Math 5411 – Mathematical Statistics I– Fall 2024
w/Nezamoddini-Kachouie

Paul Carmody
Cheatsheet for Midterm – October 21, 2024

Terms & Symbols:

X is the set values that can be attained by some random variable x.

Sample Space Ω the values that are attained by the random variable.

Discrete Random Variable a data point that can take specific list of values. That is, there is a one-to-one correspondence between elements of X and $\mathbb{N}$ and it is finite.

Continuous Random Variable a data point that comes from a range of values.

Probability, p, of a random variable x is $p : X \to [0, 1]$ and indicates the likelihood of x appearing in X. In the discrete case:

$$p(X = x) = \frac{|\{y \in X : y = x\}|}{|\Omega|}$$

Probability Mass Function, PMF a function $p : X \to [0, 1]$ indicating the values of the probabilities by element for a discrete random variable.

Cumulative Mass Function, CMF, $F(x) = p(X \le x)$ for a discrete random variable.

Probability Density Function, PDF $f : (-\infty, \infty) \to [0, 1]$ whose area under the graph is the probability by range for a continuous random variable.

$$p(a < x < b) = \int_a^b f(x)dx$$

Cumulative Density Function, CDF $F : (-\infty, \infty) \to [0, 1]$ which the cumulative probability up to the point x for a continuous random variable. That is

$$F(x) = p(X < x) = \int_{-\infty}^x f(t)dt$$

A ***Permutation*** of a discrete set, X, is a one-to-one correspondence of all elements in X. If n is the index of the elements in X then the permutation $P(n)$ is a different complete ordering of X (i.e., no replacement).

A ***Combination*** is a subset of the elements of X in a specific order (i.e., no replacement).

Conditional Probability, $P(A|B)$, read ”the probability of event A given first event B” has occurred.

$$\begin{aligned} P(A|B) &= P(A \cap B)P(B) & \qquad P(B) \neq 0 \\ P(B|A) &= P(A \cap B)P(A) & \qquad P(A) \neq 0 \end{aligned}$$

Events A and B are said to be ***independent*** if $P(A \cap B) = P(A)P(B)$. Also, typically, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Counting

$$\begin{aligned} |\Omega| &= p^n & \qquad \text{with replacement, e.g., flip a coin } n \text{ times} \\ |\Omega| &= n! & \qquad \text{without replacement} \\ \binom{n}{r} &= \frac{n!}{r!(n-r)!} & \qquad n \text{ choose } r \text{ unordered} \\ 2^n &= \sum_{i=0}^n \binom{n}{i} & \\ \binom{n}{k_1 \dots k_i} &= \frac{n!}{k_1! \cdots k_i!} & \qquad k_i \text{ form a partition on n} \end{aligned}$$

Random Variable Types:

Bernouli: $p : \{0, 1\} \to [0, 1]$, $p(x) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \end{cases}$

Uniform: $p : [a, b] \to [0, 1]$, $p(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

Binomial(n, p) indicates n trials with probability of success for each trial of p. Thus, the probability of k success in n trials is

$$p(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Negative Binomial and Geometric: first execute r successes then determine the probability of k failures.

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \qquad r \text{ successes}$$
$$P(X = k) = p(1-p)^k \qquad r = 1 \text{ implies geometric}$$

Hypergeometric: urn contains n balls r of them are black $n - r$ are not. Draw m balls with k the number of black balls drawn.

$$P(X = k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{k}}$$

Poisson: λ is the parameterized count per unit time, k is the count per unit time. *(this can be used in place of Binomial if n is very high and p is very low, $\lambda = nxp$).*

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Exponential Decay:

$$f(x) = \begin{cases} \lambda e^{-\lambda} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Gamma Distribution: α shape parameter λ scale parameter. $\beta = 1/\lambda$ is the rate parameter

$$g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} = \frac{1}{\beta^\alpha \Gamma(\alpha)} t^{\alpha-1} e^{\beta t}$$
$$\Gamma(t) = \int_0^\infty u^{x-1} e^{-u} du,\ x > 0$$
$$\Gamma(n+1) = n!$$

Normal Distributions: μ mean, σ standard deviation, ***Standard Normal Distribution*** is $\mu = 0, \sigma = 1$.

$$f(x) = \frac{1}{\sigma\sqrt{2}} e^{-(x-\mu)^2/2\sigma^2}$$
$$f(x) = \frac{1}{\sqrt{2}} e^{-x^2/2} \qquad \text{Standard Normal Distribution}$$

Expected Values and Variance:

Expected and Variance:

$$\mu = E[X] = \sum_i x_i p(x_i) \text{ and } E[X] = \int_{-\infty}^{\infty} x f(x) dx$$
$$\sigma^2 = V[X] = E[X^2] - (E(X))^2$$
$$= \sum_i x_i^2 p(x_i) - \left(\sum_i x_i p(x_i)\right)^2 \text{ and } V[X] = \int_{-\infty}^{\infty} x^2 f(x) dx - \left(\int_{-\infty}^{\infty} x f(x) dx\right)^2$$

Moment Generating Function (MFG):

$$M(t) = \sum_x e^{tx} p(x) \text{ and } M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \qquad \text{raw}$$
$$M^{(r)}(0) = E[X^r] \qquad \text{center}$$
$$E[X] = M'(0)$$
$$V[X] = M''(0) - (M'(0))^2$$

Samples and Estimators

$$\theta \in \{p, \mu, \sigma_X^2, \sigma_X\} \qquad \text{actual from population}$$

$$\hat{\theta} \in \{\hat{p} = X/N,\ \hat{\mu} = \bar{X} = \frac{\sum X_i}{N},\ \sigma_{\hat{\theta}}^2 = S^2 = E[(\bar{X} - X_i)^2],\ \sigma_{\hat{\theta}} = S\} \qquad \text{from sample}$$

$$E[X] = \mu$$

Confidence Intervals

$$z_\theta = \frac{\theta - \mu_\theta}{\sigma_\theta}$$

$$z_{0.05} = 1.644,\ z_{0.025} = 1.96,\ z_{0.01} = 2.32$$

$$CI = \mu \pm z_{\alpha/2}\sigma$$

Two Sample Confidence Intevals

$$\begin{cases} H_0 : \hat{\theta}_1 - \hat{\theta}_2 = 0 & \text{no difference} \\ H_a : |\hat{\theta}_1 - \hat{\theta}_2| > 0 & \text{one is larger than the other} \end{cases}$$

$$X = \mu_1 - \mu_2$$

$$\sigma = \sqrt{\frac{p_1 p_2}{n_1} + \frac{p_1 p_2}{n_2}}$$

Mean Squared Variance

$$\begin{aligned} \mathrm{Var}(X - x_0) &= E[(X - x_0)^2] - [E[(x - x_0)]^2 \\ MSE = E[(X - x_0)^2] &= \mathrm{Var}(X - x_0) + [E[(x - x_0)]^2 \\ &= \sigma^2 + \beta^2 \qquad \text{variance plus the bias} \end{aligned}$$

Covariance and Correlation

$$\mathrm{Cov}(X, Y) = E[(X - \mu_x)(Y - \mu_y)]$$

$$\mathrm{Cov}(V, V) = \mathrm{Var}(V)$$

$$\mathrm{Var}(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j \mathrm{Cov}(X_i, Y_j)$$

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \qquad \text{correlation coefficient}$$

Two Sample Proportion

$$\hat{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{\hat{p_1} n_i + \hat{p_2} n_2}{n_1 + n_2}$$

$$\hat{q} = 1 - \hat{p}$$

$$\hat{p_1} = \frac{x_1}{n_1},\ \hat{p_2} = \frac{x_2}{n_2}$$

$$z = \frac{(\hat{p_1} - \hat{p_2}) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}\hat{q}}{n_1} + \frac{\hat{p}\hat{q}}{n_2}}}$$