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Real Anakysis 1 (MTH5110) HWs

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1. HW1

I. This problem reviews continuity for functions on the real line.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$ if for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

(a) Show that $f(x) = x^2$ is continuous at $x = 2$.

(b) Suppose that f is continuous at a and $f(a) \neq 0$. Show that f is nonzero in some open interval containing a.

II. This problem reviews derivatives.

(a) Let $f(x) = x^n$ for some positive integer n. Using the definition of the derivative, and the binomial theorem, show that f'^{n-1}.

(b) Is the function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x \leq 0, \end{cases}$$

differentiable at $x = 0$?

III. This problem reviews sup and inf.

For any subset $A \subset \mathbb{R}$, we say that M is an *upper bound* for A if $x \leq M$ for all $x \in A$. If a set A has a finite upper bound, we say it is *bounded above*. It is a theorem about the set $\mathbb{R}$ that *for any set $A \subset \mathbb{R}$ that is bounded above, there exists a least (smallest) upper bound for A*. This least upper bound is called the *supremum* of A, and denoted $\sup A$. By definition, the number $\sup A$ has two properties:

(i) $x \leq \sup A$ for all $x \in A$ (i.e. $\sup A$ is an upper bound for M).

(ii) for any M that is an upper bound for A, we have $\sup A \leq M$.

For sets that are not bounded above, we say $\sup A = +\infty$. We often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set $\{f(x) : x \in A\}$, where f is some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted $\inf A$. It satisfies the same properties as $\sup A$ with the inequalities reversed.

(a) Find $\sup A$ and $\inf A$ for $A = (1, 2]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, and $A = \{0, 1, 2, 3, \ldots\}$.

(b) Find $\sup_{x \in (0,1)} (1 + x^2)^{-1}$.

(c) Assume that $\sup A < \infty$, and show that for any $\varepsilon > 0$, there exists $x \in A$ such that $x > \sup A - \varepsilon$.

(d) For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, and any set $A \subset \mathbb{R}$, show that $\sup_{x\in A}(f(x) + g(x)) \leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$.

IV. Section 1.1, Exercises 5, 6, 13.

Exercise 1.1.5. Let $n \geq 1$, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Verify the identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right),$$

and conclude the *Cauchy-Schwarz inequality*

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$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{1.3}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}.$$

Exercise 1.1.6. Show that $(\mathbf{R}^n, d_{l^2})$ in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5.)

Example 1.1.6 (Euclidean spaces). Let $n \geq 1$ be a natural number, and let $\mathbf{R}^n$ be the space of n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbf{R}\}.$$

We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &:= \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}. \end{aligned}$$

Exercise 1.1.13. Prove Proposition 1.1.19.

Proposition 1.1.19 (Convergence in the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc} if and only if there exists an $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.*

V. For this problem only, you do not need to give proofs. Just write the answers.

For each set, identify the boundary, interior, and closure of A, and say whether A is open, closed, both, or neither. We are working in $\mathbb{R}^2$ with the standard distance. Unless otherwise noted, the ambient space is $\mathbb{R}^2$.

(a) $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$

(b) $A = \{(1/n, 2/n) : n = 1, 2, 3, \ldots\}$ (Note: $(1/n, 2/n)$ is a vector in $\mathbb{R}^2$, not an open interval in $\mathbb{R}$.)

(c) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$, in the relative topology with respect to $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$

VI. Let (X, d) be a metric space.

(a) For a given point $x_0 \in X$, show the singleton set $\{x_0\}$ is closed.

(b) Let $x_0 \in X$ and $r > 0$. Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open.

2. HW2

I. Consider a sequence x_n of real numbers. The *limit inferior* and *limit superior* of x_n are defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right), \quad \limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right).$$

(a) Show that

$$\liminf_{n\to\infty} x_n = \sup_{n\geq 0}\left(\inf_{k\geq n} x_n\right)$$

and

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0}\left(\sup_{k\geq n} x_n\right).$$

(b) Show that $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ are well-defined for any sequence x_n. (Unlike $\lim_{n\to\infty} x_n$.) We allow values of ∞ or $-\infty$.

(c) Let x_n be a bounded sequence, and let L be the set of limit points of x_n, i.e. the set of all limits of subsequences of x_n. Show $\liminf_{n\to\infty} x_n = \inf L$ and $\limsup_{n\to\infty} x_n = \sup L$.

(d) Let x_n be a bounded sequence. Conclude using (c) that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$, with equality if and only if x_n is convergent.

II. Prove that for any (possibly uncountable) collection $(F_\alpha)_{\alpha\in A}$ of closed sets, the intersection $F = \bigcap_{\alpha\in A} F_\alpha$ is closed, in two ways:

(a) Using the fact that any union of open sets is open, and DeMorgan's Laws from set theory, which state

$$X \setminus \left(\bigcup_{\alpha\in A} E_\alpha\right) = \bigcap_{\alpha\in A} (X \setminus E_\alpha) \quad \text{and} \quad X \setminus \left(\bigcap_{\alpha\in A} E_\alpha\right) = \bigcup_{\alpha\in A} (X \setminus E_\alpha),$$

for any collection of sets $(E_\alpha)_{\alpha\in A}$.

(b) More directly, using the fact that a set G is closed if and only if for any convergent sequence (x_n) with all $x_n \in G$, the limit x is also in G.

III. (a) Let (x_n) be a Cauchy sequence in a metric space X. Show that if a subsequence (x_{n_j}) of (x_n) converges to x, then the entire sequence also converges to x.

(b) Show that the metric space

$$C^1((-1,1)) = \{f : (-1,1) \to \mathbb{R},\ f \text{ is differentiable and } f' \text{ is continuous in } (-1,1)\}$$

with the metric

$$d(f,g) = \sup_{x\in(-1,1)} |f(x) - g(x)|,$$

is not complete. (Hint: similar to the proof that the rational numbers are not complete, find a sequence in $C^1((-1,1))$ that converges in the d metric to a function that is not in $C^1((-1,1))$, and show that this sequence is Cauchy.)

IV. Let A and B be subsets of the metric space X. Which one of the following is true? Prove your conclusion:

$$(A \cup B)^\circ = A^\circ \cup B^\circ, \tag{2.1}$$

$$(A \cup B)^\circ \subset A^\circ \cup B^\circ, \qquad \text{“=” fails for some } A \text{ and } B, \tag{2.2}$$

$$(A \cup B)^\circ \supset A^\circ \cup B^\circ, \qquad \text{“=” fails for some } A \text{ and } B. \tag{2.3}$$

V. Let $C^0([a,b])$ be the space of continuous functions on $[a,b]$, with the metric $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$.

Show that the map $I : C^0([a,b]) \to \mathbb{R}$ defined by $I(f) = \int_a^b f(x)\,\mathrm{d}x$ is a continuous mapping from $C^0([a,b])$ to $\mathbb{R}$.

VI. Prove Proposition 2.3.2 in the text, in two different ways:

(a) As a consequence of Theorem 2.3.1 in the text.

(b) Directly, using the sequential definition of compactness.

Proposition 2.3.2 (Maximum principle). *Let (X,d) be a compact metric space, and let $f : X \to \mathbf{R}$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

VII. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$\lim_{|x|\to\infty} f(x) = +\infty.$$

Prove that f attains its minimum.

Recall that by definition, the limit in (**??**) meas that Given $A > 0$, there is $R > 0$ such that

$$f(x) > A \qquad \text{for all } x \notin B_R$$

in other words, $f(x) > A$ whenever $|x| \geq R$. Here, $|x| = d_2(x,0)$ and d_2 is the standard Euclidean distance defined in Example 1.4.

3. HW3

I. Let $\Omega \subset \mathbb{R}^m$, $a \in \Omega^\circ$. If $f : \Omega \to \mathbb{R}$ is continuous at$a$, $g : \Omega \to \mathbb{R}$ is differentiable at a and $g(a) = 0$, show that fg is differentiable at a. (Note that fg is the function whose value at $x \in \Omega$ is $f(x)g(x)$.)

III. Find the total derivatives (i.e. derivative matrices) of the following functions at the given points:

(a) $f(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ x_1 x_3^2 \\ \sin(x_1)e^{x_2} \\ x_1 + x_2 + x_3 \end{pmatrix}$ at $(x_1, x_2, x_3) = (1, 0, 1)$.

(b) $f(x) = \begin{pmatrix} x^2 \\ e^x \end{pmatrix}$ at $x = 3$.

(c) $f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2x_4 + \sin(x_3x_4)$ at $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$.

IV. Section 6.2, problem 2.

Exercise 6.2.2. Prove Lemma 6.2.4. (Hint: prove by contradiction. If $L_1 \neq L_2$, then there exists a vector v such that $L_1 v \neq L_2 v$; this vector must be non-zero (why?). Now apply the definition of derivative, and try to specialize to the case where $x = x_0 + tv$ for some scalar t, to obtain a contradiction.)

Lemma 6.2.4 (Uniqueness of derivatives). *Let E be a subset of $\mathbf{R}^n$, $f : E \to \mathbf{R}^m$ be a function, $x_0 \in E$ be an* interior point *of E, and let $L_1 : \mathbf{R}^n \to \mathbf{R}^m$ and $L_2 : \mathbf{R}^n \to \mathbf{R}^m$ be linear transformations. Suppose that f is differentiable at x_0 with derivative L_1, and also differentiable at x_0 with derivative L_2. Then $L_1 = L_2$.*

V. Section 6.3, problem 3 and problem 4.

Exercise 6.3.3. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by $f(x, y) := \frac{x^3}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) := 0$. Show that f is not differentiable at $(0, 0)$, despite being differentiable in every direction $v \in \mathbf{R}^2$ at $(0, 0)$. Explain why this does not contradict Theorem 6.3.8.

Exercise 6.3.4. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable function such that $f'(x) = 0$ for all $x \in \mathbf{R}^n$. Show that f is constant. (Hint: you may use the mean-value theorem or fundamental theorem of calculus for one-dimensional functions, but bear in mind that there is no direct analogue of these theorems for several-variable functions. I would not advise proceeding via first principles.) For a tougher challenge, replace the domain $\mathbf{R}^n$ by an open connected subset Ω of $\mathbf{R}^n$.

VI. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable, $\alpha \in \mathbb{R}$. If $f(tx) = t^\alpha f(x)$ for $\forall x \in \mathbb{R}^m$ and $t > 0$, we say that f is homogeneous of order α. Show that f is homogeneous of order α iff $x \cdot \nabla f(x) = \alpha f(x)$, that is

$$x^1 \partial_1 f(x) + \cdots + x^m \partial_m f(x) = \alpha f(x).$$

This equation is classically written as

$$x^1 \frac{\partial f}{\partial x^1} + \cdots + x^m \frac{\partial f}{\partial x^m} = \alpha f(x).$$

Hint: As in the development of the theory in the text, a basic idea to study multi-variable functions is to convert them into single-variable functions by restricting the variable x in a fixed direction. For example, for this problem you may consider the function $\varphi(t) = f(tx)$.

VII. (a) Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1-map,

$$|f(x) - f(y)| \geq |x - y|, \qquad \forall x, y \in \mathbb{R}^m, \tag{3.1}$$ e1

then for $\forall a \in \mathbb{R}^m$, $\det f'(a) \neq 0$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable, and assume $f(0,0) = \langle 1, 2\rangle$, and

$$Df(0,0) = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Let $g(x,y) = \langle xy^2, y+2, 2x-3y\rangle$. Find $D(g \circ f)(0,0)$.

VIII. Let $f : E \to \mathbb{R}$ be defined on some open subset $E \subset \mathbb{R}^2$, and assume the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 6.3.8 (continuity of partial derivatives implies f is differentiable) which we discussed in class.

IX. Let $F(x,y,z) = \begin{pmatrix} x+y \\ x^2y \\ z+2x \end{pmatrix}$.

(a) At what points (x_0, y_0, z_0) does F have a local inverse, i.e. a function F^{-1} defined on an open set V containing $F(x_0,y_0,z_0)$, such that $F(F^{-1}(x,y,z)) = (x,y,z)$ for all $(x,y,z) \in V$?

(b) What is $D(F^{-1})(2,1,3)$? (Hint: $F(1,1,1) = (2,1,3)$.)

X. When does the equation $x_1^2 + 2x_2^3x_3 - x_4 + \ln(1+x_4^2) = 1$ define a function $x_4 = g(x_1,x_2,x_3)$ implicitly? Find $\nabla g(1,0,-1)$.

4. HW4

I. Section 7.2, problem 2.

Exercise 7.2.2. Let A be a subset of $\mathbf{R}^n$, and let B be a subset of $\mathbf{R}^m$. Note that the Cartesian product $\{(a,b) : a \in A, b \in B\}$ is then a subset of $\mathbf{R}^{n+m}$. Show that $m^*_{n+m}(A \times B) \leq m^*_n(A)m^*_m(B)$. (It is in fact true that $m^*_{n+m}(A \times B) = m^*_n(A)m^*_m(B)$, but this is substantially harder to prove).

In Exercises 7.2.3-7.2.5, we assume that $\mathbf{R}^n$ is a Euclidean space, and we have a notion of measurable set in $\mathbf{R}^n$ (which may or may not coincide with the notion of Lebesgue measurable set) and a notion of measure (which may or may not co-incide with Lebesgue measure) which obeys axioms (i)-(xiii).

II. Section 7.4, problems 1, 4 (only parts (e) and (f)).

Exercise 7.4.1. If A is an open interval in $\mathbf{R}$, show that $m^*(A) = m^*(A \cap (0,\infty)) + m^*(A\backslash(0,\infty))$.

Exercise 7.4.4. Prove Lemma 7.4.4. (Hints: for (c), first prove that

(*e*) *Every open box, and every closed box, is measurable.*

(*f*) *Any set E of outer measure zero (i.e., $m^*(E) = 0$) is measurable.*

III. Let C be a parameterized curve in $\mathbb{R}^2$. In other words, C is the image of a function $\phi : [a,b] \to \mathbb{R}^2$. Show that, if ϕ is continuously differentiable on $[a,b]$, then C has outer measure 0.

Hint: partition $[a,b]$ into N equal subintervals, and use the Mean Value Inequality to show that the image of each subinterval is bounded in terms of N, i.e. fits inside an open rectangle of side length that can be explicitly bounded in terms of N. Add up the total 2-dimensional volume of the covering obtained in this way, and show that it can be made arbitrarily small by taking N large.

Warning: If ϕ is only continuous, then the result fails. On can construct a continuous ϕ such that

$$\phi([a,b]) = [0,1] \times [0,1].$$

V. Suppose $A_i \in \mathcal{M}$, $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$.

(a) If $m(A_1) < \infty$, show that

$$m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

(b) Show by example that if $m(A_1) = \infty$, the above conclusion may be wrong.

VI. Let $\Omega \subset \mathbb{R}^n$ be measurable, $f : \Omega \to \mathbb{R}$ is a function. If f^2 is measurable, and the set

$$A = \{x \in \Omega \mid f(x) > 0\}$$

is also measurable. Show that f is measurable.

5. HW5

I. Section 7.4, problem 10.

Exercise 7.4.10. Let $A \subseteq B \subseteq \mathbf{R}^n$. Show that if B is Lebesgue measurable with measure zero, then A is also Lebesgue measurable with measure zero.

II. Section 7.5, problem 5.

Exercise 7.5.5. Let $f : \mathbf{R}^n \to \mathbf{R}$ be Lebesgue measurable, and let $g : \mathbf{R}^n \to \mathbf{R}$ be a function which agrees with f outside of a set of measure zero, thus there exists a set $A \subseteq \mathbf{R}^n$ of measure zero such that $f(x) = g(x)$ for all $x \in \mathbf{R}^n \backslash A$. Show that g is also Lebesgue measurable. (Hint: use Exercise 7.4.10.)

III. Let $f : \Omega \to [0,\infty)$ be measurable, $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ with $\Omega_k \in \mathcal{M}$, $\Omega_k \subset \Omega_{k+1}$ for all k. Then

$$\int_{\Omega} f = \lim_{k\to\infty} \int_{\Omega_k} f.$$

Remark. If f is simple, then the result is precisely Lemma 4.27

IV. Show that $\displaystyle \lim_{n\to\infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx = \int_{[0,\infty)} e^{-x}\, dx.$

V. If $f \in L(\Omega)$, then

$$\lim_{r\to\infty} \int_{\Omega\backslash B_r} f = 0.$$

Note. Recall that B_r is the r-ball at the origin. If Ω is bounded then eventually $\Omega\backslash B_r = \emptyset$ (in this case the integral is regarded to be zero) but our Ω maybe unbounded.

V*. If $f \in L(\Omega)$, show that

$$\lim_{k\to\infty} km(\{f \geq k\}) = 0.$$

VI. Find an example of a sequence $f_n : [0,1] \to [0,\infty)$ so that strict inequality holds in Fatou's Lemma, i.e.

$$\int_{[0,1]} \liminf_{n\to\infty} f_n < \liminf_{n\to\infty} \int_{[0,1]} f_n.$$

(Hint: try a sequence where $\int_{[0,1]} f_n$ is the same positive value for every n, but $\liminf f_n \equiv 0$.)

VII. (a) Let $f \geq 0$ be integrable on $[a,b]$. Prove that the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a,b]$. (Hint: for fixed x, use the Dominated Convergence Theorem to show that $F(x+1/n) - F(x) \to 0$ and $F(x-1/n) - F(x) \to 0$ as $n \to \infty$. Then use this to prove continuity of F at x.)

(b) Assume f is Riemann integrable on $[a,b]$, and let F be defined as in (a). Show that F is differentiable almost everywhere, and the equality $F'(x) = f(x)$ is true almost everywhere.

(The same is true for any (Lebesgue) integrable function f, but this is harder to prove.)

VIII. Find an example of a uniformly bounded sequence of functions $f_n : \mathbb{R} \to [0,\infty)$ so that each f_n is Riemann integrable, but f_n converges pointwise to a function that is not Riemann integrable.

(We know this problem can't occur with the Lebesgue integral, because a pointwise limit of measurable functions is measurable.)

IX. Suppose $\rho : [0,\infty) \to \mathbb{R}$ is decreasing and continuous, $m(E) = m(B_R)$, where E is a measurable subset of $\mathbb{R}^n$ and $B_R \subset \mathbb{R}^n$ is the R-ball at the origin. Show that

$$\int_E \rho(|x|)\,dx \leq \int_{B_R} \rho(|x|)\,dx.$$