Math 5301 – Numerical Analysis– Spring 2025

w/Professor Du

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Homework #1 – January 24, 2025

Question 1 (20 points)

Using Newton's Divided Difference Table, construct a quadratic polynomial to interpolate the function $f(x) = \sin x$ at $x + 0 = 0, x_1 = \pi/4$ and $x_2 = \pi/2$.

(a) Write the polynomial in the form $P_2(x) = ax^2 + bx + c$, include the divided difference table you use.

i	x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_i, x_{i-1}, x_{i-2}]$
0	0	0		
1	$\pi/4$	0.707	$\frac{0.707}{0.785} = 0.9003$	
2	$\pi/2$	1.0	$\frac{0.2635}{-0.785}$	-0.3357

$P_2(x) = -0.3357x^2 + 0.9003x$

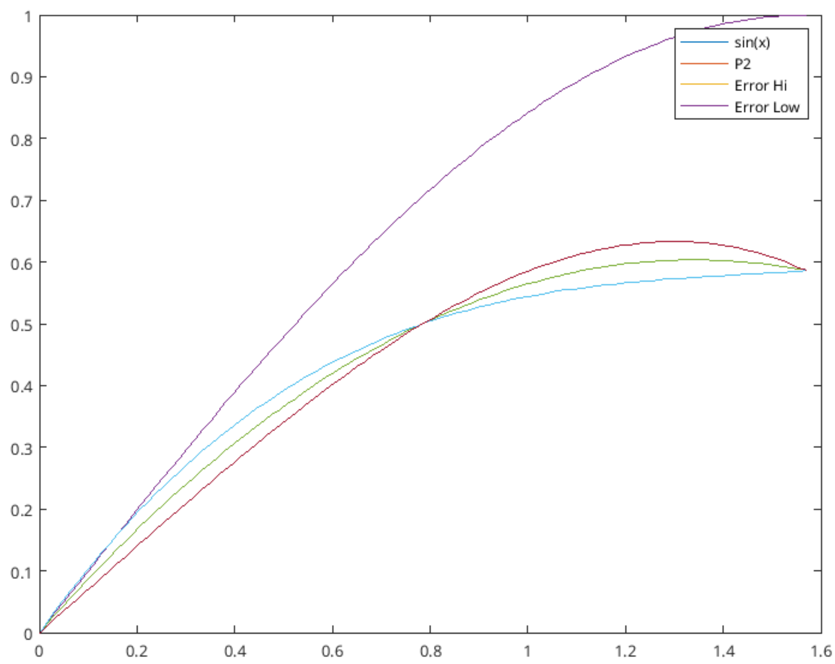
(b) Estimate the error bound for the interpolation.

Our Error Bound is

$$\begin{aligned} |\, f(x) - P_2(x)\,| &\le \frac{1}{(n+1)!}\left| f''' \right|_{\max} (x - x_0)(x - x_1)(x - x_2) \\ &\le \frac{1}{6}(x - 0)(x - \pi/4)(x - \pi/2) \end{aligned}$$

where $0 \le x \le \pi/2$

(c) Estimate (graphically) the largest real error by comparing the plots of $y = f(x)$ and $y = P_2(x)$. Attach computer generated plots.



(d) Compare the real error with the error bound computed in step (b) and comment on the comparison.

the largest real error appears to be at $\pi/2$ atabout 0.4. outside the error bound

Question 2 (20 points)

Suppose we do piecewise interpolation over equally-spaced nodes with $[1,4]$ for $f(x) = 1/x$. We would like to keep the largest error under 10^{-3}.

(a) How many nodes are required for piecewise linear interpolation?

From Theorem 3.13

$$\begin{aligned}
\text{Error Bound} &= \frac{M}{384} \max_{0\le j\le n-1} h^4 \\
\text{where } M &= \max_{a\le x\le b} f^{(4)}(x) = \max_{1\le x\le 4} f^{(4)}(x) = \max_{1\le x\le 4} \frac{4!}{x^5} = 4! = 24 \\
10^{-3} &= 24h^4 \\
h &= \sqrt[4]{\frac{0.001}{24}} \\
&= 0.803 \\
n &= \frac{4-1}{0.0803} \\
n &\approx 37
\end{aligned}$$

(b) How many nodes are required for piecewise quadratic interpolation?

$$\begin{aligned}
\text{Error Bound} &\le \frac{1}{(n+1)!} \left| f''' \right|_{\max_{a\le x\le b}} h^3 \text{ where } h = (b-a)/n \\
&\le \frac{1}{(n+1)!} \left| \frac{6}{x^4} \right|_{\max_{1\le x\le 4}} h^3 \text{ where } h = 3/n \\
10^{-3} &\le \frac{162}{n^3(n+1)!} \\
n^3(n+1)! &= \frac{0.001}{162} = 0.000006172 \\
n &\approx 3
\end{aligned}$$

(c) Use Matlab to confirm your calculation in (a).