Functional Analysis– Spring 2024

Paul Carmody
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2. Let H be a Hilbert space and $T : H \to H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*$$

$$\begin{aligned} (TT^*)^{-1} &= (T^*)^{-1}T^{-1} \\ (TT^*)^{-1}T &= (T^*)^{-1} \\ \therefore (T^*)^{-1} &\text{ exists} \end{aligned}$$

$$\begin{aligned} \langle Tx, y \rangle &= \langle x, T^*y \rangle \\ \left\langle T^{-1}Tx, y \right\rangle &= \left\langle T^{-1}x, T^*y \right\rangle \\ \langle x, y \rangle &= \left\langle T^{-1}x, T^*y \right\rangle \\ \left\langle x, (T^*)^{-1}y \right\rangle &= \left\langle T^{-1}x, (T^*)^{-1}T^*y \right\rangle \\ &= \left\langle T^{-1}x, y \right\rangle \\ &= \left\langle x, (T^{-1})^*y \right\rangle \\ (T^*)^{-1} &= (T^{-1})^* \end{aligned}$$

3. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$, show that $T_n^* \to T^*$.

$$\begin{aligned} \|T_n - T\|^2 &\geq \|T_n\|^2 - \|T\|^2 = \|T_n^*\|^2 - \|T^*\|^2 \geq \|T_n^* - T^*\|^2 \\ \text{similarly, } \|T_n^* - T^*\|^2 &\geq \|T_n^*\|^2 - \|T^*\|^2 = \|T_n\|^2 - \|T\|^2 \geq \|T_n - T\|^2 \\ \text{hence } \|T_n - T\|^2 &= \|T_n^* - T^*\|^2 \end{aligned}$$

We know that given any $N > 0$ then for all $n > N$ when $\|T_n - T\| < \epsilon$ implies that $\|T_n^* - T^*\| < \epsilon$. Therefore, $T_n^* \to T^*$.

4. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1^\perp \subset T^*(M_2^\perp)$.

   Let $x \in M_1$ and $z \in M_2^\perp$ and $x \notin \mathcal{N}(T)$.Then, $\langle Tx, z \rangle = 0$ implies $\langle x, T^*z \rangle = 0$ and either $T^*z \in \mathcal{N}(T^*)$ or $T^*z \perp x$. x is arbitrary, therefore $T^*z \perp \text{span}(M_1)$ or $T^*z \in M_1^\perp$. Thus, $T^*z \in \mathcal{N}(T^*) \cup M_1^\perp$. z is arbitrary so $T^*(M_2^\perp) = \mathcal{N}(T^*) \cup M_1^\perp$, hence, $M_1^\perp \subset T^*(M_2^\perp)$.

5. Let M_1 and M_2 in Prob. 4 be closed subspaces. Show that $T(M_1) \subset M_2$ if and only if $M_1^\perp \supset T^*(M_2^\perp)$.

6. If $M_1 = \mathcal{N}(T) = \{x \,|\, Tx = 0\}$ in Prob. 4, show that

   (a) $T^*(H_2) \subset M_1^\perp$

   (b) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$

   (c) $M_1 = [T^*(H_2)]^*$

10. **(Right shift operator)** Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the *right shift operator* to be the linear operator $T : H \to H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of T.

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4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_3 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2 \; T^* = T_1 = iT_2.$$

Show uniquness, that is, $T_1 + iT_2 = S_1 + iS_2$ implies $S_i = T_i$ and $S_2 = T_2$; here, S_1 and S_2 are self-adjoint by assumptioin.

5. On $\mathbb{C}^2$ (cf. 3.1-4) let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find T^*. Show that we have $T^*T = TT^* = 2I$. Find T_1 and T_2 as defined in prob. 4.