
Math XXXX – Independent Study: Differential Geometry, Lie Groups–
Summer 2025
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Basic Category Theory – Tom Leinster– August, 2025

Chapter 1

Foundations

Although Lie Theory is written in the abstract and most general terms, the most common examples that I've seen are with the general linear group and its associated subgroups. That is GL, SL, SO, SU, O, U. Understanding these groups as examples for illustrating the fundamentals of Lie groups and Lie algebras.

1.1 Rotation Matrices

The following notes are taken from the video “How to rotate in higher dimensions? Complex dimensions?”¹

Helpful notation: As you may know, the transpose of a *real* vector/matrix is denoted v^T . The transpose of a *complex* vector/matrix is denoted v^\dagger .

$$v^\dagger = \overline{v}^T$$

or the complex conjugate of the transpose.

SO(n), Special Orthogonal Matrices – Properties of Rotation: (R is a square matrix of degree n , U is the complex version of R). Also SU(n) Special Orthogonal Unitary (complex) matrices.

1. Rotation is linear

Real Rotate(v) = Rv .

Complex Rotate(v) = Uv . (no significant difference)

2. Preserves lengths and angles, that is, preserving the Inner Product²

Real $v \cdot v = (Rv)^T(Rv) = v^T R^T R v$. Hence, $R^T R = I$ or $R \in O(n)$.

Complex $v \cdot v = (Uv)^\dagger(Uv) = v^\dagger U^\dagger U v$. Hence, $U^\dagger U = I$ or $U \in U(n)$, or unitary matrices.

3. Preserves orientation (no flipping)

Real $\det R = 1$

Complex $\det U = 1$ implies SU(n) as the complex version of SO(n)

Note: that $R^T R$ and $U^\dagger U$ constitute quadratic matrix forms that are difficult to generalize. These four, however, form a Lie Group (or is it that each of these IS a Lie group?).

O(n)	SO(n)
U(n)	SU(n)

¹<https://www.youtube.com/watch?v=erA0jb9dSm0>

²in Abstract Algebra terminology, “the Inner Product is invariant under rotation”

Lie group		Lie algebra
$g \cdot h$ satisfies:		$[X, Y]$ satisfies:
1) <i>Closure</i>	→	1) <i>Closure</i> for all $X, Y \in \mathfrak{g}, [X, Y] \in \mathfrak{g}$
2) <i>Associativity</i>	→	2) <i>Bilinearity</i> $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ $[X, aY + bZ] = a[X, Y] + b[X, Z]$
3) <i>Identity</i>	→	3) <i>Alternativity</i> $[X, X] = 0$
4) <i>Inverse</i>	→	4) <i>Jacobi identity</i> $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Remark 1.1.1. .

Many sources have pointed out that Lie Algebras are easier to manage/manipulate than Lie Groups, which makes them the focus of study. One particular analogy was that the Lie Group is like a globe of the Earth while the Lie Algebra is the flat map that we use to represent it.

Remark 1.1.2 (MathGTP: What are the left-invariant vector spaces of \mathbb{R}^2 ?). .

- Step 1: Understand the concept of left-invariant vector fields A left-invariant vector field on a Lie group G is a vector field X such that for every $g, h \in G$, the following holds:

$$(L_g)_* X_h = X_{gh}$$

where L_g is the left translation by g and $(L_g)_*$ is its differential.

- Step 2: Identify the Lie group and its Lie algebra Here, $G = \mathbb{R}^2$, which is an abelian Lie group under addition. The Lie algebra \mathfrak{g} of \mathbb{R}^2 is also \mathbb{R}^2 , with the trivial Lie bracket $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.
- Step 3: Determine the left-invariant vector fields For \mathbb{R}^2 , left translation L_g is simply addition: $L_g(h) = g + h$. The differential $(L_g)_*$ is the identity map. Thus, a vector field X is left-invariant if:

$$X_{g+h} = X_h$$

This implies X is constant, i.e., $X = (a, b)$ for some $a, b \in \mathbb{R}$. Therefore, the left-invariant vector fields on \mathbb{R}^2 are the constant vector fields.

- Step 4: Describe the space of left-invariant vector fields The space of left-invariant vector fields on \mathbb{R}^2 is isomorphic to \mathbb{R}^2 itself, spanned by the constant basis vector fields ∂_x and ∂_y .
- Final Answer The left-invariant vector fields on \mathbb{R}^2 are the constant vector fields, and the space of left-invariant vector fields is isomorphic to \mathbb{R}^2 .

Remark 1.1.3 (MathGPT: What are the left-invariant vector fields of $GL(n, \mathbb{R})$?*)* .

- **Step 1: Understanding $GL(n, \mathbb{R})$ and Left-Invariant Vector Fields** $GL(n, \mathbb{R})$ is the general linear group of $n \times n$ invertible matrices with real entries, and it is a Lie group. A vector field X on a Lie group G is left-invariant if for every $g \in G$, the pushforward of X by the left translation L_g is equal to X . This means $(L_g)_*X = X$.
- **Step 2: Identifying the Lie Algebra** The space of left-invariant vector fields on a Lie group G is isomorphic to its Lie algebra \mathfrak{g} , which is the tangent space at the identity element, $T_e G$. For $GL(n, \mathbb{R})$, the Lie algebra, denoted as $\mathfrak{gl}(n, \mathbb{R})$, is the space of all $n \times n$ real matrices, $M(n, \mathbb{R})$.
- **Step 3: Associating Left-Invariant Vector Fields** For any matrix $A \in \mathfrak{gl}(n, \mathbb{R})$, we can associate a left-invariant vector field X_A . For any $g \in GL(n, \mathbb{R})$, the value of the vector field X_A at g is given by $X_A(g) = (L_g)_*(A)$.
- **Step 4: Calculating the Pushforward** To understand $(L_g)_*(A)$, consider a curve $\gamma(t)$ in $GL(n, \mathbb{R})$ such that $\gamma(0) = e$ and $\gamma'(0) = A$. The pushforward $(L_g)_*(A)$ is the tangent vector to the curve $L_g(\gamma(t)) = g\gamma(t)$ at $t = 0$. By computing the derivative, we find that the derivative at $t = 0$ is gA .
- **Step 5: Form of Left-Invariant Vector Fields** Therefore, the left-invariant vector field X_A associated with $A \in \mathfrak{gl}(n, \mathbb{R})$ is given by $X_A(g) = gA$. The set of all left-invariant vector fields on $GL(n, \mathbb{R})$ is $\{X_A \mid A \in M(n, \mathbb{R})\}$, where $X_A(g) = gA$.
- **Step 6: Lie Algebra Structure** These vector fields form a Lie algebra under the Lie bracket operation. If X_A and X_B are two left-invariant vector fields, then their Lie bracket $[X_A, X_B]$ is also a left-invariant vector field, corresponding to the matrix $[A, B] = AB - BA$. Thus, $[X_A, X_B] = X_{[A, B]}$.
- **Final Answer** The left-invariant vector fields of $GL(n, \mathbb{R})$ are of the form X_A , where A is an $n \times n$ real matrix, and $X_A(g) = gA$ for any $g \in GL(n, \mathbb{R})$. These vector fields span the Lie algebra of $GL(n, \mathbb{R})$, which is the space of all $n \times n$ real matrices.