Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

Paul Carmody
Assignment 2 – Februaray 4, 2025

Section 2: 1, 3, 4

2.1. Vector fields
Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x,y,z)$ the function $x^2+y^2+z^2$ on $\mathbb{R}^3$. Complute Xf.

$$\begin{aligned} Xf &= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \\ &= x\,(\,2x\,) + y\,(\,2y\,) \\ &= 2x^2 + 2y^2 \end{aligned}$$

2.3. Vector space structure on derivations at a point
Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that

(a) the sum $D + D'$ is a derivation at p.

$$\begin{aligned} (D+D')(fg) &= D(fg) + D'(fg) \\ &= (Df)g + f(Dg) + (D'f)g + f(D'g) \\ &= (Df)g + (D'f)g + f(Dg) + f(D'g) \\ &= ((D+D')f)g + f((D+D')g) \\ D + D' &\in \mathcal{D}_p(\mathbb{R}^n) \end{aligned}$$

(b) the scalar multiple cD is a derivation at p.

Let $c \in \mathbb{R}$. Then, given any $v \in T_p(\mathbb{R}^n)$ we have

$$\begin{aligned} (cD_v)f &= \sum_{i=1}^n cv^i \frac{\partial f}{\partial x^i}\bigg|_p \\ &= c\sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}\bigg|_p \\ &= cD_v(f),\ \forall v \in T_p(\mathbb{R}) \\ \therefore cD &\in \mathcal{D}_p(\mathbb{R}) \end{aligned}$$

2.4. Product of derivations

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

Counter example

$$\begin{aligned}
D_1 &= \alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} \\
D_2 &= \beta^1 \frac{\partial}{\partial x^1} + \beta^2 \frac{\partial}{\partial x^2} \\
D_1 \circ D_2 &= \alpha^1 \frac{\partial}{\partial x^1}\left(\beta^1 \frac{\partial}{\partial x^1} + \beta^2 \frac{\partial}{\partial x^2}\right) + \alpha^2 \frac{\partial}{\partial x^2}\left(\beta^1 \frac{\partial}{\partial x^1} + \beta^2 \frac{\partial}{\partial x^2}\right) \\
&= \alpha^1\beta^1 \frac{\partial^2}{\partial^2 x^1} + \alpha^1\beta^2 \frac{\partial^2}{\partial x^1 \partial x^2} + \alpha^2\beta^1 \frac{\partial^2}{\partial x^2 \partial x^1} + \alpha^2\beta^2 \frac{\partial^2}{\partial^2 x^2}
\end{aligned}$$

If we simply focus on the first term and apply it to fg

$$\begin{aligned}
\frac{\partial^2 (fg)}{\partial^2 x^1} &= \frac{\partial}{\partial x^1}\left(\frac{\partial f}{\partial x^1} g + f \frac{\partial g}{\partial x^1}\right) \\
&= \frac{\partial^2 f}{\partial^2 x^1} g + 2 \frac{\partial f}{\partial x^1}\frac{\partial g}{\partial x^1} + f \frac{\partial^2 g}{\partial^2 x^1} \\
&\neq \frac{\partial^2 f}{\partial^2 x^1} g + f \frac{\partial^2 g}{\partial^2 x^1}
\end{aligned}$$

which means that it doesn't have the Liebniz Property. Similar arguments are made with the other terms which are all additive (that is there will be no chance of eliminating terms through subtraction) thus $D_1 \circ D_2 \notin \mathcal{D}_p(\mathbb{R})$.

$$\begin{aligned}
D_1 &= \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i} \\
D_2 &= \sum_{i=1}^n \beta^i \frac{\partial}{\partial x^i} \\
D_1 \circ D_2 &= \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^n \beta^j \frac{\partial}{\partial x^j}\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha^i \beta^j \frac{\partial^2}{\partial x^i \partial x^j} \\
&= \sum_{j=1}^n \beta^j \frac{\partial}{\partial x^j}\left(\sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}\right) \\
&= D_2 \circ D_1
\end{aligned}$$

consequently $D_1 \circ D_2 - D_2 \circ D_1 = 0$ and a derivation of A.

non-commutative Algebra

$$\begin{aligned}
D_1 \circ D_2 &= \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^n \beta^j \frac{\partial}{\partial x^j}\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \alpha^i \beta^j \frac{\partial^2}{\partial x^i \partial x^j} \\
D_2 \circ D_1 &= \sum_{i=1}^n \beta^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^n \alpha^j \frac{\partial}{\partial x^j}\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \beta^i \alpha^j \frac{\partial^2}{\partial x^i \partial x^j}
\end{aligned}$$

since the algebra A is non-commutative we know that $\alpha^i\beta^j \neq \beta^j\alpha^i$ and these two are not equal. However,

$$\begin{aligned}
D(fg) &= (D_1 \circ D_2 - D_2 \circ D_1)(fg) \\
&= (D_1 \circ D_2)(fg) - (D_2 \circ D_1)(fg) \\
&= D_1((D_2 f)g + f(D_2 g)) - D_2((D_1 f)g + f(D_1 g)) \\
&= (D_1 \circ D_2)(f)g + D_2 f D_1 g + D_1 f D_2 g + f(D_1 \circ D_2)g - (D_2 \circ D_1)(f)g - D_1 f D_2 g - D_2 f(D_1 g) - f(D_2 \circ D_1)g) \\
&= (D_1 \circ D_2 - D_1 \circ D_2)(f)g + f(D_1 \circ D_2 - D_2 \circ D_1)(g) \\
&= (Df)g + f(Dg)
\end{aligned}$$

Therefore $D = (D_1 \circ D_2 - D_2 \circ D_1$ is a derivation.