Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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Section 11: The Rank of a Smooth Map – July 2, 2025

Problems

11.1. **Tangent vectors to a sphere**

The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{n=1}^{n+1}(x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

All X_p are in the tangent plane and therefore perpendicular to the Normal of that plane. Let $f(x) = \sum_{n=1}^{n+1}(x^i)^2 - 1$, then the tangent to the unit sphere $S^n = f^{-1}(0)$ in $\mathbb{R}^{n+1}$ at $(x^1, \dots, x^{n+1}) \in S^{n+1}$ we compute

$$\frac{\partial f}{\partial x^i} = 2x^i$$

at $p = (p^i, \dots, p^n)$,

$$\frac{\partial f}{\partial x^i}(p) = 2p^i$$

The equation of a tangent space at a point p is

$$\sum_{k=1}^{n+1} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) = 0$$

With the $n+1$-dimensional norm being

$$N = \left\langle\, 2p^1, \dots, 2p^{n+1} \,\right\rangle$$

any X in the tangent space will be

$$\begin{aligned} X \cdot N &= 0 \\ &= \left\langle\, a^1, \dots, a^{n+1} \,\right\rangle \cdot \left\langle\, 2p^1, \dots, 2p^{n+1} \,\right\rangle \\ &= \sum_{k=1}^{n+1} 2a^i p^{n+1} \end{aligned}$$

11.2. **Tangent vectors to a plane curve**

(a) Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the standard coordinates of $\mathbb{R}^2$ and by $\hat{x}, \hat{y}$ their restrictions to S^1. Thus, $\hat{x} = i^*x$ and $\hat{y} = i^*y$. On the upper semicircle $U = \{ (a,b) \in S^1 \mid b > 0 \}$, $\hat{x}$ is a local coordinate, so that $\frac{\partial}{\partial \hat{x}}$ is defined. Prove that for $p \in U$,

$$i_* \left(\left. \frac{\partial}{\partial \hat{x}} \right|_p \right) = \left. \left(\frac{\partial}{\partial x} + \frac{\partial \hat{y}}{\partial \hat{x}} \frac{\partial}{\partial y} \right) \right|_p$$

Thus, although $i_* : T_pS^1 \to T_p\mathbb{R}^2$ is injective, $\left. \frac{\partial}{\partial \hat{x}} \right|_p$ cannot be identified with $\left. \frac{\partial}{\partial x} \right|_p$ (Figure 11.9).

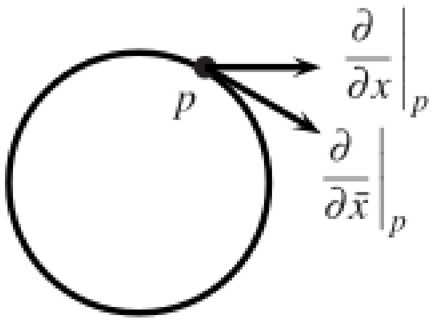

Fig. 11.9. Tangent vector $\partial/\partial\bar{x}|_p$ to a circle.

As a reminder, the pullback is defined as

$$i^*f = f \circ i$$
$$\hat{x}(p) = (i^*x)(p) = x(i(p)) = x(p)$$

that is the x-coordinate of p

$$X = \sum \left. \frac{\partial}{\partial \hat{x}} \right|_p$$
$$(i_*X)(g) = X(g \circ i)$$
$$= \sum \left. \frac{\partial g(i(x))}{\partial \hat{x}} \right|_p$$
$$= \frac{\partial g}{\partial \hat{x}}(p) + \frac{\partial g}{\partial \hat{y}}(p)$$

(b) Generalize (a) to a smooth curve C in $\mathbb{R}^2$, letting U be a chart in C on which $\hat{x}$, the restriction of x to C, is a local coordinate.

11.3. **Critical points of a smooth map on a compact manifold**

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has critical point. (*Hint*: Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be the projection to the first factor. Consider the composite map $\pi \circ f : N \to \mathbb{R}$. A second proof uses Corollary 11.6 adn the connectedness of $\mathbb{R}^m$.)

11.4. **Differential of an inclusion map**

On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a,b,c) = a \text{ and } v(a,b,c) = b.$$

So the derivations $\partial/\partial u\,|_p, \partial/\partial v\,|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u\,|_p, \partial/\partial v\,|_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\left.\frac{\partial}{\partial u}\right|_p\right) = \alpha^1 \left.\frac{\partial}{\partial x}\right|_p + \beta^1 \left.\frac{\partial}{\partial y}\right|_p + \gamma^1 \left.\frac{\partial}{\partial z}\right|_p,$$
$$i_*\left(\left.\frac{\partial}{\partial v}\right|_p\right) = \alpha^2 \left.\frac{\partial}{\partial x}\right|_p + \beta^2 \left.\frac{\partial}{\partial y}\right|_p + \gamma^2 \left.\frac{\partial}{\partial z}\right|_p,$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

11.5. **One-to-one immersion of a compact manifold**

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

11.6. **Multiplication map in** SL($n\mathbb{R}$)

Let $f : \mathrm{GL}(n, \mathbb{R})$ be the determinant map $f(A) = \det A = \det[a_{ij}]$. For $A \in \mathrm{SL}(n, \mathbb{R})$, there is at least one (k, ℓ) such that the partial derivative $\partial f/\partial a_{k\ell}(A)$ is nonzero (Example 9.13). Use Lemma 9.10 and the implicit function theorem to prove that

(a) there is a neighborhood of A in $\mathrm{SL}(n, \mathbb{R})$ in which $a_{ij}, (i, j) \neq (k, \ell)$, form a coordinate system, and $a_{k\ell}$ is a C^∞ function of the other entries $a_{ij}, (i, j \neq (k, \ell)$;

(b) the multiplication map

$$\hat{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞.