Functional Analysis– Spring 2024

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p. 290 #6, 7

6. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y.

   - $(\Rightarrow)$ $T^{-1} : \mathcal{R}(T) \to X$ is bounded. $\left\|T^{-1}y\right\| = \left\|T^{-1}\right\| \|y\|$. Given any Cauchy sequence $(y_n) \subset \mathcal{R}(T)$ must converge and let y be such that $y_n \to y$.
   - $(\Leftarrow)$ $\mathcal{R}(T)$ is closed.

7. Let $T : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, show that there are positive real numbers a and b such that $a \|x\| \leq \|Tx\| \leq b \|x\|$ for all $x \in X$.

p. 296 # 8, 9, 10

8. Let X and Y be normed spaces and let $T : X \to Y$ be a closed linear operator.

   (a) Show that the image A of a compact subset $C \subset X$ is closed in Y.

   (b) Show that the inverse image B of a compact subset $K \subset Y$ is closed in X. (Cf. Def. 2.5-1)

9. If $T : X \to Y$ is a closed linear opearator, where X and Y are normed spaces and Y is compact, show that T is bounded.

10. Let X and Y be normed spaces and X compact. If $T : X \to Y$ is a bijective closed linear operator, show that T^{-1} is bounded.

p. 246 #2, 3, 4

2. Give a simpler proof of Lemma 4.6-7 for the case tha tX is a Hilbert space.
3. If a normed space X is reflexive, show that X' is reflexive.
4. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive. (*Hint.* It can be shown that a closed subspace fo a reflexive Banach space is reflexive. Use this fact, without proving it.)

p. 268 #4, 7

4. Show that weak convergence in footnote 6 implies weak* convergence. Shwo that the converse holds if X is reflecive.

7. Let $T_n \in B(X, Y)$, where X is a Banach space. If (T_n) is strongly operator convergent, show that $(\|T_n\|)$ is bounded.