Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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Section 8: The Tangent Space – May 30, 2025

Pg. 88: Exercise 8.3 (The Differential of a mpa). Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_* : T_pN \to T_{F(p)}M$ is a linear map.

Let $[f] \in C^\infty_{F(p)}(M)$ and $f, g \in [f]$. Then,

$$\begin{aligned}(F_*(X_p))f \cdot g &= X_p(f \cdot g \circ F)\\ &= X_p\left(\,(f \circ F) \cdot (g \circ F)\,\right)\\ &= f \cdot X_p(g \circ F) + g \cdot X_p(f \circ F)\\ &= f \cdot (F_*(X_p))g + g \cdot (F_*(X_p))f\end{aligned}$$

hence F_* obeys the Liebniz Rule.

$$\begin{aligned}F_*(aX_p + b)f &= (aX_p + b)(f \circ F)\\ &= a(X_p)(f \circ F) + b(f \circ F)\\ &= a(F_*(X_p))f + (F_*(b))f\end{aligned}$$

Pg. 92: Exercise 8.14 (The Velocity Vector vs the Calculus Derivative). Let $c : (a, b) \to \mathbb{R}$ be a curve with target space $\mathbb{R}$. Verify that $c'(t) = \dot{c}(t) d/dx|_{c(t)}$.

Problems

8.1. **Differential of a map.**

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*\left(\frac{\partial}{\partial x}\Big|_p\right)$ as a linear combination of $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, and $\frac{\partial}{\partial w}$ at $F(p)$.

Let $F(x, y) = (F^1(x, y), F^2(x, y), F^3(x, y))$ then

$$J_F = \begin{bmatrix} \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial y} \\ \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial y} \\ \frac{\partial F^3}{\partial x} & \frac{\partial F^3}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}$$

$$\begin{aligned}F_*\left(\frac{\partial}{\partial x}\Big|_p\right) &= F_*(1, 0)\\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\\ &= \begin{pmatrix} 1 & 0 & y \end{pmatrix}\\ &= \frac{\partial}{\partial u} + v\frac{\partial}{\partial w}\end{aligned}$$

8.2. Differential of a linear map

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \xrightarrow[\text{iso}]{} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \langle a^1, \cdots a^n \rangle .$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identificaton of the tangent spaces as above.

Given $p = (a^1, \ldots, a^n)$ and $L : N \to M$ and $L(p) = \langle a^1, \ldots, a^n \rangle$. Given any $f : M \to \mathbb{R}$ then

$$\begin{aligned}
(L_{*,p}(X_p))f(p) &= X_p(f \circ L)(p) \\
&= \sum_{i=1}^n \frac{\partial f(L(p))}{\partial x^i}\bigg|_p \\
&= \sum_{i=1}^n \frac{\partial f(\langle a^1, \ldots, a^n \rangle)}{\partial x^i}\bigg|_p \\
&= \sum_{i=1}^n \frac{\partial f(p)}{\partial x^i}\bigg|_p \\
&= \sum_{i=1}^n \frac{\partial}{\partial x^i}\bigg|_p f(p) \\
&= L_p f(p)
\end{aligned}$$

8.3. Differential of a map

Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)\big|_{F(p)}$, find a and b in terms of x, y, and α.

Remember that F_* is linear then

$$(F_*(X_p))f = -yF_*\left(\frac{\partial}{\partial x}\right) + xF_*\left(\frac{\partial}{\partial y}\right) \tag{1}$$

The Jacobian is

$$\begin{aligned}
J_{F_*} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\
F_*\left(\frac{\partial}{\partial x}\right) &= J_{F_*}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \cos\alpha\frac{\partial}{\partial u} + \sin\alpha\frac{\partial}{\partial v} \\
F_*\left(\frac{\partial}{\partial y}\right) &= J_{F_*}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} = -\sin\alpha\frac{\partial}{\partial u} + \cos\alpha\frac{\partial}{\partial v}
\end{aligned}$$

From (1) we have

$$\begin{aligned}
(F_*(X_p))f &= -yF_*\left(\frac{\partial}{\partial x}\right) + xF_*\left(\frac{\partial}{\partial y}\right) \\
&= -y\left(\cos\alpha\frac{\partial}{\partial u} + \sin\alpha\frac{\partial}{\partial v}\right) + x\left(-\sin\alpha\frac{\partial}{\partial u} + \cos\alpha\frac{\partial}{\partial v}\right) \\
&= -(y\cos\alpha + \sin\alpha)\frac{\partial}{\partial u} + (-y\sin\alpha + x\cos\alpha)\frac{\partial}{\partial v}
\end{aligned}$$

8.4. **Transition matrix for coordinate vectors**

Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the ope set

$$U = \mathbb{R}^2 - \{\, (x,0) \,|\, x \geq 0 \,\}.$$

On U the polar coordinates r, θ are uniquley defined by

$$x = r\cos\theta$$
$$y = r\sin, r > 0, 0 < \theta < 2\pi$$

Find $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$F : U \to \mathbb{R}^2$$
$$F(r,\theta) = (r\cos\theta, r\sin\theta)$$
$$J_F = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$
$$F\left(\frac{\partial}{\partial r}\right) = J_F \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$
$$= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$
$$F\left(\frac{\partial}{\partial \theta}\right) = J_F \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}$$
$$= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

F is bijective, hence, J_F^{-1} exists and is

$$J_F^{-1} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \frac{1}{\det J_F} \begin{bmatrix} r\cos\theta & \sin\theta \\ -r\sin\theta & \cos\theta \end{bmatrix}$$
$$\det J_F = r\cos^2\theta + r\sin^\theta = r$$
$$J_F^{-1} = \begin{bmatrix} \cos\theta & \frac{\sin\theta}{r} \\ -\sin\theta & \frac{\cos\theta}{r} \end{bmatrix}$$

Then we can write $F^{-1}(x,y)$ in terms of J_F^{-1}.

8.5. **Velocity of a curve in local coordinates**

Prove Proposition 8.15.

8.6. **Velocity vector**

Let $p = (x,y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, t \in \mathbb{R}$$

is a curve with initail point p in $\mathbb{R}^2$. compute the veolocity vector $c_p'(0)$.

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} x\cos 2t - y\sin 2t \\ x\sin 2t + y\cos 2t \end{bmatrix}$$
$$\dot{c}^1(t) = -2x\sin 2t - 2y\cos 2t$$
$$\dot{c}^2(t) = 2x\cos 2t - 2y\sin 2t$$
$$c'(0) = \begin{bmatrix} -2x\cos 0 - 2y\sin 0 \\ 2x\cos 0 - 2y\sin 0 \end{bmatrix} = \begin{bmatrix} -2y \\ 2x \end{bmatrix}$$

8.7. **Tangent space to a product**

If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_1, \pi_2) : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

8.8. **Differentials of multiplication and inverse**

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\iota : G \to G$, and identity element e.

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\mu_{*,(e,e)} : T_eG \times T_eG \to T_eG,$$
$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

(*Hint:* First, computer $\mu_{*,(e,e)}(X_e, 0)$ and $\mu_{*,(e,e)}(0, Y_e)$ using Proposition 8.18)
Want to show that if $F = \mu$ and $p = (e, e)$. Thus when we write $F_*(X_p)f = X_p(f \circ F)$ we mean $\mu_{*,(e,e)}(X_{(e,e)})f = X_{(e,e)}(f \circ \mu)$

(b) Show that the differential at the identity of ι is the negative:

$$\iota_{*,(e,e)} : T_eG \to T_eG,$$
$$\iota_{*,(e,e)}(X_e) = -X_e.$$

(*Hint:* Take the differential of $\mu(c(t), (t \circ c)(t)) = e$.).

8.9. **Transforming vectors to coordinate vectors**

Let $X_1, \ldots, X_n$ be n vector fields on an open subset U of a manifold of dimensions n. Suppose that at $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ about p such that $(X_i)_p = \left(\frac{\partial}{\partial x^i}\right)_p$ for $i = 1, \ldots, n$.

8.10. **Local maxima**

A real-valued function $f : M \to \mathbb{R}$ on a manifold is said to have *local maximum* at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

(a) Prove that if a differentiable functions $f : I \to \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, then $f'(p) = 0$.

(b) Prove that a local maximum of a C^∞ fucntion $f : M \to \mathbb{R}$ is a critical pint of f. (*Hint:* Let X_p be a tangent vector in T_pM and let $c(t)$ be a curve in M starting at p with initial vector X_p. Then $f \circ c$ is a real-valued functions with a local maximum at 0. Apply (a).)