Math 725 – Advanced Linear Algebra

Paul Carmody

All about Matrices/Transformations

Important terms:

(1) minimum polynomial: The polynomial, p, with lowest degree such that $p(T) = 0$, $\forall x \in F$.
   i) The roots of the minimum polynomial are eigenvalues.
   ii) If the roots have singular multiplicity, then the matrix is diagonalizable.

(2) characteristic polynomial
   $\det(xI - T)$ forms a polynomial.
   i) the characteristic polynomial is divided by the minimum polynomial.
   ii) the characteristic polynomial and the minimum polynomial have the same roots, i.e, the same eigenvalues
   iii) if all of the factors of the characteristic polynomial are simple (i.e., have degree one) then it is the minimum polynomial.

(3) triangularizable: a matrix that has zeros below the diagonal. All matrices over the complex numbers are triangulizable.

(4) diagonalizable: a matrix that has zeros everywhere except the diagonal.

(5) **Inner Product Space** defines an inner product. The primary ability of the Inner Product is define orthogonality, orthonormal basis and norm. An inner product $\langle\ \rangle$ has the following properties.
   i) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
   ii) $\langle cv, w \rangle = c \langle v, w \rangle$ and $\langle v, dw \rangle = \overline{d} \langle w, w \rangle$
   iii) $\langle w, v \rangle = \overline{\langle v, w \rangle}$
   iv) $\langle v, v \rangle > 0$ and $\langle v, v \rangle = 0$ if $v = 0$

(6) norm: is a function $||\cdot|| : F \to \mathbb{R}$ with the following properties:
   i) $||\cdot|| \geq 0$.
   ii) $||cv|| = |c|\, ||v||$.
   iii) $||v + w|| \leq ||v|| + ||w||$ (triangular inequality).

(7) orthogonal: u, v are orthogonal is $\langle u, v \rangle = 0$ and $u \neq 0$ and $v \neq 0$.
   if Q is an *orthogonal matrix* if $QQ^T = I$.

(8) orthonomal. u, v are said to be orthonormal if they both have length one.
   Every finite dimensional inner product space has an orthonormal basis. Every linear operator T has an upper triangular matrix $[T]_B^B$ w.r.t. an orthonormal basis.

(9) orthogonal compliment. Given any set $S \subseteq V$ then $S^\perp = \{v \in V \,:\, \langle v, w \rangle = 0, \forall w \in S\}$
   Any subspace $W \subseteq V$, then $V = W \oplus W^\perp$.

(10) adjoint: $T^* \in \mathcal{L}(W, V) \to \langle Tv, w \rangle_W = \langle v, T^*w \rangle_V$
   Properties (analogous to complex arithmetic):
   i) Additive: $(S + T)^* = S^* + T^*$, $\forall S, T \in \mathcal{L}(V, W)$.
   ii) Scalar Multiplication: $(\lambda T)^* = \overline{\lambda} T^*$, $\forall T \in \mathcal{L}(V, W) \,\&\, \lambda \in F$
   iii) Multiplication anti-commutative: $(S \circ T)^* = T^* \circ S^*$, $\forall T \in \mathcal{L}(U, V) \,\&\, S \in \mathcal{L}(V, W)$
   iv) Inverse: $(T^*)^* = T$, $\forall T \in \mathcal{L}(V, V)$
   v) If $T = U_1 + iU_2$ then
      a) $U_1 = \frac{1}{2}(T + T^*)$, $U_1^* = U_1$
      b) $U_2 = \frac{1}{2}(T - T^*)$, $U_2^* = U_2$

c) Note: U_1, U_2 "look" like real numbers.

Matrices:

Given orthonormal bases B, B' on V, W, respectively. then $[T]_{B'}^{B} = A, [T^*]_{B}^{B'} = A^* \implies A^* = \overline{A^T}$, i.e, if $W = V$ then T is an operator and A is Hermitian/Symmetric.

(11) self-adjoint: $T = T^*$
  a) All $\lambda \in \mathbb{R}$ for eigenvalues of T.
  b) $\langle Tv, v \rangle \in \mathbb{R}$ even if V is complex.
  c) if $\langle Tv, v \rangle = 0$, $\forall v \in V$ then $T = 0$.

(12) normal: If $TT^* = T^*T$ then T is said to be normal. Self-adjoint implies Normal but not visa versa.
  a) If $Tv = \lambda v$ then $T^*v = \overline{\lambda} v$.
  b) T normal $\iff$ diagonalizable w.r.t. orthonormal basis.
  c) $[T]_B^B$ is hermitian, i.e, $A = [T]_B^B = \overline{A^T} = A^*$
  d) $\exists Q \to QQ^* = I$ and $A = Q^*\Lambda Q$ where Λ is a diagonal matrix consisting of eigenvalues and Q is a matrix consisting of orthonormal column eigenvectors (i.e., unitary).
  e) iff $||Tv|| = ||T^*v||$, $\forall v \in V$.
  f) eigenvectors from different eigenvalues are orthogonal to each other.

(13) unitary: a matrix made up of orthonormal column vectors.
  - the conjugate transpose is the inverse
  - determinant is one

(14) (semi-)positive definite when $\langle Tv, v \rangle > 0$, $\forall v \neq 0$ semi- implies $\langle Tv.v \rangle \geq 0$, $\forall v \neq 0$. The following are equivalent
  i) T is (semi-)positive definite.
  ii) eigenvalues of T are (semi-)positive.
  iii) $\exists R \in \mathcal{L}(V) \implies T = RR^*$.

(15) Theorem: Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{R}$ coefficients with degree $2d$ and X^T be the set of possible terms whose exponents that added up to less than or equal to $2d$. Then f is a Sum of Squares if and only if there exists a positive semi-definite matrix A such that $f = X^TAX$.

This is useful in positive-definite multi-variable polynomials. These all have even valued degrees (hence the $2d$ term), then bodes the question are they the sum of squares polynomials (i.e., equal to $\sum p_i^2$ where p_i are multivariable polynomials).

(16) Definition: The ***Singular Value Decomposition***, $\sigma_1, \sigma_2, \ldots, \sigma_r$ of $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ are the positive square roots of eigenvalues $\sigma_i = \sqrt{\lambda_i}$ where $\lambda_i \neq 0$ of the matrix $K = A^TA$ (which is positive definite, hence all $\lambda_i \geq 0$).

(17) Single Value Decomposition Theorem: $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ of rank r can be factored as $A = P\Sigma Q^T$ where
  - $P \in \mathcal{M}_{m\times r}(\mathbb{R})$ with ornormal columns (i.e, $P^TP = I_r$.
  - Σ is a diagonal matrix made up of $\sigma_1, \sigma_2, \ldots, \sigma_r$.
  - $Q^T \in \mathcal{M}_{r\times n}(\mathbb{R})$ orthonormal rows ($Q^TQ = I_r$).

(18) Pseudo-Inverse (not square): $A \in \mathcal{M}_{m\times n}(F)$ and $A = U\Sigma V^T$ then $A^+ \in \mathcal{M}_{n\times m}(F)$ and

$$A^+ = V\Sigma^{-1}U^T$$

if A is square then

$$A^+ = (A^TA)^{-1}A^T$$

(19) Definition: Given $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ then we define the ***norm*** of A as

$$||A|| := \max_{||x||=1} ||Ax||$$

Remark: if A has an SVD of $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ then $||A|| = \sigma_1$.

(20) Eichart-Mintsky-Young: Let $A = \sum_{j=1}^{r} \sigma_j u_j v_j^T$ (the Single Value Decomposition of A as the sum of rank 1 matrices). For each $1 \leq p \leq r$ let $A_p = \sum_{j=1}^{p} \sigma_j u_j v_j^T$ (a rank p matrix). Then,

$$||A - A_p|| = \min_{B \in F^{n\times n}} ||A - B|| \text{ where } \mathrm{rank}(B) \leq p$$

hence A_p is the *closest* rank p matrix to A.

Important theorems:

- Fundamental Theorem of Algebra
  Most notably that $\mathbb{C}$ is algebraically closed (i.e., all polynomials have a zero).