Math 5050 – Special Topics: Manifolds– Spring 2025

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Section 4 problems:

Within the section: 4.3 (p.37): **(A basis for 3-covectors).** Let x^1, x^2, x^3, x^4 be the coordinates in $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

$$\begin{aligned}\Phi = \{\ & dx_p^i \wedge dx_p^j \wedge dx_p^k \ : i < j < k \leq 4\ \}\\ \{\ & dx_p^1 \wedge dx_p^2 \wedge dx_p^3,\\ & dx_p^1 \wedge dx_p^2 \wedge dx_p^4,\\ & dx_p^1 \wedge dx_p^3 \wedge dx_p^4,\\ & dx_p^2 \wedge dx_p^3 \wedge dx_p^4\ \}\\ |\,\Phi\,| = & \binom{4}{3} = 4\end{aligned}$$

Within the section: 4.4 (p.38), **Wedge product of a 2-form with a 1-form.** Let ω be a 2-form and τ be a 1-form on on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z

$$\begin{aligned}(\omega \wedge \tau)(X, Y, Z) &= (\omega \otimes \tau)(X, Y, Z) - (\tau \otimes \omega)(X, Y, Z)\\ &= \omega(X)\tau(Y, Z) - \tau(X, Y)\omega(Z)\\ (\omega \wedge \tau)(X, Y, Z) &= \frac{1}{1!2!}A(\omega \otimes \tau)(X, Y, Z))\\ &= \frac{1}{2}\left(\omega(X, Y)\tau(Z) + \omega(Y, Z)\tau(X) + \omega(Z, X)\tau(Y) - \omega(Z, Y)\tau(X) - \omega(Y, X)\tau(Z) - \omega(X, Z)\tau(Y)\right)\\ &= \omega(X, Y)\tau(Z) + \omega(Y, Z)\tau(X) + \omega(Z, X)\tau(Y)\end{aligned}$$

Within the section: 4.9 (p.40) **A closed 1-form on the punctured plane.** Define a 1-form on ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx - xdy).$$

Show that ω is closed.

$$\begin{aligned}d\omega =& \frac{\partial \omega}{\partial x}dx + \frac{\partial \omega}{\partial y}dy\\ =& \left(\frac{-2x}{(x^2+y^2)^2}(-ydx - xdy) + \frac{1}{x^2+y^2}(-yd^2x - dy)\right)dx +\\ & \left(\frac{-2y}{(x^2+y^2)^2}(-ydx - xdy) + \frac{1}{x^2+y^2}(-dx - xd^2y)\right)dy\\ =& \left(\frac{-2x}{(x^2+y^2)^2}(-xdydx) + \frac{1}{x^2+y^2}(-dydx)\right) +\\ & \left(\frac{-2y}{(x^2+y^2)^2}(-ydxdy) + \frac{1}{x^2+y^2}(-dxdy)\right)\\ =& \left(\frac{2x^2}{(x^2+y^2)^2}(dydx) + \frac{1}{x^2+y^2}(-dydx)\right) +\\ & \left(\frac{2y^2}{(x^2+y^2)^2}(dxdy) + \frac{1}{x^2+y^2}(-dxdy)\right)\\ =& \frac{2x^2 - x^2 - y^2}{(x^2+y^2)^2}(dydx) + \frac{2y^2 - x^2 - y^2}{(x^2+y^2)^2}(dxdy)\\ =& \frac{2x^2 - x^2 - y^2 + 2y^2 - x^2 - y^2}{(x^2+y^2)^2}dxdy\\ =& 0\end{aligned}$$

End of the section: 1 through 6.

4.1 **A 1-form on** $\mathbb{R}^3$.

Let ω be the 1-form $zdx - dz$ and let X be the vector $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Computer $\omega(X)$ and $d(\omega)$.

$$\begin{aligned}
\omega(X) &= (zdx - dz)\,(\,y\partial/\partial x + x\partial/\partial y\,)\\
&= (zdx - dz)\,(\,y\partial/\partial x\,) + (zdx - dz)\,(\,x\partial/\partial y\,)\\
&= zy\frac{\partial}{\partial x}dx - y\frac{\partial}{\partial x}dz + zx\frac{\partial}{\partial y}dx - x\frac{\partial}{\partial y}dz \qquad \text{recall } \frac{\partial}{\partial x^i}dx^j = \delta_i^j\\
&= zy
\end{aligned}$$

$$d(\omega) = d(zdx - dz) = d(zdx) - d^2z = dz \wedge dx + z \wedge d^2x = dz \wedge dx$$

4.2 **A 2-form on** $\mathbb{R}^3$ At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\mathbf{a},\mathbf{b}) = \omega_p\left(\begin{pmatrix} a^1\\ a^2\\ a^3\end{pmatrix}, \begin{pmatrix} b^1\\ b^2\\ b^3\end{pmatrix}\right) = p^3 \det\begin{pmatrix} a^1 & b^1\\ a^2 & b^2\end{pmatrix},$$

for tangent vectors $\mathbf{a},\mathbf{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1,p^2,p^3)$. Since ω_p is an alternaing bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

$$\begin{aligned}
\omega(p) &= c_{xy}(p)(dx \wedge dy) + c_{yz}(p)(dy \wedge dz) + c_xz(p)(dx \wedge dz)\\
c_{xy}(p) &= \omega_p(e_x, e_y) = p^3\begin{pmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\end{pmatrix} = p^3\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 0\right)\\
c_{yz}(p) &= \omega_p(e_y, e_z) = p^3\begin{pmatrix} 0 & \frac{\partial}{\partial y}\\ 0 & 0\end{pmatrix} = 0\\
c_{xz}(p) &= \omega_p(e_x, e_z) = p^3\begin{pmatrix} \frac{\partial}{\partial x} & 0\\ 0 & 0\end{pmatrix} = 0
\end{aligned}$$

notice that $(dx \wedge dy)(a,b) = dx(a)dy(b) - dy(a)dx(b) = a^1b^2 - a^2b^1 = \det\begin{pmatrix} a^1 & b^1\\ a^2 & b^2\end{pmatrix}$ Thus.

$$\omega = p^3\, dx \wedge dy$$

4.3 **Exterior Calculus**.

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r,θ)-plane, not the (x,y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in of dr and $d\theta$.

$$\begin{aligned}
dx &= \cos\theta dr - r\sin\theta d\theta\\
dy &= \sin\theta dr + r\cos\theta d\theta\\
dx \wedge dy &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)\\
&= (\cos\theta dr) \wedge (\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)\\
&= (\cos\theta dr) \wedge (\sin\theta dr) + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + (r\sin\theta d\theta) \wedge (r\cos\theta d\theta)\\
&= 0 + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + 0\\
&= (\cos\theta dr) \wedge (r\cos\theta d\theta) + (\sin\theta dr) \wedge (r\sin\theta d\theta)\\
&= (r\cos^2\theta)(dr \wedge d\theta) + (r\sin^2\theta)(dr \wedge d\theta)\\
&= r(dr \wedge d\theta)
\end{aligned}$$

4.4 **Exterior Calculus**.

Suppose the standard coordiantes on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

$$\begin{aligned}
dx &= \sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta \\
dy &= \sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta \\
dz &= \cos\theta\, d\rho - \rho\sin\phi\, d\phi
\end{aligned}$$

We will attempt to cancel out any terms which have a $dx^i \wedge dx^i$ by simplifying dx, dy, and dz in the following manner

$$\begin{aligned}
dx\wedge dy\wedge dz &= (x_1\, d\rho + x_2\, d\phi + x_3\, d\theta)\wedge(y_1\, d\rho + y_2\, d\phi + y_3\, d\theta)\wedge(z_1\, d\rho + z_2\, d\phi + z_3\, d\theta) \\
&= (x_1\, d\rho\wedge y_2\, d\phi\wedge z_3\, d\theta) + (x_1\, d\rho\wedge y_3\, d\theta\wedge z_2\, d\phi) \\
&+ (x_2\, d\phi\wedge y_1\, d\rho\wedge z_3\, d\theta) + (x_2\, d\phi\wedge y_3\, d\theta\wedge z_2\, d\phi) \\
&+ (x_3\, d\theta\wedge y_1\, d\rho\wedge z_2\, d\phi) + (x_3\, d\theta\wedge y_2\, d\phi\wedge z_1\, d\rho) \\
&= (x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 + x_2y_3z_2 + x_3y_1z_2 + x_3y_2z_1)(d\rho\wedge d\phi\wedge d\theta) \\
&= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}(d\rho\wedge d\phi\wedge d\theta)
\end{aligned}$$

Solving for the determinant by expanding the bottom row

$$\begin{aligned}
\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & \end{vmatrix} &= \rho^2 \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \sin\phi\cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix} \\
&= \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix} \\
&= \rho^2\sin\phi\left(\cos\phi\begin{vmatrix} \cos\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta & \cos\theta \end{vmatrix} + \sin\phi\begin{vmatrix} \sin\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\theta \end{vmatrix}\right) \\
&= \rho^2\sin\phi\left(\cos^2\phi\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \sin^2\phi\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}\right) \\
&= \rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right) \\
&= \rho^2\sin\phi
\end{aligned}$$

That is

$$dx\wedge dy\wedge dx = (\rho^2\sin\phi)\, dr\wedge d\phi\wedge d\theta$$

4.5 **Wedge Product**. Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned}
\alpha &= a_1dx^1 + a_2dx^2 + a_3dx^3 \\
\beta &= b_1dx^2\wedge dx^3 + b_2dx^3\wedge dx^1 + b_3dx^1\wedge dx^2
\end{aligned}$$

Simplify the expression $\alpha\wedge\beta$ as much as possible.

The resulting expression $\alpha\wedge\beta\in\Omega^3(\mathbb{R}^3)$. The $\dim(\Omega^3(\mathbb{R}^3)) = 1$. Thus, there will be one term of the form $dx^1\wedge dx^2\wedge dx^3$. Further by distributing the terms of α across the terms of β and ignoring any terms where any two elements are equal, i.e., $dx^i\wedge dx^i = 0$. We will then have

$$\begin{aligned}
\alpha\wedge\beta &= a_1\, dx^1\wedge(b_1\, dx^2\wedge dx^3) + a_2\, dx^2\wedge(b_2\, dx^3\wedge dx^1) + a_3\, dx^3(b_3\, dx^1\wedge dx^2) \\
&= (a_1b_1 + a_2b_2 + a_3b_3)\, dx^1\wedge dx^2\wedge dx^3
\end{aligned}$$

4.6 **Wedge product and cross product**

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3\rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2\wedge\alpha^3 + c_2\alpha^3\wedge\alpha^1 + c_3\alpha^1\wedge\alpha^2$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3\rangle \in \mathbb{R}^3$. Show that under the correspodence, the wedge product of 1-covectors correponds to the cross product of vectors $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

First the cross product

$$\begin{aligned} v_\alpha \times v_\beta &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= i(a_2b_3 - a_3b_2) + j(a_3b_1 - a_1b_3) + k(a_1b_2 - a_2b_1) \end{aligned}$$

then the tensor product and ignoring any $a^i \wedge a^i$ terms when we expand them

$$\begin{aligned} \alpha \wedge \beta &= (a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3) \wedge (b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3) \\ &= a_1\alpha^1 \wedge b_2\alpha^2 + a_1\alpha^1 \wedge b_3\alpha^3 + a_2\alpha^2 \wedge b_1\alpha^1 + a_2\alpha^2 \wedge b_3\alpha^3 + a_3\alpha^3 \wedge b_1\alpha^1 + a_3\alpha^3 \wedge b_2\alpha^2 \\ &= \underbrace{(a_1b_2 - a_2b_1)}_{k^{\text{th}} \text{ of } v_\alpha \times v_\beta} \alpha^1 \wedge \alpha^2 + \underbrace{(a_2b_3 - a_2b_3)}_{i^{\text{th}} \text{ of } v_\alpha \times v_\beta} \alpha^2 \wedge \alpha^3 + \underbrace{(a_1b_3 - a_3b_1)}_{j^{\text{th}} \text{ of } v_\alpha \times v_\beta} \alpha^1 \wedge \alpha^3 \end{aligned}$$