1. Suppose that V is a finite dimensional vector space. Show that any linear transformation on a subspace of V can be extended to a linear transformation on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there is $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Define $T \in \mathcal{L}(V, W)$ as

$$T(v) = \begin{cases} S(v) & \text{if } v \in U \\ v & \text{otherwise} \end{cases}$$

thus

$u, v \in U$	$u \in U, v \in V \backslash U$	$u, v \in V \backslash U$
$T(u+v) = S(u+v)$	$T(u+v) = S(u) + T(v)$	$T(u+v) = T(u) + T(v)$
$= S(u) + S(V) \in U$	$= S(u) + v \in V$	$= u + v \in V$

2. Let $V = \mathcal{M}_{n \times n}(F)$, and let B be fixed matrix in V. Show that $T : V \mapsto V$ defined by $T(A) = AB - BA$ is a linear transformation.

What happens in T when you add C and D?

$$\begin{aligned} T(C+D) &= (C+D)B - B(C+D) \\ &= CB + DB - BC - BD \qquad \text{distributive law of matrix multiplication} \\ &= CB - BC + DB - BD \\ &= T(C) + T(D) \end{aligned}$$

and scalar multiplication?

$$\begin{aligned} T(cA) &= (cA)B - B(cA) \\ &= c(AB) - c(BA) \\ &= c(AB - BA) \\ &= cT(A) \end{aligned}$$

3.a) Recall that $\mathbb{C}$ is a real vector space. Find $T : \mathbb{C} \mapsto \mathbb{C}$ which is a $\mathbb{R}$-linear transformation which is not a $\mathbb{C}$-linear transformation.
b) Find a linear transformation $T : V \mapsto V$ where the range and nullspace of T are identical.
c) Find T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

4. Let T and U be two linear operators on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ and $U(x_1, x_2) = (x_1, 0)$.
a) Give a geometric interpretation for T and U.
b) Give rules for $U + T$, UT, TU, T^2, and U^2.

Extra Questions

1. Let A be an $m \times n$ matrix over F of rank k. Show that there exist a $m \times k$ matrix B and a $k \times n$ matrix C, both with rank k, where $A = BC$. Conclude that A has rank 1 if and only if $A = xy^t$ where $x \in F^m$ and $y \in F^n$.

2. Let W be the vector space of 2×2 complex Hermitian matrices. Note that W is a vector space over $\mathbb{R}$ but not over $\mathbb{C}$. Let $T : \mathbb{R}^4 \mapsto W$ be the map defined by

$$(x, y, z, t) \mapsto \begin{pmatrix} t + x & y + iz \\ y - iz & t - x \end{pmatrix}.$$

Show that T is an isomorphism.

3. We will consider the vector space $V = \mathcal{P}^{(n)}(\mathbb{R})$ of polynomials at most degree n. Let

$$[x]_k := x(x-1)(x-2)\cdots(x-k+1)$$

for $k \geq 1$ and $[x]_0 = 1$.

a) Show that $([x]_0, [x]_1, [x]_2, \ldots, [x]_n)$ is a basis of V. [Hint: argue that $[x]_k = x^k + a(k, k-1)x^{k-1} + \cdots + a(k, 1)x + a(k, 0)$ where $a(k, j)$ are integers. Construct the $(n+1) \times (n+1)$ matrix which expresses each $[x]_k$ in the basis $(1, x, x^2, \ldots, x^n)$. Show that this matrix is invertible].

b) Now prove that $x^k = \sum_{j=0}^{k} S(k, j)[x]_j$ where $S(k, j)$ are integers.

c) Show that $S(k, 0) = 0$ for $k \geq 1$. Also show that $S(k, k) = 1$ for $k \geq 0$.

d) Prove that if $1 \leq j \leq k - 1$ then

$$S(k, j) = jS(k-1, j) + S(k-1, j-1).$$

The above exercise shows that $S(k, j)$ are nonnegative integers. They are called *Stirling numbers of the second kind*.