Math 5230 – Partial Differential Equations– Fall 2025
w/Professor XXXX

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Homework #1 – September 4, 2025

Part I.

1. (a) Consider an initial value problem for th elinear tranport equation wiht a bounded, one-dimensional spatial domain:

$$\begin{cases} u_t + 3u_x = 0, & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = 0. \end{cases}$$

We assume that $g(0) = 0$, so that the initial condition and boundary condition agree at the corner $(x,t) = (0,0)$.
Find a formula for $u(x,t)$ using the same method as was seen in classe, ile., use the fact that a certian direction derivative of u is zero. What do you notice about your solution for large times?

From the Lecture:

$$u_t + bu_x = \langle b, 1 \rangle \cdot \langle u_x, u_t \rangle .$$

Define $\hat{b}$ as the vector that satisfies $b \cdot \hat{b} = 1$. Then this can also be written as

$$\hat{b}u_t + u_x = \left\langle 1, \hat{b} \right\rangle \cdot \langle u_x, u_t \rangle$$

and, from the description of the PDE

$$\hat{b}u_t + u_x = \left\langle 1, \hat{b} \right\rangle \cdot \langle u_x, u_t \rangle = 0$$

Thus, as in the lecture which emphasized the (x,t)-plane we can see a similar solution in the (t,x)-plane. Recall that we have a function $z(s)$

$$z(s) = u(x + sb, t + s)$$

Let's reparameterize z with $r = sb$ as

$$z(r) = u(x + r, t + r\hat{b})$$

Now, we differentiate with respect to r.

$$\begin{aligned} \frac{dz(r)}{dr} &= \frac{d}{dr} u(x + r, t + r\hat{b}) \\ &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}(x + r) + \frac{\partial u}{\partial t}\frac{\partial t}{\partial r}(t + r\hat{b}) \\ &= \frac{\partial u}{\partial x} + \hat{b}\frac{\partial u}{\partial t} \\ &= u_x + \hat{b}u_t = 0 \end{aligned}$$

As expected z is still constant. Then,

$$\begin{aligned} z(0) &= z(-x) \\ u(x,t) &= u(0, t - x\hat{b}) \\ &= h(t - x\hat{b}). \end{aligned}$$

Now we have two solutions for $u(x,t)$

$$\begin{aligned} u(x,t) &= g(x - bt) && \text{(from the lecture)} \\ u(x,t) &= h(t - x\hat{b}) \\ \therefore g(x - bt) &= h(t - x\hat{b}) \end{aligned}$$

is our only solution.

(b) Next, derive a solution formula for the same problem with a more general boundary condition and source term:

$$\begin{cases} u_t + 3u_x = f(x,t), & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = h(t). \end{cases}$$

We assume that $g(0) = h(0)$, so that the initial condition and boundary condition agree at the corner $(x,t) = (0,0)$.

(c) Why did we only specify the boundary condition at he left-hand boundary ($x = 0$), not the right-hand boundary ($x = 1$)? In other words, what woudl go wrong if we specified boudnary conditions on both sides, as in

$$\begin{cases} u_t + 3u_x = f(x,t), & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = h_0(t), \\ u(1,t) = h_1(t), \end{cases}$$

for some give functions $h_0(t)$ and $h_1(t)$? (We can assume $g(0) = h_0(0)$ and $g(10 = h_1(0)$, so that the intial and boundary conditions agree at the corners.)

2. In one space dimensio, Laplace's equation $\Delta u = 0$ becomes an ODE

$$u''(x) = 0$$

Describe all solutions to this ODE posed on the real line $(-\infty, \infty)$. Next, for given constants c_1, c_2, find the uniqe solution to $u''(x) = 0$ on the interval $[0,1]$ with $u(0) = c_1$ and $u(1) = c_2$.
Do these one-dimensional harmonic functions satisfy the mean value property $u(x) = \fint_{B(x,y)} u(y)dy$? Why or why not?

$$\int u''(x)dx = u'(x) + c$$

$$\int (u'(x) + c)dx = u(x) + cx + d$$

$$\therefore u(x) + cx + d = 0$$

When $u(0) = c_1$ and $u(1) = c_2$ we have

$$\begin{aligned} u(0) + c(0) + d = 0 &\implies c_1 + d = 0, d = -c_1 \\ u(1) + c(1) + d = 0 &\implies c_2 + c + d = 0 \\ c_1 = c_2 + c &\implies c = c_1 - c_2 \end{aligned}$$

That is

$$\begin{aligned} u(x) + (c_1 - c_2)x - c_1 = 0 \\ u(x) = c_1 - (c_1 - c_2)x \end{aligned}$$

is our solution. Do these one-dimensional harmonic functions satisfy the mean value property $u(x) = \fint_{B(x,y)} u(y)dy$?

$$\begin{aligned} B(x,y) &= [a,b] \\ \fint_{B(x,y)} u(y)dy &= \frac{1}{b-a}\int_a^b u(y)dy \\ &= \frac{1}{b-a}\int_a^b (c_1 - (c_1 - c_2)y)dy \\ &= \frac{1}{b-a}\left[c_1 y - \frac{c_1 - c_2}{2}y^2\right]_a^b \\ &= \frac{1}{b-a}\left[c_1(b-a) - \frac{c_1 - c_2}{2}(b-a)^2\right] \\ &= c_1 - \frac{c_1 - c_2}{2}(b-a) \\ &= \frac{2c_1 - c_1 + c_2}{2} && \text{when } [a,b] = [0,1] \\ &= \frac{c_1 + c_2}{2} \end{aligned}$$

which is the average.

3. Let $z = x + iy$ be a complex variable (x and y are real numbers). Recall that a complex function $f(z) = u(x,y) + iv(x,y)$ is *complex-differentiable* if u and v are continuously differentibable (as functiosn of x and y) and Cauchy-Reimann eqations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If f is complex-differentiable, and in addition the real and imaginary parts $u(x,y)$ and $v(x,y)$ are C^2 (twice continuously differentiable) functions, then show that u and v are harmonic.

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \\
&= 0 \\
\Delta v &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \\
&= -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \\
&= 0
\end{aligned}$$

Part II

1. Write down an explicit formula for a function u solving the initial-value problem

$$\begin{cases} u_t + b \cdot Du + c = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

2. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we definte

$$v(x) := u(Ox)\,(x \in \mathbb{R}^n)$$

then $\Delta v = 0$.

5. We say $v \in C^2(\bar{U})$ is *subharmonic* if

$$-\Delta u \leq 0 \,\text{in}\, U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v dy \;\text{ for all } B(x,y) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(d) Prove $v := |\,Du\,|^2$ is subharmonic, wherever u is harmonic.