Math 725 – Advanced Linear Algebra

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Assignment #2 – Due 9/6/23

1. Let V and W be two vector spaces over the field F. In the previous homework you showed that $V \times W$ is also a vector space. Now suppose $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$. Show that $V \times W = V_1 \times W_1 \oplus V_2 \times W_2$.

$V = V_1 \oplus V_2$ means that for every $x \in V$ there exists $v_1 \in V_1$ and $v_2 \in V_2$ such that $x = v_1 + v_2$. Similarly, $W = W_1 \oplus W_2$ means that for every $y \in W$ there exists $w_1 \in W_1$ and $w_2 \in W_2$ such that $y = w_1 + w_2$. Given any $z \in V \times W$ we can see that $z = (x, y) = (v_1 + v_2, w_1 + w_2) = (v_1, w_1) + (v_2, w_2)$. Clearly, $(v_1, w_1) \in V_1 \times W_1$ and $(v_2, w_2) \in V_2 \times W_2$. It is also clear that this is the only way to represent z thus $V \times W = V_1 \times W_1 \oplus V_2 \times W_2$.

2. Let u, v, w be three vectors in a vector space V which are linearly independent. Show that $u, u+v, u+v+w$ are also linearly independent.

u, v, w are linearly independent means that $au + bv + cw = 0$ implies that $a = b = c = 0$. Thus,

$$\begin{aligned} au + b(u+v) + c(u+v+w) &= au + bu + bv + cu + cv + cw \\ &= (a+b+c)u + (b+c)v + cw \\ \text{when } (a+b+c)u + (b+c)v + cw &= 0 \\ \text{then } a+b+c = b+c = c &= 0 \\ \therefore a = b = c &= 0 \end{aligned}$$

hence they are linearly independent.

3. An $n \times n$ matrix P is called a *permutation* matrix if P is obtained from the identity matrix I_n by a sequence of row swaps. Are the six 3×3 permutation matrices linearly independent over $\mathbb{R}$? Justify your answer.

Let's take a linear combination of all of them

$$a\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} b+f & a+d & c+e \\ a+e & c+f & b+d \\ c+d & b+e & a+f \end{pmatrix} = 0$$

we have nine equations in 6 unknown variables. These can be separated into three lists.

$$\begin{aligned} a+d = b+d = c+d = 0 &\implies a = b = c = -d \\ a+e = b+e = c+e = 0 &\implies a = b = c = -e \\ a+f = b+f = c+f = 0 &\implies a = b = c = -f \end{aligned}$$

which is true when $a = b = c = 1$ and $d = e = f = -1$ therefore they are not linearly independent over $\mathbb{R}$.

4. Prove that $\mathcal{F}(\mathbb{R},\mathbb{R})$ is infinite dimensional.

Let $\mathcal{P}^{(n)}(\mathbb{R})$ be the set of polynomial functions with degree less than or equal to n. Clearly, $\mathcal{P}^{(n)}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R},\mathbb{R})$ for all $n \in \mathbb{Z}^+$ and is a subspace. The dimension of $\mathcal{P}^{(n)}(\mathbb{R})$ is n and the range of n is infinite. Thus $\mathcal{F}(\mathbb{R},\mathbb{R})$ is infinitely dimensional.

5. Compute the dimensions of the vector spaces of $n \times n$ symmetric matrices and $n \times n$ skew-symmetric matrices by exhibiting simple bases.

Let A be a symmetric matrix. Then.

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,n} \\ A_{1,2} & A_{2,2} & A_{2,3} & \cdots & A_{2,n} \\ A_{1,3} & A_{2,3} & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{1,n} & A_{2,n} & A_{3,n} & \cdots & A_{n,n} \end{pmatrix} \Longrightarrow \begin{pmatrix} n \text{ elements} \\ n-1 \text{ elements} \\ n-2 \text{ elements} \\ \vdots \\ 1 \text{ elements} \end{pmatrix}$$

the number N of independent elements is

$$N = \sum_{i=1}^{n} i = (n+1)n/2$$

Thus, the number of linearly independent vectors to form a basis would have to be $(n+1)n/2$ which is the dimension. A basis would have the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdots$$

and so on.

Similarly, Let B be a skew-symmetric matrix. Then.

$$B = \begin{pmatrix} 0 & B_{1,2} & B_{1,3} & \cdots & B_{1,n} \\ -B_{1,2} & 0 & B_{2,3} & \cdots & B_{2,n} \\ -B_{1,3} & -B_{2,3} & 0 & \cdots & B_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ -B_{1,n} & -B_{2,n} & -B_{3,n} & \cdots & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} n-1 \text{ elements} \\ n-2 \text{ elements} \\ n-3 \text{ elements} \\ \vdots \\ 0 \text{ elements} \end{pmatrix}$$

the number M of independent elements is

$$M = \sum_{i=0}^{n-1} i = n(n-1)/2$$

Thus, the number of linearly independent vectors to form a basis would have to be $n(n-1)/2$ which is the dimension. A basis would have the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & -1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \cdots$$

and so on.

6. Let $\mathcal{P}^{(2)}(F)$ be the F-vector space of polynomials of degree at most 2, and let $\lambda \in F$ be fixed. Define

$$g_1(x) = 1,\ \ g_2(x) = x + \lambda,\ \ g_3(x) = (x+\lambda)^2.$$

Prove that $\mathcal{B} = (g_1, g_2, g_3)$ is a basis for $\mathcal{P}^{(2)}(F)$. If $f(x) = c_0 + c_1 x + c_2 x^2$ what are the coordinates of f in the basis $\mathcal{B}$?

Let $a + bx + cx^2 \in \mathcal{P}^{(2)}(F)$ where $a, b, c \in F$. Each term constitutes another polynomial that is a member of $\mathcal{P}^{(2)}(F)$. that is $a \in \mathcal{P}^{(2)}(F)$, $bx \in \mathcal{P}^{(2)}(F)$ and $cx^2 \in \mathcal{P}^{(2)}(F)$. If $\mathcal{B}$ is a basis then we should be able to find $u, v, w \in F$ each in terms of a, b, c such that

$$\begin{aligned} a + bx + cx^2 &= ug_1(x) + vg_2(x) + wg_3(x) \\ &= u + v(x+\lambda) + w(x+\lambda)^2 \\ &= u + vx + v\lambda + wx^2 + 2wx\lambda + w\lambda^2 \\ &= (u + v\lambda + w\lambda^2) + (v + 2w\lambda)x + wx^2 \end{aligned}$$

$$\begin{array}{lll} w = c & b = v + 2w\lambda & a = u + v\lambda + w\lambda^2 \\ & b = v + 2c\lambda & a = u + v\lambda + c\lambda^2 \\ & v = b - 2\lambda c & \end{array}$$

$$\begin{aligned} a &= u + (b - 2\lambda)\lambda + c\lambda^2 \\ u &= a - b\lambda + (2-c)\lambda^2 \end{aligned}$$

Hence, any vector in $\mathcal{P}^{(2)}(F)$ can be written as a linear combination of elements in $\mathcal{B}$. Further

$$\begin{aligned} ag_1(x) + bg_2(x) + cg_3(x) &= a + b\lambda + bx + cx^2 + 2c\lambda x + c\lambda^2 \\ &= a + b\lambda + c\lambda^2 + (b + 2c\lambda)x + c\lambda x^2 \\ &= 0 \text{ when} \\ & \begin{array}{lll} c = 0 & b + 2c\lambda = 0 & a + b\lambda + c\lambda^2 = 0 \\ & b = 0 & a + b\lambda = 0 \\ & & a = 0 \end{array} \end{aligned}$$

hence $\mathcal{B}$ is linearly independent thus forms a basis.

Extra Questions

1. Let F be a finite field of size $|F| = q$, and let V be an F-vector space of dimension n. In this exercise you will prove that the number of subspaces of V of dimension k is

$$\binom{n}{k}_q = \frac{(q^n - 1)\cdots(q-1)}{(q^k-1)\cdots(q-1)(q^{n-k}-1)\cdots(q-1)}.$$

The expressions $\binom{n}{k}_q$ are called q-binomial coefficients or Gaussian coefficients, and they have properties similar to those of binomial coefficients. Now let $s(n,k)$ be the number of k-dimensional subspaces of V.

a) Let $m(n,k)$ be the number of k-tuples of linearly independent vectors $(v_1, v_2, \ldots, v_k)$ in V. Show that

$$m(n,k) = (q^n - 1)(q^n - q)\cdots(q^n - q^{k-1}).$$

The first vector v_1 may be drawn from q^n possible elements minus the zero vector, hence $q^n - 1$. In order for v_2 to be linearly independent from v_1 thus it can be any vector except those that are multiples of v_1 (i.e., any xv_1 where$x \in q$) that is $q^n - q$. v_3 must be linearly independent of both v_1 and v_2 (i.e., $y((xv_1) + v_2)$ where $x, y \in q$) so v_3 must be chosen from $q^n - q^2$ and so on, until we get to the kth element which is drawn from $q^n - q^{k-1}$.

b) Each of the k-tuples in **a)** can be obtained by first selecting a subspace of dimension k and then choosing k linearly independent vectors from that subspace. Show that for any k-dimensional subspace, the number of k-tuples of linearly independent vectors from that subspace is

$$(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1}).$$

Similarly, any basis will be made up of k-tuples of vectors. The first vector may be chozen from q^k possibilities, minus the zero vector. The second vector in the basis must be linearly independent from the first so it may be chosen from $q^k - q$ remaining vectors and so on. Until the last vector which is chosen from $q^k - q^{k-1}$ possibilities

c) Show that $m(n,k) = s(n,k)(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$ and finish the proof.

$s(n,k)$ is the number of subspaces with dimension k where $m(n,k)$ represents the number of linearly independe sets of size k. We can calculate $m(n,k)$ from $s(n,k)$ by multiplying it my the number of potential first elements $q^n - 1$ then by the number of potential second elements $q^n - q$ and so on.

2. Let V be a vector space and $\mathcal{B} = \{v_i : i \in I\}$ be a basis of V. Let $\{B_1, \ldots, B_k\}$ be a partition of $\mathcal{B}$. If W is a subspace of V, is it true that

$$W = \bigoplus_{i=1}^{k} (W \cap \operatorname{span}(B_i))\,?$$

Given any $w \in W$ there exist c_j such that $w = \sum_{j \in I} c_j v_j$. And each B_i contains a disjoint subset of these v_j. Let β_i be the set of indices of the vectors which represent the basis vectors for B_i, i.e., β_i is a basis for B_i and $\{v_{\beta_j}\}$ forms a basis of B_i. Hence $w = \sum_{i=0}^{k} \sum_{j \in \beta_i} c_j v_j$ where each sum $\sum_{j \in \beta_i} c_j v_j \in B_i$.

Indeed, each of these sums are precisely $\sum_{j\in\beta_i} c_j v_j \in W \cap \mathrm{span}(B_i)$. These are clearly independent of each other thus w will be the direct sum of each as each B_i are disjoint from each other by deinition of a partition, thus $W = \bigoplus_{i=1}^{k}(W \cap \mathrm{span}(B_i))$.

3. Let V be finite dimensional vector space over an infinite field F. Prove that if $W_1, \dots, W_k$ are subspaces of V of equal dimension, then there is a subspace U of V such that $V = W_i \oplus U$ for $i = 1, \dots, k$.

4. Let E be a field and F be a subfield of E. The dimension of E as a vector space over F is denoted by $[E : F]$.
a) Let $F \subset K \subset E$ be three fields where $[K : F]$ and $[E : K]$ are finite. Show that $[E : F] = [E : K][K : F]$.
b) Show that $\mathbb{R}$ is an infinite dimensional vector space over $\mathbb{Q}$. [Hint: $\mathbb{R}$ is uncountable where $\mathbb{Q}$ is countable].