Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Section 3: 1, 2, 3, 7, 8, 9

3.1. **A basis for k-tensors**

Let V be a vector space of dimension n with basis $e_i, \dots, e_n$. Let $\alpha^1, \dots, \alpha^n$ be the dual basis in $V^\vee$ Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \dots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this show that $\dim L_k(V) = n^k$. (This problem generalizes Problem 3.1..)

Let $\Phi = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ where $i_1, \dots, i_k = 1, \dots, n$. We want to show

- WTS Φ is a linearly independent set.

$$\begin{aligned} \text{Let } x &= \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}, \text{ for some set } \{i_k\},\, i_k \in [1, n] \\ \text{and } y &= \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_k}, \text{ for some set } \{j_k\},\, j_k \in [1, n] \\ \text{where } \{i_k\} &\neq \{j_k\} \end{aligned}$$

then for any non-zero vectors $v_1, \dots, v_n \in V$ where $v_i = (v_i^1, \dots, v_i^n)$ and any $A, B \in \mathbb{R}$ where $Ax + By = 0$

$$\begin{aligned} (Ax + By)(v_1, \dots, v_k) &= A\left((\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})(v_1, \dots, v_k) \right) + B\left((\alpha^{j_1} \otimes \cdots \otimes \alpha^{j_k}(v_1, \dots, v_k) \right) \\ &= A\left(\prod_{m=1}^{k} \alpha^{i_m}(v_m) \right) + B\left(\prod_{p=1}^{k} \alpha^{j_p}(v_p) \right) \\ &= A\underbrace{\left(\prod_{m=1}^{k} v_m^{i_m} \right)}_{\neq 0} + B\underbrace{\left(\prod_{p=1}^{k} (v_p)^{j_p} \right)}_{\neq 0} \end{aligned}$$

thus $A = B = 0$ and the elements of Φ are linearly independent.

- **WTS Φ is surjective over** $L_k(V)$. Given any $f \in L_k(V)$, we can define all of the actions of f based on how it effects the basis vectors. Thus, $f(e_{i_1}, \dots, e_{i_k}) = f_{i_i, \cdots, i_k}$ where each $f_{i_1, \dots, i_k}$ is a scalar associated with the tensor product of the dual basis vectors. Therefore,

$$f = \sum_{i_1, \dots, i_k} f_{i_1, \dots, 1_k}(\alpha^{i_i} \otimes \cdots \otimes \alpha^{i_k})$$

This holds for all multi-indices $j_1, \dots, j_k$, independent of order. Hence, Φ is surjective over $L_k(V)$.