Math 5411 – Mathematical Statistics I– Fall 2024

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Homework Short #3 – October 7, 2024

Chapter 4: 2, 5, 6 (a,c,d), 14

2. If X is a discrete uniform random variable – that is, $P(X = k) = 1/n$ for $k = 1, 2, \ldots, n$–find $E(X)$ and $\mathrm{Var}(X)$.

$$\begin{aligned}
E(X) &= \sum_{k=1}^{n} kp(X = k) \\
&= \sum_{k=1}^{n} k(1/k) \\
&= \sum_{k=1}^{n} 1 \\
\mu &= n
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E[(x-\mu)^2] \\
&= \sum_{k=1}^{n} (k-\mu)^2 p(X = k) \\
&= \sum_{k=1}^{n} (k-n)^2 (1/k) \\
&= \sum_{k=1}^{n} \frac{k^2 - 2kn + n^2}{k} \\
&= \sum_{k=1}^{n} k - 2n + \frac{n^2}{k} \\
&= \frac{n(n+1)}{2} - 2n^2 + n^2 \sum_{k=1}^{n} 1/k
\end{aligned}$$

5. Let X have the density

$$f(x) = \frac{1+\alpha x}{2},\ -1 \leq x \leq 1,\ -1 \leq \alpha \leq 1$$

Find $E(X)$ and $\mathrm{Var}(X)$.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_{-1}^{1} x \frac{1+\alpha x}{2} dx \\
&= \int_{-1}^{1} \frac{x+\alpha x^2}{2} dx \\
&= \frac{1}{2} \int_{-1}^{1} x + \alpha x^2 dx \\
&= \frac{1}{2} \left[\frac{x^2}{2} + \frac{\alpha x^3}{3} \right]_{-1}^{1} \\
&= \frac{1}{2} \left(\left(\frac{1}{2} + \frac{\alpha}{3} \right) - \left(\frac{1}{2} + \frac{-\alpha}{3} \right) \right) \\
&= \frac{1}{2} \left(\frac{\alpha}{3} + \frac{\alpha}{3} \right) \\
&= \frac{\alpha}{3}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E[(x-\mu)^2] \\
&= \int_{-1}^{1} (x-\mu)^2 f(x) dx \\
&= \int_{-1}^{1} \left(x - \frac{\alpha}{3} \right)^2 \frac{1+\alpha x}{2} dx \\
&= \int_{-1}^{1} \left(x^2 - \frac{2\alpha x}{3} + \frac{\alpha^2}{9} \right) \frac{1+\alpha x}{2} dx \\
&= \frac{1}{2} \int_{-1}^{1} x^2 - \frac{2\alpha x}{3} + \frac{\alpha^2}{9} + \alpha x^3 - \frac{2\alpha^2 x^2}{3} + \frac{\alpha^3 x}{9} dx
\end{aligned}$$

remove the odd numbered exponents as they are symmetric around the origin

$$\begin{aligned}
\mathrm{Var}(X) &= \frac{1}{2} \int_{-1}^{1} x^2 + \frac{\alpha^2}{9} - \frac{2\alpha^2 x^2}{3} dx \\
&= \frac{1}{2} \int_{-1}^{1} \frac{\alpha^2}{9} - \frac{3+2\alpha^2}{3} x^2 dx \\
&= \frac{1}{2} \left[\frac{\alpha^2}{9} x - (3+2\alpha^2) x^3 \right]_{-1}^{1} \\
&= \frac{\alpha^2}{9} - 3 - 2\alpha^2 \\
&= \frac{19}{9} \alpha^2 - 3
\end{aligned}$$

6. Let X be a continuous random variable with probability density function $f(x) = 2x, 0 \leq x \leq 1$.

(a) Find $E(X)$.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 x(2x) dx \\
&= \int_0^1 2x^2 \\
&= \left[\frac{2x^3}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

(b) Use Theorem A in Section 4.1.1 to find $E(X^2)$ and compare to your answer part(b).

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
&= \int_0^1 x^2(2x) dx \\
&= \int_0^1 2x^3 \\
&= \left[\frac{x^4}{2} \right]_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(c) Find $\text{Var}(X)$ according to the definition of variance given in Section 4.2. Also, find $\text{Var}(X)$ by using Theorem B of Section 4.2.

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - \mu^2 \\
&= \frac{1}{2} - \left(\frac{2}{3} \right)^2 \\
&= \frac{1}{2} - \frac{4}{9} \\
&= \frac{5}{18}
\end{aligned}$$

14. Let X be a continuous random variable with the density function

$$f(x) = 2x,\ 0 \leq x \leq 1$$

(a) Find $E(X)$.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
&= \int_0^1 x(2x) dx \\
&= \int_0^1 2x^2 \\
&= \left[\frac{2x^3}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

(b) Find $E(X^2)$ and $\text{Var}(X)$.

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \int_0^1 x^2(2x)dx \\ &= \int_0^1 2x^3 \\ &= \left[\frac{x^4}{2}\right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{5}{18} \end{aligned}$$