Math 5411 – Mathematical Statistics I– Fall 2024
w/Nezamoddini-Kachouie

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1. Use the definition of moment generating function (MGF) and show that:
   $V[X] = d^2(M(t))/dt^2|_{t=0} - (d(M(t))/dt|_{t=0})^2$

   In general

$$\begin{aligned} M^{(r)}(0) &= E\,(\,X^r\,) \\ V[X] &= E[X^2] - (E[X])^2 \\ &= M''(0) - (\,M'(0)\,)^2 \\ &= d^2(M(t))/dt^2|_{t=0} - (d(M(t))/dt|_{t=0})^2 \end{aligned}$$

However, there are distinct definitions of MGF for discrete and continuous random variables.

Discrete Random Variables.

$$\begin{aligned} M(t) &= \sum_x e^{tx} p(x) \\ M'(t) &= \sum_x x e^{tx} p(x) \\ E[X] = M'(0) &= \sum_x x p(x) \\ M''(t) &= \sum_x x^2 e^{tx} p(x) \\ E[X^2] = M''(0) &= \sum_x x^2 p(x) \\ V[X] &= \sum_x x^2 p(x) - \left(\sum_x x p(x) \right)^2 \\ &= E[X^2] - (\,E[X]\,)^2 \\ &= M''(0) - (\,M'(0)\,)^2 \end{aligned}$$

Continuous Random Variables.

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} e^{tx} f(x)dx \\ M'(t) &= \int_{-\infty}^{\infty} x e^{tx} f(x) \\ E[X] = M'(0) &= \int_{-\infty}^{\infty} x f(x)dx \\ M''(t) &= \int_{-\infty}^{\infty} x^2 e^{tx} f(x)dx \\ E[X^2] = M''(0) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ V[X] &= \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} x f(x)dx \right)^2 \\ &= E[X^2] - (\,E[X]\,)^2 \\ &= M''(0) - (\,M'(0)\,)^2 \end{aligned}$$

2. Find the MGF for Uniform(a,b).

$$
\begin{aligned}
p(x) &= \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases} \\
M(t) &= \int_{-\infty}^{\infty} e^{tx} p(x) dx \\
&= \int_{-\infty}^{a} e^{tx} 0 dx + \int_{a}^{b} e^{tx} \frac{1}{b-a} dx + \int_{b}^{\infty} e^{tx} 0 dx \\
&= \frac{1}{t(b-a)} e^{tx} \Big|_a^b \\
&= \frac{e^{bt} - e^{at}}{t(b-a)} \\
M(t) &= \frac{be^{bt} - ae^{at}}{t(b-a)}
\end{aligned}
$$

3. Use the definitions of mean ($E[X]$) and Variance ($V[X]$) and find $E[X]$ and $V[X]$ for:

   (a) X Binomial(n,p)

$$p(X=x)=\begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$
$$E[X]=\sum_x xp(x)=0(1-p)+1p=p$$
$$E[X^2]=\sum_x x^2p(x)=0(1-p)+1p=p$$
$$V[X]=E[X^2]-(E[X])^2=p-p^2=p(1-p)$$

   (b) X Uniform(a,b)

$$p(X=x)=\begin{cases} \frac{1}{b-a} & a\le x\le b \\ 0 & \text{otherwise} \end{cases}$$
$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} xp(x) \\
&= \int_{-\infty}^{a} 0dx+\int_a^b \frac{1}{b-a}xdx+\int_b^{\infty} 0dx \\
&= \frac{x^2}{2(b-a)}\bigg|_a^b \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{b+a}{2} \\
E[X^2] &= \int_{-\infty}^{\infty} x^2p(x) \\
&= \int_{-\infty}^{a} 0dx+\int_a^b \frac{1}{b-a}x^2dx+\int_b^{\infty} 0dx \\
&= \frac{x^3}{3(b-a)}\bigg|_a^b \\
&= \frac{b^3-a^3}{3(b-a)} \\
&= \frac{1}{3}\left(b^2+ab+a^2\right) \\
V[X] &= \frac{1}{3}\left(b^2+ab+a^2\right)-\left(\frac{b+a}{2}\right)^2 \\
&= \frac{1}{3}\left(b^2+ab+a^2\right)-\frac{b^2+2ab+a^2}{4} \\
&= \frac{1}{12}(b^2-ab+a^2)
\end{aligned}$$

4. Assume the number of Hurricanes have a Poisson distribution with average of 3 hurricanes in the hurricane season (6 months from June 1 to Nov 30). Find:

   (a) Probability of having no Hurricane in a hurricane season.

$$p(k) = \frac{\lambda^k}{k!}e^{-\lambda} \text{ and } \lambda = 3$$
$$p(0) = \frac{1}{1}e^{-3} = 0.05$$

   (b) Probability of having 1 Hurricane in a hurricane season.

$$p(1) = \frac{1}{1}e^{-3} = 0.15$$

   (c) Probability of having 10 Hurricanes in a hurricane season.

$$p(k) = \frac{3^{10}}{10!}e^{-3} = \frac{59049}{3628800}0.05 = 0.0008$$

   (d) Probability of having 3 Hurricanes in the first half of a hurricane season (assume the number of hurricanes are distributed evenly over 6 months of hurricane session).

   By chopping the interval in half we are effectively changing the λ value in half or 1.5. Thus, our new formula would be

$$p(k) = \frac{1.5^k}{k!}e^{-1.5}$$
$$\text{and } p(3) = \frac{1.5^3}{3!}e^{-1.5} = \frac{3.375}{6}0.223 = 0.1255$$