Math 5102 – Linear Algebra– Fall 2024
w/Professor Penera

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Practice Final – NONE

1. Let V be a vector space over F, let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle_W$, and let $T : V \to W$ be a linear transformation. Show that

$$\langle x, y \rangle_V = \langle T(x), T(y) \rangle_W, x, y \in V$$

defines an inner product of V if and only if T is one-to-one.

Define $\langle \cdot, \cdot \rangle_V$ in this way

$$\begin{aligned} x &\notin N(T) \\ 0 &= \langle x, 0 \rangle = \langle T(x), T(0) \rangle = 0 \\ T(0) &= 0 \\ 0 &= \langle T(x), T(y) \rangle \to T(y) \in N(T) \end{aligned}$$

2. Let V be an inner produce space over F and let W bve a subspace of V. Show that $(W^\perp)^\perp = W$.

Primarily, this is a 'set based' equation, thus we must show subset and superset in order to show equality

- $\subseteq$ that is, let $v \in (W^\perp)^\perp$ then for every $x \in W^\perp, \langle x, v \rangle = 0$. Let $y \in W$ then $\langle x, y \rangle = \langle x, v \rangle = 0$. Then $2 \langle x, y + v \rangle = 0$. x is arbitrary and non-zero therefore $y + v = 0$ and $v \in W$.
- $\supseteq$

3. Let V be an inner product space over F and let $\beta = \{v_1, \ldots, v_n\}$ be an orthornormal basis for V. Show that

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}$$

for all $x, y \in V$.

$$\begin{aligned} y &= \sum \langle y_i, v_i \rangle \\ x &= \sum \langle x, v_i \rangle \\ \langle x, y \rangle &= \left\langle x, \sum \langle y_i, v_i \rangle \right\rangle = \sum \langle x, \langle y_i, v_i \rangle \rangle = \end{aligned}$$

4. Let V be an inner product space and let T be an invertible linear operator on V. Show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

$$\begin{aligned} I &= TT^{-1} \\ I^* &= (TT^{-1})^* = T^*(T^{-1})^* \\ (T^*)^{-1} I &= (T^*)^{-1} T^* (T^{-1})^* \\ (T^*)^{-1} &= (T^{-1})^* \end{aligned}$$

5. Let $V = W \oplus W^\perp$ and let T be the projection on W along $W^\perp$. Show that $T^* = T$.

$$\begin{aligned} R(T) \perp W &\to R(T) \subseteq W \\ &\forall x \in V, \langle x, y \rangle \end{aligned}$$