Functional Analysis– Spring 2024

Paul Carmody
Assignment #4– April 4, 2024

p. 175 #2, 4, 5, 10.

#2. Show that if the orthogonal dimension of Hilbert Space H is finite, it equals the dimension of H regarded as a vector space; conversely, if the latter is finite, show that so is the former.

Assuming that the Hilbert Dimension of HIlbert space H is finite, $\dim H = n$. Let the orthonormal family $(e_\alpha)_{\alpha\in A} \in H$ for some set A. Further let there exists a countable subset of A' such $\left\langle e_{\alpha_j}, e_{\alpha_k} \right\rangle = \delta_{jk}$ for all $j,k \in [1..n]$ and that span$\{e_{\alpha_k}\}$ is dense and equal to H. Further, if $y \perp \text{span}\{e_{\alpha_k}\}$ then $y = 0$. Therefore, given any $x \in H$, $x = \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$. We can see, then, that every $x \in H$ is a linear combination of $\{e_{\alpha_k}\}$. Hence $\{e_{\alpha_k}\}$ forms a basis. There must be n elements in A', hence the vector space dimension is the same as the Hilbert space dimension.

Assuming that we have a finite dimensional vector space X. Then there exists an orthonormal basis $\{e_k\} \in X$. Define an Inner Product on $X, \langle \cdot, \cdot \rangle$. Clearly, $\langle e_j, e_k \rangle = \delta_{jk}$. Also, given any $x \in X$ such that $\langle x, e_k \rangle = 0$ for all $k \in [1, n]$ we can see that $x = 0$ as all e_k are linearly independent from each other. Hence, span e_k is dense in X. X must be a Hilbert space with dimension n.

#4 Derive from (3) the following formula (which is often called the *Parseval relation*).

$$\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$$

Given an orthonormal basis $\{e_k\}_{k=1}^\infty$ on H we can define $x \in H$ as

$$\begin{aligned}
x &= \sum_k \langle x, e_k \rangle e_k \\
\|x\|^2 &= |\langle x, x \rangle| \\
&= \left| \left\langle \sum_k \langle x, e_k \rangle e_k, \sum_j \langle x, e_j \rangle e_j \right\rangle \right| \\
&= \sum_k \sum_j |\langle \langle x, e_k \rangle e_k, \langle x, e_j \rangle e_j \rangle| \\
&= \sum_k \sum_j \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \langle e_k, e_j \rangle \right| \\
&= \sum_k \sum_j \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \delta_{jk} \right| \\
&= \sum_k \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \right|
\end{aligned}$$

replacing the right x in the Inner Product with y and we get

$$\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$$

#5 Show that an orthonormal family $(e_\kappa), \kappa \in I$, in a Hilbert Space H is total if and only if the relation in Prob. 4 holds for every x and y in H.

($\Rightarrow$) Let an orthonormal family $(e_\kappa), \kappa \in I$, in a Hilbert Space H be total. Let $x, y \in H$ we know that we can represent them as $x = \sum_k \langle x, e_\kappa \rangle e_\kappa$ and $y = \sum_\iota \langle y, e_\kappa \rangle e_\iota$ where $\kappa, \iota \in I$. Thus

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_\kappa \langle x, e_\kappa \rangle e_\kappa, \sum_\iota \langle y, e_\iota \rangle e_\iota \right\rangle \\
&= \sum_\kappa \sum_\iota \langle \langle x, e_\kappa \rangle e_\kappa, \langle y, e_\iota \rangle e_\iota \rangle \\
&= \sum_\kappa \sum_\iota \langle x, e_\kappa \rangle \overline{\langle y, e_\iota \rangle} \langle e_\kappa, e_\iota \rangle \\
&= \sum_\kappa \sum_\iota \langle x, e_\kappa \rangle \overline{\langle y, e_\iota \rangle} \delta_{\kappa\iota} \\
&= \sum_\kappa \langle x, e_\kappa \rangle \overline{\langle y, e_\kappa \rangle}
\end{aligned}$$

x, y are arbitrary therefore true for all elements of H.

($\Leftarrow$)Assuming that this is true for all $x, y \in H$ and we have an orthonormal set $(e_\kappa) \in H$ where $\kappa \in I$. We can see that the same steps can be executed in reverse indicating that all $x \in H$ can be represented as $x = \sum_k \langle x, e_\kappa \rangle e_\kappa$. This indicates that $\text{span}\{e_\kappa\} = H$. Let $z \in H$ such that $z \in (e_\kappa)^\perp$ then $\langle z, e_\kappa \rangle = 0$ for all $\kappa \in I$. We know that $z = \sum_\kappa \langle z, e_\kappa \rangle e_\kappa$. Therefore $z = 0$ and $(e_\kappa)^\perp = \{0\}$, hence, $\text{span}\{e_\kappa\}$ must be dense.

#10 Let M be a subset of a Hilbert space H, and let $v, w \in H$. Suppose that $\langle v, x \rangle = \langle w, x \rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$ show that M is total in H.

$$\begin{aligned}
\text{let } z \in M^\perp, \ &\langle z, x \rangle = 0, \forall x \in M \\
\langle v, x \rangle &= \langle w, x \rangle + \langle z, x \rangle \\
&= \langle w + z, x \rangle \\
v &= w + z \\
v - w &= z
\end{aligned}$$

$v - w = 0$ thus $z = 0$. Since z was arbitrary $M^\perp = \{0\}$ and therefore $\text{span}\{M\}$ is dense in H.

p. 194 #2, 4, 5, 7.
#2 **(Space ℓ^2)** Show that every bounded linear functional f on ℓ^2 can be represented in the form

$$f(x) = \sum_{j=1}^{\infty} \xi_j \overline{\zeta_j} \qquad [z = (\zeta_j) \in \ell^2].$$

Given $x = (\xi_n) \in \ell^2$. Let f be a bounded linear functional. Then, by the Reisz Representation Theorem

$$\begin{aligned} f &: \ell^2 \to K & K &= \mathbb{R} \text{ or } \mathbb{C} \\ f(x) &= \langle x, z \rangle & &\text{for some } z = (\zeta_n) \in \ell^2 \\ &= \langle (\xi_n), (\zeta_m) \rangle \\ &= \sum_{k=1}^{\infty} \xi_k \overline{\zeta_k} \end{aligned}$$

#4 Consider Prob. 3. If the mapping $X \to X'$ given by $z \mapsto f$ is surjective, show that X must be a Hilbert space.

Given z then f is a bounded linear functional on X with $\|z\| = \|f\|$.

$$z \mapsto f(x) = \langle x, z \rangle$$

#5 Show that the dual space of the real space ℓ^2 is ℓ^2. (Use 3.8-1.)

Let $f \in \ell^{2'}$. Then there exists $z = (\zeta_n) \in \ell^2$ such that $f(x) = \langle x, z \rangle$ for all $x \in \ell^2$. Hence, for $x = (\xi_n)$ we have

$$\begin{aligned} f(x) &= \langle x, z \rangle \\ &= \langle (\xi_n), (\zeta_m) \rangle \\ &= \sum_{k=1}^{\infty} \xi_k \zeta_k \end{aligned}$$

which converges due to Cauchy-Schwartz, thus $f \in \ell^2$ for all $f \in \ell^{2'}$.

#7 Show that the dual space H' of a Hilbert space H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_1$ defined by

$$\langle f_z, f_v \rangle_1 = \overline{\langle z, v \rangle} = \langle v, z \rangle,$$

where $f_z(x) = \langle x, z \rangle$, etc.

$$\begin{aligned} f_z(x) &= \langle x, z \rangle \\ f_v(x) &= \langle x, v \rangle \end{aligned}$$

Show that the inner product has all of the properties. Let $f_u, f_v, f_w \in H'$ and $u, v, w \in H$ and $f_u(x) = \langle x, u \rangle$, $f_v(x) = \langle x, v \rangle$ and $f_w(x) = \langle x, w \rangle$ for all $x \in H$.

(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

$$(f_u + f_v)(x) = f_u(x) + f_v(x) = \langle x, u \rangle + \langle x, v \rangle = \langle x, u + v \rangle$$

$$\begin{aligned} \therefore \langle f_u + f_v, f_w \rangle_1 &= \langle w, u + v \rangle \\ &= \langle w, u \rangle + \langle w, v \rangle \\ &= \langle f_u, f_w \rangle_1 + \langle f_v, f_w \rangle_1 \end{aligned}$$

(IP2) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

$$\langle \alpha f, g \rangle =$$

(IP3) $\langle\, x, z\,\rangle = \overline{\langle\, z, x\,\rangle}$

$$\begin{aligned}\text{Since } \langle\, f_u, f_v\,\rangle_1 = \langle\, v, u\,\rangle \text{ and } \langle\, f_v, f_u\,\rangle_1 = \langle\, u, v\,\rangle \\ \overline{\langle\, f_v, f_u\,\rangle_1} = \overline{\langle\, u, v\,\rangle} = \langle\, v, u\,\rangle = \langle\, f_u, f_v\,\rangle_1\end{aligned}$$

(IP4) $\langle\, x, x\,\rangle \geq 0$ and $\langle\, x, x\,\rangle = 0 \iff x = 0$.

$$\begin{gathered}\langle\, f_v, f_v\,\rangle_1 = \langle\, v, v\,\rangle = \|v\| \geq 0 \\ \langle\, f_v, f_v\,\rangle_1 = 0 \iff v = 0 \\ \text{thus } f_v(x) = \langle\, x, 0\,\rangle = 0, \forall x \in H\end{gathered}$$

Hence f is the zero function.

Finally, let (z_k) be a convergent sequence in H and $z_n, z_m \in (z_k)$ be any two elements. The representative elements in H' are f_{z_n}, f_{z_m}, respectively. We know that $\|z_n - z_m\| < \epsilon$ for some number $\epsilon > 0$

$$\begin{aligned}\|z_n - z_m\|^2 &= \langle\, z_n - z_m, z_n - z_m\,\rangle \\ &= \|z_n\| + \|z_m\| - 2\,\langle\, z_n, z_m\,\rangle \\ &= \|f_{z_n}\| + \|f_{z_m}\| - 2\,\langle\, f_{z_m}, f_{z_n}\,\rangle \\ &= \|f_{z_n}, f_{z_m}\|^2 \\ \therefore\ \|f_{z_n} - f_{z_m}\| < \epsilon\end{aligned}$$

and H' is complete.