Math 5050 – Special Topics: Manifolds– Fall 2025

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Section 7: Quotients – May 17, 2025

Pg. 77: Exercise 7.11 (Real projective space as a quotient of a sphere).* For $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, let $||x|| = \sqrt{\sum_i (x^i)^2}$ be the modulus of x. Prove that the map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{||x||}$$

induces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n/\sim$. Where

$$x \sim y \iff x = \pm y,\ x, y \in S^n$$

(*Hint:* Find an inverse map

$$\bar{g} : S^n/\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Given the relation above $x \sim y \iff x = \pm y,\ x, y \in S^n$. Define

$$\bar{g}([x]) = [x]$$

Note that on the left $[x] \in S^n/\sim$ and $[x] \in \mathbb{R}P^n$. For clarity,[1]

$$[a] \in S^n/\sim \implies [a] = \{a, -a\} \text{ where } a \in S^n$$
$$[b] \in \mathbb{R}P^n \implies [b] = \{x \in \mathbb{R}^{n+1} | \, x = \alpha b,\ \forall \alpha \in \mathbb{R}\} \text{ where } b \in \mathbb{R}^{n+1}$$

Notice that

$$\begin{aligned}
\bar{f}([b]) = [f(b)] &= \left[\frac{b}{||b||}\right] \in S^n/\sim \\
\bar{g}\left(\left[\frac{b}{||b||}\right]\right) &= [b] \\
\therefore \bar{g} \circ \bar{f} &= \mathbb{I} \\
\text{and } \bar{g}([a]) &= [a] \in \mathbb{R}P^n \\
\bar{f}\,(\,\bar{g}([a])\,) &= \left[\frac{[a]}{||[a]||}\right] = [f(a)] = [a] \in S^n/\sim \\
\therefore \bar{f} \circ \bar{g} &= \mathbb{I}
\end{aligned}$$

$\bar{f}$ is continuous because f is continuous. $\bar{g}$ is continuous because it is a mapping of one identity to another. Therefore, $\bar{f}$ is a homeomorphism.

$$\begin{tikzcd}
S \arrow[r, dash, "f"] \arrow[d, dash, "\pi"'] & Y \\
S/\sim \arrow[ur, dash, "\hat{f}"'] &
\end{tikzcd}$$

Problems

7.1. **Image of the inverse image of a map**

Let $f : X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if f is surjective, then $f(f^{-1}(B)) = B$.

$\subseteq$: Let $b \in B$ and $a \in X$ such that $f(a) = b$. Then, $a \in f^{-1}(b)$, thus a is an arbitrary point in $f^{-1}(B)$. We know that $f(a) \in f(X)$ and $f(a) \in B$, therefore $f(a) \in B \cap f(X)$ and $f(f^{-1}(B)) \subseteq B \cap f(X)$.

$\supseteq$: Let $b \in B \cap f(X)$. Since $b \in f(X)$, there exists $a \in X$ such that $f(a) = b$ and since $b \in B$ then $b = f(a) \in f(f^{-1}(b)) \subseteq f^{-1}(B)$. Therefore $b \in f(f^{-1}(B))$

[1] $\mathbb{R}P^n \equiv \mathbb{R}^{n+1}/r(x, y)$ where $r(x, y)$ is the relation that is true when x, y, p are colinear.

7.2. **Real projective plane**

Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. In the notation of example 7.13, prove that the induced map $f : H^2/\sim \to S^2/\sim$ is a homeomorphism. (*Hint:* Imitate Propostion 7.3.) Let H^2 be the upper hemisphere and S^2 be the unit sphere

$$H^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1, z \geq 0\}$$
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$$

These two are homeomorphic to each via

$$\varphi : S^2 \mapsto H^2$$
$$\varphi(x,y,z) = (x,y,|z|)$$

and its inverse

$$\psi : H^2 \mapsto S^2$$
$$\psi(x,y,z) = (x,y,z)$$

Define the relations

$$(x,y,z) \sim (-x,-y,-z) \to x^2+y^2 = z^2, \, \forall (x,y,z) \in S^2$$
$$(x,y,z) \sim (x,y,z) \to \sqrt{x^2+y^2} = z, \, \forall (x,y,z) \in H^2$$

Then we induce $f : H^2/\sim \to S^2/\sim$ as

$$f([(x,y,z)]) = [(x,y,z)]$$

7.3. **Closedness of the diagonal of a Hausdorff Space**

Deduce Theorem 7.7 from Corollary 7.8 (*Hint:* To prove that if $S/\sim$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi : S \to S/\sim$. To prove the reverse implication, use the openness of π.)

7.4. **Quotient of a sphere with antipodal points indentified**

Let S^n be the unit sphere centered at the oringin $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by indentifying antipodal points:

$$x \sim y \iff x = \pm y, \; x, y, \in S^n$$

(a) Show that $\sim$ is an open equivalence relation.

(b) Apply Theorem 7.7 and Corallary 7.8 to prove that the quotient space $S^n/\sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \cong S^n/\sim$.

7.5. **Orbit space of a continuous group action**

Suppose a right action of a topological group G on a topological space S is continuous, this simply imeans that the map $S \times G \to S$ describing teh action is continuous. Define two points x, y of S to bequivalent if they are in teh same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be teh quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi : S \to S/G$ is an open map. (This problem generlaizes Proposition 7.14, in which $G = R^\times = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^\times$ is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplicatin is written on teh right.)

7.6. **Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$.**

Let the addtivei group $2\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that eh orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

7.7. **The circle asa quotient space**

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of circles S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \coprod B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = p\bar{h}i_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.

(b) The complex exponetial $\mathbb{R} \to S^1, t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1, F([t]) = e^{it}$. Prove that F is C^∞.

(c) Prove that $F : \mathbb{R}/2\pi/Z \to S^1$ is a diffeomorphism.

7.8. **The Grassmanian $G(k,n)$**

7.9. **Compactness of real projective space**

Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint:* Use Exercise 7.11.)