Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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Section 7: Quotients – May 17, 2025

Pg. 77: Exercise 7.11 (Real projective space as a quotient of a sphere).* For $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, let $||x|| = \sqrt{\sum_i (x^i)^2}$ be the modulus of x. Prove that the map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{||x||}$$

induces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n/\sim$. Where

$$x \sim y \iff x = \pm y,\ x, y \in S^n$$

(*Hint:* Find an inverse map

$$\bar{g} : S^n/\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Given the relation above $x \sim y \iff x = \pm y,\ x, y \in S^n$. Define

$$\bar{g}([x]) = [x]$$

Note that on the left $[x] \in S^n/\sim$ and $[x] \in \mathbb{R}P^n$. For clarity,[1]

$$[a] \in S^n/\sim \implies [a] = \{a, -a\} \text{ where } a \in S^n$$
$$[b] \in \mathbb{R}P^n \implies [b] = \{x \in \mathbb{R}^{n+1} | \, x = \alpha b,\ \forall \alpha \in \mathbb{R}\} \text{ where } b \in \mathbb{R}^{n+1}$$

Notice that

$$\bar{f}([b]) = [f(b)] = \left[\frac{b}{||b||}\right] \in S^n/\sim$$
$$\bar{g}\left(\left[\frac{b}{||b||}\right]\right) = [b]$$
$$\therefore \bar{g} \circ \bar{f} = \mathbb{I}$$
$$\text{and } \bar{g}([a]) = [a] \in \mathbb{R}P^n$$
$$\bar{f}\left(\bar{g}([a])\right) = \left[\frac{[a]}{||[a]||}\right] = [f(a)] = [a] \in S^n/\sim$$
$$\therefore \bar{f} \circ \bar{g} = \mathbb{I}$$

$\bar{f}$ is continuous because f is continuous. $\bar{g}$ is continuous because it is a mapping of one identity to another. Therefore, $\bar{f}$ is a homeomorphism.

$$\begin{tikzcd} S \arrow[r, dash, "f"] \arrow[d, dash, "\pi"'] & Y \\ S/\sim \arrow[ur, dash, "\hat{f}"'] & \end{tikzcd}$$

Problems

7.1. **Image of the inverse image of a map**

Let $f : X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if f is surjective, then $f(f^{-1}(B)) = B$.

$\subseteq$: Let $b \in B$ and $a \in X$ such that $f(a) = b$. Then, $a \in f^{-1}(b)$, thus a is an arbitrary point in $f^{-1}(B)$. We know that $f(a) \in f(X)$ and $f(a) \in B$, therefore $f(a) \in B \cap f(X)$ and $f(f^{-1}(B)) \subseteq B \cap f(X)$.

$\supseteq$: Let $b \in B \cap f(X)$. Since $b \in f(X)$, there exists $a \in X$ such that $f(a) = b$ and since $b \in B$ then $b = f(a) \in f(f^{-1}(b)) \subseteq f^{-1}(B)$. Therefore $b \in f(f^{-1}(B))$

[1] $\mathbb{R}P^n \equiv \mathbb{R}^{n+1}/r(x, y)$ where $r(x, y)$ is the relation that is true when x, y, p are colinear.

7.2. **Real projective plane**

Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. In the notation of example 7.13, prove that the induced map $f : H^2/\sim \to S^2/\sim$ is a homeomorphism. (*Hint:* Imitate Propostion 7.3.) Let H^2 be the upper hemisphere and S^2 be the unit sphere

$$H^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1, z \geq 0\}$$
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$$

These two are homeomorphic to each via

$$\varphi : S^2 \mapsto H^2$$
$$\varphi(x,y,z) = (x,y,|z|)$$

and its inverse

$$\psi : H^2 \mapsto S^2$$
$$\psi(x,y,z) = (x,y,z)$$

Define the relations

$$(x,y,z) \sim (-x,-y,-z) \to x^2+y^2 = z^2, \forall (x,y,z) \in S^2$$
$$(x,y,z) \sim (x,y,z) \to \sqrt{x^2+y^2} = z, \forall (x,y,z) \in H^2$$

Then we induce $f : H^2/\sim \to S^2/\sim$ as

$$f([(x,y,z)]) = [(x,y,z)]$$

7.3. **Closedness of the diagonal of a Hausdorff Space**

Deduce Theorem 7.7 from Corollary 7.8 (*Hint:* To prove that if $S/\sim$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi : S \to S/\sim$. To prove the reverse implication, use the openness of π.)

Assuming Corollary 7.8, that a Topological Spacee is Hausdorf if and only if the diagonal Δ in $S \times S$ closed. Thus, $R = S \times S - \Delta$ must be open. That means there exists and $U \times V \subset S \times S - \Delta$ where $U, V \in S$ such that $U \cap V = \emptyset$. Since π is continuous, let $A = \pi^{-1}(U)$ and $B = \pi^{-1}(V)$. Clearly $A \cap B = \emptyset$. Thus, $x \in A$ and $y \in B$ means that $x \neq y$. Let $\sim \equiv =$. Then, there exists $[x],[y] \in S/\sim$ where $[x] \cap [y] = \emptyset$

7.4. **Quotient of a sphere with antipodal points indentified**

Let S^n be the unit sphere centered at the origin $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by indentifying antipodal points:

$$x \sim y \iff x = \pm y, \; x, y, \in S^n$$

(a) Show that $\sim$ is an open equivalence relation.

Given an open set U and $x \in U$. Then there exists $\epsilon > 0$ such that the n dimensional ball $B(x,\epsilon) = \{y : |y - x| < \epsilon\} \subset U$. Let $z \in B(x,\epsilon)$ and $z \sim z'$. Then, $z = \pm z'$ and $|z - x| < \epsilon$ implies that $| \pm z' - x| < \epsilon$ or $z' \in B(x,\epsilon)$. z, z' are arbitrary points in $B(x,\epsilon)$ thus the open ball is mapped to itself and is open. Therefore $\sim$ is open.

(b) Apply Theorem 7.7 and Corallary 7.8 to prove that the quotient space $S^n/\sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \cong S^n/\sim$.

From (a), $\sim$ is an open relation. WTS that the graph R of $\sim$ in $S \times S$ is closed, thus, from 7.8, $S^n/\sim$ is Hausdorff. We can do this by showing that the diagonal of R, $\Delta = \{(x,x) \in S^n \times S^n\}$, is closed.
Let $P = \Delta^c = S^n \times S^n - \Delta$. Let $(U,V) \subseteq P$ for open sets $U, V \subseteq S$. If $U \cap V = \emptyset$ then $U \sim V = \emptyset$ is open. Otherwise, $U \cap V$ is open. Define $Q = \{(x,y) \in U \cap V \,|\, x \sim y\}$. Since $\sim$ is an open relations and $U \cap V$ is an open set, Q must be open. $Q \subseteq P$. Now, let $N = \{(x,y) \in P - Q\}$. Then $x \neq y$ and $x \nsim y$.

7.5. **Orbit space of a continuous group action.**

Suppose a right action of a topological group G on a topological space S is continuous, this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi : S \to S/G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = R^{\times} = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^{\times}$ is commutative, a left $\mathbb{R}^{\times}$-action becomes a right $\mathbb{R}^{\times}$-action if scalar multiplication is written on the right.)

†Want to show that given an open set $U \in S$, then $\pi^{-1}(\pi(U)) \in S$ is open given that there exists $f : S \times G \to S$ which is continuous.

$$\begin{aligned}\pi^{-1}(\pi(U)) &= \{x \in S \,|\, \pi(x) \in \pi(U)\} \\ &= \{x \in S \,|\, [x] \cap U \neq \emptyset\} \\ &= \bigcup_{g \in G} Ug\end{aligned}$$

Thus, this mapping $\pi^{-1} \circ \pi$ on an open set is a set of elements in S based on xg for all $x \in U$ and $g \in G$. Now $f(x, g) : S \times G \to S$ is continuous and $f(x, g) = xg$. Thus $f^{-1}(U)$ is open for all $g \in G$. Thus each Ug is open in S. $(\pi^{-1} \circ \pi)(U)$ is the union of open sets and is therefore open.

7.6. **Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$.**

Let the additive group $2\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $\mathbb{R}/2\pi\mathbb{Z}$ is a smooth manifold.

Primary Option: Let $\mathbb{R} = S$ and $G = 2\pi\mathbb{Z}$ and $[x] = \{x \in \mathbb{R} : x + 2\pi n, \forall n \in \mathbb{Z}\}$.
Second Option: Must show that it is 2nd countable, Hausdorff and there exists an atlas covering the set.

2nd countable.
Yes. The basis for $\mathbb{R}$ is a basis for $\mathbb{R}/2\pi\mathbb{Z}$.

Hausdorff.
Yes. Every open set in $\mathbb{R}/2\pi\mathbb{Z}$ is open in $\mathbb{R}$ and every point as well. Thus, Hausdorff.

An Atlas Exists.
Yes. Let $U_i = (\pi i, \pi i + 2\pi)$ and $\phi_i(x) = x - 2\pi i$. And $\bigcup_{i \in \mathbb{Z}} U_i = \mathbb{R}$ and $\phi_i \in \mathbb{C}^\infty$

7.7. **The circle as a quotient space**

(a) Let $\{(U_\alpha, \phi_\alpha)_{\alpha=1}^2\}$ be the atlas of circles S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \coprod B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.
From Example 5.7

$$\begin{aligned}U_1 &= \{e^{it} \in \mathbb{C} \,|\, -\pi < t < \pi\} \text{ and } \phi_1(e^{it}) = t,\ -\pi < t < \pi \\ U_2 &= \{e^{it} \in \mathbb{C} \,|\, 0 < t < 2\pi\} \text{ and } \phi_2(e^{it}) = t,\ 0 < t < 2\pi \\ A &= \{e^{it} \,|\, -\pi < t < 0\} \\ B &= \{e^{it} \,|\, 0 < t < \pi\} \\ \bar{\phi}_1(e^{i(t+2\pi n)}) &= [\phi_1(e^{it}e^{2\pi n})] = [\phi_1(e^{it})] = [t],\ -\pi < t < \pi \\ \bar{\phi}_2(e^{it}) &= [\phi_2(e^{it})] = [t],\ 0 < t < 2\pi \\ \bar{\phi}(e^{it}) &= [t] \in \mathbb{R}/2\pi\mathbb{Z},\ \forall t \in \mathbb{R}\end{aligned}$$

(b) The complex exponential $\mathbb{R} \to S^1, t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1, F([t]) = e^{it}$. Prove that F is C^∞.

$$\begin{tikzcd}\mathbb{R} \arrow[r, "f"] \arrow[d, "\pi"'] & S^1 \\ \mathbb{R}/2\pi\mathbb{Z} \arrow[ur, "F"'] & \end{tikzcd}$$

$f : \mathbb{R} \to S^1, t \mapsto e^{it}$ is smooth, and $\pi : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}, t \mapsto [t]$ is smooth, from the diagram above $f = F \circ \pi$. Therefore F is smooth.

(c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

7.8. **The Grassmanian $G(k, n)$**

7.9. **Compactness of real projective space**

Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint:* Use Exercise 7.11.)