoring out monomial factors until none are left.

EXAMPLE 8

Identifying Common Monomial Factors

Polynomial	Common Monomial Factor	Remaining Factor	Factored Form
$2x + 4$	2	$x + 2$	$2x + 4 = 2(x + 2)$
$3x - 6$	3	$x - 2$	$3x - 6 = 3(x - 2)$
$2x^2 - 4x + 8$	2	$x^2 - 2x + 4$	$2x^2 - 4x + 8 = 2(x^2 - 2x + 4)$
$8x - 12$	4	$2x - 3$	$8x - 12 = 4(2x - 3)$
$x^2 + x$	x	$x + 1$	$x^2 + x = x(x + 1)$
$x^3 - 3x^2$	x^2	$x - 3$	$x^3 - 3x^2 = x^2(x - 3)$
$6x^2 + 9x$	$3x$	$2x + 3$	$6x^2 + 9x = 3x(2x + 3)$

Notice that once all common monomial factors have been removed from a polynomial, the remaining factor is either a prime polynomial of degree 1 or a polynomial of degree 2 or higher. (Do you see why?)

The list of special products (2) through (6) given earlier provides a list of factoring formulas when the equations are read from right to left. For example, equation (2) states that if the polynomial is the difference of two squares, $x^2 - a^2$, it can be factored into $(x - a)(x + a)$. The following example illustrates several factoring techniques.

EXAMPLE 9

Factoring Polynomials

Factor completely each polynomial.

(a) $x^4 - 16$ (b) $x^3 - 1$ (c) $9x^2 - 6x + 1$

(d) $x^2 + 4x - 12$ (e) $3x^2 + 10x - 8$ (f) $x^3 - 4x^2 + 2x - 8$

Solution

(a) $x^4 - 16 = \underbrace{(x^2 - 4)(x^2 + 4)}_{\text{Difference of squares}} = \underbrace{(x - 2)(x + 2)(x^2 + 4)}_{\text{Difference of squares}}$

(b) $x^3 - 1 = \underbrace{(x - 1)(x^2 + x + 1)}_{\text{Difference of cubes}}$

(c) $9x^2 - 6x + 1 = \underbrace{(3x - 1)^2}_{\text{Perfect square}}$

(d) $x^2 + 4x - 12 = (x + 6)(x - 2)$

The sum of 6 and -2 is 4. The product of 6 and -2 is -12.

(e) $3x^2 + 10x - 8 = (3x - 2)(x + 4)$

$12x - 2x = 10x$; $3x^2$; -8

(f) $x^3 - 4x^2 + 2x - 8 = \underbrace{(x^3 - 4x^2) + (2x - 8)}_{\text{Group terms}}$

$= \underbrace{x^2(x - 4) + 2(x - 4)}_{\text{Distributive Property}} = \underbrace{(x^2 + 2)(x - 4)}_{\text{Distributive Property}}$

COMMENT The technique used in part (f) is called **factoring by grouping**.

Now Work PROBLEMS 85, 101, AND 135

6 Complete the Square

The idea behind completing the square in one variable is to "adjust" an expression of the form $x^2 + bx$ to make it a perfect square. Perfect squares are trinomials of the form

$$x^2 + 2ax + a^2 = (x + a)^2 \quad \text{or} \quad x^2 - 2ax + a^2 = (x - a)^2$$

For example, $x^2 + 6x + 9$ is a perfect square because $x^2 + 6x + 9 = (x + 3)^2$. And $p^2 - 12p + 36$ is a perfect square because $p^2 - 12p + 36 = (p - 6)^2$.

So how do we "adjust" $x^2 + bx$ to make it a perfect square? We do it by adding a number. For example, to make $x^2 + 6x$ a perfect square, add 9. But how do we know to add 9? If we divide the coefficient of the first-degree term, 6, by 2, and then square the result, we obtain 9. This approach works in general.

Completing the Square of $x^2 + bx$

- Identify the coefficient of the first-degree term, namely b.
- Multiply b by $\frac{1}{2}$ and then square the result. That is, compute $\left(\frac{1}{2}b\right)^2$.
- Add $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$ to get $x^2 + bx + \left(\frac{1}{2}b\right)^2 = \left(x + \frac{b}{2}\right)^2$

WARNING To use $\left(\frac{1}{2}b\right)^2$ to complete the square, the coefficient of the x^2 term must be 1. ■

EXAMPLE 10 **Completing the Square**

Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2}\cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y + 4)^2$
$x^2 + 12x$	$\left(\frac{1}{2}\cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x + 6)^2$
$a^2 - 20a$	$\left(\frac{1}{2}\cdot(-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a - 10)^2$
$p^2 - 5p$	$\left(\frac{1}{2}\cdot(-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

Notice that the factored form of a perfect square is either

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \quad \text{or} \quad x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$$

Now Work PROBLEM 125

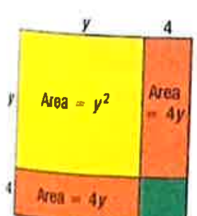


Figure 22

Are you wondering why we refer to making an expression a perfect square as "completing the square"? Look at the square in Figure 22. Its area is $(y + 4)^2$. The yellow area is y^2 and each orange area is $4y$ (for a total area of $8y$). The sum of these areas is $y^2 + 8y$. To complete the square, we need to add the area of the green region, which is $4 \cdot 4 = 16$. As a result, $y^2 + 8y + 16 = (y + 4)^2$.

A.3 Assess Your Understanding

Concepts and Vocabulary

1. The polynomial $3x^4 - 2x^3 + 13x^2 - 5$ is of degree __4__. The leading coefficient is __3__.
2. $(x^2 - 4)(x^2 + 4) =$ __$x^4 - 16$__.
3. $(x - 2)(x^2 + 2x + 4) =$ __$x^3 - 8$__.
4. ***True or False*** $4x^{-2}$ is a monomial of degree -2. False
5. ***True or False*** $(x + a)(x^2 + ax + a) = x^3 + a^3$. False
6. ***True or False*** The polynomial $x^2 + 4$ is prime. True
7. ***True or False*** $3x^3 - 2x^2 - 6x + 4 = (3x - 2)(x^2 + 2)$. False
8. To complete the square of the expression $x^2 + 5x$, you would __add__ the number __$\frac{25}{4}$__.
9. To check division, the expression that is being divided, the dividend, should equal the product of the __quotient__ and the __divisor__ plus the __remainder__.
10. ***Multiple Choice*** The monomials that make up a polynomial are called which of the following?
    (a) terms **(b)** variables **(c)** factors **(d)** coefficients a
11. ***Multiple Choice*** Choose the degree of the monomial $3x^4$.
    (a) 3 **(b)** 7 **(c)** 4 **(d)** 2 c
12. ***Multiple Choice*** Choose the best description of $x^2 - 64$.
    (a) Prime **(b)** Difference of two squares
    (c) Difference of two cubes **(d)** Perfect Square b
13. ***Multiple Choice*** Choose the complete factorization of
    $$4x^2 - 8x - 60$$
    (a) $2(x + 3)(x - 5)$ **(b)** $4(x^2 - 2x - 15)$
    (c) $(2x + 6)(2x - 10)$ **(d)** $4(x + 3)(x - 5)$ d
14. ***Multiple Choice*** To complete the square of $x^2 + bx$, use which of the following?
    (a) $(2b)^2$ **(b)** $2b^2$ **(c)** $\left(\frac{1}{2}b\right)^2$ **(d)** $\frac{1}{2}b^2$ c

Skill Building

*In Problems 15–24, tell whether the expression is a monomial. If it is, name the variable and the coefficient and give the degree of the monomial. If it is not a monomial, state why not.**

15. $2x^3$ 16. $-4x^2$ 17. $\frac{8}{x}$ 18. $-2x^{-3}$ 19. $-2x^3 + 5x^2$
20. $6x^5 - 8x^2$ 21. $\frac{8x}{x^2 - 1}$ 22. $-\frac{2x^2}{x^3 + 1}$ 23. $x^2 + 2x - 5$ 24. $3x^2 + 4$

In Problems 25–34, tell whether the expression is a polynomial. If it is, give its degree. If it is not, state why not.

25. $3x^2 - 5$ Yes; 2 26. $1 - 4x$ Yes; 1 27. 5 Yes; 0 28. $-\pi$ Yes; 0 29. $3x^2 - \frac{5}{x}$ No
30. $\frac{3}{x} + 2$ No 31. $2y^3 - \sqrt{2}$ Yes; 3 32. $10z^2 + z$ Yes; 2 33. $\frac{x^2 + 5}{x^3 - 1}$ No 34. $\frac{3x^3 + 2x - 1}{x^2 + x + 1}$ No

In Problems 35–60, add, subtract, or multiply, as indicated. Express your answer as a single polynomial in standard form.

35. $(x^2 + 4x + 5) + (3x - 3)$ $x^2 + 7x + 2$
36. $(x^3 + 3x^2 + 2) + (x^2 - 4x + 4)$
37. $(x^3 - 2x^2 + 5x + 10) - (2x^2 - 4x + 3)$
38. $(x^2 - 3x - 4) - (x^3 - 3x^2 + x + 5)$
39. $6(x^3 + x^2 - 3) - 4(2x^3 - 3x^2)$ $-2x^3 + 18x^2 - 18$
40. $8(4x^3 - 3x^2 - 1) - 6(4x^3 + 8x - 2)$
41. $9(y^2 - 3y + 4) - 6(1 - y^2)$ $15y^2 - 27y + 30$
42. $8(1 - y^3) + 4(1 + y + y^2 + y^3)$
43. $x(x^2 + x - 4)$ $x^3 + x^2 - 4x$
44. $4x^2(x^3 - x + 2)$
45. $(x + 2)(x + 4)$ $x^2 + 6x + 8$
46. $(x + 3)(x + 5)$ $x^2 + 8x + 15$
47. $(2x + 5)(x + 2)$ $2x^2 + 9x + 10$
48. $(3x + 1)(2x + 1)$ $6x^2 + 5x + 1$
49. $(x - 7)(x + 7)$ $x^2 - 49$
50. $(x - 1)(x + 1)$ $x^2 - 1$
51. $(2x + 3)(2x - 3)$ $4x^2 - 9$
52. $(3x + 2)(3x - 2)$ $9x^2 - 4$
53. $(x + 4)^2$ $x^2 + 8x + 16$
54. $(x - 5)^2$ $x^2 - 10x + 25$
55. $(2x - 3)^2$ $4x^2 - 12x + 9$
56. $(3x - 4)^2$ $9x^2 - 24x + 16$
57. $(x - 2)^3$ $x^3 - 6x^2 + 12x - 8$
58. $(x + 1)^3$ $x^3 + 3x^2 + 3x + 1$
59. $(2x + 1)^3$ $8x^3 + 12x^2 + 6x + 1$
60. $(3x - 2)^3$

In Problems 61–76, find the quotient and the remainder. Check your work by verifying that

$$\textit{Quotient} \cdot \textit{Divisor} + \textit{Remainder} = \textit{Dividend}$$

61. $4x^3 - 3x^2 + x + 1$ divided by $x + 2$
62. $3x^3 - x^2 + x - 2$ divided by $x + 2$
63. $4x^3 - 3x^2 + x + 1$ divided by x^2 $4x - 3$; R: $x + 1$
64. $3x^3 - x^2 + x - 2$ divided by x^2 $3x - 1$; R: $x - 2$
65. $5x^4 - 3x^2 + x + 1$ divided by $x^2 + 2$
66. $5x^4 - x^2 + x - 2$ divided by $x^2 + 2$
67. $4x^5 - 3x^2 + x + 1$ divided by $2x^3 - 1$
68. $3x^5 - x^2 + x - 2$ divided by $3x^3 - 1$ x^2; R: $x - 2$
69. $2x^4 - 3x^3 + x + 1$ divided by $2x^2 + x + 1$
70. $3x^4 - x^3 + x - 2$ divided by $3x^2 + x + 1$
71. $-4x^3 + x^2 - 4$ divided by $x - 1$
72. $-3x^4 - 2x - 1$ divided by $x - 1$

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

73. $1 - x^2 + x^4$ divided by $x^2 + x + 1$

74. $1 - x^2 + x^4$ divided by $x^2 - x + 1$

75. $x^3 - a^3$ divided by $x - a$ $x^2 + ax + a^2$; R: 0

76. $x^5 - a^5$ divided by $x - a$

In Problems 77–124, factor completely each polynomial. If the polynomial cannot be factored, say it is prime.

77. $x^2 - 36$ $(x + 6)(x - 6)$
78. $x^2 - 9$ $(x + 3)(x - 3)$
79. $2 - 8x^2$
80. $3 - 27x^2$
81. $x^2 + 11x + 10$
82. $x^2 + 5x + 4$
83. $x^2 - 10x + 21$
84. $x^2 - 6x + 8$
85. $4x^2 - 8x + 32$
86. $3x^2 - 12x + 15$
87. $x^2 + 4x + 16$
88. $x^2 + 12x + 36$
89. $15 + 2x - x^2$
90. $14 + 6x - x^2$
91. $3x^2 - 12x - 36$
92. $x^3 + 8x^2 - 20x$
93. $y^4 + 11y^3 + 30y^2$
94. $3y^3 - 18y^2 - 48y$
95. $4x^2 + 12x + 9$
96. $9x^2 - 12x + 4$
97. $6x^2 + 8x + 2$
98. $8x^2 + 6x - 2$
99. $x^4 - 81$
100. $x^4 - 1$
101. $x^6 - 2x^3 + 1$
102. $x^6 + 2x^3 + 1$
103. $x^7 - x^5$
104. $x^8 - x^5$
105. $16x^2 + 24x + 9$
106. $9x^2 - 24x + 16$
107. $5 + 16x - 16x^2$
108. $5 + 11x - 16x^2$
109. $4y^2 - 16y + 15$
110. $9y^2 + 9y - 4$
111. $1 - 8x^2 - 9x^4$
112. $4 - 14x^2 - 8x^4$
113. $x(x + 3) - 6(x + 3)$
114. $5(3x - 7) + x(3x - 7)$
115. $(x + 2)^2 - 5(x + 2)$
116. $(x - 1)^2 - 2(x - 1)$
117. $(3x - 2)^3 - 27$
118. $(5x + 1)^3 - 1$ $5x(25x^2 + 15x + 3)$
119. $3(x^2 + 10x + 25) - 4(x + 5)$
120. $7(x^2 - 6x + 9) + 5(x - 3)$
121. $x^3 + 2x^2 - x - 2$
122. $x^3 - 3x^2 - x + 3$
123. $x^4 - x^3 + x - 1$ $(x - 1)(x + 1)(x^2 - x + 1)$
124. $x^4 + x^3 + x + 1$ $(x + 1)^2(x^2 - x + 1)$

In Problems 125–130, determine the number that should be added to complete the square of each expression. Then factor each expression.

125. $x^2 + 10x$ 25; $(x + 5)^2$
126. $p^2 + 14p$ 49; $(p + 7)^2$
127. $y^2 - 6y$ 9; $(y - 3)^2$
128. $x^2 - 4x$ 4; $(x - 2)^2$
129. $x^2 - \frac{1}{2}x$ $\frac{1}{16}$; $\left(x - \frac{1}{4}\right)^2$
130. $x^2 + \frac{1}{3}x$ $\frac{1}{36}$; $\left(x + \frac{1}{6}\right)^2$

Applications and Extensions

In Problems 131–140, expressions that occur in calculus are given. Factor completely each expression.

131. $2(3x + 4)^2 + (2x + 3)\cdot 2(3x + 4)\cdot 3$ $2(3x + 4)(9x + 13)$
132. $5(2x + 1)^2 + (5x - 6)\cdot 2(2x + 1)\cdot 2$ $(2x + 1)(30x - 19)$
133. $2x(2x + 5) + x^2\cdot 2$ $2x(3x + 5)$
134. $3x^2(8x - 3) + x^3\cdot 8$ $x^2(32x - 9)$
135. $2(x + 3)(x - 2)^3 + (x + 3)^2\cdot 3(x - 2)^2$
136. $4(x + 5)^3(x - 1)^2 + (x + 5)^4\cdot 2(x - 1)$
137. $(4x - 3)^2 + x\cdot 2(4x - 3)\cdot 4$ $3(4x - 3)(4x - 1)$
138. $3x^2(3x + 4)^2 + x^3\cdot 2(3x + 4)\cdot 3$ $3x^2(3x + 4)(5x + 4)$
139. $2(3x - 5)\cdot 3(2x + 1)^3 + (3x - 5)^2\cdot 3(2x + 1)^2\cdot 2$
140. $3(4x + 5)^2\cdot 4(5x + 1)^2 + (4x + 5)^3\cdot 2(5x + 1)\cdot 5$
141. Show that $x^2 + 4$ is prime.
142. Show that $x^2 + x + 1$ is prime.

Explaining Concepts: Discussion and Writing

143. Explain why the degree of the product of two nonzero polynomials equals the sum of their degrees.
144. Explain why the degree of the sum of two polynomials of different degrees equals the larger of their degrees.
145. Give a careful statement about the degree of the sum of two polynomials of the same degree.
146. Do you prefer to memorize the rule for the square of a binomial $(x + a)^2$ or to use FOIL to obtain the product? Write a brief position paper defending your choice.
147. Make up a polynomial that factors into a perfect square.
148. Explain to a fellow student what you look for first when presented with a factoring problem. What do you do next?

A.4 Synthetic Division

OBJECTIVE 1 Divide Polynomials Using Synthetic Division (p. A31)

1 Divide Polynomials Using Synthetic Division

To find the quotient as well as the remainder when a polynomial of degree 1 or higher is divided by $x - c$, a shortened version of long division, called **synthetic division**, makes the task simpler.

To see how synthetic division works, first consider long division for dividing the polynomial $2x^3 - x^2 + 3$ by $x - 3$.

$$
\begin{array}{r|rrrrl}
 & 2x^2 & +\,5x & +\,15 & & \leftarrow \text{Quotient} \\ \cline{2-5}
x-3 & 2x^3 & -\,x^2 & & +\,3 & \\
 & 2x^3 & -\,6x^2 & & & \\ \cline{2-3}
 & & 5x^2 & & & \\
 & & 5x^2 & -\,15x & & \\ \cline{3-4}
 & & & 15x & +\,3 & \\
 & & & 15x & -\,45 & \\ \cline{4-5}
 & & & & 48 & \leftarrow \text{Remainder}
\end{array}
$$

✓**Check:** Divisor · Quotient + Remainder

$$
\begin{aligned}
&= (x - 3)(2x^2 + 5x + 15) + 48 \\
&= 2x^3 + 5x^2 + 15x - 6x^2 - 15x - 45 + 48 \\
&= 2x^3 - x^2 + 3
\end{aligned}
$$

The process of synthetic division arises from rewriting the long division in a more compact form, using simpler notation. For example, in the long division above, the terms in blue are not really necessary because they are identical to the terms directly above them. With these terms removed, we have

$$
\begin{array}{r|rrrr}
 & 2x^2 & +\,5x & +\,15 & \\ \cline{2-5}
x-3 & 2x^3 & -\,x^2 & & +\,3 \\
 & & -\,6x^2 & & \\ \cline{2-3}
 & & 5x^2 & & \\
 & & & -\,15x & \\ \cline{3-5}
 & & & 15x & \\
 & & & & -\,45 \\ \cline{4-5}
 & & & & 48
\end{array}
$$

Most of the x's that appear in this process can also be removed, provided that we are careful about positioning each coefficient. In this regard, we will need to use 0 as the coefficient of x in the dividend, because that power of x is missing. Now we have

$$
\begin{array}{r|rrrr}
 & 2x^2 & +\,5x & +\,15 & \\ \cline{2-5}
x-3 & 2 & -\,1 & 0 & 3 \\
 & & -\,6 & & \\ \cline{2-3}
 & & 5 & & \\
 & & & -\,15 & \\ \cline{3-5}
 & & & 15 & \\
 & & & & -\,45 \\ \cline{4-5}
 & & & & 48
\end{array}
$$

We can make this display more compact by moving the lines up until the numbers in blue align horizontally.

$$
\begin{array}{r|rrrrl}
 & 2x^2 & +\,5x & +\,15 & & \text{Row 1} \\ \cline{2-5}
x-3 & 2 & -\,1 & 0 & 3 & \text{Row 2} \\
 & & -\,6 & -\,15 & -\,45 & \text{Row 3} \\ \cline{2-5}
 & \bigcirc & 5 & 15 & 48 & \text{Row 4}
\end{array}
$$

Because the leading coefficient of the divisor is always 1, the leading coefficient of the dividend will also be the leading coefficient of the quotient. So we place the leading coefficient of the quotient, 2, in the circled position. Now, the first three numbers in row 4 are precisely the coefficients of the quotient, and the last number in row 4 is the remainder. Since row 1 is not really needed, we can compress the process to three rows, where the bottom row contains both the coefficients of the quotient and the remainder.

$$\begin{array}{r|rrrrl} x-3 & 2 & -1 & 0 & 3 & \text{Row 1} \\ & & -6 & -15 & -45 & \text{Row 2 (subtract)} \\ \hline & 2 & 5 & 15 & 48 & \text{Row 3} \end{array}$$

Recall that the entries in row 3 are obtained by subtracting the entries in row 2 from those in row 1. Rather than subtracting the entries in row 2, we can change the sign of each entry and add. With this modification, our display will look like this:

$$\begin{array}{r|rrrrl} x-3 & 2 & -1 & 0 & 3 & \text{Row 1} \\ & & 6 & 15 & 45 & \text{Row 2 (add)} \\ \hline & 2 & 5 & 15 & 48 & \text{Row 3} \end{array}$$

Notice that the entries in row 2 are three times the prior entries in row 3. Our last modification to the display replaces the $x - 3$ by 3. The entries in row 3 give the quotient and the remainder, as shown next.

$$\begin{array}{r|rrrrl} 3 & 2 & -1 & 0 & 3 & \text{Row 1} \\ & & 6 & 15 & 45 & \text{Row 2 (add)} \\ \hline & 2 & 5 & 15 & 48 & \text{Row 3} \end{array}$$

Quotient: $2x^2 + 5x + 15$ Remainder: 48

Let's go through an example step by step.

EXAMPLE 1 Using Synthetic Division to Find the Quotient and Remainder

Use synthetic division to find the quotient and remainder when

$$x^3 - 4x^2 - 5 \quad \text{is divided by} \quad x - 3$$

Solution

STEP 1: Write the dividend in descending powers of x. Then copy the coefficients, remembering to insert a 0 for any missing powers of x.

$$\begin{array}{rrrrl} 1 & -4 & 0 & -5 & \text{Row 1} \end{array}$$

STEP 2: Insert the usual division symbol. In synthetic division, the divisor is of the form $x - c$, and c is the number placed to the left of the division symbol. Here, since the divisor is $x - 3$, insert 3 to the left of the division symbol.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \text{Row 1} \end{array}$$

STEP 3: Bring the 1 down two rows, and enter it in row 3.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \text{Row 1} \\ & \downarrow & & & & \text{Row 2} \\ \hline & 1 & & & & \text{Row 3} \end{array}$$

STEP 4: Multiply the latest entry in row 3 by 3, and place the result in row 2, one column over to the right.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \text{Row 1} \\ & & 3 & & & \text{Row 2} \\ \hline & 1 & & & & \text{Row 3} \end{array}$$

STEP 5: Add the entry in row 2 to the entry above it in row 1, and enter the sum in row 3.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \text{Row 1} \\ & & 3 & & & \text{Row 2} \\ \hline & 1 & -1 & & & \text{Row 3} \end{array}$$

(continued)

STEP 6: Repeat Steps 4 and 5 until no more entries are available in row 1.

$$\begin{array}{r|rrrrl} 3 & 1 & -4 & 0 & -5 & \textbf{Row 1} \\ & & 3 & -3 & -9 & \textbf{Row 2} \\ \hline & 1 & -1 & -3 & -14 & \textbf{Row 3} \end{array}$$

STEP 7: The final entry in row 3, the -14, is the remainder; the other entries in row 3, the 1, -1, and -3, are the coefficients (in descending order) of a polynomial whose degree is 1 less than that of the dividend. This is the quotient. That is,

$$\text{Quotient} = x^2 - x - 3 \qquad \text{Remainder} = -14$$

✓ **Check:** Divisor · Quotient + Remainder

$$= (x-3)(x^2 - x - 3) + (-14)$$
$$= (x^3 - x^2 - 3x - 3x^2 + 3x + 9) + (-14)$$
$$= x^3 - 4x^2 - 5 = \text{Dividend}$$

Let's do an example in which all seven steps are combined.

EXAMPLE 2 **Using Synthetic Division to Verify a Factor**

Use synthetic division to show that $x + 3$ is a factor of

$$2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$$

Solution The divisor is $x + 3 = x - (-3)$, so place -3 to the left of the division symbol. Then the row 3 entries will be multiplied by -3, entered in row 2, and added to row 1.

$$\begin{array}{r|rrrrrrl} -3 & 2 & 5 & -2 & 2 & -2 & 3 & \textbf{Row 1} \\ & & -6 & 3 & -3 & 3 & -3 & \textbf{Row 2} \\ \hline & 2 & -1 & 1 & -1 & 1 & 0 & \textbf{Row 3} \end{array}$$

Because the remainder is 0, we have

Divisor · Quotient + Remainder

$$= (x+3)(2x^4 - x^3 + x^2 - x + 1) = 2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$$

As we see, $x + 3$ is a factor of $2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$.

As Example 2 illustrates, the remainder after division gives information about whether the divisor is, or is not, a factor. We say more about this in Chapter 4.

Now Work PROBLEMS 9 AND 19

A.4 Assess Your Understanding

Concepts and Vocabulary

1. To check division, the expression that is being divided, the dividend, should equal the product of the __quotient__ and the __divisor__ plus the __remainder__.
2. To divide $2x^3 - 5x + 1$ by $x + 3$ using synthetic division, the first step is to write __$-3\overline{)2 \quad 0 \quad -5 \quad 1}$__.
3. ***Multiple Choice*** Choose the division problem that cannot be done using synthetic division.
   (a) $2x^3 - 4x^2 + 6x - 8$ is divided by $x - 8$
   (b) $x^4 - 3$ is divided by $x + 1$
   (c) $x^5 + 3x^2 - 9x + 2$ is divided by $x + 10$
   (d) $x^4 - 5x^3 + 3x^2 - 9x + 13$ is divided by $x^2 + 5$ d
4. ***Multiple Choice*** Choose the correct conclusion based on the following synthetic division:
$$\begin{array}{r|rrrr} -5 & 2 & 3 & -38 & -15 \\ & & -10 & 35 & 15 \\ \hline & 2 & -7 & -3 & 0 \end{array}$$
   (a) $x + 5$ is a factor of $2x^3 + 3x^2 - 38x - 15$
   (b) $x - 5$ is a factor of $2x^3 + 3x^2 - 38x - 15$
   (c) $x + 5$ is not a factor of $2x^3 + 3x^2 - 38x - 15$
   (d) $x - 5$ is not a factor of $2x^3 + 3x^2 - 38x - 15$ a

5. *True or False* In using synthetic division, the divisor is always a polynomial of degree 1, whose leading coefficient is 1. True

6. *True or False*

$$\begin{array}{r|rrrr} -2 & 5 & 3 & 2 & 1 \\ & & -10 & 14 & -32 \\ \hline & 5 & -7 & 16 & -31 \end{array}$$ means $\dfrac{5x^3 + 3x^2 + 2x + 1}{x + 2} = 5x^2 - 7x + 16 + \dfrac{-31}{x + 2}$. True

Skill Building

In Problems 7–18, use synthetic division to find the quotient and remainder when:

7. $x^3 - 7x^2 + 5x + 10$ is divided by $x - 2$
8. $x^3 + 2x^2 - 3x + 1$ is divided by $x + 1$
9. $3x^3 + 2x^2 - x + 3$ is divided by $x - 3$
10. $-4x^3 + 2x^2 - x + 1$ is divided by $x + 2$
11. $x^5 - 4x^3 + x$ is divided by $x + 3$
12. $x^4 + x^2 + 2$ is divided by $x - 2$
13. $4x^6 - 3x^4 + x^2 + 5$ is divided by $x - 1$
14. $x^5 + 5x^3 - 10$ is divided by $x + 1$
15. $0.1x^3 + 0.2x$ is divided by $x + 1.1$
16. $0.1x^2 - 0.2$ is divided by $x + 2.1$
17. $x^5 - 32$ is divided by $x - 2$
18. $x^5 + 1$ is divided by $x + 1$

In Problems 19–28, use synthetic division to determine whether $x - c$ is a factor of the given polynomial.

19. $4x^3 - 3x^2 - 8x + 4$; $x - 2$ No
20. $-4x^3 + 5x^2 + 8$; $x + 3$ No
21. $2x^4 - 6x^3 - 7x + 21$; $x - 3$ Yes
22. $4x^4 - 15x^2 - 4$; $x - 2$ Yes
23. $5x^6 + 43x^3 + 24$; $x + 2$ Yes
24. $2x^6 - 18x^4 + x^2 - 9$; $x + 3$ Yes
25. $x^5 - 16x^3 - x^2 + 19$; $x + 4$ No
26. $x^6 - 16x^4 + x^2 - 16$; $x + 4$ Yes
27. $3x^4 - x^3 + 6x - 2$; $x - \frac{1}{3}$ Yes
28. $3x^4 + x^3 - 3x + 1$; $x + \frac{1}{3}$ No

Applications and Extensions

29. Find the sum of a, b, c, and d if

$$\frac{x^3 - 2x^2 + 3x + 5}{x + 2} = ax^2 + bx + c + \frac{d}{x + 2}$$ -9

Explaining Concepts: Discussion and Writing

30. When dividing a polynomial by $x - c$, do you prefer to use long division or synthetic division? Does the value of c make a difference to you in choosing? Give reasons.

A.5 Rational Expressions

OBJECTIVES
1. Reduce a Rational Expression to Lowest Terms (p. A35)
2. Multiply and Divide Rational Expressions (p. A36)
2. Add and Subtract Rational Expressions (p. A37)
4. Use the Least Common Multiple Method (p. A39)
5. Simplify Complex Rational Expressions (p. A40)

1 Reduce a Rational Expression to Lowest Terms

If we form the quotient of two polynomials, the result is called a **rational expression**. Some examples of rational expressions are

(a) $\dfrac{x^3 + 1}{x}$ (b) $\dfrac{3x^2 + x - 2}{x^2 + 5}$ (c) $\dfrac{x}{x^2 - 1}$ (d) $\dfrac{xy^2}{(x - y)^2}$

Expressions (a), (b), and (c) are rational expressions in one variable, x, whereas (d) is a rational expression in two variables, x and y.

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

Rational expressions are described in the same manner as rational numbers. In expression (a), the polynomial $x^3 + 1$ is the **numerator**, and x is the **denominator**. When the numerator and denominator of a rational expression contain no common factors (except 1 and -1), we say that the rational expression is **reduced to lowest terms**, or **simplified**.

The polynomial in the denominator of a rational expression cannot be equal to 0 because division by 0 is not defined. For example, for the expression $\frac{x^3+1}{x}$, x cannot take on the value 0. The domain of the variable x is $\{x \mid x \neq 0\}$.

A rational expression is reduced to lowest terms by factoring the numerator and the denominator completely and canceling any common factors using the Cancellation Property:

Cancellation Property

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \quad \text{if } b \neq 0, c \neq 0 \tag{1}$$

EXAMPLE 1 **Reducing a Rational Expression to Lowest Terms**

Reduce to lowest terms: $\dfrac{x^2+4x+4}{x^2+3x+2}$

Solution Begin by factoring the numerator and the denominator.

$$x^2 + 4x + 4 = (x+2)(x+2)$$
$$x^2 + 3x + 2 = (x+2)(x+1)$$

WARNING Use the Cancellation Property only with rational expressions written in factored form. Be sure to cancel only common factors, not common terms!

Since a common factor, $x + 2$, appears, the original expression is not in lowest terms. To reduce it to lowest terms, use the Cancellation Property:

$$\frac{x^2+4x+4}{x^2+3x+2} = \frac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x+1)} = \frac{x+2}{x+1} \qquad x \neq -2, -1$$

EXAMPLE 2 **Reducing Rational Expressions to Lowest Terms**

Reduce each rational expression to lowest terms.

(a) $\dfrac{x^3-8}{x^3-2x^2}$ (b) $\dfrac{8-2x}{x^2-x-12}$

Solution

(a) $$\frac{x^3-8}{x^3-2x^2} = \frac{\cancel{(x-2)}(x^2+2x+4)}{x^2\cancel{(x-2)}} = \frac{x^2+2x+4}{x^2} \qquad x \neq 0, 2$$

(b) $$\frac{8-2x}{x^2-x-12} = \frac{2(4-x)}{(x-4)(x+3)} = \frac{2(-1)\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} = \frac{-2}{x+3} \qquad x \neq -3, 4$$

Now Work PROBLEM 7

2 Multiply and Divide Rational Expressions

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers.

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational expressions, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(2)}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad \text{if } b \neq 0, c \neq 0, d \neq 0 \qquad \textbf{(3)}$$

In using equations (2) and (3) with rational expressions, be sure first to factor each polynomial completely so that common factors can be canceled. Leave your answer in factored form.

EXAMPLE 3 **Multiplying and Dividing Rational Expressions**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x^2 - 2x + 1}{x^3 + x} \cdot \dfrac{4x^2 + 4}{x^2 + x - 2}$ (b) $\dfrac{\frac{x + 3}{x^2 - 4}}{\frac{x^2 - x - 12}{x^3 - 8}}$

Solution (a)

$$\frac{x^2 - 2x + 1}{x^3 + x} \cdot \frac{4x^2 + 4}{x^2 + x - 2} = \frac{(x - 1)^2}{x(x^2 + 1)} \cdot \frac{4(x^2 + 1)}{(x + 2)(x - 1)}$$

$$= \frac{(x - 1)^{\cancel{2}}(4)\cancel{(x^2 + 1)}}{x\cancel{(x^2 + 1)}(x + 2)\cancel{(x - 1)}}$$

$$= \frac{4(x - 1)}{x(x + 2)} \qquad x \neq -2, 0, 1$$

(b)

$$\frac{\frac{x + 3}{x^2 - 4}}{\frac{x^2 - x - 12}{x^3 - 8}} = \frac{x + 3}{x^2 - 4} \cdot \frac{x^3 - 8}{x^2 - x - 12}$$

$$= \frac{x + 3}{(x - 2)(x + 2)} \cdot \frac{(x - 2)(x^2 + 2x + 4)}{(x - 4)(x + 3)}$$

$$= \frac{\cancel{(x + 3)}\cancel{(x - 2)}(x^2 + 2x + 4)}{\cancel{(x - 2)}(x + 2)(x - 4)\cancel{(x + 3)}}$$

$$= \frac{x^2 + 2x + 4}{(x + 2)(x - 4)} \qquad x \neq -3, -2, 2, 4$$

Now Work PROBLEMS 15 AND 21

In Words
To add (or subtract) two rational expressions with the same denominator, keep the common denominator and add (or subtract) the numerators.

3 Add and Subtract Rational Expressions

The rules for adding and subtracting rational expressions are the same as the rules for adding and subtracting rational numbers. If the denominators of two rational expressions to be added (or subtracted) are equal, then add (or subtract) the numerators and keep the common denominator.

If $\frac{a}{b}$ and $\frac{c}{b}$ are two rational expressions, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \qquad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b} \qquad \text{if } b \neq 0 \tag{4}$$

EXAMPLE 4 **Adding and Subtracting Rational Expressions with Equal Denominators**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{2x^2-4}{2x+5} + \dfrac{x+3}{2x+5} \quad x \neq -\dfrac{5}{2}$ (b) $\dfrac{x}{x-3} - \dfrac{3x+2}{x-3} \quad x \neq 3$

Solution (a) $$\frac{2x^2-4}{2x+5} + \frac{x+3}{2x+5} = \frac{(2x^2-4)+(x+3)}{2x+5}$$
$$= \frac{2x^2+x-1}{2x+5} = \frac{(2x-1)(x+1)}{2x+5}$$

(b) $$\frac{x}{x-3} - \frac{3x+2}{x-3} = \frac{x-(3x+2)}{x-3} = \frac{x-3x-2}{x-3}$$
$$= \frac{-2x-2}{x-3} = \frac{-2(x+1)}{x-3}$$

Now Work PROBLEM 23

If the denominators of two rational expressions to be added or subtracted are not equal, we can use the general formulas for adding and subtracting rational expressions.

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational expressions, then

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{ad+bc}{bd} \qquad \text{if } b \neq 0, d \neq 0 \tag{5a}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d} = \frac{ad-bc}{bd} \qquad \text{if } b \neq 0, d \neq 0 \tag{5b}$$

EXAMPLE 5 **Adding and Subtracting Rational Expressions with Unequal Denominators**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x-3}{x+4} + \dfrac{x}{x-2} \quad x \neq -4, 2$ (b) $\dfrac{x^2}{x^2-4} - \dfrac{1}{x} \quad x \neq -2, 0, 2$

Solution (a) $$\frac{x-3}{x+4} + \frac{x}{x-2} \underset{(5a)}{=} \frac{x-3}{x+4} \cdot \frac{x-2}{x-2} + \frac{x+4}{x+4} \cdot \frac{x}{x-2}$$
$$= \frac{(x-3)(x-2)+(x+4)(x)}{(x+4)(x-2)}$$
$$= \frac{x^2-5x+6+x^2+4x}{(x+4)(x-2)} = \frac{2x^2-x+6}{(x+4)(x-2)}$$

(b) $$\frac{x^2}{x^2-4} - \frac{1}{x} = \frac{x^2}{x^2-4}\cdot\frac{x}{x} - \frac{x^2-4}{x^2-4}\cdot\frac{1}{x} = \frac{x^2(x) - (x^2-4)(1)}{(x^2-4)\cdot x}$$

(5b)

$$= \frac{x^3 - x^2 + 4}{x(x-2)(x+2)}$$

Now Work PROBLEM 25

4 Use the Least Common Multiple Method

If the denominators of two rational expressions to be added (or subtracted) have common factors, we usually do not use the general rules given by equations (5a) and (5b). Just as with fractions, we use the **least common multiple (LCM) method**. The LCM method uses the polynomial of least degree that has each denominator polynomial as a factor.

The LCM Method for Adding or Subtracting Rational Expressions

The Least Common Multiple (LCM) Method requires four steps:

Step 1: Factor completely the polynomial in the denominator of each rational expression.

Step 2: The LCM of the denominators is the product of each unique factor, with each of these factors raised to a power equal to the greatest number of times that the factor occurs in any denominator.

Step 3: Write each rational expression using the LCM as the common denominator.

Step 4: Add or subtract the rational expressions using equation (4).

We begin with an example that requires only Steps 1 and 2.

EXAMPLE 6 Finding the Least Common Multiple

Find the least common multiple of the following pair of polynomials:

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

Solution

Step 1: The polynomials are already factored completely as

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

Step 2: Start by writing the factors of the left-hand polynomial. (Or you could start with the one on the right.)

$$x(x-1)^2(x+1)$$

Now look at the right-hand polynomial. Its first factor, 4, does not appear in our list, so we insert it.

$$4x(x-1)^2(x+1)$$

The next factor, $x-1$, is already in our list, so no change is necessary. The final factor is $(x+1)^3$. Since our list has $x+1$ to the first power only, we replace $x+1$ in the list by $(x+1)^3$. The LCM is

$$4x(x-1)^2(x+1)^3$$

Notice that the LCM is, in fact, the polynomial of least degree that contains $x(x-1)^2(x+1)$ and $4(x-1)(x+1)^3$ as factors.

EXAMPLE 7 **Using the Least Common Multiple to Add Rational Expressions**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{x}{x^2 + 3x + 2} + \frac{2x - 3}{x^2 - 1} \qquad x \neq -2, -1, 1$$

Solution STEP 1: Factor completely the polynomials in the denominators.

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$
$$x^2 - 1 = (x - 1)(x + 1)$$

STEP 2: The LCM is $(x + 2)(x + 1)(x - 1)$. Do you see why?

STEP 3: Write each rational expression using the LCM as the denominator.

$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x + 2)(x + 1)} = \frac{x}{(x + 2)(x + 1)} \cdot \frac{x - 1}{x - 1} = \frac{x(x - 1)}{(x + 2)(x + 1)(x - 1)}$$

Multiply numerator and denominator by $x - 1$ to get the LCM in the denominator

$$\frac{2x - 3}{x^2 - 1} = \frac{2x - 3}{(x - 1)(x + 1)} = \frac{2x - 3}{(x - 1)(x + 1)} \cdot \frac{x + 2}{x + 2} = \frac{(2x - 3)(x + 2)}{(x - 1)(x + 1)(x + 2)}$$

Multiply numerator and denominator by $x + 2$ to get the LCM in the denominator

STEP 4: Now add by using equation (4).

$$\frac{x}{x^2 + 3x + 2} + \frac{2x - 3}{x^2 - 1} = \frac{x(x - 1)}{(x + 2)(x + 1)(x - 1)} + \frac{(2x - 3)(x + 2)}{(x + 2)(x + 1)(x - 1)}$$

$$= \frac{(x^2 - x) + (2x^2 + x - 6)}{(x + 2)(x + 1)(x - 1)}$$

$$= \frac{3x^2 - 6}{(x + 2)(x + 1)(x - 1)} = \frac{3(x^2 - 2)}{(x + 2)(x + 1)(x - 1)}$$

Now Work PROBLEM 29

5 Simplify Complex Rational Expressions

When sums and/or differences of rational expressions appear as the numerator and/or denominator of a quotient, the quotient is called a **complex rational expression**.*
For example,

$$\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \quad \text{and} \quad \frac{\frac{x^2}{x^2 - 4} - 3}{\frac{x - 3}{x + 2} - 1}$$

are complex rational expressions. To **simplify** a complex rational expression means to write it as a rational expression reduced to lowest terms. This can be accomplished in either of two ways.

*Some texts use the term **complex fraction**.

Simplifying a Complex Rational Expression

OPTION 1: Treat the numerator and denominator of the complex rational expression separately, performing whatever operations are indicated and simplifying the results. Follow this by simplifying the resulting rational expression.

OPTION 2: Find the LCM of the denominators of all rational expressions that appear in the complex rational expression. Multiply the numerator and denominator of the complex rational expression by the LCM and simplify the result.

We use both options in the next example. By carefully studying each option, you can discover situations in which one may be easier to use than the other.

EXAMPLE 8 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{1}{2}+\dfrac{3}{x}}{\dfrac{x+3}{4}} \qquad x \neq -3, 0$

Solution *Option 1:* First, perform the indicated operation in the numerator, and then divide.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\text{Rule for adding quotients}}{=} \frac{\frac{1\cdot x+2\cdot 3}{2\cdot x}}{\frac{x+3}{4}} = \frac{\frac{x+6}{2x}}{\frac{x+3}{4}} \underset{\text{Rule for dividing quotients}}{=} \frac{x+6}{2x}\cdot\frac{4}{x+3}$$

$$\underset{\text{Rule for multiplying quotients}}{=} \frac{(x+6)\cdot 4}{2\cdot x\cdot(x+3)} = \frac{2\cdot 2\cdot(x+6)}{2\cdot x\cdot(x+3)} = \frac{2(x+6)}{x(x+3)}$$

Option 2: The rational expressions that appear in the complex rational expression are

$$\frac{1}{2}, \quad \frac{3}{x}, \quad \frac{x+3}{4}$$

The LCM of their denominators is $4x$. Multiply the numerator and denominator of the complex rational expression by $4x$ and then simplify.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\substack{\text{Multiply the}\\ \text{numerator and}\\ \text{denominator by } 4x.}}{=} \frac{4x\cdot\left(\frac{1}{2}+\frac{3}{x}\right)}{4x\cdot\left(\frac{x+3}{4}\right)} \underset{\substack{\text{Use the Distributive Property}\\ \text{in the numerator.}}}{=} \frac{4x\cdot\frac{1}{2}+4x\cdot\frac{3}{x}}{\frac{4x\cdot(x+3)}{4}}$$

$$= \frac{\cancel{2}\cdot 2x\cdot\frac{1}{\cancel{2}}+4\cancel{x}\cdot\frac{3}{\cancel{x}}}{\frac{\cancel{4}x\cdot(x+3)}{\cancel{4}}} \underset{\text{Simplify.}}{=} \frac{2x+12}{x(x+3)} \underset{\text{Factor.}}{=} \frac{2(x+6)}{x(x+3)}$$

EXAMPLE 9 **Simplifying a Complex Rational Expression**

Simplify: $\dfrac{\dfrac{x^2}{x-4}+2}{\dfrac{2x-2}{x}-1} \quad x \neq 0, 2, 4$

Solution We use Option 1.

$$\frac{\dfrac{x^2}{x-4}+2}{\dfrac{2x-2}{x}-1} = \frac{\dfrac{x^2}{x-4}+\dfrac{2(x-4)}{x-4}}{\dfrac{2x-2}{x}-\dfrac{x}{x}} = \frac{\dfrac{x^2+2x-8}{x-4}}{\dfrac{2x-2-x}{x}}$$

$$= \frac{\dfrac{(x+4)(x-2)}{x-4}}{\dfrac{x-2}{x}} = \frac{(x+4)\cancel{(x-2)}}{x-4}\cdot\frac{x}{\cancel{x-2}}$$

$$= \frac{x(x+4)}{x-4}$$

Now Work PROBLEM 33

A.5 Assess Your Understanding

Concepts and Vocabulary

1. When the numerator and denominator of a rational expression contain no common factors (except 1 and −1), the rational expression is in lowest terms.
2. LCM is an abbreviation for least common multiple.
3. ***True or False*** The rational expression $\dfrac{2x^3-4x}{x-2}$ is reduced to lowest terms. True
4. ***True or False*** The LCM of $2x^3+6x^2$ and $6x^4+4x^3$ is $4x^3(x+1)$. False
5. ***Multiple Choice*** Choose the statement that is not true. Assume $b \neq 0, c \neq 0$, and $d \neq 0$ as necessary.
   (a) $\dfrac{ac}{bc}=\dfrac{a}{b}$ (b) $\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{a+c}{b}$
   (c) $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{ad-bc}{bd}$ (d) $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{ac}{bd}$ d
6. ***Multiple Choice*** Choose the rational expression that simplifies to −1.
   (a) $\dfrac{a-b}{b-a}$ (b) $\dfrac{a-b}{a-b}$ (c) $\dfrac{a+b}{a-b}$ (d) $\dfrac{b-a}{b+a}$ a

Skill Building

In Problems 7–14, reduce each rational expression to lowest terms.

7. $\dfrac{3x+9}{x^2-9}$ $\dfrac{3}{x-3}$
8. $\dfrac{4x^2+8x}{12x+24}$ $\dfrac{x}{3}$
9. $\dfrac{x^2-2x}{3x-6}$ $\dfrac{x}{3}$
10. $\dfrac{15x^2+24x}{3x^2}$ $\dfrac{5x+8}{x}$
11. $\dfrac{24x^2}{12x^2-6x}$ $\dfrac{4x}{2x-1}$
12. $\dfrac{x^2+4x+4}{x^2-4}$ $\dfrac{x+2}{x-2}$
13. $\dfrac{y^2-25}{2y^2-8y-10}$ $\dfrac{y+5}{2(y+1)}$
14. $\dfrac{3y^2-y-2}{3y^2+5y+2}$ $\dfrac{y-1}{y+1}$

In Problems 15–36, perform the indicated operation and simplify the result. Leave your answer in factored form.

*15. $\dfrac{3x+6}{5x^2}\cdot\dfrac{x}{x^2-4}$
16. $\dfrac{3}{2x}\cdot\dfrac{x^2}{6x+10}$ $\dfrac{3x}{4(3x+5)}$
*17. $\dfrac{4x^2}{x^2-16}\cdot\dfrac{x^3-64}{2x}$
*18. $\dfrac{12}{x^2+x}\cdot\dfrac{x^3+1}{4x-2}$
19. $\dfrac{\frac{8x}{x^2-1}}{\frac{10x}{x+1}}$ $\dfrac{4}{5(x-1)}$
20. $\dfrac{\frac{x-2}{4x}}{\frac{x^2-4x+4}{12x}}$ $\dfrac{3}{x-2}$
21. $\dfrac{\frac{4-x}{4+x}}{\frac{4x}{x^2-16}}$ $\dfrac{(x-4)^2}{4x}$
22. $\dfrac{\frac{3+x}{3-x}}{\frac{x^2-9}{9x^3}}$ $\dfrac{9x^3}{(x-3)^2}$
*23. $\dfrac{x^2}{2x-3}-\dfrac{4}{2x-3}$
*24. $\dfrac{3x^2}{2x-1}-\dfrac{9}{2x-1}$
*25. $\dfrac{x}{x^2-4}+\dfrac{1}{x}$
*26. $\dfrac{x-1}{x^3}+\dfrac{x}{x^2+1}$

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

27. $\dfrac{x}{x^2 - 7x + 6} - \dfrac{x}{x^2 - 2x - 24}$

28. $\dfrac{x}{x - 3} - \dfrac{x + 1}{x^2 + 5x - 24}$

29. $\dfrac{4x}{x^2 - 4} - \dfrac{2}{x^2 + x - 6}$

30. $\dfrac{3x}{x - 1} - \dfrac{x - 4}{x^2 - 2x + 1}$

31. $\dfrac{3}{(x - 1)^2(x + 1)} + \dfrac{2}{(x - 1)(x + 1)^2}$

32. $\dfrac{2}{(x + 2)^2(x - 1)} - \dfrac{6}{(x + 2)(x - 1)^2}$

33. $\dfrac{1 + \dfrac{1}{x}}{1 - \dfrac{1}{x}}$ $\dfrac{x + 1}{x - 1}$

34. $\dfrac{4 + \dfrac{1}{x^2}}{3 - \dfrac{1}{x^2}}$ $\dfrac{4x^2 + 1}{3x^2 - 1}$

35. $\dfrac{\dfrac{x - 2}{x + 2} + \dfrac{x - 1}{x + 1}}{\dfrac{x}{x + 1} - \dfrac{2x - 3}{x}}$

36. $\dfrac{\dfrac{2x + 5}{x} - \dfrac{x}{x - 3}}{\dfrac{x^2}{x - 3} - \dfrac{(x + 1)^2}{x + 3}}$

Applications and Extensions

In Problems 37–44, expressions that occur in calculus are given. Reduce each expression to lowest terms.

37. $\dfrac{(2x + 3) \cdot 3 - (3x - 5) \cdot 2}{(3x - 5)^2}$ $\dfrac{19}{(3x - 5)^2}$

38. $\dfrac{(4x + 1) \cdot 5 - (5x - 2) \cdot 4}{(5x - 2)^2}$ $\dfrac{13}{(5x - 2)^2}$

39. $\dfrac{x \cdot 2x - (x^2 + 1) \cdot 1}{(x^2 + 1)^2}$ $\dfrac{(x + 1)(x - 1)}{(x^2 + 1)^2}$

40. $\dfrac{x \cdot 2x - (x^2 - 4) \cdot 1}{(x^2 - 4)^2}$ $\dfrac{x^2 + 4}{(x + 2)^2(x - 2)^2}$

41. $\dfrac{(3x + 1) \cdot 2x - x^2 \cdot 3}{(3x + 1)^2}$ $\dfrac{x(3x + 2)}{(3x + 1)^2}$

42. $\dfrac{(2x - 5) \cdot 3x^2 - x^3 \cdot 2}{(2x - 5)^2}$ $\dfrac{x^2(4x - 15)}{(2x - 5)^2}$

43. $\dfrac{(x^2 + 1) \cdot 3 - (3x + 4) \cdot 2x}{(x^2 + 1)^2}$ $-\dfrac{(x + 3)(3x - 1)}{(x^2 + 1)^2}$

44. $\dfrac{(x^2 + 9) \cdot 2 - (2x - 5) \cdot 2x}{(x^2 + 9)^2}$ $-\dfrac{2(x^2 - 5x - 9)}{(x^2 + 9)^2}$

45. **The Lensmaker's Equation** The focal length f of a lens with index of refraction n is

$$\frac{1}{f} = (n - 1)\left[\frac{1}{R_1} + \frac{1}{R_2}\right]$$

where R_1 and R_2 are the radii of curvature of the front and back surfaces of the lens. Express f as a rational expression. Evaluate the rational expression for $n = 1.5$, $R_1 = 0.1$ meter, and $R_2 = 0.2$ meter.

46. **Electrical Circuits** An electrical circuit contains three resistors connected in parallel. If the resistance of each is R_1, R_2, and R_3 ohms, respectively, their combined resistance R is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Express R as a rational expression. Evaluate R for $R_1 = 5$ ohms, $R_2 = 4$ ohms, and $R_3 = 10$ ohms.

Explaining Concepts: Discussion and Writing

47. The following expressions are called **continued fractions:**

$$1 + \frac{1}{x}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{x}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}}}, \quad \ldots$$

Each simplifies to an expression of the form

$$\frac{ax + b}{bx + c}$$

Trace the successive values of a, b, and c as you "continue" the fraction. Can you discover the patterns that these values follow? Go to the library and research Fibonacci numbers. Write a report on your findings.

48. Explain to a fellow student when you would use the LCM method to add two rational expressions. Give two examples of adding two rational expressions, one in which you use the LCM and the other in which you do not.

49. Which of the two methods given in the text for simplifying complex rational expressions do you prefer? Write a brief paragraph stating the reasons for your choice.

A.6 Solving Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Factoring Polynomials (Section A.3, pp. A27–A28)
- Zero-Product Property (Section A.1, p. A4)
- Square Roots (Section A.1, pp. A9–A10)
- Absolute Value (Section A.1, pp. A5–A6)

Now Work the 'Are You Prepared?' problems on page A51.

OBJECTIVES
1. Solve Equations by Factoring (p. A46)
2. Solve Equations Involving Absolute Value (p. A46)
3. Solve a Quadratic Equation by Factoring (p. A47)
4. Solve a Quadratic Equation by Completing the Square (p. A48)
5. Solve a Quadratic Equation Using the Quadratic Formula (p. A49)

An **equation in one variable** is a statement in which two expressions, at least one containing the variable, are equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable. Unless otherwise restricted, the admissible values of the variable are those in the domain of the variable. These admissible values of the variable, if any, that result in a true statement are called **solutions**, or **roots**, of the equation. To **solve an equation** means to find all the solutions of the equation.

For example, the following are all equations in one variable, x:

$$x + 5 = 9 \qquad x^2 + 5x = 2x - 2 \qquad \frac{x^2 - 4}{x + 1} = 0 \qquad \sqrt{x^2 + 9} = 5$$

The first of these statements, $x + 5 = 9$, is true when $x = 4$ and false for any other choice of x. That is, 4 is the solution of the equation $x + 5 = 9$. We also say that 4 **satisfies** the equation $x + 5 = 9$, because, when 4 is substituted for x, a true statement results.

Sometimes an equation will have more than one solution. For example, the equation

$$\frac{x^2 - 4}{x + 1} = 0$$

has -2 and 2 as solutions.

Usually, we write the solutions of an equation as a set, called the **solution set** of the equation. For example, the solution set of the equation $x^2 - 9 = 0$ is $\{-3, 3\}$.

Some equations have no real solution. For example, $x^2 + 9 = 5$ has no real solution, because there is no real number whose square, when added to 9, equals 5.

An equation that is satisfied for every value of the variable for which both sides are defined is called an **identity**. For example, the equation

$$3x + 5 = x + 3 + 2x + 2$$

is an identity, because this statement is true for any real number x.

One method for solving an equation is to replace the original equation by a succession of **equivalent equations**, equations having the same solution set, until an equation with an obvious solution is obtained.

For example, consider the following succession of equivalent equations:

$$2x + 3 = 13$$
$$2x = 10$$
$$x = 5$$

We conclude that the solution set of the original equation is $\{5\}$.

How do we obtain equivalent equations? In general, there are five ways.

Procedures That Result in Equivalent Equations

- Interchange the two sides of the equation:

  Replace $3 = x$ by $x = 3$

- Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on:

  Replace $(x + 2) + 6 = 2x + (x + 1)$

  by $x + 8 = 3x + 1$

- Add or subtract the same expression on both sides of the equation:

  Replace $3x - 5 = 4$

  by $(3x - 5) + 5 = 4 + 5$

- Multiply or divide both sides of the equation by the same nonzero expression:

  Replace $\dfrac{3x}{x-1} = \dfrac{6}{x-1} \quad x \neq 1$

  by $\dfrac{3x}{x-1} \cdot (x - 1) = \dfrac{6}{x-1} \cdot (x - 1)$

- If one side of the equation is 0 and the other side can be factored, then write it as the product of factors:

  Replace $x^2 - 3x = 0$

  by $x(x - 3) = 0$

WARNING Squaring both sides of an equation does not necessarily lead to an equivalent equation. For example, $x = 3$ has one solution, but $x^2 = 9$ has two solutions, -3 and 3.

Whenever it is possible to solve an equation in your head, do so. For example,

- The solution of $2x = 8$ is 4.
- The solution of $3x - 15 = 0$ is 5.

Now Work PROBLEM 15

EXAMPLE 1 Solving an Equation

Solve the equation: $3x - 5 = 4$

Solution Replace the original equation by a succession of equivalent equations.

$$\begin{aligned} 3x - 5 &= 4 \\ (3x - 5) + 5 &= 4 + 5 && \text{Add 5 to both sides} \\ 3x &= 9 && \text{Simplify} \\ \frac{3x}{3} &= \frac{9}{3} && \text{Divide both sides by 3} \\ x &= 3 && \text{Simplify} \end{aligned}$$

The last equation, $x = 3$, has the solution 3. All these equations are equivalent, so 3 is the only solution of the original equation, $3x - 5 = 4$.

✓**Check:** Check the solution by substituting 3 for x in the original equation.

$$3x - 5 = 3 \cdot 3 - 5 = 9 - 5 = 4$$

The solution checks. The solution set is $\{3\}$.

Now Work PROBLEMS 29 AND 35

1 Solve Equations by Factoring

If an equation can be written as a product of factors equal to 0, use the Zero-Product Property to set each factor equal to 0 and solve the resulting equations.

EXAMPLE 2

Solving Equations by Factoring

Solve the equations: (a) $x^3 = 4x$ (b) $x^3 - x^2 - 4x + 4 = 0$

Solution (a) Begin by collecting all terms on one side. This results in 0 on one side and an expression to be factored on the other.

$$x^3 = 4x$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0 \quad \text{Factor}$$
$$x(x - 2)(x + 2) = 0 \quad \text{Factor again.}$$
$$x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{Use the Zero-Product Property.}$$
$$x = 0 \quad \text{or} \quad x = 2 \quad \text{or} \quad x = -2 \quad \text{Solve for } x.$$

Check: $x = -2$: $(-2)^3 = -8$ and $4(-2) = -8$ $\quad$ -2 is a solution.

$x = 0$: $0^3 = 0$ and $4 \cdot 0 = 0$ $\quad$ 0 is a solution.

$x = 2$: $2^3 = 8$ and $4 \cdot 2 = 8$ $\quad$ 2 is a solution.

The solution set is $\{-2, 0, 2\}$.

(b) Group the terms of $x^3 - x^2 - 4x + 4 = 0$ as follows:

$$(x^3 - x^2) - (4x - 4) = 0$$

Factor out x^2 from the first grouping and 4 from the second.

$$x^2(x - 1) - 4(x - 1) = 0$$

This reveals the common factor $(x - 1)$, so

$$(x^2 - 4)(x - 1) = 0$$
$$(x - 2)(x + 2)(x - 1) = 0 \quad \text{Factor again.}$$
$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{Use the Zero-Product Property.}$$
$$x = 2 \qquad x = -2 \qquad x = 1 \quad \text{Solve for } x.$$

Check:

$x = -2$: $(-2)^3 - (-2)^2 - 4(-2) + 4 = -8 - 4 + 8 + 4 = 0$ $\quad$ -2 is a solution.

$x = 1$: $1^3 - 1^2 - 4(1) + 4 = 1 - 1 - 4 + 4 = 0$ $\quad$ 1 is a solution.

$x = 2$: $2^3 - 2^2 - 4(2) + 4 = 8 - 4 - 8 + 4 = 0$ $\quad$ 2 is a solution.

The solution set is $\{-2, 1, 2\}$.

Now Work PROBLEMS 39 AND 45

2 Solve Equations Involving Absolute Value

On the real number line, the absolute value of a equals the distance from the origin to the point whose coordinate is a. For example, there are two points whose distance from the origin is 5 units, -5 and 5. So the equation $|x| = 5$ has the solution set $\{-5, 5\}$.

EXAMPLE 3 **Solving an Equation Involving Absolute Value**

Solve the equation: $|x + 4| = 13$

Solution There are two possibilities:

$$x + 4 = 13 \quad \text{or} \quad x + 4 = -13$$
$$x = 9 \quad \text{or} \quad x = -17$$

The solution set is $\{-17, 9\}$.

Now Work PROBLEM 51

3 Solve a Quadratic Equation by Factoring

DEFINITION Quadratic Equation

A **quadratic equation** is an equation equivalent to one of the form

$$ax^2 + bx + c = 0 \qquad (1)$$

where a, b, and c are real numbers and $a \neq 0$.

A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form.**

Sometimes, a quadratic equation is called a **second-degree equation** because, when it is in standard form, the left side is a polynomial of degree 2.

When a quadratic equation is written in standard form, it may be possible to factor the expression on the left side into the product of two first-degree polynomials. Then, using the Zero-Product Property and setting each factor equal to 0, the resulting linear equations can be solved to obtain the solutions of the quadratic equation.

EXAMPLE 4 **Solving a Quadratic Equation by Factoring**

Solve the equation: $2x^2 = x + 3$

Solution Put the equation $2x^2 = x + 3$ in standard form by subtracting x and 3 from both sides.

$$2x^2 = x + 3$$
$$2x^2 - x - 3 = 0 \qquad \text{Subtract } x \text{ and 3 from both sides}$$

The left side may now be factored as

$$(2x - 3)(x + 1) = 0 \qquad \text{Factor.}$$

so that

$$2x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \qquad \text{Use the Zero-Product Property}$$
$$x = \frac{3}{2} \qquad x = -1 \qquad \text{Solve.}$$

The solution set is $\left\{-1, \frac{3}{2}\right\}$.

When the left side factors into two linear equations with the same solution, the quadratic equation is said to have a **repeated solution.** This solution is also called a **root of multiplicity 2,** or a **double root.**

EXAMPLE 5 **Solving a Quadratic Equation by Factoring**

Solve the equation: $9x^2 - 6x + 1 = 0$

Solution This equation is already in standard form, and the left side can be factored.

$$9x^2 - 6x + 1 = 0$$
$$(3x - 1)(3x - 1) = 0 \quad \text{Factor.}$$

so

$$x = \frac{1}{3} \quad \text{or} \quad x = \frac{1}{3} \quad \text{Solve for } x.$$

This equation has only the repeated solution $\frac{1}{3}$. The solution set is $\left\{\frac{1}{3}\right\}$.

Now Work PROBLEM 69

The Square Root Method

Suppose that we wish to solve the quadratic equation

$$x^2 = p \qquad (2)$$

where $p > 0$ is a positive number. Proceeding as in the earlier examples,

$$x^2 - p = 0 \quad \text{Put in standard form.}$$
$$(x - \sqrt{p})(x + \sqrt{p}) = 0 \quad \text{Factor (over the real numbers)}$$
$$x = \sqrt{p} \quad \text{or} \quad x = -\sqrt{p} \quad \text{Solve}$$

We have the following result:

If $x^2 = p$ and $p > 0$, then $x = \sqrt{p}$ or $x = -\sqrt{p}$. **(3)**

When statement (3) is used, it is called the **Square Root Method.** In statement (3), note that if $p > 0$ the equation $x^2 = p$ has two solutions, $x = \sqrt{p}$ and $x = -\sqrt{p}$. We usually abbreviate these solutions as $x = \pm\sqrt{p}$, which is read as "x equals plus or minus the square root of p."

For example, the two solutions of the equation

$$x^2 = 4$$

are

$$x = \pm\sqrt{4} \quad \text{Use the Square Root Method}$$

and, since $\sqrt{4} = 2$, we have

$$x = \pm 2$$

The solution set is $\{-2, 2\}$.

Now Work PROBLEM 83

4 Solve a Quadratic Equation by Completing the Square

EXAMPLE 6 **Solving a Quadratic Equation by Completing the Square**

Solve by completing the square: $2x^2 - 8x - 5 = 0$

Solution First, rewrite the equation so that the constant is on the right side.

$$2x^2 - 8x - 5 = 0$$
$$2x^2 - 8x = 5 \quad \text{Add 5 to both sides.}$$

Next, divide both sides by 2 so that the coefficient of x^2 is 1. (This enables us to complete the square at the next step.)

$$x^2 - 4x = \frac{5}{2}$$

Finally, complete the square by adding $\left[\frac{1}{2}(-4)\right]^2 = 4$ to both sides.

$$x^2 - 4x + 4 = \frac{5}{2} + 4 \qquad \text{Add 4 to both sides}$$

$$(x-2)^2 = \frac{13}{2} \qquad \text{Factor on the left; simplify on the right.}$$

$$x - 2 = \pm\sqrt{\frac{13}{2}} \qquad \text{Use the Square Root Method}$$

$$x - 2 = \pm\frac{\sqrt{26}}{2} \qquad \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}$$

$$x = 2 \pm \frac{\sqrt{26}}{2}$$

NOTE If we wanted an approximation, say rounded to two decimal places, of these solutions, we would use a calculator to get $\{-0.55, 4.55\}$.

The solution set is $\left\{2 - \frac{\sqrt{26}}{2}, 2 + \frac{\sqrt{26}}{2}\right\}$.

Now Work PROBLEM 87

5 Solve a Quadratic Equation Using the Quadratic Formula

We can use the method of completing the square to obtain a general formula for solving any quadratic equation

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

NOTE There is no loss in generality to assume that $a > 0$, since if $a < 0$ we can multiply by -1 to obtain an equivalent equation with a positive leading coefficient.

As in Example 6, rearrange the terms as

$$ax^2 + bx = -c \qquad a > 0$$

Since $a > 0$, divide both sides by a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now the coefficient of x^2 is 1. To complete the square on the left side, add the square of $\frac{1}{2}$ times the coefficient of x; that is, add

$$\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$

to both sides. Then

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \qquad (4)$$

Provided that $b^2 - 4ac \geq 0$, we can now use the Square Root Method to get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \text{The square root of a quotient equals the quotient of the square roots. Also, } \sqrt{4a^2} = 2a \text{ since } a > 0$$

(*continued*)

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Add } -\frac{b}{2a} \text{ to both sides.}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Combine the quotients on the right.}$$

What if $b^2 - 4ac$ is negative? Then equation (4) states that the left expression (a real number squared) equals the right expression (a negative number). Since this is impossible for real numbers, we conclude that if $b^2 - 4ac < 0$, the quadratic equation has no *real* solution. (We discuss quadratic equations for which the quantity $b^2 - 4ac < 0$ in detail in the next section.)

THEOREM Quadratic Formula

Consider the quadratic equation

$$\boxed{ax^2 + bx + c = 0 \qquad a \neq 0}$$

- If $b^2 - 4ac < 0$, this equation has no real solution.
- If $b^2 - 4ac \geq 0$, the real solution(s) of this equation is (are) given by the **quadratic formula**:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \qquad (5)$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation, because its value tells us whether the equation has real solutions. In fact, it also tells us how many solutions to expect.

Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$:

- If $b^2 - 4ac > 0$, there are two unequal real solutions.
- If $b^2 - 4ac = 0$, there is a repeated solution, a double root.
- If $b^2 - 4ac < 0$, there is no real solution.

When asked to find the real solutions of a quadratic equation, always evaluate the discriminant first to see if there are any real solutions.

EXAMPLE 7 Solving a Quadratic Equation Using the Quadratic Formula

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 - 5x + 1 = 0$$

Solution The equation is in standard form, so compare it to $ax^2 + bx + c = 0$ to find a, b, and c.

$$3x^2 - 5x + 1 = 0$$
$$ax^2 + bx + c = 0 \quad a = 3, b = -5, c = 1$$

With $a = 3$, $b = -5$, and $c = 1$, evaluate the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$$

Since $b^2 - 4ac > 0$, there are two real solutions, which can be found using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{13}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\frac{5 - \sqrt{13}}{6}, \frac{5 + \sqrt{13}}{6}\right\}$.

EXAMPLE 8 **Solving a Quadratic Equation Using the Quadratic Formula**

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 + 2 = 4x$$

Solution The equation, as given, is not in standard form.

$$3x^2 + 2 = 4x$$

$$3x^2 - 4x + 2 = 0 \quad \text{Put in standard form.}$$

$$ax^2 + bx + c = 0 \quad \text{Compare to standard form}$$

With $a = 3$, $b = -4$, and $c = 2$, the discriminant is

$$b^2 - 4ac = (-4)^2 - 4 \cdot 3 \cdot 2 = 16 - 24 = -8$$

Since $b^2 - 4ac < 0$, the equation has no real solution.

Now Work PROBLEMS 93 AND 99

SUMMARY

Steps for Solving a Quadratic Equation

To solve a quadratic equation, first put it in standard form:

$$ax^2 + bx + c = 0$$

Then:

STEP 1: Identify a, b, and c.

STEP 2: Evaluate the discriminant, $b^2 - 4ac$.

STEP 3:
- If the discriminant is negative, the equation has no real solution.
- If the discriminant is zero, the equation has one repeated real solution, a double root.
- If the discriminant is positive, the equation has two distinct real solutions.

If you can easily spot factors, use the factoring method to solve the equation. Otherwise, use the quadratic formula or the method of completing the square.

A.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Factor: $x^2 - 5x - 6$ (pp. A27–A28) $(x - 6)(x + 1)$

2. Factor: $2x^2 - x - 3$ (pp. A27–A28) $(2x - 3)(x + 1)$

3. The solution set of the equation $(x - 3)(3x + 5) = 0$ is ______. (p. A4) $\left\{-\frac{5}{3}, 3\right\}$

4. *True or False* $\sqrt{x^2} = |x|$. (pp. A9–A10) **True**

Concepts and Vocabulary

5. An equation that is satisfied for every choice of the variable for which both sides are defined is called a(n) <u>identity</u>.

*6. **True or False** The solution of the equation $3x - 8 = 0$ is $\frac{3}{8}$.

7. **True or False** Some equations have no solution. True

8. To solve the equation $x^2 + 5x = 0$ by completing the square, you would <u>add</u> the number <u>$\frac{25}{4}$</u> to both sides.

9. The quantity $b^2 - 4ac$ is called the <u>discriminant</u> of a quadratic equation. If it is <u>negative</u>, the equation has no real solution.

10. **True or False** Quadratic equations always have two real solutions. False

11. **True or False** If the discriminant of a quadratic equation is positive, then the equation has two solutions that are negatives of one another. False

12. **Multiple Choice** An admissible value for the variable that makes the equation a true statement is called a(n) ______ of the equation.
(a) identity (b) solution (c) degree (d) model b

13. **Multiple Choice** A quadratic equation is sometimes called a ______-degree equation.
(a) first (b) second (c) third (d) fourth b

14. **Multiple Choice** Which of the following quadratic equations is in standard form?
(a) $x^2 - 7x = 5$ (b) $9 = x^2$
(c) $(x + 5)(x - 4) = 0$ (d) $0 = 5x^2 - 6x - 1$ d

Skill Building

In Problems 15–80, solve each equation.

15. $3x = 21$ $\{7\}$
16. $3x = -24$ $\{-8\}$
17. $5x + 15 = 0$ $\{-3\}$
18. $3x + 18 = 0$ $\{-6\}$
19. $2x - 3 = 5$ $\{4\}$
20. $3x + 4 = -8$ $\{-4\}$
21. $\frac{1}{3}x = \frac{5}{12}$ $\left\{\frac{5}{4}\right\}$
22. $\frac{2}{3}x = \frac{9}{2}$ $\left\{\frac{27}{4}\right\}$
23. $6 - x = 2x + 9$ $\{-1\}$
24. $3 - 2x = 2 - x$ $\{1\}$
*25. $2(3 + 2x) = 3(x - 4)$
26. $3(2 - x) = 2x - 1$ $\left\{\frac{7}{5}\right\}$
*27. $8x - (2x + 1) = 3x - 10$
*28. $5 - (2x - 1) = 10$
29. $\frac{1}{2}x - 4 = \frac{3}{4}x$ $\{-16\}$
30. $1 - \frac{1}{2}x = 5$ $\{-8\}$
31. $0.9t = 0.4 + 0.1t$ $\{0.5\}$
32. $0.9t = 1 + t$ $\{-10\}$
33. $\frac{2}{y} + \frac{4}{y} = 3$ $\{2\}$
34. $\frac{4}{y} - 5 = \frac{5}{2y}$ $\left\{\frac{3}{10}\right\}$
35. $(x + 7)(x - 1) = (x + 1)^2$ $\{2\}$
36. $(x + 2)(x - 3) = (x - 3)^2$ $\{3\}$
37. $z(z^2 + 1) = 3 + z^3$ $\{3\}$
38. $w(4 - w^2) = 8 - w^3$ $\{2\}$
39. $x^2 = 9x$ $\{0, 9\}$
40. $x^3 = x^2$ $\{0, 1\}$
41. $t^3 - 9t^2 = 0$ $\{0, 9\}$
42. $4z^3 - 8z^2 = 0$ $\{0, 2\}$
43. $\frac{3}{2x - 3} = \frac{2}{x + 5}$ $\{21\}$
44. $\frac{-2}{x + 4} = \frac{-3}{x + 1}$ $\{-10\}$
45. $x^3 + 2x^2 - 4x - 8 = 0$ $\{-2, 2\}$
46. $x^3 + 50 = 2x^2 + 25x$ $\{-5, 2, 5\}$
*47. $\frac{2}{x - 2} = \frac{3}{x + 5} + \frac{10}{(x + 5)(x - 2)}$
*48. $\frac{1}{2x + 3} + \frac{1}{x - 1} = \frac{1}{(2x + 3)(x - 1)}$
49. $|2x| = 6$ $\{-3, 3\}$
50. $|3x| = 12$ $\{-4, 4\}$
51. $|2x + 3| = 5$ $\{-4, 1\}$
52. $|3x - 1| = 2$ $\left\{-\frac{1}{3}, 1\right\}$
53. $|1 - 4t| = 5$ $\left\{-1, \frac{3}{2}\right\}$
54. $|1 - 2z| = 3$ $\{-1, 2\}$
55. $|-2x| = 8$ $\{-4, 4\}$
56. $|-x| = 1$ $\{-1, 1\}$
57. $|-2|x = 4$ $\{2\}$
58. $|3|x = 9$ $\{3\}$
*59. $|x - 2| = -\frac{1}{2}$
*60. $|2 - x| = -1$
61. $|x^2 - 4| = 0$ $\{-2, 2\}$
62. $|x^2 - 9| = 0$ $\{-3, 3\}$
63. $|x^2 - 2x| = 3$ $\{-1, 3\}$
64. $|x^2 + x| = 12$ $\{-4, 3\}$
*65. $|x^2 + x - 1| = 1$
*66. $|x^2 + 3x - 2| = 2$
67. $x^2 = 4x$ $\{0, 4\}$
68. $x^2 = -8x$ $\{-8, 0\}$
*69. $z^2 + 4z - 12 = 0$
*70. $v^2 + 7v + 12 = 0$
*71. $2x^2 - 5x - 3 = 0$
*72. $3x^2 + 5x + 2 = 0$
*73. $x(x - 7) + 12 = 0$
74. $x(x + 1) = 12$ $\{-4, 3\}$
75. $4x^2 + 9 = 12x$ $\left\{\frac{3}{2}\right\}$
76. $25x^2 + 16 = 40x$ $\left\{\frac{4}{5}\right\}$
77. $6x - 5 = \frac{6}{x}$ $\left\{-\frac{2}{3}, \frac{3}{2}\right\}$
78. $x + \frac{12}{x} = 7$ $\{3, 4\}$
*79. $\frac{4(x - 2)}{x - 3} + \frac{3}{x} = \frac{-3}{x(x - 3)}$
80. $\frac{5}{x + 4} = 4 + \frac{3}{x - 2}$

In Problems 81–86, solve each equation by the Square Root Method.

81. $x^2 = 25$ $\{-5, 5\}$
82. $x^2 = 36$ $\{-6, 6\}$
83. $(x - 1)^2 = 4$ $\{-1, 3\}$
84. $(x + 2)^2 = 1$ $\{-3, -1\}$
85. $(2y + 3)^2 = 9$ $\{-3, 0\}$
86. $(3x - 2)^2 = 4$ $\left\{0, \frac{4}{3}\right\}$

*Due to space restrictions, answers to these exercises may be found in the Answers in the back of the text.

In Problems 87–92, solve each equation by completing the square.

87. $x^2 + 4x = 21$ $\{-7, 3\}$

88. $x^2 - 6x = 13$

89. $x^2 - \frac{1}{2}x - \frac{3}{16} = 0$ $\left\{-\frac{1}{4}, \frac{3}{4}\right\}$

90. $x^2 + \frac{2}{3}x - \frac{1}{3} = 0$ $\left\{-1, \frac{1}{3}\right\}$

91. $3x^2 + x - \frac{1}{2} = 0$

92. $2x^2 - 3x - 1 = 0$

In Problems 93–104, find the real solutions, if any, of each equation. Use the quadratic formula.

93. $x^2 - 4x + 2 = 0$

94. $x^2 + 4x + 2 = 0$

95. $x^2 - 5x - 1 = 0$

96. $x^2 + 5x + 3 = 0$

97. $2x^2 - 5x + 3 = 0$

98. $2x^2 + 5x + 3 = 0$

99. $4y^2 - y + 2 = 0$

100. $4t^2 + t + 1 = 0$

101. $4x^2 = 1 - 2x$

102. $2x^2 = 1 - 2x$

103. $x^2 + \sqrt{3}\,x - 3 = 0$

104. $x^2 + \sqrt{2}\,x - 2 = 0$

In Problems 105–110, use the discriminant to determine whether each quadratic equation has two unequal real solutions, a repeated real solution, or no real solution without solving the equation.

105. $x^2 - 5x + 7 = 0$

106. $x^2 + 5x + 7 = 0$

107. $9x^2 - 30x + 25 = 0$

108. $25x^2 - 20x + 4 = 0$

109. $3x^2 + 5x - 8 = 0$

110. $2x^2 - 3x - 4 = 0$

Applications and Extensions

In Problems 111–116, solve each equation. The letters a, b, and c are constants.

111. $ax - b = c, \quad a \neq 0$ $\quad x = \frac{b + c}{a}$

112. $1 - ax = b, \quad a \neq 0$ $\quad x = \frac{1 - b}{a}$

113. $\frac{x}{a} + \frac{x}{b} = c, \quad a \neq 0, b \neq 0, a \neq -b$

114. $\frac{a}{x} + \frac{b}{x} = c, \quad c \neq 0$ $\quad x = \frac{a + b}{c}$

115. $\frac{1}{x - a} + \frac{1}{x + a} = \frac{2}{x - 1}$ $\quad x = a^2$

116. $\frac{b + c}{x + a} = \frac{b - c}{x - a}, \quad c \neq 0, \quad a \neq 0$ $\quad x = \frac{ab}{c}$

Problems 117–122 list some formulas that occur in applications. Solve each formula for the indicated variable.

117. **Electricity** $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R $\quad R = \frac{R_1R_2}{R_1 + R_2}$

118. **Finance** $A = P(1 + rt)$ for r $\quad r = \frac{A - P}{Pt}$

119. **Mechanics** $F = \frac{mv^2}{R}$ for R $\quad R = \frac{mv^2}{F}$

120. **Chemistry** $PV = nRT$ for T $\quad T = \frac{PV}{nR}$

121. **Mathematics** $S = \frac{a}{1 - r}$ for r $\quad r = \frac{S - a}{S}$

122. **Mechanics** $v = -gt + v_0$ for t $\quad t = \frac{v_0 - v}{g}$

123. Show that the sum of the roots of a quadratic equation is $-\frac{b}{a}$.

124. Show that the product of the roots of a quadratic equation is $\frac{c}{a}$.

125. Find k so that the equation $kx^2 + x + k = 0$ has a repeated real solution.

126. Find k so that the equation $x^2 - kx + 4 = 0$ has a repeated real solution. $\quad k = -4$ **or** 4

127. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the negatives of the real solutions of the equation $ax^2 - bx + c = 0$. Assume that $b^2 - 4ac \geq 0$.

128. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the reciprocals of the real solutions of the equation $cx^2 + bx + a = 0$. Assume that $b^2 - 4ac \geq 0$.

Explaining Concepts: Discussion and Writing

129. Which of the following pairs of equations are equivalent? Explain. **b**

(a) $x^2 = 9; \quad x = 3$

(b) $x = \sqrt{9}; \quad x = 3$

(c) $(x - 1)(x - 2) = (x - 1)^2; \quad x - 2 = x - 1$

130. The equation

$$\frac{5}{x + 3} + 3 = \frac{8 + x}{x + 3}$$

has no solution, yet when we go through the process of solving it, we obtain $x = -3$. Write a brief paragraph to explain what causes this to happen.