Functional Analysis– Spring 2024

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p. 126 #8 Show that the dual space of the space c_0 is ℓ^1.(Cf. Prob. 1 in Sec. 2.3.)

Want to show that

1. every element of c_0' is an element of ℓ^1
   Let (e_k) be the unique Shauder basis for ℓ^1 where $e_k = (\delta_{jk})$. Let $x = (\xi_j) \in c_0$, that is $\lim_{j\to\infty} \xi_j = 0$ which has the unique representation $x = \sum_{j=1}^{\infty} \xi_j e_j$. Let $f \in c_0'$, that is $f : c_0 \to \mathbb{R}$ which is linear and bounded. Therefore,

$$f(x) = \sum_{j=1}^{\infty} \xi_j f(e_j)$$
$$|f(e_j)| \leq \|f\| \, \|e_j\| = \|f\|$$
$$\|f(x)\| \leq \|f\| \left| \sum_{j=1}^{\infty} \xi_j \right| \leq \|f\| \sum_{j=1}^{\infty} |\xi_j| = \|f\| \, \|x\|_{\ell^1}$$

   which means that $f \in \ell^1$.

2. that the norm over c_0' is the norm over ℓ^1. want to show that $|f(x)| = \|x\|$. Let $\gamma = \sup_j f(e_j)$

$$|f(x)| = \left| \sum_{j=1}^{\infty} \xi_j f(e_j) \right| \leq \gamma \sum_{j=1}^{\infty} |\xi_j| = \gamma \, \|x\|$$

p. 135 #9 Prove

$$\operatorname{Re} \langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 \right)$$
$$\operatorname{Im} \langle x, y \rangle = \frac{1}{4} \left(\|x+iy\|^2 - \|x-iy\|^2 \right)$$

p. 141 #7-10,

7. Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| = \|x - \alpha y\|$ (see Fig. 25.)

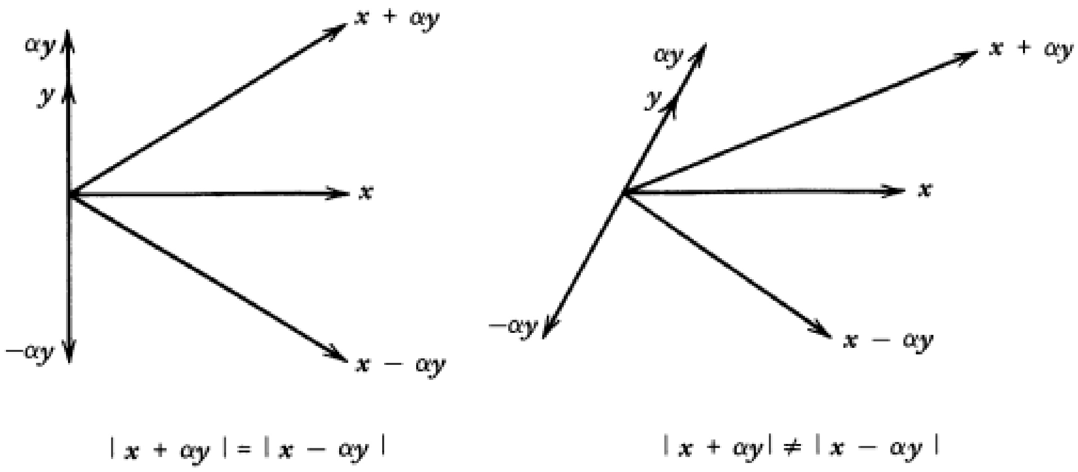

Fig. 25. Illustration of Prob. 7 in the Euclidean plane $\mathbf{R}^2$

8. Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

9. Let V be the vector space of all continuous complex-valued functions on $J - [a, b]$. Let $X_1 = (V, \|\cdot\|_\infty)$, where $\|x\|_\infty = \max_{t \in J} |x(t)|$; and let $X_2 = (v, \|\cdot\|_2)$, where

$$\|x\|_2 = \langle x, x \rangle^{1/2}, \quad \langle x, y \rangle = \int_a^b x(t)\overline{y(t)}dt$$

Show that the identity mapping $x \mapsto x$ of X_1 onto X_2 is continuous.
(It is not a homeomorphism. X_2 is not complete.)

10. **(Zero Operator)** Let $T : XtoX$ be a bounded linear operator on a complex inner product space X. If $\langle Tx, x \rangle = 0$ for al $x \in X$, show that $T = 0$.
Show that this does not hold in the case of a real inner product space. *Hint.* Consider a rotation of the Euclidiean plane.

p. 150 #2, 3a, 6,

2. Show that the subset $M = \{y = (\eta_j) \mid \sum \eta_j = 1\}$ of complex space $\mathbb{C}^n$ (cf 3.1-4) is complete and convex. Find the vector of minimum norm in M.

3. (a) Show that the vector space X of all real-valued continuous functions on $[-1, a]$ is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on $[-1, 1]$.

6. Show that $Y = \{x \mid x = (\xi_j) \in \ell^2, \xi_{2n} = 0, n \in \mathbb{N}\}$ is a closed subspace of ℓ^2 and find $Y^\perp$. What is $Y^\perp$ if $Y = \operatorname{span}\{e_1, \cdots, e_n\} \subset \ell^2$, where $e_j = (\delta_{jk})$?