

Math 5102 – Linear Algebra– Fall 2024

w/Professor Pendera

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7. Let $S = \{0, 1\}$ and $F = R$. If $\mathcal{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

- $f = g$
 $f(0) = 2(0) + 1 = 1$ and $g(0) = 1 + 4(0) - 2(0)^2 = 1 \rightarrow f(0) = g(0)$
 $f(1) = 2(1) + 1 = 3$ and $g(1) = 1 + 4(1) - 2(1)^2 = 1 + 4 - 2 = 3 \rightarrow f(1) = g(1)$
 $f = g$ since $f(s) = g(s)$ for all $s \in S$
- $f + g = h$

$$\begin{aligned}(f + g)(t) &= f(t) + g(t) \\ &= 2t + 1 + 1 + 4t - 2t^2 \\ &= 2 + 6t - 2t^2\end{aligned}$$

$$(f + g)(0) = 2 + 6(0) - 2(0)^2 = 2$$

$$(f + g)(1) = 2 + 6(1) - 2(1)^2 = 6$$

$$h(0) = 5^0 + 1 = 2$$

$$h(1) = 5^1 + 1 = 6$$

$$(f + g)(0) = h(0) \text{ and } (f + g)(1) = h(1)$$

$$\therefore f + g = h \text{ since } (f + g)(t) = h(t), \forall t \in S$$

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of additions and scalar multiplication defined in Example 3.

From Example 3 we have

$$(f + g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c[f(s)], \forall s \in \mathbb{R}$$

From the additive rule of differentiation $(f + g)'(s) = f'(s) + g'(s)$ which implies that $(f + g) \in V$ and from the multiplicative rule of differentiation we know that $(cf)'(s) = c[f'(s)]$ which implies that $cf \in V$. Hence V is a vector space.

11. Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F . Prove that V is a vector space over F . (V is called the **zero vector space**).

Given any $v, w \in V$ we have $v + w = 0 + 0 = 0 \in V$ and for all $c \in F$, $cv = c(0) = 0 \in V$. Therefore V is a vector space.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. (See Example 5 for the definition of a sequence.) For $\{a_n\}, \{b_n\} \in V$ and any real number t , define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \text{ and } t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space. over \mathbb{R} .

Given any two real sequences $\{a_n\}, \{b_n\} \in V$, we can see that the sequence $\{a_n + b_n\}$ is made up of real numbers, namely $a_i + b_i$ for all $i \in \mathbb{N}$, making it a real sequence. We can also see that $\{ca_n\}$ is made up of real numbers, namely ca_i for all $i \in \mathbb{N}$, making it a real sequence. Hence, V is a vector space.

21. Let V and W be vector spaces over a field F . Let

$$Z = \{ (v, w) : v \in V \text{ and } w \in W \}$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1)$$

Given $v_1, v_2 \in V$ and $w_1, w_2 \in W$ we can see that by definition $z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$. We want to show that $z_1 + z_2 \in Z$ and $cz_1 \in Z$ for all $c \in F$.

Well, $z_1 + z_2 = (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and since V, W are vector spaces we know that $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$. Therefore, $z_1 + z_2 \in Z$.

Additionally, $cz_1 = c(v_1, w_1) = (cv_1, cw_1)$ and we know that $cv_1 \in V$ and $cw_1 \in W$ therefore $cz_1 \in Z$.

Hence, Z is a vector space.

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3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m \times n}(F)$ and any $a, b \in F$.

$$\begin{aligned} A^t &\implies [A_{ij}]^t = [A_{ji}] \\ (aA + bB)^t &= [(aA + bB)_{ji}] \\ &= [aA_{ji} + bB_{ji}] \\ &= a[A_{ji}] + b[B_{ji}] \\ &= aA^t + bB^t \end{aligned}$$

4. Prove that $(A^t)^t = A$ for each $A \in M_{m \times n}(F)$.

Let $B = A^t = [A_{ji}]$ and $B^t = [A_{ji}]^t = [A_{ij}] = A$.

5. Prove that $A + A^t$ is symmetric for any square matrix A .

$A = [A_{ij}]$ and $A^t = [A_{ji}]$. Therefore $A + A^t = [A_{ij}] + [A_{ji}] = [A_{ij} + A_{ji}]$. Given any $i, j \in [1, \dots, n]$ we can see that $(A + A^t)_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = (A + A^t)_{ji}$, hence symmetric.

10. Prove that $W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 + a_2 + \dots + a_n = 0\}$ is a subspace of F^n , but $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 \text{ is not}\}$.

Given $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in W_1$ we can see that $a + b = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ and $a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n = a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n = 0 + 0 = 0$ therefore $a + b \in W_1$. Similarly, $ca = c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$ and $ca_1 + ca_2 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c(0) = 0$. Therefore W_1 is a vector space.

However, let $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in W_2$ we can see that $a + b = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ and $a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n = a_1 + a_2 + \dots + a_n + b_1 + b_2 + \dots + b_n = 1 + 1 = 2$ which means that $a + b \notin W_2$.

15. Is the set of all differentiable real-valued functions defined on \mathbb{R} a subspace of $C(\mathbb{R})$? Justify your answer.

Yes. If $f, g \in C(\mathbb{R})$ are differentiable, by the addition rule so is $f + g$, i.e., $(f + g)' = f' + g'$, and so is cf , i.e., $(cf)' = cf'$, for all $c \in F$.

19. Let W_1 and W_2 be subspaces of a vector space V . Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Let $w_1 \in W_1, w_2 \in W_2$ and specifically $w_1 \notin W_2$. Clearly, $w_1, w_2 \in W_1 \cap W_2$, if $x = w_1 + w_2 \in W_1 \cup W_2$ then $x \in W_1, x \in W_1 \cap W_2$, or $x \in W_2$. Cases:

$x \in W_1$: this implies that both $w_1, w_2 \in W_1$, hence $W_2 \subseteq W_1$.

$x \in W_1 \cap W_2$: this also implies that both $w_1, w_2 \in W_1$, hence $W_2 \subseteq W_1$.

$x \in W_2$: this is a contradiction from our assumption that $w_1 \notin W_2$.