

Topology without Tears

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Chapter 1

Topology Spaces

1.1 Topology – Exercises

- Let $x = \{a, b, c, d, e, f\}$. Determine whether or not each of the following collections of subsets of X is a topology on X :
 - $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$;
No, $\{a, f\} \cap \{b, f\} = \{f\} \notin \mathcal{T}$.
 - $\mathcal{T}_2 = \{X, \emptyset, \{a, b, f\}, \{a, b, d\}, \{a, b, d, f\}\}$;
No, $\{a, b, f\} \cap \{a, b, d\} \notin \mathcal{T}$.
 - $\mathcal{T}_3 = \{X, \emptyset, \{f\}, \{e, f\}, \{a, f\}\}$;
No, $\{e, f\} \cup \{a, f\} = \{a, e, f\} \notin \mathcal{T}$.
- Let $X = \{a, b, c, d, e, f\}$. Which of the following collections of subsets of X is a topology on X ? (Justify your answer.)
 - $\mathcal{T}_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$;
 - $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$;
 - $\mathcal{T}_3 = \{X, \emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}\}$;
- If $X = \{a, b, c, d, e, f\}$, and \mathcal{T} is the discrete topology on X , which of the following statements are true?
 - $X \in \mathcal{T}$; **YES** (b) $\{X\} \in \mathcal{T}$; **NO** (c) $\{\emptyset\} \in \mathcal{T}$; **NO** (d) $\emptyset \in \mathcal{T}$; **YES**
 - $\emptyset \in X$; **NO** (f) $\{\emptyset\} \in X$; **NO** (g) $\{a\} \in \mathcal{T}$; **YES** (h) $a \in \mathcal{T}$; **NO**
 - $\emptyset \subseteq X$; **YES** (j) $\{a\} \in X$; **NO** (k) $\{\emptyset\} \subseteq X$; **YES** (l) $a \in X$; **YES**
 - $X \subseteq \mathcal{T}$; **YES** (n) $X \subseteq \mathcal{T}$; **YES** (o) $\{X\} \subseteq \mathcal{T}$; **NO** (p) $a \subseteq \mathcal{T}$; **NO**
- Let (X, \mathcal{T}) be any topological space. Verify that *the intersection of any finite number of members of \mathcal{T} is a member of \mathcal{T}* .
- Let \mathbb{R} be the set of all real numbers. Prove that each of the following collections of subsets of \mathbb{R} is a topology
 - \mathcal{T}_1 consists of \mathbb{R}, \emptyset , and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;
 - \mathcal{T}_2 consists of \mathbb{R}, \emptyset , and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;
 - \mathcal{T}_3 consists of \mathbb{R}, \emptyset , and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;
- \mathcal{T}_1 consists of \mathbb{N}, \emptyset , and every set $\{1, 2, \dots, n\}$, for n any positive integer. (This is called *initial segment topology*).
 - \mathcal{T}_2 consists of \mathbb{N}, \emptyset , and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the *final segment topology*).
- List all possible topologies on the following sets:
 - $X = \{a, b\}$;
 - $Y = \{a, b, c\}$;
- Let X be an infinite set and \mathcal{T} a topology on X . If every infinite subset of X is in \mathcal{T} , prove that \mathcal{T} is the discrete topology.

9. Let \mathbb{R} be the set of all real numbers. Precisely three of the following ten collections are subsets of \mathbb{R} that are topologies. Identify these and justify your answer.

- (i) \mathcal{T}_1 consists of \mathbb{R}, \emptyset , and every interval (a, b) , for a and b any real numbers where $a < b$.
- (ii) \mathcal{T}_2 consists of \mathbb{R}, \emptyset and every interval $(-r, r)$, for r any positive real number.
- (iii) \mathcal{T}_3 consists of \mathbb{R}, \emptyset , and every interval $(-r, r)$, for r any positive rational number;
- (iv) \mathcal{T}_4 consists of \mathbb{R}, \emptyset , and every interval $[-r, r]$, for r any positive rational number;
- (v) \mathcal{T}_5 consists of \mathbb{R}, \emptyset , and every interval $(-r, r)$, for r any positive irrational number;
- (vi) \mathcal{T}_6 consists of \mathbb{R}, \emptyset , and every interval $[-r, r]$, for r any positive irrational number;
- (vii) \mathcal{T}_7 consists of \mathbb{R}, \emptyset , and every interval $[-r, r)$, for r any positive real number;
- (viii) \mathcal{T}_8 consists of \mathbb{R}, \emptyset , and every interval $(-r, r]$, for r any positive real number;
- (ix) \mathcal{T}_9 consists of \mathbb{R}, \emptyset , and every interval $[-r, r]$, and every interval $(-1, r)$, for r any positive real number;
- (x) \mathcal{T}_{10} consists of \mathbb{R}, \emptyset , every interval $[-n, n]$, and every interval $(-r, r)$, for n any positive integer and r any positive real number.

1.2 Open Sets - Exercises

1. List all 64 subsets of the set X in Example 1.1.2. Write down, next to each set, whether it is (i) clopen, (ii) neither open nor closed; (iii) open but not closed; (iv) closed but not open.

Example 1.1.2: Let $X = \{a, b, c, d, e, f\}$ and

$$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}.$$

- size one

$\{a\}$,clopen $\{b\}$,neither $\{c\}$,neither $\{d\}$,neither $\{e\}$,neither $\{f\}$,neither

- size two

$\{a, b\}$,neither $\{a, c\}$ $\{a, d\}$ $\{a, e\}$ $\{a, f\}$
 $\{b, c\}$ $\{b, d\}$ $\{b, e\}$ $\{b, f\}$
 $\{c, d\}$,open $\{c, e\}$ $\{d, f\}$
 $\{d, e\}$ $\{d, f\}$
 $\{e, f\}$

- size three

$\{a, b, c\}$ $\{a, b, d\}$ $\{a, b, e\}$ $\{a, b, f\}$
 $\{a, c, d\}$,open $\{a, c, e\}$ $\{a, c, f\}$
 $\{a, d, e\}$ $\{a, d, f\}$
 $\{a, e, f\}$
 $\{b, c, d\}$ $\{b, c, e\}$ $\{b, c, f\}$
 $\{b, d, e\}$ $\{b, d, f\}$
 $\{b, e, f\}$
 $\{c, d, e\}$ $\{c, d, f\}$
 $\{c, e, f\}$
 $\{d, e, f\}$

- size four

$\{a, b, c, d\}$ $\{a, b, c, e\}$ $\{a, b, c, f\}$
 $\{a, b, d, e\}$ $\{a, b, d, f\}$
 $\{a, b, e, f\}$
 $\{b, c, d, e\}$ $\{b, c, d, f\}$
 $\{c, d, e, f\}$

- size five

$\{a, b, c, d, e\}$ $\{a, b, c, d, f\}$
 $\{a, b, c, e, f\}$
 $\{a, b, d, e, f\}$
 $\{a, c, d, e, f\}$
 $\{b, c, d, e, f\}$,clopen

- size six

$\{a, b, c, d, e, f\}$,open

2. Let (X, \mathcal{T}) be a topological space with the property that every subset is closed. Prove that it is a discrete space.

$$\begin{aligned} S \subseteq X &\implies X \setminus S \text{ is open} \implies X \setminus S \in \mathcal{T} \\ T \in \mathcal{T} &\implies X \setminus T \text{ is closed} \implies T \subseteq X \end{aligned}$$

3. Observe that if (X, \mathcal{T}) is a discrete space or an indiscrete space, then every open set is a clopen set. Find a topology \mathcal{T} on the set $X = \{a, b, c, d\}$ which is not discrete and is not indiscrete but has the property that every open set is clopen.

Let $\mathcal{T} = \{X, \emptyset, \{a\}, \{b, c, d\}\}$

4. Let X be an infinite set. If \mathcal{T} is a topology on X such that every infinite subset of X is closed, prove that \mathcal{T} is the discrete topology.

$$\begin{aligned} S \subseteq X \text{ and } |S| = \infty \\ |X \setminus S| < \infty \implies X \setminus S \text{ is open} \end{aligned}$$

there are an infinite number of finite subsets whose complement is infinite and closed. These are precisely what make up a discrete topology.

5. Let X be an infinite set and \mathcal{T} a topology on X with the property that the only infinite subset of X which is open is X itself. Is (X, \mathcal{T}) necessarily an indiscrete space?
6. (i) Let \mathcal{T} be a topology on a set X such that \mathcal{T} consists of precisely four sets; that is, $\mathcal{T} = \{X, \emptyset, A, B\}$, where A and B are non-empty distinct proper subsets of X . [A is a **proper subset** of X means that $A \subseteq X$ and $A \neq X$. This is denoted by $A \subset X$.] Prove that A and B must satisfy exactly one of the following conditions.

$$(a) B = X \setminus A; (b) A \subset B; (c) B \subset A;$$

[Hint. Firstly show that A and B must satisfy at least one of the conditions and then show that they cannot satisfy more than one of the conditions.]

- (ii) Using (i) list all topologies on $X = \{1, 2, 3, 4\}$ which consist of exactly four sets.
7. (i) As recorded in http://en.wikipedia.org/wiki/Finite_topological_space, the number of distinct topologies on a set with $n \in \mathbb{N}$ points can be very large even for small n ; namely when $n = 2$, there are 4 topologies; when $n = 3$, there are 29 topologies; when $n = 4$, there are 355 topologies; when $n = 5$, there are 6942 topologies etc. Using mathematical induction, prove that as n increases the number of topologies increases.
- (ii) Using mathematical induction prove that if the finite set X has $n \in \mathbb{N}$ then it has at least $(n - 1)!$ distinct topologies.
- (iii) If X is any infinite set of cardinality \aleph , prove that there are at least 2^{\aleph} distinct topologies on X . Deduce that every infinite set has an uncountable number of distinct topologies on it.

1.3 Finite Closed Topology – Exercises

1. Let f be a function from a set X into a set Y . Then we stated in Example 1.3.9 that

$$f^{-1}\left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} f^{-1}(B_j) \quad (1.1)$$

and

$$f^{-1}\left(B_1 \cap B_2\right) = f^{-1}(B_1) \cap f^{-1}(B_2) \quad (1.2)$$

for any subsets B_j of Y and any index set J .

- (a) Prove that (1.1) is true

$$\begin{aligned} &\text{Let } y \in \bigcup_{j \in J} B_j \\ &\exists k \in J \rightarrow y \in B_k \\ &f^{-1}(y) \in f^{-1}\left(\bigcup_{j \in J} B_j\right) \text{ and } f^{-1}(y) \in f^{-1}(B_k) \\ &f^{-1}(B_k) \subseteq f^{-1}\left(\bigcup_{j \in J} B_j\right) \end{aligned}$$

since there MUST be a k for each y then it must be that all $\cup_{j \in J} f^{-1}(B_j) \subseteq f^{-1}\left(\bigcup_{j \in J} B_j\right)$

- (b) Prove that (1.2) is true.
 (c) Find (concrete) sets A_1, A_2, X , and Y and a function $f : X \rightarrow Y$ such that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$, where $A_1 \subseteq X$ and $A_2 \subseteq X$.
2. Is the topology \mathcal{T} described in Exercises 1.1 #6 (ii) the finite-closed topology?
 \mathcal{T}_2 consists of \mathbb{N}, \emptyset , and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the **final segment topology**.)

T_1 -spaces

3. A topological space (X, \mathcal{T}) is said to be a **T_1 -space** if every singleton set $\{x\}$ is closed in (X, \mathcal{T}) . Show that precisely two of the following nine topological spaces are T_1 -spaces. (Justify your answer).
- (i) a discrete space.
 - (ii) an indiscrete space with at least two points.
 - (iii) an infinite set with the finite-closed topology.
 - (iv) Example 1.1.2;
 - (v) Exercise 1.1 #5 (i)
 \mathcal{T}_1 consists of \mathbb{R}, \emptyset , and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;
 - (vi) Exercise 1.1 #5 (ii)
 \mathcal{T}_2 consists of \mathbb{R}, \emptyset , and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;
 - (vii) Exercise 1.1 #5 (iii)
 \mathcal{T}_3 consists of \mathbb{R}, \emptyset , and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;
 - (viii) Exercise 1.1 #6 (i)
 \mathcal{T}_1 consists of \mathbb{N}, \emptyset , and every set $\{1, 2, \dots, n\}$, for n any positive integer. (This is called **initial segment topology**).
 - (ix) Exercise 1.1 #6 (ii)
 \mathcal{T}_2 consists of \mathbb{N}, \emptyset , and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the **final segment topology**.)

4. Let \mathcal{T} be the finite-closed topology on a set X . If \mathcal{T} is also the discrete topology, prove that the set X is finite.

T_0 -space and the Sierpinski Space

5. A topological space (X, \mathcal{T}) is said to be a **T_0 -space** if for each pair of distinct points a, b in X , either there exist an open set containing a and not b , or there exists an open set containing b and not a .
- (i) Prove that every T_1 -space is a T_0 -space.
 - (ii) Which of (i) – (iv) in Exercise 3 above are T_0 -spaces?
 - (iii) Put a topology \mathcal{T} on the set $X = \{0, 1\}$ so that (X, \mathcal{T}) will be a T_0 -space but not a T_1 -space. [known as the **Sierpinski space**.]
 - (iv) Prove that each of the topological spaces described in Exercise 1.1 #6 is a T_0 -space.

Countable-Closed Topology

6. Let X be any infinite set. The **countable-closed topology** is defined to be the topology having as its closed sets X and all countable subsets of X . Prove that this is indeed a topology on X .
7. Let \mathcal{T}_1 and \mathcal{T}_2 be two topologies on a set X . Prove each of the following statements.
- (i) \mathcal{T}_3 is defined by $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$, then \mathcal{T}_3 is not necessarily a topology on X .
 - (ii) If \mathcal{T}_4 is defined by $\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2$, then \mathcal{T}_4 is a topology on X .
 - (iii) If (X, \mathcal{T}_1) and (X, \mathcal{T}_2) are T_1 -spaces, then (X, \mathcal{T}_4) is a T_1 -space.
 - (iv) If (X, \mathcal{T}_1) and (X, \mathcal{T}_2) are T_0 -spaces, then (X, \mathcal{T}_4) is not necessarily a T_0 -space.
 - (v) If $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$ are topologies on a set X , the $\mathcal{T} = \bigcap_{i=1}^n \mathcal{T}_i$ is a topology on X .
 - (vi) If for each $i \in I$, for some index set I , each \mathcal{T}_i is a topology on the set X , then $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$ is a topology on X .

Distinct T_1 -topologies on a Finite Set

8. In Wikipedia [//en.wikipedia.org/wiki/Finite_topological_space](https://en.wikipedia.org/wiki/Finite_topological_space), as we noted in Exercise 1.2 #7, it says that the number of topologies on a finite set with $n \in \mathbb{N}$ points can be quite large, even for small n . This is also true even for T_0 -spaces; for $n = 5$, there are 4231 distinct T_0 -spaces. Prove, using mathematical induction, that as n increases, the number of T_0 -spaces increases.
9. A topological space (X, \mathcal{T}) is said to be a **door space** if every subset of X is either an open set or a closed set (or both).
- (i) Is a discrete space a door space?
 - (ii) Is an indiscrete space a door space?
 - (iii) If X is an infinite set and \mathcal{T} is the finite-closed topology, is (X, \mathcal{T}) a door space?
 - (iv) Let X be the set $\{a, b, c, d\}$. Identify those topologies \mathcal{T} on X which make it into a door space.

Saturated Sets

10. A subset S of a topological space (X, \mathcal{T}) is said to be **saturated** if it is an intersection of open sets in (X, \mathcal{T}) .
- (i) Verify that every open set is a saturated set.
 - (ii) Verify that in a T_1 -space every set is saturated set.
 - (iii) Give an example of a topological space which has at least one subset which is not saturated.
 - (iv) Is it true that if the topological space (X, \mathcal{T}) is such that every subset is saturated, then (X, \mathcal{T}) is a T_1 -space?

Chapter 2

The Euclidean Topology

2.1 Euclidian Space – Exercises

1. Prove that if $a, b \in \mathbb{R}$ with $a < b$ then neither $[a, b)$ nor $(a, b]$ is an open subset of \mathbb{R} . Also show that neither is a closed subset of \mathbb{R} .

In the case of $[a, b)$ there is no set $a \in (x, y)$ because $x < a$ implies that $x + \frac{|x-a|}{2}$ would have to be a member of $[a, b)$ which it cannot. Similarly for $(a, b]$.

2. Prove that the sets $[a, \infty)$ and $(-\infty, a]$ are closed subsets of \mathbb{R} .

The complement of $[a, \infty)$ is $(-\infty, a)$ which is open and similarly for $(-\infty, a]$.

3. Show, by example, that the union of an infinite number of closed subsets of \mathbb{R} is not necessarily a closed subset of \mathbb{R} .

Define $S_i = [1/i, 1]$ then $\mathcal{S} = \cup_{i=1}^{\infty} S_i$. Obviously, given any $n \in \mathbb{N}$ there is a closed set $S_n = [1/n, 1]$ and there exists $(1/(n+1), 1) \subseteq \mathcal{S}$ such that $1/n \in (1/(n+1), 1)$ hence \mathcal{S} must be open.

4. Prove each of the following statements.

- (i) The set \mathbb{Z} of all integers is not an open set of \mathbb{R} .
- (ii) The set \mathbb{P} of all prime numbers is a closed subset of \mathbb{R} but not an open subset of \mathbb{R} .
- (iii) The set \mathbb{I} of all irrational numbers is neither a closed subset nor an open subset of \mathbb{R} .

5. If F is a non-empty finite subset of \mathbb{R} , show that F is closed in \mathbb{R} but that F is not open in \mathbb{R} .
6. If F is non-empty countable subset of \mathbb{R} , prove that F is not an open set, but that F may or may not be a closed set depending on the choice of F .
7. (i) Let $S = \{0, 1, 1/2, 1/3, 1/4, 1/5, \dots, 1/n, \dots\}$. Prove that the set S is closed in the euclidean topology on \mathbb{R} .
(ii) Is the set $T = \{1, 1/2, 1/3, 1/4, 1/5, \dots, 1/n, \dots\}$ closed in \mathbb{R} ?
(iii) Is the set $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \dots, n\sqrt{2}, \dots\}$ closed in \mathbb{R} ?

F_σ -Sets and G_δ -sets.

8. (i) Let (X, \mathcal{T}) be a topological space.

A subset S of X is said to be an F_σ set if it is the **union of a countable number of closed sets**.

Prove that all open intervals (a, b) and all closed intervals $[a, b]$ are F_σ -sets in \mathbb{R} .

All open intervals: Define $n \in \mathbb{N}$ then we can see that

$$(a, b) = \bigcup_{n=2}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right].$$

The left side is made up of countable number of closed sets.

All closed intervals are a countable union of closed intervals.

- (ii) Let (X, \mathcal{T}) be topological space.

A subset T of X is said to be a G_δ -**set** if it is the **intersection of a countable number of open sets**. Prove that all open intervals (a, b) and all closed intervals $[a, b]$ are G_δ -sets in \mathbb{R} .

All open intervals: every $(a, b) = (a, b) \cap \emptyset$ both of which are open.

All closed intervals: Define $n \in \mathbb{N}$ then

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n} \right)$$

(iii) Prove that the set \mathbb{Q} of rationals is an F_σ -set in \mathbb{R} .

For every $q \in \mathbb{Q}$ there exists $a, b \in \mathbb{Z}$ such that $a = \frac{a}{b}$. There are countable number of elements for a and countable number of elements for b , the union of which leads to countable number of elements for \mathbb{Q} . Each of these elements is closed, hence F_σ .

(iv) Verify that the complement of an F_σ -set is a G_δ -set and the complement of a G_δ -set is an F_σ -set.

Given any $(a, b) \in F_\sigma$ then

$$\begin{aligned} (a, b)^C &= (-\infty, a] \cup [b, \infty) \\ &= \left(\bigcup_{i=1}^{\infty} \left(\infty, a - \frac{1}{i} \right) \right) \cup \left(\bigcup_{j=1}^{\infty} \left(+\frac{1}{j} \right) \right) \end{aligned}$$

2.2 Basis for a Topology – Exercises

1. In this exercise you will prove that disc $\{\langle x, y \rangle, : x^2 + y^2 < 1\}$ is an open set of \mathbb{R}^2 , and then that every open disc in the plane is an open set.
 - (i) Let $\langle a, b \rangle$ be any point in the disc $D = \{\langle x, y \rangle : x^2 + y^2 < 1\}$. Put $r = \sqrt{a^2 + b^2}$. Let $R_{\langle a, b \rangle}$ be the open rectangle with vertices at the points $\langle a \pm \frac{1-r}{8}, b \pm \frac{1-r}{8} \rangle$. Verify that $R_{\langle a, b \rangle} \subset D$.
 - (ii) Using (i) show that

$$D = \bigcup_{\langle a, b \rangle \in D} R_{\langle a, b \rangle}.$$
 - (iii) Deduce from (ii) that D is an open set in \mathbb{R}^2 .
 - (iv) Show that every disc $\{\langle x, y \rangle : (x - a)^2 + (y - b)^2 < c^2, a, b, c \in \mathbb{R}\}$ is open in \mathbb{R}^2 .
2. In this exercise you will show that the collection of all open discs in \mathbb{R}^2 is a basis for a topology on \mathbb{R}^2 . [Later we shall see that this is the euclidean topology.]
 - (i) Let D_1 and D_2 be any open discs in \mathbb{R}^2 with $D_1 \cap D_2 \neq \emptyset$. If $\langle a, b \rangle$ is any point in $D_1 \cap D_2$, show that there exists an open disc $D_{\langle a, b \rangle}$ with center $\langle a, b \rangle$ such that $D_{\langle a, b \rangle} \subset D_1 \cap D_2$. [Hint: draw a picture and use a method similar to that of Exercise 1 (i).]
 - (ii) Show that

$$D_1 \cap D_2 = \bigcup_{\langle a, b \rangle \in D_1 \cap D_2} D_{\langle a, b \rangle}$$
 - (iii) Using (ii) and Proposition 2.2.8, prove that the collection of all open discs in \mathbb{R}^2 is a basis for a topology on \mathbb{R}^2 .
3. Let \mathcal{B} be a collection of all open intervals (a, b) in \mathbb{R} with $a < b$ and a and b rational numbers. Prove that \mathcal{B} is a basis for euclidean topology on \mathbb{R} . [Compare this with Proposition 2.2.1 and Example 2.2.3 where a and b were not necessarily rational.]

Second Axiom of Countability

4. A topological space (X, \mathcal{T}) is said to satisfy the **second axiom of countability** or to be **second countable** if there exists a basis \mathcal{B} for \mathcal{T} , where \mathcal{B} consists of only a countable number of sets.
 - (i) Using Exercise 3 above show that \mathbb{R} satisfies the second axiom of countability.
 - (ii) Prove that the discrete topology on an uncountable set does not satisfy the second axiom of countability. [Hint: It is not enough to show that one particular basis is uncountable. You must prove that every basis for this topology is uncountable.]
 - (iii) Prove that \mathbb{R}^n satisfies the second axiom of countability, for each positive integer n .
 - (iv) Let (X, \mathcal{T}) be the set of all integers with the finite-closed topology. Does the space (X, \mathcal{T}) satisfy the second axiom of countability?
5. Prove the following statements:
 - (i) Let m and c be real numbers. Then the line $L = \{\langle x, y \rangle : y = mx + c\}$ is a closed subset of \mathbb{R}^2 .
 - (ii) Let \mathbb{S}^1 be the unit circle given by $\mathbb{S}^1 = \{\langle x, y \rangle \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then \mathbb{S}^1 is a closed subset of \mathbb{R}^2 .
 - (iii) Let \mathbb{S}^n be the unit n -sphere given by

$$\mathbb{S}^n = \{\langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 = 1\}.$$
 Then \mathbb{S}^n is closed subset of \mathbb{R}^{n+1} .
 - (iv) Let B^n be the closed unit n -ball given by

$$B^n = \{\langle x_1, x_2, \dots, x_n \rangle \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\}.$$
 Then B^n is a closed subset of \mathbb{R}^n .
 - (v) the curve $C = \{\langle x, y \rangle \in \mathbb{R}^2 : xy = 1\}$ is a closed subset of \mathbb{R}^2 .

Product Topology

6. Let \mathcal{B}_1 be a basis for a topology \mathcal{T}_1 on a set X and \mathcal{B}_2 a basis for a topology \mathcal{T}_2 on a set Y . the set $X \times Y$ consists of all ordered pairs $\langle x, y \rangle, x \in X$ and $y \in Y$. Let \mathcal{B} be the collection of subsets of $X \times Y$ consisting of all the sets $B_1 \times B_2$ where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Prove that \mathcal{B} is a basis for a topology on $X \times Y$. the topology so defined is called the **product topology** on $X \times Y$.
7. Using Exercise 3 above and Exercise 2.1 #8, prove that every open subset of \mathbb{R} is an F_σ -set and a G_δ -set.

2.3 Basis for a Given Topology

- Determine whether or not each of the following collections is a basis for the euclidean topology on \mathbb{R}^2 .
 - the collection of all "open" squares with sides parallel to the axes;
 - the collection of all "open" discs;
 - the collection of all "open" squares;
 - the collection of all "open" rectangles;
 - the collection of all "open" triangles";
- Let \mathcal{B} be a basis for a topology \mathcal{T} on a non-empty set X . If \mathcal{B}_1 is a collection of subsets of X such that $\mathcal{T} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}$, prove that \mathcal{B}_1 is also a basis for \mathcal{T} .
 - Deduce from (i) that there exist an uncountable number of distinct bases for the euclidean topology on \mathbb{R} .
- Let $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$. As seen in Example 2.3.1, \mathcal{B} is a basis for a topology \mathcal{T} on \mathbb{R} and \mathcal{T} is not the euclidean topology on \mathbb{R} . nevertheless, show that each interval (a, b) is open in $(\mathbb{R}, \mathcal{T})$.
- Let $C[0, 1]$ be the set of all continuous real-valued functions on $[0, 1]$.

- Show that the collection \mathcal{M} , where $\mathcal{M} = \{M(f, \epsilon) : f \in C[0, 1] \text{ and } \epsilon \text{ is a positive real number}\}$ and

$$M(f, \epsilon) = \left\{ g : g \in C[0, 1] \text{ and } \int_0^1 |f - g| < \epsilon \right\},$$

is a basis for a topology \mathcal{T}_1 on $C[0, 1]$

- Show that the collection \mathcal{U} , where $\mathcal{U} = \{U(f, \epsilon) : f \in C[0, 1] \text{ and } \epsilon \in \mathbb{R}^+\}$ and

$$U(f, \epsilon) = \left\{ g : g \in C[0, 1] \text{ and } \sup_{x \in [0, 1]} |f(x) - g(x)| < \epsilon \right\},$$

is a basis for a topology \mathcal{T}_2 on $C[0, 1]$

- Prove that $\mathcal{T}_1 \neq \mathcal{T}_2$.

Subbasis for a Topology

- Let (X, \mathcal{T}) be a topological space. A non-empty collection \mathcal{S} of open subsets of X is said to be **subbasis** of \mathcal{T} if the collection of all finite intersections of members of \mathcal{S} forms a basis for \mathcal{T} .
 - Prove that the collection of all open intervals of the form (a, ∞) or $(-\infty, b)$ is a subbasis for the euclidean topology on \mathbb{R} .
 - Prove the $\mathcal{S} = \{\{a\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ is a subbasis for the topology \mathcal{T}_1 of Example 1.1.2.
- Let \mathcal{S} be a subbasis for a topology \mathcal{T} on the set \mathbb{R} . (See Exercise 5 above.) If all of the closed intervals $[a, b]$, with $a < b$, are in \mathcal{S} , prove that \mathcal{T} is the discrete topology.
- Let X be a set with at least two elements and \mathcal{S} the collections of all $X \setminus \{x\}, x \in X$. Prove that \mathcal{S} is a subbasis for the finite-closed topology on X .
- Let X be any infinite set and \mathcal{T} the discrete topology on X . Find a subbasis \mathcal{S} for \mathcal{T} such that \mathcal{S} does not contain any singleton sets.
- Let \mathcal{S} be a collection of all straight lines in the plane \mathbb{R}^2 . If \mathcal{S} is a subbasis for a topology \mathcal{T} on the set \mathbb{R}^2 , what is the topology?
- Let \mathcal{S} be a collection of all straight lines in the plane which are parallel to the X-axis. If \mathcal{S} is a subbasis for a topology \mathcal{T} on \mathbb{R}^2 , describe the open set in $(\mathbb{R}^2, \mathcal{T})$.
- Let \mathcal{S} be a collection of all circles in the plane. If \mathcal{S} is a subbasis for a topology \mathcal{T} on \mathbb{R}^2 , describe the open sets in $(\mathbb{R}^2, \mathcal{T})$.
- Let \mathcal{S} be the collection of all circles in the plane which have their centres on the X-axis. If \mathcal{S} is a subbasis for a topology \mathcal{T} on \mathbb{R}^2 , describe the open sets in $(\mathbb{R}^2, \mathcal{T})$.

Chapter 50

Important Definitions

50.1 Basic and T-spaces

Definition 50.1.1 (Discrete/Indiscrete Topologies). Let X be any non-empty set and let \mathcal{T} be the collection of all subsets of X . Then topological space (X, \mathcal{T}) is called *the discrete topology* on the set X .

Let X be any non-empty set and is called *the indiscrete topology* and $(X, \mathcal{T}) \rightarrow \mathcal{T} = \{X, \emptyset\}$.

Definition 50.1.2 (Finite-Closed Topologies). Let X be any non-empty set. A topology \mathcal{T} on X is called the *finite-closed topology or the cofinite topology* if the closed subsets of X are X and all finite subsets of X ; that is, the open sets are \emptyset and all subsets of X which have finite complements.

Definition 50.1.3 (Countable-Closed Topologies).

Definition 50.1.4 (T_0 Spaces). A topological space (X, \mathcal{T}) is said to be a T_0 -space if for each pair of distinct points a, b in X , either there exist an open set containing a and not b , or there exists an open set containing b and not a .

Definition 50.1.5 (T_1 Spaces). A topological space (X, \mathcal{T}) is said to be a T_1 -space if every singleton set $\{x\}$ is closed in (X, \mathcal{T}) .

Definition 50.1.6 (T_2 Space or Hausdorff). A topological space (X, \mathcal{T}) is said to be Hausdorff (or a T_2 -space) if given any pair of distinct points a, b in X there exist open sets U and V such that $a \in U, b \in V$, and $U \cap V = \emptyset$.

Definition 50.1.7 (T_3 Spaces, Regular Space). A topological space (X, \mathcal{T}) is said to be a regular space if for any closed subset A of X and any point $x \in X \setminus A$, there exist open sets U and V such that $x \in U, A \subseteq V$, and $U \cap V = \emptyset$. If (X, \mathcal{T}) is regular and a T_1 -space, then it is said to be a T_3 -space.

Definition 50.1.8 (Dense). Let A be a subset of a topological space (X, \mathcal{T}) . Then, A is said to be *dense in X* or *everywhere dense in X* if $\overline{A} = X$.

50.2 Homeomorphisms

Remark 50.2.1. Preserved by Homeomorphisms

1. T_0 -space;
2. T_1 -space;
3. T_2 -space;
4. Regular space;
5. T_3 -space;
6. Second Axiom of Countability.
7. Separable Space;
8. Discrete Space;
9. Indiscrete Space;
10. Finite-closed topology;
11. Countable-closed topology;

Remark 50.2.2. two spaces (X, \mathcal{T}) and (Y, \mathcal{T}_1) cannot be homeomorphic if X and Y have different cardinalities (e.g. X is countable and Y is uncountable) or if \mathcal{T} and \mathcal{T}_1 have different cardinalities.

50.3 Countability and Topologies

Definition 50.3.1 (Countable-closed topology). Let X be any infinite set. The *countable-closed topology* is defined to be the topology having as its closed sets X and all countable subsets of X .

Definition 50.3.2 (Second Axiom of Countability). A topological space (X, \mathcal{T}) is said to satisfy the *second axiom of countability* or to be *second countable* if there exists a basis \mathcal{B} for \mathcal{T} , where \mathcal{B} consists of only a countable number of sets.

Definition 50.3.3 (F_σ set). Let (X, \mathcal{T}) be a topological space.

A subset S of X is said to be an F_σ *set* if it is the **union of a countable number of closed sets**.

Definition 50.3.4 (G_δ -set). Let (X, \mathcal{T}) be topological space.

A subset T of X is said to be a G_δ -*set* if it is the **intersection of a countable number of open sets**.

Definition 50.3.5 (Product Topology). Let \mathcal{B}_1 be a basis for a topology \mathcal{T}_1 on a set X and \mathcal{B}_2 a basis for a topology \mathcal{T}_2 on a set Y . the set $X \times Y$ consists of all ordered pairs $\langle x, y \rangle, x \in X$ and $y \in Y$. Let \mathcal{B} be the collection of subsets of $X \times Y$ consisting of all the sets $B_1 \times B_2$ where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Prove that \mathcal{B} is a basis for a topology on $X \times Y$. the topology so defined is called the *product topology* on $X \times Y$.

Definition 50.3.6 (Subbasis). Let (X, \mathcal{T}) be a topological space. A non-empty collection \mathcal{S} of open subsets of X is said to be *subbasis* of \mathcal{T} if the collection of all finite intersections of members of \mathcal{S} forms a basis for \mathcal{T} .

50.4 Miscellaneous

Definition 50.4.1 (Door Space). A topological space (X, \mathcal{T}) is said to be a *door space* if every subset of X is either an open set or a closed set (or both)

Definition 50.4.2 (Saturated Topological Space). A subset S of a topological space (X, \mathcal{T}) is said to be **saturated** if it is an intersection of open sets in (X, \mathcal{T}) .