Functional Analysis– Spring 2024

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4. Let p be defined on a vector space X and satisfy (1) and (2). Show that for any given $x_0 \in X$ there is a linear functional $\tilde{f}$ on X such that $\tilde{f}(x_0) = p(x_0)$ and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

   Let $f \in X'$ and $f(x_0) = p(x_0)$. Clearly, f is defined on the subspace spanned by x_0, that is, f is linear and $f(\alpha x_0) = \alpha f(x_0)$. The Hahn-Banach Theorem says that there exists an extension of f, namely, $\tilde{f} \in X'$ such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

7. Give another proof of Theorem 4.3-3 in the case of a Hilbert space.

   Theorem 4.3-3a: (Bounded linear functionals, Hilbert). Let X be a Hibert space and let $x_0 \neq 0$ be any element in X. Then there exists a bounded linear functional $\tilde{f}$ on X such that

$$\left\|\tilde{f}\right\| = 1, \qquad \tilde{f}(x_0) = \|x_0\|$$

   Proof: Let $x_0 \in X$, then Z is the subspace spanned by x_0. Any Cauchy sequence in Z will converge because that same sequence is in X which is complete. Hence. Z is also complete. We know that, for any $f_g \in Z'$ there exists a $g \in Z$ such that $f_g(x) = \langle x, g \rangle$ and $\|f_g\| = 1$. By Hahn-Bannach, there exists an extension $\tilde{f} \in X'$ such that $\left\|\tilde{f}\right\| = 1$ and $|f(x_0)| = \left\|\tilde{f}\right\| \|x_0\| = \|x\|$.

8. Let X be a normed space and X' its dual space. If $X \neq \{0\}$, show that X' cannot be $\{0\}$.

   Let $f(x) = \|x\|$, this is linear by definition. Therefore, $f \in X'$. We can see that when $x \neq 0$ that $f(x) \neq 0$. Therefore f is not the zero function and $X' \neq \{0\}$.

9. Show that for a separable normed space X, theorem 4.3-2 can be proved directly, without the use of Zorn's Lemma (which was used indirectly, namely, in the proof of Theorem 4.2-1).

   We still need a function p defined over all of X. We can still use the p defined in the proof, that is

$$p(x) = \|f\|_Z \, \|x\|$$

   and we know that it satisfies conditiosn (1) and (2) as well by

$$\begin{aligned} p(x+y) &= \|f\|_Z \, \|x+y\| \leq \|f\|_Z \, (\, \|x\| + \|y\| \,) = p(x) + p(y) \\ p(\alpha x) &= \|f\|_Z \, \|\alpha x\| = |\alpha| \, \|f\|_Z \, \|x\| = \alpha \, \|x\| \end{aligned}$$

   What we need now is an $\tilde{f}$ which is a maximal function such that $\tilde{f} \leq p(x)$ for all $x \in X$. Let f_1 be a linear extension of f over $\mathcal{D}(f_1)$. That is $f_1(x) = f(x)$ for all $x \in Z$. We know that f_1 exists because at the very least $f_1 = f$ and $\mathcal{D}(f_1) = Z$. When $x_1 \in X \backslash Z$ we now have $Z \subset \mathcal{D}(f_1)$. We can repeat this for f_2 giving us $x_2, \mathcal{D}(f_2)$, and so on. For any number n we have a set (f_i) such that $f(x) \leq f_i(x) \leq p(x)$ for all $x \in \bigcup_{i=1}^{n} \mathcal{D}(f_i)$. Since, X is dense we know that there exists an f_{n+1} and x_{n+1}, thus $\bigcup_{i=1}^{\infty} \mathcal{D}(f_i) = X$. Let $f_i \to \tilde{f}$. Thus $\tilde{f}(x) = f(x)$ when $x \in Z$ and $\tilde{f}(x) \leq p(x)$ when $x \in X$. Further, each f_i is an extension of f we know that $\|f\|_Z = \|f_i\|_{\mathcal{D}(f_i)}$ thus $\|f\|_Z = \left\|\tilde{f}\right\|_X$. □

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10. **(Space c_0)** Let $y = (\eta_j), \eta_j \in \mathbb{C}$, be such that $\sum \xi_j \eta_j$ converges for every $x = (\xi_j) \in c_0$, where $c_0 \in l^\infty$ is the subspace of all complex sequences converge to zero. Show that $\sum |\eta_j| < \infty$. (Use 4.7-3)

    Let $T : c_0 \to \mathbb{C}$ such that $T_n(x) = \sum_{i=1}^{n} \xi_i \eta_i$. $(T_n(x))$ is bounded, thus, (T_n) is bounded. By the Unified Boundedness Theorem there exists $c > 0$ such that $\|T_n\| < c$ for all $n \in \mathbb{N}$. For each n there is a sequence $x_n = (\mu_1, \mu_2, \ldots, \mu_n, \ldots)$ where μ_i are on the unit sphere, that is $\|\mu_i\| = 1$ and $\mu_i \eta_i = \eta_i$. Since $\|T_n\| \leq c$ for all $n \in N$, we have $T_n(x_n) = \sum_{i=1}^{n} \|\mu_i \eta_i\| = \sum_{i=1}^{n} \|\eta_i\| < c$. Since this is true for all n, y is convergent.

11. Let X be a Banach space, Y a normed space and $T_n \in B(X, Y)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. Show that $(\|T_n\|)$ is bounded.

    For any $j \in \mathbb{N}$ we know that T_j is Cauchy in Y. Therefore, given any convergent sequence $(x_k) \in X$ we know that $(T_j x_k)$ converges, there must be some number $y_j \in Y$ such $\|T_j x_k\| = \|T_j x_k\| \|x_k\| \leq \|T_j\| \|x_k\|$ for all x_k. That is, there exists c_j such that $\|T_j\| \leq c_j$. By Uniform Boundedness Thoeorem, there exists a c such that $\|T_n\| \leq c$ for all $n \in \mathbb{N}$, thus $(\|T_n\|)$ is bounded.

13. If (x_n) in a Banach space X is such that $(f(x_n))$ is bounded for all $f \in X'$, show that $(\|x_n\|)$ is bounded.

    Let $g : X \to X''$ be such that $g(x)f = f(x)$ and $f \in X'$. Then $|g(x)f| \leq |f(x)| \leq \|f\| \|x\|$ and hence bounded. Further, $(g(x_n)f)$ is bounded because $(f(x_n))$ is bounded for all $f \in X'$. X is complete therefore, $(|g(x_n)f|)$ is bounded implies that $(\|x_n\|)$ is bounded.

14. if X and Y are Banach spaces and $T_n \in B(X, Y), n = 1, 2, \cdots$, show that equivalent statements are:

    (a) $(\|T_n\|)$ is bounded.

    (b) $(\|T_n x\|)$ is bounded for all $x \in X$.

    (c) $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$.

    Let's show that $B(X, Y)$ is complete. Let $T_n \in B(X, Y)$ be a Cauchy sequence. For each $x \in X$, we have

    $$\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\|,$$

    which shows that $(T_n x)$ is Cauchy in Y. Since Y is complete, there is a $y \in Y$ such that $T_n x \to y$. We can see that $T : X \to Y$ where $T_n x = y$ forms a linear map. For any $\epsilon > 0$, let N_ϵ be such that $\|T_n - T_m\| < \epsilon/2$ for all $n, m \geq N_\epsilon$. Whenever, $n \geq N_\epsilon$, for each $x \in X$, there is an $m_x \geq N_\epsilon$ such that $\|T_{m_x} x - Tx\| \leq \epsilon/2$. If $\|x\| = 1$ we have

    $$\|T_n x - Tx\| \leq \|T_n x - T_{m_x} x\| + \|T_{m_x} x - Tx\| \leq \epsilon$$

    If follows that as $n \geq N_\epsilon$, then

    $$\|Tx\| \leq \|T_n x\| + \|Tx - T_n x\| \leq \|T_n\| + \epsilon$$

    for all x with $\|x\| = 1$, so T is bounded. If follows that $\lim_{n \to \infty} \|T_n - T\| = 0$. Therefore, $T_n \to T$ and $B(X, Y)$ is complete.

    (a) $Tn \to T$ thus $\|T_n\| \to \|T\|$ thus $(\|T_n\|)$ is bounded.

    (b) $\|T_n x\| \leq \|T_n\| \|x\|$ for all $x \in X$ and $\|T_n\|$ is bounded thus $(\|T_n x\|)$ is bounded.

    (c) Y is a Banach space Y' is also a Banach space. Thus, given any $g \in Y'$ and any convergent sequence $x_n \in X$ then $(|g(x_n)|)$ converges. Let $x_n = T_n x_0$ for some fixed $x_0 \in X$ and we can see that $(|g(T_n x_0)|)$ converges. Since x_0 and g are arbitrary we see that $(|g(T_n x)|)$ converges for all $x \in X$ and $g \in Y'$.