

Math 725 – Advanced Linear Algebra
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 Assignment #6 – Due 10/06/23

1. Let A be an $n \times n$ matrix with entries from a field F (actually, from any commutative ring R). Let

$$D(A) = A_{i_1, j_1} A_{i_2, j_2} \cdots A_{i_n, j_n}.$$

Prove that $D(A)$ is an n -linear function if and only if i_1, i_2, \dots, i_n are distinct.

Let there be two indices s, t such that $i_s = i_t$. Then, attempting to verify that $D(A)$ is n -linear on the i_s^{th} column, i.e., multiplying the i_s^{th} column by a scalar c will be equivalent to $cD(A)$. That is (WTLOG we can assume that $s < t$)

$$\begin{aligned} D(cA) &= A_{i_1, j_1} \cdots cA_{i_s, j_s} \cdots A_{i_t, j_t} \cdots A_{i_n, j_n} \\ &= A_{i_1, j_1} \cdots cA_{i_s, j_s} \cdots cA_{i_s, j_s} \cdots A_{i_n, j_n} \\ &= c^2 D(A) \text{ which indicates that it is not } n\text{-linear} \end{aligned}$$

Further, there must be n elements and there cannot be any duplicates so essentially the set of indices i_k is permutation on the numbers $1, \dots, n$.

2. Consider a Vandermonde matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{pmatrix}.$$

Show that the determinant of this matrix is equal to $\prod_{1 \leq i < j \leq n} (t_j - t_i)$. Conclude that a Vandermonde matrix is invertible if and only if t_1, \dots, t_n are distinct. [Hint: induction on n]

For $n = 2$ Let $A_2 = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \end{pmatrix}$ and $\det A_2 = t_2 - t_1$. Now, assuming that it is true for n we want to show that it is also true for $n + 1$. Let A be an $(n + 1) \times (n + 1)$ Vandermonde Matrix and use the right most column, A_{n+1} to evaluate determinant. Each of the reduced $n \times n$ matrices, $A(i|n)$, will have the form

$$A(k|n) = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{k-1} & t_{k-1}^2 & \cdots & t_{k-1}^{n-1} \\ 1 & t_{k+1} & t_{k+1}^2 & \cdots & t_{k+1}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+1} & t_{n+1}^2 & \cdots & t_{n+1}^{n-1} \end{pmatrix}$$

thus giving the determinant as

$$\begin{aligned}
 \det(A) &= \sum_{k=1}^{n+1} (-1)^{i+n} t_k^n \det(A(k|n)) \\
 &= \sum_{k=1}^{n+1} (-1)^{i+n} t_k^n \prod_{1 \leq i < j \leq n+1, i, j \neq k} (t_j - t_i) \\
 \text{WTS} &= \prod_{1 \leq i < j \leq n+1} (t_j - t_i)
 \end{aligned}$$

3.a) A matrix $A \in \mathcal{M}_{n \times n}(F)$ is called skew-symmetric if $A^t = -A$. If $F = \mathbb{C}$ and n is odd, show that $\det(A) = 0$ if A is skew-symmetric.

Remember that the $\det(A(i|j)) = \det(A(j|i))$. Let E_j be the determinant based on rows and E'_j be the determinant based on columns then

$$\begin{aligned}
 E_j(A) &= \sum_{i=1}^n (-1)^{i+j} A_{ij} D(A(i|j)) \\
 E'_j(A) &= \sum_{i=1}^n (-1)^{i+j} A_{ji} D(A(j|i))
 \end{aligned}$$

then for every pair i, j there are two terms $A_{ij} D(A(i|j)) = -A_{ji} D(A(j|i))$ each being the negative of the other and canceling each other out, that is, unless n is odd where there will be an additional outstanding term that is not canceled out.

b) A matrix $A \in \mathcal{M}_{n \times n}(F)$ is called orthogonal if $A^t A = I$. Show that $\det(A) = \pm 1$ if A is orthogonal.

$$\begin{aligned}
 A^t A &= I \\
 \det(A^t A) &= \det(I) \\
 \det(A^t) \det(A) &= 1 \\
 \det(A) &= \det(A^t)^{-1} \\
 &= \det(A)^{-1} \\
 &= \pm 1
 \end{aligned}$$

c) A matrix $A \in \mathcal{M}_{n \times n}(\mathbb{C})$ is called unitary if $A^* A = I$. Here A^* is the conjugate transpose of A defined as $A_{ij}^* = \bar{A}_{ji}$ where \bar{z} is the conjugate of a complex number z . Show that $|\det(A)| = 1$ if A is unitary.

4. Let A be an $m \times n$ matrix with entries from a field F . For simplicity, we assume $m \leq n$. Now for any $k \leq m$ let $I = \{i_1, i_2, \dots, i_k\}$ where $1 \leq i_1 < i_2 < \dots < i_k \leq m$ and $J = \{j_1, j_2, \dots, j_k\}$ where $1 \leq j_1 < j_2 < \dots < j_k \leq n$. We define $A(I, J)$ as the $k \times k$ matrix obtained from A by using the rows of A indexed by I and the columns of A indexed by J . The determinant of $A(I, J)$ is called a k -minor of A . Show that $\text{rank}(A) = k$ if and only if there exists a nonzero k -minor of A and all $k+1$ -minors of A are zero.

5.a) Let $A \in \mathcal{M}_{n \times n}(F)$. Show that there are at most n scalars $c \in F$ such that $\det(cI - A) = 0$.

Let $B = \det(cI - A)$ then

$$B = \begin{pmatrix} c - A_{11} & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & c - A_{22} & \cdots & -A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & c - A_{nn} \end{pmatrix}$$

all A_{ij} are constants. Thus the only terms with variables in them will be along the diagonal. Essentially the trace of B will determine the potential values for c . And,

$$\text{tr}(B) = (c - A_{11})(c - A_{22}) \cdots (c - A_{nn})$$

which will be a polynomial in c of degree n . The zeros of this polynomial will determine values of c that bring the determinant to zero. Hence there are at most n values of c where $\det(cI - A) = 0$.

b) Let $A, B \in \mathcal{M}_{n \times n}(F)$ where A is invertible. Show that there are at most n scalars $c \in F$ such that $cA - B$ is not invertible.

6. Let $A \in \mathcal{M}_{n \times n}(F)$ where A is in block-diagonal form: $\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$, where A_j is a $r_j \times r_j$ matrix. Show that $\det(A) = \det(A_1) \det(A_2) \cdots \det(A_k)$.

Step 1: We know that

$$E_j(A) = \sum_{i=1}^n (-1)^{i+j} A_{ij} D_{ij}(A)$$

and that this is the same for all j . Since A_1 is a 1×1 matrix we can see that

$$E_1(A) = A_1 D_{11}(A)$$

Step 2: removing the first row and first column and designating $D_{11}(A) = A'_1$ and replacing A_2 with a matrix

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A'_1 = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \mathbf{0} & \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

$$E_1(A'_1) = a_{11} D_{11}(A'_1) - a_{12} D_{12}(A'_1)$$

this gives us two matrices

$$D_{11}(A'_1) = \begin{pmatrix} a_{22} & 0 & \cdots & 0 \\ \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

$$D_{12}(A'_1) = \begin{pmatrix} a_{21} & 0 & \cdots & 0 \\ \mathbf{0} & A_3 & \cdots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \cdots & A_n \end{pmatrix}$$

remove the first two rows and first two columns of A'_1 and designate it A'_2 and from Step 1 we know that

$$D_{11}(A'_1) = a_{22} D_{11}(A'_2)$$

$$D_{12}(A'_1) = a_{21} D_{11}(A'_2)$$

$$E_1(A'_1) = (a_{11} a_{22} - a_{12} a_{21}) D_{11}(A'_2)$$

$$= D(A_2) D_{11}(A'_2)$$

and $E_1(A) = D(A_1) D(A_2) D_{11}(A'_2)$

Step 3: Each of the three terms in $E_1(A'_2)$ will generate a matrix with the same form as A'_1 . That is, each will have a 2×2 matrix in the first rows and first columns followed by zeros and a matrix A'_3 containing the remainder of A . Just as in Step 2 each of these will generate a term of the form $D_{ij}(A'_2)D_{11}(A'_3)$ That is,

$$E_1(A'_2) = D(A_3)D_{11}(A'_3)$$

$$\text{and } E_1(A) = D(A_1)D(A_2)D(A_3)D_{11}(A'_3)$$

Next Step: it should be clear now that the 4×4 matrix will generate four 3×3 matrices which we apply Step 3 to each and will sum to be a determinant for A_4 multiplied by the determinant of the remaining matrix. Extrapolating to n we get $D_{11}(A_n) = D(A_n)$

$$E_1(A) = D(A_1)D(A_2) \cdots D(A_n)$$

Extra Questions

1. In this exercise you will prove the famous Cramer's rule. For this let $Ax = b$ be a system of equations where $A \in \mathcal{M}_{n \times n}(F)$ and $b \in F^n$. Using the observation that $(\text{adj} A)A = \det(A)I$ show that $\det(A)x_i = \sum_{j=1}^n (-1)^{i+j} \det(A(j|i))b_j$ where $A(j|i)$ is obtained by deleting the j th row and i th column of A . Prove that the right-hand side of this expression is equal to $\det(B_i)$ where B_i is obtained from A by replacing the i th column of A with the vector b . Conclude that if $\det(A) \neq 0$ the unique solution to $Ax = b$ is obtained by $x_i = \frac{\det(B_i)}{\det(A)}$.

2. Let A be an $n \times (n+1)$ matrix of rank n . The rank nullity theorem tells us that $\text{null}(A)$ is generated by a single nonzero vector. Show that this nonzero vector can be taken to be

$$(+\det(A^{(1)}), -\det(A^{(2)}), +\det(A^{(3)}), \dots, (-1)^{n+1} \det(A^{(n+1)}))$$

where $A^{(j)}$ is the $n \times n$ matrix obtained from A by deleting its j th column. [Hint: without loss of generality we can assume that a generating vector's last coordinate is equal to 1. Now use Cramer's rule.]

3. We have an efficient algorithm for solving the linear equation $Ax = b$. For simplicity let's assume that A is an $n \times n$ invertible matrix. However, there is one more thing one needs to worry about in computer science. What if the *size* of the solution is so big, that even if we have an efficient algorithm to produce the solution, it will consume all of the memory? Algorithms are considered to be good if they are efficient and the output they are expected to deliver is a polynomial function (instead of, say, an exponential function) in the input size of the problem. Here is how we define the input size of the linear system $Ax = b$. Note that, ultimately, a computer computes with rational numbers that are close approximations to real numbers. So if p/q is a rational number expressed in lowest terms the size of p/q is defined to be (roughly) $\log p + \log q$. This is approximately the number of bits needed to be used to represent p/q inside the computer. The size D of the input $Ax = b$ is the sum of sizes of all input numbers in the matrix A and b . Now using Cramer's rule and properties of determinants prove that the size of the unique solution to $Ax = b$ is at most a polynomial function in D .

4. For $m \leq n$ let A be an $m \times n$ matrix and B is an $n \times m$ matrix. For $I \subset \{1, 2, \dots, n\}$ of cardinality m , we let A_I be the $m \times m$ submatrix of A whose columns are those of A indexed by I , and we let B_I be the $m \times m$ submatrix of B whose rows are those of B indexed by I . Prove that

$$\det(AB) = \sum_{|I|=m} \det(A_I) \det(B_I).$$

Note that when $m = n$ this proves the product formula for determinants.