Functional Analysis– Spring 2024

Paul Carmody

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p. 81 #7. If $\dim Y < \infty$ in Riesz's lemma 2.5-4, show that one can even choose $\theta = 1$.

p. 101 #3, 5, 6, 7, 8, 9.

3. If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

5. Show that the operator $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_i) = Tx, \eta_j = \xi/j, x = (\xi_j)$, is linear and bounded.

6. **(Range)** Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \to Y$ need not be closed in Y. *Hint.* Use T in Prob 5.

7. **(Inverse operator)** Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that
$$\|Tx\| \geq b\, \|x\| \quad \text{for all } x \in X$$
show that then $T^{-1} : Y \to x$ exists and is bounded.

8. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded. *Hint.* Use T in Prob. 5.

9. Let $T : C[0,1] \to C[0,1]$ be defined by
$$y(t) = \int_0^1 x(\tau) d\tau.$$
Find $\mathcal{R}(T)$ and $T^{-1} : \mathcal{R}(T) \to C[0,1]$. Is T^{-1} linear and bounded?

p. 109 #2, 3, 4.

2. Show that the functionals defined on $C[a,b]$ by

$$f_1(x) = \int_a^b x(t) y_0(t)\,dt \qquad (y_o \in C[a,b])$$
$$f_2(x) = \alpha x(a) + \beta x(b) \qquad (\alpha, \beta \text{ fixed})$$

   are linear and bounded.

3. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^0 x(t)\,dt - \int_0^1 x(t)\,dt.$$

4. Show that

$$f_1(x) = \max_{t \in J} x(t)$$
$$J = [a,b]$$
$$f(2) = \min_{t \in J} x(t)$$

   define functionals on $C[a,b]$. Are they linear? Bounded?