

Math 5110 – Real Analysis I– Fall 2024  
w/Professor Liu

Paul Carmody  
Homework #1 – September 9, 2024

I. This problem review continuity for functions on real line.

We say a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *continuous* at a point  $a \in \mathbb{R}$  if for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|x - a| < \delta$  implies  $|f(x) - f(a)| < \epsilon$ .

(a) Show that  $f(x) = x^2$  is continuous at  $x = 2$ .

Given an  $\epsilon > 0$ , when  $|f(x) - 4| < \epsilon$ ,  $|x^2 - 4| < \epsilon$ . Let  $\delta < \sqrt{\epsilon + 4}$

If

$$(2 + \delta)^2 - 4 < \epsilon$$

$$(2 + \delta)^2 < \epsilon + 4$$

$$2 + \delta < \sqrt{\epsilon + 4}$$

$$\delta < \sqrt{\epsilon + 4} - 2$$

(b) Suppose that  $f$  is continuous at  $a$  and  $f(a) \neq 0$ . Show that  $f$  is nonzero in some open interval containing  $a$ .

Since  $f$  is continuous at  $a$  and  $f(a) \neq 0$  then for every  $\epsilon > 0$  such that when  $|f(x) - f(a)| < \epsilon$ . Without loss of generality, assume  $f(a) > 0$ . Choose  $\epsilon < f(a)$  then  $0 < f(a) - \epsilon < f(x) < f(a) + \epsilon$ . Therefore,  $f(x) \neq 0$

II. This problem review derivatives.

- (a) Let  $f(x) = x^n$  for some positive integer  $n$ . Using the definition of the derivative, and the binomial theorem, show that  $f'(x^n) = nx^{n-1}$ .

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^n$$

thus

$$f'(x^n) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sum_{k=0}^n \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

remove the first entry from the summation

$$f'(x^n) = \lim_{h \rightarrow 0} \frac{x^n + \sum_{k=1}^n \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sum_{k=1}^n \binom{n}{k} x^{n-k} h^k}{h}$$

$$= \sum_{k=1}^n \lim_{h \rightarrow 0} \frac{\binom{n}{k} x^{n-k} h^k}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\binom{n}{1} x^{n-1} h}{h} + \lim_{h \rightarrow 0} \frac{\binom{n}{2} x^{n-2} h^2}{h} + \cdots + \lim_{h \rightarrow 0} \frac{\binom{n}{n} x^0 h^n}{h}$$

$$= nx^{n-1} + \lim_{h \rightarrow 0} \frac{\binom{n}{2} x^{n-2} h^2}{h} + \cdots + \lim_{h \rightarrow 0} \frac{\binom{n}{n} x^0 h^n}{h}$$

the  $\lim_{h \rightarrow 0} \frac{h^k}{h} = \lim_{h \rightarrow 0} h^{k-1} = 0$  for all  $k \geq 1$  therefore

$$f'(x^n) = nx^{n-1}$$

- (b) Is the function

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x > 0 \end{cases}$$

differentiable at  $x = 0$ .

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} -x^2 = 0$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$f(x)$  is both continuous and differentiable at  $x = 0$ .

## III. This problem reviews sup and inf.

For any subset  $A \subset \mathbb{R}$ , we say that  $M$  is an *upper bound* for  $A$  if  $x \leq M$  for all  $x \in A$ . If a set  $A$  has a finite upper bound, we say it is *bounded above*. It is a theorem about the set  $\mathbb{R}$  that *for any set  $A \subset \mathbb{R}$  that is bounded above, there exists a least (smallest) upper bound for  $A$* . This least upper bound is called supremum of  $A$ , and denoted  $\sup A$ . By definition, the number  $\sup A$  has two properties.

- (i)  $x \leq \sup A$  for all  $x \in A$  (i.e.,  $\sup A$  is an upper bound for  $A$ ).
- (ii) for any  $M$  that is an upper bound for  $A$ , we have  $\sup A \leq M$ .

For sets that are not bounded above, we say that  $\sup A = +\infty$ . we often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set  $\{f(x) : x \in A\}$ , where  $f$  is a some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted  $\inf A$ . It satisfies the same properties as  $\sup A$  with the inequalities reversed.

- (a) Find  $\sup A$  and  $\inf A$  for  $A = (1, 2]$ ,  $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ , and  $A = \{0, 1, 2, 3, \dots\}$ .

- $A = (1, 2]$ ,  $\sup A = 2$  and  $\inf A = 1$
- $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ ,  $\sup A = 1$ , and  $\inf A = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .
- $A = \{0, 1, 2, 3, \dots\}$ .  $\sup A = \lim_{n \rightarrow \infty} n = \infty$ , and  $\inf A = 0$

- (b) Find  $\sup_{x \in (0,1)} (1+x^2)^{-1}$

Let  $f(x) = (1+x^2)^{-1}$ . On the interval  $(0, 1)$  we can see that it is strictly decreasing ( $f'(x) = -2(1+x^2)^{-2}(2x) < 0$  for all  $x \in (0, 1)$ ), that is  $a < b \implies f(a) > f(b)$ . Thus,  $\sup_{x \in (0,1)} f(x) = f(0) = (1+0^2)^{-1} = 1$ .

- (c) Assume that  $\sup A < \infty$ , and show that for every  $\epsilon > 0$ , there exists  $x \in A$  such that  $x > \sup A - \epsilon$ .

Given any  $\epsilon > 0$  let  $x > \sup A - \epsilon$ . If  $x \notin A$  then  $x$  is an upper bound of  $A$ , i.e.,  $x \in M$  and  $x < \sup A$ , but that violates proper (ii). Hence,  $x \in A$ .

- (d) For any two functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , and any set  $A \subset \mathbb{R}$ , show that  $\sup_{x \in A} (f(x) + g(x)) \leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$ .

$$\begin{aligned} f(x) &\leq \sup_{x \in A} f(x) \text{ and } g(x) \leq \sup_{x \in A} g(x), \forall x \in x \in A \\ \therefore f(x) + g(x) &\leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x), \forall x \in A \\ \text{and } \sup_{x \in A} (f(x) + g(x)) &\leq \sup_{x \in A} f(x) + \sup_{x \in A} g(x) \end{aligned}$$

## IV. Section 1.1, Exercise 5, 6, 13.

*Exercise 1.1.5.* Let  $n \geq 1$ , and let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers verify the identity

$$\left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right), \quad (1.3)$$

and conclude *Cauchy-Schwarz inequality*

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2}$$

Let's start by expanding the center term

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \sum_{i=1}^n \sum_{j=1}^n ((a_i b_j)^2 + (a_j b_i)^2 - 2 a_i b_j a_j b_i) \\ &= \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2 + \sum_{i=1}^n b_i^2 \sum_{j=1}^n a_j^2 - 2 \sum_{i=1}^n a_i b_i \sum_{j=1}^n a_j b_j \\ &= 2 \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) - 2 \left( \sum_{i=1}^n a_i b_i \right)^2 \end{aligned}$$

Equation 1.3 then becomes

$$\begin{aligned} \left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\ \left( \sum_{i=1}^n a_i b_i \right)^2 + \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) - \left( \sum_{i=1}^n a_i b_i \right)^2 &= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \end{aligned}$$

which is true. Since

$$\begin{aligned} \left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 &= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\ \left( \sum_{i=1}^n a_i b_i \right)^2 &= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 \\ \therefore \left| \sum_{i=1}^n a_i b_i \right| &\leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2} \end{aligned}$$

Let's start by taking the square of the distance from  $a + b$  to zero using the  $\ell^2$ .

$$\begin{aligned} d_{\ell^2}(a + b, 0)^2 &= \sum_{i=1}^n (a_i + b_i)^2 \\ &= \sum_{i=1}^n (a_i^2 + b_i^2 + 2a_i b_i) \\ &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \end{aligned}$$

apply Cauchy-Schwarz and factor.

$$\begin{aligned}
 d_{\ell^2}(a+b, 0)^2 &\leq d_{\ell^2}(a, 0) + d_{\ell^2}(b, 0) + 2 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{j=1}^n b_j^2 \right)^{1/2} \\
 &\leq d_{\ell^2}(a, 0) + d_{\ell^2}(b, 0) + 2 (d_{\ell^2}(a, 0) \cdot d_{\ell^2}(b, 0))^{1/2} \\
 &\leq \left( d_{\ell^2}(a, 0)^{1/2} + d_{\ell^2}(b, 0)^{1/2} \right)^2
 \end{aligned}$$

Expand the  $\ell^2$  metrics and take the square root of both sides and

$$\left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2}$$

*Exercise 1.1.6* Show that  $(\mathbb{R}^n, d_{l^2})$  in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5)

**Example 1.1.6** (Euclidean spaces). Let  $n \geq 1$  be a natural number, and let  $\mathbb{R}^n$  be the space of  $n$ -tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}$$

We define the *Euclidean metric* (also called the  $l^2$  metric)  $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \\ &= \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \end{aligned}$$

We must prove that  $d_{l^2}$  is symmetric, positive definite and that the triangle inequality holds.

Symmetric: show that  $d_{l^2}(x, y) = d_{l^2}(y, x)$ .

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} \\ &= \left( \sum_{i=1}^n (y_i - x_i)^2 \right)^{1/2} \\ &= d_{l^2}((y_1, \dots, y_n), (x_1, \dots, x_n)) \end{aligned}$$

Positive Definite: show that  $d_{l^2}(x, y) \geq 0$  and  $d_{l^2}(x, y) = 0 \rightarrow x = y$ .

The square root is taken as a positive value.  $d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) = 0$  implies that each  $x_i - y_i = 0$  for all  $i = 1, \dots, n$  therefore  $x = y$ .

Triangle Inequality: show that  $d_{l^2}(x, z) \leq d_{l^2}(x, y) + d_{l^2}(y, z)$

Exercise 1.1.5 proves the triangle inequality replacing  $a_i = x_i$  and  $b_i = y_i$ .

*Exercise 1.1.13* Prove Proposition 1.1.19.

**Proposition 1.1.19** (Convergence in the discrete metric). *Let  $X$  be any set, and let  $d_{\text{disc}}$  be the discrete metric on  $X$ . Let  $(x^{(n)})_{n=m}^{\infty}$  be a sequence of points in  $X$ , and let  $x$  be a point in  $X$ . Then  $(x^{(n)})_{n=m}^{\infty}$  convergent to  $x$  with respect to the discrete metric  $d_{\text{disc}}$  if and only if there exists  $N \geq m$  such that  $x^{(n)} = x$  for all  $n \geq N$ .*

Remember that:

$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

( $\Rightarrow$ ) assume that  $x^{(n)} \rightarrow x$  under  $d_{\text{disc}}$ . Then, for any  $\epsilon > 0$  there exists  $n \in \mathbb{N}$  such that  $d_{\text{disc}}(x^{(n)}, x) < \epsilon$ .

Clearly,  $d_{\text{disc}}(x^{(n)}, x)$  can be equal to either 1 or 0. Thus,  $d_{\text{disc}}(x^{(n)}, x) = 0$  or  $d_{\text{disc}}(x^{(n)}, x) = 1$  and hence true for all  $n > N$ .

( $\Leftarrow$ ) assume that  $\exists N > m$  such that when  $n > N$ ,  $x^{(n)} = x$ . Given any  $\epsilon > 0$  and  $n > N$  we can see that  $d_{\text{disc}}(x^{(n)}, x) = 0 < \epsilon$ . Therefore  $x^{(n)} \rightarrow x$ .

V. For this problem only, you do not need to give proofs. Just write the answers.

For each set, identify the boundary, interior, and closure of  $A$ , and say whether  $A$  is open, closed, both or neither. We are working in  $\mathbb{R}^2$  with standard distance. Unless otherwise noted, the ambient space is  $\mathbb{R}^2$ .

- (a)  $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$ .

Boundary:  $\partial A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$

Interior:  $A^\circ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$

Closure:  $\overline{A} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1\}$

$A$  is open.

- (b)  $A = \{(1/n, 2/n) : n = 1, 2, 3, \dots\}$  (Note:  $(1/n, 2/n)$  is a vector in  $\mathbb{R}^2$ , not an open interval in  $\mathbb{R}$ .)

Boundary:  $\partial A = A$

Interior:  $A^\circ = \emptyset$

Closure:  $\overline{A} = A$

$A$  is closed.

- (c)  $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$ , in the relative topology with respect to  $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ .

Boundary:  $\partial_Y A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, d(x, 0) = 1\}$  the right semi-circle combined with the y-axis from 1 to -1.

Interior:  $A^\circ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) < 1\}$

Closure:  $\overline{A} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, d(x, 0) \leq 1\}$

$A$  is closed relative to  $Y$ .

VI. Let  $(X, d)$  be a metric space.

- (a) For a given point  $x_0 \in X$ , show the singleton set  $\{x_0\}$  is closed.

let  $E = \{x_0\}^c = X \setminus \{x_0\}$ . Given any  $x \in E$  and  $0 < \epsilon < |x_0 - x|$  we can easily see that there exists a ball  $B = B_d(x, \epsilon)$  such that  $B \cap \{x_0\} = \emptyset$ . Further,  $\partial E = \{x_0\}$  and  $\{x_0\} \notin E$  thus  $E$  is open. This implies that  $E^c = \{x_0\}$  is closed.

- (b) Let  $x_0 \in X$  and  $r > 0$ . Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open.

Let  $E = B(x_0, r)$ ,  $x \in E$  and  $0 < \epsilon_x < r - d(x_0, x)$ . Then, for any  $y \in B(x, \epsilon_x)$  we can see that  $d(x_0, y) < d(x_0, x) + \epsilon_x < r$  therefore  $y \in E$  and the open ball  $B(x, \epsilon_x) \subset E$ .

$$\bigcup_{x \in E} B(x, \epsilon_x) = E$$

then  $B(x, \epsilon_x) \subset E, \forall x \in E$

$$\bigcup_{x \in E} B(x, \epsilon_x) = E$$

thus  $B(x_0, r)$  is the union of open balls.