Math 5301 – Numerical Analysis– Spring 2025
w/Professor Du

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Homework #2 – February 8, 2025

Question1. (20 points)

Construct a Natual Cubic Spline for $f(x) = \frac{1}{1+5x^2}$ over $[-3,3]$ with 61 equally spaced nodes

(a) Explicity form the linear systerm (matrix and right hand side vector).

$h = 0.1$ and $n = 61$ we have

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0.1 & 2(0.1+0.1) & 0.1 & 0 & \cdots & 0 \\ 0 & 0.1 & 2(0.1+0.1) & 0.1 & \cdots & 0 \\ 0 & 0 & 0.1 & 2(0.1+0.1) & \cdots & 0 \\ \vdots & \vdots & \vdots, & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0.1 & 2(0.1+0.1) & 0.1 \\ 0 & \cdots & \cdots & 0 & 0.1 & 2(0.1+0.1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0.1 & 0.4 & 0.1 & 0 & \cdots & 0 \\ 0 & 0.1 & 0.4 & 0.1 & \cdots & 0 \\ 0 & 0 & 0.1 & 0.4 & \cdots & 0 \\ \vdots & \vdots & \vdots, & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0.1 & 0.4 & 0.1 \\ 0 & \cdots & \cdots & 0 & 0.1 & 0.4 \end{bmatrix}$$

$$\frac{3}{h} = \frac{3}{0.1} = 30$$

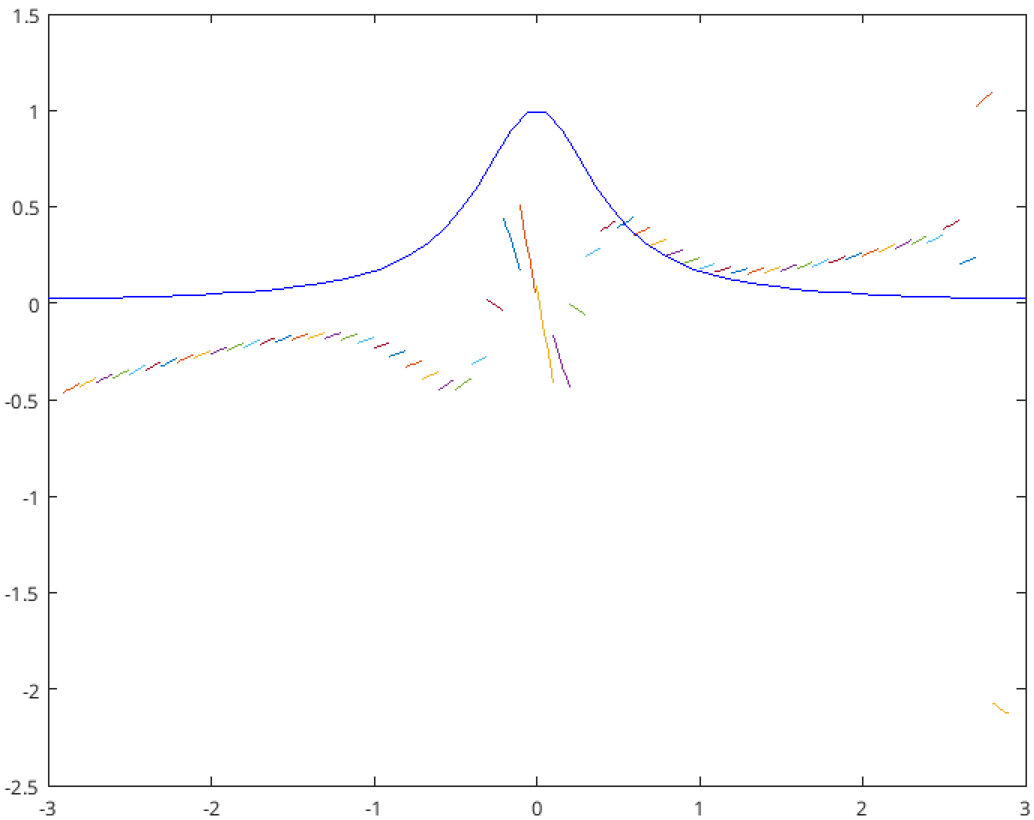
$$b = \begin{bmatrix} 0 \\ 30(f(x_2) - f(x_1)) - 30(f(x_1) - f(x_0)) \\ \vdots \\ 30(f(x_n) - f(x_{n-1})) - 30(f(x_{n-1}) - f(x_{n-2})) \end{bmatrix} = \begin{bmatrix} 0 \\ 30(f(x_2) - 2f(x_1) + f(x_0)) \\ \vdots \\ 30(f(x_n) - 2f(x_{n-1}) + f(x_{n-2})) \end{bmatrix}$$

(b) Solve the linear system using Thomas Algorithm. Print out the solutions.

```
x = linspace(-3,3,60);
y = 1./(1+5*x.^2);

% initialize the working arrays
alpha = linspace(-3,3,60);
l = linspace(-3,3,60);
u = linspace(-3,3,60);
z = linspace(-3,3,60);

% initialize the return values

a = y;
b = linspace(-3,3,60);
c = linspace(-3,3,60);
d = linspace(-3,3,60);

h = 0.1;
n = 61;
for i = 2:1:n-2
        alpha(i) = 30*( y(i+1) - 2*y(i) + y(i-1) );
end
l(1) = 1;
u(1) = 0;
z(1) = 0;

for i = 2:1:n-1
```

```
        l(i) = 0.4 - 0.1*u(i-1);
        u(i) = 0.1/l(i);
        z(i) = (alpha(i) - 0.1*z(i-1)) / l(i);
end

l(n)=1;
z(n)=0;
c(n)=0;

% plot f(x) in blue
plot(x,y,'-b');
hold on;

for j = n-2:-1:1
        c(j) = z(j) - u(j)*c(j+1);
        b(j) = ( a(j+1) - a(j) )/0.1 - 0.1*(c(j+1) + 2*c(j) )/3;
        d(j) = ( c(j+1) - c(j) )/30;
end

%now plot each of the S_j splines within their intervals

for j=2:1:n-2
        left = -3 + (j-1)*0.1;
        right = -3 + j*0.1;
        xj=[left:0.01:right];
        p = [a(j) b(j) c(j) d(j)];
        plot(xj, polyval(p, xj));
        %yj=a(j) + b(j).*xj + c(j)*xj.^2 + d(j)*xj.^3;
        %plot(xj,yj,"-r");
        clear xj yj p;
end
```

(c) Plot the cubic spline together with $f(x)$ and estimate the largest error.



Question 2. (20 points)
For $f(x) = |x|$ over $[-1.5, 1.5]$, find the best polynomial of order 5 to interpolate the function.

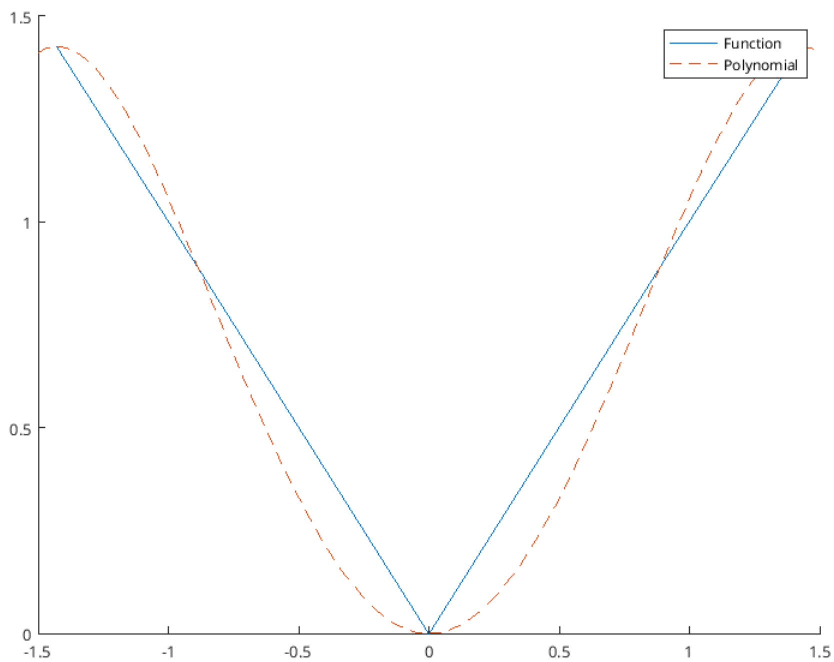
(a) Identify all nodes used to construct the polynomial.

The formula for n nodes on an interval $[a, b]$ is

$$\begin{aligned} x_k &= \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\cos\left(\frac{2k-1}{2n}\pi\right),\ k = 1, \ldots, n \\ x_k &= \frac{1}{2}(-1.5+1.5) + \frac{1}{2}(-1.5-1.5)\cos\left(\frac{2k-1}{2\cdot 5}\pi\right),\ k = 1, \ldots, 5 \\ &= -1.5 \cdot \cos\left(\frac{2k-1}{2\cdot 5}\pi\right),\ k = 1, \ldots, 5 \end{aligned}$$

$$\begin{aligned} x_1 &= -1.5 \cdot \cos\left(\frac{1}{10}\pi\right) = -1.4266 \\ x_2 &= -1.5 \cdot \cos\left(\frac{3}{10}\pi\right) = -0.8817 \\ x_3 &= -1.5 \cdot \cos\left(\frac{5}{10}\pi\right) = 0.0000 \\ x_4 &= -1.5 \cdot \cos\left(\frac{7}{10}\pi\right) = 0.8817 \\ x_5 &= -1.5 \cdot \cos\left(\frac{9}{10}\pi\right) = 1.4266 \end{aligned}$$

(b) Plot the polynomial together with $y = f(x)$ and estimate the largest error.



(c) Compare the largest error with the error bound.

The error bound for using Chebyschev points is

$$\begin{aligned} \max_{[-1,1]} |f(x) - P(x)| &\leq \frac{1}{2^n(n+1)!} \max_{[-1,1]} |f^{(n)}(x)| \\ &\leq \frac{1}{2^5 6!} = 0.000043403 \end{aligned}$$