Math 725 – Advanced Linear Algebra

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Assignment #8 – Due 11/3/23

1.a) Let V be a vector space with an inner product $\langle \, , \, \rangle$ on it. Prove that if $\langle v, w\rangle = 0$ for all $w \in V$ then $v = 0$.

$\langle v + u, v \,\rangle = \langle v, v \,\rangle + \langle u, v \,\rangle = 0$ which implies that $\langle v, v \,\rangle = 0$ which implies that $v = 0$

b) Let V be a vector space with an inner product $\langle \, , \, \rangle$ on it. Prove that if $\langle v, w\rangle = \langle u, w\rangle$ for all $w \in V$ then $v = u$.

$\langle u + v, w \,\rangle = \langle u, w \,\rangle + \langle v, w \,\rangle = 2 \,\langle u, w \,\rangle = \langle 2u, w \,\rangle$ which implies that $u + v = 2u$ and $v = u$

c) Now let V be a finite dimensional inner product space and let $\mathcal{B} = \{w_1, \ldots, w_n\}$ be a basis of V. Prove that for given scalars $c_1, \ldots, c_n$ there exists a unique $v \in V$ such that $\langle v, w_i\rangle = c_i$ for $i = 1, \ldots, n$.

Suppose that there are two such vectors, u, v such that $\langle u, w_i \,\rangle = \langle v, w_i \,\rangle = c_i$ for all $i = 1, \ldots, n$. Then $\langle u, w_i \,\rangle - \langle v, w_i \,\rangle = \langle u - v, w_i \,\rangle = 0$ for all $i = 1, \ldots, n$. $w_i \neq 0$ for all $i = 1, \ldots, n$ because each of these are basis vectors. Therefore, $u - v = 0$ and $u = v$.

2. Let $V = \mathcal{P}^{(n)}(\mathbb{R})$. Show that

$$\langle a_0 + a_1 x + \cdots + a_n x^n \,,\, b_0 + b_1 x + \cdots + b_n x^n \rangle \;=\; \sum_{i,j} \frac{a_i b_j}{i+j+1}$$

is an inner product on V. [Hint: Consider $\int_0^1 f(t)g(t)\,dt$].

$$\begin{aligned}\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) &= \sum_{i=0}^{n}\left(a_i x^i\left(\sum_{j=0}^{n} b_j x^j\right)\right)\\ &= \sum_{i,j=0}^{n} a_i b_j x^{i+j}\end{aligned}$$

If we use the inner product as defined as

$$\begin{aligned}\langle , \rangle &= \int_0^1 f(t)g(t)\,dt\\ \langle a_0 + a_1 x + \cdots + a_n x^n \,,\, b_0 + b_1 x + \cdots + b_n x^n \rangle &= \int_0^1 \sum_{i,j=0}^{n} a_i b_j t^{i+j} dt\\ &= \sum_{i,j=0}^{n} \frac{a_i b_j}{i+j+1} t^{i+j+1}\Bigg|_0^1\\ &= \sum_{i,j=0}^{n} \frac{a_i b_j}{i+j+1} a_i b_j\end{aligned}$$

3.a) Let $V = \mathcal{P}^{(3)}(\mathbb{R})$ with the inner product $\langle f\,,\, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$. Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$ to obtain the first few *Legendre polynomials*.

Let $\{w_1, w_2, w_3, w_4\} = \{1, x, x^2, x^3\}$ then

$$u_j = w_j - \langle w_j, v_1 \rangle\, v_2 - \cdots - \langle w_j, v_{j-1} \rangle \qquad v_j = \frac{u_j}{||u_j||}$$

$$u_1 = 1 \qquad v_1 = \frac{u_1}{||u_1||} = 1$$

$$\begin{aligned}
u_2 &= w_2 - \langle w_2, v_1 \rangle\, v_1 \\
&= x - \langle x, 1 \rangle \\
&= x - \int_{-1}^{1} x\,dx \\
&= x \qquad\qquad v_2 = \frac{u_2}{||u_2||} = x \\
u_3 &= w_3 - \langle w_3, v_1 \rangle\, v_1 - \langle w_3, v_2 \rangle\, v_2 \\
&= x^2 - \int_{-1}^{1} x^2 dx - \int_{-1}^{1} x^2 \cdot x\,dx \\
&= x^2 - \frac{1}{3}x^3|_{-1}^{1} - \frac{1}{4}x^4|_{-1}^{1} \\
&= x^2 - \frac{2}{3},\ ||u_3|| = \sqrt{1 + 2^2/3^2} = 1/3 \qquad\qquad v_3 = \frac{u_3}{||u_3||} = 3(x^2 - 1) \\
u_4 &= w_4 - \langle w_4, v_1 \rangle\, v_1 - \langle w_4, v_2 \rangle\, v_2 - \langle w_4, v_3 \rangle\, v_3 \\
&= x^3 - \int_{-1}^{1} x^3\,dx - \int_{-1}^{1} x^3 \cdot x\,dx - \int_{-1}^{1} x^3\left(3(x^2 - 1)\right)\,dx \\
&= x^3 - \frac{2}{5} - \int_{-1}^{1} (3x^5 - 3x^3)\,dx \\
&= x^3 - \frac{2}{5},\ ||u_4|| = \sqrt{1 + \frac{4}{25}} = \frac{\sqrt{29}}{5} \qquad\qquad v_4 = \frac{u_4}{||u_4||} = \frac{5(x^3 - 2)}{\sqrt{29}}
\end{aligned}$$

b) Now use the inner product $\int_{-\infty}^{+\infty} f(t)g(t)e^{-t^2}\,dt$ on V and apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3\}$ to obtain the first few *Hermite polynomials.*

$$u_1 = w_1 \qquad\qquad v_1 = \frac{u_1}{||u_1|} = 1$$

$$\begin{aligned}
u_2 &= w_2 - \langle w_2, v_1 \rangle\, v_1 \\
&= x - \int_{-\infty}^{+\infty} te^{-t^2}\,dt \\
&= x + \int_{\infty}^{\infty} \frac{1}{2} e^{v'}\,dv', \text{ where } v' = -t^2,\ dv' = 2t\,dt \\
&= x - \frac{1}{2}(e^{-t^2})|_{-\infty}^{\infty}
\end{aligned} \qquad\qquad v_2 = \frac{u_2}{||u_2||} = x$$

$$\begin{aligned}
u_3 &= w_3 - \langle w_3, v_1 \rangle\, v_1 - \langle w_3, v_2 \rangle\, v_2 \\
&= x^2 - \left\langle x^2, 1 \right\rangle - \left\langle x^2, x \right\rangle x \\
&= x^2 - \int_{-\infty}^{+\infty} t^2 e^{-t^2}\,dt - \left(\int_{-\infty}^{+\infty} t^2 \cdot t\, e^{-t^2}\,dt \right)
\end{aligned}$$

4. Let $\{v_1, \ldots, v_n\}$ be an orthonormal set of vectors in an inner product space V. Prove that $\sum_{i=1}^{n} |\langle w, v_i\rangle|^2 \leq ||w||^2$ for any $w \in V$, and the equality holds if and only if $w = \sum_{i=1}^{n}\langle w, v_i\rangle v_i$.

$$\begin{aligned}
||w|| &= \langle w, w\,\rangle \\
||w||^2 &= \left(\sum_{i=1}^{n} \langle w, v_i\,\rangle\right)^2 \\
&= \sum_{i=1}^{n} \langle w, v_i\,\rangle^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \langle w, v_i\,\rangle \langle w, v_j\,\rangle \\
&\geq \sum_{i=1}^{n} \langle w, v_i\,\rangle^2
\end{aligned}$$

5. Let $V = \mathcal{M}_{n\times n}(\mathbb{C})$ with an inner product defined by $\langle A, B \rangle := \operatorname{tr}(AB^*)$. Determine the orthogonal complement of the subspace of diagonal matrices in V.

Let D be the subspace of diagonal matrices. Then, $D^\perp = \{A \in \mathcal{M}_{n\times n}(\mathbb{C}) : \langle A, B \rangle = 0, \text{ for all } B \in D\}$. That is

$$\begin{aligned}
\langle A, B \rangle &= \operatorname{tr}(AB^*) \\
&= \operatorname{tr}\left([A_{ij}\overline{B_{ji}}]\right) \\
&= \sum_{i=1}^{n} A_{ii}\overline{B_{ii}} \\
&= 0
\end{aligned}$$

since $B \in D$ we must allow that $\overline{B_{ii}} \neq 0$ therefore each $A_{ii} = 0$. Thus, $D^\perp$ must be the set of matrices with zeros along the diagonal.

6. Let W be a subspace of a finite dimensional inner product space V, and let E be the orthogonal projection operator onto W. Prove that $\langle Ev, w\rangle = \langle v, Ew\rangle$ for all $v, w \in V$.

Let $B = \{b_1, \ldots, b_n\}$ be the orthonormal basis such that $[E]_B^B$ is upper triangular and $v = \{v_1, \ldots, v_n\} = \sum_{i=1}^n v_i b_i, w = \{w_1, \ldots, w_n\} = \sum_{i=1}^n w_i b_i$ under this basis. When $Ev = [E]_B^B v = \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j b_j$. Then

$$\begin{aligned}
\langle Ev, w\rangle &= \left\langle \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j b_j, w \right\rangle \\
&= \sum_{i=1}^n \left\langle \sum_{j=i}^n E_{ij} v_j b_j, w \right\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j \langle b_j, w\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j \left\langle b_j, \sum_{k=1}^n w_k b_k \right\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j \sum_{k=1}^n w_k \langle b_j, b_k\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j \sum_{k=1}^n w_k \delta_{jk} \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} v_j w_j
\end{aligned}$$

in a similar manner

$$\begin{aligned}
\langle v, Ew\rangle &= \left\langle v, \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j b_j \right\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j \langle v, b_j\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j \left\langle \sum_{k=1}^n v_k b_k, b_j \right\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j \sum_{k=1}^n v_k \langle b_k, b_j\rangle \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j \sum_{k=1}^n v_k \delta_{jk} \\
&= \sum_{i=1}^n \sum_{j=i}^n E_{ij} w_j v_j
\end{aligned}$$

7. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis in an inner product space V, and let T be a linear operator with $A = [T]_{\mathcal{B}}^{\mathcal{B}}$. Prove that $A_{ij} = \langle Tv_j, v_i \rangle$.

Given any $x \in V, x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$. Then,

$$\begin{aligned}
T(x) &= T(\sum_{i=1}^{n} \langle x, v_i \rangle v_i) \\
&= \sum_{i=1}^{n} T(\langle x, v_i \rangle v_i) \\
&= \sum_{i=1}^{n} \left\langle \sum_{j=1}^{n} T(x_j v_j), v_i \right\rangle v_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x_j \langle T(x_j), v_i \rangle v_i \\
T(x)_i &= \sum_{j=1}^{n} x_j \langle T(x_j), v_i \rangle v_i \\
T(x) &= Ax \\
&= \left[\sum_{j=1}^{n} A_{ij} x_j \right] \\
T(x)_i &= \sum_{j=1}^{n} A_{ij} x_i \\
A_{ij} &= \langle T(x_j), v_i \rangle
\end{aligned}$$

Extra Questions

1. Let V and W be two inner product spaces (with their own inner products) and let $T : V \mapsto W$ be a linear transformation. We will attempt to measure the *size* of T as follows. We let the *norm* of T to be $||T|| := \sup_{||x||=1} ||Tx||$. Prove the following about the norm of T:
a) $||cT|| = |c|||T||$ for any scalar c,
b) if S is another linear transformation then $||S + T|| \leq ||S|| + ||T||$.

2. Let T be as in the question above. Show the following:
a) $||Tx|| \leq ||T||||x||$ for any $x \in V$.
b) Let S be yet another inner product space and $U : W \mapsto S$ be a linear transformation. Then $||S \circ T|| \leq ||S||||T||$.

3. Let V be a finite-dimensional inner product space and T a linear operator on V that is invertible. Let S be another linear operator on V that is "close" to T in the following sense: $||T - S|| \leq 1/||T^{-1}||$. Prove that S is also invertible. [Hint: let $U = T - S$ and factor $S = T \circ (I - T^{-1} \circ U)$. Argue that you just have to show that $(I - T^{-1} \circ U)$ is invertible, hence it is enough to prove that the nullspace of $(I - T^{-1} \circ U)$ is trivial. Prove this by contradiction.]

4.) Let V be a finite-dimensional inner product space and T a linear operator on V. First of all, note that we can replace sup with max in the definition of the norm of T: $||T|| := \max_{||x||=1} ||Tx||$.
a) Now let λ be an eigenvalue of T. Show that $||T|| \geq |\lambda|$.
b) We define $r(T) = \max_i |\lambda_i|$ to be the *spectral radius* of T where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of T. Conclude that $||T|| \geq r(T)$.
c) Also show that $||T^j||^{1/j} \geq r(T)$ for any integer $j \geq 1$. One can show (though it requires some work) that $r(T) = \lim_{j \mapsto \infty} ||T^j||^{1/j}$.