# Functional Analysis– Spring 2024

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**#2**. Show that if the orthogonal dimension of Hilbert Space $H$ is finite, it equals the dimension of $H$ regarded as a vector space; conversely, if the latter is finite, show that so is the former.

Assuming that the Hilbert Dimension of HIlbert space $H$ is finite, $\dim H = n$. Let the orthonormal family $(e_\alpha)_{\alpha \in A} \in H$ for some set $A$. Further let there exists a countable subset of $A'$ such $\left\langle e_{\alpha_j}, e_{\alpha_k} \right\rangle = \delta_{jk}$ for all $j, k \in [1..n]$ and that $\operatorname{span}\{e_{\alpha_k}\}$ is dense and equal to $H$. Further, if $y \perp \operatorname{span}\{e_{\alpha_k}\}$ then $y = 0$. Therefore, given any $x \in H$, $x = \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$. We can see, then, that every $x \in H$ is a linear combination of $\{e_{\alpha_k}\}$. Hence $\{e_{\alpha_k}\}$ forms a basis. There must be $n$ elements in $A'$, hence the vector space dimension is the same as the Hilbert space dimension.

Assuming that we have a finite dimensional vector space $X$. Then there exists an orthonormal basis $\{e_k\} \in X$. Define an Inner Product on $X, \langle \cdot, \cdot \rangle$. Clearly, $\langle e_j, e_k \rangle = \delta_{jk}$. Also, given any $x \in X$ such that $\langle x, e_k \rangle = 0$ for all $k \in [1, n]$ we can see that $x = 0$ as all $e_k$ are linearly independent from each other. Hence, $\operatorname{span} e_k$ is dense in $X$. $X$ must be a Hilbert space with dimension $n$.

**#4** Derive from (3) the following formula (which is often called the *Parseval relation*).

$$\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$$

Given an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ on $H$ we can define $x \in H$ as

$$\begin{aligned}
x &= \sum_k \langle x, e_k \rangle e_k \\
\|x\|^2 &= | \langle x, x \rangle | \\
&= \left| \left\langle \sum_k \langle x, e_k \rangle e_k, \sum_j \langle x, e_j \rangle e_j \right\rangle \right| \\
&= \sum_k \sum_j | \langle \langle x, e_k \rangle e_k, \langle x, e_j \rangle e_j \rangle | \\
&= \sum_k \sum_j \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \langle e_k, e_j \rangle \right| \\
&= \sum_k \sum_j \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \delta_{jk} \right| \\
&= \sum_k \left| \langle x, e_k \rangle \overline{\langle x, e_j \rangle} \right|
\end{aligned}$$

replacing the right $x$ in the Inner Product with $y$ and we get

$$\langle x, y \rangle = \sum_k \langle x, e_k \rangle \overline{\langle y, e_k \rangle}$$

#5 Show that an orthonormal family $(e_\kappa), \kappa \in I$, in a Hilbert Space $H$ is total if and only if the relation in Prob. 4 holds for every $x$ and $y$ in $H$.

($\Rightarrow$) Let an orthonormal family $(e_\kappa), \kappa \in I$, in a Hilbert Space $H$ be total. Let $x, y \in H$ we know that we can represent them as $x = \sum_k \langle\, x, e_\kappa \,\rangle\, e_\kappa$ and $y = \sum_\iota \langle\, y, e_\kappa \,\rangle\, e_\iota$ where $\kappa, \iota \in I$. Thus

$$\begin{aligned}
\langle\, x, y \,\rangle &= \left\langle \sum_\kappa \langle\, x, e_\kappa \,\rangle\, e_\kappa,\ \sum_\iota \langle\, y, e_\iota \,\rangle\, e_\iota \right\rangle \\
&= \sum_\kappa \sum_\iota \langle\, \langle\, x, e_\kappa \,\rangle\, e_\kappa,\ \langle\, y, e_\iota \,\rangle\, e_\iota \,\rangle \\
&= \sum_\kappa \sum_\iota \langle\, x, e_\kappa \,\rangle\, \overline{\langle\, y, e_\iota \,\rangle}\, \langle\, e_\kappa,\ e_\iota \,\rangle \\
&= \sum_\kappa \sum_\iota \langle\, x, e_\kappa \,\rangle\, \overline{\langle\, y, e_\iota \,\rangle} \delta_{\kappa\iota} \\
&= \sum_\kappa \langle\, x, e_\kappa \,\rangle\, \overline{\langle\, y, e_\kappa \,\rangle}
\end{aligned}$$

$x, y$ are arbitrary therefore true for all elements of $H$.

($\Leftarrow$)Assuming that this is true for all $x, y \in H$ and we have an orthonormal set $(e_\kappa) \in H$ where $\kappa \in I$. We can see that the same steps can be executed in reverse indicating that all $x \in H$ can be represented as $x = \sum_k \langle\, x, e_\kappa \,\rangle\, e_\kappa$. This indicates that $\text{span}\{e_\kappa\} = H$. Let $z \in H$ such that $z \in (e_\kappa)^\perp$ then $\langle\, z, e_\kappa \,\rangle = 0$ for all $\kappa \in I$. We know that $z = \sum_\kappa \langle\, z, e_\kappa \,\rangle\, e_\kappa$. Therefore $z = 0$ and $(e_\kappa)^\perp = \{0\}$, hence, $\text{span}\{e_\kappa\}$ must be dense.

**#10** Let $M$ be a subset of a Hilbert space $H$, and let $v, w \in H$. Suppose that $\langle\, v, x \,\rangle = \langle\, w, x \,\rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$ show that $M$ is total in $H$.

$$\begin{aligned}
\text{let } z \in M^\perp,\ \langle\, z, x \,\rangle &= 0,\ \forall x \in M \\
\langle\, v, x \,\rangle &= \langle\, w, x \,\rangle + \langle\, z, x \,\rangle \\
&= \langle\, w + z, x \,\rangle \\
v &= w + z \\
v - w &= z
\end{aligned}$$

$v - w = 0$ thus $z = 0$. Since $z$ was arbitrary $M^\perp = \{0\}$ and therefore $\text{span}\{M\}$ is dense in $H$.

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#2 **(Space $\ell^2$)** Show that every bounded linear functional $f$ on $\ell^2$ can be represented in the form

$$f(x) = \sum_{j=1}^{\infty} \xi_j \overline{\zeta_j} \qquad [z = (\zeta_j) \in \ell^2].$$

Given $x = (\xi_n) \in \ell^2$. Let $f$ be a bounded linear functional. Then, by the Reisz Representation Theorem

$$\begin{aligned} f &: \ell^2 \to K & K &= \mathbb{R} \text{ or } \mathbb{C} \\ f(x) &= \langle\, x, z \,\rangle & \text{for some } z &= (\zeta_n) \in \ell^2 \\ &= \langle\, (\xi_n), (\zeta_m) \,\rangle \\ &= \sum_{k=1}^{\infty} \xi_k \overline{\zeta_k} \end{aligned}$$

#4 Consider Prob. 3. If the mapping $X \to X'$ given by $z \mapsto f$ is surjective, show that $X$ must be a Hilbert space.

Given $z$ then $f$ is a bounded linear functional on $X$ with $\|z\| = \|f\|$.

$$z \mapsto f(x) = \langle\, x, z \,\rangle$$

#5 Show that the dual space of the real space $\ell^2$ is $\ell^2$. (Use 3.8-1.)

Let $f \in \ell^{2'}$. Then there exists $z = (\zeta_n) \in \ell^2$ such that $f(x) = \langle\, x, z \,\rangle$ for all $x \in \ell^2$. Hence, for $x = (\xi_n)$ we have

$$\begin{aligned} f(x) &= \langle\, x, z \,\rangle \\ &= \langle\, (\xi_n), (\zeta_m) \,\rangle \\ &= \sum_{k=1}^{\infty} \xi_k \zeta_k \\ &= \langle\, (\zeta_m), (\xi_n) \,\rangle \\ &= \langle\, z, x \,\rangle \end{aligned}$$

#7 Show that the dual space $H'$ of a Hilbert space $H$ is a Hilbert space with inner product $\langle\, \cdot, \cdot \,\rangle_1$ defined by

$$\langle\, f_x, f_v \,\rangle_1 = \overline{\langle\, z, v \,\rangle} = \langle\, v, z \,\rangle\,,$$

where $f_z(x) = \langle\, x, z \,\rangle$, etc.