Functional Analysis– Spring 2024

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6. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.

   - ($\Rightarrow$) $T^{-1} : \mathcal{R}(T) \to X$ is bounded. Given any Cauchy sequence $(x_n) \in X$ we know that $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. Further, we know that $\|Tx_n - Tx_m\| \leq \|T\| \|x_n - x_m\|$ which implies that $\|Tx_n - Tx_m\| \to 0$ as $n, m \to \infty$. Let x be such that $Tx_n \to x$ as $n \to \infty$. Clearly, $x \in X$ because X is complete and $Tx \in \mathcal{R}(T)$ and $\|Tx_n - Tx\| \leq \|T\| \|x_n - x\| \to 0$ as $n \to \infty$. Thus, $\mathcal{R}(T)$ is closed.
   - ($\Leftarrow$) $\mathcal{R}(T)$ is closed. Given any sequence $(y_n) \in \mathcal{R}(T)$ we know that it converges. Let y be such that $y_n \to y$ as $n \to \infty$. Since T is injective then $T^{-1} : \mathcal{R}(T) \to X$ is a function. Let $x_i = T^{-1}(y_i)$ for all $i \in \mathbb{N}$ and $x = T^{-1}y$. $\|y_n - y\| \to 0$ thus $\left\|T^{-1}(y_n - y)\right\| = \left\|T^{-1}(y_n) - T^{-1}(y)\right\| = \|x_n - x\| \leq \left\|T^{-1}\right\| \|y_n - y\| \to 0$ as $n \to \infty$.

     $$\begin{aligned}\left\|T^{-1}(y_n - y)\right\| &= \left\|T^{-1}(y_n) - T^{-1}(y)\right\| \\ &= \|x_n - x\| \\ &\leq \left\|T^{-1}\right\| \|y_n - y\| \\ &\leq \left\|T^{-1}\right\| \|T\| \|x_n - x\| \to 0 \text{ as } n \to \infty\end{aligned}$$

     T is bounded.

7. Let $T : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, show that there are positive real numbers a and b such that $a \|x\| \leq \|Tx\| \leq b \|x\|$ for all $x \in X$.

   T bounded means that there exists b such that $\|Tx\| \leq b \|x\|$ for all $x \in X$.. T^{-1} bounded means that there exists positive number c such that $\left\|T^{-1}y\right\| \leq c \|y\|$ for all $y \in Y$. Let x be such that $Tx = y$. Then $\left\|T^{-1}y\right\| = \|x\| \leq c \|Tx\|$. Let $a = 1/c$ then $a \|x\| \leq \|Tx\| \leq b \|x\|$.

p. 296 # 8, 9, 10

8. Let X and Y be normed spaces and let $T : X \to Y$ be a closed linear operator.

   (a) Show that the image A of a compact subset $C \subset X$ is closed in Y.

   Let $(x_n) \in C$. Since C is compact, let α be the ordered set of integers such that $(x_i)_{i\in\alpha}$ converges and let $x_{\alpha_i} \to x$ as $i \to \infty$. Then $T(x_{\alpha_i}) \in A$ for all $i \in \mathbb{N}$. Since T is a closed linear operator and C is compact (hence closed) the set $\mathcal{G}(T) = \{(x, y) \,|\, x \in C, y \in A\}$ must also be closed. Therefore $((x_{\alpha_i}, Tx_{\alpha_i})) \in \mathcal{G}(T)$ as $i \to \infty$ so must $(x, Tx) \in \mathcal{G}(T)$ which means that $Tx \in A$. Hence A is closed in Y.

   (b) Show that the inverse image B of a compact subset $K \subset Y$ is closed in X. (Cf. Def. 2.5-1)

   Let $(y_n) \in K$ and let $\alpha \subset \mathbb{N}$ be an ordered set of indices such that (y_{α_n}) converges and let $y = (y_{\alpha_n})$. Then $\|(y_{\alpha_n}) - y\| \to 0$ as $n \to \infty$. Thus, $\left\|T^{-1}y_{\alpha_n} - T^{-1}y\right\| = \left\|T^{-1}(y_{\alpha_n} - y)\right\| \leq \left\|T^{-1}\right\| \|y_{\alpha_n} - y\| \to 0$ as $n \to \infty$. Sine T^{-1} is closed $T^{-1}y \in B$ and B is closed.

9. If $T : X \to Y$ is a closed linear opearator, where X and Y are normed spaces and Y is compact, show that T is bounded.

   Let (x_n) be a sequence in X then since Y is compact (Tx_n) converges and let $y = Tx_n$ as $n \to \infty$ and let x be such that $Tx = y$. Thus $\|Tx_n - y\| = \|Tx_n - Tx\| \leq M \|x_n - x\| \to 0$. Thus, T is bounded.

10. Let X and Y be normed spaces and X compact. If $T : X \to Y$ is a bijective closed linear operator, show that T^{-1} is bounded.

    Let $A \subset X$ be closed and bounded. T is bijective implies that $T^{-1}TA = A$ thus $(T^{-1})^{-1}(C) = T(C) \subset Y$ which is compact, that implies that T^{-1} is continuous and hence, bounded.

p. 246 #2, 3, 4

2. Give a simpler proof of Lemma 4.6-7 for the case that X is a Hilbert space.

   Let $\tilde{f}(x) = \delta \langle x, x_0 \rangle / \|x_0\|$. When $x \in Y$, $\tilde{f}(x) = 0$ and when $x = x_0$, $\tilde{f}(x_0) = \delta$.

3. If a normed space X is reflexive, show that X' is reflexive.

   X reflexive implies that $C_X : X \to X''$ defined as $x \mapsto g_x(f) = f(x)$ is isomorphic. Let $C_{X'} : X' \to X^{(3)}$ where $X^{(3)}$ is the dual-dual of X'. Let $h \in X^{(3)}$ and define $\tilde{h} \in X'$ by $\tilde{h}(f) = h(C_X(f))$ for all $f \in X$. Then, for all $g \in X''$, we have $C_{X'}(\tilde{h})(g) = g(\tilde{h}) = \tilde{h}(C_X^{-1}(g)) = h(g)$. That is, $C_{X'}(\tilde{h}) = h$ which implies that $C_{X'}$ is surjective and hence bijective, thus an isomorphism.

4. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive. (*Hint.* It can be shown that a closed subspace fo a reflexive Banach space is reflexive. Use this fact, without proving it.)

   ($\Rightarrow$) see exercise 3

   ($\Leftarrow$) X' is reflexive. Then the cannoncial embedded mapping, $C : X \to X''$, maps all of X onto X'', that is $C(X)$ is a subspace of X'' ismorphic to X. Hence, $C(X)$ is a Banach Space and closed. Thus, by the Hint, $C(X)$ is reflexive and, being isomorphic, makes X reflexive.

p. 268 #4, 7

4. Show that weak convergence in footnote 6 implies weak* convergence. Show that the converse holds if X is reflexive.

   Let $(f_n) = f$ be a weak* convergent sequence of functions in X' and let X be reflexive. Therefore $\|f_n(x) - f(x)\| \to 0$ for each $x \in X$. If we choose $g_x \in X''$ be associated with x. Then we can say $\|g_x(f_n(x) - f(x))\| = \|g_x(f_n)(x) - g_x(f)(x)\| \to \|g_x(0)(x)\|$ which is true for all $x \in X$. Thus $\|g_x(f_n) - g_x(f)\| \to 0$ which is strongly convergent.

7. Let $T_n \in B(X, Y)$, where X is a Banach space. If (T_n) is strongly operator convergent, show that $(\,\|T_n\|)$ is bounded.

   Let x be any fixed member or X. Then, $\|T_n x - Tx\| \to 0$ as $n \to \infty$.

$$\begin{aligned} \|T_n x - Tx\| &\leq \|T_n x\| - \|Tx\| \\ &\leq \|T_n\|\,\|x\| - \|T\|\,\|x\| \\ &\leq (\,\|T_n\| - \|T\|\,)\,\|x\| \\ \therefore\ \|T_n\| - \|T\| &\to 0 \end{aligned}$$

   hence bounded.