Math 5110 – Real Analysis I– Fall 2024
w/Professor Liu

Paul Carmody
Homework #2 – September 18, 2024

I. Consider a sequence x_n of real numbers. The *limit inferiior* and *limit superior* of x_n are defined by

$$\liminf_{n\to\infty} x_n = \lim_{n->\infty}\left(\inf_{k\geq n} x_k\right),\ \limsup_{n\to\infty} x_n = \lim_{n->\infty}\left(\sup_{k\geq n} x_k\right)$$

(a) Show that

$$\liminf_{n\to\infty} x_n = \sum_{n\geq 0}\left(\inf_{k\geq n} x_n\right)$$

and

$$\limsup_{n\to\infty} x_n = \inf\left(\sup_{k\geq n} x_n\right)$$

(b) Show that $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are well-defined for any sequence x_n. (Unlike $\lim_{n\to\infty} x_n$.) We allow values of ∞ and $-\infty$

(c) Let x_n be a bounded sequence, and let L be the set of limit points of x_n, i.e., the set of all limits of subsequences of x_n. Show $\liminf_{n\to\infty} x_n = \inf L$ and $\limsup_{n\to\infty} = \sup L$.

(d) Let x_n be a bounded sequence. Conclude using (c) that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, with equality if and only if x_n is convergent.

II. Prove that for any (possibly uncountable) collection $(F_\alpha)_{\alpha\in A}$ of closed sets, the intersection $F = \bigcup_{\alpha\in A} F_\alpha$ is closed, in two ways.

(a) Using the fact that any union of open sets is open, and DeMorgan's Laws from set theory, which state

$$X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A}(\,X\backslash E_\alpha\,) \text{ and } X\backslash\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A}(\,X\backslash E_\alpha\,)$$

for all collection of sets $(E_\alpha)_{\alpha\in A}$

(b) More directly, using the fact that a set G is closed if and only if for any convergent sequence (x_n) with all $x_n \in G$, the limit x is also in G.

III. (a) Let (x_n) be a Cauchy sequence in a metric space X. Show that if a subsequence (x_{n_j}) of x_n converges to x, then the entire sequence also converges to x.

(b) Show that the metric space

$$C^1((-1,1)) = \{f : (-1,1) \to \mathbb{R}, f \text{ is differentiable and } f' \text{ is continuous in } (1,-1)\}$$

with the metric

$$d(f,g) = \sup_{x\in(-1,1)} |f(x) - g(x)|,$$

is not complete. (Hint: similar to the proof that the rational numbers are not complete, find a sequence $C'((-1,1))$ that converges in d metric to a function that is not in $C^1((-=1,1))$, and show that this sequence is Cauchy.)

IV. Let A and B be subsets of the metric space X. which one of the following is true?

$$(A\cup B)^o = A^o \cup B^o, \tag{2.1}$$
$$(A\cup B)^o \subset A^o \cup B^o, \quad \text{"="} \text{ fails for some } A \text{ and } B \tag{2.2}$$
$$(A\cup B)^o \supset A^o \cup B^o, \quad \text{"="} \text{ fails for some } A \text{ and } B \tag{2.3}$$

V. Leet $C^0([a,b])$ be the space of continuous functions on $[a,b]$, with the metric $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$.

Show that the map $I : C^0([a,b]) \to \mathbb{R}$ defined by $I(f) = \int_a^b f(x)dx$ is continuous mapping from $C^0([a,b])$ to $\mathbb{R}$.

VI. Prove Propostion 2.3.2 in the text, in two different ways.:

a) As a consequence of Theorem 2.3.1 in text.

b) Directly, using the sequential definition of compactness. **Proposition 2.3.2** (Maximum principle). *Let* (X, d) *be a compact metric space, and let* $f : X \to \mathbb{R}$ *be a contnuous function. Then* f *is bounded. Furthermore,* f *attains its maximum at some point* $x_{\max} \in X$*, and also attains its minimum at some point* $x_{\min} \in X$*.*

VII. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\lim_{|x| \to \infty} = +\infty$$

Prove that f attains its minimum.

Recall that by definition, the limit in (??) means that Given $A > 0$, there is $R > 0$ such that

$$f(x) > A \text{ for all } x \notin B_R$$

in other words, $f(x) > A$ whenever $|x| \geq R$. Here, $|x| = d_2(x, 0)$ and d_2 is the standard Euclidean distance defined in Example 1.4.