Math 5411 – Mathematical Statistics I– Fall 2024
w/Nezamoddini-Kachouie

Paul Carmody
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1. Review the attached document below and show that sample variance (with $n-1$ in denominator) is unbiased.

(quoting from the text on page 3)
We start with our definition of S^2

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

To find the *mean of* S^2, we'll the difference between our observation using two steps.

1. X_i to the sample mean $\hat{X}$
2. from the sample mean to the distribution mean.

that is

$$X_i - \mu = (X_i - \hat{X}) + (\hat{X} - \mu)$$

thus

$$\begin{aligned}
\sum_{i=1}^{n}\left(X_i - \mu\right)^2 &= \sum_{i=1}^{n}(X_i - \hat{X}) + (\hat{X} - \mu) \\
&= \sum_{i=1}^{n}\left(X_i - \mu\right)^2 + 2\sum_{i=1}^{n}\left(X_i - \mu\right)\left(\hat{X} - \mu\right) + \sum_{i=1}^{n}\left(\hat{X} - \mu\right)^2 \\
&= \sum_{i=1}^{n}\left(X_i - \mu\right)^2 + 2\left(\hat{X} - \mu\right)\sum_{i=1}^{n}\left(X_i - \mu\right) + \sum_{i=1}^{n}\left(\hat{X} - \mu\right)^2 \qquad \text{Note: } \hat{X} = \mu \\
&= \sum_{i=1}^{n}\left(X_i - \mu\right)^2 + n\left(\hat{X} - \mu\right)^2 \qquad (1)
\end{aligned}$$

Using (1) in the definition of S^2 we have

$$\begin{aligned}
S^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 + \left(\hat{X} - \mu\right)^2
\end{aligned}$$

Examining the *expected value* of variance, keeping in mind its linear properties, we have

$$\begin{aligned}
E[S^2] &= E\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \mu\right)^2 + \left(\hat{X} - \mu\right)^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left[\,(X_i - \mu)^2\,\right] - E\left[\,(\bar{X} - \mu)^2\,\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(X_i) - \mathrm{Var}(\hat{X}) \\
&= \frac{1}{n}n\sigma^2 - \frac{\sigma^2}{n} \\
&= \frac{n-1}{n}\sigma^2
\end{aligned}$$

this is NOT the same as the *true* variance σ^2. Hence S^2 is a biased estimator for σ^2. However, if we define the unbiased variance, S_u^2 as

$$\begin{aligned}
S_u^2 &= \frac{n}{n-1}S^2 \\
E\left[\,S_u^2\,\right] &= E\left[\frac{n}{n-1}S^2\right] \\
&= \frac{n}{n-1}E\left[\,S^2\,\right] \\
&= \frac{n}{n-1}\left(\frac{n-1}{n}\sigma^2\right) \\
&= \sigma^2
\end{aligned}$$

2. Find the variance of estimator of probability of success, $\hat{p} = X/n$, where X is the number of successes and n is the sample size. Notice that X has Binomial distribution.

$$\begin{aligned}
V(\hat{p}) &= V(X/n) \\
&= \frac{1}{n^2}E\left[\,(X-\mu)^2\,\right] \\
&= \frac{1}{n^2}V(X) \\
&= \frac{1}{n^2}\left(\,E[X^2] - E[X]^2]\,\right) \\
&= E[X^2/n^2] - (\mu/n)^2 \\
&= E[\hat{p}^2] - E[\hat{p}]^2
\end{aligned}$$