Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Definitions

1. ***Diffeomorphism***: If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then f is said to be a ***diffeomorphism***. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be ***diffeomorphic*** to each other.
2. ***Tangent Space*** at a point p. The set of all vectors rooted at p, written as $T_p(\mathbb{R}^n)$.
   Let $p = (x^1, \ldots, x^n)$. The directional deriviative for each component would be described as
   $$\left.\frac{\partial}{\partial x^1}\right|_p$$
   $$\text{notice } \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta^i_j, \forall p.$$
   that is perpendicular and form a orthogonal basis. Thus, a Tangent Vector is also called a "Derivation".
3. ***Derivations***: any operation that supports the Liebniz Rule $D(fg) = (Df)g + fDg$.
4. ***Derivation Space***. $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p. This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ defined as
   $$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p. \end{aligned}$$
5. ***Germ***: equivalence class of functions whose derivatives around a point are the same.
6. Vector Field vs Vector Space.
   - ***A Vector Field*** a function that assigns a vector to every point in the subset U.
     $$\begin{aligned} f : (U \subset \mathbb{R}^m) &\to T_p(\mathbb{R}^n) \\ X &\mapsto X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p. \end{aligned}$$
     consider a^i as coefficent functions. We say that X is C^∞ on U if $a^i \in C^\infty$, $\forall i = 1, \ldots, n$.
   - ***A Vector Space*** is any abstraciton that is closed under addition and scalar multiplication.
7. ***Dual Basis and Dual Space***. The ***Dual Basis*** is a set of functions $\alpha^i : V \to \mathbb{R}$
   $$\begin{aligned} \alpha^i &: V \to \mathbb{R} \\ \alpha^i(e_j) &= \delta^i_j \end{aligned}$$
   the ***Dual Space*** $V^\vee$ is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called ***Functionals (Analysis)/1-Covectors (Differential Geometry)***.
8. ***Multi-Linear Functions and Vector Space of k-tensors*** $L_k(V)$ Let V be a vector space and V^k be k-tuples of vectors in V. A ***K-linear map or k-tensor*** $f : V^k \to \mathbb{R}$ such that each i^{th} component is linear. The vector space of all k-tensors on V is denoted $L_k(V)$.
   Permuting Mult-linear Functions. Given any permutation $\sigma \in S_k$
   $$f(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$
   e.g., $f(x, y, z) = xyz \to f(z, x, y) = zxy$. FYI: if x, y, z are from non-commutative rings (i.e., matrices) then we must be aware of the $\text{sgn}(\sigma)$.
9. Left R-***Module***: An Abelian group R with a scalar multiplication map:
   $$\mu : R \times A \to A$$
   usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a
   (i) (associative) $(rs)a = r(sa)$.

(ii) (identity) $1a = a$ (1 is a multiplicative identity).
(iii) (distributivity) $(r+s)a = ra + sa$ and $r(a+b) = ra + rb$.

If R is a field then R-module is precisely a vector space over R.

A ***K-Algebra over a field*** K is also a ring A that is also a vector space over K such that the ring multiplication satisfies homogeniety (scalar distributes over vector multipliation to only one of the operators).

A ***graded Algebra*** is an algebra A over a field K if it can be writte as the direct sum

$$A = \bigoplus_{i=0}^{\infty} A^i$$

of vector spaces over K such that the mupltiplication map sends $A^k \times A^l \to A^{k+l}$

10. The set of all C^∞-vector fields on U, denoted by $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the $C^\infty(U)$ ring.

$$\mathfrak{X}(U) = \{\, X : V \to V \mid X \in C^\infty(U) \,\} \text{ where } V = (\mathbb{R} \text{ or } \mathbb{C})^n$$

11. ***Derivation:*** A ***derivation*** on an algebra A is a K-multilinear function $D : A \to A$ such that

$$D(ab) = (Da)b + aDb,\ \forall a, b \in A$$

known as the ***Liebniz Rule***.

The set of all derivations on A forms a vector space, $\mathrm{Der}(C^\infty(U))$. Thus a $C^\infty(U)$ vector field gives rise to a derivation of the algebra $C^\infty(U)$. Thus the mapping

$$\varphi : \mathfrak{X}(U) \to \mathrm{Der}(C^\infty(U))$$
$$X \mapsto (f \mapsto Xf)$$

this map is an isomorphism of vector spaces.

12. **Exterior Algebras $\Lambda(V)$.** The exterior algebra $\Lambda(V)$ is obtained by imposing an **anti-commutative** relation:

$$v \otimes w + w \otimes v = 0,\ \forall v, w \in V$$

this means that the quotient algebra is:

$$\Lambda(V) = T(V) / \langle\, v \otimes w + w \otimes v \,\rangle.$$

Where $T(V)$ is the **tensor algebra**

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

13. **Symmetric Algebras $S(V)$.** The symmetric algebra $S(V)$ is obtained by imposing an **commutative** relation:

$$v \otimes w - w \otimes v = 0,\ \forall v, w \in V$$

this means that the quotient algebra is:

$$S(V) = T(V) / \langle\, v \otimes w - w \otimes v \,\rangle.$$

14. ***Tensor Product*** The tensor product between two 1-covectors, $f, g : V \to \mathbb{R}$ is the 2-covector $f \otimes g$.

$$(f \otimes g)(u, v) = f(u)g(v)$$

. In general, the tensor product of a k-covector $p : V^k \to \mathbb{R}$ with a l-covector $q : v^l \to \mathbb{R}$ is the $(k+l)$-covector $p \otimes q : V^{k+l} \to \mathbb{R}$.

$$(p \otimes q)(u, v) = p(u)q(v),\ \forall u \in V^k,\ v \in V^l$$

15. ***Tensor Product(?)*** is an operator on $v \in V$ and $u \in U$ where

$$v \otimes u : V \times U \to V \oplus U$$
$$(v \otimes u)_{i \cdot j} = v_i \cdot u_j,\ \forall i = 1, \ldots, \dim(V),\ j = 1, \ldots, \dim(U)$$

Given two vector spaces V, W with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

α_i, β_j are all scalars. The real issue is the behavior of unit basis vectors v_i, w_j and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

16. ***Wedge Product***

Between two covectors Let $f, g \in L_1(V)$ then for all $u, v \in V$

$$(f \wedge g)(u, v) = (f \otimes g)(u, v) - (g \otimes f)(u, v) = f(u)g(v) - f(v)g(u)$$

Between mulitple 1-covectors.

$$\begin{aligned}(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \ldots, v_k) &= \det[\alpha^1(v_j)] \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)})\end{aligned}$$

Between k-covector and l-covector. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A_{k+l}(V)$$

or explicitly

$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f\left(v_{\sigma 1}, \ldots, v_{\sigma_k}\right) g\left(v_{\sigma_{k+1}}, \ldots, v_{\sigma k+l}\right)$$

Anticommutative. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$(f \wedge g) = (-1)^{kl} g \wedge f$$

17. **Differential k-Forms**

1-forms, covectors

$$\begin{aligned}(dx^i)\left(\frac{\partial}{\partial x^j}\bigg|_p\right) &= \frac{\partial}{\partial x^j}\bigg|_p x^i = \delta^i_j \\ (df)_p(X_p) = X_p f &= \sum a^i(p) \frac{\partial f}{\partial x^i}\bigg|_p = \sum \frac{\partial}{\partial x^i} dx^i\end{aligned}$$

18. $\Omega^k(U)$**, Vector space of C^∞ k-forms on U**.

$\Omega^0 = A_0(T_p(\mathbb{R}^n)) = C^\infty(U)$, e.g., $f \in \Omega^0$ then $f : V \to \mathbb{R}$ is a functional/covector/1-tensor.
Elements of 1-form $\Omega^1 = A_1(T_p(\mathbb{R}^n))$. For example, when $n = 3$

$$f dx + g dy + h dz, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

Elements of 2-form $\Omega^2 = A_2(T_p(\mathbb{R}^n))$. For example, when $n = 3$[1]

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

if $n = 4$, that is coordinates for u, v, w, x. Each form is derived from these bases
- 0-form $\Omega^0(\mathbb{R}^4) \in \mathbb{R}$
- 1-forms $\Omega^1(\mathbb{R}^4)$ summing du, dv, dw, dx,
- 2-forms $\Omega^2(\mathbb{R}^4)$ summing $du \wedge dv$, $du \wedge dw$, $du \wedge dx$, $dv \wedge dw$, $dv \wedge dx$, $dw \wedge dx$,
- 3-forms $\Omega^3(\mathbb{R}^4)$ summing $du \wedge dv \wedge dw \mid du \wedge dw \wedge dx \mid du \wedge dv \wedge dx \mid dv \wedge dw \wedge dx$
- 4-form $\Omega^4(\mathbb{R}^4)$ $du \wedge dv \wedge dw \wedge dx$.

Also, $U \subseteq \mathbb{R}^n$ **then** $k < n$. k-forms for $k > n$ are zero. Further $|\Omega^k(\mathbb{R}^n)| = \binom{k}{n}$ and $\left|\bigcup_k \Omega^k(\mathbb{R}^n)\right| = 2^n$ and think of $\Omega^*(U) = \bigcup_k \Omega^k(\mathbb{R}^n)$

Direct Sum. $\Omega^*(U) = \bigoplus_k \Omega^k(U)$ is an anti-commutative graded algebra over $\mathbb{R}$.
Since one can multiply C^∞ k-forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k-forms is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$ and $\Omega^*(U)$ is also a module over C^∞ of C^∞ functions.

19. **Wedge Product of k-form.** Recall: $dx^i \wedge dx^i = 0$ for all $i = 1, \ldots, n$. Therefore, $\wedge$ only makes sense to be defined on *disjoint indice-lists*, that is, $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$ such that $I \cap J = \emptyset$. Then,

$$\begin{aligned}\wedge : \Omega^k(U) \times \Omega^l(U) &\to \Omega^{k+l}(U) \\ (\omega, \tau) &\mapsto (\omega \wedge \tau) = \sum_{I,J} a_I b_J dx^I \wedge dx^J.\end{aligned}$$

where $\omega = \sum_I a_I dx^I, \tau = \sum_J b_J, dx^J$.

[1]NOTE the cyclic order of the indices x, y, z. Switching any one of these will flip the sign.

20. **the Exterior Derivative**. If $k \geq 1$ and if $\omega = \sum_I a_i dx^I \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$ and

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_J \frac{\partial a_I}{\partial x_J} dx^J \right) \wedge dx^I$$

Example: Let $\omega \in \Omega^1(\mathbb{R}^2)$ and $\omega = f dx + g dy$, $f, g \in C^\infty(\mathbb{R}^2)$.

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\ &= (g_x - f_y)\, dx \wedge dy \end{aligned}$$

Definition: Let $\bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K. An ***anti-derivation*** of the graded algebra A is a K-linear map $D : A \to A$ such that $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a D b$$

Proposition 4.7: Three Criterion for an Exterior Derivation

i) The ***exterior derivation*** $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

ii) $d^2 = 0$.

iii) If $f \in \mathbb{C}^\infty$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

NOTE: "In a typical school, there would be graduate level courses on Smooth Manifods and another on Remannian Manifolds."

Q: What is the difference between $\mathfrak{X}(U)$ and $C^\infty(U)$?

The difference between $\mathfrak{X}(U)$ and $C^\infty(U)$ lies in the types of objects they contain:
1. **$C^\infty(U)$: The Space of Smooth Functions** - $C^\infty(U)$ consists of all smooth (infinitely differentiable) real-valued functions defined on an open subset U of a manifold M. - Elements of $C^\infty(U)$ are scalar functions $f : U \to \mathbb{R}$. - These functions can be added and multiplied pointwise, forming an algebra over $\mathbb{R}$.

2. **$\mathfrak{X}(U)$: The Space of Smooth Vector Fields** - $\mathfrak{X}(U)$ consists of all smooth vector fields on U. - A vector field X assigns to each point $p \in U$ a tangent vector $X_p \in T_pM$, smoothly varying with p. - Vector fields act as derivations on smooth functions, meaning they satisfy the Leibniz rule:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(U).$$

- The space $\mathfrak{X}(U)$ forms a module over $C^\infty(U)$, meaning smooth functions can scale vector fields: if $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then fX is also a vector field.

Key Differences

Feature	$C^\infty(U)$	$\mathfrak{X}(U)$
Elements	Smooth scalar functions $f : U \to \mathbb{R}$	Smooth vector fields $X : U \to TM$
Algebraic Structure	Commutative algebra (pointwise multiplication)	Module over $C^\infty(U)$, noncommutative under Lie bracket
Operations	Addition, multiplication	Addition, scalar multiplication by $C^\infty(U)$, Lie bracket $[X, Y]$

In summary, $C^\infty(U)$ consists of smooth functions, while $\mathfrak{X}(U)$ consists of smooth vector fields, which act as differential operators on $C^\infty(U)$.

Compare and contrast.

Set	Dim	index	basis	Delta
$L_1(U)$	n	$i = 1, \ldots, n$	α^i	$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
$L_k(U)$	n^k	$I, J \in \{\underbrace{i_i, \ldots, i_k}_{k \text{ times}}\}$, $i_k \in [1, \ldots, n]$	$\alpha^I = \alpha^{i_1} \otimes \alpha^{i_2} \otimes \cdots \otimes \alpha^k$	
$A_k(U)$	$\binom{n}{k}$	$I, J \in \{\underbrace{i_i, \ldots, i_k}_{k \text{ times}}\}$, $i_1 < i_2 < \ldots i_k \in [1, n]$	$\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^k$	$\delta_I^J = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$

Supersets

Symbol	Name (set of)	Definition	Example
$\Omega^0(U)$	0-forms	{ scalar fields }	$f : V \to \mathbb{R}$ $f(x, y, z)$
$\Omega^1(U)$	1-forms	{ 1-forms, vector fields }	$d\omega(v) = A(v)dx + B(v)dy + C(v)dx$ $A, B, C : V \to \mathbb{R}$
$\Omega^k(U)$	k-forms	{ k-forms }	$\cdots + dx^1 \wedge \cdots \wedge dx^k + \cdots$
$\Omega^*(U)$	sum of k-forms	$\{\ x = \sum y \,\vert\, y \in \oplus_k \Omega^k(U)\ \}$	$Adx + Bdx \wedge dy + Cdx \wedge dy \wedge dz,\ A, B, C : V \to \mathbb{R}$
$\mathfrak{X}(U)$	vector fields on U	$\{X \to \exists f : U \to U\}$	
$C^\infty(U)$	smooth functions on U		
$X_p = T_p(U)$	a vector field at p	$\{v \in U \,\vert\, v = p + x \text{ for some } x \in U\}$	

Map of $\Omega^k(\mathbb{R}^3)$

$$\begin{tikzcd}
\Omega^0(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^1(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^2(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^3(U) \arrow[d, "\cong"'] \\
C^\infty(U) \arrow[r, "\mathrm{grad}"'] & \mathfrak{X}(U) \arrow[r, "\mathrm{curl}"'] & \mathfrak{X}(U) \arrow[r, "\mathrm{div}"'] & C^\infty(U).
\end{tikzcd}$$

Shorthand

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_I a_I = \sum_{n=1}^{k} a_{i_n}$$

$$\sum_{I,J} a_I b_J = \sum_{n=1}^{k} a_{i_n} \sum_{m=1}^{k} b_{j_m}$$

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_I^J = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_k}^{j_k} = \begin{cases} 1 & i_n = j_n, \forall n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Definition 0.0.1 (Exact and Closed k-forms)**.** A k-form ω on U is ***closed*** if $d\omega = 0$; it is ***exact*** if there is a $(k-1)$-form τ such that $\omega = d\tau$ on U. Since $d(d\tau) = 0$, every exact form is closed.

Definition 0.0.2 (de Rham Cohomology)**.** .

The ***k^{th}-cohomology of*** U is defined as the quotient vector space

$$H^k(U) = \frac{\{\text{closed k-forms}\}}{\{\text{exact k-forms}\}}$$

That is, each element is a vector space forming an equivalence class of k-forms.

Examples of de Rham Cohomology

De Rham cohomology provides a way to study the topology of smooth manifolds using differential forms. Below are some key examples illustrating how to compute and interpret de Rham cohomology groups.

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Example 1: Euclidean Space $\mathbb{R}^n$

For $M = \mathbb{R}^n$, we claim that the de Rham cohomology is:

$$H^k_{\mathrm{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(\mathbb{R}^n)$:**
   - The 0-forms are just smooth functions f.
   - A function is closed if $df = 0$, meaning f is constant.
   - Every constant function is not only closed but also exact since $f = d(fx)$.
   - The space of closed 0-forms is $\mathbb{R}$ (constant functions), and there are no exact forms to quotient out.
   - So, $H^0_{\mathrm{dR}}(\mathbb{R}^n) = \mathbb{R}$.
2. **$H^k_{\mathrm{dR}}(\mathbb{R}^n)$ for $k > 0$:**
   - Any closed k-form ω is locally exact due to **Poincaré's lemma**.
   - That is, every closed form is of the form $\omega = d\eta$, meaning it contributes nothing to cohomology.
   - Thus, $H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ for $k > 0$.
     This result reflects the fact that $\mathbb{R}^n$ is **contractible**, so it has trivial topology.

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Example 2: The Circle S^1

For $M = S^1$, we find:

$$H^k_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(S^1) = \mathbb{R}$**
   - Smooth functions f that satisfy $df = 0$ are constant.
   - Thus, $H^0_{\mathrm{dR}}(S^1) = \mathbb{R}$.
2. **$H^1_{\mathrm{dR}}(S^1) = \mathbb{R}$**
   - Consider the 1-form $\omega = d\theta$, where θ is the angular coordinate.
   - $d\omega = 0$, so ω is closed.
   - Is ω exact? If $\omega = d\eta$ for some η, then $d\eta = d\theta$, but no globally defined function η exists on S^1 satisfying this.
   - So ω represents a **nontrivial cohomology class**, giving $H^1_{\mathrm{dR}}(S^1) = \mathbb{R}$.
3. **$H^k_{\mathrm{dR}}(S^1) = 0$ for $k \geq 2$**
   - There are no nontrivial 2-forms on a 1-dimensional manifold.
   Interpretation

- The nontrivial $H^1_{\mathrm{dR}}(S^1)$ reflects the existence of a **loop** in S^1.

- This cohomology detects the ability to define a **non-exact closed form**, related to the winding number.

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****Example 3: The 2-Sphere S^2** For $M = S^2$:**

$$H^k_{\mathrm{dR}}(S^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ 0, & k = 1. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(S^2) = \mathbb{R}$**

   - As always, closed 0-forms are constant functions, so $H^0_{\mathrm{dR}}(S^2) = \mathbb{R}$

2. **$H^1_{\mathrm{dR}}(S^2) = 0$**
   - Any closed 1-form is exact by a higher-dimensional **Poincaré lemma**, so $H^1_{\mathrm{dR}}(S^2) = 0$.

3. **$H^2_{\mathrm{dR}}(S^2) = \mathbb{R}$**

   - The standard volume form $\omega = \sin\theta \, d\theta \wedge d\phi$ is closed.

   - It is not exact, because there is no 1-form η such that $d\eta = \omega$ (this follows from **Stokes' theorem**).

   - So ω represents a generator of $H^2_{\mathrm{dR}}(S^2)$.

Interpretation

- $H^1_{\mathrm{dR}}(S^2) = 0$ reflects that there are no **nontrivial loops** (all loops contract).
- $H^2_{\mathrm{dR}}(S^2) = \mathbb{R}$ corresponds to the existence of a volume form, a global topological feature.

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****Example 4: The Torus $T^2 = S^1 \times S^1$****

For T^2, the de Rham cohomology groups are:

$$H^k_{\mathrm{dR}}(T^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ \mathbb{R} \oplus \mathbb{R}, & k = 1, \\ 0, & k > 2. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(T^2) = \mathbb{R}$** (constant functions).

2. **$H^1_{\mathrm{dR}}(T^2) = \mathbb{R} \oplus \mathbb{R}$**
   - The torus has two independent 1-forms: $d\theta_1$ and $d\theta_2$, corresponding to the two loops in T^2.

3. **$H^2_{\mathrm{dR}}(T^2) = \mathbb{R}$**
4. The volume form $d\theta_1 \wedge d\theta_2$ represents a nontrivial 2-class.

Interpretation

- The rank of $H^1_{\mathrm{dR}}(T^2)$ reflects the **two independent loops** in the torus.

- The nontrivial $H^2_{\mathrm{dR}}(T^2)$ corresponds to the existence of a **volume form**.

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Summary Table

Manifold M	$H^0_{dR}(M)$	$H^2_{dR}(M)$	$H^1_{dR}(M)$
$\mathbb{R}^n$	$\mathbb{R}$	0	0
S^1	$\mathbb{R}$	$\mathbb{R}$	0
S^2	$\mathbb{R}$	0	$\mathbb{R}$
T^2	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$

§8 Tangent Space.

Definition 8.1.3 (Tangent vector)**.** A ***tangent vector*** at a point p is a derivation at p.

Remark 8.1.4. In general, x^i, y^j are coordinates for manifolds N, M and r^i are coordinates associated with charts. Thus $p = (x^1, \ldots, x^n)$ and $\phi(p) = (r^1, \ldots, r^k)$ where $k \leq n$.

Definition 8.1.5 (Push-back and Push-forward)**.** .

Given the mapping between manifolds N, M as $\varphi : N \to M$ with charts $\phi : N \to \mathbb{R}$ and $f' : M \to \mathbb{R}$. We define the ***push-back*** $\varphi^* f'(p)$ as [2]

$$\varphi^* f'(p) : N \to \mathbb{R}$$
$$\varphi^* f'(p) = f' \circ \varphi(p)$$

Note: this is a functional from the original manifold N to the reals. Let's think of it as a short-cut from $N \to \mathbb{R}$ through M via φ. Note that the push-back operates on a chart.

Let's define the ***push-forward***. If M and N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map $F_* : T_pM \to T_{F(p)}N$, called the ***pushforward*** associated with F, by

$$(F_*X)(f) = X(f \circ F).$$

Note that if $f \in C^\infty$, then $f \circ F \in C^\infty$, so $X(f \circ F)$ makes sense.

Remark 8.1.6 (**MathGPT:** the gradient, the differential and the pushforward). .

1. Step 1: Understanding the Gradient

   The gradient on a Riemannian manifold is a vector field derived from a scalar function. It requires a Riemannian metric to convert the differential (a covector field) into a vector field. The gradient points in the direction of the greatest rate of increase of the function.

2. Step 2: Understanding the Differential

   The differential of a smooth map is a linear map between tangent spaces. For a scalar function, it is a covector field mapping tangent vectors to scalars. For a general map between manifolds, it maps tangent vectors from the domain to the codomain's tangent space. It does not require a Riemannian metric.

3. Step 3: Understanding the Pushforward

   The pushforward is often synonymous with the differential of a map, but it specifically refers to transforming vector fields or tensor fields from one manifold to another. It requires the map to be a diffeomorphism for a well-defined global pushforward of vector fields.

[2] beware the f-prime, f', is NOT the derivative

4. Final Answer

The gradient is a vector field derived from a scalar function using a Riemannian metric. The differential is a linear map between tangent spaces, fundamental for approximating smooth maps. The pushforward transforms vector fields between manifolds, often requiring the map to be a diffeomorphism.

Remark 8.1.7 (**Identities of Push-foward**). .

1. **Functoriality/Chain Rule for Pushforward of Vector Fields**

If $f : M \to N$ and $g : N \to P$ are smooth mas between manifolds, and $X \subseteq M$ is a vector field, the push-forward satisfies ***The Chain Rule***

$$(g \circ f)_* X = g_*(f_* X)$$

This means pushing forward by a composition of maps is the same as pushing forward by each map sequentially. This is a direct generalization of the chain rule in calculus. Note: the above notation can be deceptive, keep in mind that $(g \circ f)_*(p) : T_p(M) \to T_{g(f(p))}(P)$.

2. **Linearity of the Pushforward**

The push-forward of a vector field is a ***linear map between tangent-spaces***:

$$f_*(aX + bY) = af_*X + bf_*Y$$

where a and b are scalars and X and Y are vector fields on M.

3. **Pushforward and Lie Brackets (for Diffeomorphisms)**

if $f : M \to N$ is a diffeomorphism and X, Y are vector fields on M, the ***the pushforward of the Lie Bracket*** is the Lie bracket of their pushforwards:

$$f_*[X, Y] = [f_*X, f_*Y]$$

This highlights how the pushforward respects the Lie algebra structure, particularly for Lie groups and their associated Lie algebras.

4. **Pushforward and Flows**

The pushforward of a vector field is closely related to the flows generated by vector fields.[3] If ψ_t is the flow generated by a vector field X, then the pushforward of a vector field Y by the flow can be used to define the Lie derivative:

$$L_X Y = \frac{d}{dt}|_{t=0}(\psi_t)_* Y$$

5. **Relationship with Lie Derivative (Alternative)**

[3]It's a one-parameter family of diffeomorphisms (smooth maps with smooth inverses) that move points along the flow lines of a vector field. It is not a term with a formal meaning, but is typically understood to refer to parabolic partial differential equations.

Another way to express the relationship between the Lie derivative and pushforward involves the pullback of a vector field by a flow. The ***Lie derivative*** can be written as:

$$L_X Y = \lim_{t \to 0} \frac{(\psi_{-t})_* Y - Y}{t}$$

This identity reveals the Lie derivative as the infinitesimal change in Y as it is flowed along X

6. **In Summary**

   These identities are fundamental in differential geometry. They demonstrate how the pushforward operation relates to other important concepts like compositions of maps, linearity, Lie brackets, and flows. Understanding these relationships is crucial for working with vector fields and other differential geometric objects.