Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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Section 15: Lie Groups – July 31, 2025

Notes:

Definition 0.0.1 (Lie Group)**.** A ***Lie Group*** is a smooth manifold where the group operations (multiplication and inverse) are smooth between manifolds.

More precisely, the group operations of multiplication $\mu : G \times G \to G$ and inverse $\iota : G \to G$ such that

$$\mu(g,h) = gh$$
$$\iota(g) = g^{-1}$$

and both $\mu, \iota \in C^\infty$.

Perhaps a more accurate description is that we have a group, therefore it has an operator and inverse, and it is a Lie Group when they are smooth.

Definition 0.0.2 (Lie subGroup)**.** .
A ***Lie subgroup*** of a Lie group G is

(i) an abstract group H that is,

(ii) an *immersed* submanifold via the inclusion map such that

(iii) the group operations on H are C^∞

Remark 0.0.3 (Miscellaneous Concepts).

$$\pi_0(\mathrm{GL}_2(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$$
$$\pi_1(\mathrm{GL}_2(\mathbb{R})) \cong \pi(\mathrm{SO}_2(\mathbb{R})) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Remark 0.0.4 (Group Homomorphism and Left-Multiplication). .
A map $F : H \to G$ between two Lie groups H and G is a ***Lie Group Homomorphism*** if it is a C^∞ map (smooth) and a group homomorphism. The group homomorphism condition means that for all $h, x \in H$,[1]

$$F(hx) = F(h)F(x).$$

This may be rewritten in the functional notation as

$$F \circ \ell_h = \ell_{F(h)} \circ F, \ \forall h \in H.$$

Remark 0.0.5 (Intuitive ideas regarding Push-back and Pull-forward). .

[1] keep in mind that the group need not be commutative, hence the definition of a left-multiplication function.

[From StackExchante[2]] The fundamental intuition is that it doesn't matter which manifold you do your calculations in, you get the same result either way.

This was already clear in the case of coordinate charts; calculus on a manifold is often defined in terms of what you get using coordinates to map the problem over to Euclidean space. The point is that the idea extends to more general manifolds than just Euclidean space.

Given a vector on M1 and a scalar field on M2 , there are two ways you might combine them to get a directional derivative: either pull the problem back to M1 or push it forward to M2 . The identity you cite is the one that asserts you get the same answer both ways.

Anyways, I think there is an extremely compelling algebraic rationale for this.

Suppose you are doing calculus in one variable x, then later you decide you need a second independent variable y. This changes absolutely nothing about the calculations you've done — if y doesn't appear in any of your calculations, everything is as if it didn't exist at all.

I.e. $d\sin(x) = \cos(x)dx$ is always true; it doesn't matter whether or not you have any other variables and whether or not x is dependent with any of them.

Consider the case where ϕ is the projection onto the first component map $\mathbb{R}^2 \to \mathbb{R}$, using standard coordinates on both.

The pullback ϕ_* on scalar fields is precisely the "add in the variable y" operation. The push forward ϕ^* on vectors expresses the fact only the x-direction matters. It's clear that if we have $v \in T\mathbb{R}^2$ and $f \in C^1(\mathbb{R})$, then we expect

$$(\phi_*)(f) = v(\phi^* f)$$

because both formulas are expressing exactly the same operation.

Examples[3]

Dimension Zero

Lie Groups $\cong$ Discrete groups $\implies$ classification: hopeless. However, given

$$G_0 \longrightarrow G \longrightarrow G/G_0$$

where G_0 is the **connected component** (i.e., a Lie subgroup and a normal group containing the identity) and G/G_0 is discrete (that is G is separated into two parts). G/G_0 being discrete offers no new informations. G_0 being connected can be classified. For example,

$$\mathbb{R}_{>0} \longrightarrow \mathbb{R}^\times \longrightarrow \{+,-\}$$

$R_{>0}$ is a connected component, $\mathbb{R}^\times$ a group, $\{+,-\}$ a discrete subgroup.

The theory on Lie Groups focuses on 'connected' Lie Groups, thus, for this example G_0 ...

One Dimensional Examples

- $\mathbb{R}$ with addition.
- $\mathbb{R}^\times$ with multiplication.
- $S^1 \subseteq \mathbb{C}$, i.e., $|z| = 1$

[2] https://math.stackexchange.com/questions/2445738/intuitive-explanation-for-the-relation-between-push-fo
[3] Video from YouTube https://www.youtube.com/watch?v=pAuRWd8dpvE&t=1875s

There exists isomorphisms

$$\exp : \mathbb{R}_{>0} \to \mathbb{R}^{\times}$$
$$x \mapsto e^{2\pi i x} : \mathbb{R}_{>0} \to S^1 \qquad\qquad \text{local isomorphism}$$

Any connected Lie group is isomorphic to $\mathbb{R}_{>0}$ under addition or S^1. In general, $\mathbb{R}^1/$(discrete subgroup), e.g., $\mathbb{R}_{>0} = \mathbb{R}^1/0$ and $S_1 = \mathbb{R}^1/\mathbb{Z}$, respectively.

Two Dimensional Examples

Simply take all of the possible examples from One Dimension and mix and match them to two dimensions. Each is abelian and has the form $\mathbb{R}^2/$(discrete group)

- $\mathbb{R}^1 \times \mathbb{R}^1 \implies$ plane. Equal to $\mathbb{R}^2/0$.
- $\mathbb{R}^1 \times S^1 \implies$ cylinder. Equal to $\mathbb{R}^2/\mathbb{Z}$.
- $S^1 \times S^1 \implies$ torus. Equal to $\mathbb{R}^2/\mathbb{Z}^2$.

Any connected abelian subgroup can be written in this form, i.e., is isomorphic to, $\mathbb{R}^m \times (S^1)^n \cong \mathbb{R}^{m+n}/$(discrete group).

Non-abelian connected Lie subgroups of dimension two

- **Afine transformation of the Reals,** $ax + b$
  $\mathbb{R} \to \mathbb{R}$, as in

  $$x \mapsto ax + b \implies \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \qquad\qquad a \neq 0$$

  This is not abelian (?) but is a "solvable Lie Group". By ***solvable*** we mean that given a chain of normal subgroups, G_i.

  $$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n \subseteq G$$

  where G_i/G_{i-1} are all abelian. Then, G_1 is the group of elements over b with $a = 1$ and G_2 being the whole group. In this particular case, the length of the chain is one.

Three Dimensional Examples

- **Special Linear and Projected Special Linear Lie groups**
  The single most useful Lie Group of them all $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, Special Linear Group (the set of two by two matrices with determinant 1). Also, $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm\mathbb{I}_2\}$, known as the projective space on $\mathbb{R}^2$, the group of automorphisms of the upper half plan in complex analysis.
- **The sphere** S^3
  Also known as *the group of unit quaternions.*

  $$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|, x_1^2 + x_2^2 + x_3^2 + x^4 = 1\}$$

  Notice that S^2 is NOT a Lie group, further, S_0, S_1, S_3 are the **only** spheres that are Lie groups. These correspond to the dimensions of the identity of $S_0 \to \dim(\mathbb{I}_{\mathbb{R}} = 0) = 0$, $S_1 \to \dim(\mathbb{I}_{\mathbb{C}} = i) = 1$, $S_1 \to \dim(\mathbb{I}_{\mathbb{H}}) = 3$
- **Heisenberg Group**
  Taking the upper triangule 3×3 matrices quotiented with the discrete group of the center.

  $$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad\qquad n \in \mathbb{Z}$$

  or $H/H_{\mathbb{Z}}$

This is an example of a "common way of constructing Lie groups by taking the quotient of one Lie Group by the discrete center of another." Notice that $H \cong \mathbb{R}^3$ is simply connected (think G_0) and $H_{\mathbb{Z}}$ is NOT simply connected, known as *the fundamental group given by* $\mathbb{Z}$ (to be discussed later).

"The idea of this is that you can simplify the classification of Lie groups to the classification of Simply Connected Lie Groups quotiented with the discrete subgroup of the center."

"The Heisenberg Group is an example of a nilpotent group." We define ***nilpotent*** as follows. Let $G_{i+1} = G_i/Z(G_i)$, that is, each G_i is the quotient of the previous G_i with its own center. If at the end of this progression, $G_n = I$ then G is said to be nilpotent." Simply connected nilpotent Lie subgroups are isormorphic to closed upper triangular subgroup of matrices.

6: the Lorentz Group

The group of rotations of spacetime

Dimension 8

SU_3 of 3×3 unitary matrices. That is, determinant 1. "The Eightfold Way" a simple Lie Group.

Dimension 10

Poincare Group set of all translations and rotations of spacetime. More or less a product of simple groups that is abelian and solvable.

Complex Simple Lie Groups

The "classical groups"

- Special Linear $\mathrm{SL}_n(\mathbb{C}), n \geq 1$
- Orthogonal $O_n(\mathbb{C}), n \geq 3$
- Simplectic $\mathrm{Sp}_n(\mathbb{C}), n \geq 1$

Killing found five more: $G_2, F_4, \varepsilon_6, \varepsilon_7, \varepsilon_8$ whose dimensions are 14, 52, 78, 133, 248, respectively.

Lastly

"We can simplify a lot of material on a Lie Group by studying the tangent space. This takes us into our next subject, a Lie Algebra."

Exercises

Exercise 15.2, Pg. 164 (Left Multiplicaton)

For an element a in a Lie Group G, prove that the left multiplication $\ell_a : G \to G$ is a diffeomorphism.

Rewording the problem: Prove that $\ell_a^{-1} = \ell_{a^{-1}}$.

Since $a \in G, \exists a^{-1} \in G$ and

$$\begin{aligned} \ell_a(x) &= \mu(a,x) \text{ and } \ell_{a^{-1}} = \mu(a^{-1},x) \\ (\ell_a \circ \ell_{a^{-1}})(x) &= \mu(a, \mu(a^{-1},x)) \\ &= \mu(\mu(a,a^{-1}),x) && \text{associativity} \\ &= \mu(e,x) \\ &= x \end{aligned}$$

Thus. $\ell_a^{-1} = \ell_{a^{-1}}$. Every step in the above progression is smooth, therefore a diffeomorphism.

Problems

15.1. Matrix exponential

For $X \in \mathbb{R}^{n\times n}$, defined the partial sum $s_m = \sum_{k=0}^{m} X^k/k!$.

(a) Show that for $\ell \geq m$,

$$\| s_\ell - s_m \| \leq \sum_{k=m+1}^{\ell} \| X \|^k / k!.$$

$$\begin{aligned} \| s_\ell - s_m \| &= \left\| \sum_{k=0}^{\ell} X^k/k! - \sum_{k=0}^{m} X^k/k! \right\| \\ &= \left\| \sum_{k=m+1}^{\ell} X^k/k! \right\| \\ &\leq \sum_{k=m+1}^{\ell} \| X^k \| / k! \end{aligned}$$

(b) Conclude that s_m is a Cauchy sequence in $\mathbb{R}^{n\times n}$ and therefore converges to a matrix, which we denote by e^X. This gives another way of showing that $\sum_{k=0}^{\infty} X^k/k!$ is convergent, without using the comparison test or the theorem that absolute convergence implies convergence in a complete normed vector space.

15.2. Identity component of a Lie group

The *identity component* G_0 of a Lie group G is the connected component of the identity element e in G. Let μ and t be the multiplication map and the inverse map of G.

(a) For any $x \in G_0$, show that $\mu(\{x\} \times G_0) \subset G_0$. (*Hint:* Apply Proposition A.43.)
In other words, demonstrate that G_0 is a subgroup.

(b) Show that $i(G_0) \subset G_0$.
Let $g \in G_0$ then $i(g) = g^{-1}$ and $\mu(g, i(g)) = \mu(g, g^{-1}) = e$, thus $g^{-1} \in G_0$.

(c) Show that G_0 is an open subset of G. (*Hint:* Apply Problem A.16.)

(d) Prove that G_0 is itself a Lie group.

15.3. Product rule for matrix-valued functions

Let (a, b) be an open interval in $\mathbb{R}$. Suppose $A : (a, b) \to \mathbb{R}^{m\times n}$ and $B : (a, b) \to \mathbb{R}^{n\times p}$ are $m \times n$ and $n \times p$ matrices respectively whose entries are differentiable functions of $t \in (a, b)$. Prove that for $t \in (a, b)$,

$$\frac{d}{dt} A(t)B(t) = A'(t)B(t) + A(t)B'(t)$$

where $A'(t) = (dA/dt)(t)$ and $B'(t) = (dB/dt)(t)$.

$$
\begin{aligned}
AB &= \left[\sum_{k=1}^{n} a_{ik} b_{jk}\right] \\
\frac{d}{dt}AB &= \frac{d}{dt}\left[\sum_{k=1}^{n} a_{ik} b_{jk}\right] \\
&= \left[\sum_{k=1}^{n} \frac{d}{dt}(a_{ik} b_{jk})\right] \\
&= \left[\sum_{k=1}^{n} \frac{d}{dt}(a_{ik}) b_{jk} + \sum_{k=1}^{n} + a_{ik}\frac{d}{dt}(b_{jk})\right] \\
&= \left[\sum_{k=1}^{n} \frac{d}{dt}(a_{ik}) b_{jk}\right] + \left[\sum_{k=1}^{n} + a_{ik}\frac{d}{dt}(b_{jk})\right] \\
&= A'B + AB'
\end{aligned}
$$

15.4. Open subgroup of a connected Lie group

Prove that an open subgroup H of a connected Lie group G is equal to G.

IMHO, this is an abuse of terminology. It might be better to say "Prove that an open *with respect to G* subgroup H of a connected Lie group G is equal to G". For example, let $G = \mathbb{R}^2$ and $H = (-1, 1)$. Clearly, H is open and connected and $G \neq H$. However, H is open in $\mathbb{R}$ but *with respect to G*, every point in H is a boundary point. That is, let $h \in H$ then there exists an open set about h ***in*** G which has elements in H and elements in G. Consquently, H is open *with respect* $\mathbb{R}$ *but not to G*, the set that it appears to be defined by.

In order for H to be an open subgroup *with respect to G* it cannot have a dimension less than G. It must also be immersive, therefore, it must be G.

This 'problem' is deceptive in that 'open' on a 'subgroup' must be in referrence to the ambient space and not in its definition. Its kind of like saying "Starry Starry night by Vincent van Gogh is beautiful now prove that it isn't real."

15.5. Differential of the multiplicative map

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, and let $\ell_a : G \to G$ and $r_b : G \to G$ be left and right multiplication by a and $b \in G$, respectively. Show that the differential of μ at $(a, b) \in G \times G$ is

$$\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_*(X_a) + (\ell_a)_*(Y_b) \text{ for } X_a \in T_a(G),\, Y_b \in T_b(G)$$

15.6. Differential of the inverse map

let G be a Lie group with mulitplication map $\mu : G \times G \to G$, inveerse map $i : G \to G$, and identity e. Show that the differential of the inverse map at $a \in G$,

$$i_{*,a} : T_a G \to T_{a^{-1}} G,$$

is given by

$$i_{*,a}(Y_a) = -(r_{a^{-1}})_*(\ell_{a^{-1}})_* Y_a,$$

where $(r_{a^{-1}})_* = (r_{a^{-1}})_{*,e}$ and $(\ell_{a^{-1}})_* = (\ell_{a^{-1}})_{*,a}$. (The differential of the inverse at the identity was calculated in Problem 8.8(b).)

15.7. Differential of the determinant map at A

Show that the differential of the determinant map $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ at $A \in \mathrm{GL}(n,\mathbb{R})$ is given by

$$\det{}_{*,A}(AX) = (\det A)\operatorname{tr} X, \quad \text{for } X \in \mathbb{R}^{n\times n}. \tag{15.7}$$

$$\begin{aligned}
\det{}_{*,A}(AX) &= \frac{d}{dt}\det(e^{tAX})\bigg|_{t=0} \\
&= \frac{d}{dt}e^{t\operatorname{tr}(AX)}\bigg|_{t=0} \\
&= \operatorname{tr}(AX)e^{t\operatorname{tr}(AX)}\big|_{t=0}
\end{aligned}$$

Once again, some abuse of terminology for the unawares. This is $\det_{*,A}$ which is specifically at A then we evaluate it at AX.

15.8. Special linear group

Use Problem 15.7 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

15.9. Structure of a General Linear Group

(1) $r \in \mathbb{R}^\times : \mathbb{R} - \{0\}$, let M_r be the $n \times n$ matrix

$$M_r = \begin{bmatrix} r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{pmatrix} re_1 & e_2 & \dots & e_n \end{pmatrix}$$

where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$. Prove that the map

$$\begin{aligned} f : \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^\times, \\ A &\mapsto \big(AM_{a/\det A}, \det A\big). \end{aligned}$$

is a diffeomorphism.

(2) The *center* $Z(G)$ of a group G is the subgroup of elements $g \in G$ that commute with all elements of G:

$$Z(G) = \{g \in G \,|\, gx = xg, \forall x \in G\}.$$

Show that the center of $\mathrm{GL}(2,\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$, corresponding to the subgroup of scalar matrices, and that the center of $\mathrm{S}(2,\mathbb{R}) \times \mathbb{R}^\times$ is isomorphic to $\{\pm 1\} \times \mathbb{R}^\times$. The group $\mathbb{R}^\times$ has two elements of order 2, while the group $\{\pm 1\} \times \mathbb{R}^\times$ has four elements of order 2. Since their center are not isomorphic, $\mathrm{G}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times$ are not isomorphic groups.

(3) Show that

$$\begin{aligned} h : \mathrm{GL}(3,\mathbb{R}) &\to \mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^\times \\ A &\mapsto \big((\det A)^{1/3}, \det A\big) \end{aligned}$$

is a Lie group isomorphism.

The same arguments in (b) and (c) prove that the n even, the two Lie groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^{\times}$ are not isomorphic as groups, while for n odd, tey are isomorphic as Lie Groups.

15.10. Orthogonal Group

Show that the orthogonal group $O(n)$ is compact by proving the following two statements

(a) $O(n)$ is a closed subset of $\mathbb{R}^{n\times n}$.

(b) $O(n)$ is a bounded subset of $\mathbb{R}^{n\times n}$

15.11. Special orthogonal group $\mathrm{SO}(2)$

The *special orthogonal group* $\mathrm{SO}(2)$ is defined to be the subgroup of $O(n)$ consisting of matrices of determinant 1. Show that every matrix $A \in \mathrm{SO}(2)$ can be written in the form

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. Then prove that $\mathrm{SO}(2)$ is diffeomorphic to the cirlce S^1.

15.12. Unitary group

The *unitary group* $U(n)$ is defined to be

$$U(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) \,|\, \overline{A}^T A = I\}$$

where $\overline{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugatin every entry of $A : (\overline{A})_{ij} = \overline{a_{ij}}$. Show that $U(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{C})$ and that $\dim U(n) = n^2$.

15.13. Special unitary group $\mathrm{SU}(2)$

The *special unitary group* $\mathrm{SU}(n)$ is defined by the subgroup $U(n)$ consisting of matrics of determinant 1.

(a) Show that $\mathrm{SU}(2)$ can also be described as a the set

$$\mathrm{SU}(n) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in \mathbb{C}^{2\times 2} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\}$$

(*Hint:* Write out the condition $A^{-1} = \bar{A}^T$ in terms of the entries of A.)

(b) Show that $\mathrm{SU}(2)$ is diffeomorphic to the three-dimensional sphere

$$S^3 = \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \,\}$$

15.14. A matrix exponential

Compute $\exp \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

15.15. Symplectic group

This problem requires a knowledge of quaternions as in Appendix E. Let $\mathbb{H}$ be the skew field of quaternions. the *symplectic group* $\mathrm{Sp}(n)$ is defined to be

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n,\mathbb{H}) \,|\, \bar{A}^T A = I\}$$

where $\bar{A}$ denotes the quaternionic conjugate of A. Show that $\mathrm{Sp}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{H})$ and compute its dimension.

15.16. Complex Symplectic Group

Let J be the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The *complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$ is defined to be

$$\mathrm{Sp}(2n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) \mid A^T J A = J\}.$$

Show that $\mathrm{Sp}(2n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{C})$ and compute its dimension. (*Hint:* Mimic Example 15.6. It si crucial to choose the correct trarget space for the map $f(A) = A^T J A$.)