Math 725 – Advanced Linear Algebra
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Assignment #6 – Due 10/06/23

1. Let A be an $n \times n$ matrix with entries from a field F (actually, from any commutative ring R). Let

$$D(A) = A_{i_1,j_1} A_{i_2,j_2} \cdots A_{i_n,j_n}.$$

Prove that $D(A)$ is an n-linear function if and only if $i_1, i_2, \ldots, i_n$ are distinct.

Let there be two indices s, t such that $i_s = i_t$. Then, attempting to verify that $D(A)$ is n-linear on the ${i_s}^{\text{th}}$ column, i.e., multpling the ${i_s}^{\text{th}}$ column by a scalar c will be equivalent to $cD(A)$. That is (W1LOG we can assume that $s < t$)

$$
\begin{aligned}
D(cA) &= A_{i_1,j_1} \cdots cA_{i_s,j_s} \cdots A_{i_t,j_t} \cdots A_{i_n,j_n} \\
&= A_{i_1,j_1} \cdots cA_{i_s,j_s} \cdots cA_{i_s,j_s} \cdots A_{i_n,j_n} \\
&= c^2 D(A) \text{ which indicates that it is not } n\text{-linear}
\end{aligned}
$$

Further, there must be n elements and there cannot be any duplicates so essentially the set of indices i_k is permutation on the numbers $1, \ldots, n$.

2. Consider a Vandermonde matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{pmatrix}.$$

Show that the determinant of this matrix is equal to $\prod_{1 \leq i < j \leq n}(t_j - t_i)$. Conclude that a Vandermonde matrix is invertible if and only if $t_1, \ldots, t_n$ are distinct. [Hint: induction on n]

For $n = 2$ Let $A_2 = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \end{pmatrix}$ and $\det A_2 = t_2 - t_1$. Now, assuming that it is true for n we want to show that it is also true for $n + 1$. Let's start by taking our $(n+1) \times (n+1)$ Vandermonde matrix and using column reduction reduce all but the first column to 0.

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 1 & t_2 & t_2^2 & \cdots & t_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+1} & t_{n+1}^2 & \cdots & t_{n+1}^n \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & t_2 - t_1 & t_2^2 - t_1^2 & \cdots & t_2^n - t_1^n t_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+1} - t_1 & t_{n+1}^2 - t_1^2 t_{n+1}^2 & \cdots & t_{n+1}^n - t_1^n t_{n+1}^{n-1} \end{pmatrix}$$

Expanding by the first row and factoring $t_i - t_1$ from the ith row for $i = 2, \ldots, n + 1$ we find that

$$\det(A) = \prod_{i=2}^{n+1}(t_i - t_1) \det \begin{pmatrix} 1 & t_2 & \cdots & t_2^{n-1} \\ 1 & t_3 & \cdots & t_3^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+1} & \cdots & t_{n+1}^{n-1} \end{pmatrix}$$

because we assume that it works for n we can say that

$$\prod_{i=2}^{n+1}(t_i - t_1) \prod_{2 \leq i < j \leq n+1}(t_j - t_i) = \prod_{1 \leq i < j \leq n+1}(t_j - t_i)$$

And, if any two elements $t_i = t_j$ then $t_j - t_i = 0$ thus making the entire product, i.e., determinant, zero.

3.a) A matrix $A \in \mathcal{M}_{n \times n}(F)$ is called skew-symmetric if $A^t = -A$. If $F = \mathbb{C}$ and n is odd, show that $\det(A) = 0$ if A is skew-symmetric.

Notice that since the determinant is n-linear we get $\det(kA) = k^n \det(A)$. This is evident when we factor k out of each column of A.

Suppose $\det(A) \neq 0$. Then A is invertible and $AA^{-1} = I$. Hence

$$I = AA^{-1}$$
$$1 = \det(A) \det(A^{-1})$$
$$= \det(A^t) \det(A^{-1})$$
$$= \det(-A) \det(A^{-1})$$
$$= (-1)^n \det(A) \det(A^{-1})$$

which is only true when n is even. Otherwise, when n is odd $\det(A) = 0$

b) A matrix $A \in \mathcal{M}_{n \times n}(F)$ is called orthogonal if $A^t A = I$. Show that $\det(A) = \pm 1$ if A is orthogonal.

$$A^t A = I$$
$$\det(A^t A) = \det(I)$$
$$\det(A^t) \det(A) = 1$$
$$\det(A) = \det(A^t)^{-1}$$
$$= \det(A)^{-1}$$
$$= \pm 1$$

c) A matrix $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ is called unitary if $A^* A = I$. Here A^* is the conjugate transpose of A defined as $A^*_{ij} = \bar{A}_{ji}$ where $\bar{z}$ is the conjugate of a complex number z. Show that $|\det(A)| = 1$ if A is unitary.

Let $B = A^* A$ then the diagonal entries will be of the form

$$B_{ii} = \sum_{j=1}^{n} A^*_{ij} A_{ji} = \sum_{j=1}^{n} (a_{ij} - ib_{ij})(a_{ji} + ib_{ji}) = \sum_{j=1}^{n} (a_{ij}^2 + b_{ij}^2) = 1$$

This is essentially the value $A_i \cdot A_i = |A_i|^2$ (where A_i is a column vector of A). Similarly, off diagonal entries are effectively $A_i \cdot A_j = 0$ which means that they're orthogonal to each other and linearly independent. So, the magnitude of each vector A_i is 1 and they are orthogonal to each other, we call this an orthonormal basis which makes up all of A and hence $|\det(A)| = 1$

4. Let A be an $m \times n$ matrix with entries from a field F. For simplicity, we assume $m \leq n$. Now for any $k \leq m$ let $I = \{i_1, i_2, \ldots, i_k\}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $J = \{j_1, j_2, \ldots, j_k\}$ where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. We define $A(I, J)$ as the $k \times k$ matrix obtained from A by using the rows of A indexed by I and the columns of A indexed by J. The determinant of $A(I, J)$ is called a k-minor of A. Show that $\text{rank}(A) = k$ if and only if there exists a nonzero k-minor of A and all $k+1$-minors of A are zero.

Let $k = 2$. Then, given any $A(I, J)$ we'll have

$$A(I, J) = \begin{pmatrix} a_{I_1 J_1} & a_{I_1 J_2} \\ a_{I_2 J_1} & a_{I_2 J_2} \end{pmatrix}$$

$$\det(A(I, J)) = a_{I_1 J_1} a_{I_2 J_2} - a_{I_1 J_2} a_{I_2 J_1}$$

if this equals zero then

$$a_{I_1 J_1} a_{I_2 J_2} - a_{I_1 J_2} a_{I_2 J_1} = 0$$

$$a_{I_1 J_1} \frac{a_{I_2 J_2}}{a_{I_2 J_1}} - a_{I_1 J_2} = 0$$

which implies that the two 2-dimensional vectors represented here are linearly dependent. If this is true for ALL combinations of I, J then all of the vectors represented by the columns of A are linearly dependent and further, if there is just one combination where $\det(A(I, J)) \neq 0$ then the corresponding 2-dimensional vectors must be linearly independent. This implies that $\text{rank}(A)$ is at least 2.

Assuming that the $\text{rank}(A) = k$ and that there exists indices for I, J such that $B = A(I, J)$ and $\det(A(I, J)) \neq 0$. At this point, $\text{rank}(B) = k$. Expand B to include another row and another column from A and call it B'. The additional column will be linearly dependent on the columns of B. That is, through a series of column manipulations we can fill the $k + 1^{\text{th}}$ extra column of B' with zeros. These manipulations will have no effect on $\det(B')$ and that will be zero. Hence, all expanded matrices will have a determinant of zero.

5.a) Let $A \in \mathcal{M}_{n \times n}(F)$. Show that there are at most n scalars $c \in F$ such that $\det(cI - A) = 0$.

Let $B = \det(cI - A)$ then

$$B = \begin{pmatrix} c - A_{11} & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & c - A_{22} & \cdots & -A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & c - A_{nn} \end{pmatrix}$$

all A_{ij} are constants. This will generate an expression of terms each with n factors which is essentially a polynomial in c with degree at most n. This can be represented by $(c - b_1)(c - b_2) \cdots (c - b_n)$ for some constants $b_i, i = 1, \ldots, n$. Any one of these will set the entire polynomial to zero. Since each b_i need not be unique, we say "at most" n values for c.

b) Let $A, B \in \mathcal{M}_{n \times n}(F)$ where A is invertible. Show that there are at most n scalars $c \in F$ such that $cA - B$ is not invertible.

$$A^{-1}(cA - B) = A^{-1}cA - A^{-1}B = cA^{-1}A - A^{-1}B = cI - A^{-1}B$$

$A^{-1}B$ is an $n \times n$ matrix. Thus $cA - B$ will have no more than n values for c which make $\det(cA - B) = 0$, hence non-invertible.

6. Let $A \in \mathcal{M}_{n \times n}(F)$ where A is in block-diagonal form: $\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$, where A_j is a $r_j \times r_j$

matrix. Show that $\det(A) = \det(A_1) \det(A_2) \cdots \det(A_k)$.

Step 1: We know that

$$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$$

and that this is the same for all j. Since A_1 is a 1×1 matrix we can see that

$$E_1(A) = A_1 D_{11}(A)$$

Step 2: removing the first row and first column and designating $D_{11}(A) = A_1'$ and replacing A_2 with a matrix

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A_1' = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \mathbf{0} & \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

$$E_1(A_1') = a_{11} D_{11}(A_1') - a_{12} D_{12}(A_1')$$

this gives us two matrices

$$D_{11}(A_1') = \begin{pmatrix} a_{22} & 0 & \cdots & 0 \\ \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

$$D_{12}(A_1') = \begin{pmatrix} a_{21} & 0 & \cdots & 0 \\ \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

remove the first two rows and first two columns of A_1' and designate it A_2' and from Step 1 we know that

$$D_{11}(A_1') = a_{22} D_{11}(A_2')$$
$$D_{12}(A_1') = a_{21} D_{11}(A_2')$$
$$E_1(A_1') = (a_{11}a_{22} - a_{12}a_{21}) D_{11}(A_2')$$
$$= \det(A_2) D_{11}(A_2')$$
and $E_1(A) = \det(A_1) \det(A_2) D_{11}(A_2')$

Step 3: Each of the three terms in $E_1(A_2')$ will generate a matrix with the same form as A_1'. That is, each will have a 2×2 matrix in the first rows and first columns followed by zeros and a matrix A_3' containing the remainder of A. Just as in Step 2 each of these will generate a term of the form $D_{ij}(A_2')D_{11}(A_3')$ That is,

$$E_1(A_2') = \det(A_3)D_{11}(A_3')$$

$$\text{and } E_1(A) = \det(A_1)\det(A_2)\det(A_3)D_{11}(A_3')$$

Next Step: it should be clear now that the 4×4 matrix will generate four 3×3 matrices which we apply Step 3 to each and will sum to be a determinant for A_4 multiplied by the determinant of the remaining matrix. Extrapolating to n we get $D_{11}(A_n) = \det(A_n)$

$$E_1(A) = \det(A_1)\det(A_2)\cdots\det(A_n)$$

Extra Questions

1. In this exercise you will prove the famous Cramer's rule. For this let $Ax = b$ be a system of equations where $A \in \mathcal{M}_{n \times n}(F)$ and $b \in F^n$. Using the observation that $(\text{adj}A)A = \det(A)I$ show that $\det(A)x_i = \sum_{j=1}^{n}(-1)^{i+j}\det(A(j|i))b_j$ where $A(j|i)$ is obtained by deleting the jth row and ith column of A. Prove that the right-hand side of this expression is equal to $\det(B_i)$ where B_i is obtained from A by replacing the ith column of A with the vector b. Conclude that if $\det(A) \neq 0$ the unique solution to $Ax = b$ is obtained by $x_i = \frac{\det(B_i)}{\det(A)}$.

2. Let A be an $n \times (n+1)$ matrix of rank n. The rank nullity theorem tells us that null(A) is generated by a single nonzero vector. Show that this nonzero vector can be taken to be

$$(+\det(A^{(1)}), -\det(A^{(2)}), +\det(A^{(3)}), \dots, (-1)^{n+1}\det(A^{(n+1)}))$$

where $A^{(j)}$ is the $n \times n$ matrix obtained from A by deleting its jth column. [Hint: without loss of generality we can assume that a generating vector's last coordinate is equal to 1. Now use Cramer's rule.]

3. We have an efficient algorithm for solving the linear equation $Ax = b$. For simplicity let's assume that A is an $n \times n$ invertible matrix. However, there is one more thing one needs to worry about in computer science. What if the *size* of the solution is so big, that even if we have an efficient algorithm to produce the solution, it will consume all of the memory? Algorithms are considered to be good if they are efficient and the output they are expected to deliver is a polynomial function (instead of, say, an exponential function) in the input size of the problem. Here is how we define the input size of the linear system $Ax = b$. Note that, ultimately, a computer computes with rational numbers that are close approximations to real numbers. So if p/q is a rational number expressed in lowest terms the size of p/q is defined to be (roughly) $\log p + \log q$. This is approximately the number of bits needed to be used to represent p/q inside the computer. The size D of the input $Ax = b$ is the sum of sizes of all input numbers in the matrix A and b. Now using Cramer's rule and properties of determinants prove that the size of the unique solution to $Ax = b$ is at most a polynomial function in D.

4. For $m \leq n$ let A be an $m \times n$ matrix and B is an $n \times m$ matrix. For $I \subset \{1, 2, \dots, n\}$ of cardinality m, we let A_I be the $m \times m$ submatrix of A whose columns are those of A indexed by I, and we let B_I be the $m \times m$ submatrix of B whose rows are those of B indexed by I. Prove that

$$\det(AB) = \sum_{|I|=m} \det(A_I)\det(B_I).$$

Note that when $m = n$ this proves the product formula for determinants.