Functional Analysis– Spring 2024

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p. 224 #4, 7, 8, 9,

4. Let p be defined on a vector space X and satisfy (1) and (2). Show that for any given $x_0 \in X$ there is a linear functional $\tilde{f}$ on X such that $\tilde{f}(x_0) = p(x_0)$ and $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

   Let $f \in X'$ and $f(x_0) = p(x_0)$. Clearly, f is defined on the subspace spanned by x_0, that is, f is linear and $f(\alpha x_0) = \alpha f(x_0)$. The Hahn-Banach Theorem says that there exists an extension of f, namely, $\tilde{f} \in X'$ such that $|\tilde{f}(x)| \leq p(x)$ for all $x \in X$.

7. Give another proof of Theorem 4.3-3 in the case of a Hilbert space.

   Theorem 4.3-3a: (Bounded linear functionals, Hilbert). Let X be a Hibert space and let $x_0 \neq 0$ be any element in X. Then there exists a bounded linear functional $\tilde{f}$ on X such that

   $$\left\|\tilde{f}\right\| = 1, \qquad \tilde{f}(x_0) = \|x_0\|$$

   Proof: Let $x_0 \in X$, then Z is the subspace spanned by x_0. Any Cauchy sequence in Z will converge because that same sequence is in X which is complete. Hence. Z is also complete. We know that, for any $f_g \in Z'$ there exists a $g \in Z$ such that $f_g(x) = \langle x, g \rangle$ and $\|f_g\| = 1$. By Hahn-Bannach, there exists an extension $\tilde{f} \in X'$ such that $\left\|\tilde{f}\right\| = 1$ and $|f(x_0)| = \left\|\tilde{f}\right\| \, \|x_0\| = \|x\|$.

8. Let X be a normed space and X' its dual space. If $X \neq \{0\}$, show that X' cannot be $\{0\}$.

   Let $f(x) = \|x\|$, this is linear by definition. Therefore, $f \in X'$. We can see that when $x \neq 0$ that $f(x) \neq 0$. Therefore f is not the zero function and $X \neq \{0\}$.

9. Show that for a separable normed space X, theorem 4.3-2 can be proved directly, without the use of Zorn's Lemma (which was used indirectly, namely, in the proof of Theorem 4.2-1).

p. 255 #10, 11, 13, 14,

10. **(Space** $c - 0$**)** Let $y = (\eta_j), \eta_j \in \mathbb{C}$, be such that $\sum \xi_j \eta_j$ converges for every $x = (\xi_j) \in c_0$, where $c_0 \in l^\infty$ is the subspace of all complex sequences converge to zero. Show that $\sum |\eta_j| < \infty$. (Use 4.7-3)

11. Let X be a Banach space, Y a normed space and $T_n \in B(X, YT)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. Show that $(\|T_n\|)$ is bounded.

13. If (x_n) in a Banach space X is such that $(f(x_n))$ is bounded for all $f \in X'$, show that $(\|x_n\|)$ is bounded.

14. if X and Y are Banach spaces and $T_n \in B(X, Y), n = 1, 2, \cdots$, show that equivalent statements are:
    (a) $(\|T_n\|)$ is bounded.
    (b) $(\|T_n x\|)$ is bounded for all $x \in X$.
    (c) $(|g(T_n x)|$ is bounded for all $x \in X$ and all $g \in Y'$.