Functional Analysis– Spring 2024

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p. 126 #8 Show that the dual space of the space c_0 is ℓ^1.(Cf. Prob. 1 in Sec. 2.3.)

Want to show that

1. every element of c_0' is an element of ℓ^1
   Let (e_k) be the unique Shauder basis for ℓ^1 where $e_k = (\delta_{jk})$. Let $x = (\xi_j) \in c_0$, that is $\lim_{j\to\infty} \xi_j = 0$ which has the unique representation $x = \sum_{j=1}^{\infty} \xi_j e_j$. Let $f \in c_0'$, that is $f : c_0 \to \mathbb{R}$ which is linear and bounded. Therefore,
$$f(x) = \sum_{j=1}^{\infty} \xi_j f(e_j)$$
$$|f(e_j)| \le \|f\| \, \|e_j\| = \|f\|$$
$$\|f(x)\| \le \|f\| \left| \sum_{j=1}^{\infty} \xi_j \right| \le \|f\| \sum_{j=1}^{\infty} |\xi_j| = \|f\| \, \|x\|_{\ell^1}$$
   which means that $f \in \ell^1$.
2. that the norm over c_0' is the norm over ℓ^1. want to show that $|f(x)| = \|x\|$. Let $\gamma = \sup_j f(e_j)$
$$|f(x)| = \left| \sum_{j=1}^{\infty} \xi_j f(e_j) \right| \le \gamma \sum_{j=1}^{\infty} |\xi_j| = \gamma \, \|x\|$$

p. 135 #9 Prove

$$\operatorname{Re} \langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right)$$
$$\operatorname{Im} \langle x, y \rangle = \frac{1}{4} \left(\|x+iy\|^2 - \|x-iy\|^2 \right)$$

Remember that

$$\begin{aligned}
\|x+y\|^2 &= \|x\|^y + \|y\|^2 + 2\operatorname{Re}\langle x, y \rangle \\
\text{and} \quad \|x+y\|^2 + \|x-y\|^2 &= 2\,\|x\|^2 + 2\,\|y\|^2 \\
\text{thus } 2\operatorname{Re}\langle x, y \rangle &= \|x+y\|^2 - \|x\|^2 - \|y\|^2 \\
&= \|x+y\|^2 - (\|x\|^2 + \|y\|^2) \\
&= \|x+y\|^2 - \frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2 \right) \\
4\operatorname{Re}\langle x, y \rangle &= \|x+y\|^2 - \|x-y\|^2 \\
\operatorname{Re}\langle x, y \rangle &= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right)
\end{aligned}$$

Notice that

$$\begin{aligned}
\text{Notice} \quad \|x+iy\|^2 + \|x-iy\|^2 &= 2\,\|x\|^2 + 2\,\|y\|^2 \\
\|x-iy\|^2 &= \langle x-iy, x-iy \rangle \\
&= \langle x, x-iy \rangle + \langle iy, x-iy \rangle \\
&= \langle x, x \rangle + \langle x, iy \rangle - \langle iy, x \rangle + \langle iy, iy \rangle \\
&= \|x\|^2 + |i|^2\,\|y\|^2 - i\langle x, y \rangle - i\overline{\langle x, y \rangle} \\
&= \|x\|^2 + \|y\|^2 + 2\operatorname{Im}\langle x, y \rangle \\
2\operatorname{Im}\langle x, y \rangle &= \|x-iy\|^2 - \left(\|x\|^2 + \|y\|^2 \right) \\
&= \|x-iy\|^2 - \frac{1}{2}\left(\|x+iy\|^2 + \|x-iy\|^2 \right) \\
4\operatorname{Im}\langle x, y \rangle &= \|x-iy\|^2 - \|x+iy\|^2
\end{aligned}$$

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7. Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ (see Fig. 25.)

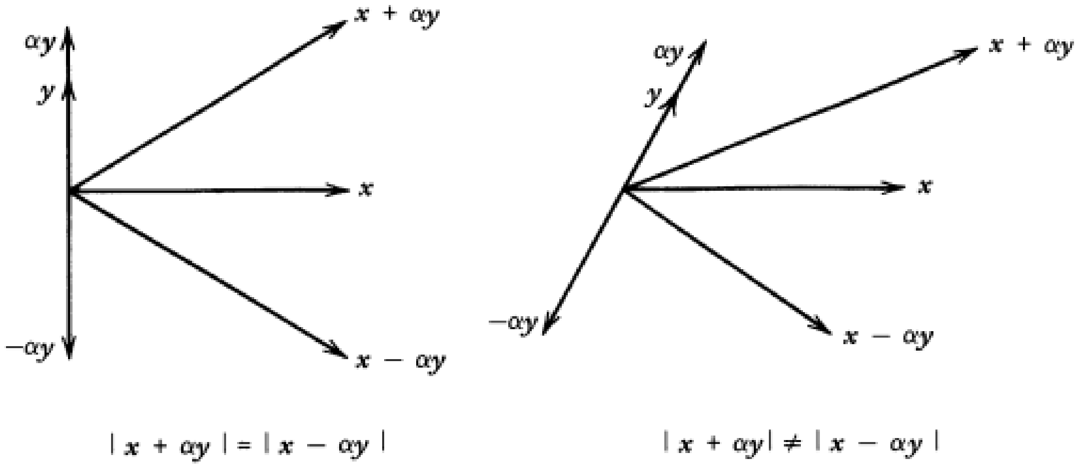

Fig. 25. Illustration of Prob. 7 in the Euclidean plane $\mathbf{R}^2$

Assuming that $x \perp y$ then $\langle x, y \rangle = 0$ and

$$\begin{aligned}
\|x + \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \, \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \\
&= \|x\|^2 + |\alpha|^2 \, \|y\|^2 \\
\|x - \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \, \|y\|^2 - 2|\alpha| \operatorname{Re} \langle x, y \rangle \\
&= \|x\|^2 + |\alpha|^2 \, \|y\|^2
\end{aligned}$$

Asuming that they are equal

$$\begin{aligned}
\|x + \alpha y\|^2 - \|x - \alpha y\|^2 &= \Big(\|x\|^2 + |\alpha|^2 \, \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \Big) - \Big(\|x\|^2 + |\alpha|^2 \, \|y\|^2 - 2 \, | \, \alpha \, | \operatorname{Re} \langle x, y \rangle \Big) \\
&= 4|\alpha| \operatorname{Re} \langle x, y \rangle
\end{aligned}$$

which can only be zero when $\operatorname{Re} \langle x, y \rangle = 0$ or $x \perp y$.

8. Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

Assuming that $x \perp y$ then

$$\begin{aligned}
\|x + \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \, \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \\
&= \|x\|^2 + |\alpha|^2 \, \|y\|^2
\end{aligned}$$

$|\alpha| \geq 0$ for all α as well as $\|y\| \geq 0$ for all y. Thus, $\|x + \alpha y\| \geq \|x\|$.
Assuming that this statement is true then for all $\alpha \in \mathbb{C}$ and $y \in Y$

$$\begin{aligned}
\|x\|^2 + |\alpha|^2 \, \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle &\geq \|x\|^2 \\
|\alpha|^2 \, \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle &\geq 0 \\
|\alpha|^2 \, \|y\|^2 &\geq -2|\alpha| \operatorname{Re} \langle x, y \rangle
\end{aligned}$$

the left side is positive and the right side is negative therefore $\operatorname{Re} \langle x, y \rangle = 0$ and $x \perp y$.

9. Let V be the vector space of all continuous complex-valued functions on $J = [a, b]$. Let $X_1 = (V, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{t \in J} |x(t)|$; and let $X_2 = (V, \|\cdot\|_2)$, where

$$\|x\|_2 = \langle x, x \rangle^{1/2}, \quad \langle x, y \rangle = \int_a^b x(t)\overline{y(t)}dt$$

Show that the identity mapping $x \mapsto x$ of X_1 onto X_2 is continuous.
(It is not a homeomorphism. X_2 is not complete.)

Let $T : X_1 \to X_2$ such that $Tx = x$. We want to show that given any $\epsilon > 0$ when $\|Tx - Ty\| < \epsilon$, then $\|x - y\| \leq \delta$ for some $\delta > 0$ dependent on ϵ.

$$\begin{aligned}
\epsilon > \|Tx - Ty\|_2^2 &= \langle x - y, x - y \rangle \\
&= \int_a^b (x(t) - y(t))\overline{(x(t) - y(t))}dt \\
&= \int_a^b \mathrm{Re}\,(x(t) - y(t))^2 + \mathrm{Im}\,(x(t) - y(t))^2\,dt \\
&\geq \int_a^b \|x(t) - y(t)\|_2^2\,dt
\end{aligned}$$

by the Extreme Value Theorem, there exists $p \in J$ such that $x(p) \geq x(t) - y(t), \forall t \in J$ i.e., $|x(p) - y(p)| = \|x - y\|_\infty$. Thus.

$$\int_a^b \|x(t) - y(t)\|_2^2\,dt \leq (b - a)(x(p) - y(p))^2 = (b - a)\,\|x - y\|_\infty^2 < \delta$$

10. **(Zero Operator)** Let $T : X \to X$ be a bounded linear operator on a complex inner product space X. If $\langle Tx, x \rangle = 0$ for all $x \in X$, show that $T = 0$.
Show that this does not hold in the case of a real inner product space. *Hint.* Consider a rotation of the Euclidean plane.

$$\begin{aligned}
0 &= \langle T(x + y), x + y \rangle \\
&= \langle Tx + Ty, x + y \rangle \\
&= \langle Tx, x + y \rangle + \langle Ty, x + y \rangle \\
&= \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle \\
&= \langle Tx, y \rangle + \langle Ty, x \rangle
\end{aligned}$$

The Inner Product is positive definite, which means that the only way that the right side can be equal to zero is if $Tx = 0$ for $x \in x$.

p. 150 #2, 3a, 6,

2. Show that the subset $M = \{y = (\eta_j) \mid \sum \eta_j = 1\}$ of complex space $\mathbf{C}^n$ (cf 3.1-4) is complete and convex. Find the vector of minimum norm in M.

$$\langle x, y \rangle = \sum \xi_j \overline{\eta_j}$$
$$\|x\|^2 = \sum \xi_j \overline{\xi_j}$$

3. (a) Show that the vector space X of all real-valued continuous functions on $[-1, 1]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$.

Define the inner product over the set of continuous functions on $[-1, 1]$ as

$$\langle x, y \rangle = \int_{-1}^{1} x(t)y(t)dt$$

Let E be the set of even functions. That is for all $f \in E$ then $f(x) = f(-x)$. Let $g \in E^{\perp}$ then

$$\begin{aligned} 0 = \langle f, g \rangle &= \int_{-1}^{1} f(t)g(t)dt \\ &= \int_{-1}^{0} f(t)g(t)dt + \int_{0}^{1} f(t)g(t)dt \end{aligned}$$

notice that the function $h(t) = f(t)g(t)$ on $t \in [-1, 1]$ is odd. Yet, f is even, thus g must be odd. Since g is arbitrary $E^{\perp}$ is filled with odd functions. And we know that $X = E \oplus E^{\perp}$.

6. Show that $Y = \{x \mid x = (\xi_j) \in \ell^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of ℓ^2 and find $Y^{\perp}$. What is $Y^{\perp}$ if $Y = \text{span}\{e_1, \cdots, e_n\} \subset \ell^2$, where $e_j = (\delta_{jk})$?

Let $x = (\xi_j), y = (\eta_j) \in Y$. Then, $x + \alpha y = (\xi_j + \alpha\eta_j)$. Whenever, j is even $\xi_j = \eta_j = 0$ and $\xi_j + \alpha\eta_j = 0$ thus $x + \alpha y \in Y$. Further, let $x_n = (\xi_j^n) \in Y$ be a convergent sequence and $x_n \to x$. NOTE: ξ^n is NOT an exponent but an index. Then

$$\|x_n - x\|^2 = \sum_{i=1}^{\infty} (\xi_i^n - \xi_i)^2$$

when i is even we have zero. Each of the terms on the right must be positive. Thus, $\lim_{n\to\infty} \|x_n - x\|^2 = 0$ which implies that $\xi_n^m \to \xi_i$ for all $i \in \mathbb{N}$. Thus, $x \in Y$ and Y is closed.