Math 5110 – Real Analysis I– Fall 2024
w/Professor Liu

Paul Carmody
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I. Consider a sequence x_n of real numbers. The *limit inferior* and *limit superior* of x_n are defined by

$$\liminf_{n\to\infty} x_n = \lim_{n->\infty}\left(\inf_{k\geq n} x_k\right), \ \limsup_{n\to\infty} x_n = \lim_{n->\infty}\left(\sup_{k\geq n} x_k\right)$$

(a) Show that

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0}\left(\inf_{k\geq n} x_k\right)$$

and

$$\limsup_{n\to\infty} x_n = \inf_{k\geq 0}\left(\sup_{k\geq n} x_k\right)$$

First, $\liminf_{n\to\infty}$: Let $y_n = \inf_{k\geq n} x_k$. Then, given any $j > k, y_k \leq y_j$. That is, y_n is a bounded increasing sequence. All $y_n \leq \sup_{n\to\infty} y_n$ Thus, the $\lim_{n\to\infty} y_n = \sup_{n\geq 0} y_n$.
Next, $\limsup_{n\to\infty}$: Let $z_n = \sup_{k\geq n} x_k$. Then, given any $j > k, z_k \geq z_j$. That is, z_n is a bounded decreasing sequence. All $z_n \geq \inf_{n\to\infty} z_n$ Thus, the $\lim_{n\to\infty} z_n = \inf_{n\geq 0} z_n$.

(b) Show that $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are well-defined for any sequence x_n. (Unlike $\lim_{n\to\infty} x_n$.) We allow values of ∞ and $-\infty$
Using (y_n) from (a), that must exist one and only one value for $\lim_{n\to\infty} y_n$ as it is bounded and increasing, thus its limit is well-defined. Similarly, for (z_n).

(c) Let x_n be a bounded sequence, and let L be the set of limit points of x_n, i.e., the set of all limits of subsequences of x_n. Show $\liminf_{n\to\infty} x_n = \inf L$ and $\limsup_{n\to\infty} = \sup L$.
Let L be the set of limit points for x_n. Then, for any $w \in L$ there is a $(w_k) \in (x_n)$ subsequence such that $\lim_{k\to\infty} w_k = w$. The $\inf_{k\to\infty} w_k \geq \inf L \geq \liminf_{n\to\infty} x_n$. However, from (a) we can see that

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0}\left(\inf_{k\geq n} x_k\right)$$

therefore $\liminf_{n\to\infty} x_n \geq \inf L$ thus $\liminf_{n\to\infty} x_n = \inf L$.
Similarly, for $\limsup_{n\to\infty} x_n$.

(d) Let x_n be a bounded sequence. Conclude using (c) that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, with equality if and only if x_n is convergent.
By definition, $\inf L \leq \sup L$ therefore $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Therefore, from (a)

$$\sup_{n\geq 0}\left(\inf_{k\geq n} x_k\right) \leq \inf_{n\geq 0}\left(\sup_{k\geq n} x_k\right)$$

Now using (y_n) and (z_n) from (a) we can see that we have

$$\sup_{n\geq 0} y_n \leq \inf_{n\geq 0} z_n$$

we have a bounded increasing sequence on the left less than a bounded decreasing sequence on the right. They can only be equal if they converge to the same value.

II. Prove that for any (possibly uncountable) collection $(F_\alpha)_{\alpha\in A}$ of closed sets, the intersection $F = \bigcup_{\alpha\in A} F_\alpha$ is closed, in two ways.

(a) Using the fact that any union of open sets is open, and DeMorgan's Laws from set theory, which state

$$X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A}(X\backslash E_\alpha) \text{ and } X\backslash\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A}(X\backslash E_\alpha)$$

for all collection of sets $(E_\alpha)_{\alpha\in A}$

Given that every open set, $E \in X$ is the union of other open sets $\bigcup_{\alpha\in A} E_\alpha$ for some index set A (whether countable or uncountable). We know that the complement is closed and the complement can be expressed as

$$\begin{aligned} E^c &= X\backslash E \\ &= X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) \\ &= \bigcap_{\alpha\in A} (\,X\backslash E_\alpha\,) \end{aligned}$$

each E_α is the complement of an open set, hence they are closed. Thus, E^c which is closed is made up of the intersection of closed sets.

(b) More directly, using the fact that a set G is closed if and only if for any convergent sequence (x_n) with all $x_n \in G$, the limit x is also in G.

Let $F, G \in X$ be closed sets and let $(x_n) \subset G$ and $(y_n) \subset F$ both be convergent sequences. Further, we let $(x_n), (y_n) \subset G\cap F$. Not that F closed means that $(x_n) \in F$ implies that $\lim_{n\to\infty} x_n \in F$, thus $\lim_{n\to\infty} x_n \in G\cap F$ and a similar argument can be made for y_n and G. Thus sequences contained in $G\cap F$ must also contain their limits and $G\cap F$ is closed. This can extend to any number of intersections.

III. (a) Let (x_n) be a Cauchy sequence in a metric space X. Show that if a subsequence (x_{n_j}) of x_n converges to x, then the entire sequence also converges to x.

Let (x_n) be Cauchy and let (x_{n_j}) be a convergent subsequence of (x_n). Then, there exists for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that whenever $j, k > N$, $|x_{n_j} - x_{n_k}| < \epsilon$. Let $M = \min\{n_j, n_k\}$. We can see that $|x_m - x_k| < \epsilon$. $x_m, x_k \in x_n$ and x_n is Cauchy, therefore all this is true for all elements of $m, k > M$, hence (x_n) converges.

(b) Show that the metric space

$$C^1\,(\,(-1,1)\,) = \{f : (-1,1) \to \mathbb{R}, f \text{ is differentiable and } f' \text{ is continuous in } (1,-1)\}$$

with the metric

$$d(f,g) = \sup_{x\in(-1,1)} |f(x) - g(x)|$$

is not complete. (Hint: similar to the proof that the rational numbers are not complete, find a sequence $C^1\,(\,(-1,1)\,)$ that converges in d metric to a function that is not in $C^1\,(\,(-1,1)\,)$, and show that this sequence is Cauchy.)

Let $f_n(x) = x^{\frac{1}{2n+1}}$. We can see that given any $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $n, m > N$ the distance

$$\begin{aligned} d(f_n, f_m) &= \sup_{x\in(-1,1)} |f_n(x) - f_m(x)| \\ &= \sup_{x\in(-1,1)} |x^{1/2n+1} - x^{1/2m+1}| \\ &< \epsilon \end{aligned}$$

The functions are all differentiable and their derivatives are continuous, but

$$\lim_{n\to\infty} x^{\frac{1}{2n+1}} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}, \forall x \in (-1,1)$$

which is not a member of $C^1\,(\,(-1,1)\,)$

IV. Let A and B be subsets of the metric space X. which one of the following is true?

$$(A\cup B)^o = A^o \cup B^o, \tag{2.1}$$

$$(A\cup B)^o \subset A^o \cup B^o, \quad \text{"=" fails for some } A \text{ and } B \tag{2.2}$$

$$(A\cup B)^o \supset A^o \cup B^o, \quad \text{"=" fails for some } A \text{ and } B \tag{2.3}$$

(2.3) Consider $X = \mathbb{R}^3$ and A is the open unit disc in the X-Y plane centered at the origin and B is the open unit disc in the Y-Z plane centered at the origin. $(A\cup B)^o \supset A^o \cup B^o$.

V. Let $C^0([a,b])$ be the space of continuous functions on $[a,b]$, with the metric $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$.

Show that the map $I : C^0([a,b]) \to \mathbb{R}$ defined by $I(f) = \int_a^b f(x)dx$ is continuous mapping from $C^0([a,b])$ to $\mathbb{R}$.

I is continuous if for ever $\epsilon > 0$ there exists $\delta > 0$ such that $d(I(f), I(g)) < \epsilon$ whenever $d(f,g) < \delta$. Or

$$\begin{aligned}
d(I(f), I(g)) &= \sup_{x\in[a,b]} |I(f(x)) - I(g(x))| \\
&= \sup_{x\in[a,b]} \left| \int_a^b f(x)dx - \int_a^b g(x)dx \right| \\
&= \sup_{x\in[a,b]} \left| \int_a^b f(x) - g(x)dx \right| \\
&= \sup_{x\in[a,b]} \int_a^b |\, f(x) - g(x)\,| \, dx \\
&\leq \int_a^b \sup_{x\in[a,b]} |\, f(x) - g(x)\,| \, dx \\
&\leq \int_a^b d(f,g)dx \\
&\leq d(f,g)[b-a]
\end{aligned}$$

Thus when $\epsilon > 0$ choose $\delta > [b-a]d(f,g)$. Hence, I is continuous.

VI. **Proposition 2.3.2** (Maximum principle). *Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a contnuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{\max} \in X$, and also attains its minimum at some point $x_{\min} \in X$. .*
Prove Propostion 2.3.2 in the text, in two different ways.:

a) As a consequence of Theorem 2.3.1 in text.
Let $f : X \to \mathbb{R}$ be a continuous function on a compact set X. Then, by 2.3.1, $f(X)$ is a compact set. Every compact set in $\mathbb{R}$ is an interval. Let $\langle\, a, b\,\rangle$ be that interval, that is, $f : X \to \langle\, a, b\,\rangle$. If f were unbounded, then there would exist an $x \in X$ such that $f(x) \notin \langle\, a, b\,\rangle$ which cannot happen. Therefore, there must exists values in the domain $x_{\min}$ and $x_{\max}$ which are the maximum and minimum values of f, namely, a, b, respectively.

b) Directly, using the sequential definition of compactness.
Let $(x_n) \in X$ be any sequence in the compact space X. Being compact, (x_n) must converge and $\lim_{n\to\infty} x_n = x \in X$. Let $f : X \to \mathbb{R}$ be a continuous function. x_n converges implies that $f(x_n)$ also converges. Therefore, $\lim_{n\to\infty} f(x_n) = f(x)$ and is finite (otherwise f would not be continuous). Therefore, there exists an upper and lower bound of f. Let L be the lower bound and $(y_n) \in f(X)$ be a sequence such that $\lim_{n\to\infty} y_n = L$. Then, let z_i be such that $f(z_i) = y_i$ for all i. Then, we have a sequence $(z_n) \in X$ which must converge. Thus, $\lim_{n\to\infty} f(z_n) = L$ and $\lim_{n\to\infty} z_n = x_{\min}$.
Similarly, for $x_{\max}$.

VII. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\lim_{|x|\to\infty} f(x) = +\infty$$

Prove that f attains its minimum.

Recall that by definition, the limit in (??) means that Given $A > 0$, there is $R > 0$ such that

$$f(x) > A \text{ for all } x \notin B_R$$

in other words, $f(x) > A$ whenever $|x| \geq R$. Here, $|x| = d_2(x, 0)$ and d_2 is the standard Euclidean distance defined in Example 1.4.

Given any $A > 0$ there exists an $R > 0$ such that $f(x) > A$ whenever $|x| > R$. Therefore, $f(x) \leq A$ whenever $|x| < R$. $f(x)$ is bounded on B_R. Hence there exists an interval $\langle\, a, b\,\rangle \in \mathbb{R}$ such that $F(B_R) \subset \langle\, a, b\,\rangle$. Therefore $f(x)$ is continuous on an interval, i.e., a compact set, and assumes a greatest and least value for some $x_{\min}, x_{\max} \in B_R$.