Functional Analysis– Summer 2023

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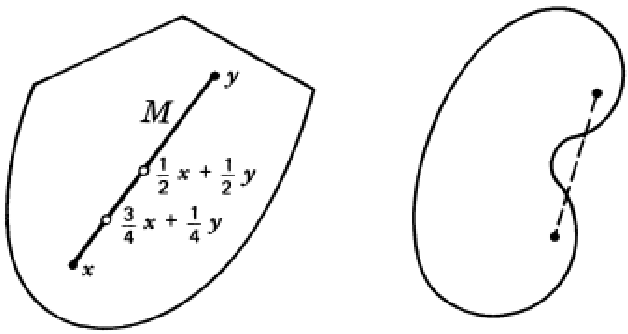
Page. 65 #11, **(Convex set, segment)** A subset A of a vector space X is said to be *convex* if $x, y \in A$ implies

$$M = \{z \in Z \mid z = \alpha x + (1-\alpha)y,\, 0 \le \alpha \le 1\} \subset A$$

M is called *a closed segment* with *boundary points* x and y; any other $z \in M$ is called an *interior point* of M. Show that the *closed unit ball*

$$\tilde{B}(0;1) = \{x \in X \mid ||x|| \le 1\}$$

in a normed space X is convex.



(a) Convex *(b)* Not convex

Let, $x, y \in \tilde{B}(0;1)$ which implies that $||x|| \le 1$ and $||y|| \le 1$. Given any point $m \in M$ there exists α where $0 \le \alpha \le 1$, such that $m = \alpha x + (1-\alpha)y$. Thus, $||m|| = ||\alpha x + (1-\alpha)y||$

$$\begin{aligned}
\|m\| &= \|\alpha x + (1-\alpha)y\| \\
&\le ||\alpha x|| + ||(1-\alpha)y|| \\
&\le |\alpha|\,||x|| + |1-\alpha|\,||y|| \\
\text{Let } p &= \max(||x||, ||y||) \\
||m|| &\le |\alpha|\,p + |1-\alpha|\,p \\
&\le (|\alpha| + |(1-\alpha)|)p \\
&\le p \\
\therefore m &\in \tilde{B}(0;1)
\end{aligned}$$

x, y are arbitrary points and m is an arbitrary point between them. Hence, $\tilde{B}(0;1)$ must be convex.

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1. Show that $c \subset \ell^\infty$ is a vector space of ℓ^∞ (cf. 1.5-3) and so is c_0, the space of all sequences of scalars converging to zero.

   Given any $x, y \in \ell^\infty$ and c_x, c_y are bounds for these sequences with $x = (\eta_j) \leq c_x$ and $y = (\xi_j) \leq c_y$. Then given any $\alpha \in \mathbb{C}$ we have

$$\begin{aligned} \alpha(x+y) &= \alpha(\eta_j + \xi_j)_{j=1}^\infty && \text{component-wise addition} \\ &= (\alpha\eta_j + \alpha\xi_j)_{j=1}^\infty \\ |\alpha\eta_j + \alpha\xi_j|_{j=1}^\infty &\leq |\alpha|(c_x + c_y) \end{aligned}$$

   thus we have a new bounded sequence, that is $\alpha(x+y) \in \ell^\infty$. Thus, ℓ^∞ is a vector space.

   Notice that if $c_x = c_y = 0$ that $|\eta_j + \xi_j| \leq c_x + c_y = 0$ for all $1 \leq j < \infty$, thus $x + y \in c_0$.

2. Show that c_0 in Prob 1 is a *closed* subspace of ℓ^∞, so that c_0 is complete by 1.5-2 and 1.4-7.

   Let $x, y \in \ell^\infty \backslash c_0$ each converges to real numbers c_x, c_y, respectively. Note that c_x, c_y are strictly greater than zero. Thus, $d(x, y) \leq \max(c_x, c_y)$ and is distinctly not zero. Hence, given any $\epsilon > 0$ there exists $B(x; \epsilon) \subset \ell^\infty \backslash c_0$. Thus $\ell^\infty \backslash c_0$ must be open which indicates that c_0 must be closed.

3. In ℓ^∞, let Y be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of ℓ^∞ but not a closed subspace.

   Let $x = (\eta_m), y = (\xi_m) \in \ell^\infty$ such that I_x, I_y each represent a list of indices where $x_i \neq 0$ when $i \in I_x$ and similarly to I_y. Then, we can see that $x + y$ will be the sequence $(z_m) = (\eta_m + \xi_m)$. We can see that when $j \in I_x \cup I_y$ that $z_j \neq 0$. Hence, I_z (the set of indices for non-zero entries in z) will be $I_z = I_x \cup I_y$. Thus, $z \in Y$. Given any $\alpha > 0$ we can see that it has no effect on I_x, thus $(\alpha\eta_m) \in Y$. Since it is closed under addition and scalar multiplication it must be a vector space.

   Let $x_1 = \{1, 0, \dots\} \in Y$, that is, it has 1 in the first component. And $x_2 = \{1, 1, 0 \dots\}$ and so on. The general term x_i means that this the first i components are 1 and the remaining are zero. Clearly, $x_i \in Y$ for all $i \in \mathbb{N}$. However, the limit point of x_i as i increases without bound is $x_\infty = \{1, 1, \dots\}$ with one repeating forever. $x_\infty \notin Y$ hence a sequence in Y does not contain its limit points which means that Y is not closed.

8. If in a normed space X, absolute convergence of any series always implies convergence of that series, show that X is complete.

   Suppose that X is not complete. Then there exists $x = (\eta_m) \in X$ that is an absolutely convergent series and converges but does NOT have a convergent subsequence. $\sum_{i=1}^{\infty} \|\eta_i\| < \infty$, also $\exists s$ such that $s = \lim_{n \to \infty} \eta_n$

9. Show that in a Banach space, an absolutely convergent series is convergent.

10. **(Schauder basis)** Show that if a normed space has a Shauder basis, it is separable.

11. Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for ℓ^p, where $1 \leq p < +\infty$.

15. **(Product of normed spaces)** If $(X_1, ||\cdot||_1)$ and $(X_2, ||\cdot||_2)$ are normed spaces, show that the product vector space $X = X_1 \times X_2$ (cf. prob 13, Sec 2.1) becomes a normed space if we define

$$||x|| = \max(\,||x_1||_1, ||x_2||_2\,) \text{ where } x = (x_1, x_2).$$

Page. 76 #1. Give examples of subspaces of ℓ^∞ and ℓ^2 which are not closed.