Math 5111 – Real Analysis II– Sprint 2025

w/Professor Liu

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Continuity is a property unlike most that we encounter in understanding functions. Typically, we start with the domain and apply it to the function and get a result in the range. A function is continuous, however, when the area around the range has the same properties as the area around the domain.

1. **continuous functions** defined using epsilon/delta. Which becomes increasingly difficult when our domain and range go beyond the Real numbers (i.e., multiple dimensions, complex numbers, sets that aren't compact).

2. **Topological definition of continuous functions** which is that an open set in the range comes from an open set in the domain. This skips over the concept of measure by simply redefining "open sets" in more abstract terms (an open set is a member of a topology). A metric topology has a way of measuring distance but it isn't necessary for this definition.

3. Analogously speaking, **a function is measurable** (a.k.a., continuous) when a measurable (a.k.a., open) set in the range comes from a measurable (a.k.a., open) set in the domain. The sets are defined by the Lebesgue Outer Measure (basically, 'not a point', which is much like but not the same as an open set in topology).

4. Definitions of "superset" topology and σ-algebra. These two concepts are so similar that they might be the same thing.

   (a) A **topology** is defined as the collection where at least the set and the empty set as members, as well as subsets where all intersections (countable) and all unions (even uncountable) are also members. "Open sets" are simply the members of the topology.

   (b) A σ-algebra is defined as having the set itself, all compliments of its members (which are subsets), and countable unions of its members.[1] The set and σ-algebra combine to make a **Measurable Space** and its members are called **Measurable Sets** (analogous to "open sets").

   (c) If f maps from a measurable space to a topological space and each open set in the range comes from a measurable set in the domain (i.e., $f^{-1}(V)$ maps open set V from a measurable set through f) then f is said to be **Measurable Function**.

   (d) These two, σ-algebra and topology, appear to me to be logically equivalent.

   (e) Put more simply: topology is defined by arbitrary unions and countable intersections of *open sets* and a σ-algebra is defined by countable unions of *measurable sets.*

5. Further, a **Borel Set** is an element of the smallest possible σ-algebra generated by the set. It is, consequently, measurable (in the Lebesgue sense, i.e., 'not a point'). And a function is **Borel Measurable** when Borel Sets are mapped from Borel Sets.

Rules for Composing different types of these functions.

	Nomen.[2]	$f : X \to Y$	$g : Y \to Z$	$g \circ f(x) = g(f(x)) = z$
standard	open inteval	continuous	continuous	continuous
topology[3]	open	continuous	continuous	continuous
measurable	measurable	measurable	continuous	measurable
measurable	measurable	measurable	measurable	measurable
Borel Measurable	Borel	Borel measurable	continuous	???
Borel Measurable	Borel	continuous	Borel measurable	???
Borel Measurable	Borel	measurable	Borel mapping	measurable
Borel Measurable	Borel	Boreal measurable	Borel mapping	Borel measurable

[1] σ signifies infinite unions and δ (Not shown) would signify infinite intersectios

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Definition 0.0.1 (**Simple Integration**). Where s is a simple function, that is, $\exists\{s_i\}, i = 1, \ldots, n$ and E_i such that $x \in E_i \implies s(x) = s_i$.

$$\int_X s d\mu = \sum_{i=1}^{n} s_i \mu(X \cap E_i)$$

Theorem 0.0.2. *Given any positive measurable function f there exists a sequence of simple measurable functions $\{s_i\}$ such that $s_i \to f$.*

Definition 0.0.3 (**Integration of Positive Function**). Given $f : X \to [0, \infty] \in \mathfrak{M}, E \in \mathfrak{M}(X)$

$$\int_E f d\mu = \sup \int_E s d\mu$$

supremum over all simple functions $0 \leq s < f$.

Theorem 0.0.4. *Let E_i be a partition on X then*

$$\int_X f d\mu = \int_X \sum_i f \chi_{E_i} d\mu$$

Theorem 0.0.5 (Lebesque Monotone Convergence Theorem)**.** *Given an increasing sequence of measurable functions $f_n : X \to [0, \infty] \in \mathfrak{M}$ where $f_n \to f$. Then,*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Theorem 0.0.6 (Fatou's Lemma)**.** *Given an increasing sequence of measurable functions $f_n : X \to [0, \infty] \in \mathfrak{M}$ where $f_n \to f$. Then,*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$