Math 5102 – Linear Algebra– Fall 2024
w/Professor Penera

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Homework #5 – NONE

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4. Define

$$T : M_{2\times 1}(R) \to P_2(R) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Computer $[T]_\beta^\gamma$.

8. Let V be an n-dimensional vector space with an ordere basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

9. Let V be the vector space of complex numbes over the field R. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$. (Recall by Exersise 38 of Section 2.1 that T is not linear if V is regarded as a vector space over the field C.)

13. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W If $R(t) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is linearly independent subset of $\mathcal{L}(V, W)$.

16. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

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3. Let $g(x) = 3 + x$. Let $T : P_2(R) \to P_2(R)$ adn $U : P_2(R) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

   Let β and γ be the standard ordered bases of $P_2(R)$ and $\mathbb{R}^3$, respectively

   (a) Compute $[U]_\beta^\gamma, [T]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your results.

   (b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to veriy your results.

9. Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

11. Let V be a vector space, and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

12. Let V, W and Z be vector spaces, and let $: V \to W$ and $U :: W \to Z$ be linear.

   (a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

   (b) Proive that if UT is onto, then U is onto. Must T also be onto?

   (c) Prove that if U and T are one-to-one and onto, then UT is also.

13. Let A and$B$ be $n \times n$ matrices. Recall that the trace of A is defined by

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$$

   Prove that $\text{tr}(AB) = \text{tr}(BA)$ and $\text{tr}(A) = \text{tr}(A^t)$