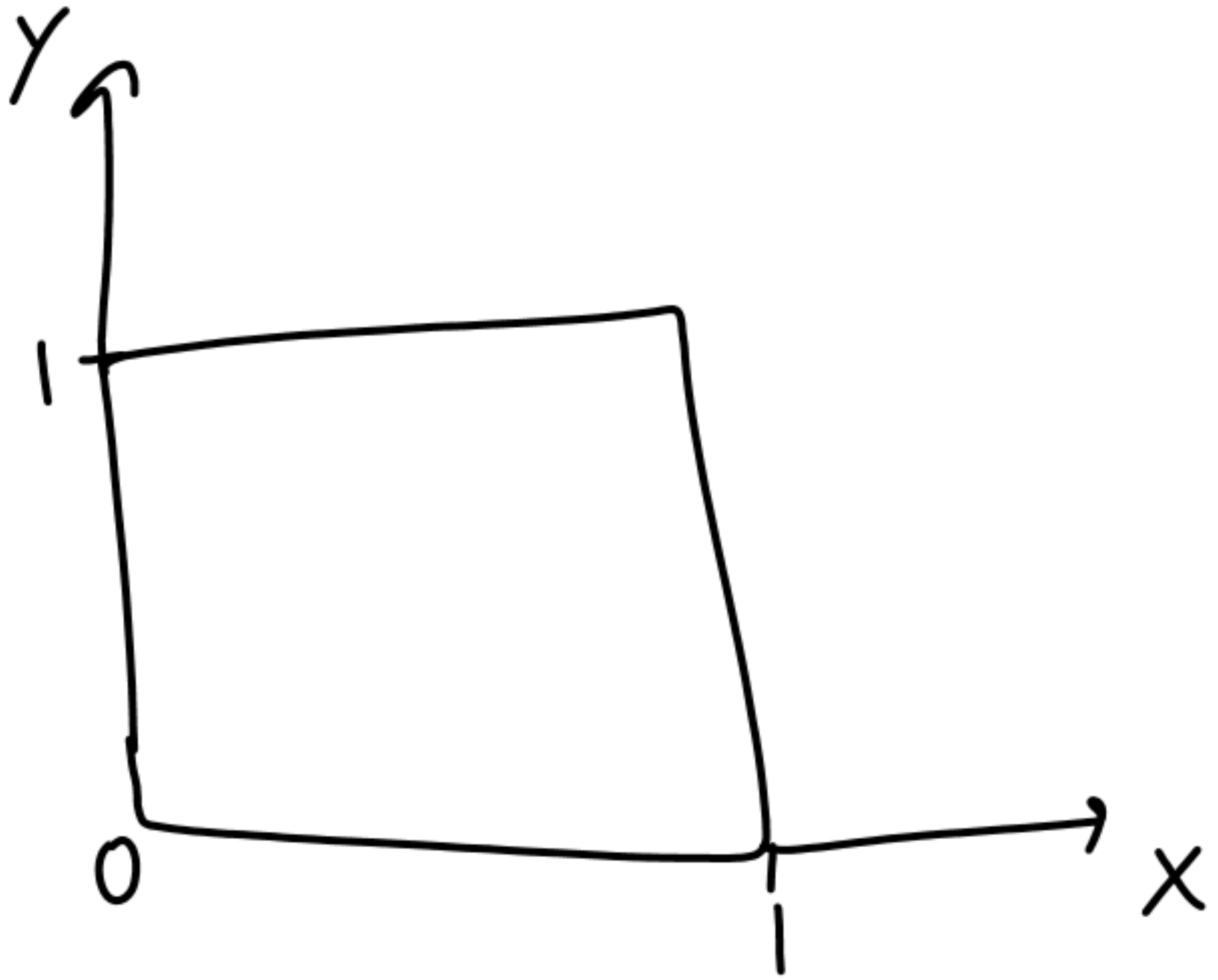
Examples of topics:

① $$\begin{cases} \boxed{u_t - \Delta u = 0}, & t>0,\ 0 \le x \le 1,\ 0 \le y \le 1 \\ u(t,x,y) = 0 & \text{if } x=0 \text{ or } x=1 \text{ or } y=0 \text{ or } y=1 \\ u(0,x,y) = g(x,y) \end{cases}$$

$$u_t = \partial_t u = \frac{\partial u}{\partial t}$$

$$\Delta u = \nabla^2 u = u_{xx} + u_{yy}$$

$\uparrow$
Laplacian

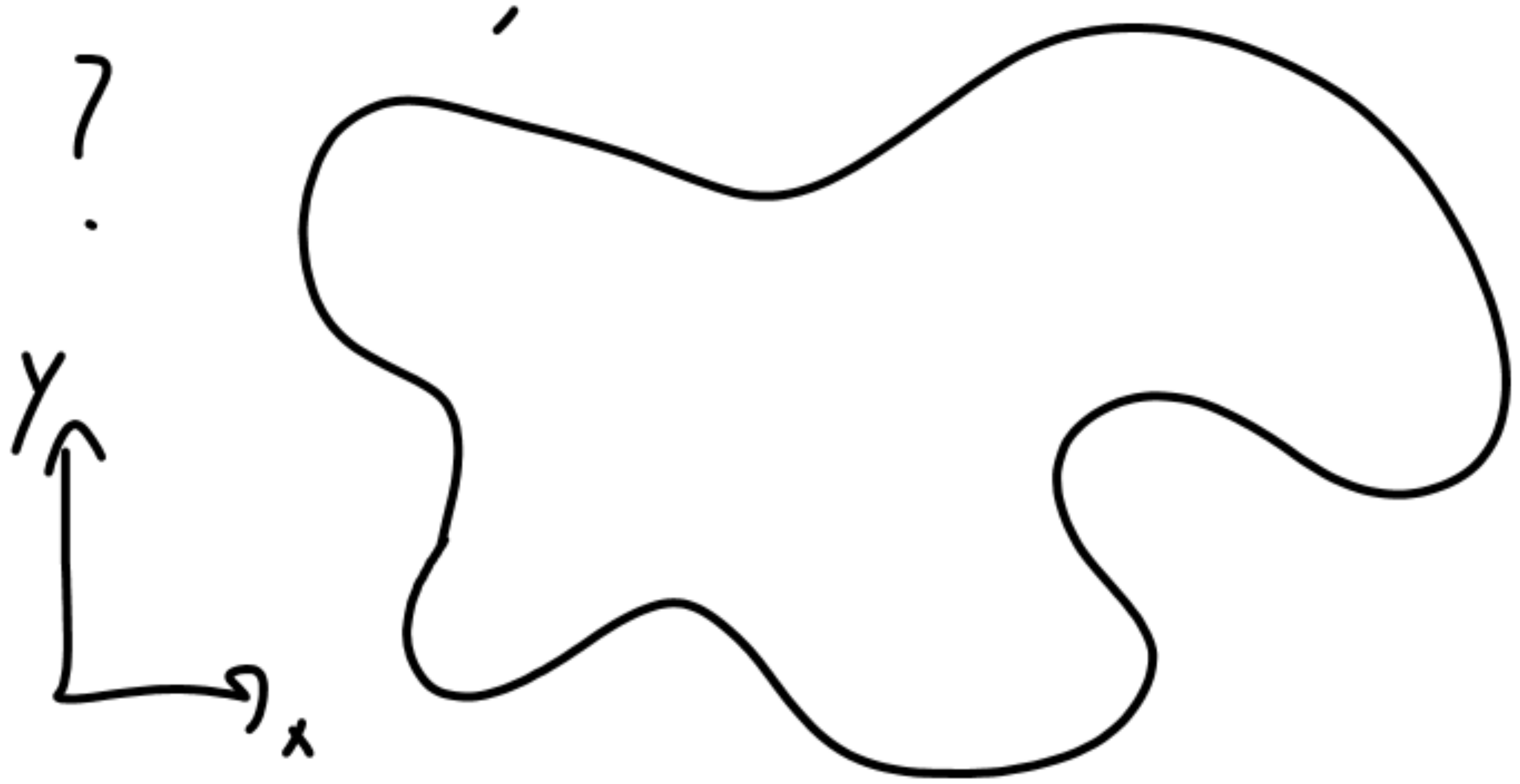


Separation of Variables:

$$u(t,x,y) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} a_{mk}\, e^{-\lambda_{mk} t} \sin(m\pi x)\sin(k\pi y)$$

↳ Fourier coefficients determined by $g$.

Q: What if we replace the square $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ with a less symmetric shape?



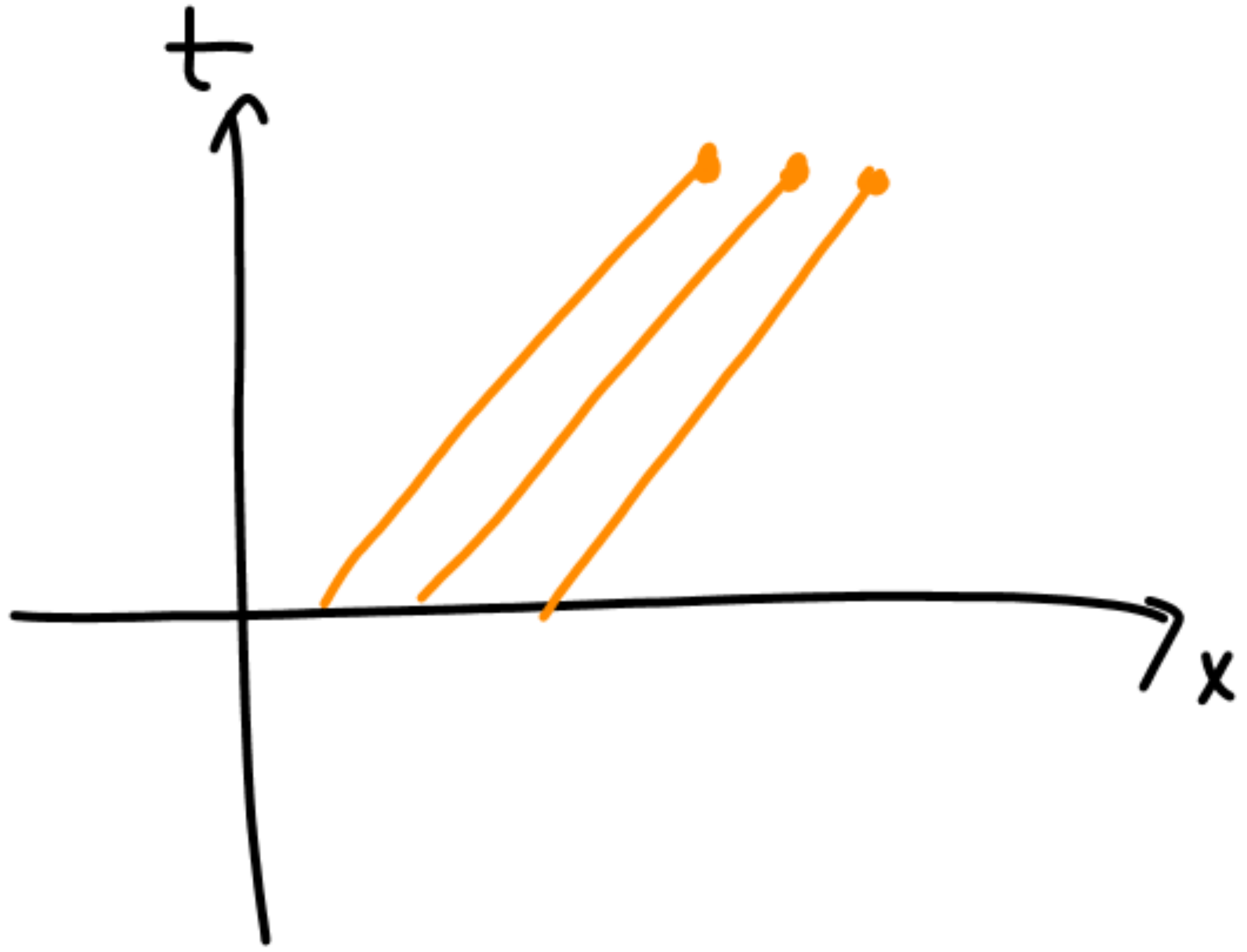
We need a way to represent solutions that don't rely on symmetry.

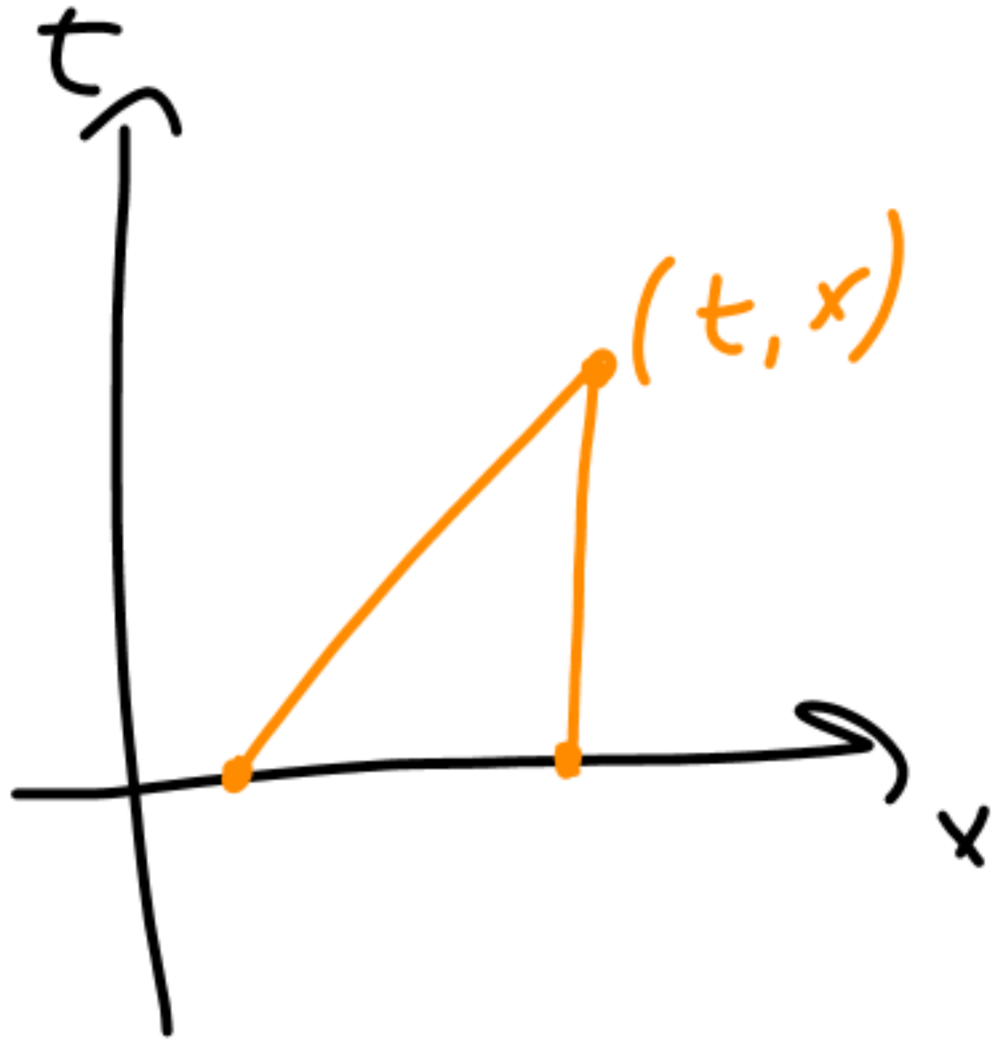
$$F(t, u, u_t, u_x) = 0$$

② Nonlinear Conservation Laws

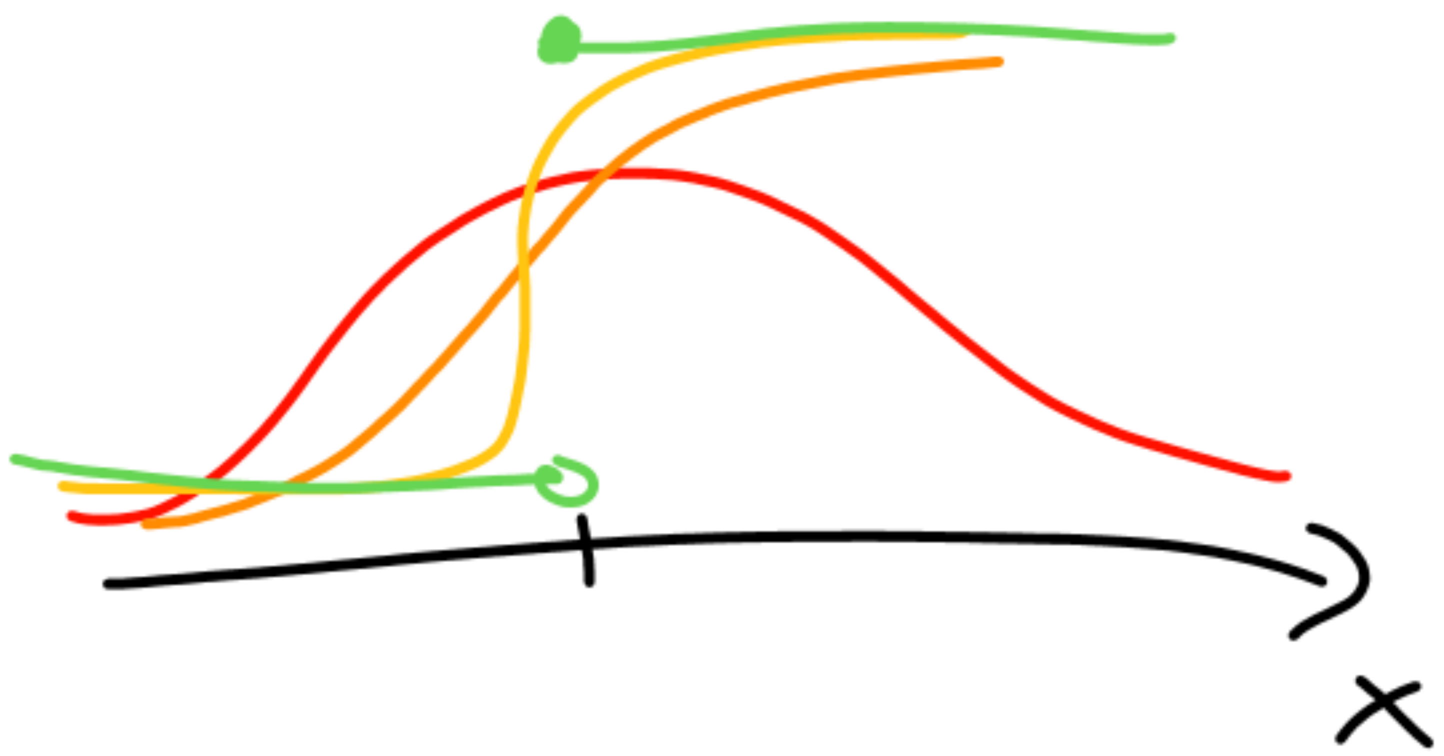
$$\begin{cases} u_t + u u_x = 0 \\ u(0,x) = g(x) \end{cases} \qquad t \geq 0, \quad -\infty < x < \infty$$

$$u_t + 3u_x = 0$$



When characteristics cross, the solution can have a shock. Then the PDE




$$u_t + u u_x = 0$$

doesn't make sense "pointwise".

Need a notion of <u>weak solution</u>

## ③ Qualitative Properties of Solutions

Compare

$$u_t - \Delta u = 0 \qquad \text{Heat}$$

$$u_{tt} - \Delta u = 0 \qquad \text{Wave}$$

The heat eqn <u>smooths out</u> the initial data, but the wave eqn doesn't.

Relatedly, the heat equation features infinite speed of propagation, but the wave eqn has finite speed of propagation. This is hard to see from Separation of Variables formulas

Plan: Fundamental Theory of:

Ch 2 { ① Linear Transport Equation
② Laplace's Equation
③ Heat Equation
④ Wave Equation }

Ch 3 { ⑤ Nonlinear conservation laws }

# Linear Transport Equation

$$\boxed{u_t + \vec{b} \cdot Du = 0}$$

$Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$

Unknown function $u(\vec{x}, t)$ $\quad t \geq 0$

$\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$

"in" $\quad$ n-dim Euclidean space

Initial condition:

$u(\vec{x}, 0) = g(\vec{x})$

This is an example of a continuity equation:

$$\boxed{u_t + \nabla \cdot \vec{j} = \sigma}$$

$u$ = density

$\vec{j}$ = flux density, $\quad \sigma$ = sources & sinks

$\vec{b} = (b_1, \dots, b_n)$

$u_t + D \cdot \vec{j} = \sigma$

If $\vec{j} = \vec{b}u$, with $\vec{b} \in \mathbb{R}^n$ a fixed vector,
and $\sigma = 0$, then $\nabla \cdot (\vec{b}u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\vec{b}u)_i$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u b_i)$$

$$= \sum_{i=1}^{n} b_i u_{x_i}$$

$$= \vec{b} \cdot Du$$

Dot product: $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

$Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n})$

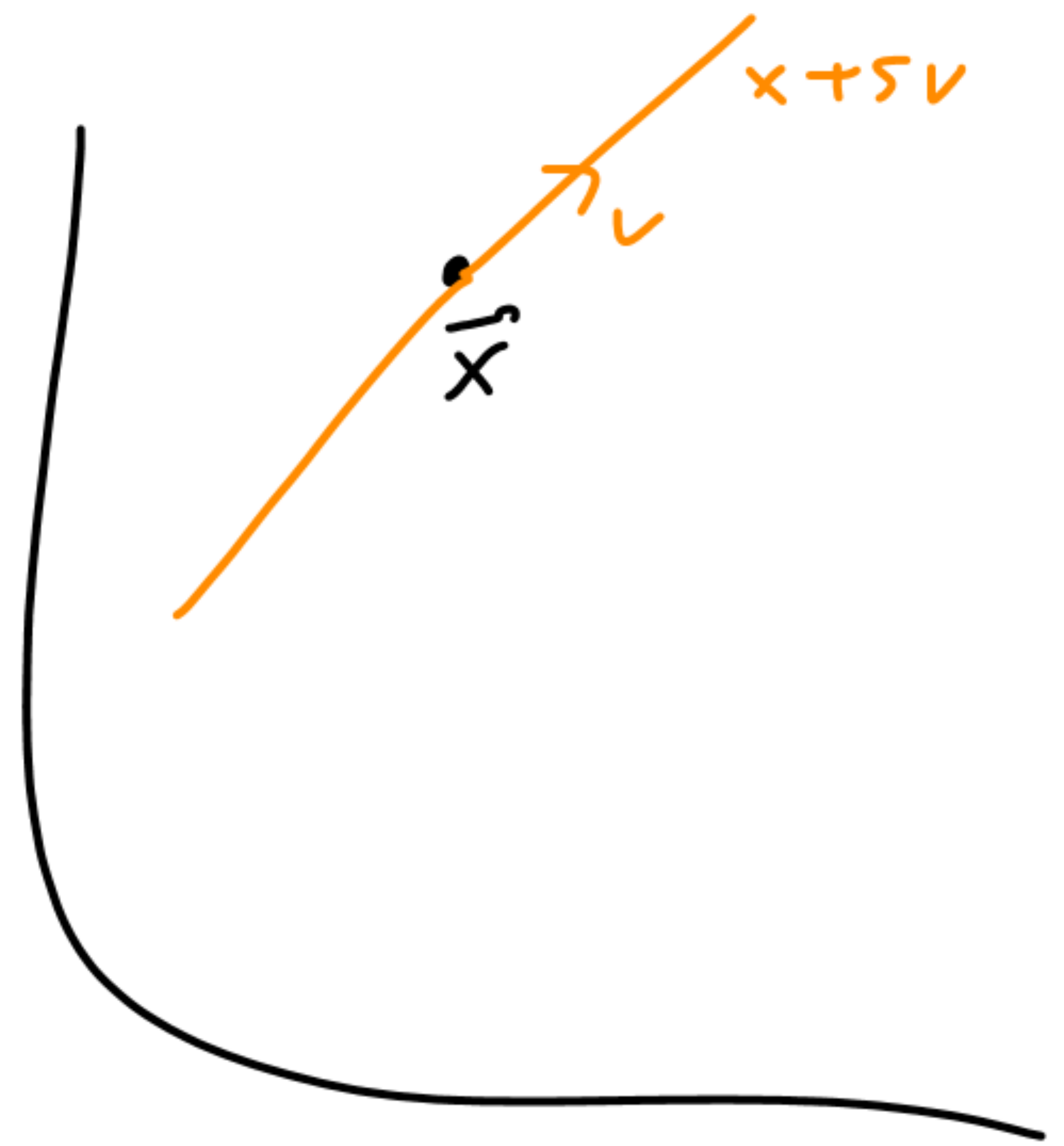
$u_t + \vec{b} \cdot Du$

---

1D case:

$u(x,t)$ solves

$$\begin{cases} u_t + b u_x = 0 & \text{for some } b \in \mathbb{R} \\ u(x,0) = g(x) \end{cases}$$

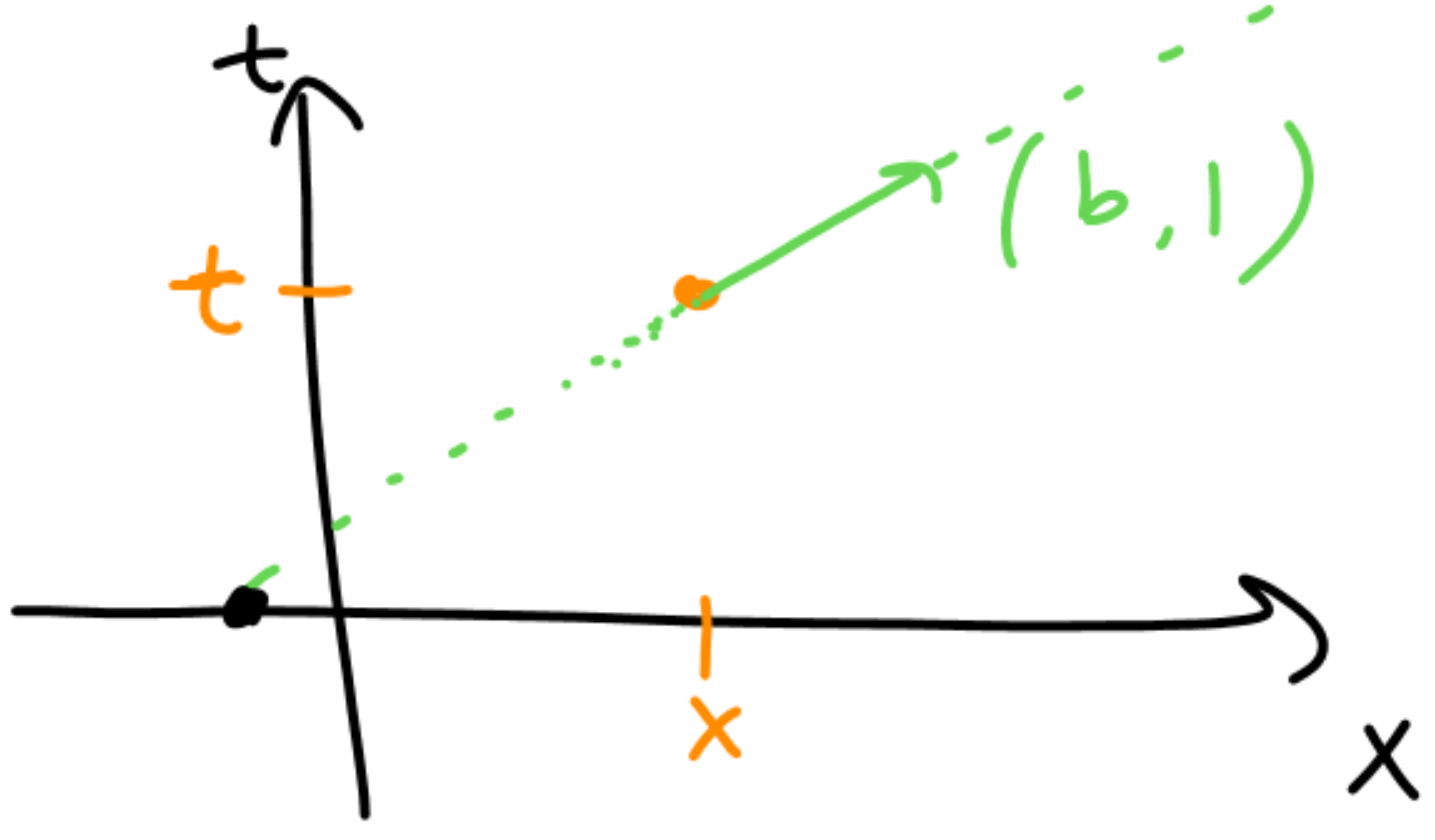



Realize: $u_t + b u_x$ is a directional derivative of $u$ in $(t,x)$ space

(Recall: A directional derivative of $f(\vec{x})$ is $\frac{d}{ds} f(x+sv)\Big|_{s=0} = Df(x) \cdot \vec{v}$ )

$$u_t + bu_x = (b, 1) \cdot \underbrace{(u_x, u_t)}_{= Du}$$

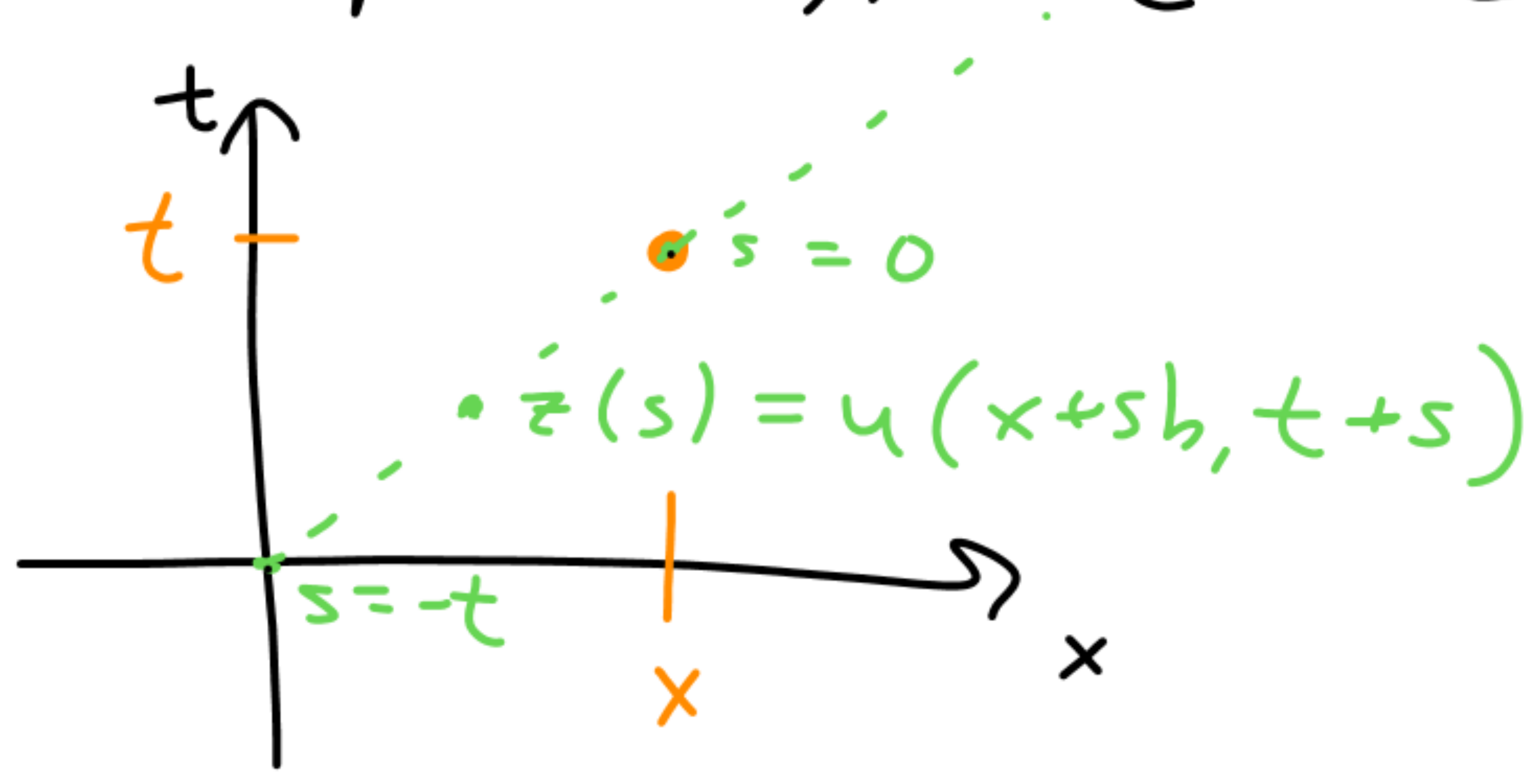
$= Du$, with $u$ a function of $t$ & $x$




The PDE $u_t + bu_x = 0$ says that $u$ is constant along lines in $(t,x)$ space parallel to $(b,1)$.

More precisely, let $z(s) = u\big(\overbrace{x+sb,\ t+s}^{\text{line in } \mathbb{R}^2}\big)$

($x$ & $t$ are fixed)




The line $(x+sb,\ t+s)$ intersects the horizontal axis when $t+s=0 \longrightarrow s=-t$

$z(s)$ is constant in $s$:

$$\frac{d}{ds} z(s) = \frac{d}{ds} u(x+sb,\ t+s) = \frac{\partial u}{\partial x}\frac{\partial}{\partial s}(x+sb) + \frac{\partial u}{\partial t}\frac{\partial}{\partial s}(t+s)$$

$$= b\,u_x + u_t = 0 \quad \text{(from the PDE)}$$

$z(s)$ is constant: $z(0) = z(-t)$

Recall $z(s) = u(x+sb,\ t+s)$

$$u(x,t) = u(x-bt,0) = g(x-bt)$$

Solution: $\boxed{u(x,t) = g(x-bt)}$

Check: Initial Condition: $u(x,0) = g(x)$ $\checkmark$

PDE: $u_t + b u_x = g'(x-bt)(-b) + b g'(x-bt)$

$$= 0 \quad \checkmark$$

$g(x)$

We've shown that $u(x,t) = g(x-bt)$ solves the PDE pointwise <u>if</u> $g$ is $C^1$ (continuously differentiable).

If $g$ is not $C^1$, we can still write $u(x,t) = g(x-bt)$ as a "weak solution".