Math 5111 – Real Analysis II– Sprint 2025
w/Professor Perera

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Practice Test 2 – March 26, 2025

1. Show that if $f \in L^1(X)$ and $E_1 \subset E_2 \subset \cdots$ is a sequence of mesaurable sets such that $\bigcup_n E_n = X$, then

$$\lim_{n\to\infty} \int_{E_n} f d\mu = \int_X f d\mu$$

Let $g_i = f\chi_{E_i}$ then

$$\begin{aligned} \lim_{n\to\infty} g_i = \lim_{n\to\infty} f\chi_{E_i} = f \lim_{n\to\infty} \chi_{E_i} = f, \text{ over all } X \\ \int_X g_i d\mu = \int_X f\chi_{E_i} d\mu = \int_{E_i} fd \\ \lim_{n\to\infty} \int_X g_i d\mu = \lim_{n\to\infty} \int_X f\chi_{E_i} d\mu = \lim_{n\to\infty} \int_{E_i} f d\mu \end{aligned}$$

2. Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are lower semicontinuous functions, then $f + g$ is lower semicontinuous.

f, g are lower semi-continuous means that $\liminf_{x\to x_0} f(x) \geq f(x_0)$ for all $x_0 \in X$. Then

$$\begin{aligned} \liminf_{x\to x_0} f(x) + g(x) &= \liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) \\ &\geq f(x_0) + g(x_0) \end{aligned}$$

for all $x_0 \in X$. Therefore $f + g$ is lower semicontinuous.

3. Show that if $f \in L^1(X)$, then for each $\epsilon > 0, \exists\delta > 0$ such that $\int_E |f|d\mu < \epsilon$ whenever $\mu(E) < \delta$.

4. Show that if X is locally compact Hausdorff space and λ and μ are outer regular Borel measures on X such that $\lambda(V) = \mu(V)$ for all $V \subset X$, then $\lambda = \mu$.

λ, μ are outer regular means that given any $V \subset X$ then $\lambda(V) = \inf\{\lambda(U) \,|\, V \in U \text{ and open }\}$. Let V_i be a sequence of open sets such that $V \subset V_i$ for all i and $\cap_i V_i = V$. Let $\lambda_i = \lambda(V_i)$ and $\mu_i = \mu(V_i)$. WTS $|\lambda_i - \mu_i| \to 0$