Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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Homework #1 – September 2, 2024

I. This problem review continuity for functions on real line.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(a) Show that $f(x) = x^2$ is continuous at $x = 2$.
Given an $\epsilon > 0$, when $|f(x) - 4| < \epsilon$, $|x^2 - 4| < \epsilon$. Let $\delta < \sqrt{\epsilon + 4}$

(b) Suppose that f is continuous at a and $f(a) \neq 0$. Show that f is nonzero in some open interval containing a.
Since f is continuous at a and $f(a) \neq 0$ then for every $\epsilon > 0$ such that when $|f(x) - f(a)| < \epsilon$ or $f(x) = f(a) \pm \epsilon$. Without loss of generality, we can also say $f(a) \neq \epsilon$. Since both $f(a)$ and ϵ are both non-zero, $f(x) \neq 0$.

II. This problem review derivatives.

(a) Let $f(x) = x^n$ for some positive integer n. Using the definition of the derivative, and the binomial theorem, show that f'^{n-1}.

(b) Is the function

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x \leq 0 \end{cases}$$

differentiable at $x = 0$.

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$f(x)$ is both continuous and differentiaable at $x = 0$.

III. This problem reviews sup and inf.

For any subset $A \subset \mathbb{R}$, we say that M is an *upper bound* for A if $x \leq M$ for all $x \in A$. If a set A has a finite upper bound, we say it is *boundared above*. It is a theorem about the set $\mathbb{R}$ that *for any set $A \subset \mathbb{R}$ that is bounded above, there exists a least (smallest) upper bound for A*. This least upper bound is called supermum of A, and denoted $\sup A$. By definition, the number $\sup A$ has two properties.

(i) $x \leq \sup A$ for all $x \in A$ (i.e., $\sup A$ is an upper bound for M).
(ii) for any M that is an upper bound for A, we have $\sup A \leq M$.

For sets that are not bounded above, we say that $\sup A = +\infty$. we often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set $\{f(x) : x \in A\}$, where f is a some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted $\inf A$. It satisfies the same properties as $\sup A$ with the inequalities reversed.

(a) Find $\sup A$ and $\inf A$ for $A = (1, 2]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, and $A = \{0, 1, 2, 3, \ldots\}$.

- $A = (1, 2]$, $\sup A = 2$ and $\inf A = 1$
- $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $\sup A = 1$, and $\inf A = \lim_{n \to \infty} \frac{1}{n} = 0$.
- $A = \{0, 1, 2, 3, \ldots\}$. $\sup A = \lim_{n \to \infty} = \infty$, and $\inf A = 0$

(b) Find $\sup_{x \in (0,1)} (1 + x^2)^{-1}$
Let $f(x) = (1 + x^2)^{-1}$. On the interval $(0, 1)$ we can see that it is strictly decreasing, that is $a < b \implies f(a) > f(b)$. Thus, $\sup_{x \in (0,1)} f(x) = f(0) = (1 + 0^2)^{-1} = 1$.

(c) Assume that $\sup A < \infty$, and show that for every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$.
Given any $\epsilon > 0$ let $x > \sup A - \epsilon$. If $x \notin A$ then x is an upper bound of A, i.e., $x \in M$ and $x < \sup A$, but that violates proper (ii). Hence, $x \in A$.

(d) For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, and any set $A \subset \mathbb{R}$, show that $\sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$.

$$f(x) \le \sup_{x\in A} f(x) \text{ and } g(x) \le \sup_{x\in A} g(x), \forall x \in x \in A$$
$$\therefore f(x)+g(x) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x), \forall x \in A$$
$$\text{and } \sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$$

IV. Section 1.1, Exercise 5, 6, 13.

Exercise 1.1.5. Let $n \ge 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers verify the identity

$$\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right), \tag{1.3}$$

and conclude *Cauchy-Schwarz inequality*

$$\left|\sum_{i=1}^n a_1 b_i\right| \le \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{j=1}^n b_j^2\right)^{1/2}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^n (a_i+b_i)^2\right)^{1/2} \le \left(\sum_{i=1}^n a_i^2\right)^{1/2} + \left(\sum_{i=j}^n b_j^2\right)^{1/2}$$

Let's start by expanding the center term

$$\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2 + \sum_{i=1}^n b_i^2 \sum_{j=1}^n a_i^2 - 2\sum_{i=1}^n a_i b_i \sum_{j=1}^n a_j b_j$$
$$= 2\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) - 2\left(\sum_{i=1}^n a_i b_i\right)^2$$

Equation 1.3 then becomes

$$\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$
$$\left(\sum_{i=1}^n a_i b_i\right)^2 + \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

which is true. Since

$$\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$
$$\left(\sum_{i=1}^n a_i b_i\right)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) - \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2$$
$$\therefore \left|\sum_{i=1}^n a_i b_i\right| \le \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{j=1}^n b_j^2\right)^{1/2}$$

Exercise 1.1.6 Show that $(\mathbb{R}^n, d_{l^2})$ in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5)

Example 1.1.6 (Euclidean spaces). Let $n \ge 1$ be a natural number, and let $\mathbb{R}^n$ be the space of n-tupes of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^{n} (x_i - y_y)^2 \right) \end{aligned}$$

Exercise 1.1.13 Prove Proposition 1.1.19. **Proposition 1.1.19** (Convergence in a the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X. let $(x^{(x=n)})_{n=m}^{\infty}$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^{\infty}$ convergent to x with respect to the discrete metric d_{disc} if and only if there exists $N \geq m$ sucht ath $x^{(n)} = x$ for all $n \geq N$.*

For this problem only, you do not need to give proofs. just write the answers.

For each set, idenitfy the boundary, interior, and closure of A, and say whether A is open, closed, both or neither. We are working in $\mathbb{R}^2$ with the standard distance. Unless othewise noted, the ambient space is $\mathbb{R}^2$.

(a) $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$

(b) $A = \{(1/n, 2/n) : n = 1, 2, 3, \ldots\}$ (Note: $(1/n, 2/n)$ is a vector in $\mathbb{R}^2$, not an open intevarl in $\mathbb{R}$.).

(c) $A \ \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$, in the relative topology with respect to $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$.

Let (X, d) be a metric space.

(a) For a given piont $x_0 \in X$, show the singleton set $\{x_0\}$ is closed.

(b) Let $x_0 \in X$ and $r > 0$. Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open