Math 5110 – Real Analysis I– Fall 2024
w/Professor Liu

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Homework #2 – September 18, 2024

I. Consider a sequence x_n of real numbers. The *limit inferior* and *limit superior* of x_n are defined by

$$\liminf_{n\to\infty} x_n = \lim_{n->\infty}\left(\inf_{k\geq n} x_k\right),\ \limsup_{n\to\infty} x_n = \lim_{n->\infty}\left(\sup_{k\geq n} x_k\right)$$

(a) Show that

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0}\left(\inf_{k\geq n} x_l\right)$$

and

$$\limsup_{n\to\infty} x_n = \inf_{k\geq n}\left(\sup_{k\geq n} x_n\right)$$

If $\{x_n\}$ were not bounded then this questions has no value. Assuming, then, that x_n is bounded above by its least upper bound, M, and below by its most lower bound, L. We need to be aware of a repeating sequence or subsequence. For example, $x_n = c$ where c or $x_n = \{1/2, c, 1/4, c, 1/6, c, 1/8, \dots\}$. These sequences are also bounded and form trivial solutions to this question.
We will focus, then, on the convergent subsequence, $\{x_{\alpha_k}\}$ where $\alpha_k \in A$ and A is an infinite list of indices. We have two types of these convergent sequences those that coverge increasingly and those that converge decreasingly.

Converge Increasingly. These subsequences may have many values but as $n \to \infty$ we see that $|x_{\alpha_n} - M| \to 0$ and without loss of generality, allow it to be Cauchy.
Claim: given any $n \geq 0$, the $\inf_{k\geq n} x_{\alpha_k} = x_{\alpha_n}$. We can see that given any $\epsilon > 0$ there is $N \in \mathbb{N}$ we have $|x_{\alpha_n} - x_{\alpha_k}| < \epsilon$ whenever $N < n < k$. That is, the subsequence is increasing, thus, $x_{\alpha_n} < x_{\alpha_k}$ for all $k \geq n$. Thus $\inf_{k\geq n} x_{\alpha_k} = x_{\alpha_n}$. Further, a $n \to \infty$ the infimum $\inf_{k\geq n} x_{\alpha_k}$ increases thus the supremum of these values is

$$\liminf_{n\to\infty} x_{\alpha_n} = \sup_{n\geq 0}\left(\inf_{k\geq n} x_{\alpha_k}\right)$$

(b) Show that $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are well-defined for any sequence x_n. (Unlike $\lim_{n\to\infty} x_n$.) We allow values of ∞ and $-\infty$

(c) Let x_n be a bounded sequence, and let L be the set of limit points of x_n, i.e., the set of all limits of subsequences of x_n. Show $\liminf_{n\to\infty} x_n = \inf L$ and $\limsup_{n\to\infty} = \sup L$.

(d) Let x_n be a bounded sequence. Conclude using (c) that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, with equality if and only if x_n is convergent.

II. Prove that for any (possibly uncountable) collection $(F_\alpha)_{\alpha\in A}$ of closed sets, the intersection $F = \bigcup_{\alpha\in A} F_\alpha$ is closed, in two ways.

(a) Using the fact that any union of open sets is open, and DeMorgan's Laws from set theory, which state

$$X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A}(\,X\backslash E_\alpha\,) \text{ and } X\backslash\left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A}(\,X\backslash E_\alpha\,)$$

for all collection of sets $(E_\alpha)_{\alpha\in A}$
Given that every open set, $E \in X$ is the union of other open sets $\bigcup_{\alpha\in A} E_\alpha$ for some index set A (whether countable or uncountable). We know that the complement is closed and the complement can be expressed as

$$\begin{aligned} E^c &= X\backslash E \\ &= X\backslash\left(\bigcup_{\alpha\in A} E_\alpha\right) \\ &= \bigcap_{\alpha\in A}(\,X\backslash E_\alpha\,) \end{aligned}$$

each E_α is the complement of an open set, hence they are closed. Thus, E^c which is closed is made up of the intersection of closed sets.

(b) More directly, using the fact that a set G is closed if and only if for any convergent sequence (x_n) with all $x_n \in G$, the limit x is also in G.
Let $F, G \in X$ be closed sets and let $(x_n) \subset G$ and $(y_n) \subset F$ both be convergent sequences. Further, we let $(x_n), (y_n) \subset G \cap F$. Not that F closed means that $(x_n) \in F$ implies that $\lim_{n\to\infty} x_n \in F$, thus $\lim_{n\to\infty} x_n \in G \cap F$ and a similar argument can be made for y_n and G. Thus sequences contained in $G \cap F$ must also contain their limits and $G \cap F$ is closed. This can extend to any number of intersections.

III. (a) Let (x_n) be a Cauchy sequence in a metric space X. Show that if a subsequence (x_{n_j}) of x_n converges to x, then the entire sequence also converges to x.

(b) Show that the metric space

$$C^1((-1,1)) = \{f : (-1,1) \to \mathbb{R}, f \text{ is differentiable and } f' \text{ is continuous in } (1,-1)\}$$

with the metric

$$d(f,g) = \sup_{x\in(-1,1)} |f(x) - g(x)|$$

is not complete. (Hint: similar to the proof that the rational numbers are not complete, find a sequence $C'((-1,1))$ that converges in d metric to a function that is not in $C^1((-=1,1))$, and show that this sequence is Cauchy.) ,r to the proof that the rational numbers are not complete, find a sequence $C'((-1,1))$ that converges in d metric to a function that is not in $C^1((-=1,1))$, and show that this sequence is Cauchy.)

IV. Let A and B be subsets of the metric space X. which one of the following is true?

$$(A \cup B)^o = A^o \cup B^o, \tag{2.1}$$
$$(A \cup B)^o \subset A^o \cup B^o, \quad \text{"=" fails for some } A \text{ and } B \tag{2.2}$$
$$(A \cup B)^o \supset A^o \cup B^o, \quad \text{"=" fails for some } A \text{ and } B \tag{2.3}$$

V. Let $C^0([a,b])$ be the space of continuous functions on $[a,b]$, with the metric $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$.

Show that the map $I : C^0([a,b]) \to \mathbb{R}$ defined by $I(f) = \int_a^b f(x)dx$ is continuous mapping from $C^0([a,b])$ to $\mathbb{R}$.
I is continuous if for ever $\epsilon > 0$ there exists $\delta > 0$ such that $d(I(f), I(g)) < \epsilon$ whenever $d(f,g) < \delta$. Or

$$\begin{aligned}
d(I(f), I(g)) &= \sup_{x\in[a,b]} |I(f(x)) - I(g(x))| \\
&= \sup_{x\in[a,b]} \left| \int_a^b f(x)dx - \int_a^b g(x)dx \right| \\
&= \sup_{x\in[a,b]} \left| \int_a^b f(x) - g(x)dx \right| \\
&= \sup_{x\in[a,b]} \int_a^b |\, f(x) - g(x) \,| \, dx \\
&\le \int_a^b \sup_{x\in[a,b]} |\, f(x) - g(x) \,| \, dx \\
&\le \int_a^b d(f,g)dx \\
&\le d(f,g)[b-a]
\end{aligned}$$

Thus when $\epsilon > 0$ choose $\delta < [b-a]d(f,g)$. Hence, I is continuous.

VI. Prove Propostion 2.3.2 in the text, in two different ways.:

a) As a consequence of Theorem 2.3.1 in text.

b) Directly, using the sequential definition of compactness. **Proposition 2.3.2** (Maximum principle). *Let (X,d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a contnuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{\max} \in X$, and also attains its minimum at some point $x_{\min} \in X$.*

VII. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\lim_{|x|\to\infty} f(x) = +\infty$$

Prove that f attains its minimum.
Recall that by definition, the limit in (??) means that Given $A > 0$, there is $R > 0$ such that

$$f(x) > A \text{ for all } x \notin B_R$$

in other words, $f(x) > A$ whenever $|x| \ge R$. Here, $|x| = d_2(x,0)$ and d_2 is the standard Euclidean distance defined in Example 1.4.