Math 5050 – Special Topics: Manifolds– Spring 2025

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Assignment 7 – April 30, 2025

Exercise 8.3: (The Differential of a Map). Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_* : T_pN \to T_{F(p)}M$ is a linear map.

Let f be a germ at $F(p)$. Then

$$(F_*(X_p))f = X_p(f \circ F) \in \mathbb{R}, \text{ for } f \in C^\infty_{F(p)}(M)$$

Need to show that F_* has the Liebniz Condition that is, given f, g of the same germ at $F(p)$ then

$$\begin{aligned}
(F_*(X_p))fg &= X_p(fg \circ F) \\
&= X_p((f \circ F)(g \circ F)) \\
&= (g \circ F)X_p(f \circ F) + (f \circ F)X_p(g \circ F) \\
&= (g \circ F)(F_*(X_p))f + (f \circ F)(F_*(X_p))g
\end{aligned}$$

8.2. **Differential of a linear map**. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$ there is a canonical identification $T_p(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \langle a^1, \dots, a^n \rangle$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Let $[L^j_k]$ be the Jacobian of L. From equation 8.2 of the test

$$L_{*,p}\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_k L^j_k \frac{\partial}{\partial x^k}\bigg|_{F(p)} = \frac{\partial F^i}{\partial x^j}(p)$$

applying any vector $v = \sum\limits_{i=1}^n a_i \frac{\partial}{\partial x^i}$ we have

$$\begin{aligned}
L_{*,p}(v) &= \sum_{i=1}^n a_i L_{*.p}\left(\frac{\partial}{\partial x^i}\bigg|_p\right) \\
&= \sum_{i=1}^n a_i \sum_{k=1}^m L^i_k \frac{\partial}{\partial x^k}\bigg|_{F(p)} \\
&= \sum_{k=1}^m \left(\sum_{i=1}^n a_i L^i_k \frac{\partial}{\partial x^k}\bigg|_{F(p)}\right) \\
&= \sum_{k=1}^m L^k(v) \\
&= L(v)
\end{aligned}$$

8.3. **Differental on a map**

Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Let $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\right)\big|_{F(p)}$, find a and b in terms of x, y, and α.

Since F is linear its Jacobian is constant and at p we have

$$\begin{aligned}
p = (x, y) &= \begin{bmatrix} -y \\ x \end{bmatrix} \\
F_*(X_p) &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} -y \\ x \end{bmatrix} \\
&= \begin{bmatrix} -y\cos\alpha - x\sin\alpha \\ -y\sin\alpha + x\cos\alpha \end{bmatrix} \\
&= (-y\cos\alpha - x\sin\alpha)\frac{\partial}{\partial u} + (-y\sin\alpha + x\cos\alpha)\frac{\partial}{\partial V}
\end{aligned}$$

therefore $a = -y\cos\alpha - x\sin\alpha$ and $b = -y\sin\alpha + x\cos\alpha$

8.4. **Transition matrix for coordinate vectors**

Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the open set

$$U = \mathbb{R}^2 - \{(x,0)|x \geq 0\}.$$

On U the polar coordinates r, θ are uniquely defined by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, r > 0, 0 < \theta < 2\pi \end{aligned}$$

find $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Given any $f : U \to \mathbb{R}$ We have

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y} \end{aligned}$$

that is the expression may be expressed as

$$\begin{aligned} \frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \end{aligned}$$