1. Suppose that V is a finite dimensional vector space. Show that any linear transformation on a subspace of V can be extended to a linear transformation on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there is $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Define $T \in \mathcal{L}(V, W)$ as

$$T(v) = \begin{cases} S(v) & \text{if } v \in U \\ v & \text{otherwise} \end{cases}$$

thus

$u, v \in U$	$u \in U, v \in V \backslash U$	$u, v \in V \backslash U$
$T(u+v) = S(u+v)$	$T(u+v) = S(u) + T(v)$	$T(u+v) = T(u) + T(v)$
$= S(u) + S(V) \in U$	$= S(u) + v \in V$	$= u + v \in V$

2. Let $V = \mathcal{M}_{n \times n}(F)$, and let B be fixed matrix in V. Show that $T : V \mapsto V$ defined by $T(A) = AB - BA$ is a linear transformation.

What happens in T when you add C and D?

$$\begin{aligned} T(C+D) &= (C+D)B - B(C+D) \\ &= CB + DB - BC - BD && \text{distributive law of matrix multiplication} \\ &= CB - BC + DB - BD \\ &= T(C) + T(D) \end{aligned}$$

and scalar multiplication?

$$\begin{aligned} T(cA) &= (cA)B - B(cA) \\ &= c(AB) - c(BA) \\ &= c(AB - BA) \\ &= cT(A) \end{aligned}$$

3.a) Recall that $\mathbb{C}$ is a real vector space. Find $T : \mathbb{C} \mapsto \mathbb{C}$ which is a $\mathbb{R}$-linear transformation which is not a $\mathbb{C}$-linear transformation.

b) Find a linear transformation $T : V \mapsto V$ where the range and nullspace of T are identical.

When $\text{null}(T) = \text{range}(T)$ we have $\{v \in V : T(v) = 0\} = \{T(v) : v \in V\}$. That is, given any $v \in V, T(v) = 0$. T is by definition the zero transformation.

c) Find T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

Let M be the matrix associated with T, that is, $Tx = Mx$ and N be the matrix assoicated with U, that is, $Ux = Nx$. Then, $TU = MN$. Now let

$$\begin{aligned}
M &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
N &= \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\
MN &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\
&= \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} = 0 \\
\text{thus } & \begin{array}{c|c} aw = -by & ax = -bz \\ cw = -dy & cx = -dz \end{array} \\
\text{but } NM &= \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} wa + xc & wb + xd \\ ya + zc & yb + zd \end{pmatrix} \neq 0 \\
\text{thus } & \begin{array}{c|c} aw \neq -cx & bw \neq -dx \\ ay \neq -cz & bd \neq -dz \end{array} \\
\text{Let } M &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\
MN &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1-1 & 1-1 \\ 1-1 & 1-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
NM &= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+1 & 1+1 \\ -1+(-1) & -1+(-1) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}
\end{aligned}$$

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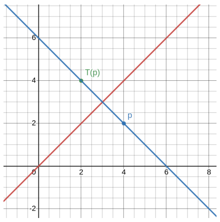
4. Let T and U be two linear operators on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (x_2, x_1)$ and $U(x_1, x_2) = (x_1, 0)$.

a) Give a geometric interpretation for T and U.

T transposes the right and left values which essentially maps every point across the line $x_2 = x_1$ distance from that line. That is, if a point P has a perpendicular distance m to the line $x_1 = x_2$ then $T(P)$ will be the same distance away on the other side of that line.



U provides the projection onto the x_1 axis which is either directly above or below the point.

b) Give rules for $U + T$, UT, TU, T^2, and U^2.

$U + T$, $(U + T)(x_1, x_2) = U(x_1, x_2) + T(x_1, x_2) = (x_1, 0) + (x_2, x_1) = (x_1 + x_2, x_1)$

UT, $UT(x_1, x_2) = U(T(x_1, x_2)) = U(x_2, x_1) = (x_2, 0)$

TU, $TU(x_1, x_2) = T(U(x_1, x_2)) = T(x_1, 0) = (0, x_1)$

T^2, $T^2(x_1, x_2) = T(T(x_1, X_2)) = T(x_2, x_1) = (x_1, x_2)$ the identity map.

U^2, $U^2(x_1, x_2) = U(U(x_1, X_2)) = U(x_1, 0) = (x_1, 0)$ $U^2 = U$ and is not the identity map.

Extra Questions

1. Let A be an $m \times n$ matrix over F of rank k. Show that there exist a $m \times k$ matrix B and a $k \times n$ matrix C, both with rank k, where $A = BC$. Conclude that A has rank 1 if and only if $A = xy^t$ where $x \in F^m$ and $y \in F^n$.

2. Let W be the vector space of 2×2 complex Hermitian matrices. Note that W is a vector space over $\mathbb{R}$ but not over $\mathbb{C}$. Let $T : \mathbb{R}^4 \mapsto W$ be the map defined by

$$(x, y, z, t) \mapsto \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix}.$$

Show that T is an isomorphism.

$$\begin{aligned} \text{null}(T) &= \left\{ (x, y, z, t) \mapsto \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ & t+x = 0, t = -x \text{ and } t-x = 0, t = x \implies t = x = 0 \\ & y-iz = 0, y = iz \text{ and } y+iz = 0, y = -iz \implies y = z = 0 \\ \text{null}(T) &= \{(0,0,0,0)\} \end{aligned}$$

which means that T is injective.

Let w be any element in $w \in W$ and for $a, b, c, d \in \mathbb{R}$

$$w = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$$

Find t, x such that $t + x = a$ and $t - x = c$ or $2t = a + c$ or $t = \frac{a+c}{2}$ and $2x = a - c$ or $x = \frac{a-c}{2}$. Similarly, find y, z such that $y - iz = b - ic$ and $y + iz = b + ic$ or $y = b, z = c$. Thus,

$$T\left(\frac{a-c}{2}, b, c, \frac{a+c}{2}\right) = w$$

which is true for all $w \in W$ which means that range$(T) = W$ and is surjective.

A transformation that is both injective and surjective is an isomorphism.

3. We will consider the vector space $V = \mathcal{P}^{(n)}(\mathbb{R})$ of polynomials at most degree n. Let

$$[x]_k := x(x-1)(x-2)\cdots(x-k+1)$$

for $k \geq 1$ and $[x]_0 = 1$.

a) Show that $([x]_0, [x]_1, [x]_2, \ldots, [x]_n)$ is a basis of V. [Hint: argue that $[x]_k = x^k + a(k, k-1)x^{k-1} + \cdots + a(k,1)x + a(k,0)$ where $a(k, j)$ are integers. Construct the $(n+1) \times (n+1)$ matrix which expresses each $[x]_k$ in the basis $(1, x, x^2, \ldots, x^n)$. Show that this matrix is invertible].

b) Now prove that $x^k = \sum_{j=0}^{k} S(k, j)[x]_j$ where $S(k, j)$ are integers.

c) Show that $S(k, 0) = 0$ for $k \geq 1$. Also show that $S(k, k) = 1$ for $k \geq 0$.

d) Prove that if $1 \leq j \leq k-1$ then

$$S(k, j) = jS(k-1, j) + S(k-1, j-1).$$

The above exercise shows that $S(k, j)$ are nonnegative integers. They are called *Stirling numbers of the second kind*.