Math XXXX – Independent Study: Manifolds– Summer 2025
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An Introduction to Lie Algebras– August, 2025

Chapter 1

Introduction

Definition 1.0.1 (Lie Bracket)**.** We define the Lie Bracket, $[\cdot, \cdot]$ as a bilinear operation

$$[\cdot, \cdot] : L \times L \to L$$

with the following properties

$$[x, x] = 0 \tag{L1}$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{L2}$$

Definition 1.0.2 (Derivation of A)**.** Given an algebra A over a field F, a ***derivation*** $D : A \to A$ is defined by

$$D(ab) = aD(b) + D(A)b, \forall a, b, \in A$$

We denote $\operatorname{Der} A$ as the set of all derivations of A.

We definte the ***inner derivation*** of the Lie Algebra L, denoted as $\operatorname{IDer} L$ as the set of all $\operatorname{ad} x : L \to L$ which are derivations.

1.1 Exercises

1.1 (Pg 2.)

(a) Show that $[v, 0] = 0 = [0, v]$ for all $v \in L$.

$$\begin{aligned} [v, v] &= 0 \\ [v, v] - [v, 0] &= 0 - [v, 0] \\ [v - v, v - 0] &= [0, v] \\ [0, v] &= [v, 0] \end{aligned}$$

but $[0, v] = -[v, 0]$ for all v therefore $[0, v] = 0$.

(b) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent on F.
Want to show that $ax + by = 0$ implies that $a, b = 0$.

$$\begin{aligned} \text{Let } ax + by &= 0 \\ by &= -ax \implies y = cx, \text{ for some } c \\ [x, y] &= [x, cx] = c[x, x] = 0 \end{aligned}$$

but $[x, y] \neq 0$ therefore $c = 0$ and x, y are linearly independent.

1.2 (Pg 2.) Convince yourself that $\wedge$ is bilinear. Then check that the Jacobi Identity holds. *Hint:* if $x \cdot y$ denotes the dot product of $x, y \in \mathbb{R}^3$, then

$$x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z, \; \forall x, y, z \in \mathbb{R}^3.$$

***wedge* is bilinear.**
Given $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ we have

$$\begin{aligned}
x \wedge y &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \\
(x + (0, b, 0)) \wedge y &= ((x_2 + b)y_3 - (x_3 + 0)y_2, (x_3 + 0)y_1 - (x_1 + 0)y_3, (x_1 + 0)y_2 - (x_2 + b)y_1) \\
&= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) + (by_3, 0, -by_1) \\
&= x \wedge y + (0, b, 0) \wedge y
\end{aligned}$$

Therefore additive on the left for the middle coordinate. Each argument is independent of coordinate so is true for $(a, 0, 0)$ and $(0, 0, c)$ and can be easily seen when used on the write (e.g., $x \wedge (y + (0, b, 0))$.
The Jacobi Identity
Want to show

$$x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (y \wedge x) = 0 \tag{1.1}$$

from the hint

$$x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z$$

and from (1)

$$\begin{aligned}
x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (y \wedge x) &= (x \cdot z)y - (x \cdot y)z \\
&+ (y \cdot x)z - (y \cdot z)x \\
&+ (z \cdot y)x - (z \cdot x)y \\
&= (\,(x \cdot z) - (z \cdot x)\,)\,y \\
&+ (\,-(x \cdot y) + (y \cdot x)\,)\,z \\
&+ (\,-(y \cdot z) + (z \cdot y)\,)\,x \\
&= 0
\end{aligned}$$

1.3 (Pg 2.) Supoose that V is a finite-dimensional vector space over F. Write $\mathrm{gl}(V)$ for the set of all linear maps from V to V. This is again a vector space over F, and it becomes a Lie algebra, known as the *general linear algebra*, if we define the Lie bracket $[--, --]$ by

$$[x, y] := x \circ y - y \circ x, \; \forall x, y \in \mathrm{gl}(V),$$

where $\circ$ deontes the composition of maps. Check that the Jacobi Identity holds.

Given $R, S, T \in \mathrm{gl}(V)$ there exists matrix $A, B, C \in \mathcal{M}_{n \times n}(F)$ where $n = \dim V$ and $Rx = Ax$, $Sx = Bx$, $Tx = Cx$, $\forall x \in V$. Further remember that $R \circ S = AB$ (similar for the other two transormations) for all $x \in v$. Then

$$\begin{aligned}
[R, [S, T]] + [S, [T, R]] + [T, [R, S]] &= (R \circ (S \circ T - T \circ S) - (S \circ T - T \circ S) \circ R) \\
&+ (S \circ (T \circ R - R \circ T) - (T \circ R - R \circ T) \circ S) \\
&+ (T \circ (R \circ S - S \circ R) - (R \circ S - S \circ R) \circ T) \\
&= (A(BC - CB) - (BC - CB)A) \\
&+ (B(CA - AC) - (CA - AC)B) \\
&+ (C(AB - BC) - (AB - BA)C)
\end{aligned}$$

by rearranging the terms we can see that they all cancel out. Most notably this is done *without commuting*. It is important to remember that, in general, $R \circ S \neq S \circ R$.

1.4 Let $b(n, F)$ be the upper triangular matrices in $\mathrm{gl}(n, F)$. (A matrix x is said to be upper triangular if $x_{ij} = 0$ whnever $i > j$.) This is a Lie algebra with the same Lie bracket as $\mathrm{gl}(n, F)$.

Similarly, let $n(n, F)$ be the strictly upper triangular matrices in $\mathrm{gl}(n, F)$. (A matrix x i said to be strictly upper triangular if $x_{ij} = 0$ whenever $i \geq j$.) Again this is a Lie algebra with the same Lie bracket as $\mathrm{gl}(n, F)$.

Verify these assertions.

Let $b(n, F) = \{A \in \mathrm{gl}(n, F) \,|\, A = [x_{ij}],\, i > j \to x_{ij} = 0\}$. Define

$$[x, y] := x \circ y - y \circ x,\ \forall x, y \in b(n, F),$$

The only question that needs to be answered is ... Given $S, T \in (n, F)$ is $S \circ T \in b(n, F)$. Let $A, B \in \mathcal{M}_{n\times n}(F)$ and $T(x) = Ax, S(x) = Bx, \forall x \in F$. Then $(T \circ S)(x) = ABx$. Is $AB \in b(n, F)$.

$$\text{Let } A = [a_{ij}] \text{ and } B = [b_ij]$$
$$AB = \left[x_{ij} = \sum_{k=1}^{n} a_{ik} b_{k_j} \right]$$

If $i > j$ then x_{ij}

1.5 (Pg 4) Find $Z(L)$ when $L = \mathrm{sl}(2, F)$. You should find the answer depends on the characteristic of F.

Let $\mathrm{sl}(n, F)$ be the subspace of $GL(n, F)$ consisting of all matrices whose trace is zero, i.e., $\mathrm{sl}(n, F) = \left\{ A \in \mathcal{M}_{n\times n}(F) \,\middle|\, \sum_{i=1}^{n} a_{ii} = 0 \right\}$. This is known as *Special Linear Algebra* on square matrices.

When is $\sum_{i=1}^{n} a_{ii} = 0$ for all $a_{ii} \in F$? OR $a_{11} + a_{22} = 0$?.

Notice, for example, that on the discrete field $F = \mathbb{Z}/\mathbb{Z}5$, $2 + 3 = 0$. Thus, when $L = \mathrm{sl}(2, \mathbb{Z}/\mathbb{Z}p)$ where p is prime, $Z(L)$ will have elements where $a_{11} + a_{22} = p$.

1.6 (Pg 5.) Show that if $\varphi : L_1 \to L_2$ is a homormorphism, then the kernel of $\varphi, \ker\varphi$, is an ideal of L_1, and the image of $\varphi, \mathrm{im}\,\varphi$, is a Lie subalgebra of L_2.

Show that the kernel is an ideal. Let $h, k \in \ker\varphi$ such that $h \neq k$. Then $\varphi(k) = \varphi(h) = 0$.

$$\varphi(a - b) = \varphi(a) - \varphi(b) = 0$$
$$\therefore a - b \in \ker\varphi$$

which makes it a group under addition. Now we need to show that it is closed under multiplication, that is, $ra \in \ker\varphi$ for all $r \in L$. Let $r \in L$ then

$$\varphi(ra) = \varphi(r)\varphi(a) = 0$$
$$\therefore ra \in \ker\varphi$$

Show that the image is a subalgebra. We need to show three things:

Closed under addition (group condition).

Let $u, v \in \mathrm{im}\,\varphi$ then there exists $x, y \in L_1$ such that $\varphi(x) = u, \varphi(y) = v$.
Then $\varphi(x + y) = \varphi(x) + \varphi(y) = u + v \in \mathrm{im}\,\varphi$.
Therefore closed under addition.

closed under scalar multipication (ring condition).

Let $r, a \in \operatorname{im}\varphi$. Then there exists $x, y \in L_1$ such that $\varphi(x) = r, \varphi(y) = x$.
Then $\varphi(xy) = \varphi(x)\varphi(y) = ra \in \operatorname{im}\varphi$
Therefore closed under scalar multiplication.

closed under Lie bracket (subalgebra condition).

Let $u, v \in \operatorname{im}\varphi$ then there exists $x, y \in L_1$ such that $\varphi(x) = u, \varphi(y) = v$.
Then

$$\begin{aligned}
\varphi([x+y, x+y]) &= \varphi\left([x,x] + [x,y] + [y,x] + [y,y] \right) \\
&= \varphi\left([x,y] + [y,x] \right) \\
&= \varphi\left([x,y] \right) + \varphi\left([y,x] \right) \\
\varphi\left([x,y] \right) &= -\varphi\left([y,x] \right) \\
[\varphi(x+y), \varphi(x+y)] &= [\varphi(x)+\varphi(y), \varphi(x)+\varphi(y)] \\
&= [u+v, u+v] \\
&= [u,u] + [u,v] + [v,u] + [v,v] \\
&= [u,v] + [v,u] \\
[u,v] &= -\, [v,u]
\end{aligned}$$

therefore closed under Lie Bracket.

1.7 (Pg 6.) Let L be a Lie algebra. Show that the Lie bracket is associative, this is $[x, [y,z]] = [[x,y], z]$ for all $x, y, z \in L$, if and only if for all $a, b \in L$ the commutator $[a, b]$ lies in $Z(L)$.

1.8 (Pg 6) Let D and E be derivations on algebra A.

(i) Show that $[D, E] = D \circ E - E \circ D$ is also a derivation.

$$\begin{aligned}
(D \circ E)(ab) &= D\left(aE(b) - E(a)b \right) \\
&= D(aE(b)) - D(E(a)b) \\
&= aD(E(b)) - D(a)E(b) - E(a)D(b) + D(E(a))b \\
&= aD(E(b)) + D(E(a))b - D(a)E(b) - E(a)D(b)
\end{aligned}$$

We can switch D and E to computer $E \circ D$

$$(E \circ D)(ab) = aE(D(b)) + E(D(a))b - E(a)D(b) - D(a)E(b)$$

taking the difference

$$\begin{aligned}
(D \circ E)(ab) - (E \circ D)(ab) &= aD(E(b)) + D(E(a))b - D(a)E(b) - E(a)D(b) \\
&\quad - \left(aE(D(b)) + E(D(a))b - E(a)D(b) - D(a)E(b) \right)
\end{aligned}$$

$$\begin{aligned}
[D,E]\,(ab) &= a\,[D,E]\,(b) - [D,E]\,(a)b \\
&= a(D \circ E)(b) - ((D \circ E)(a))b - \left(a(E \circ D)(b) - (E \circ D)(a)b \right) \\
[D,E]\,(ab) &= (D \circ E)(ab) - (E \circ D)(ab) \\
&= D(E(ab)) - E(D(ab)) \\
&= D\left(aE(b) - E(a)b \right) - E\left(aD(b) - D(a)b \right) \\
&= D(aE(b)) - D(E(a)b) - E(aD(b)) + E(D(a)b) \\
&= aD(E(b)) - E(b)D(a) \\
&- E(a)D(b) + D(E(a))b \\
&- aE(D(b)) + E(a)D(b) \\
&+ D(a)E(b) - E(D(a))b \\
&= a(D(E(b)) - E(D(b)) - (E(b))D(a)
\end{aligned}$$

(ii) Show that $D \circ E$ need not be a derivation. (see example).

1.9 (Pg 7.) Let L_1 and L_2 be Lie algebras. Show that L_1 is isomorphic to L_2 if and only if there is a basis B_1 of L_1 and a basis B_2 of L_2 such that the structure constants of L_1 with respect to B_1 are equal to the structure constants of L_2 with respect to B_2.

($\Rightarrow$)Assuming that $L_1 \underset{\text{iso}}{\longrightarrow} L_2$. Define $f : L_1 \to L_2$ to be that isomorphism. Let $B_1 = (x_1, \ldots, x_n)$ be the basis vectors for L_1. Then,

$$\begin{aligned} f([x_i, x_j]) &= f\left(\sum_{k=1}^{n} a_{ij}^k x_k\right) \\ &= \sum_{k=1}^{n} a_{ij}^k f(x_k) \end{aligned} \tag{1.6}$$

since f is isomorphic, it is also injective and surjective. Thus, each $f(x_k)$ is unique. Further, given any $i, j \in [1, \ldots, n]$ we know that x_i, x_j are linearly independent. Thus,

$$0 = Ax_i + Bx_j \implies A = B = 0 \text{ and}$$
$$f(0) = 0 = f(Ax_i + Bx_j) = Af(x_i) + Bf(x_j)$$

therefore, $f(x_i), f(x_j)$ are linearly independent and thus, form a basis. From (1.6) we see that it has the same Structure Constants.

1.10 (Pg 7.) Let L be a Lie algebra with basis $(x_1, \ldots, x_n)$. What condition does the Jacobi identity impose on the structure constants a_{ij}^k?

We have three brackets for the Jacobi Identity that start with

$$\begin{aligned} [x_i, x_j] &= \sum_{k=1}^{n} a_{ij}^k x_k \\ [x_e, x_f] &= \sum_{k=1}^{n} a_{ef}^k x_k \\ [x_b, x_c] &= \sum_{k=1}^{n} a_{bc}^k x_k \\ [x_i, [x_e, x_f]] &= \left[x_i, \sum_{k=1}^{n} a_{ef}^k x_k\right] \\ &= \sum_{k=1}^{n} a_{ef}^k \left[x_i, x_k\right] \\ &= \sum_{k=1}^{n} a_{ef}^k \sum_{l=1}^{n} a_{ik}^l x_l \end{aligned}$$

Since, the x_i are linearly independeent we can examining each element l independently that is

$$[x_i, [x_e, x_f]]_l = \sum_{k=1}^{n} a_{ef}^k a_{ik}^l x_l$$

cycling through the other terms of the Jacobi identity we get

$$\begin{aligned} [x_e, [x_f, x_i]]_l &= \sum_{k=1}^{n} a_{fi}^k a_{ek}^l x_l \\ [x_f, [x_i, x_e]]_l &= \sum_{k=1}^{n} a_{ei}^k a_{fk}^l x_l \end{aligned}$$

The Jacobi Identity means that the sum of the coefficiints of these terms must be zero that is

$$0 = \sum_{k=1}^{n} a_{ef}^k a_{ik}^l + \sum_{k=1}^{n} a_{fi}^k a_{ek}^l g + \sum_{k=1}^{n} a_{ei}^k a_{fk}^l$$

1.11 (Pg 8.) Let L_1 and L_2 be two abelian Lie algebras. Show that L_1 and L_2 are isomorphic if and only if they have the same dimension.

If L_1 and L_2 are abelian then since $[x, y] = -[y, x]$ then $[x, y] = 0$ for all $x, y, \in L_1$ or L_2. Consequently, these are vector spaces that are isomorphic to the each other and, hence, have the same dimension.

1.12 Find the structure constants of $\mathrm{sl}(2, F)$ with respect to the basis given by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie Bracket for $\mathrm{sl}(2, F)$ is $[X, Y] = XY - YX$. Thus,

$$\begin{aligned}
[e, f] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= h \\
[f, h] &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \\
&= 2f \\
[h, e] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \\
&= -2e
\end{aligned}$$

Thus,

$$\begin{aligned}
a_{ii}^k &= 0, \forall k = 1, 2, 3 \\
[e, f] &= a_{12}^1 e + a_{12}^2 f + a_{12}^3 h = h \to a_{12}^3 = 1 \\
[f, h] &= a_{23}^1 e + a_{23}^2 f + a_{23}^3 h = 2f \to a_{23}^2 = 2 \\
[h, e] &= a_{31}^1 e + a_{31}^2 f + a_{31}^3 h = -2e \to a_{31}^1 = -2
\end{aligned}$$

all else are zero.

1.13 Prove $\mathrm{sl}(2, \mathbb{C})$ has no non-trivial ideals.

1.14 Let L by the 3-dimensional *complex* Lie algebra with basis (x, y, z) and Lie bracket defined by

$$[x, y] = z,\ [y, z] = x,\ [z, x] = y$$

(Here L is the“complexification” of the 3-dimensional real Lie algebra $\mathbb{R}^3_\wedge$.)

(i) Show that L is isomorphic to the Lie subalgebra of $\mathrm{gl}(3, \mathbb{C})$ consistent for all 3×3 antisymmetric matrices with entries in $\mathbb{C}$.

Let $U = \{A = \mathrm{gl}(3, N) : A \text{ is an anti-symmetric matrix }\}$. Thus for any $A \in U$ there exists $a, b, c \in \mathbb{C}$ such that

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

which have three linearly independent elements

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Verify

$$\begin{aligned}
[x, y] &= xy - yx \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= z
\end{aligned}$$

$$\begin{aligned}
[y, z] &= yz - zy \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= x
\end{aligned}$$

$$
\begin{aligned}
[z,x] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= y
\end{aligned}
$$

(ii) Find an explicit isomorphism $\mathrm{sl}(2,\mathbb{C}) \underset{\text{iso}}{\longrightarrow} L$.

1.15 Let S be an $n \times n$ matrix with entries in a field F. Define

$$\mathrm{gl}_S(n,F) = \{x \in \mathrm{gl}(n,F) \,:\, x^tS = -Sx\}.$$

(i) Show that $\mathrm{gl}_S(n,F)$ is a Lie subalgebra of $\mathfrak{gl}(n,F)$.

Additive Group

Let $x, y \in \mathrm{gl}_S(n,F)$, then

$$(x+y)^tS = x^tS + y^tS = -Sx - Sy = -S(x+y)$$

Multicative property.

Let $x \in \mathrm{gl}_S(n,F)$ then $x^tS = -Sx$ and $rx^tS = -Sxr$ for all $r, \in F$

Lie Bracket

Let $x, y \in \mathrm{gl}_S(n,F)$ then

$$
\begin{aligned}
[x,y] &= xy - yx \\
[x,y]^tS &= (xy-yx)^tS \\
&= (xy)^tS - (yx^t)S \\
&= y^tx^tS - x^ty^tS \\
&= -y^tSx + x^tSy \\
&= Syx - Sxy \\
&= S(yx - xy) \\
&= -S[x,y]
\end{aligned}
$$

(ii) Find $\mathrm{gl}_S(2,\mathbb{R})$ if $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let $x \in \mathrm{gl}_S(2, \mathbb{R})$ and

$$\begin{aligned}
x &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
x^t S &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \\
Sx &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix} \\
0 = x^t S + Sx &= \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + \begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & a+d \\ 0 & b \end{pmatrix} \\
x &= \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \\
&= a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

(iii) Does there exist a matrix S such that $\mathrm{gl}_S(2, \mathbb{R})$ is equal to the set of all diagonal matrices in $\mathrm{gl}(2, \mathbb{R})$.

Let $A \in \mathrm{gl}(2, \mathbb{R})$ be a diagonal matrix.

$$\begin{gathered}
\text{Let } A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\
\text{Let } S = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \\
A^t S + SA = AS + SA \to AS = -SA \\
au = -ua \text{ and } bz = -zb
\end{gathered}$$

No, no such S exists.

(iv) Find a matrix S such that $\mathrm{gl}_S(3, \mathbb{R})$ is isomorphic to the Lie algebra $\mathbb{R}^3_\wedge$ defined in §1.2, Example 1.

Hint: Part (i) of Exercise 1.14 is relevant.

Let x, y, z be a basis of $\mathbb{R}^3$. We want to find $\phi : \mathbb{R}^3 \to \mathbb{R}^3_\wedge$.

Let $X, Y \in \mathrm{gl}_S(3, \mathbb{R})$ and $\phi : \mathrm{gl}_S(3, \mathbb{R}) \to \mathbb{R}^3_\wedge$ such that

$$\begin{aligned}
\phi([X, Y]) &= [\phi(X), \phi(Y)] = \phi(X) \wedge \phi(Y) \\
\phi(XY - YX) &= \phi(X) \wedge \phi(Y)
\end{aligned}$$

Notice that

$$\begin{aligned}
(XY)^t S &= Y^t X^t S = -Y^t SX = SYX \\
\text{and } [X, Y]^t S &= (XY - YS)^t S \\
&= (XY)^t S - (YX)^t S \\
&= SYX - SXY \\
&= S(YX - XY) \\
&= -S[X, Y]
\end{aligned}$$

$$\phi(X^t S) = \phi(-SX) = -\phi(S)\phi(X)$$

1.16 Show, by giving an example, that if F is a field of characteristic 2, there are algebras over F which statisfy (L1') and (L2) but are not Lie algebras.

1.17 Let V be an n-dimensional complex vector space and let $L = \mathrm{gl}(V)$. Suppose that $x \in L$ is diagonalisable, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $\mathrm{ad}\, x \in \mathrm{gl}(L)$ is also diagonalisable and that its eigenvalues are $\lambda_i - \lambda_j$ for $1 \leq i, j \leq n$.

1.18 Let L be a Lie algebra. We saw in §1.6, Example 1.2(2) that the maps $\mathrm{ad}\, x : L \to L$ for $x \in L$ are derivations of L; these are known as *inner derivations*. Show that if $\mathrm{IDER}\, L$ is the set of inner derivations of L, then $\mathrm{IDER}\, L$ is an ideal of $\mathrm{DER}\, L$.

1.19 Let A be an algebra and let $\delta : A \to A$ be a derivation. Prove that δ satisfies the Leibniz rule

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y),\ \forall x, y \in A.$$

This resembles the binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

Consider an inductive proof and consider $\delta^0(x) = x$
Show true for $n = 1$.

$$\begin{aligned}\delta(xy) &= \binom{1}{0}\delta^0(x)\delta(y) + \binom{1}{1}\delta(x)\delta^0(y) \\ &= x\delta(y) + \delta(x)y\end{aligned}$$

which is the Liebniz rule.
Show true for $n+1$. Now, assuming that this is true for some number n, we must show that it is also true for $n+1$. Thus, starting with n we'll calculate $\delta(\delta^n(xy)) = \delta^{n+1}(xy)$.

$$\begin{aligned}\delta^n(xy) &= \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y),\ \forall x, y \in A. \\ \delta(\delta^n(xy)) &= \delta\left(\sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y)\right) \\ &= \sum_{r=0}^{n} \binom{n}{r} \delta\left(\delta^r(x)\delta^{n-r}(y)\right) \qquad (*)\end{aligned}$$

Let us focus on the term in the summation

$$\begin{aligned}\delta\left(\delta^r(x)\delta^{n-r}(y)\right) &= \delta^r(x)\delta(\delta^{n-r}(y)) + \delta(\delta^r(x))\delta^{n-r}(y) \\ &= \delta^r(x)\delta^{n-r+1}(y) + \delta^{r+1}(x)\delta^{n-r}(y).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{r=0}^{n} \binom{n}{r} \delta\left(\delta^r(x)\delta^{n-r}(y)\right) &= \sum_{r=0}^{n} \binom{n}{r} \left(\delta^r(x)\delta^{n-r+1}(y) + \delta^{r+1}(x)\delta^{n-r}(y)\right) \\ &= \sum_{r=0}^{n} \left(\binom{n}{r} + \binom{n}{r-1}\right) \delta^r(x)\delta^{n-r+1}(y)\end{aligned}$$

when $r = 0$ we have

$$\begin{aligned}r = 0 &\to x\delta^{n+1}(y) + \delta(x)\delta^n(y) \\ r = n &\to \delta^n(x)\delta(y) + \delta^{n+1}(x)y\end{aligned}$$

From combinatorics we have the identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

and we have

$$\begin{aligned}\delta^{n+1}(xy) &= x\delta^{n+1}(y) + \delta(x)\delta^{n}(y) \\ &\quad + \sum_{r=0}^{n} \binom{n+1}{r} \delta^{r}(x)\delta^{n-r+1}(y) \\ &\quad + \delta^{n}(x)\delta(y) + \delta^{n+1}(x)y \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} \delta^{r}(x)\delta^{n-r+1}(y)\end{aligned}$$

Thus, by Mathematical Induction, our assertion is true □

Chapter 2

Ideals and Homomorphisms

Operations that work on Ideals

Addition: $I + J = \{x + y \,:\, x \in I, y \in J\}$ is an ideal.

Lie Bracket: $[I, J] = \text{span}\{[x, y] \,|\, x \in I, y \in J\}$ is an ideal.

Quotient: $L/I = \{z + I \,:\, z \in L\}$ is a quotient algebra.

Notes:

Correspondence: $L \supset J \supset I$, where I, J are ideals of L. Then, J/I is an ideal of L/I.

Also, if K is an ideal of L/I and $J = \{z \in L : z + I \in K\}$ (i.e., J is the set of cosets of K in I) then J is an ideal of L and $J \supset I$.

2.1 Exercises

2.1 (Pg. 11) Show that $I + J$ is an ideal of L where

$$I + J = \{x + y \,:\, x \in I, y \in J\}.$$

Let $z \in L$ and $x, y \in I + J$ then there exists $x_I, y_I \in I$ and $x_J, y_J \in J$ such that $x = x_I + x_J$ and $y = y_I + y_J$ then from (L2) we have

$$[\underbrace{[y, x]}_{\in I+J}, z] = \underbrace{[x, [y, z]]}_{\in I} + \underbrace{[y, [z, x]]}_{\in J} \in I + J$$

2.2 (Pg. 12) Show that $\text{sl}(2, \mathbb{C})' = \text{sl}(2, \mathbb{C})$.

Let $L = \text{sl}(2, \mathbb{C})$ and $X \in [L, L]$. Then, there exist $A, B \in L$ such that $[A, B] = X$ thus

$$X = [A, B] = AB - BA$$

$AB \in L$ and $BA \in L$ therefore $X \in L$.

2.3 (Pg. 13)

(i) Show that the Lie Bracket defined in L/I is bilinear and satisfies the axioms (L1) and (L2).

Define the Lie Bracket of two cosets as

$$[w + I, z + I] = [w, z] + I, , \forall w, z \in L$$

where the bracket on the right side is the Lie Bracket defined for L. Thus, let $a, b \in L$ then we have

$$\begin{aligned}[][a+w+I, b+z+I] &= [a+w, b+z]+I \\ &= [a,b]+[a,z]+[w,b]+[w,z]+I\end{aligned}$$

the four Lie Brackets add up to a single element in L and is therefore true. Thus, this Lie Bracket is bilinear.

(ii) Show that the linear transformation $\pi : L \to L/I$ which takes an element $z \in L$ to its coset $z+I$ is a homomorphism of a Lie Algebras.

Need to show that

$$\pi([x,y]) = [\pi(x), \pi(y)]$$

I prefer to call elements of L/I equivalence classes. That is L/I is partitioned into equivalence classes (cosets) and its elements are these subsets. The proper notation for sucn and element would be $[x] \in L/I$ where x is a representative element of the equivalence class containing x. Thus $\pi(x) = [x] = \{x+I\}$.

$$\begin{aligned}\pi(x) &= [x] = \{x+I\} \\ [\pi(x), \pi(y)] &= [\,[x],[y]\,] \\ &= [\,\{\,x+I\,\}, \{\,y+I\,\}\,] \\ &= [x,y]+I \\ &= [\,[x,y]\,]\end{aligned}$$

or the equivalence class of the Lie Bracket of the left hand side.

2.4 (Pg. 14) Show that if L is a Lie Algebra then $L/Z(L)$ is isomorphic to a subalgebra of $\mathrm{gl}(L)$.

$Z(L) = \{x \in L : [x,y] = 0 \text{ for all } y \in L\}$. Therefore, $[x] \in L/Z(L) = \{y \in L : y = x+z, z \in Z(L)\}$. $Z(L)$ is an ideal. Thus, $[x] = x + Z(L)$. Let $\varphi : L/Z(L) \to \mathrm{gl}(L)$ be a homomorphism. Then $x, y \in Z(L)$ implies that $\varphi([x,y]) = \ker\varphi$. From the first isomorphism theorem, $L/\ker\varphi = L/Z(L) \cong \operatorname{Im}\varphi$.

2.5 Show that if $z \in L'$ then $\operatorname{tr}\operatorname{ad} z = 0$.

The thing to remember is that every $z \in L'$ is a linear combination of Lie Brackets. Thus

$$\begin{aligned}z &= \sum_k [x_k, y_k] \\ \operatorname{tr}\operatorname{ad} z &= \sum_k \operatorname{tr}\operatorname{ad}([x_k, y_k]) \\ \text{or each } \operatorname{tr}\operatorname{ad}([x_k,y_k]) &= 0, \forall k\end{aligned}$$

That is,

$$\begin{aligned}\operatorname{ad}([x_k,y_k]) &= \operatorname{ad} x_k \circ \operatorname{ad} y_k - \operatorname{ad} y_k \circ \operatorname{ad} x_k = 0 \\ \therefore \operatorname{tr}\operatorname{ad} z &= 0\end{aligned}$$

2.6 Suppose L_1 and L_2 are Lie algebras. let $L := \{(x_1, x_2) : x_i \in L_i\}$ be the direct sum of their underlying vector spaces, e.g., $L = L_1 \oplus L_2$. Show that if we define

$$[(x_1,x_2),(y_1,y_2)] := (\,[x_1,y_1]\,,[x_2,y_2]\,)$$

then L becomes a Lie algebra, the *direct sum* of L_1 and L_2, $L = L_1 \oplus L_2$.

(i) Prove that $\mathrm{gl}(2,\mathbb{C})$ is isomorphic to the direct sum of $\mathrm{sl}(2,\mathbb{C})\oplus\mathbb{C}$, the 1-dimensional complex abelien Lie algebra.
Let $\varphi : \mathrm{gl}(2,\mathbb{C}) \to \mathrm{sl}(2,\mathbb{C})\oplus\mathbb{C}$ be a surjective transformation. Then

$$\begin{aligned} \dim \mathrm{gl}(2,\mathbb{C}) &= \dim\ker\varphi + \dim \operatorname{range}\varphi \\ \dim\ker\varphi &= \dim \mathrm{gl}(2,\mathbb{C}) - \dim(\mathrm{sl}(2,\mathbb{C})\oplus\mathbb{C}) \\ &= n^2 - n^2 = 0 \end{aligned}$$

The dimension of the kernel of φ is 0 therefore φ is a bijection implying an isomorphsim.

(ii) Show that if $L = L_1 \oplus L_2$ then $Z(L) = Z(L_1)\oplus Z(L_2)$ and $L' = L_1' \oplus L_2'$. Formulate a general version for a direct sum $L_1 \oplus \cdots \oplus L_k$.
1: Show $Z(L) = Z(L_1)\oplus Z(L_2)$.
For any $u \in L$ there exists $u_1 \in L_1$ and $u_2 \in L_2$ such that $u = (u_1, u_2)$. If $z \in Z(L)$ then $[z, u] = 0$.

$$\begin{aligned} [z,u] &= [(z_1,z_2),(u_1,u_2)] \\ &= (\,[z_1,u_1],[z_2,u_2]\,) \\ \therefore [z_1,u_1] &= 0 \text{ and } [z_2,u_2] = 0 \end{aligned}$$

for any u. Thus, $z_1 \in Z(L_1)$ and $z_2 \in Z(L_2)$.
2: Show $L' = L_1' \oplus L_2'$.
Let $z \in L$ then there exists a linear combination of commutators $[x_k, y_k]$ equal to zero

$$z = \sum_k [x_k, y_k]$$

There exist $a_k, b_k \in L_1$ and $c_k, d_k \in L_2$ such that $x_k = (a_k, c_k)$ and $y_k = (b_k, d_k)$. then

$$\begin{aligned} z &= \sum_k [(a_k,c_k),(b_k,d_k)] \\ &= \sum_k ([a_k,b_k],[c_k,d_k]) \\ &= \left(\sum_k [a_k,b_k], \sum_k [c_k,d_k] \right) \\ &\in L_1 \oplus L_2 \end{aligned}$$

Thus

$$L = \bigoplus_k L_k \implies Z(L) = \bigoplus_k Z(L_k) \text{ and } L' = \bigoplus_k L_k'$$

(iii) Are the summands in the direct sum decomposition of a Lie Algebra uniquely determined? *Hint:* If you think that the answer is yes, now might be a good time to read §16.4 in Appendix A on the "diagonal fallacy". The next question looks at this point in more detail.

2.7 Suppose $L = L_1 \oplus L_2$ is the direct sum of two Lie algebras.

(i) Show that $\{(x_1, 0) : x_1 \in L_1\}$ is an ideal of L isomorphic to L_1 and that $\{(0, x_2) : x_2 \in L_2\}$ is an ideal of L isomorphic to L_2. Show that the projections $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ are Lie algebra homomorphisms.
Show the L_1 isomorphism.
Let $u = (u_1, u_2) \in L$. Then $N_1 = \{(x_1, 0) : x_1 \in L_1\}$ and $x = (x_1, x_2) \in N_1$ then $[u, x] = [(u_1, u_2), (x_1, 0)] = ([u_1, x_1], [u_2, 0]) = ([u_1, x_1], 0) \in N_1$ and therefore an ideal. Also, allow

$\varphi : N_1 \to L_1$. Let $a, b \in \ker \varphi$. Then $\varphi(a+b) = \varphi(a) + \varphi(b) = (0,0)$ implies that $a_1 = b_1$ or $a = b$. Thus, φ is an isomorphism.
A similar argument for the L_2 isomorphism.
Proejctions:
Given any $x, y \in L$

$$\begin{aligned} p_1([x,y]) &= p_1([x_1,y_1],[x_2,y_2]) \\ &= [x_1,y_1] \end{aligned}$$

thus $p_1([x,y]) \in L_1$. A similar argument for L_2.
Now suppose that L_1 and L_2 do not have any non-trivial proper ideals.

(ii) Let J be a proper ideal of L. Show that $J \cap L_1 = 0$ and $J \cap L_2 = 0$, then the projection $p_1 : J \to L_1$ and $p_2 : J \to L_2$ are isomorphisms.

(iii) Deduce that if L_1 and L_2 are not isomorphic as Lie algebras, then $L_1 \oplus L_2$ has only two non-trivial proper ideals.

(iv) Assume that the ground field is infintie. Show that if $L_1 \cong L_2$ and L_1 is 1-dimensional, then $L_1 \oplus L_2$ has infinitely many different ideals.

2.8 Let L_1 and L_2 be Lie algebras, and let $\varphi : L_1 \to L_2$ be a surjective Lie algebra homomorphism. True or False:

(a) $\varphi(L_1') = L_2'$;
Let $x, y \in L_1$ then $[x,y] \in L_1'$. Then $\varphi([x,y]) = [\varphi(x), \varphi(y)] \in L_2'$. Therefore $\varphi(L_1') \subseteq L_2'$. Then, we know that given any $u, v \in L_2$ there is $[u,v] \in L_2'$ and there exist $x, y \in L_1$ such that $\varphi(x) = u, \varphi(y) = v$. Thus, $[u,v] = [\varphi(x), \varphi(y)] = \varphi([x,y]$ and $L_2' \subseteq \varphi(L_1')$.
TRUE

(b) $\varphi(Z(L_1)) = Z(L_2)$;
Let $u \in Z(L_1)$. Then for any $x \in L_1$ we have $\varphi(0) = 0$ or $\varphi(u) = 0$ therefore $\varphi(Z(L_1)) \subseteq Z(L_2)$
Given any $v \in Z(L_2)$ then any $y \in L_2$ implies that $[v,y] = 0$ and there exist $w, z \in L_1$ such that $\varphi(w) = v$ and $\varphi(z) = y$. Then $[v,y] = [\varphi(w), \varphi(z)] = \varphi([w,z]) \in Z(L_1)$ because $w \in Z(L_1)$. Therefore, $Z(L_2) \subseteq \varphi(Z(L_1))$.

(c) $h \in L_1$ and ad_h is diagonalisable then $\mathrm{ad}_{\varphi(h)}$ is diagonalisable.
Notice that

$$\begin{aligned} \mathrm{ad}_h(x) &= [h,x] \\ \varphi(\mathrm{ad}_h(x)) &= \varphi([h,x]) \\ &= [\varphi(h), \varphi(x)] \\ &= \mathrm{ad}_{\varphi(h)}(\varphi(x)) \end{aligned}$$

2.9 For each pair of the following Lie algebras over $\mathbb{R}$, decide whether or not they are isomorphic:

(i) the Lie algebra $R^3_\wedge$ where the Lie bracket is given by the vector product;
In other words, compare the wedge with the cross product. Define a Lie Algebra $\mathbb{R}^3_\times = \mathbb{R}^3$ with $[x,y] = x \times y$ for all $x, y \in \mathbb{R}^3_\times$. Let $\varphi : \mathbb{R}^3_\vee \to \mathbb{R}^3_\times$. Then, $u, v \in \mathbb{R}^3_\vee$ and $x, y \in \mathbb{R}^3_\times$ such that $\varphi(u) = x, \varphi(v) = y$.

$$\varphi([u,v]) = [\varphi(u), \varphi(v)] = [x,y]$$
$$\varphi(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - v_1u_2) = (\, x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2\,)$$

which is true if φ is the identity

(ii) the upper triangular 2×2 matices over $\mathbb{R}$;

(iii) the strict upper triangular 3×3 matrices over $\mathbb{R}$;

(iv) $L = \{x \in \mathrm{gl}(3, \mathbb{R}) : x^t = -x\}$.

Hint: Use Exercises 1.15 and 2.8.

2.10 Let F be a field. Show that the derived algebra of $\mathrm{gl}(n, F)$ is $\mathrm{sl}(n, F)$

Show that $\mathrm{gl}(n, F)' \subseteq \mathrm{sl}(n, F)$ Let $u \in \mathrm{gl}(n, F)'$ then there exist $x, y \in \mathrm{gl}(n, F)$ such that $u = [x, y]$

$$\begin{aligned} \operatorname{tr} u &= \operatorname{tr}[x, y] \\ &= \operatorname{tr}(xy - yx) \\ &= \operatorname{tr}(xy) - \operatorname{tr}(yx) \\ &= 0 \end{aligned}$$
$$\therefore u \in \mathrm{sl}(n, F)$$

Show that $\mathrm{gl}(n, F)' \supseteq \mathrm{sl}(n, F)$ Let $u \in \mathrm{sl}(n, F)$ then $\operatorname{tr} u = 0$

2.11 In Exercise 1.15, we defined the Lie Algebra $\mathrm{gl}_S(n, F)$ over a field F where S is an $n \times n$ matrix with entries in F.

Suppose that $T \in \mathrm{gl}(n, F)$ is another $n \times n$ matrix such that $T = P^t S P$ for some invertible $n \times n$ matrix $P \in \mathrm{gl}(n, F)$ (Equivalently, the bilinear forms defined by S and T are congruent.) Show that the Lie algebras $\mathrm{gl}_S(n, F)$ and $\mathrm{gl}_T(n, F)$ are isomorphic.

2.12 Let S be an $n \times n$ invertible matrix with entries in $\mathbb{C}$. Show that if $x \in \mathrm{gl}_S(n, \mathbb{C})$, then $\operatorname{tr} x = 0$

2.13 Let I be an ideal of a Lie Algebra L. Let B be the centraliser of I in L; that is

$$B = C_L(I) = \{x \in L : [x, a] = 0, \ \forall a \in I\}$$

Show that B is an ideal of L. Now suppose that

(a) $Z(I) = 0$, and

(b) if $D : I \to I$ is a derivation, then $D = \operatorname{ad} x$ for some $x \in I$.

Show that $L = I \oplus B$.

(c) Recall that if L is Lie algebra, we defined L' to be the subspace spanned by the commutators $[x, y]$ for $x, y \in L$. The purpose of this execise, which may safely be skipped on first reading, is to show that the *set* of commutators may not even be a vector space (and so certainly not an ideal of L.).

Let $\mathbb{R}[x, y]$ denote the ring of all real polynomials in two variables. Let L be the set of all matrices of the form

$$A\,(\,f(x), g(y), h(x, y)\,) = \begin{pmatrix} 0 & f(x) & h(x, y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Prove L is a Lie algebra with usual commutator bracket. (In contrast to all the Lie algebras seen so fro, L is infinite-dimensional.)

(ii) Prove that

$$[\,A\,(\,f_1(x), g_1(y), h_1(x, y)\,)\,, A\,(\,(f_2(x), g_2(y), h_2(x, y)\,)\,] = A(0, 0, f_1(x)g_2(x) - f_2(x)g_1(y)).$$

Hence describe L'.

(iii) Show that if $h(x, y) = s^2 + xy + y^2$, then $A(0, m0, h(x, y))$ is not a commutator.

Chapter 3

Low Dimensional Lie Aglebras

Dimension 1

Given a single basis vector e then by definition the Lie bracket must be $[e, e] = 0$ making the entire 1-dimensional vector space Abelian.

$$\mathfrak{g} \cong \mathbb{R}$$

Dimension 2

Given a basis e_1, e_2 .

1. **Abelian**
   The Lie Breacket must be $[e_1, e_2] = 0$ for all elements of the vector space (plane).
2. **Non-Abelian (solvable)**
   The Lie Bracket must be $[e_1, e_2] = e_2$. This algebra is solvable but not nilpotent (i.e., $A^n = 0$ for some n).

Dimension 3

Given a basis e_1, e_2, e_3.

1. **Abelian**
   All Lie brackets are zero.
2. **Heisenberg Algebra**
   The Lie bracket is $[e_1, e_2] = e_3$. This becomes nilpotent.
3. **Solvable (non-nilpotent)**

   Type 1:

   $$[e_1, e_2] = e_3 \text{ and } [e_1, e_3] = e_2$$

   Type 2:

   $$[e_1, e_2] = -e_3 \text{ and } [e_1, e_3] = e_2$$

4. **Simple Lie Algebras**
   (a) $\mathfrak{so}(3)$

   $$[e_1, e_2] = e_3,\ [e_2, e_3] = e_1 \text{ and } [e_3, e_1] = e_2$$

   (b) $\mathfrak{sl}(2, \mathbb{R})$
   Consider the basis h, e, f

   $$[h, e] = 2e,\ [h, f] = -2f \text{ and } [e, f] = h$$

3.1 Exercises

3.1. Let V be a vector space and let φ be an endomorpism of V. Let L have underlying vector space $V \oplus \operatorname{span}\{x\}$. Show that if we define the Lie bracket on L by $[y, z] = 0$ and $[x, y] = \varphi(y)$ for $y, z \in V$, then L is a Lie algebra and $\dim L' = \operatorname{rank}\varphi$. (For a more general construction, see Exercise 3.9 below.)

If $u_1, u_2 \in L$ then there exist $y_1, y_2 \in V$ and $x_1, x_1 \in \operatorname{Span}(x)$ such that $u_1 = (y_1, x_1)$ and $u_2 = (y_2, x_2)$ then

$$\begin{aligned}
[u_1, u_2] &= [(y_1, x_1), (y_2, x_2)] \\
&= [y_1, x_1] + [y_1, x_2] + [y_2, x_1] + [y_2, x_2] \\
&= 2\varphi(y_1) + 2\varphi(y_2)
\end{aligned}$$

The only elements of L that are non-zero are the ones indicated by x and hence φ. Therefore, $\dim L' = \operatorname{rank}\varphi$, i.e., the number of linearly independent rows of its matrix of transformation.

3.2. With the notation of §3.2.3, show that the Lie algebra L_μ is isomorphic to L_ν if and only if either $\mu = \nu$ or $\mu = \nu^{-1}$.

From 3.1, let $V_\mu, V_/=$ be spanned by $[y_\mu, z_\mu], [y_\nu, z_\nu]$ and $x_\mu, x\nu$ be the such that $L_\mu = V_\mu \oplus \operatorname{span}\{x\}, L_\nu = V_\nu \oplus \operatorname{span}\{x\}$. We can see that the Lie bracket for each L_μ is $[x_\mu, y_\nu] = y$ (from the text) and $[y_m u, z_\mu] = 0$, from the lemma. This Lie Bracket is precisely the Lie bracket from 3.1 for both μ, ν. There rank must each be 2, therefore an isomorphisms exists $\varphi : L_\mu \to L_\nu$.

Let $\varphi : L_\mu \to L_\nu$ be this isomorphism. Thus, given any $u \in L_\mu$, $v \in L_\nu$

$$\begin{aligned}
\varphi(u) = A_\mu u &\text{ and } \varphi^{-1}(v) = A^{-1}v \\
A_\mu &= \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \\
I &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \\
&= \begin{pmatrix} a & b\mu \\ c & d\mu \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & d\mu \end{pmatrix} \\
\therefore d = \mu^{-1}
\end{aligned}$$

A^{-1} is the $\operatorname{ad} x_\nu$.

3.3. Find out where each of the following 3-dimensional complex Lie algebras appears in our classification:

(i) $\mathrm{gl}_S(3, \mathbb{C})$, where $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;

Remember that

$$\mathrm{gl}_S(n, F) = \{u \in \mathrm{gl}(n, F) \, : \, u^t S = -Su\}.$$

Let's determine a basis for $\mathrm{gl}_S(3,\mathbb{C})$. Let $u \in \mathrm{gl}_S(3,\mathbb{C})$ then

$$u^tS = -Su$$
$$\text{Let } u = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$
$$-Su = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$
$$= \begin{pmatrix} -a & -b & -c \\ d & e & f \\ g & h & k \end{pmatrix}$$
$$u^tS = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} a & d & -g \\ b & e & -h \\ c & f & -k \end{pmatrix}$$

Which leads us to the following conclusions:

$$a = k = 0$$
$$-b = d \text{ and } b = d \to b = d = 0$$
$$g = c$$
$$e = e$$
$$f = -h \text{ and } f = h \to f = h = 0$$

Thus,

$$u = \begin{pmatrix} 0 & 0 & c \\ 0 & e & 0 \\ c & 0 & 0 \end{pmatrix}$$
$$= c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{gl}_S(3,\mathbb{C})$ is 2 dimensional. This is not Abelian therefore it is non-abelian solvable.

(ii) the Lie subalgebra of $\mathrm{gl}(3,\mathbb{C})$ spanned by the matrices

$$\mu = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where λ, μ, ν are fixed complex numbers;

$$\mu v = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \lambda v$$
$$vw = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$
$$\mu w = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu \\ 0 & 0 & 0 \end{pmatrix} = \nu w$$

(iii) $\left\{ \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$;

(iv) $\left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$.

3.4. Suppose L is a vector space with basis x, y and that a bilinear operation $[-,-]$ on L is defined such that $[u, u] = 0$ for aqll $u \in L$. Show that the Jacobi Identity holds and hence L is a Lie algebra.

3.5. Show that over $\mathbb{R}$ the Lie algebras $\mathrm{sl}(2, \mathbb{R})$ and $\mathbb{R}^3_\wedge$ are not isomorphic. *Hint*: Prove tht there is no non-zero $x \in \mathbb{R}^3_\wedge$ such that the map $\operatorname{ad} x$ is diagonalisable.

3.6. Show that over $\mathbb{R}$ there are exactly two non-isomophic 3-dimensional Lie algrebras with $L' = L$.

3.7. Let L be a non-abelian Lie algebra. Show that $\dim Z(L) \leq \dim L - 2$.

3.8. let L be the 3-dimensional Heisenberg Lie algebra defined over field F. Show that $\operatorname{Der} L$ is 6-dimensional. Identify the inner derivations (as defined in Exercise 1.18) and show that he quotient $\operatorname{Der} L / \operatorname{IDer} L$ is isomorphic to $\mathrm{gl}(2, F)$.

3.9. Suppose that I is an ideal of a Lie algebra L and that there is a subalgebra S of L such that $L = S \oplus I$.

(i) Show that the map $\theta : S \to \mathrm{gl}(I)$ defined by $\theta(s)x = [s, x]$ is a Lie algebra homomorphism from S into $\operatorname{Der} I$.

We say that L is *semidirect product* of I by S. (the reader may have seen the analogous construction for groups.)

(ii) Show conversely that given Lie aglebras S and I and a Lie algebra homomorphism $\theta : S \to \operatorname{Der} I$, the vector space $S \oplus I$ may be made into a Lie algebra by defining

$$[(s_1, x_1), (s_2, x_2)] = ([s_1, s_2], [x_1, x_2] + \theta(s_1)x_2 - \theta(x_2)x_1)$$

for $s_1, s_2 \in S$, and $x_1, x_2 \in I$, and that this Lie algebra is a semidirect product of I by S. (The direct sum construction introduced in Exercise 2.6 is the special case where $\theta(s) = 0$ for all $s \in S$.)

(iii) Show that the Lie algebras in Exercise 3.1 may be constructed as semidirect products.

(iv) Investigate necessary and sufficient condtiions for two semidirect products to be isomorphic.

3.10. Find, up to isomorphism, all Lie algebras with a 1-dimensional derived algebra.

Chapter 4

Solvable Lie Algebras

Remark 4.0.1 (**MathGPT: Describe the difference between an Ideal and a derived Lie algebra.**). .
Solution:

Step 1: **Define an ideal of a Lie Algebra.**

An ideal of a Lie algebra is a subalgebra that interacts with the Lie bracket in a specific way. For a Lie algebra $\mathfrak{g}$ over a field F, a subsapce $\mathfrak{i} \subset \mathfrak{g}$ is an ideal if and only $x \in \mathfrak{g}$ and $y \in \mathfrak{i}$, the Lie bracket $[x, y]$ is in i. This condition is expressed as $[\mathfrak{g}, \mathfrak{i}]$. Ideals are analogous to normal subgroups in group theory and are essential for construcitng quotient Lie algebras.

Step 2: **Define a Derived Lie Algebra**

The derived Lie algebra, also known as the commutator subalgebra, of a Lie algebra $\mathfrak{g}$ is denoted $\mathfrak{g}'$ or $[\mathfrak{g}, \mathfrak{g}]$. It is the subspace spanned by all Lie brackets of elements in $\mathfrak{g}$, as shown using the Jacobi identity.

Step 3: **Difference in Generality.**

An ideal is a type of subspace with a specific property related to the Lie bracket, while the derived Lie algebra is a specific construction that results in an ideal. Every derived Lie algebra is an ideal, but not every ideal is a derived algebra.

Step 4: An ideal $\mathfrak{i}$ is defined by the property $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ and can be any subspace satisfying this condition. The derived Lie algebra $\mathfrak{g}'$ is defined as the set of all possible Lie brackets of elements within $\mathfrak{g}$ and is uniquely determined by $\mathfrak{g}$.

Step 5: **Difference in Purpose/Role**

Ideals are used to define quotient Lie algebras and help understand the structure of Lie algebras, similar to normal subgroups in group theory. The derived Lie algebra $\mathfrak{g}'$ measures how "non-abelian" a Lie algebra is. If $\mathfrak{g}' = \{0\}$, then $\mathfrak{g}$ is abelian. The derrived series of all Lie algebra is used to defien solvable and nilpotent Lie algebras.

Step 6: **Examples**

Consider the Lie algebra $\mathfrak{gl}(n, F)$ of all $n \times n$ matrices over a field F, with the Lie bracket $[A, B] = AB - BA$.

- **Ideal:** The set of scalar matrices, $\mathfrak{s} = \{cI \mid c \in F\}$, forms an ideal of $\mathfrak{gl}(n, F)$. This ideal is not the derived Lie algebras.
- **Derived Lie Algebra:** the derived Lie algebra $\mathfrak{gl}(n, F)$ is $\mathfrak{sl}(n, F)$, the Lie algebra of $n \times n$ matrices with trace zero. This is because the trace of a commutator is always zero, and any matricx with trace zero can be written as a commutator. $\mathfrak{sl}(n, F)$ is an ideal of $\mathfrak{gl}(n, F)$.

Final Answer In summary, while the derived Lie algebra is always an ideal, an ideal is a more general concept. The derived Lie algebra is a specific ideal that captures the "commutativity" properties of the Lie algebra itself.

Remark 4.0.2. **Simple Algebras:** Special Linear, Special Orthogonal and Symplectic Lie Algebras

Special Linear, $\mathfrak{sl}(n, \mathbb{C})$

Description:

Teh Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ consists of all $n \times n$ complex matrices with trace zero, $\operatorname{tr} A = 0$. The Lie bracket is defined by the commutator $[A, B] = AB - BA$.

Dimensions:

The dimensions of $\mathfrak{sl}(n, \mathbb{C})$ is $n^2 - 1$, as the trace condition imposes one linear constraint on the n^2 entries of a matrix.

Importance:

- If $L = \mathfrak{gl}(n, \mathbb{C})$ then $L' = \mathfrak{sl}(n, \mathbb{C})$.
- Fundamental Representation: It is the Lie algebra of the specal linear group $SL(n, \mathbb{C})$ which acts on $\mathbb{C}^n$.
- Simple Lie Algebra: For $n \geq 2$, it is a simple Lie algebar, correspoidng ot the A-series of the Carten classification.
- Physics: It appears in quantum mechanics and quantum field theory, describing symmetries.

The Special Orthogonal Lie Algebra, $\mathfrak{so}(n, \mathbb{C})$

Description:

The Lie Algebra $\mathfrak{so}(n, \mathbb{C})$ consists of $n \times n$ complex skew-symmetric matrices. The Lie bracket is the commutator $[A, B] = AB - BA$.

Dimension:

The dimesion is $\frac{n(n-1)}{2}$, as skew-symmetric matrices have z ero diagonal entries and $A_{aj} = -A_{ji}$ for $i \neq j$.

Impotrance:

- Orthogonal Group: It si the Lie algebra of the special orthogoanl gorup $SO(n, \mathbb{C})$, preserving a symmetric bilinear form.
- Simple Lie Algebra: For $n \geq 3$ and $n \neq 4$, it is simple, corresponging to the B-series and D-series in the Cartan classification.
- Geometry and Physics: It is crucial in geometery and physics, related to rotattions and Lorentz transformations.

The Symplectic Lie Algebra, $\mathfrak{sp}(2n, \mathbb{C})$**.**

Description:

The Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ consists $2n \times 2n$ complex matrices A staisfying $A^T J + JA = 0$ where J is a standard skew-symmetric matrix.

Dimension:

The dimension is $n(2n + 1)$, derived fromt the conditiosn on the matrix blocks P, Q, R, S.

Importance:

- Symplectic Group: It is the Lie algebra of the symplectic group $Sp(2n, \mathbb{C})$, preserving skew-symmetric bilinear form.
- Simple Lie Algebra: For $n \geq 1$, it is simple, corresponding to the C-series in the Cartan classification.
- Hamiltonian Mechanics and Quantum Mechancis: It is fundamental in Hamiltonian mechanics and quantum mechancis, realted to symplectic manifolds and canonical transformations.

4.1 Exercises

4.1 Suppose that $\varphi : L_1 \to L_2$ is a surjective homomorphism of Lie algebras. Show that

$$\varphi\left(L_1^{(k)}\right) = (L_2)^{(k)}.$$

A proof by induction. The initial case

$$\begin{aligned}\varphi(L_1') &= \varphi([L_1, L_1]) \\ &= [\varphi(L_1), \varphi(L_1)]\end{aligned}$$

Since φ is surjective $\varphi(L_1) = L_2$ hence

$$\varphi(L_1') = [L_2, L_2] = L_2'$$

The inductive case.

$$\begin{aligned}\text{assume } \varphi(L_1^{(k)}) &= (L_2)^{(k)} \\ \varphi(L_1^{(k+1)}) &= \varphi([L_1^{(k)}, L_1^{(k)}]) \\ &= \left[\varphi(L_1^{(k)}), \varphi(L_1^{(k)})\right] \\ &= [(L_2)^{(k)}, (L_2)^{(k)}] \\ &= (L_2)^{(k+1)}\end{aligned}$$

4.2 Let $x \in \mathfrak{gl}(2\ell, \mathbb{C})$. Show that x belongs to $\mathfrak{sp}(2\ell, \mathbb{C})$ if and only if it is of the form

$$x = \begin{pmatrix} m & p \\ q & -m \end{pmatrix}$$

where p and q are symmetric. Hence find the dimension of $\mathfrak{sp}(2\ell, \mathbb{C})$. (See Exercise 12.1 for the other families)

4.3 Use lemma 4.4 to show that if L is a Lie algebra then L is solvable if and only if ad L is a solvable subalgebra of gl(L). Show that this result also holds if we replace "solvable" with "nilpotent".

4.4 Let $L = \mathfrak{n}(n, F)$, the Lie algebra of strictly upper triangular $n \times n$ matrices over a field F. Show that L^k has a basis consisting of all the matrix units e_{ij} with $j - i > k$. Hence show that L is nilpotent. What is the smallest m such that $L^m = 0$?

Let $L \in \mathfrak{n}(n, F)$.

$$L^1 = L' = [L, L]$$

given any $A, B \in L$, $[A, B] = AB - BA$. Given any $i, j < n$, first we can see that $A_{ij} = 0$ whenever $i \geq j$ (the diagonal and lower triangle of the matrix). Further,

$$\begin{aligned} AB &= \left[\sum_{k=1}^{n} A_{ik}B_{kj}\right] \\ &= \left[\sum_{k=j}^{n} A_{ik}B_{kj}\right] \\ \text{or } i < j+1 &\implies AB_{ij} = BA_{ij} = 0 \end{aligned}$$

that is, the elements along the diagonal one row up (or one column to the right) are all zero. Similarly, let $C = AB$ then for some $D \in L$ we get a similar argument, that is $i < j + 2$ implies that $DC_{ij} = CD_{ij} = 0$. $C \in [L, L]$ and $D \in [L, L'] = L^2$, this process can be repeated as long as $j - i < k$, that is $k = n - 1$. When $k = n - 1$ we have a zero indicating that L is nilpotent at $m = n - 1$

4.5 Let $L = \mathfrak{b}(n, F)$ be the Lie algebra of upper triangular $n \times n$ matrices over a field F.

(i) Show that $L' = \mathfrak{n}(n, F)$.
Let $A, B \in \mathfrak{b}(n, F)$. Looking only at the elements on the diagonal, we can see that,

$$\begin{aligned} (AB)_{ii} = \sum_{k=1}^{n} A_{ik}B_{ki} = \sum_{k=1}^{n} B_{ik}A_{ki} = (BA)_{ii} \\ \therefore (AB)_{ii} - (BA)_{ii} = 0 \implies [A, B] \in \mathfrak{n}(n, F) \end{aligned}$$

(ii) More generally, show that $L^{(k)}$ has a basis consisting of all the matrix units e_{ij} with $j - i \geq 2^{k-1}$. (the commutator formula for the e_{ij} given in §1.2 will be helpful.)

(iii) Hence show that L is solvable. What is the smallest m such that $L^{(m)} = 0$?

(iv) Show that if $n \geq 2$ then L is not nilpotent.

4.6 Show that a Lie algebra is semisimple if and only if it has no non-zero abelian ideals. (this was the original definition of semisimplicity given Wilhelm Killing.)

4.7 Prove directly that $\mathfrak{sl}(n, \mathbb{C})$ is a simple Lie algebra $n \geq 2$.

4.8 Let L be a Lie algebra over a field F such that $[[a, b], b] = 0$ for all $a, b \in L$, (or equivalently, $(\operatorname{ad} b)^2 = 0$ for all $b \in L$).

(i) Suppose the characteristic of F is not 3. Show that then $L^4 = 0$. *Hint:* Show first that the Lie brackets $[[x, y], z]$ are alterenating; that is,

$$[[x, y], z] = -[[y, x], z], \; [[x, y], z] = -[[x, z], y]$$

for all $x, y, z \in L$.

4.9 The purpose of this exercise is to give some idea why the families of Lie algebra are given the names that we have used. We shall not need to refer to this exercise later; some basic group theory is needed . . .

4.10 Let F be a field. Exercise 2.11 shows that if $S, T \in \mathrm{gl}(n, F)$ are congruent matrices (that is, there exists an invertible matrix P such that $T = P^t SP$), then $\mathrm{gl}_S(n, F) \cong \mathrm{gl}_T(n, F)$. Does the converse hold when $F = \mathbb{C}$? For a challenge, think about other fields.