Topology without Tears

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Contents

Chapter 1

Topology Spaces

1.1 Topology – Exercises

1. Let $x = \{a, b, c, d, e, f\}$. Determine whether or not each of the following collections of subsets of X is a topology on X:

   (a) $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, f\}, \{b, f\}, \{a, b, f\}\}$;
   No, $\{a, f\} \cap \{b, f\} = \{f\} \notin \mathcal{T}$.

   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a, b, f\}, \{a, b, d\}, \{a, b, d, f\}\}$;
   No, $\{a, b, f\} \cap \{a, b, d\} \notin \mathcal{T}$.

   (c) $\mathcal{T}_3 = \{X, \emptyset, \{f\}, \{e, f\}, \{a, f\}\}$;
   No, $\{e, f\} \cup \{a, f\} = \{a, e, f\} \notin \mathcal{T}$.

2. Let $X = \{a, b, c, d, e, f\}$. Which of the following collections of subsets of X is a topology on X? (Justify your answer.)

   (a) $\mathcal{T}_1 = \{X, \emptyset, \{c\}, \{b, d, e\}, \{b, c, d, e\}, \{b\}\}$;

   (b) $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b, d, e\}, \{a, b, d\}, \{a, b, d, e\}\}$;

   (c) $\mathcal{T}_3 = \{X, \emptyset, \{b\}, \{a, b, c\}, \{d, e, f\}, \{b, d, e, f\}\}; \}$

3. If $X = \{a, b, c, d, e, f\}$, and $\mathcal{T}$ is the discrete topology on X, which of hte following statements are true?
   (a) $X \in \mathcal{T}$; **YES** (b) $\{X\} \in \mathcal{T}$; NO (c) $\{\emptyset\} \in \mathcal{T}$; NO (d) $\emptyset \in \mathcal{T}$; **YES**
   (e) $\emptyset \in X; NO$ (f) $\{\emptyset\} \in X; NO$ (g) $\{a\} \in \mathcal{T}$**;** **YES** (h) $a \in \mathcal{T}$; NO
   (i) $\emptyset \subseteq X$; **YES** (j) $\{a\} \in X$; NO (k) $\{\emptyset\} \subseteq X$; **YES** (l) $a \in X$; **YES**
   (m) $X \subseteq \mathcal{T}$; **YES** (n) $X \subseteq \mathcal{T}$; YES (o) $\{X\} \subseteq \mathcal{T}$; NO (p) $a \subseteq \mathcal{T}$; NO

4. Let $(X, \mathcal{T})$ be any topological space. Verify that ***the intersection of any finite number of members of $\mathcal{T}$ is a member of $\mathcal{T}$.***

5. Let $\mathbb{R}$ be the set of all real numbers. Prove that each of the following collections of subsets of $\mathbb{R}$ is a topology

   (i) $\mathcal{T}_1$ consists of $\mathbb{R}, \emptyset$, and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;

   (ii) $\mathcal{T}_2$ consists of $\mathbb{R}, \emptyset$, and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;

   (iii) $\mathcal{T}_3$ consists of $\mathbb{R}, \emptyset$, and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;

6. (i) $\mathcal{T}_1$ consists of $\mathbb{N}, \emptyset$, and every set $\{1, 2, \ldots, n\}$, for n any positive integer. (This is called ***initial segment topology***).

   (ii) $\mathcal{T}_2$ consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \ldots\}$, for n any positive integer. (This is called the ***final segment topology***.)

7. List all possible topologies on the following sets:

   (a) $X = \{a, b\}$;

   (b) $Y = \{a, b, c\}$;

8. Let X be an infinite set and $\mathcal{T}$ a topology on X. If every infinte subset of X is in $\mathcal{T}$, prove that $\mathcal{T}$ is the discrete topology.

9. Let $\mathbb{R}$ be the set of all real numbers Precisely three of the followin ten collections are subsets of $\mathbb{R}$ are topologies. Identify these and justifey your answer.

   (i) $\mathcal{T}_1$ consists of $\mathbb{R}, \emptyset$, and every interval (a, b), for a and b any real numbers where $a < b$.

   (ii) $\mathcal{T}_2$ consists of $\mathbb{R}, \emptyset$ and every interval $(-r, r)$, for r any positive real number.

   (iii) $\mathcal{T}_3$ consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r)$, for r any positive rational number;

   (iv) $\mathcal{T}_4$ consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, for r any positive rational number;

   (v) $\mathcal{T}_5$ consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r)$, for r any positive irrational number;

   (vi) $\mathcal{T}_6$ consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, for r any positive irrational number;

   (vii) $\mathcal{T}_7$ consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r)$, for r any positive real number;

   (viii) $\mathcal{T}_8$ consists of $\mathbb{R}, \emptyset$, and every interval $(-r, r]$, for r any positive real number;

   (ix) $\mathcal{T}_9$ consists of $\mathbb{R}, \emptyset$, and every interval $[-r, r]$, and every interval $(-1, r)$, for r any positive real number;

   (x) $\mathcal{T}_{10}$ consists of $\mathbb{R}, \emptyset$, every internval $[-n, n]$, and every interval $(-r, r)$, for n any positive intger and r any positive real number.

1.2 Open Sets - Exercises

1. List all 64 subsets of the set X in Example 1.1.2. Write down, next to each set, whether it is (i) clopen, (ii) neither open nor closed; (iii) open but not closed; (iv) closed but not open.

   Example 1.1.2: Let $X = \{a, b, c, d, e, f\}$ and

$$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}.$$

- size one

  $\{a\}$,clopen $\{b\}$,neither $\{c\}$,neither $\{d\}$,neither $\{e\}$,neither $\{f\}$,neither

- size two

$\{a, b\}$,neither	$\{a, c\}$	$\{a, d\}$	$\{a, e\}$	$\{a, f\}$
$\{b, c\}$	$\{b, d\}$	$\{b, e\}$	$\{b, f\}$	
$\{c, d\}$,open	$\{c, e\}$	$\{d, f\}$		
$\{d, e\}$	$\{d, f\}$			
$\{e, f\}$				

- size three

$\{a, b, c\}$	$\{a, b, d\}$	$\{a, b, e\}$	$\{a, b, f\}$
$\{a, c, d\}$,open	$\{a, c, e\}$	$\{a, c, f\}$	
$\{a, d, e\}$	$\{a, d, f\}$		
$\{a, e, f\}$			
$\{b, c, d\}$	$\{b, c, e\}$	$\{b, c, f\}$	
$\{b, d, e\}$	$\{b, d, f\}$		
$\{b, e, f\}$			
$\{c, d, e\}$	$\{c, d, f\}$		
$\{c, e, f\}$			
$\{d, e, f\}$			

- size four

$\{a, b, c, d\}$	$\{a, b, c, e\}$	$\{a, b, c, f\}$
$\{a, b, d, e\}$	$\{a, b, d, f\}$	
$\{a, b, e, f\}$		
$\{b, c, d, e\}$	$\{b, c, d, f\}$	
$\{c, d, e, f\}$		

- size five

$\{a, b, c, d, e\}$	$\{a, b, c, d, f\}$
$\{a, b, c, e, f\}$	
$\{a, b, d, e, f\}$	
$\{a, c, d, e, f\}$	
$\{b, c, d, e, f\}$,clopen	

- size six

  $\{a, b, c, d, e, f\}$,open

2. Let $(X, \mathcal{T})$ be a topological space with the property that every subset is closed. Prove that it is a discrete space.

$$S \subseteq X \implies X \backslash X \text{ is open } \implies X \backslash S \in \mathcal{T}$$
$$T \in \mathcal{T} \implies X \backslash T \text{ is closed } \implies T \subseteq X$$

3. Observe that if $(X, \mathcal{T})$ is a discrete space or an indiscrete space, then every open set is a clopen set. Find a topology $\mathcal{T}$ on the set $X = \{a, b, c, d\}$ which is not discrete and is not indiscrete but has the property that every open set is clopen.

   Let $\mathcal{T} = \{X, \emptyset, \{a\}, \{b, c, d\}\}$

4. Let X be an infinite set. If $\mathcal{T}$ is a topology on X such that every infinite subset of X is closed, prove that $\mathcal{T}$ is the discrete topology.

$$S \subseteq X \text{ and } |S| = \infty$$
$$|X \backslash S| < \infty \implies X \backslash S \text{ is open}$$

   there are an infinite number of finite subsets whose compliment is infinite and closed. These are precisely what make up a discrete topology.

5. Let X be an infinite set and $\mathcal{T}$ a topology on X with the property that the only infinite subset of X which is open is X itself. Is $(X, \mathcal{T})$ necessarily an indiscrete space?

6. (i) Let $\mathcal{T}$ be a topology on a set X such that $\mathcal{T}$ consists of precisely for sets; that is, $\mathcal{T} = \{X, \emptyset, A, B\}$, where A and B are non-empty distinct proper subsets of X. [A is a ***proper subset*** of X means that $A \subseteq X$ and $A \neq X$. This si denoted by $A \subset X$.] Prove that A and B must satisfy exactly one of the following conditions.

   (a) $B = X \backslash A$; (b) $A \subset B$; (c) $B \subset A$;

   [Hint. Firstly show that A and B must satisfy at least one of the conditions and then show that they cannot satisfy more than one of the conditions.]

   (ii) Using (i) list all topologies on $X = \{1, 2, 3, 4\}$ which consist of exactly four sets.

7. (i) A recorded in http://en.wikipedia.org/wiki/Finite_topological_space, the number of distinct topologies on a set with $n \in \mathbb{N}$ points can be very large even for small n; namely when $n = 2$, there are 4 topologies; when $n = 3$, there are 29 topologies: when $n = 4$, there are 355 topologies; when $n = 5$, there are 6942 topologies etc. Using mathematical induction, prove that as n increases the number of topologies increases.

   (ii) Using mathematical induction prove that if the finite set X has $n \in \mathbb{N}$ then it has at least $(n-1)!$ distinct topolgies.

   (iii) If X is any infintie set of cardinality $\mathfrak{N}$, prove that there are at least $2^{\mathfrak{N}}$ distinct topologies on X. Deduce that every infinite set has an uncountable number of distinct topolgies on it.

1.3 Finite Closed Topology – Exercises

1. Let f be a function from a set X into a set Y. Then we stated in Example 1.3.9 that

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j) \tag{1.1}$$

and

$$f^{-1}\left(B_1 \bigcap B_2\right) = f^{-1}(B_1) \cap f^{-1}(B_2) \tag{1.2}$$

for any subsets B_j of Y and any index set J.

(a) Prove that (1.1) is true

$$\text{Let } y \in \bigcup_{j\in J} B_j$$
$$\exists k \in J \to y \in B_k$$
$$f^{-1}(y) \in f^{-1}\left(\bigcup_{j\in J} B_j\right) \text{ and } f^{-1}(y) \in f^{-1}(B_k)$$
$$f^{-1}(B_k) \subseteq f^{-1}\left(\bigcup_{j\in J} B_j\right)$$

since there MUST be a k for each y then it must be that all $\cup_{j\in J} f^{-1}(B_j) \subseteq f^{-1}\left(\bigcup_{j\in J} B_j\right)$

(b) Prove that (1.2) is true.

(c) Find (concrete) sets A_1, A_2, X, and Y and a function f :X→ Y such that $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$, where $A_1 \subseteq X$ and $A_2 \subseteq X$.

2. Is the topology $\mathcal{T}$ described in Exercises 1.1 #6 (ii) the finite-closed topology?

$\mathcal{T}_2$ consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the ***final segment topology***.)

T_1*-spaces*

3. A topological space $(X, \mathcal{T})$ is said to be a T_1**-space** if every singleton set $\{x\}$ is closed in $(X, \mathcal{T})$. Show that precisely two of the following nine topological spaces are T_1-spaces. (Justify your answer).

(i) a discrete space.

(ii) an indiscrete space with at least two points.

(iii) an infinite set with the finite-closed topology.

(iv) Exampe 1.1.2;

(v) Exercise 1.1 #5 (i)

$\mathcal{T}_1$ consists of $\mathbb{R}, \emptyset$, and every interval $(-n, n)$, for n any positive integer, where $(-n, n)$ denotes the set $\{x \in \mathbb{R} : -n < x < n\}$;

(vi) Exercise 1.1 #5 (ii)

$\mathcal{T}_2$ consists of $\mathbb{R}, \emptyset$, and every interval $[-n, n]$, for n any positive integer, where $[-n, n]$ denotes the set $\{x \in \mathbb{R} : -n \leq x \leq n\}$;

(vii) Exercise 1.1 #5 (iii)

$\mathcal{T}_3$ consists of $\mathbb{R}, \emptyset$, and every interval $[n, \infty)$, for n any positive integer, where $[n, \infty)$ denotes the set $\{x \in \mathbb{R} : n \leq x\}$;

(viii) Exercise 1.1 #6 (i)

$\mathcal{T}_1$ consists of $\mathbb{N}, \emptyset$, and every set $\{1, 2, \dots, n\}$, for n any positive integer. (This is called ***initial segment topology***).

(ix) Exercise 1.1 #6 (ii)

$\mathcal{T}_2$ consists of $\mathbb{N}, \emptyset$, and every $\{n, n+1, \dots\}$, for n any positive integer. (This is called the ***final segment topology***.)

4. Let $\mathcal{T}$ be the finite-closed topology on a set X. If $\mathcal{T}$ is also the discrete topology, prove that the set X is finite.

T_0*-space and the Sierpinsi Space*

5. A topological space $(X, \mathcal{T})$ is said to be a T_0**-space** if for each pair of distinct points a, b in X, either there exist an open seet containing a and not b, or there exists an open set containing b and not a.

   (i) Prove that every T_1-space is a T_0-space.

   (ii) Which of $(i) - (iv)$ in Exercise 3 above are T_0-spaces?

   (iii) Put a topology $\mathcal{T}$ on the set $X = \{0, 1\}$ so that $(X, \mathcal{T})$ will be a T_0-space but not a T_1-space. [known as the **Sierpinski space**.]

   (iv) Prove that each of the topological spaces described in Exercise 1.1 #6 is a T_0-space.

Countable-Closed Topology

6. Let X be any infinite set. The ***countable-closed topology*** is defined to be the topology having as its closed sets X and all countable subsets of X. Prove that this is indeed a topology on X.

7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. Prove each of the following statements.

   (i) $\mathcal{T}_3$ is definted by $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$, then $\mathcal{T}_3$ is not necessarily a topology on X.

   (ii) If $\mathcal{T}_4$ is defined by $\mathcal{T}_4 = \mathcal{T}_1 \cap \mathcal{T}_2$, then $\mathcal{T}_4$ is a topology on X.

   (iii) If $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are T_1-spaces, then $(X, \mathcal{T}_4)$ is a T_1-space.

   (iv) If $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are T_0-spaces, then $(X, \mathcal{T}_4)$ is not necessarily a T_0-space.

   (v) If $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ are topologies on a set X, the $\mathcal{T} = \bigcap_{i=1}^{n} \mathcal{T}_i$ is a toplogy on X.

   (vi) If for each $i \in I$, for some index set I, each $\mathcal{T}_i$ is a topology on the set X, then $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$ is a topology on X.

Distinct T_1*-topologies on a Finite Set*

8. In Wikipedia //enwikipedia.org/wiki/Finite_topological_space, as we noted in Exercise 1.2 #7, it says that the number of topologies on a finite set with $n \in \mathbb{N}$ points can be quite large, even for small n. This is also true even for T_0-spaces; for $n = 5$, ther are 4231 distinct T_0-spaces. Prove, using mathemtaical induction, that as n increases, the number of T_0-spaces increases.

9. A topological space (X, T) is said to be a ***door space*** if every subset of X is either an open set or a closed set (or both).

   (i) Is a discrete space a door space?

   (ii) Is an indiscrete space a door space?

   (iii) If X is an infintie set and $\mathcal{T}$ is the finite-closed topology, is $(X, \mathcal{T})$ a door space?

   (iv) Let X be the set $\{a, b, c, d\}$. Identify those topologies $\mathcal{T}$ on X which make it into a door space.

Saturated Sets

10. A subset S of a topological space $(X, \mathcal{T})$ is said to be **saturated** if it is an intersection of open sets in $(X, \mathcal{T})$.

    (i) Verify that every open set is a saturated set.

    (ii) Verify that in a T_1-space every set is saturated set.

    (iii) Give an example of a topological space which has atleast one subset which is not saturated.

    (iv) Is it true that if the topological sapce $(X, \mathcal{T})$ is such that every subset is saturated, then $(X, \mathcal{T})$ is a T_1-space?

Chapter 2

The Euclidean Topology

2.1 Euclidian Space – Exercises

1. Prove that if $a, b \in \mathbb{R}$ with $a < b$ then neither $[a, b)$ nor $(a, b]$ is an open subset of $\mathbb{R}$. Also show that neither is a closed subset of $\mathbb{R}$.

   In the case of $[a, b)$ there is no set $a \in (x, y)$ because $x < a$ implies that $x + \frac{|x-a|}{2}$ would have to be a member of $[a, b)$ which it cannot. Similarly for $(a, b]$.

2. Prove that the sets $[a, \infty)$ and $(-\infty, a]$ are closed subsets of $\mathbb{R}$.

   The composite of $[a, \infty)$ is $(-\infty, a)$ which is open and similarly for $(-\infty, a]$.

3. Show, by example, that the union of an infinite number of closed subsets of $\mathbb{R}$ is not necessarily a closed subset of $\mathbb{R}$.

   Define $S_i = [1/i, 1]$ then $\mathcal{S} = \cup_{i=1}^{\infty} S_i$. Obviously, given any $n \in \mathbb{N}$ there is a closed set $S_n = [1/n, 1]$ and there exists $(1/(n+1), 1) \subseteq \mathcal{S}$ such that $1/n \in (1/(n+1), 1)$ hence $\mathcal{S}$ must be open.

4. Prove each of the following statements.

   (i) The set $\mathbb{Z}$ of all integers is not an open set of $\mathbb{R}$.

   (ii) The set $\mathbb{P}$ of all prime numbers is a closed subset of $\mathbb{R}$ but not an open subset of $\mathbb{R}$.

   (iii) The set $\mathbb{I}$ of all irrational numbers is neither a closed subset nor an open subset of $\mathbb{R}$.

5. If F is a non-empty finite subset of $\mathbb{R}$, show that F is closed in $\mathbb{R}$ but that F is not open in $\mathbb{R}$.

6. if F is non-empty countable subset of $\mathbb{R}$, prove that F is not an open set, but that F may or may not be a closed set depending on the choice of F.

7. (i) Let $S = \{0, 1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots\}$. Prove that the set S is closed in the euclidean topology on $\mathbb{R}$.

   (ii) Is the set $T = \{1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/n, \ldots\}$ closed in $\mathbb{R}$?

   (iii) Is the set $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \ldots, n\sqrt{2}, \ldots\}$ closed in $\mathbb{R}$?

F_σ*-Sets and* G_δ*-sets.*

8. (i) Let $(X, \mathcal{T})$ be a topological space.

   A subset S of X is said to be an F_σ ***set*** if it is the **union of a countable number of closed sets**.

   Prove that all open intervals (a, b) and all closed intervals $[a, b]$ are F_σ-sets in $\mathbb{R}$.

   All open intervals: Define $n \in \mathbb{N}$ then we can see that

   $$(a, b) = \bigcup_{n=2}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right].$$

   The left side is made up of countable number of closed sets.

   All closed intervals are a countable union of closed intervals.

   (ii) Let $(X, \mathcal{T})$ be topological space.

A subset T of X is said to be a G_δ***-set*** if it is the **intersection of a countable number of open sets**.

Prove that all open intervals (a,b) and all closed intervals $[a,b]$ are G_δ-sets in $\mathbb{R}$.

All open intervals: every $(a,b) = (a,b) \cap \emptyset$ both of which are open.

All closed intervals: Define $n \in \mathbb{N}$ then

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n} \right)$$

(iii) Prove that the set $\mathbb{Q}$ of rationals is an F_σ-set in $\mathbb{R}$.
For every $q \in \mathbb{Q}$ there exists $a, b \in \mathbb{Z}$ such that $a = \frac{a}{b}$. There are countable number of elements for a and countable number of elements for b, the union of which leads to countable number of elements for $\mathbb{Q}$. Each of these elements is closed, hence F_σ.

(iv) Verify that the complement of an F_σ-set is a G_δ-set and the complement of a G_δ-set is an F_σ-set.
Given any $(a,b) \in F_\sigma$ then

$$\begin{aligned}(a,b)^C &= (-\infty, a] \cup [b, \infty) \\ &= \left(\bigcup_{i+1}^{\infty} \left(\infty, a - \frac{1}{i} \right) \right) \cup \left(\bigcup_{j=1}^{\infty} \left(\pm\frac{1}{j} \right) \right)\end{aligned}$$

2.2 Basis for a Topology – Exercises

1. In this exercise you will prove that disc $\{\langle x, y\rangle, : x^2 + y^2 < 1\}$ is an open set of $\mathbb{R}^2$, and then that every open disc in the plane is an open set.

   (i) Let $\langle a, b\rangle$ be any point in the disc $D = \{\langle x, y\rangle : x^2 + y^2 < 1\}$. Put $r = \sqrt{a^2 + b^2}$. Let $R_{\langle a,b\rangle}$ be the open rectangle with vertices at the points $\langle a \pm \frac{1-r}{8}, b \pm \frac{1-r}{8}\rangle$. Verify that $R_{\langle a,b\rangle} \subset D$.

   (ii) Using (i) show that
   $$D = \bigcup_{\langle a,b\rangle \in D} R_{\langle a,b\rangle}.$$

   (iii) Deduce from (ii) that D is an open set in $\mathbb{R}^2$.

   (iv) Show that every disc $\{\langle x, y\rangle : (x - a)^2 + (y - b)^2 < c^2, a, b, c \in \mathbb{R}\}$ is open in $\mathbb{R}^2$.

2. In this exercise you will show that the collection of all open discs in $\mathbb{R}^2$ is a basis for a topology on $\mathbb{R}^2$. [Later we shall see that this is the euclidean topology.]

   (i) Let D_1 and D_2 be any open discs in $\mathbb{R}^2$ wtih $D_1 \cap D_2 \neq \emptyset$. If $\langle a, b\rangle$ is any point in $D_1 \cap D_2$, show that there exists an open disc $D_{\langle a,b\rangle}$ with center $\langle a, b\rangle$ such that $D_{\langle A,B\rangle} \subset D_1 \cap D_2$. [Hint: draw a picture and use a method similar to that of Exercsie 1 (i).]

   (ii) Show that
   $$D_1 \cap D_2 = \bigcup_{\langle a,b\rangle \in D_1 \cap D_2} D_{\langle a,b\rangle}$$

   (iii) Using (ii) and Proposition 2.2.8, prove that the collection of all open discs in $\mathbb{R}^2$ is a basis for a topology on $\mathbb{R}^2$.

3. Let $\mathcal{B}$ be a collection of all open intervals (a, b) in $\mathbb{R}$ with $a < b$ and a and b rational numbers. Prove that $\mathcal{B}$ is a basis for euclidean topology on $\mathbb{R}$.[Compare this with Proposition 2.2.1 and Example 2.2.3 where a and b where not necessarily rational.]

Second Axiom of Countability

4. A topological space $(X, \mathcal{T})$ is said to satisfy the ***second axiom of countability*** or to be ***second countable*** if there exists a basis $\mathcal{B}$ for $\mathcal{T}$, where $\mathcal{B}$ consists of only a countable number of sets.

   (i) Using Exercise 3 above show that $\mathbb{R}$ satisfies the second axiom of countability.

   (ii) Prove that the discrete topology on an uncountable set does not satisfy the second axiom of countability. [Hint: It is not enough to show that one particular basis is uncountable. You must prove that every basis for this topology is uncountable.]

   (iii) Prove that $\mathbb{R}^n$ satisfies the second axiom of countability, for each positive integer n.

   (iv) Let $(X, \mathcal{T})$ be the set of all integers with the finite-closed topology. Does the space $(X, \mathcal{T})$ satisfy the second axiom of countability?

5. Prove the following statements:

   (i) Let m and c be real numbers. Then the line $L = \{\langle x, y\rangle : y = mx + c\}$ is a closed subset of $\mathbb{R}^2$

   (ii) Let $\mathbb{S}^1$ be the unit circle given by $\mathbb{S}^1 = \{\langle x, y\rangle \in \mathbb{R}^2 : X^2 + y^2 = 1\}$. Then $\mathbb{S}^1$ is a closed subset of $\mathbb{R}^2$.

   (iii) Let $\mathbb{S}^n$ be the unit n-sphere give by
   $$\mathbb{S}^n = \{\langle x_1, x_2, \dots, x_n\rangle \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$
   Then $\mathbb{S}^n$ is closed subset of $\mathbb{R}^{n+1}$.

   (iv) Let B^n be the closed unit n-ball give by
   $$B^n = \{\langle x_1, x_2, \dots, x_n\rangle \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}.$$
   Then B^n is a closed subset of $\mathbb{R}^n$.

   (v) the curve $C = \{\langle x, y\rangle \in \mathbb{R}^2 : xy = 1\}$ is a closed subset of $\mathbb{R}^2$

Product Topology

6. Let $\mathcal{B}_1$ be a basis for a topology $\mathcal{T}_1$ on a set X and $\mathcal{B}_2$ a basis for a topology T_2 on a set Y. the set $X \times Y$ consists of all ordered pairs $\langle x, y\rangle$,X$\in X$ and $y \in Y$. Let $\mathcal{B}$ be the collection of subsets of $X \times Y$ consisting of all the sets $B_1 \times B_2$ where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Prove that $\mathcal{B}$ is a basis for a topology on $X \times Y$. the topology so defined is called the ***product topology*** on $X \times Y$.

7. Using Exercise 3 above and Exercise 2.1 #8, prove that every open subset of $\mathbb{R}$ is an F_σ-set and a G_δ-set.

2.3 Basis for a Given Topology

1. Determine whether or not aeach of the following collections is a basis for the euclidean topology on $\mathbb{R}^2$.

   (i) the collection of all "open" squares with sides parallel to the axes;

   (ii) the collection of all "open" discs;

   (iii) the collection of all "open" squares;

   (iv) the colection of all "open" rectangles;

   (v) the collection of all "open" triangles";

2. (i) Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on a non-0empty set X. If $\mathcal{B}_1$ is a collection of subsets of X sucht that $\mathcal{T} \subseteq \mathcal{B}_1 \subseteq \mathcal{B}$, prove that $\mathcal{B}_1$ is also a basis for $\mathcal{T}$.

   (ii) Deduce from (i) that there exist an uncountable number of distinct bases for the euclidean topology on $\mathbb{R}$.

3. Let $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$. As seen in Example 2.3.1, $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}$ and $\mathcal{T}$ is <u>not</u> the eucliden topology on $\mathbb{R}$. nevfertheless, show that each interval (a, b) is open in $(\mathbb{R}, \mathcal{T})$.

4. Let $C[0, 1]$ be the set of all continuous real-valued functions on $[0, 1]$.

   (i) Show that the collection $\mathcal{M}$, where $\mathcal{M} = \{M(f, \epsilon) : f \in C[0, 1]$ and ϵ is a positive real number$\}$ and

   $$M(f, \epsilon) = \left\{ g : g \in C[0, 1] \text{ and } \int_0^1 |f - g| < \epsilon \right\},$$

   is a basis for a topology $\mathcal{T}_1$ on $C[0, 1]$

   (ii) Show that the collection $\mathcal{U}$, where $\mathcal{U} = \{\, U(f, \epsilon) : f \in C[0, 1]$ and $\epsilon \in \mathbb{R}^+ \,\}$ and

   $$M(f, \epsilon) = \left\{ g : g \in C[0, 1] \text{ and } \sup_{x \in [0,1]} |f(x) - g(x)| < \epsilon \right\},$$

   is a basis for a topology $\mathcal{T}_2$ on $C[0, 1]$

   (iii) Prove that $\mathcal{T}_1 \neq \mathcal{T}_2$.

Subbasis for a Topology

5. Let $(X, \mathcal{T})$ be a topological space. A non-empty collection $\mathcal{S}$ of open subsets of X is said to be ***subbasis*** of $\mathcal{T}$ if the collection of all finite intersections fo members of $\mathcal{S}$ forms a basis for $\mathcal{T}$.

   (i) Prove that the collection of all open intervals of the form (a, ∞) or $(-\infty, b)$ is a subbasis for the euclidean topology on $\mathbb{R}$.

   (ii) Prove the $\mathcal{S} = \{\{a\}, \{a, c, d\}, \{b, c, d, e, f\}\}$ is a subbasis for the topology $\mathcal{T}_1$ of Example 1.1.2.

6. Let $\mathcal{S}$ be a subbasis for a topology $\mathcal{T}$ on the set $\mathbb{R}$. (See Exercise 5 above.) If all of the closed intervals $[a, b]$, with $a < b$, are in $\mathcal{S}$, prove that $\mathcal{T}$ is the discrete topology.

7. Let X be a set with at least two elements and $\mathcal{S}$ the colections of all $X \backslash \{x\}$,X$\in X$. Prove that $\mathcal{S}$ is a subbasis for the finite-closed topology on X.

8. Let X be any infinite set and $\mathcal{T}$ the discrete topology on X. Find a subbasis $\mathcal{S}$ for $\mathcal{T}$ such that $\mathcal{S}$ does not contain any singleton sets.

9. Let $\mathcal{S}$ be a collection of all straight lines in the plane $\mathbb{R}^2$. If $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on the set $\mathbb{R}^2$, what is the topology?

10. Let $\mathcal{S}$ be a collection of all straight lines in the plane which are parallel to the X-axis. If $\mathcal{S}$ is a subbasis for4 a topology $\mathcal{T}$ on $\mathbb{R}^2$, describe the open set in $(\mathbb{R}^2, \mathcal{T})$.

11. Let $\mathcal{S}$ be a collection of all circles in the plane. If $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on $\mathbb{R}^2$, describe the open sets in $(\mathbb{R}^2, \mathcal{T})$.

12. Let $\mathcal{S}$ be the collection of all circles in the plane which have their centres on the X-axis. If $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on $\mathbb{R}^2$, describe the open sets in $(\mathbb{R}^2, \mathcal{T})$.

Chapter 3

Important Definitions

3.1 Basic and T-spaces

Definition 3.1.1 (Discrete/Indiscrete Topologies)**.** Let X be any non-empty set and let $\mathcal{T}$ be the collection of all subsets of X . Then topological space $(X, \mathcal{T})$ is called ***the discrete topology*** on the set X.

Let X be any non-empty set and is called ***the indiscrete topology*** and $(X, \mathcal{T}) \to \mathcal{T} = \{X, \emptyset\}$.

Definition 3.1.2 (Finite-Closed Topologies)**.** Let X be any non-empty set. A topology $\mathcal{T}$ on X is called the ***finite-closed topology or the cofinite topology*** if the closed subsets of X are X and all finite subsets of X ; that is, the open sets are $\emptyset$ and all subsets of X which have finite complements.

Definition 3.1.3 (Countable-Closed Topologies)**.**

Definition 3.1.4 (T_0 Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be a **T_0-space** if for each pair of distinct points a, b in X, either there exist an open set containing a and not b, or there exists an open set containing b and not a.

Definition 3.1.5 (T_1 Spaces)**.** A topological space $(X, \mathcal{T})$ is said to be a **T_1-space** if every singleton set $\{x\}$ is closed in $(X, \mathcal{T})$.

Definition 3.1.6 (T_2 Space or Hausdorf)**.** A topological space $(X, \mathcal{T})$ is said to be Hausdorff (or a T_2-space) if given any pair of distinct points a, b in X there exist open sets U and V such that $a \in U, b \in V$, and $U \cap V = \emptyset$.

Definition 3.1.7 (T_3 Spaces, Regular Space)**.** A topological space $(X, \mathcal{T})$ is said to be a regular space if for any closed subset A of X and any point $x \in X \backslash A$, there exist open sets U and V such that $x \in U, A \subseteq V$, and $U \cap V = \emptyset$. If $(X, \mathcal{T})$ is regular and a T_1-space, then it is said to be a T_3-space

3.2 Homeomorphisms

Remark 3.2.1. Preserved by Homeomorphisms

1. T_0-space;
2. T_1-space;
3. T_2-space;
4. Regular space;
5. T_3-space;
6. Second Axiom of Countability.
7. Separable Space;
8. Discrete Space;
9. Indiscrete Space;
10. Finite-closed topology;
11. Countable-closed topology;

Remark 3.2.2. two spaces $(X, \mathcal{T}) and (Y, \mathcal{T}_1)$ cannot be homeomorphic if X and Y have different cardinalities (e.g. X is countable and Y is uncountable)or if $\mathcal{T}$ and $\mathcal{T}_1$ have different cardinalities.

3.3 Countability and Topologies

Definition 3.3.1 (Countable-closed topology)**.** Let X be any infinite set. The ***countable-closed topology*** is defined to be the topology having as its closed sets X and all countable subsets of X.

Definition 3.3.2 (Second Axiom of Countability)**.** A topological space $(X, \mathcal{T})$ is said to satisfy the ***second axiom of countability*** or to be ***second countable*** if there exists a basis $\mathcal{B}$ for $\mathcal{T}$, where $\mathcal{B}$ consists of only a countable number of sets.

Definition 3.3.3 (F_σ set)**.** Let $(X, \mathcal{T})$ be a topological space.

A subset S of X is said to be an F_σ ***set*** if it is the **union of a countable number of closed sets**.

Definition 3.3.4 (G_δ-set)**.** Let $(X, \mathcal{T})$ be topological space.

A subset T of X is said to be a G_δ***-set*** if it is the **intersection of a countable number of open sets**.

Definition 3.3.5 (Product Topology)**.** Let $\mathcal{B}_1$ be a basis for a topology $\mathcal{T}_1$ on a set X and $\mathcal{B}_2$ a basis for a topology T_2 on a set Y. the set $X \times Y$ consists of all ordered pairs $\langle x, y \rangle, x \in X$ and $y \in Y$. Let $\mathcal{B}$ be the collection of subsets of $X \times Y$ consisting of all the sets $B_1 \times B_2$ where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. Prove that $\mathcal{B}$ is a basis for a topology on $X \times Y$. the topology so defined is called the ***product topology*** on $X \times Y$.

Definition 3.3.6 (Subbasis)**.** Let $(X, \mathcal{T})$ be a topological space. A non-empty collection $\mathcal{S}$ of open subsets of X is said to be ***subbasis*** of $\mathcal{T}$ if the collection of all finite intersections fo members of $\mathcal{S}$ forms a basis for $\mathcal{T}$.

3.4 Miscellaneous

Definition 3.4.1 (Door Space)**.** A topological space (X, T) is said to be a ***door space*** if every subset of X is either an open set or a closed set (or both)

Definition 3.4.2 (Saturated Topological Space)**.** A subset S of a topological space $(X, \mathcal{T})$ is said to be **saturated** if it is an intersection of open sets in $(X, \mathcal{T})$.