Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

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Section 9: Submanifolds – June 11, 2025

Problems

9.1. **Regular values**

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find all values $c \in \mathbb{R}$ for which the level $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

Stated another way: find all of the $p \in \mathbb{R}^2$ such that $f(p) = c$ and p is regular (i.e., not a critical point). Regular points are points that have a non-zero Jacobian. Thus,

$$\begin{aligned}
J(f) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left(3x^2 - 6y, -6x + 2y \right) \\
J(f) &= 0 \\
x^2 - 3y &= 0 \text{ and } y = 3x \\
3x &= \frac{x^2}{3} \\
\frac{x^2}{3} - 3x &= 0 \implies x = 0, 6 \\
y &= 0, 18
\end{aligned}$$

thus, these points $(0,0), (6,18)$ are the critical points which will NOT be in a smooth manifold (non-regular points). Therefore, the values of c are which do not work are

$$\begin{aligned}
f(0,0) &= 0 \\
f(6,18) &= 6^3 - 6(6)(18) + 18^2 = -108
\end{aligned}$$

Thus $c \in \mathbb{R} \backslash \{0, -108\}$

9.2. **Solution set of one equation.**

Let x, y, z, w be the standard coordinates on $\mathbb{R}^4$. Is the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in $\mathbb{R}^4$ a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology).

The Jacobian is essentially the gradient.

$$\begin{aligned}
\text{Let } f(x, y, z, w) &= x^5 + y^5 + z^5 + w^5 - 1 \\
\nabla f(x, y, z, w) &= (5x^4, 5y^4, 5z^4, 5w^4) \\
\nabla f = 0 &\implies (0, 0, 0, 0)
\end{aligned}$$

but $(0, 0, 0, 0)$ is not a solution of f. Therefore the solution space is smooth.

9.3. **Solution set of two equations.**

Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \; z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $F(x, y, z) = (x^3 + y^3 + z^3 - 1,\ z - xy)$. Then

$$J(F) = \begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ -y & -x & 1 \end{bmatrix}$$

are these two linearly independent? Is there a λ such that

$$\begin{aligned} 3x^2 &= \lambda(-y),\ 3y^2 = \lambda(-x),\ 3z^2 = \lambda \\ -3x^2/y &= -3y^2/x \to x^3 = y^3 \to x = y \\ 3x^2 &= \lambda(-x) \to \lambda = -3x \\ -3x &= 3z^2 \to -x = z^2 \text{ and } z = xy = x^2 \implies x = 0, 1 \end{aligned}$$

Thus, $(0, 0, 0)$ is a critical point but doesn't exist in the range of F_x therefore the solution set is a submanifold.

9.4. **Regular submanifolds**

Suppose that a subset S of $\mathbb{R}^2$ that the property that locally on S one of the coordinates is C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$. (Note that the unit circle defined by $x^2 + y^2 = 1$ has this property at every point of the circle, there is a neighborhood in which y is a C^∞ function of x or x is a C^∞ function of y.)

9.5. **Graph of a smooth function**

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

$$\Gamma(f) = \{\, (\, x, y, f(x, y)\,) \in \mathbb{R}^3 \,\}$$

is a regular submanifold of $\mathbb{R}^3$.

Redefine Γ as

$$\begin{aligned} \Gamma(x, y, z) &= z - f(x, y) = 0 \\ J(\Gamma) &= \left[\ \tfrac{\partial \Gamma}{\partial x} \quad \tfrac{\partial \Gamma}{\partial y} \quad \tfrac{\partial \Gamma}{\partial z}\ \right] \\ &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, 1 \right] \end{aligned}$$

which is never zero. Therefore, there are no critical points.

9.6. **Euler's formula** A polynomial $F(x_0, \ldots, x_n) \in \mathbb{R}[x_0, \ldots, x_n]$ is *homogenous of degree* k if it is a linear combination of monomials $x_9^{i_o} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^{n} i + j = k$. Let $F(x_0, \ldots, x_n)$ be a homogenous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_o, \ldots, tx_n) = t^k F(x_0, \ldots, x_n).$$

Show that

$$\sum +i = 0^n x_i \frac{\partial F}{\partial x_i} = kF.$$

9.7. **Smooth projectve hypersurface**

On the projective space $\mathbb{R}P^n$ a hmogenous polynomial $F(x_0, \ldots, x_n)$ of degree k is not a function, since its value at a point $[a_0, \ldots, a_n]$ is not unique. However, the zero set in $\mathbb{R}P^n$ of a homogenous polynomial $F(x_0, \ldots, x_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0,\ \forall t \in R^\times : \mathbb{R} - \{0\}$$

The zero set of finitely many homogenous polynomials in $\mathbb{R}P^q$ is called a *real projective variety.* A projective variety defined by a single homogeneous polynomial of degree k is called a *hypersurface* of degree k. Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if $\partial F/\partial x_0, \partial F/\partial x_1$ and $\partial F/\partial x_2$ are simultaneously zero on $Z(F)$. (*Hint:* The standard coordinates on U_0 which is homeomorpic to $\mathbb{R}^2$, are $x = x_1/x_0, y = x_2/x_0$ (see Subsection 7.7). In $U_0, F(x_0, x_1, x_2) = x_0^t F(1, x_1/x_0, x_2/x_0) = x_0^k F(1, x, y)$. Define $f(x, y) = F(1, x, y)$. Then f and F have the same zero set in U_0.)

9.8. **Product of regular submanifolds**

If S_1 is a regular submanifold of the manifold M_i for $i = 1, 2$, prove that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.

9.9. **Complex special linear group**

The complex special linear group $\mathrm{SL}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of $n \times n$ complex matrices of determinant 1. Show that $\mathrm{SL}(n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$ and determine its dimension. (This problem requries a rudimentary knowledge of complex analysis.)

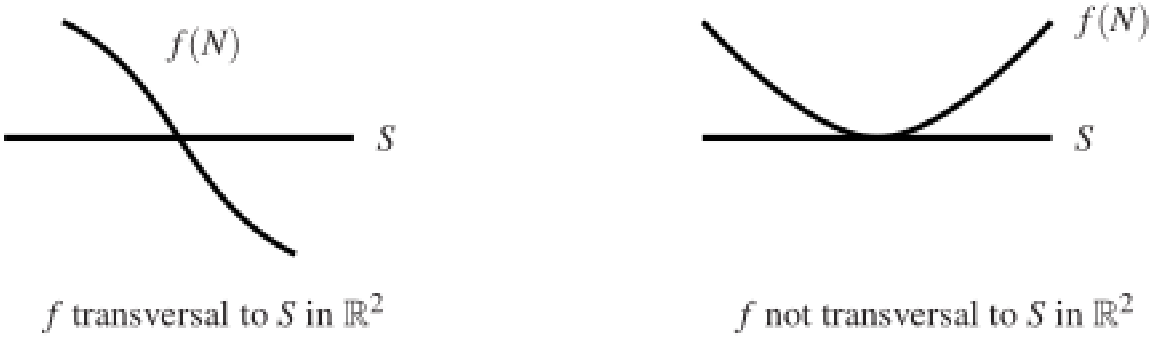

Fig. 9.5. Transversality.

9.10. **The transversality theorem**

A C^∞ map $f : N \to M$ is said to be *transversal* to a submanifold $S \subset M$ (Figure 9.5) if for every $p \in f^{-1}(S)$.

$$f_*(T_pN) + T_{f(p)}S = T_{f(p)}M.$$

(If A and B are subspaces of a vector space, their sum $A + B$ is the subspace consisting all $a + b$ with $a \in A$ and $b \in B$. The sum need not be a diret sum.) The goal of this exercise is to prove that the ***transversality theorem:*** if a C^∞ map $f : N \to M$ is transveral to a regular submanifold S of codimension k in M, then $f^{-1}(S)$ is a regular submanifold of codimension k in N.

When S consists of a single point c, transversality of f to S simply means that $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is especially useful in giving conditions under which the intersection of two submanifolds is a submanifold.

Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z(x^{m-l+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = (x^{m-k+1}, \ldots, x^m).$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = g \circ f^{-1}(0)$.

(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of teh function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.

(c) Prove the transversality theorem.