Math 1001

Test 1, on 2/13/25, will cover sections 2.2, 2.4, 2.5, 2.6, 3.1, 3.2. *Solutions to * are on the back.*

0*. (a) For $x, a \neq 0$ and $x \neq \pm a$, simplify $\frac{x^{-1}+a^{-1}}{x^{-2}-a^{-2}}$.

(b) For $x \neq 1, -1$, simplify $\frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$.

1*. Evaluate $\displaystyle\lim_{x\to 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{x}$.

2. Evaluate $\displaystyle\lim_{x\to 4} \frac{16-x^2}{\sqrt{x}-2}$

3. Evaluate $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{|x|} - \frac{1}{x}\right)$.

4. Evaluate $\displaystyle\lim_{x\to 3} \frac{\frac{2}{\sqrt{x}} - \frac{2}{\sqrt{3}}}{x-3}$

5. Evaluate $\displaystyle\lim_{x\to 1} \frac{2x^2-5x+3}{1-x}$.

6*. Evaluate $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{3x^2-29x+7}}{2-5x}$.

7. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x\cos(2x)}$.

8*. Evaluate $\displaystyle\lim_{x\to 1} \frac{x^3+x^2-2x}{x^2+2x-3}$.

9. Evaluate $\displaystyle\lim_{x\to 0} \frac{\pi+\sin x}{\cos x - \frac{\pi}{2}}$.

10. Find the horizontal asymptote and vertical asymptotes for $f(x) = \frac{1-5x^2}{x^2-3x+2}$. Calculate $\lim_{x\to+\infty} \frac{1-5x^2}{x^2-3x+2}$ (as in #5).

11. Give the limit definition of what it means for a function f to be *continuous* at a point c.

12*. For which b is $f(x) = \begin{cases} \frac{x^2-1}{x-1} & x \neq 1 \\ b+7 & x = 1 \end{cases}$ continuous at 1?

13. Give the limit definition of what it means for a function f to be *differentiable* at a point c.

14*. For which b is $f(x) = \begin{cases} bx & x < 0 \\ x^2-3x & x \geq 0 \end{cases}$ differentiable at 0?

15. Is the function $f(x) = \begin{cases} x^2+1 & x \leq 2 \\ x+3 & x > 2 \end{cases}$ continuous at 2? Is it differentiable at 2?

16*. Let $f(x) = \frac{3}{x+2} - x$. Using **only the limit definition of derivative**, calculate $f'(1)$. Then give the equation of the line tangent to the graph of f at the point $(1, 0)$.

17. Let $f(x) = \dfrac{x}{1+x^2}$. Using **only the limit definition of derivative**, calculate $f'(0)$.

18*. At which points on the graph of $f(x) = 2x^3+3x^2-12x+1$ is the tangent line horizontal?

Solutions to select exercises

0. Solution.

(a)

$$\frac{x^{-1}+a^{-1}}{x^{-2}-a^{-2}}=\frac{\frac{1}{x}+\frac{1}{a}}{\frac{1}{x^2}-\frac{1}{a^2}}=\frac{\frac{a+x}{ax}}{\frac{a^2-x^2}{a^2x^2}}=\frac{a+x}{ax}\cdot\frac{x^2a^2}{a^2-x^2}=\frac{a+x}{ax}\cdot\frac{x^2a^2}{(a-x)(a+x)}=\frac{xa}{a-x}.$$

(b)

$$\frac{1}{2}\left(\frac{1}{1-x}-\frac{1}{1+x}\right)=\frac{1}{2}\left(\frac{1+x-(1-x)}{(1-x)(1+x)}\right)=\frac{1}{2}\cdot\left(\frac{2x}{1-x^2}\right)=\frac{x}{1-x^2}.$$

1. Solution.

$$\lim_{x\to 0}\frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}=\lim_{x\to 0}\frac{\frac{x+1+x-1}{x^2-1}}{x}=\lim_{x\to 0}\frac{2x}{(x^2-1)x}=\lim_{x\to 0}\frac{2}{x^2-1}=-2.$$

6. Solution. Factoring out the highest power in both the numerator and denominator gives, for $x<0$

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-29x+7}}{2-5x}=\lim_{x\to-\infty}\frac{\sqrt{x^2\left(3-\frac{29}{x}+\frac{7}{x^2}\right)}}{x(\frac{2}{x}-5)}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\cdot\sqrt{3-\frac{29}{x}+\frac{7}{x^2}}}{x(\frac{2}{x}-5)}=\lim_{x\to-\infty}\frac{|x|\cdot\sqrt{3-\frac{29}{x}+\frac{7}{x^2}}}{x(\frac{2}{x}-5)}=\lim_{x\to-\infty}\frac{(-x)\cdot\sqrt{3-\frac{29}{x}+\frac{7}{x^2}}}{x(\frac{2}{x}-5)}$$

$$=\lim_{x\to-\infty}\frac{(-1)\cdot\sqrt{3-\frac{29}{x}+\frac{7}{x^2}}}{\frac{2}{x}-5}=\frac{\sqrt{3}}{5}.$$

8. Solution.

$$\lim_{x\to 1}\frac{x^3+x^2-2x}{x^2+2x-3}=\lim_{x\to 1}\frac{x(x^2+x-2)}{(x+3)(x-1)}=\lim_{x\to 1}\frac{x(x+2)(x-1)}{(x+3)(x-1)}=\lim_{x\to 1}\frac{x(x+2)}{x+3}=\frac{3}{4}.$$

12. Solution. For which b is $\lim\limits_{x\to 1}f(x)=f(1)$? Since $f(1)=b+7$, if f is continuous at 1, then

$$b+7=f(1)=\lim_{x\to 1}f(x)=\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}(x+1)=2.\qquad\text{That is,}\quad b=-5.$$

14. Solution. For which b does $f'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{f(h)}{h}$ exist? Since

$$\lim_{h\to 0+}\frac{f(h)}{h}=\lim_{h\to 0+}\frac{h^2-3h}{h}=\lim_{h\to 0+}(h-3)=-3\qquad\textit{and}\qquad\lim_{h\to 0-}\frac{f(h)}{h}=\lim_{h\to 0-}\frac{bh}{h}=\lim_{h\to 0-}b=b,$$

f differentiable at 0 if and only if $b=-3$.

16. Solution. Since $f(x)=\frac{3}{x+2}-x$, $f(1)=\frac{3}{1+2}-1=0$. By definition

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{f(1+h)}{h}=\lim_{h\to 0}\frac{\frac{3}{3+h}-(1+h)}{h}=\lim_{h\to 0}\frac{3-(1+h)(3+h)}{(3+h)h}=\lim_{h\to 0}\frac{3-(3+4h+h^2)}{(3+h)h}$$

$$=\lim_{h\to 0}\frac{-(4h+h^2)}{(3+h)h}=\lim_{h\to 0}\frac{-h(4+h)}{(3+h)h}=\lim_{h\to 0}\frac{-(4+h)}{3+h}=\frac{-4}{3}.$$

The tangent line has equation $y=f'(1)(x-1)+f(1)=\frac{-4}{3}(x-1)$.

18. Solution. The tangent line to the graph $f(x)=2x^3+3x^2-12x+1$ will be horizontal at precisely those points $(a,f(a))$, where $f'(a)=0$. Since $f'(x)=6x^2-6x-12=6(x^2-x-2)=6(x-2)(x+1)$, we have $f'(2)=0$ and $f'(-1)=0$. Thus the tangent line will be horizontal at the points $(2,f(2))=(2,5)$ and $(-1,f(-1))=(-1,14)$.