Math XXXX – Independent Study: Manifolds– Summer 2025

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An Introduction to Lie Algebras– August, 2025

Chapter 1

Introduction

Definition 1.0.1 (Lie Bracket)**.** We define the Lie Bracket, $[\cdot,\cdot]$ as a bilinear operation

$$[\cdot,\cdot] : L \times L \to L$$

with the following properties

$$[x, x] = 0 \tag{L1}$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{L2}$$

1.1 Exercises

1.1 (Pg 2.)

(a) Show that $[v, 0] = 0 = [0, v]$ for all $v \in L$.

$$\begin{aligned} [v, v] &= 0 \\ [v, v] - [v, 0] &= 0 - [v, 0] \\ [v - v, v - 0] &= [0, v] \\ [0, v] &= [v, 0] \end{aligned}$$

but $[0, v] = -[v, 0]$ for all v therefore $[0, v] = 0$.

(b) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent on F.
Want to show that $ax + by = 0$ implies that $a, b = 0$.

$$\begin{aligned} \text{Let } ax + by = 0& \\ by = -ax &\implies y = cx, \text{ for some } c \\ [x, y] = [x, cx] &= c[x, x] = 0 \end{aligned}$$

but $[x, y] \neq 0$ therefore $c = 0$ and x, y are linearly independent.

1.2 (Pg 2.) Convince yourself that $\wedge$ is bilinear. Then check that the Jacobi Identity holds. *Hint:* if $x \cdot y$ denotes the dot product of $x, y \in \mathbb{R}^3$, then

$$x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z, \forall x, y, z \in \mathbb{R}^3.$$

***wedge* is bilinear.**
Given $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ we have

$$\begin{aligned} x \wedge y &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \\ (x + (0, b, 0)) \wedge y &= ((x_2 + b)y_3 - (x_3 + 0)y_2, (x_3 + 0)y_1 - (x_1 + 0)y_3, (x_1 + 0)y_2 - (x_2 + b)y_1) \\ &= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) + (by_3, 0, -by_1) \\ &= x \wedge y + (0, b, 0) \wedge y \end{aligned}$$

Therefore additive on the left for the middle coordinate. Each argument is independent of coordinate so is true for $(a, 0, 0)$ and $(0, 0, c)$ and can be easily seen when used on the write (e.g., $x \wedge (y + (0, b, 0))$.

The Jacobi Identity

Want to show

$$x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (y \wedge x) = 0 \tag{1.1}$$

from the hint

$$x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z$$

and from (1)

$$\begin{aligned} x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (y \wedge x) &= (x \cdot z)y - (x \cdot y)z \\ &+ (y \cdot x)z - (y \cdot z)x \\ &+ (z \cdot y)x - (z \cdot x)y \\ &= (\,(x \cdot z) - (z \cdot x)\,)\,y \\ &+ (\,-(x \cdot y) + (y \cdot x)\,)\,z \\ &+ (\,-(y \cdot z) + (z \cdot y)\,)\,x \\ &= 0 \end{aligned}$$

1.3 (Pg 2.) Supoose that V is a finite-dimensional vector space over F. Write $\mathrm{gl}(V)$ for the set of all linear maps from V to V. This is again a vector space over F, and it becomes a Lie algebra, known as the *general linear algebra*, if we define the Lie bracket $[--, --]$ by

$$[x, y] := x \circ y - y \circ x, \ \forall x, y \in \mathrm{gl}(V),$$

where $\circ$ deontes the composition of maps. Check that the Jacobi Identity holds.

Given $R, S, T \in \mathrm{gl}(V)$ there exists matrix $A, B, C \in \mathcal{M}_{n\times n}(F)$ where $n = \dim V$ and $Rx = Ax$, $Sx = Bx$, $Tx = Cx$, $\forall x \in V$. Further remember that $R \circ S = AB$ (similar for the other two transormations) for all $x \in v$. Then

$$\begin{aligned} [R, [S, T]] + [S, [T, R]] + [T, [R, S]] &= (R \circ (S \circ T - T \circ S) - (S \circ T - T \circ S) \circ R) \\ &+ (S \circ (T \circ R - R \circ T) - (T \circ R - R \circ T) \circ S) \\ &+ (T \circ (R \circ S - S \circ R) - (R \circ S - S \circ R) \circ T) \\ &= (A(BC - CB) - (BC - CB)A) \\ &+ (B(CA - AC) - (CA - AC)B) \\ &+ (C(AB - BC) - (AB - BA)C) \end{aligned}$$

by rearranging the terms we can see that they all cancel out. Most notably this is done *without commuting*. It is important to remember that, in general, $R \circ S \neq S \circ R$.

1.4 Let $b(n, F)$ be the upper triangular matrices in $\mathrm{gl}(n, F)$. (A matrix x is said to be upper triangular if $x_{ij} = 0$ whnever $i > j$.) This is a Lie algebra with the same Lie bracket as $\mathrm{gl}(n, F)$.

Similarly, let $n(n, F)$ be the strictly upper triangular matrices in $\mathrm{gl}(n, F)$. (A matrix x i said to be strictly upper triangular if $x_{ij} = 0$ whenever $i \geq j$.) Again this is a Lie algebra with teh same Lie bracket as $\mathrm{gl}(n, F)$.

Verify these assertions.

Let $b(n,F) = \{A \in \mathrm{gl}(n,F) \mid A = [x_{ij}],\ i > j \to x_{ij} = 0\}$. Define

$$[x,y] := x \circ y - y \circ x,\ \forall x,y \in b(n,F),$$

The only question that needs to be answered is ... Given $S,T \in (n,F)$ is $S \circ T \in b(n,F)$. Let $A,B \in \mathcal{M}_{n\times n}(F)$ and $T(x) = Ax, S(x) = Bx, \forall x \in F$. Then $(T \circ S)(x) = ABx$. Is $AB \in b(n,F)$.

$$\text{Let } A = [a_{ij}] \text{ and } B = [b_ij]$$
$$AB = \left[x_{ij} = \sum_{k=1}^{n} a_{ik} b_{k_j} \right]$$

If $i > j$ then x_{ij}

1.5 (Pg 4) Find $Z(L)$ when $L = \mathrm{sl}(2,F)$. You should find the answer depends on the characteristic of F.

Let $\mathrm{sl}(n,F)$ be the subspace of $GL(n,F)$ consisting of all matrices whose trace is zero, i.e., $\mathrm{sl}(n,F) = \left\{ A \in \mathcal{M}_{n\times n}(F) \;\middle|\; \sum_{i=1}^{n} a_{ii} = 0 \right\}$. This is known as *Special Linear Algebra* on square matrices.

When is $\sum_{i=1}^{n} a_{ii} = 0$ **for all** $a_{ii} \in F$**? OR** $a_{11} + a_{22} = 0$?.

Notice, for example, that on the discrete field $F = \mathbb{Z}/\mathbb{Z}5$, $2 + 3 = 0$. Thus, when $L = \mathrm{sl}(2, \mathbb{Z}/\mathbb{Z}p)$ where p is prime, $Z(L)$ will have elements where $a_{11} + a_{22} = p$.

1.6 (Pg 5.) Show that if $\varphi : L_1 \to L_2$ is a homormorphism, then the kernel of φ, $\ker\varphi$, is an ideal of L_1, and the image of φ, $\mathrm{im}\,\varphi$, is a Lie subalgebra of L_2.

Show that the kernel is an ideal. Let $h,k \in \ker\varphi$ such that $h \neq k$. Then $\varphi(k) = \varphi(h) = 0$.

$$\varphi(a-b) = \varphi(a) - \varphi(b) = 0$$
$$\therefore a - b \in \ker\varphi$$

which makes it a group under addition. Now we need to show that it is closed under multiplication, that is, $ra \in \ker\varphi$ for all $r \in L$. Let $r \in L$ then

$$\varphi(ra) = \varphi(r)\varphi(a) = 0$$
$$\therefore ra \in \ker\varphi$$

Show that the image is a subalgebra. We need to show three things:

Closed under addition (group condition).
Let $u,v \in \mathrm{im}\,\varphi$ then there exists $x,y \in L_1$ such that $\varphi(x) = u, \varphi(y) = v$.
Then $\varphi(x+y) = \varphi(x) + \varphi(y) = u + v \in \mathrm{im}\,\varphi$.
Therefore closed under addition.

closed under scalar multipication (ring condition).
Let $r,a \in \mathrm{im}\,\varphi$. Then there exists $x,y \in L_1$ such that $\varphi(x) = r, \varphi(y) = x$.
Then $\varphi(xy) = \varphi(x)\varphi(y) = ra \in \mathrm{im}\,\varphi$
Therefore closed under scalar multiplication.

closed under Lie bracket (subalgebra condition).
Let $u,v \in \mathrm{im}\,\varphi$ then there exists $x,y \in L_1$ such that $\varphi(x) = u, \varphi(y) = v$.

Then

$$\begin{aligned}
\varphi([x+y,x+y]) &= \varphi\,(\,[x,x]+[x,y]+[y,x]+[y,y]\,)\\
&= \varphi\,(\,[x,y]+[y,x]\,)\\
&= \varphi\,(\,[x,y]\,)+\varphi\,(\,[y,x]\,)\\
\varphi\,(\,[x,y]\,) &= -\varphi\,(\,[y,x]\,)\\
[\varphi(x+y),\varphi(x+y)] &= [\varphi(x)+\varphi(y),\varphi(x)+\varphi(y)]\\
&= [u+v,u+v]\\
&= [u,u]+[u,v]+[v,u]+[v,v]\\
&= [u,v]+[v,u]\\
[u,v] &= -\,[v,u]
\end{aligned}$$

therefore closed under Lie Bracket.

1.7 (Pg 6.) Let L be a Lie algebra. Show that the Lie bracket is associative, this is $[x,[y,z]] = [[x,y]\,,z]$ for all $x,y,z \in L$, if and only if for all $a,b \in L$ the commutator $[a,b]$ lies in $Z(L)$.

1.8 (Pg 6) Let D and E be derivations on algebra A.

(i) Show that $[D,E] = D \circ E - E \circ D$ is also a derivation.

$$\begin{aligned}
(D\circ E)(ab) &= D\,(\,aE(b)-E(a)b\,)\\
&= D(aE(b))-D(E(a)b)\\
&= aD(E(b))-D(a)E(b)-E(a)D(b)+D(E(a))b\\
&= aD(E(b))+D(E(a))b-D(a)E(b)-E(a)D(b)
\end{aligned}$$

We can switch D and E to computer $E \circ D$

$$(E\circ D)(ab) = aE(D(b))+E(D(a))b-E(a)D(b)-D(a)E(b)$$

taking the difference

$$\begin{aligned}
(D\circ E)(ab)-(E\circ D)(ab) &= aD(E(b))+D(E(a))b-D(a)E(b)-E(a)D(b)\\
&\quad -\,(\,aE(D(b))+E(D(a))b-E(a)D(b)-D(a)E(b)\,)
\end{aligned}$$

$$\begin{aligned}
[D,E]\,(ab) &= a\,[D,E]\,(b)-[D,E]\,(a)b\\
&= a(D\circ E)(b)-((D\circ E)(a))b-(\,a(E\circ D)(b)-(E\circ D)(a)b\,)\\
[D,E]\,(ab) &= (D\circ E)(ab)-(E\circ D)(ab)\\
&= D(E(ab))-E(D(ab))\\
&= D\,(\,aE(b)-E(a)b\,)-E\,(\,aD(b)-D(a)b\,)\\
&= D(aE(b))-D(E(a)b)-E(aD(b))+E(D(a)b)\\
&= aD(E(b))-E(b)D(a)\\
&\quad -E(a)D(b)+D(E(a))b\\
&\quad -aE(D(b))+E(a)D(b)\\
&\quad +D(a)E(b)-E(D(a))b\\
&= a(D(E(b))-E(D(b))-(E(b))D(a)
\end{aligned}$$

(ii) Show that $D \circ E$ need not be a derivation. (see example).

1.9 (Pg 7.) Let L_1 and L_2 be Lie algebras. Show that L_1 is isomorphic to L_2 if and only if there is a basis B_1 of L_1 and a basis B_2 of L_2 such that the structure constants of L_1 with respect to B_1 are equal to the structure constants of L_2 with respect to B_2.

($\Rightarrow$)Assuming that $L_1 \underset{\text{iso}}{\longrightarrow} L_2$. Define $f : L_1 \to L_2$ to be that isomorphism. Let $B_1 = (x_1, \ldots, x_n)$ be the basis vectors for L_1. Then,

$$\begin{aligned} f([x_i, x_j]) &= f\left(\sum_{k=1}^{n} a_{ij}^k x_k\right) \\ &= \sum_{k=1}^{n} a_{ij}^k f(x_k) \end{aligned} \tag{1.6}$$

since f is isomorphic, it is also injective and surjective. Thus, each $f(x_k)$ is unique. Further, given any $i, j \in [1, \ldots, n]$ we know that x_i, x_j are linearly independent. Thus,

$$0 = Ax_i + Bx_j \implies A = B = 0 \text{ and}$$
$$f(0) = 0 = f(Ax_i + Bx_j) = Af(x_i) + Bf(x_j)$$

therefore, $f(x_i), f(x_j)$ are linearly independent and thus, form a basis. From (1.6) we see that it has the same Structure Constants.

1.10 (Pg 7.) Let L be a Lie algebra with basis $(x_1, \ldots, x_n)$. What condition does the Jacobi identity impose on the structure constants a_{ij}^k?

We have three brackets for the Jacobi Identity that start with

$$\begin{aligned} [x_i, x_j] &= \sum_{k=1}^{n} a_{ij}^k x_k \\ [x_e, x_f] &= \sum_{k=1}^{n} a_{ef}^k x_k \\ [x_b, x_c] &= \sum_{k=1}^{n} a_{bc}^k x_k \\ [x_i, [x_e, x_f]] &= \left[x_i, \sum_{k=1}^{n} a_{ef}^k x_k\right] \\ &= \sum_{k=1}^{n} a_{ef}^k \, [x_i, x_k] \\ &= \sum_{k=1}^{n} a_{ef}^k \sum_{l=1}^{n} a_{ik}^l x_l \end{aligned}$$

Since, the x_i are linearly independeent we can examining each element l independently that is

$$[x_i, [x_e, x_f]]_l = \sum_{k=1}^{n} a_{ef}^k a_{ik}^l x_l$$

cycling through the other terms of the Jacobi identity we get

$$\begin{aligned} [x_e, [x_f, x_i]]_l &= \sum_{k=1}^{n} a_{fi}^k a_{ek}^l x_l \\ [x_f, [x_i, x_e]]_l &= \sum_{k=1}^{n} a_{ei}^k a_{fk}^l x_l \end{aligned}$$

The Jacobi Identity means that the sum of the coefficiints of these terms must be zero that is

$$0 = \sum_{k=1}^{n} a^k_{ef} a^l_{ik} + \sum_{k=1}^{n} a^k_{fi} a^l_{ek} g + \sum_{k=1}^{n} a^k_{ei} a^l_{fk}$$

1.11 (Pg 8.) Let L_1 and L_2 be two abelian Lie algebras. Show that L_1 and L_2 are isomorphic if and only if they have the same dimension.

If L_1 and L_2 are abelian then since $[x,y] = -[y,x]$ then $[x,y] = 0$ for all $x,y,\in L_1$ or L_2. Consequently, these are vector spaces that are isomorphic to the each other and, hence, have the same dimension.

1.12 Find the structure constants of $\mathrm{sl}(2,F)$ with respect to the basis given by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie Bracket for $\mathrm{sl}(2,F)$ is $[X,Y] = XY - YX$. Thus,

$$\begin{aligned}
[e,f] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= h \\
[f,h] &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \\
&= 2f \\
[h,e] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \\
&= -2e
\end{aligned}$$

Thus,

$$\begin{aligned}
a^k_{ii} &= 0, \forall k = 1,2,3 \\
[e,f] &= a^1_{12}e + a^2_{12}f + a^3_{12}h = h \to a^3_{12} = 1 \\
[f,h] &= a^1_{23}e + a^2_{23}f + a^3_{23}h = 2f \to a^2_{23} = 2 \\
[h,e] &= a^1_{31}e + a^2_{31}f + a^3_{31}h = -2e \to a^1_{31} = -2
\end{aligned}$$

all else are zero.

1.13 Prove $\mathrm{sl}(2,\mathbb{C})$ has no non-trivial ideals.

1.14 Let L by the 3-dimensional *complex* Lie algebra with basis (x, y, z) and Lie bracket defined by

$$[x, y] = z,\ [y, z] = x,\ [z, x] = y$$

(Here L is the"complexification" of the 3-dimensional real Lie algebra $\mathbb{R}^3_\wedge$.)

(i) Show that L is isomorphic to the Lie subalgebra of $\mathrm{gl}(3, \mathbb{C})$ consistent for all 3×3 antisymmetric matrices with entries in $\mathbb{C}$.

Let $U = \{A = \mathrm{gl}(3, N) : A \text{ is an anti-symmetric matrix }\}$. Thus for any $A \in U$ there exists $a, b, c \in \mathbb{C}$ such that

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

which have three linearly independent elements

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Verify

$$\begin{aligned}
[x, y] &= xy - yx \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\
&= z
\end{aligned}$$

$$\begin{aligned}
[y, z] &= yz - zy \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= x
\end{aligned}$$

$$[z,x] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
$$= y$$

(ii) Find an explicit isomorphism $\mathrm{sl}(2,\mathbb{C}) \underset{\text{iso}}{\longrightarrow} L$.

1.15 Let S be an $n \times n$ matrix with entries in a field F. Define

$$\mathrm{gl}_S(n,F) = \{x \in \mathrm{gl}(n,F) \,:\, x^t S = -Sx\}.$$

(i) Show that $\mathrm{gl}_S(n,F)$ is a Lie subalgebra of $\mathfrak{gl}(n,F)$.

Additive Group

Let $x, y \in \mathrm{gl}_S(n,F)$, then

$$(x+y)^t S = x^t S + y^t S = -Sx - Sy = -S(x+y)$$

Multicative property.

Let $x \in \mathrm{gl}_S(n,F)$ then $x^t S = -Sx$ and $rx^t S = -Sxr$ for all $r, \in F$

Lie Bracket

Let $x, y \in \mathrm{gl}_S(n,F)$ then

$$\begin{aligned} [x,y] &= xy - yx \\ [x,y]^t S &= (xy - yx)^t S \\ &= (xy)^t S - (yx^t)S \\ &= y^t x^t S - x^t y^t S \\ &= -y^t Sx + x^t Sy \\ &= Syx - Sxy \\ &= S(yx - xy) \\ &= -S[x,y] \end{aligned}$$

(ii) Find $\mathrm{gl}_S(2,\mathbb{R})$ if $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let $x \in \mathrm{gl}_S(2,\mathbb{R})$ and

$$\begin{aligned} x &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ x^t S &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \\ Sx &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix} \\ 0 = x^t S + Sx &= \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} + \begin{pmatrix} b & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & a+d \\ 0 & b \end{pmatrix} \\ x &= \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \end{aligned}$$

(iii) Does there exist a matrix S such that $\mathrm{gl}_S(2,\mathbb{R})$ is equal to the set of all diagonal matrices in $\mathrm{gl}(2,\mathbb{R})$.

Let $A \in \mathrm{gl}(2,\mathbb{R})$ be a diagonal matrix.

$$\text{Let } A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
$$\text{Let } S = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$$
$$A^tS + SA = AS + SA \to AS = -SA$$
$$au = -ua \text{ and } bz = -zb$$

No, no such S exists.

(iv) Find a matrix S such that $\mathrm{gl}_S(3,\mathbb{R})$ is isomorphic to the Lie algebra $\mathbb{R}^3_\wedge$ defined in §1.2, Example 1.

Hint: Part (i) of Exercise 1.14 is relevant.

Let x, y, z be a basis of $\mathbb{R}^3$. We want to find $\phi : \mathbb{R}^3 \to \mathbb{R}^3_\wedge$.

Let $X, Y \in \mathrm{gl}_S(3,\mathbb{R})$ and $\phi : \mathrm{gl}_S(3,\mathbb{R}) \to \mathbb{R}^3_\wedge$ such that

$$\phi([X,Y]) = [\phi(X),\phi(Y)] = \phi(X) \wedge \phi(Y)$$
$$\phi(XY - YX) = \phi(X) \wedge \phi(Y)$$

Notice that

$$\begin{aligned} (XY)^tS &= Y^tX^tS = -Y^tSX = SYX \\ \text{and } [X,Y]^tS &= (XY - YS)^tS \\ &= (XY)^tS - (YX)^tS \\ &= SYX - SXY \\ &= S(YX - XY) \\ &= -S[X,Y] \end{aligned}$$

$$\phi(X^tS) = \phi(-SX) = -\phi(S)\phi(X)$$

1.16 Show, by giving an example, that if F is a field of characteristic 2, there are algebras over F which statisfy (L1') and (L2) but are not Lie algebras.

1.17 Let V be an n-dimensional complex vector space and let $L = \mathrm{gl}(V)$. Suppose that $x \in L$ is diagonalisable, with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that $\mathrm{ad}\, x \in \mathrm{gl}(L)$ is also diagonalisable and that its eigenvalues are $\lambda_i - \lambda_j$ for $1 \le i, j \le n$.

1.18 Let L be a Lie algebra. We saw in §1.6, Example 1.2(2) that the maps $\mathrm{ad}\, x : L \to L$ for $x \in L$ are derivations of L; these are known as *inner derivations*. Show that if $\mathrm{IDER}\, L$ is the set of inner derivations of L, then $\mathrm{IDER}\, L$ is an ideal of $\mathrm{DER}\, L$.

1.19 Let A be an algebra and let $\delta : A \to A$ be a derivation. Prove that δ satisfies the Leibniz rule

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y), \ \forall x, y \in A.$$

This resembles the binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

Consider an inductive proof and consider $\delta^0(x) = x$
Show true for $n = 1$.

$$\begin{aligned}\delta(xy) &= \binom{1}{0}\delta^0(x)\delta(y) + \binom{1}{1}\delta(x)\delta^0(y)\\ &= x\delta(y) + \delta(x)y\end{aligned}$$

which is the Liebniz rule.
Show true for $n+1$. Now, assuming that this is true for some number n, we must show that it is also true for $n+1$. Thus, starting with n we'll calculate $\delta(\delta^n(xy)) = \delta^{n+1}(xy)$.

$$\begin{aligned}\delta^n(xy) &= \sum_{r=0}^{n}\binom{n}{r}\delta^r(x)\delta^{n-r}(y),\ \forall x, y \in A.\\ \delta(\delta^n(xy)) &= \delta\left(\sum_{r=0}^{n}\binom{n}{r}\delta^r(x)\delta^{n-r}(y)\right)\\ &= \sum_{r=0}^{n}\binom{n}{r}\delta\left(\delta^r(x)\delta^{n-r}(y)\right) \qquad (*)\end{aligned}$$

Let us focus on the term in the summation

$$\begin{aligned}\delta\left(\delta^r(x)\delta^{n-r}(y)\right) &= \delta^r(x)\delta(\delta^{n-r}(y)) + \delta(\delta^r(x))\delta^{n-r}(y)\\ &= \delta^r(x)\delta^{n-r+1}(y) + \delta^{r+1}(x)\delta^{n-r}(y).\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{r=0}^{n}\binom{n}{r}\delta\left(\delta^r(x)\delta^{n-r}(y)\right) &= \sum_{r=0}^{n}\binom{n}{r}\left(\delta^r(x)\delta^{n-r+1}(y) + \delta^{r+1}(x)\delta^{n-r}(y)\right)\\ &= \sum_{r=0}^{n}\left(\binom{n}{r} + \binom{n}{r-1}\right)\delta^r(x)\delta^{n-r+1}(y)\end{aligned}$$

when $r = 0$ we have

$$\begin{aligned}r = 0 &\to x\delta^{n+1}(y) + \delta(x)\delta^n(y)\\ r = n &\to \delta^n(x)\delta(y) + \delta^{n+1}(x)y\end{aligned}$$

From combinatorics we have the identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

and we have

$$\begin{aligned}\delta^{n+1}(xy) &= x\delta^{n+1}(y) + \delta(x)\delta^n(y)\\ &+ \sum_{r=0}^{n}\binom{n+1}{r}\delta^r(x)\delta^{n-r+1}(y)\\ &+ \delta^n(x)\delta(y) + \delta^{n+1}(x)y\\ &= \sum_{r=0}^{n+1}\binom{n+1}{r}\delta^r(x)\delta^{n-r+1}(y)\end{aligned}$$

Thus, by Mathematical Induction, our assertion is true $\square$

Chapter 2

Ideals and Homomorphisms

Operations that work on Ideals

Addition: $I + J = \{x + y \,:\, x \in I, y \in J\}$ is an ideal.

Lie Bracket: $[I, J] = \mathrm{span}\{[x, y] \,|\, x \in I, y \in J\}$ is an ideal.

Quotient: $L/I = \{z + I \,:\, z \in L\}$ is a quotient algebra.

Notes:

Correspondence: $L \supset J \supset I$, where I, J are ideals of L. Then, J/I is an ideal of L/I.

Also, if K is an ideal of L/I and $J = \{z \in L : z + I \in K\}$ (i.e., J is the set of cosets of K in I) then J is an ideal of L and $J \supset I$.

2.1 Exercises

2.1 (Pg. 11) Show that $I + J$ is an ideal of L where

$$I + J = \{x + y \,:\, x \in I, y \in J\}.$$

Let $z \in L$ and $x, y \in I + J$ then there exists $x_I, y_I \in I$ and $x_J, y_J \in J$ such that $x = x_I + x_J$ and $y = y_I + y_J$ then from (L2) we have

$$[\underbrace{[y, x]}_{\in I+J}, z] = \underbrace{[x, [y, z]]}_{\in I} + \underbrace{[y, [z, x]]}_{\in J} \in I + J$$

2.2 (Pg. 12) Show that $\mathrm{sl}(2, \mathbb{C})' = \mathrm{sl}(2, \mathbb{C})$.

Let $L = \mathrm{sl}(2, \mathbb{C})$ and $X \in [L, L]$. Then, there exist $A, B \in L$ such that $[A, B] = X$ thus

$$X = [A, B] = AB - BA$$

$AB \in L$ and $BA \in L$ therefore $X \in L$.

2.3 (Pg. 13)

(i) Show that the Lie Bracket defined in L/I is bilinear and satisfies the axioms (L1) and (L2).

Define the Lie Bracket of two cosets as

$$[w + I, z + I] = [w, z] + I, , \forall w, z \in L$$

where the bracket on the right side is the Lie Bracket defined for L. Thus, let $a, b \in L$ then we have

$$\begin{aligned}[a + w + I, b + z + I] &= [a + w, b + z] + I \\ &= [a, b] + [a, z] + [w, b] + [w, z] + I\end{aligned}$$

the four Lie Brackets add up to a single element in L and is therefore true. Thus, this Lie Bracket is bilinear.

(ii) Show that the linear transformation $\pi : L \to L/I$ which takes an element $z \in L$ to its coset $z + I$ is a homomorphism of a Lie Algebras.

Need to show that

$$\pi([x, y]) = [\pi(x), \pi(y)]$$

I prefer to call elements of L/I equivalence classes. That is L/I is partitioned into equivalence classes (cosets) and its elements are these subsets. The proper notation for sucn and element would be $[x] \in L/I$ where x is a representative element of the equivalence class containing x. Thus $\pi(x) = [x] = \{x + I\}$.

$$\begin{aligned}\pi(x) &= [x] = \{x + I\} \\ [\pi(x), \pi(y)] &= [\,[x], [y]\,] \\ &= [\,\{\,x + I\,\}, \{\,y + I\,\}\,] \\ &= [x, y] + I \\ &= [\,[x, y]\,]\end{aligned}$$

or the equivalence class of the Lie Bracket of the left hand side.

2.4 (Pg. 14) Show that if L is a Lie Algebra then $L/Z(L)$ is isomorphic to a subalgebra of $\mathrm{gl}(L)$.

$Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$. Therefore, $[x] \in L/Z(L) = \{y \in L : y = x + z, z \in Z(L)\}$. $Z(L)$ is an ideal. Thus, $[x] = x + Z(L)$. Let $\varphi : L/Z(L) \to \mathrm{gl}(L)$ be a homomorphism. Then $x, y \in Z(L)$ implies that $\varphi([x, y]) = \ker\varphi$. From the first isomorphism theorem, $L/\ker\varphi = L/Z(L) \cong \mathrm{Im}\,\varphi$.

2.5 Show that if $z \in L'$ then $\mathrm{tr}\,\mathrm{ad}\,z = 0$.

The thing to remember is that every $z \in L'$ is a linear combination of Lie Brackets. Thus

$$\begin{aligned}z &= \sum_k [x_k, y_k] \\ \mathrm{tr}\,\mathrm{ad}\,z &= \sum_k \mathrm{tr}\,\mathrm{ad}([x_k, y_k]) \\ \text{or each } \ \mathrm{tr}\,\mathrm{ad}([x_k, y_k]) &= 0, \forall k\end{aligned}$$

That is,

$$\begin{aligned}\mathrm{ad}([x_k, y_k]) &= \mathrm{ad}\,x_k \circ \mathrm{ad}\,y_k - \mathrm{ad}\,y_k \circ \mathrm{ad}\,x_k = 0 \\ \therefore\ \mathrm{tr}\,\mathrm{ad}\,z &= 0\end{aligned}$$

2.6 Suppose L_1 and L_2 are Lie algebras. let $L := \{(x_1, x_2) : x_i \in L_i\}$ be the direct sum of their underlying vector spaces, e.g., $L = L_1 \oplus L_2$. Show that if we define

$$[(x_1, x_2), (y_1, y_2)] := (\,[x_1, y_1]\,, [x_2, y_2]\,)$$

then L becomes a Lie algebra, the *direct sum* of L_1 and L_2, $L = L_1 \oplus L_2$.

(i) Prove that $\mathrm{gl}(2,\mathbb{C})$ is isomorphic to the direct sum of $\mathrm{sl}(2,\mathbb{C})\oplus\mathbb{C}$, the 1-dimensional complex abelien Lie algebra.
Let $\varphi : \mathrm{gl}(2,\mathbb{C}) \to \mathrm{sl}(2,\mathbb{C}) \oplus \mathbb{C}$ be a surjective transformation. Then

$$\begin{aligned} \dim \mathrm{gl}(2,\mathbb{C}) &= \dim\ker\varphi + \dim\operatorname{range}\varphi \\ \dim\ker\varphi &= \dim \mathrm{gl}(2,\mathbb{C}) - \dim(\mathrm{sl}(2,\mathbb{C})\oplus\mathbb{C}) \\ &= n^2 - n^2 = 0 \end{aligned}$$

The dimension of the kernel of φ is 0 therefore φ is a bijection implying an isomorphsim.

(ii) Show that if $L = L_1 \oplus L_2$ then $Z(L) = Z(L_1)\oplus Z(L_2)$ and $L' = L_1' \oplus L_2'$. Formulate a general version for a direct sum $L_1 \oplus \cdots \oplus L_k$.
1: Show $Z(L) = Z(L_1) \oplus Z(L_2)$.
For any $u \in L$ there exists $u_1 \in L_1$ and $u_2 \in L_2$ such that $u = (u_1, u_2)$. If $z \in Z(L)$ then $[z, u] = 0$.

$$\begin{aligned} [z,u] &= [(z_1,z_2),(u_1,u_2)] \\ &= (\,[z_1,u_1],[z_2,u_2]\,) \\ \therefore [z_1,u_1] &= 0 \text{ and } [z_2,u_2] = 0 \end{aligned}$$

for any u. Thus, $z_1 \in Z(L_1)$ and $z_2 \in Z(L_2)$.
2: Show $L' = L_1' \oplus L_2'$.
Let $z \in L$ then there exists a linear combination of commutators $[x_k, y_k]$ equal to zero

$$z = \sum_k [x_k, y_k]$$

There exist $a_k, b_k \in L_1$ and $c_k, d_k \in L_2$ such that $x_k = (a_k, c_k)$ and $y_k = (b_k, d_k)$. then

$$\begin{aligned} z &= \sum_k [(a_k,c_k),(b_k,d_k)] \\ &= \sum_k ([a_k,b_k],[c_k,d_k]) \\ &= \left(\sum_k [a_k,b_k], \sum_k [c_k,d_k] \right) \\ &\in L_1 \oplus L_2 \end{aligned}$$

Thus

$$L = \bigoplus_k L_k \implies Z(L) = \bigoplus_k Z(L_k) \text{ and } L' = \bigoplus_k L_k'$$

(iii) Are the summands in the direct sum decomposition of a Lie Algebra uniquely determined? *Hint:* If you think that the answer is yes, now might be a good time to read §16.4 in Appendix A on the "diagonal fallacy". The next question looks at this point in more detail.

2.7 Suppose $L = L_1 \oplus L_2$ is the direct sum of two Lie algebras.

(i) Show that $\{(x_1, 0) : x_1 \in L_1\}$ is an ideal of L isomorphic to L_1 and that $\{(0, x_2) : x_2 \in L_2\}$ is an ideal of L isomorphic to L_2. Show that the projections $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ are Lie algebra homomorphisms.
Show the L_1 isomorphism.
Let $u = (u_1, u_2) \in L$. Then $N_1 = \{(x_1, 0) : x_1 \in L_1\}$ and $x = (x_1, x_2) \in N_1$ then $[u, x] = [(u_1, u_2), (x_1, 0)] = ([u_1, x_1], [u_2, 0]) = ([u_1, x_1], 0) \in N_1$ and therefore an ideal. Also, allow

$\varphi : N_1 \to L_1$. Let $a, b \in \ker \varphi$. Then $\varphi(a+b) = \varphi(a) + \varphi(b) = (0,0)$ implies that $a_1 = b_1$ or $a = b$. Thus, φ is an isomorphism.
A similar argument for the L_2 isomorphism.
Proejctions:
Given any $x, y \in L$

$$\begin{aligned} p_1([x,y]) &= p_1([x_1,y_1],[x_2,y_2]) \\ &= [x_1,y_1] \end{aligned}$$

thus $p_1([x,y]) \in L_1$. A similar argument for L_2.
Now suppose that L_1 and L_2 do not have any non-trivial proper ideals.

(ii) Let J be a proper ideal of L. Show that $J \cap L_1 = 0$ and $J \cap L_2 = 0$, then the projection $p_1 : J \to L_1$ and $p_2 : J \to L_2$ are isomorphisms.

(iii) Deduce that if L_1 and L_2 are not isomorphic as Lie algebras, then $L_1 \oplus L_2$ has only two non-trivial proper ideals.

(iv) Assume that the ground field is infintie. Show that if $L_1 \cong L_2$ and L_1 is 1-dimensional, then $L_1 \oplus L_2$ has infinitely many different ideals.

2.8 Let L_1 and L_2 be Lie algebras, and let $\varphi : L_1 \to L_2$ be a surjective Lie algebra homomorphism. True or False:

(a) $\varphi(L_1') = L_2'$;

(b) $\varphi(Z(L_1)) = Z(L_2)$;

(c) $h \in L_2$ and $\operatorname{ad} h$ is diagonalisable then $\operatorname{ad} \varphi(h)$ is diagonalisable.

2.9 For each pair of the following Lie algebras over $\mathbb{R}$, decide whether or not they are isomorphic:

(i) the Lie algebra $R^3_\wedge$ where the Lie bracket is given by the vector product;

(ii) the upper triangular 2×2 matices over $\mathbb{R}$;

(iii) the strict upper triangular 3×3 matrices over $\mathbb{R}$;

(iv) $L = \{x \in \mathrm{gl}(3, \mathbb{R}) : x^t = -x\}$.
Hint: Use Exercises 1.15 and 2.8.

2.10 Let F be a field. Show that the derived algebra of $\mathrm{gl}(n, F)$ is $\mathrm{sl}(n, F)$

2.11 In Exercise 1.15, we defined the Lie Algebra $\mathrm{gl}_S(n, F)$ over a field F where S is an $n \times n$ matrix with entries in F.

Suppose that $T \in \mathrm{gl}(n, F)$ is another $n \times n$ matrix such that $T = P^t S P$ for some invertible $n \times n$ matrix $P \in \mathrm{gl}(n, F)$ (Equivalently, the bilinear forms defined by S and T are congruent.) Show that the Lie algebras $\mathrm{gl}_S(n, F)$ and $\mathrm{gl}_T(n, F)$ are isomorphic.

2.12 Let S be an $n \times n$ invertible matrix with entries in $\mathbb{C}$. Show that if $x \in \mathrm{gl}_S(n, \mathbb{C})$, then $\operatorname{tr} x = 0$

2.13 Let I be an ideal of a Lie Algebra L. Let B be the centraliser of I in L; that is

$$B = C_L(I) = \{x \in L : [x, a] = 0, \forall a \in I\}$$

Show that B is an ideal of L. Now suppose that

(a) $Z(I) = 0$, and

(b) if $D : I \to I$ is a derivation, then $D = \operatorname{ad} x$ for some $x \in I$.
Show that $L = I \oplus B$.

(c) Recall that if L is Lie algebra, we defined L' to be the subspace spanned by the commutators $[x, y]$ for $x, y \in L$. The purpose of this execise, which may safely be skipped on first reading, is to show that the *set* of commutators may not even be a vector space (and so certainly not an ideal of L.).

Let $\mathbb{R}[x, y]$ denote the ring of all real polynomials in two variables. Let L be the set of all matrices of the form

$$A\left(\,(f(x), g(y), h(x, y)\,\right) = \begin{pmatrix} 0 & f(x) & h(x,y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Prove L is a Lie algebra with usual commutator bracket. (In contrast to all the Lie algebras seen so fro, L is infinite-dimensional.)

(ii) Prove that

$$[\,A\left(\,(f_1(x), g_1(y), h_1(x, y)\,\right), A\left(\,(f_2(x), g_2(y), h_2(x, y)\,\right)] = A(0, 0, f_1(x)g_2(x) - f_2(x)g_1(y)).$$

Hence describe L'.

(iii) Show that if $h(x, y) = s^2 + xy + y^2$, then $A(0, m0, h(x, y))$ is not a commutator.