Solutions to Homeworks

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Contents

1. Previous final

1. (6 points) Prove that any nonempty open subset $A \subset \mathbb{R}^n$ has outer measure $m^*(A) > 0$.

Proof. Since $A \neq \emptyset$, there is $a \in A$. But A is open, $B_r(a) \subset A$ for some $r > 0$. Let $\delta = r/\sqrt{n}$,

$$I_\delta(a) = \prod_{i=1}^{n} \left(a^i - \delta, a^i + \delta\right),$$

then $I_\delta(a) \subset B_r(a)$. Hence

$$m^*(A) \geq m^*(B_r(a)) \geq m^*(I_\delta(a)) = |I_\delta(a)| = \left(\frac{2r}{\sqrt{n}}\right)^n > 0.$$

2. (8 points) Find the value of

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin(x/n)}{1+x^2}\, \mathrm{d}x.$$

Hint: Use a convergence theorem for the Lebesgue integral.

Proof. Let $f_n(x) = \dfrac{\sin(x/n)}{1+x^2}$, then for all $x \in \mathbb{R}$,

$$f_n(x) \to 0 =: f(x), \qquad |f_n(x)| \leq \frac{1}{1+x^2} =: g(x).$$

Since $g \in L(\mathbb{R})$, by Lebesgue dominated theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{\sin(x/n)}{1+x^2} dx = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} f(x)\, dx = 0.$$

3. (8 points) Let $f_n : [0,1] \to \mathbb{R}$ be continuous, and assume $f_n \to f$ uniformly on $[0,1]$. Prove that f is continuous.

Proof. Given $a \in [a,b]$. To prove that f is continuous at a, take $\{x_n\} \subset [0,1]$, $x_n \to a$. Since $f_n \rightrightarrows f$ on $[0,1]$, for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|f_N - f|_\infty = \sup_{x \in [0,1]} |f_N(x) - f(x)| < \varepsilon.$$

Hence

$$\begin{aligned} |f(x_n) - f(a)| &\le |f(x_n) - f_N(x_n)| + |f_N(x_n) - f_N(a)| + |f_N(a) - f(a)| \\ &\le 2|f_N - f|_\infty + |f_N(x_n) - f_N(a)| < 2\varepsilon + |f_N(x_n) - f_N(a)|. \end{aligned}$$

Consequently, because f_N is continuous at a, we get

$$\begin{aligned} \overline{\lim_{n\to\infty}} |f(x_n) - f(a)| &\le \overline{\lim_{n\to\infty}} \{2\varepsilon + |f_N(x_n) - f_N(a)|\} \\ &= 2\varepsilon + \overline{\lim_{n\to\infty}} |f_N(x_n) - f_N(a)| = 2\varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$ we deduce $f(x_n) \to f(a)$. Hence f is continuous at a.

4. (a) (6 points, no explanation needed) Define the set $E \subset \mathbb{R}^2$ by

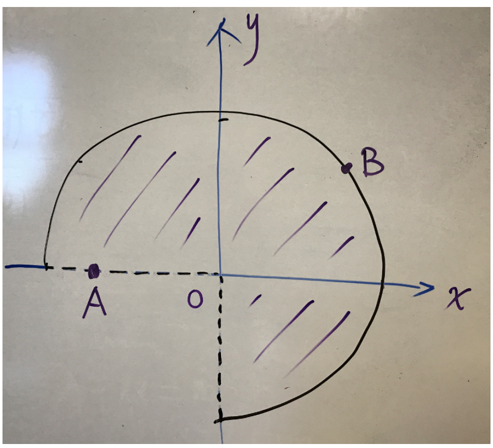
$$E = \{(x,y) \in \mathbb{R}^2 : x > 0, x^2 + y^2 \le 1\} \cup \{(x,y) \in \mathbb{R}^2 : y > 0, x^2 + y^2 \le 1\}.$$

Sketch E, and find its closure, interior, and boundary. Is E open, closed, both, or neither? Is E connected? Is E compact?

We are using the standard topology on $\mathbb{R}^2$ in this part of the question.

Proof. The set E is sketched below:

(1) $\overline{E}$ is the 3/4 disk with the 3/4 circle and the two dashed radius;
(2) E° is the 3/4 disk without the 3/4 circle and the two radius;
(3) ∂E is the union of the 3/4 circle and the two radius.



(1) E is not open ($B \in E$ but $B \notin B^\circ$);
(2) E is not closed ($A \in \partial E$, but $A \notin E$), hence E is not compact;
(3) E is connected.

(b) (4 points, no explanation needed) Now consider the same set E, but in the subspace topology with respect to the set

$$Y = \{(x,y) \in \mathbb{R}^2 : x > 0\} \cup \{(x,y) \in \mathbb{R}^2 : y > 0\}.$$

Sketch Y. Find the boundary, closure, and interior of E in the metric space Y with the subspace metric. In this metric, is E open, closed, both, or neither?

Proof. In Y,

(1) the bdry of E is the $3/4$ circle;
(2) the closure of E is E itself, thus E is closed;
(3) the interier of E is the union of the $3/4$ disk (without the $3/4$ circle) and the two dashed radius.

5. (6 points each)

(a) Let A be an $m \times n$ matrix, and define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x) = Ax$. Using the definition of differentiability, show that $Df(x) = A$ for every $x \in \mathbb{R}^n$.

Proof. Take an $x \in \mathbb{R}^n$. For $h \in \mathbb{R}^n$, we have

$$\frac{f(x+h) - f(x) - Ah}{|h|} = \frac{A(x+h) - A(x) - Ah}{|h|} = \frac{0}{|h|} \to 0$$

as $h \to 0$. By definition, f is differentiable at x and $f'(x) = A$, or we write $Df(x) = A$.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function, and assume $\det(Df(\mathbf{x}_0)) = 0$ for some $\mathbf{x}_0 \in \mathbb{R}^n$. Show that, if f has a local inverse function f^{-1} in a neighborhood of x_0, then f^{-1} is **not** differentiable at $f(x_0)$.

Proof. Observe that $f^{-1} \circ f = \mathbf{1}_{\mathbb{R}^n}$ is linear map determined by the identity matrix I_n. By (a) we have

$$(\mathbf{1}_{\mathbb{R}^n})'(x_0) = I_n.$$

If f^{-1} is differentiable at $f(x_0)$, we may apply the chain role to get

$$\begin{aligned} I_n = (\mathbf{1}_{\mathbb{R}^n})'(x_0) &= \left(f^{-1} \circ f\right)'(x_0) \\ &= \left[\left(f^{-1}\right)'(f(x_0))\right]\left[f'(x_0)\right], \end{aligned}$$

the right hand side is the product of two matrix. Because $\det[f'(x_0)]$, we deduce

$$\begin{aligned} 1 = \det I_n &= \det\left\{\left[\left(f^{-1}\right)'(f(x_0))\right]\left[f'(x_0)\right]\right\} \\ &= \det\left[\left(f^{-1}\right)'(f(x_0))\right] \cdot \det\left[f'(x_0)\right] = 0, \end{aligned}$$

a contradiction.

6. (8 points) Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let $f, g : \Omega \to \mathbb{R}$ be measurable functions. Prove that $\min(f, g)$ is also a measurable function.

Proof. By assumptions, $F : \Omega \to \mathbb{R}^2$ given by $F(x) = (f(x), g(x))$, is measurable. If $g : \mathbb{R}^2 \to \mathbb{R}$,

$$g(u, v) = \min\{u, v\},$$

is continuous, then $\min(f, g) = g \circ F : \Omega \to \mathbb{R}$ is continuous (Lemma 4.16 in Lecture Notes).

To prove that g is continuous at (a, b) for given $(a, b) \in \mathbb{R}^2$, take a sequence $\{(u_n, v_n)\} \subset \mathbb{R}^2$ with $(u_n, v_n) \to (a, b)$, we have (by properties of limits of numerical sequences)

$$\begin{aligned} g(u_n, v_n) = \min\{u_n, v_n\} &= \frac{u_n + v_n - |u_n - v_n|}{2} \\ &\to \frac{a + b - |a - b|}{2} = \min\{a, b\} = g(a, b). \end{aligned}$$

7. (a) (6 points) Let $\Omega \subset \mathbb{R}^n$ be a measurable set with $m(\Omega) < \infty$. If $f : \Omega \to [0, \infty]$ is such that $\int_\Omega f^2 < \infty$, prove that $\int_\Omega f < \infty$ also.
Hint: Try decomposing Ω into two parts based on the size of f. You are not allowed to use Hölder's inequality.

Proof. If $x \in \{f \geq 1\}$, then $f(x) \leq f^2(x)$; if $x \in \{f < 1\}$, then $f(x) < 1$. Hence

$$\begin{aligned} \int_\Omega f = \int_{\{f<1\}} f + \int_{\{f\geq 1\}} f &\leq \int_{\{f<1\}} 1 + \int_{\{f\geq 1\}} f^2 \\ &\leq \int_\Omega 1 + \int_\Omega f^2 = m(\Omega) + \int_\Omega f^2 < \infty. \end{aligned}$$

(b) (4 points) Show by example that the conclusion of (a) can fail if $m(\Omega) = \infty$, i.e. find a function f with $\int_\Omega f^2 < \infty$ and $\int_\Omega f = \infty$. (You could take $\Omega = \mathbb{R}$, for example.)

Proof. Let $\Omega = (1, \infty)$, $f : \Omega \to \mathbb{R}$, $f(x) = 1/x$. Then

$$\int_\Omega f^2 = \int_1^\infty \frac{dx}{x^2} = 1 < \infty, \qquad \int_\Omega f = \int_1^\infty \frac{dx}{x} = \infty.$$

2. Final Exam

1. (6 points) Prove that any nonempty open subset $A \subset \mathbb{R}^n$ has outer measure $m^*(A) > 0$.

Proof. Since $A \neq \emptyset$, there is $a \in A$. But A is open, $B_r(a) \subset A$ for some $r > 0$. Let $\delta = r/\sqrt{n}$,

$$I_\delta(a) = \prod_{i=1}^n \left(a^i - \delta, a^i + \delta\right),$$

then $I_\delta(a) \subset B_r(a)$. Hence

$$m^*(A) \geq m^*(B_r(a)) \geq m^*(I_\delta(a)) = |I_\delta(a)| = \left(\frac{2r}{\sqrt{n}}\right)^n > 0.$$

2. (8 points) Let $f : [1, \infty) \to \mathbb{R}$ be integrable, show that

$$\lim_{k\to\infty} \int_1^\infty \frac{f(x)}{x^k}\, dx = 0.$$

Hint: Use a convergence theorem for the Lebesgue integral.

Proof. Let $f_k(x) = x^{-k} f(x)$, then $f_k \to \chi_{\{1\}}$ on $[1, \infty)$, $|f_k| \le f$. By Lebesgue dominated theorem,

$$\int_1^\infty \frac{f(x)}{x^k} dx = \int_1^\infty f_k \to \int_1^\infty \chi_{\{1\}} = 0.$$

3. (8 points) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, and assume $f_n \to f$ uniformly on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

Proof. Since $f_n \rightrightarrows f$ on $[0, 1]$, that is

$$|f_n - f|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0,$$

we have

$$\left| \int_0^1 f_n - \int_0^1 f \right| \le \int_0^1 |f_n(x) - f(x)|\, dx \le \int_0^1 |f_n - f|_\infty = |f_n - f|_\infty \to 0.$$

That is the desired result.

4. (a) (6 points, no explanation needed) Define the set $E \subset \mathbb{R}^2$ by

$$E = \{(x, y) \in \mathbb{R}^2 : x > 0, |x|+|y| \le 1\} \cup \{(x, y) \in \mathbb{R}^2 : y > 0, |x|+|y| \le 1\}.$$

Sketch E, and find its closure, interior, and boundary. Is E open, closed, both, or neither? Is E connected? Is E compact?
We are using the standard topology on $\mathbb{R}^2$ in this part of the question.

(b) (4 points, no explanation needed) Now consider the same set E, but in the subspace topology with respect to the set

$$Y = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

Sketch Y. Find the boundary, closure, and interior of E in the metric space Y with the subspace metric. In this metric, is E open, closed, both, or neither?

5. (6 points each)

(a) Let A be an $n \times n$ ***symmetric*** matrix, Ω be an open subset of $\mathbb{R}^n$. The function $f : \Omega \to \mathbb{R}$ is defined by

$$f(x) = \tfrac{1}{2} Ax \cdot x.$$

Suppose $a \in \Omega$, $h \in \mathbb{R}^n$ is a vector. Find $\nabla_h f(a)$, the directional derivative of f at a in the direction h.

Proof. By definition, let $\varphi(t) = f(a + th)$, then

$$\begin{aligned}
\nabla_h f(a) = \varphi'(0) &= \left.\frac{d}{dt}\right|_{t=0} \varphi(t) = \left.\frac{d}{dt}\right|_{t=0} f(a+th) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{1}{2} A(a+th) \cdot (a+th) - \frac{1}{2} Aa \cdot a \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{1}{2} (Aa + tAh) \cdot (a+th) - \frac{1}{2} Aa \cdot a \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{t}{2} (Aa \cdot h + Ah \cdot a) + \frac{t^2}{2} Ah \cdot h \right) \\
&= \frac{1}{2} (Aa \cdot h + Ah \cdot a) \\
&= \frac{1}{2} \left(Aa \cdot h + h \cdot A^{\mathrm{T}} a \right) = Aa \cdot h
\end{aligned}$$

because $A = A^{\mathrm{T}}$.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at $x_0 \in \mathbb{R}^n$, and assume $\det(Df(x_0)) = 0$. Show that, if f has a local inverse function f^{-1} in a neighborhood of x_0, then f^{-1} is **not** differentiable at $f(x_0)$.

Proof. Observe that $f^{-1} \circ f = \mathbf{1}_{\mathbb{R}^n}$ is linear map determined by the identity matrix I_n. We have

$$(\mathbf{1}_{\mathbb{R}^n})'(x_0) = I_n.$$

If f^{-1} is differentiable at $f(x_0)$, we may apply the chain role to get

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I_n = (\mathbf{1}_{\mathbb{R}^n})'(x_0) &= \left(f^{-1} \circ f \right)'(x_0) \\
&= \left[\left(f^{-1} \right)' (f(x_0)) \right] \left[f'(x_0) \right],
\end{aligned}$$

the right hand side is the product of two matrix. Because det $[f'(x_0)]$, we deduce

$$\begin{aligned}
1 = \det I_n &= \det \left\{ \left[\left(f^{-1} \right)' (f(x_0)) \right] \left[f'(x_0) \right] \right\} \\
&= \det \left[\left(f^{-1} \right)' (f(x_0)) \right] \cdot \det \left[f'(x_0) \right] = 0,
\end{aligned}$$

a contradiction.

6. $(3 + 5$ points)

(a) Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined below is continuous:

$$g(u, v) = \max\{u, v\}.$$

Hint: $\max\{u, v\} = \dfrac{u + v + |u - v|}{2}$.

Proof. To prove that g is continuous at (a, b) for given $(a, b) \in \mathbb{R}^2$, take a sequence $\{(u_n, v_n)\} \subset \mathbb{R}^2$ with $(u_n, v_n) \to (a, b)$, we have (by properties of limits of numerical sequences)

$$\begin{aligned} g(u_n, v_n) &= \max\{u_n, v_n\} = \frac{u_n + v_n + |u_n - v_n|}{2} \\ &\to \frac{a + b + |a - b|}{2} = \max\{a, b\} = g(a, b). \end{aligned}$$

(b) Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let $f, g : \Omega \to \mathbb{R}$ be measurable functions. Prove that $\max\{f, g\}$ is also a measurable function.

Proof. By assumptions, $F : \Omega \to \mathbb{R}^2$ given by $F(x) = (f(x), g(x))$, is measurable. By (a), we know that $g : \mathbb{R}^2 \to \mathbb{R}$,

$$g(u, v) = \max\{u, v\},$$

is continuous, then $\max(f, g) = g \circ F : \Omega \to \mathbb{R}$ is continuous (Lemma 4.16 in Lecture Notes).

7. (a) (6 points) Let $\Omega \subset \mathbb{R}^n$ be a measurable set with $m(\Omega) < \infty$. If $f : \Omega \to [0, \infty]$ is such that $\int_\Omega f^3 < \infty$, prove that $\int_\Omega f < \infty$ also.
*Hint: Try decomposing Ω into two parts based on the size of f. You are **not allowed** to use Hölder's inequality.*

Proof. If $x \in \{f \geq 1\}$, then $f(x) \leq f^3(x)$; if $x \in \{f < 1\}$, then $f(x) < 1$. Hence

$$\begin{aligned} \int_\Omega f &= \int_{\{f<1\}} f + \int_{\{f\geq 1\}} f \leq \int_{\{f<1\}} 1 + \int_{\{f\geq 1\}} f^3 \\ &\leq \int_\Omega 1 + \int_\Omega f^3 = m(\Omega) + \int_\Omega f^3 < \infty. \end{aligned}$$

(b) (4 points) Show by example that the conclusion of (a) can fail if $m(\Omega) = \infty$, i.e. find a function f with $\int_\Omega f^3 < \infty$ and $\int_\Omega f = \infty$. (You could take $\Omega = (0, \infty)$, for example.)

Proof. Let $\Omega = (1, \infty)$, $f : \Omega \to \mathbb{R}$, $f(x) = 1/x$. Then

$$\int_\Omega f^3 = \int_1^\infty \frac{dx}{x^3} < \infty, \qquad \int_\Omega f = \int_1^\infty \frac{dx}{x} = \infty.$$