Math 5411 – Mathematical Statistics I– Fall 2024
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Homework #8 – October 28, 2024

1, 2, 3, 4(a,b,d,e)

1. Consider two random variables X and Y with joint PMF given in Table 5.5.3.

	$Y=2$	$Y=4$	$Y=5$
$X=1$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$
$X=2$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{8}$
$X=3$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{12}$

(a) Find $P(X \leq 2, Y \leq 4)$.

$$\begin{aligned}
P(X \leq 2, Y \leq 4) &= P(X=1, Y \leq 4) + P(X=2, Y \leq 4) \\
&= P(X=1, Y=2) + P(X=2, Y=2) + P(X=1, Y=4) + P(X=2, Y=4) \\
&= \frac{1}{12} + \frac{1}{6} + \frac{1}{24} + \frac{1}{12} \\
&= \frac{2}{24} + \frac{4}{24} + \frac{1}{24} + \frac{1}{24} \\
&= \frac{8}{24} = \frac{1}{3}
\end{aligned}$$

(b) Find the marginal PMFs of X and Y.

$$\begin{aligned}
P(X=1) &= \sum_{y=2,4,5} P(X=1, Y=y) \\
&= P(X=1, Y=2) + P(X=1, Y=4) + P(X=1, Y=5) \\
&= \frac{1}{12} + \frac{1}{24} + \frac{1}{24} \\
&= \frac{4}{24} = \frac{1}{6} \\
P(X=2) &= \sum_{y=2,4,5} P(X=2, Y=y) \\
&= P(X=2, Y=2) + P(X=2, Y=4) + P(X=2, Y=5) \\
&= \frac{1}{6} + \frac{1}{12} + \frac{1}{8} \\
&= \frac{9}{24} = \frac{3}{8} \\
P(X=3) &= \sum_{y=2,4,5} P(X=3, Y=y) \\
&= P(X=3, Y=2) + P(X=3, Y=4) + P(X=3, Y=5) \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} \\
&= \frac{11}{24}
\end{aligned}$$

$$
\begin{aligned}
P(Y=2) &= \sum_{x=1,2,3} P(X=x, Y=2) \\
&= P(X=1,Y=2) + P(X=2,Y=2) + P(X=3,Y=2) \\
&= \frac{1}{12} + \frac{1}{6} + \frac{1}{4} \\
&= \frac{6}{12} = \frac{1}{2} \\
P(Y=4) &= \sum_{x=1,2,3} P(X=x, Y=4) \\
&= P(X=1,Y=4) + P(X=2,Y=4) + P(X=3,Y=4) \\
&= \frac{1}{24} + \frac{1}{12} + \frac{1}{8} \\
&= \frac{6}{24} = \frac{1}{4} \\
P(Y=4) &= \sum_{x=1,2,3} P(X=x, Y=5) \\
&= P(X=1,Y=5) + P(X=2,Y=5) + P(X=3,Y=5) \\
&= \frac{1}{24} + \frac{1}{8} + \frac{1}{12} \\
&= \frac{6}{24} = \frac{1}{4}
\end{aligned}
$$

(c) Find $P(Y=2|X=1)$.

$$P(Y=2|X=1) = P(X=1, Y=2) = \frac{1}{12}$$

(d) Are X and Y independent? Yes

2. I have a bag containing 40 blue marbles and 60 red marbles. I choose 10 marbles (without replacement) at random. Let X be the number of blue marbles and Y be the number of read marbles. Find the joint PMF of X and Y.

This is a hypergeometric distribution in one variable. Let b be the number of blue marbles. then

$$P(X=b, Y=10-b) = \frac{\binom{40}{b}\binom{60}{10-b}}{\binom{100}{10}} \text{ for } b = 1 \dots 10$$

3. Let X and Y be two independent discrete random variables with the same CDFs F_X and F_Y. Define

$$
\begin{aligned}
Z &= \max(X, Y) \\
W &= \min(X, Y)
\end{aligned}
$$

Find CDFs of Z and W.

When assessing the maximum we can see that when $Y > X$ the probability of X is included. Thus,

$$
\begin{aligned}
F(Z=n) &= P(\max(X,Y) \le n) \\
&= P(X \le n, Y \le n) \\
&= P(X \le n)P(Y \le n) \\
&= F_X(n)F_Y(n)
\end{aligned}
$$

When assessuing the minimum we can see that when $Y < X$ there is still a greater probability that X could occur, thus we must assess this based on $1 - \min(X,Y)$ and switch the direction of integration which would include the greater value probability.

$$
\begin{aligned}
F(W=n) &= P(\min(X,Y) \le n) \\
&= 1 - P(\min(X,Y) \ge n) \\
&= 1 - P(X \ge n)P(Y \ge n) \\
&= 1 - (1 - P(X \le n))(1 - P(Y \le n) \\
&= 1 - (1 - F_X(n))(1 - F_Y(n)) \\
&= 1 - (1 - F_X(n) - F_Y(n) + F_X(n)F_Y(n)) \\
&= F_X(n) + F_Y(n) - F_X(n)F_Y(n)
\end{aligned}
$$

4. Let X and Y be two discrete random variables, with range

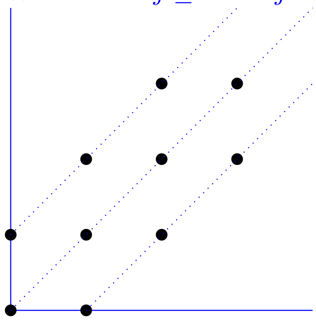
$$R_{XY} = \{\, (i,j) \in \mathbb{Z}^2 \,|\, i,j \geq 0,\ |i-j| \leq 1 \,\}$$

and joint PMF

$$P_{XY}(i,j) = \frac{1}{6 \cdot 2^{\min(i,j)}}, \quad \text{for } (i,j) \in \mathbb{R}_{XY}$$

(a) Pictorially show R_{XY} in the $x-y$ plane.

We have $i-j \leq 1$ and $j-i \leq 1$.



(b) Find the marginal PMFs $P_X(i), P_Y(j)$.

$$P_X(X=0) = P_{XY}(X=0,Y=0) + P_{XY}(X=0,Y=1) = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$P_X(X=1) = P_{XY}(X=1,Y=0) + P_{XY}(X=1,Y=1) + P_{XY}(X=1,Y=2) = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$$
$$P_X(X=2) = P_{XY}(X=2,Y=1) + P_{XY}(X=2,Y=2) + P_{XY}(X=2,Y=3) = \frac{1}{12} + \frac{1}{24} + \frac{1}{24} = \frac{1}{6}$$

each $P(X)$ had three terms and in general is defined for $n \geq 0$ as

$$P_X(n) = \begin{cases} \frac{1}{3} & n = 0 \\ \frac{1}{3 \cdot 2^{n-1}} & n \geq 1 \end{cases}$$

$P(Y)$ is the same.

(c) Find $P(X = Y \,|\, X < 2)$.

$$\begin{aligned} P(X=Y\,|\,X<2) &= P(X=Y\,|\,Y=1) + P(X=Y\,|\,Y=0) \\ &= P(X=1\,|\,Y=1) + P(X=0\,|\,Y=0) \\ &= \frac{1}{12} + \frac{1}{6} = \frac{1}{4} \end{aligned}$$

(d) skip

(e) $P(X=Y)$.

$$P(X=Y) = \sum_{x=0}^{\infty} \frac{1}{6 \cdot 2^{\min(x,x)}} = \sum_{x=0}^{\infty} \frac{1}{6 \cdot 2^x} = \frac{1}{6} \sum_{x=0}^{\infty} \frac{1}{2^x} = \frac{2}{6} = \frac{1}{3}$$