Topology without Tears

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Contents

Chapter 1

Topology Spaces

1.1 Topology – Exercises

1. Let $x = \{a, b, c, d, e, f\}$. Determine whether or not each of the following collections of subsets of X is a topology on X:

   (a) $\tau_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$;
   No, $\{a, f\} \cap \{b, f\} = \{f\} \notin \tau$.

   (b) $\tau_2 = \{X, \emptyset, \{a, b, f\}, \{a, b, d\}, \{a, b, d, f\}\}$;
   No, $\{a, b, f\} \cap \{a, b, d\} \notin \tau$.

   (c) $\tau_3 = \{X, \emptyset, \{f\}, \{e, f\}, \{a, f\}\}$;
   No, $\{e, f\} \cup \{a, f\} = \{a, e, f\} \notin \tau$.

2. Let $X = \{a, b, c, d, e, f\}$. Which of the following collections of subsets of X is a topology on X? (Justify your answer.)

   (a) $\tau_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$;

   (b) $\tau_2 = \{X, \emptyset, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$;

   (c) $\tau_3 = \{X, \emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}\};\}$

3. If $X = \{a, b, c, d, e, f\}$, and τ is the discrete topology on X, which of hte following statements are true?
   (a) $X \in \tau$; YES (b) $\{X\} \in \tau$; ??? (c) $\{\emptyset\} \in \tau$; ??? (d) $\emptyset \in \tau$; YES
   (e) $\emptyset \in X; NO$ (f) $\{\emptyset\} \in X; NO$ (g) $\{a\} \in \tau; YES$ (h) $a \in \tau$; NO
   (i) $\emptyset \in X$; NO (j) $\{a\} \in X$; NO (k) $\{\emptyset\} \subseteq X$; YES (l) $a \in X$; YES
   (m) $X \subseteq \tau$; YES (n) $\{a\} \subseteq \tau$; YES (o) $\{X\} \subseteq \tau$; YES (p) $a \subseteq \tau$; NO

4. Let (X, τ) be any topological space. Verify that ***the intersection of any finite number of members of*** τ ***is a member of*** τ.

5. Let $\mathbb{R}$ be the set of all real numbers. Prove that each of the following collections of subsets of $\mathbb{R}$ is a topology

   (i) τ_1 consists of $\mathbb{R}, \emptyset$, and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;

   (ii) τ_2 consists of $\mathbb{R}, \emptyset$, and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;

   (iii) τ_3 consists of $\mathbb{R}, \emptyset$, and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;

6. (i) τ_1 consists of $\mathbb{N}, \emptyset$, and every set $\{1, 2, \ldots, n\}$, for n any positive integer. (This is called ***initial segment topology***).

   (ii) τ_2 consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \ldots\}$, for n any positive integer. (This is called the ***final segment topology***.)

7. List all possible topologies on the following sets:

   (a) $X = \{a, b\}$;

   (b) $Y = \{a, b, c\}$;

8. Let X be an infinite set and τ a topology on X. If every infinte subset of X is in τ, prove that τ is the discrete topology.

9. Let $\mathbb{R}$ be the set of all real numbers Precisely three of the followin ten collections are subsets of $\mathbb{R}$ are topologies. Identify these and justifey your answer.

   (i) τ_1 consists of $\mathbb{R}, \emptyset$, and every interval (a, b), for a and b any real numbers where $a < b$.

   (ii) τ_2 consists of $\mathbb{R}, \emptyset$ and every interval $(-r, r)$, for r any positive real number.

   (iii) τ_3 consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r)$, for r any positive rational number;

   (iv) τ_4 consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, for r any positive rational number;

   (v) τ_5 consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r)$, for r any positive irrational number;

   (vi) τ_6 consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, for r any positive irrational number;

   (vii) τ_7 consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r)$, for r any positive real number;

   (viii) τ_8 consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r]$, for r any positive real number;

   (ix) τ_9 consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, and every interval $(-1, r)$, for r any positive real number;

   (x) τ_{10} consists of $\mathbb{R}, \emptyset$, every internval $[-n, n]$, and every interval $(-r, r)$, for n any positive intger and r any positive real number.

1.2 Open Sets - Exercises

1. List all 64 subsets of the set X in Example 1.1.2. Write down, next to each set, whether it is (i) clopen, (ii) neither open nor closed; (iii) open but not closed; (iv) closed but not open.

   Example 1.1.2: Let $X = \{a, b, c, d, e, f\}$ and

   $$\tau_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}.$$

   - size one

     $\{a\}$,clopen $\{b\}$,neither $\{c\}$,neither $\{d\}$,neither $\{e\}$,neither $\{f\}$,neither

   - size two

     $\{a, b\}$,neither $\{a, c\}$ $\{a, d\}$ $\{a, e\}$ $\{a, f\}$
     $\{b, c\}$ $\{b, d\}$ $\{b, e\}$ $\{b, f\}$
     $\{c, d\}$,open $\{c, e\}$ $\{d, f\}$
     $\{d, e\}$ $\{d, f\}$
     $\{e, f\}$

   - size three

     $\{a, b, c\}$ $\{a, b, d\}$ $\{a, b, e\}$ $\{a, b, f\}$
     $\{a, c, d\}$,open $\{a, c, e\}$ $\{a, c, f\}$
     $\{a, d, e\}$ $\{a, d, f\}$
     $\{a, e, f\}$
     $\{b, c, d\}$ $\{b, c, e\}$ $\{b, c, f\}$
     $\{b, d, e\}$ $\{b, d, f\}$
     $\{b, e, f\}$
     $\{c, d, e\}$ $\{c, d, f\}$
     $\{c, e, f\}$
     $\{d, e, f\}$

   - size four

     $\{a, b, c, d\}$ $\{a, b, c, e\}$ $\{a, b, c, f\}$
     $\{a, b, d, e\}$ $\{a, b, d, f\}$
     $\{a, b, e, f\}$
     $\{b, c, d, e\}$ $\{b, c, d, f\}$
     $\{c, d, e, f\}$

   - size five

     $\{a, b, c, d, e\}$ $\{a, b, c, d, f\}$
     $\{a, b, c, e, f\}$
     $\{a, b, d, e, f\}$
     $\{a, c, d, e, f\}$
     $\{b, c, d, e, f\}$,clopen

   - size six
     $\{a, b, c, d, e, f\}$,open

2. Let (X, τ) be a topological space with the property that every subset is closed. Prove that it is a discrete space.

$$S \subseteq X \implies X \backslash X \text{ is open } \implies X \backslash S \in \tau$$
$$T \in \tau \implies X \backslash T \text{ is closed } \implies T \subseteq X$$

3. Observe that if (X, τ) is a discrete space or an indiscrete space, then every opne set is a clopen set. Find a topology τ on the set $X = \{a, b, c, d\}$ which is not discrete and is not indiscrete but has the property that every open set is clopen.

   Let $\tau = \{X, \emptyset, \{a\}, \{b, c, d\}\}$

4. Let X be an infinite set. If τ is a topology on X such that every infinite subset of X is closed, prove that τ is the discrete topology.

$$S \subseteq X \text{ and } |S| = \infty$$
$$|X \backslash S| < \infty \implies X \backslash S \text{ is open}$$

   there are an infinite number of finite subsets whose compliment is infinite and closed. These are precisely what make up a discrete topology.

5. Let X be an infinite set and τ a topology on X with the property that the only infinite subset of X which is open is X itself. Is (X, τ) necessarily an indiscrete space?

6. (i) Let τ be a topology on a set X such that τ consists of precisely for sets; that is, $\tau = \{X, \emptyset, A, B\}$, where A and B are non-empty distinct proper subsets of X. [A is a ***proper subset*** of X means that $A \subseteq X$ and $A \neq X$. This si denoted by $A \subset X$.] Prove that A and B must satisfy exactly one of the following conditions.

   (a) $B = X \backslash A$; (b) $A \subset B$; (c) $B \subset A$;

   [Hint. Firstly show that A and B must satisfy at least one of the conditions and then show that they cannot satisfy more than one of the conditions.]

   (ii) Using (i) list all topologies on $X = \{1, 2, 3, 4\}$ which consist of exactly four sets.

7. (i) A recorded in http://en.wikipedia.org/wiki/Finite_topological_space, the number of distinct topologies on a set with $n \in \mathbb{N}$ points can be very large even for small n; namely when $n = 2$, there are 4 topologies; when $n = 3$, there are 29 topologies: when $n = 4$, there are 355 topologies; when $n = 5$, there are 6942 topologies etc. Using mathematical induction, prove that as n increases the number of topologies increases.

   (ii) Using mathematical induction prove that if the finite set X has $n \in \mathbb{N}$ then it has at least $(n-1)!$ distinct topolgies.

   (iii) If X is any infintie set of cardinality $\mathfrak{N}$, prove that there are at least $2^{\mathfrak{N}}$ distinct topologies on X. Deduce that every infinite set has an uncountable number of distinct topolgies on it.

1.3 Finite Closed Topology – Exercises

1. Let f be a function from a set X into a set Y. Then we stated in Example 1.3.9 that

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j) \tag{1.1}$$

and

$$f^{-1}\left(B_1 \bigcap B_2\right) = f^{-1}(B_1) \cap f^{-1}(B_2) \tag{1.2}$$

for any subsets B_j of Y and any index set J.

(a) Prove that (1.1) is true

$$\begin{aligned}
&\text{Let } y \in \bigcup_{j\in J} B_j \\
&\exists k \in J \to y \in B_k \\
&f^{-1}(y) \in f^{-1}\left(\bigcup_{j\in J} B_j\right) \text{ and } f^{-1}(y) \in f^{-1}(B_k) \\
&f^{-1}(B_k) \subseteq f^{-1}\left(\bigcup_{j\in J} B_j\right)
\end{aligned}$$

since there MUST be a k for each y then it must be that all $\cup_{j\in J} f^{-1}(B_j) \subseteq f^{-1}\left(\bigcup_{j\in J} B_j\right)$

(b) Prove that (1.2) is true.

(c) Find (concrete) sets A_1, A_2, X, and Y and a function $f : X \to Y$ such that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$, where $A_1 \subseteq X$ and $A_2 \subseteq X$.

2. Is the topology τ described in Exercises 1.1 #6 (ii) the finite-closed topology?

τ_2 consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the ***final segment topology***.)

T_1-*spaces*

3. A topological space (X, τ) is said to be a T_1**-space** if every singleton set $\{x\}$ is closed in (X, τ). Show that precisely two of the following nine topological spaces are T_1-spaces. (Justify your answer).

(i) a discrete space.

(ii) an indiscrete space with at least two points.

(iii) an infinite set with the finite-closed topology.

(iv) Exampe 1.1.2;

(v) Exercise 1.1 #5 (i)

τ_1 consists of $\mathbb{R}, \emptyset$, and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;

(vi) Exercise 1.1 #5 (ii)

τ_2 consists of $\mathbb{R}, \emptyset$, and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;

(vii) Exercise 1.1 #5 (iii)

τ_3 consists of $\mathbb{R}, \emptyset$, and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;

(viii) Exercise 1.1 #6 (i)

τ_1 consists of $\mathbb{N}, \emptyset$, and every set $\{1, 2, \dots, n\}$, for n any positive integer. (This is called ***initial segment topology***).

(ix) Exercise 1.1 #6 (ii)

τ_2 consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the ***final segment topology***.)

4. Let τ be the finite-closed topology on a set X. If τ is also the discrete topology, prove that the set X is finite.

T_0*-space and the Sierpinsi Space*

5. A topological space (X, τ) is said to be a T_0**-space** if for each pair of distinct points a, b in X, either there exist an open seet containing a and not b, or there exists an open set containing b and not a.

   (i) Prove that every T_1-space is a T_0-space.

   (ii) Which of $(i) - (iv)$ in Exercise 3 above are T_0-spaces?

   (iii) Put a topology τ on the set $X = \{0, 1\}$ so that (X, τ) will be a T_0-space but not a T_1-space. [known as the **Sierpinski space**.]

   (iv) Prove that each of the topological spaces described in Exercise 1.1 #6 is a T_0-space.

Countable-Closed Topology

6. Let X be any infinite set. The ***countable-closed topology*** is defined to be the topology having as its closed sets X and all countable subsets of X. Prove that this is indeed a topology on X.

7. Let τ_1 and τ_2 be two topologies on a set X. Prove each of the following statements.

   (i) τ_3 is definted by $\tau_3 = \tau_1 \cup \tau_2$, then τ_3 is not necessarily a topology on X.

   (ii) If τ_4 is defined by $\tau_4 = \tau_1 \cap \tau_2$, then τ_4 is a topology on X.

   (iii) If (X, τ_1) and (X, τ_2) are T_1-spaces, then (X, τ_4) is a T_1-space.

   (iv) If (X, τ_1) and (X, τ_2) are T_0-spaces, then (X, τ_4) is not necessarily a T_0-space.

   (v) If $\tau_1, \tau_2, \ldots, \tau_n$ are topologies on a set X, the $\tau = \bigcap_{i=1}^{n} \tau_i$ is a toplogy on X.

   (vi) If for each $i \in I$, for some index set I, each τ_i is a topology on the set X, then $\tau = \bigcap_{i \in I} \tau_i$ is a topology on X.

Distinct T_1*-topologies on a Finite Set*

8. In Wikipedia `//enwikipedia.org/wiki/Finite_topological_space`, as we noted in Exercise 1.2 #7, it says that the number of topologies on a finite set with $n \in \mathbb{N}$ points can be quite large, even for small n. This is also true even for T_0-spaces; for $n = 5$, ther are 4231 distinct T_0-spaces. Prove, using mathemtaical induction, that as n increases, the number of T_0-spaces increases.

9. A topological space (X, T) is said to be a ***door space*** if every subset of X is either an open set or a closed set (or both).

   (i) Is a discrete space a door space?

   (ii) Is an indiscrete space a door space?

   (iii) If X is an infintie set and τ is the finite-closed topology, is (X, τ) a door space?

   (iv) Let X be the set $\{a, b, c, d\}$. Identify those topologies τ on X which make it into a door space.

Saturated Sets

10. A subset S of a topological space (X, τ) is said to be **saturated** if it is an intersection of open sets in (X, τ).

    (i) Verify that every open set is a saturated set.

    (ii) Verify that in a T_1-space every set is saturated set.

    (iii) Give an example of a topological space which has atleast one subset which is not saturated.

    (iv) Is it true that if the topological sapce (X, τ) is such that every subset is saturated, then (X, τ) is a T_1-space?

Chapter 2

The Euclidean Topology

2.1 Euclidian Space – Exercises

1. Prove that if $a, b \in \mathbb{R}$ with $a < b$ then neither $[a, b)$ nor $(a, b]$ is an open subset of $\mathbb{R}$. Also show that neither is a closed subset of $\mathbb{R}$.

   In the case of $[a, b)$ there is no set $a \in (x, y)$ because $x < a$ implies that $x + \frac{|x-a|}{2}$ would have to be a member of $[a, b)$ which it cannot. Similarly for $(a, b]$.

2. Prove that the sets $[a, \infty)$ and $(-\infty, a]$ are closed subsets of $\mathbb{R}$.

   The composite of $[a, \infty)$ is $(-\infty, a)$ which is open and similarly for $(-\infty, a]$.

3. Show, by example, that the union of an infinite number of closed subsets of $\mathbb{R}$ is not necessarily a closed subset of $\mathbb{R}$.

   Define $S_i = [1/i, 1]$ then $\mathcal{S} = \cup_{i=1}^{\infty} S_i$. Obviously, given any $n \in \mathbb{N}$ there is a closed set $S_n = [1/n, 1]$ and there exists $(1/(n+1), 1) \subseteq \mathcal{S}$ such that $1/n \in (1/(n+1), 1)$ hence $\mathcal{S}$ must be open.

4. Prove each of the following statements.

   (i) The set $\mathbb{Z}$ of all integers is not an open set of $\mathbb{R}$.

   (ii) The set $\mathbb{P}$ of all prime numbers is a closed subset of $\mathbb{R}$ but not an open subset of $\mathbb{R}$.

   (iii) The set $\mathbb{I}$ of all irrational numbers is neither a closed subset nor an open subset of $\mathbb{R}$.

5. If F is a non-empty finite subset of $\mathbb{R}$, show that F is closed in $\mathbb{R}$ but that F is not open in $\mathbb{R}$.

6. if F is non-empty countable subset of $\mathbb{R}$, prove that F is not an open set, but that F may or may not be a closed set depending on the choice of F.

7. (i) Let $S = \{0, 1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots\}$. Prove that the set S is closed in the euclidean topology on $\mathbb{R}$.

   (ii) Is the set $T = \{1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots\}$ closed in $\mathbb{R}$?

   (iii) Is the set $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots, n\sqrt{2}, \ldots\}$ closed in $\mathbb{R}$?

F_σ*-Sets and* G_δ*-sets.*

8. (i) Let (X, τ) be a toplogical space. A subset S of X is said to be an F_σ ***set*** if it is the union of a countable number of closed sets. Prove that all open intervals (a, b) and all closed intervals $[a, b]$ are F_σ-sets in $\mathbb{R}$.

   (ii) Let (X, τ) be topological space. A subset T of X is said to be a G_δ***-set*** if it is the intersection of a countable number of open sets. Prove that all open intervals (a, b) and all closed intervalse $[a, b]$ are G_δ-sets in $\mathbb{R}$.

   (iii) Prove that the set $\mathbb{Q}$ of rationasl is an F_σ-set in $\mathbb{R}$.

   (iv) Verify that the complement of an F_σ-set is a G_δ-set and the complement of a G_δ-set is an F_σ-set.