# Math 5050 – Special Topics: Manifolds– Spring 2025

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## Definitions

1. ***Diffeomorphism***: If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then $f$ is said to be a ***diffeomorphism***. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be ***diffeomorphic*** to each other.
2. ***Tangent Space*** at a point $p$. The set of all vectors rooted at $p$, written as $T_p(\mathbb{R}^n)$.
3. ***Derivations***: any operation that supports the Liebniz Rule ($D(fg) = (Df)g + fDg$).
4. ***Derivation Space***. $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at $p$. This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ defined as
$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$$
$$v \mapsto D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$
5. ***Germ***: equivalence class of functions whose derivatives around a point are the same.
6. Vector Field vs Vector Space.
   - ***A Vector Field*** a function that assigns a vector to every point in the subset $U$.
   $$f : (U \subset \mathbb{R}^m) \to T_p(\mathbb{R}^n)$$
   $$X \mapsto X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p.$$
   consider $a^i$ as coeffecent functions. We say that $X$ is $C^\infty$ on $U$ if $a^i \in C^\infty$, $\forall i = 1, \ldots, n$.
   - ***A Vector Space*** is any abstraciton that is closed under addition and scalar multiplication.
7. ***Dual Basis and Dual Space***. The ***Dual Basis*** is a set of functions $\alpha^i : V \to \mathbb{R}$
$$\alpha^i : V \to \mathbb{R}$$
$$\alpha^i(e_j) = \delta^i_j$$
the ***Dual Space*** $V^\vee$ is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called ***Functionals (Analysis)/1-Covectors (Differential Geometry)***.
8. ***Multi-Linear Functions*** Let $V$ be a vector space and $V^k$ be $k$-tuples of vectors in $V$. A ***$K$-linear map or $k$-tensor*** $f : V^k \to \mathbb{R}$ such that each $i^{\text{th}}$ component is linear. The vector space of all $k$-tensors on $V$ is denoted $L_k(V)$.

   **Permuting Mult-linear Functions**. Given any permutation $\sigma \in S_k$
$$f(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$
e.g., $f(x, y, z) = xyz \to f(z, x, y) = zxy$. FYI: if $x, y, z$ are from non-commutative rings (i.e., matrices) then we must be aware of the $\operatorname{sgn}(\sigma)$.
9. Left $R$-***Module***: An Abelian group $R$ with a scalar multiplication map:
$$\mu : R \times A \to A$$
usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a
   (i) (associative) $(rs)a = r(sa)$.
   (ii) (identity) $1a = a$ (1 is a multiplicative identity).
   (iii) (distributivity) $(r + s)a = ra + sa$ and $r(a + b) = ra + rb$.

   If $R$ is a field then $R$-module is precisely a vector space over $R$.

   A ***$K$-Algebra over a field*** $K$ is also a ring $A$ that is also a vector space over $K$ such that the ring multiplication satisfies homogeniety (scalar distributes over vector multipliation to only one of the operators).

   A ***graded Algebra*** is an algebra $A$ over a field $K$ if it can be writte as the direct sum
$$A = \bigoplus_{i=0}^{\infty} A^i$$
of vector spaces over $K$ such that the mupltiplication map sends $A^k \times A^l \to A^{k+l}$

10. The set of all $C^\infty$-vector fields on $U$, denoted by $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the $C^\infty(U)$ ring.

$$\mathfrak{X}(U) = \{\, X : V \to V \mid X \in C^\infty(U)\,\} \text{ where } V = (\mathbb{R} \text{ or } \mathbb{C})^n$$

11. ***Derivation:*** A ***derivation*** on an algebra $A$ is a $K$-multilinear function $D : A \to A$ such that

$$D(ab) = (Da)b + aDb, \forall a, b \in A$$

known as the ***Liebniz Rule***.

The set of all derivations on $A$ forms a vector space, $\mathrm{Der}(C^\infty(U))$. Thus a $C^\infty(U)$ vector field gives rise to a derivation of the algebra $C^\infty(U)$. Thus the mapping

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \mathrm{Der}(C^\infty(U)) \\ X &\mapsto (f \mapsto Xf) \end{aligned}$$

this map is an isomorphism of vector spaces.

12. **Exterior Algebras $\Lambda(V)$.** The exterior algebra $\Lambda(V)$ is obtained by imposing an **anti-commutative** relation:

$$v \otimes w + w \otimes v = 0, \forall v, w \in V$$

this means that the quotient algebra is:

$$\Lambda(V) = T(V)/\langle\, v \otimes w + w \otimes v \,\rangle\,.$$

Where $T(V)$ is the **tensor algebra**

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

13. **Symmetric Algebras $S(V)$.** The symmetric algebra $S(V)$ is obtained by imposing an **commutative** relation:

$$v \otimes w - w \otimes v = 0, \forall v, w \in V$$

this means that the quotient algebra is:

$$S(V) = T(V)/\langle\, v \otimes w - w \otimes v \,\rangle\,.$$

14. ***Tensor Product*** The tensor product between two 1-covectors, $f, g : V \to \mathbb{R}$ is the 2-covector $f \otimes g$.

$$(f \otimes g)(u, v) = f(u)g(v)$$

. In general, the tensor product of a $k$-covector $p : V^k \to \mathbb{R}$ with a $l$-covector $q : v^l \to \mathbb{R}$ is the $(k+l)$-covector $p \otimes q : V^{k+l} \to \mathbb{R}$.

$$(p \otimes q)(u, v) = p(u)q(v), \forall u \in V^k,\, v \in V^l$$

15. ***Tensor Product(?)*** is an operator on $v \in V$ and $u \in U$ where

$$\begin{aligned} v \otimes u &: V \times U \to V \oplus U \\ (v \otimes u)_{i \cdot j} &= v_i \cdot u_j,\, \forall i = 1, \ldots, \dim(V),\, j = 1, \ldots, \dim(U) \end{aligned}$$

Given two vector spaces $V, W$ with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

$\alpha_i, \beta_j$ are all scalars. The real issue is the behavior of unit basis vectors $v_i, w_j$ and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

16. ***Wedge Product***

**Between two covectors** Let $f, g \in L_1(V)$ then for all $u, v \in V$

$$(f \wedge g)(u, v) = (f \otimes g)(u, v) - (g \otimes f)(u, v) = f(u)g(v) - f(v)g(u)$$

**Between mulitple 1-covectors.**

$$\begin{aligned}(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \ldots, v_k) &= \det[\alpha^1(v_j)] \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)})\end{aligned}$$

**Between $k$-covector and $l$-covector**. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A_{k+l}(V)$$

or explicitly

$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f\left(v_{\sigma 1}, \ldots, v_{\sigma_k}\right) g\left(v_{\sigma_{k+1}}, \ldots, v_{\sigma k+l}\right)$$

**Anticommutative.** Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$(f \wedge g) = (-1)^{kl} g \wedge f$$

17. **Differential k-Forms**

    **1-forms, covectors**

$$(dx^i)\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p x^i = \delta^i_j$$

$$(df)_p(X_p) = X_p f = \sum a^i(p) \frac{\partial f}{\partial x^i}\bigg|_p = \sum \frac{\partial}{\partial x^i} dx^i$$

18. $\Omega^k(U)$**, Vector space of $C^\infty$ $k$-forms on U**.

    $\Omega^0 = A_0(T_p(\mathbb{R}^n)) = C^\infty(U)$, e.g., $f \in \Omega^0$ then $f : V \to \mathbb{R}$ is a functional/covector/1-tensor.

    Elements of 1-form $\Omega^1 = A_1(T_p(\mathbb{R}^n))$. For example, when $n = 3$

$$f dx + g dy + h dz, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

    Elements of 2-form $\Omega^2 = A_2(T_p(\mathbb{R}^n))$. For example, when $n = 3$[1]

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

    if $n = 4$, that is coordinates for $u, v, w, x$. Each form is derived from these bases

    0-form $\Omega^0(\mathbb{R}^4) \in \mathbb{R}$

    1-forms $\Omega^1(\mathbb{R}^4)$ summing $du, dv, dw, dx,$

    2-forms $\Omega^2(\mathbb{R}^4)$ summing $du \wedge dv$, $du \wedge dw$, $du \wedge dx$, $dv \wedge dw$, $dv \wedge dx$, $dw \wedge dx$,

    3-forms $\Omega^3(\mathbb{R}^4)$ summing $du \wedge dv \wedge dw \mid du \wedge dw \wedge dx \mid du \wedge dv \wedge dx \mid dv \wedge dw \wedge dx$

    4-form $\Omega^4(\mathbb{R}^4)$ $du \wedge dv \wedge dw \wedge dx$.

    Also, $U \subseteq \mathbb{R}^n$ **then** $k < n$. $k$-forms for $k > n$ are zero. Further $|\Omega^k(\mathbb{R}^n)| = \binom{k}{n}$ and $\left|\bigcup_k \Omega^k(\mathbb{R}^n)\right| = 2^n$ and think of $\Omega^*(U) = \bigcup_k \Omega^k(\mathbb{R}^n)$

    **Direct Sum.** $\Omega^*(U) = \bigoplus_k \Omega^k(U)$ is an anti-commutative graded algebra over $\mathbb{R}$.

    Since one can multiply $C^\infty$ $k$-forms by $C^\infty$ functions, the set $\Omega^k(U)$ of $C^\infty$ $k$-forms is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$ and $\Omega^*(U)$ is also a module over $C^\infty$ of $C^\infty$ functions.

19. **Wedge Product of $k$-form.** Recall: $dx^i \wedge dx^i = 0$ for all $i = 1, \ldots, n$. Therefore, $\wedge$ only makes sense to be defined on *disjoint indice-lists*, that is, $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$ such that $I \cap J = \emptyset$. Then,

$$\begin{aligned}\wedge : \Omega^k(U) \times \Omega^l(U) &\to \Omega^{k+l}(U) \\ (\omega, \tau) &\mapsto (\omega \wedge \tau) = \sum_{I,J} a_I b_J dx^I \wedge dx^J.\end{aligned}$$

    where $\omega = \sum_I a_I dx^I, \tau = \sum_J b_J, dx^J$.

[1]NOTE the cyclic order of the indices $x, y, z$. Switching any one of these will flip the sign.

20. **the Exterior Derivative**. If $k \geq 1$ and if $\omega = \sum_I a_i dx^I \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$ and

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left( \sum_J \frac{\partial a_I}{\partial x_J} dx^J \right) \wedge dx^I$$

*Example:* Let $\omega \in \Omega^1(\mathbb{R}^2)$ and $\omega = f dx + g dy,\ f, g \in C^\infty(\mathbb{R}^2)$.

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left( \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\ &= \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\ &= (g_x - f_y)\, dx \wedge dy \end{aligned}$$

**Definition:** Let $\bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field $K$. An ***anti-derivation*** of the graded algebra $A$ is a $K$-linear map $D : A \to A$ such that $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a Db$$

**Proposition 4.7: Three Criterion for an Exterior Derivation**

i) The ***exterior derivation*** $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

ii) $d^2 = 0$.

iii) If $f \in \mathbb{C}^\infty$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

**NOTE:**"In a typical school, there would be graduate level courses on Smooth Manifods and another on Remannian Manifolds."

Q: What is the difference between $\mathfrak{X}(U)$ and $C^\infty(U)$?

The difference between $\mathfrak{X}(U)$ and $C^\infty(U)$ lies in the types of objects they contain:
1. **$C^\infty(U)$: The Space of Smooth Functions** - $C^\infty(U)$ consists of all smooth (infinitely differentiable) real-valued functions defined on an open subset $U$ of a manifold $M$. - Elements of $C^\infty(U)$ are scalar functions $f : U \to \mathbb{R}$. - These functions can be added and multiplied pointwise, forming an algebra over $\mathbb{R}$.

2. **$\mathfrak{X}(U)$: The Space of Smooth Vector Fields** - $\mathfrak{X}(U)$ consists of all smooth vector fields on $U$. - A vector field $X$ assigns to each point $p \in U$ a tangent vector $X_p \in T_pM$, smoothly varying with $p$. - Vector fields act as derivations on smooth functions, meaning they satisfy the Leibniz rule:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(U).$$

- The space $\mathfrak{X}(U)$ forms a module over $C^\infty(U)$, meaning smooth functions can scale vector fields: if $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $fX$ is also a vector field.

**Key Differences**

| Feature | $C^\infty(U)$ | $\mathfrak{X}(U)$ |
|---|---|---|
| Elements | Smooth scalar functions $f : U \to \mathbb{R}$ | Smooth vector fields $X : U \to TM$ |
| Algebraic Structure | Commutative algebra (pointwise multiplication) | Module over $C^\infty(U)$, noncommutative under Lie bracket |
| Operations | Addition, multiplication | Addition, scalar multiplication by $C^\infty(U)$, Lie bracket $[X, Y]$ |

In summary, $C^\infty(U)$ consists of smooth functions, while $\mathfrak{X}(U)$ consists of smooth vector fields, which act as differential operators on $C^\infty(U)$.

Compare and contrast.

| Set | Dim | index | basis | Delta |
|---|---|---|---|---|
| $L_1(U)$ | $n$ | $i = 1, \ldots, n$ | $\alpha^i$ | $\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ |
| $L_k(U)$ | $n^k$ | $I, J \in \{\underbrace{i_i, \ldots, i_k}_{k \text{ times}}\}$, $i_k \in [1, \ldots, n]$ | $\alpha^I = \alpha^{i_1} \otimes \alpha^{i_2} \otimes \cdots \otimes \alpha^k$ | |
| $A_k(U)$ | $\binom{n}{k}$ | $I, J \in \{\underbrace{i_i, \ldots, i_k}_{k \text{ times}}\}$, $i_1 < i_2 < \ldots i_k \in [1, n]$ | $\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^k$ | $\delta_I^J = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$ |

Supersets

| Symbol | Name (set of) | Definition |
|---|---|---|
| $\Omega^0(U)$ | 0-forms | $\{$ scalar fields $\}$ |
| $\Omega^1(U)$ | 1-forms | $\{$ 1-forms, vector fields $\}$ |
| $\Omega^k(U)$ | $k$-forms | $\{$ $k$-forms $\}$ |
| $\Omega^*(U)$ | sum of $k$-forms | $\{ x = \sum y \mid y \in \oplus_k \Omega^k(U) \}$ |
| $\mathfrak{X}(U)$ | vector fields on $U$ | $\{X \to \exists f : U \to U\}$ |
| $C^\infty(U)$ | smooth functions on $U$ | |
| $X_p = T_p(U)$ | a vector field at $p$ | $\{v \in U \mid v = p + x \text{ for some } x \in U\}$ |

Shorthand

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_I a_I = \sum_{n=1}^{k} a_{i_n}$$

$$\sum_{I,J} a_I b_J = \sum_{n=1}^{k} a_{i_n} \sum_{m=1}^{k} b_{j_m}$$

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_I^J = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_k}^{j_k} = \begin{cases} 1 & i_n = j_n, \forall n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$