Functional Analysis– Spring 2024

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2. Let H be a Hilbert space and $T : H \to H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*$$

$$\begin{aligned}
\langle Tx, y \rangle &= \langle x, T^*y \rangle \\
\left\langle T^{-1}Tx, y \right\rangle &= \left\langle T^{-1}x, T^*y \right\rangle \\
\langle x, y \rangle &= \left\langle T^{-1}x, T^*y \right\rangle \\
\left\langle x, (T^*)^{-1}y \right\rangle &= \left\langle T^{-1}x, (T^*)^{-1}T^*y \right\rangle \\
&= \left\langle T^{-1}x, y \right\rangle \\
&= \left\langle x, (T^{-1})^*y \right\rangle \\
(T^*)^{-1} &= (T^{-1})^*
\end{aligned}$$

3. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$, show that $T_n^* \to T^*$.

$$\begin{aligned}
\|T_n - T\|^2 \geq \|T_n\|^2 - \|T\|^2 = \|T_n^*\|^2 - \|T^*\|^2 &\geq \|T_n^* - T^*\|^2 \\
\text{similarly, } \|T_n^* - T^*\|^2 \geq \|T_n^*\|^2 - \|T^*\|^2 = \|T_n\|^2 - \|T\|^2 &\geq \|T_n - T\|^2 \\
\text{hence } \|T_n - T\|^2 = \|T_n^* - T^*\|^2
\end{aligned}$$

We know that given any $N > 0$ then for all $n > N$ when $\|T_n - T\| < \epsilon$ implies that $\|T_n^* - T^*\| < \epsilon$. Therefore, $T_n^* \to T^*$.

4. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1^\perp \subset T^*(M_2^\perp)$.

Let $x \in M_1$ and $z \in M_2^\perp$ and $x \notin \mathcal{N}(T)$.Then, $\langle Tx, z \rangle = 0$ implies $\langle x, T^*z \rangle = 0$ and either $T^*z \in \mathcal{N}(T^*)$ or $T^*z \perp x$. x is arbitrary, therefore $T^*z \perp \text{span}(M_1)$ or $T^*z \in M_1^\perp$. Thus, $T^*z \in \mathcal{N}(T^*) \cup M_1^\perp$. z is arbitrary so $T^*(M_2^\perp) = \mathcal{N}(T^*) \cup M_1^\perp$, hence, $M_1^\perp \subset T^*(M_2^\perp)$.

5. Let M_1 and M_2 in Prob. 4 be closed subspaces. Show that $T(M_1) \subset M_2$ if and only if $M_1^\perp \supset T^*(M_2^\perp)$.

($\Rightarrow$) Assuming that $T(M_1) \subset M_2$. We can see that $H_1 = M_1 \oplus M_1^\perp$. Then, let $x \in H_1, x = a + b$ for some $a \in M_1$ and $b \in M_1^\perp$ and $z \in M_2^\perp$ such that $z \neq 0$. Then,

$$\begin{aligned}
\langle Tx, z \rangle &= \langle Ta, z \rangle + \langle Tb, z \rangle \\
&= \langle Tb, z \rangle \\
&= \langle b, T^*z \rangle \\
z \neq 0 &\implies T^*z \in M_1^\perp
\end{aligned}$$

z is arbitrary, thus $T^*(M_2^\perp) \subset M_1^\perp$.

($\Leftarrow$) Assuming that $M^\perp \supset T^*(M_2^\perp)$. We can see that $H_2 = M_2 \oplus M_2^\perp$. Then, let $z \in H_2, z = c + d$ for some $c \in M_2$ and $d \in M_2^\perp$ and $x \in M_1$ and $x \neq 0$. Then,

$$\begin{aligned}
\langle x, T^*z \rangle &= \langle x, T^*c \rangle + \langle x, T^*d \rangle \\
&= \langle x, T^*c \rangle \\
&= \langle Tx, c \rangle \\
x \neq 0 &\implies Tx \in M_2
\end{aligned}$$

x is arbitrary, thus $T(M_1) \subset M_2$

6. If $M_1 = \mathcal{N}(T) = \{x \,|\, Tx = 0\}$ in Prob. 4, show that

   (a) $T^*(H_2) \subset M_1^\perp$

   $\mathcal{N}(T)$ is a closed vector space and by Prob 5 we can see that if $M_2 = \{0\}$ then $M_2^\perp = H_2$. Thus, $T^*(M_2^\perp) = T^*(H_2) \subset M_1^\perp$.

(b) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$
Let $x \in H_1 \backslash \mathcal{N}(T)$ and $z \in \mathcal{N}(T^*)$ then $0 = \langle x, T^*z \rangle = \langle Tx, z \rangle$ implies that $Tx \in \mathcal{N}(T^*)^\perp$.
Since $H_1 = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$ given any $x \in \mathcal{N}(T)$ then $Tx = 0 \in \mathcal{N}(T^*)$ or $x \in \mathcal{N}(T)^\perp$ then $Tx \in \mathcal{N}(T^*)^\perp$, then $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$.

(c) $M_1 = [T^*(H_2)]^\perp$
Let $z \in H_2 \backslash \mathcal{N}(T^*)$ and $x \in \mathcal{N}(T)$ then $0 = \langle Tx, z \rangle = \langle x, T^*z \rangle$ implies that $T^*z \in \mathcal{N}(T)^\perp$.
Since $H_2 = \mathcal{N}(T^*) \oplus \mathcal{N}(T^*)^\perp$ given any $z \in \mathcal{N}(T^*)$ then $T^*z = 0 \in \mathcal{N}(T)$ or $z \in \mathcal{N}(T^*)^\perp$ then $T^*z \in \mathcal{N}(T)^\perp$, then $[T^*(H_2)]^\perp = \mathcal{N}(T)$.

10. **(Right shift operator)** Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the *right shift operator* to be the linear operator $T : H \to H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of T.

The *right shift operator* gets its name by shifting the element back one position.
The range of $\mathcal{R}(T)$ is the set of total orthornormal sequences.
The null space is the first element if $e_1 = 1, 0, 0, 0, \ldots$ for clearly $Te_1 = (0)$.
The norm $\|T\| = \sup\limits_{x \in H,\, \|x\|=1} Tx = 1$
The adjoint, T^*, note that

$$\langle Te_i, e_j \rangle = \langle e_{i+1}, e_j \rangle = \begin{cases} 1 & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} \langle e_i, T^*e_j \rangle &= \langle Te_i, e_j \rangle \\ &\implies T^*e_j = e_{j-1} \end{aligned}$$

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4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \text{ and } T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2 \text{ and } T^* = T_1 - iT_2.$$

Show uniqueness, that is, $T_1 + iT_2 = S_1 + iS_2$ implies $S_1 = T_1$ and $S_2 = T_2$; here, S_1 and S_2 are self-adjoint by assumption.

$$\begin{aligned}
T_1 &= \frac{1}{2}(T + T^*)\\
\langle T_1 x, y \rangle &= \left\langle \left(\frac{1}{2}(T + T^*)\right) x, y \right\rangle\\
&= \frac{1}{2} \langle Tx + T^*x, y \rangle\\
&= \frac{1}{2} (\langle Tx, y \rangle + \langle T^*x, y \rangle)\\
&= \frac{1}{2} (\langle x, T^*y \rangle + \langle x, Ty \rangle)\\
&= \frac{1}{2} \langle x, T^*y + Ty \rangle\\
&= \left\langle x, \frac{1}{2} (T^*y + Ty) \right\rangle\\
&= \langle x, T_1 y \rangle
\end{aligned}$$

$$\begin{aligned}
T_2 &= \frac{1}{2i}(T - T^*)\\
\langle x, T_2 y \rangle &= \left\langle x, \frac{1}{2i} (T - T^*) y \right\rangle\\
&= \frac{-1}{2} \langle x, Ty - T^*y \rangle\\
&= \frac{-1}{2} (\langle x, Ty \rangle - \langle x, T^*y \rangle)\\
&= \frac{-1}{2} (\langle T^*x, y \rangle - \langle Tx, y \rangle)\\
&= \frac{-1}{2} \langle T^*x - Tx, y \rangle\\
&= \left\langle \frac{-1}{2} (T^*x - Tx), y \right\rangle\\
&= \langle T_2 x, y \rangle
\end{aligned}$$

$$\begin{aligned}
T_1 + iT_2 &= \frac{1}{2}(T + T^*) + \frac{1}{2i}(T - T^*)\\
&= \frac{1}{2}T + \frac{1}{2}T^* + \frac{1}{2}T - \frac{1}{2}T^*\\
&= T
\end{aligned}$$

$$\begin{aligned}
T_1 - iT_2 &= \frac{1}{2}(T + T^*) - \frac{1}{2i}(T - T^*)\\
&= \frac{1}{2}T + \frac{1}{2}T^* - \frac{1}{2}T + \frac{1}{2}T^*\\
&= T^*
\end{aligned}$$

5. On $\mathbb{C}^2$ (cf. 3.1-4) let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find T^*. Show that we have $T^*T = TT^* = 2I$. Find T_1 and T_2 as defined in prob. 4.

$$\begin{aligned}
T(a + bi) &= a + ib + i(a - ib) = a + b + i(a + b) = (a + b)(1 + i)\\
\langle T(a + bi), c + di \rangle &= \langle (a + b)(1 + i), c + di \rangle = (a + b) \langle 1 + i, c + di \rangle = (a + b)(c + d)\\
(a + b)(c + d) &= \langle a + bi, 1 + i \rangle (c + d) = \langle a + bi, (c + d)(1 + i) \rangle\\
T^*(c + di) &= (c + d)(1 + i) = (c + id) + i(c - id)\\
T^* &= T\\
\|T1\|^2 &= \langle T1, T1 \rangle = \langle 1 + i, 1 + i \rangle = 2\\
\|T\| &= \sqrt{2} = \|T^*\|\\
\|TT^*\| &= 2\\
TT^* &= 2I\\
T_1 &= \frac{1}{2} ((1 + i) + (1 + i)) = 1 + i = T\\
T_2 &= \frac{1}{2i} ((1 + i) - (1 + i)) = 0
\end{aligned}$$