Math 5111 – Real Analysis II– Sprint 2025

w/Professor Perera

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Homework (not graded) – May 2025

Pg 30, 1

Does there exist an infinite σ-algebra which has only countably many members?

First we establish the smallest possible measurable set containing a point, B_x. Then demonstrate that these are unique and distinct (that is, $x \neq y \implies B_x \cap B_y = \emptyset$). Then assuming that there are countably many B_x show that there are still some missing points not in the $\bigcup_{\forall x} B_x$.

Pg 30, 2

Prove an analogue of Theorem 1.8 for n functions.

Pg 30, 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for for every rational r, then f is measurable.

Let $E_r = \{x : f(x) \geq r\}$ each of which is measurable and $E = \bigcup_{r\in\mathbb{Q}} E_r$. E is measurable because it is the countable union of measurable sets. Given any measurable set $I \in \mathbb{R}$ we can clearly see that $f(E) \cap I \neq \emptyset$ because $\mathbb{Q}$ is dense.

$$f(E) = f\left(\bigcup_{r\in I\cap\mathbb{Q}} E_r\right)$$

We are saying that I is measurable, therefore it can be made up of the union of disjoint measurable sets A and B where $B = I \cap f(E)$ and $A = I \backslash B$. $f^{-1}(I) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. We can see that $f^{-1}(B)$ is measurable, How do we show that $f^{-1}(A)$ is?

Notice, $E_r \in \mathfrak{M} \implies E_r^c \in \mathfrak{M}$. Now, we can try to connect $f^{-1}(A)$ to these sets.

Pg 30, 4

Let $\{a_n\}$ and $\{b_n\}$ be sequencesin $[-\infty, \infty]$, and prove the following assertions:

(a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

$$\sup_{n\to\infty}\{-a_n\} = -\inf_{n\to\infty}\{a_n\}$$
$$\limsup_{n\to\infty}\{-a_n\} = \lim_{n\to\infty}\sup_{k>n}\{-a_k\} = \lim_{n\to\infty}\left(-\inf_{k>n}\{a_k\}\right) = -\liminf_{n\to\infty}\{a_n\}$$

(b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ provided none of the sums is $\infty - \infty$.

$$\limsup_{n\to\infty}\{a_n + b_n\} = \lim_{n\to\infty}\sup_{k>n}\{a_k + b_k\}$$

keep mind that $a_k + b_k$ could be zero, but $\sup a_k + \sup b_k$ could only be zero after evaluating both sequences. Thus,

$$\sup_{k>n}\{a_k + b_k\} \leq \sup_{k>n}\{a_k\} + \sup_{k>n}\{b_k\}, \forall n$$
$$\lim_{n\to\infty}\sup_{k>n}\{a_k + b_k\} \leq \lim_{n\to\infty}\sup_{k>n}\{a_k\} + \sup_{k>n}\{b_k\}, \forall n$$
$$\limsup_{n\to\infty}\{a_n + b_n\} \leq \limsup_{n\to\infty}\{a_n\} + \limsup_{n\to\infty}\{b_n\}$$

(c) if $a_n \leq b_n$, for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$$

Show by example that strick inequalty can hold in (b).

Pg 30, 5

(a) Suppose $f : X \to [-\infty, \infty]$ and $g; X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)|$$

are measurable.

Given any $y \in X$ let $s_y = \{x : f(x) < g(y)\}$. f is measurable implies that s_y is measurable. Now taking the intersection over all $y \in X$, that is

$$\bigcup_{y\in X} s_y = \{x : f(x) < g(x)\} \in \mathfrak{M}$$

(b) Prove that the set of points at which a sequence of measurable real-value functions converges (to a finite limit) is measurable.

Let $f, f_n : X \to [0, \infty] \in \mathfrak{M}$ and let $f_n \to f, \forall x \in X$. Given any measurable set $p \in X$ the sequence of sets $\{f_i(p)\}$ are all measurable. Further, since $f_i(x) \to f(x), \forall x \in X$ we can say $f_i(p) \to f(p), \forall x \in p$. Thus, for any measurable set P and $\exists p \in \mathfrak{M}, n \to f_n(p) = P$.

Given any measurable set $P \subset [-\infty, \infty]$ then $f_n^{-1}(P) \in \mathfrak{M}, \forall n$. Let $p_n \subset X$ such that $p_n = f_n^{-1}(P)$. Then, $\cap_n f_n(p_n) = \cap_n f(p_n) = P$. Thus, f is measurable.

Pg 30, 6

Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E and E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Pg 30, 7

Suppose $f_n : X \to [0, \infty]$ is measurable fo $rn = 1, 2, \ldots, f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0, f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

$$\lim_{n\to\infty} \int_X f_n d\mu = \lim_{n\to\infty} \sup \int_X s_n d\mu$$

where each s_n are the simple function less than f_n for all x. These are all less than f and whose $\limsup_{n\to\infty} s_n = \limsup_{n\to\infty} S_n$ of simple functions less than f. thus

$$\limsup_{n\to\infty} \int_X s_n d\mu = \limsup_{n\to\infty} \int S_n d\mu = \int_x f d\mu$$

Pg 30, 8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Fatou's Lemma given any any $f_n : X \to [0, \infty) \in \mathfrak{M}$ then

$$\int_X \liminf_{n\to\infty} f_x d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

The sequence of functions does not converge but oscilates. Thus, LHS goes to zero and the RHS goes to 1.

Pg 30, 9

Suppose μ is a postiive measure on X, $f : X \to [0, \infty]$ is measurable $\int_X f d\mu = c$, where $0 < c < \infty$, and α is constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatous lemma can be applied.

Pg 30, 10

Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

and show that the hypothesis"$\mu(X) < \infty$" cannot be omitted.

Pg 30, 11

Show that

$$A = \bigcap_{nj=1}^{\infty} \bigcup_{k=n}^{\infty} E^k$$

in Theorem 1.41 and hence prove the theorme without any reference to integration.

Pg 30, 12

Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Pg 30, 13

Show that proposition 1.24(c) is also true when $c = \infty$.

Page 56, 9, 10

Pg 56, 9

Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$, and

$$\lim_{n\to\infty} \int_0^1 f_n(x) dx = 0$$

but such that the sequence $\{f_n(x)\}$ convergs for no $x \in [0, 1]$.

Pg 56, 10

If $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$ such that $0 \leq f_n \leq 1$ and such that $f_n(x) \to 0$ as $n \to \infty$, for every $x \in [0, 1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x) dx = 0.$$

Try to prove this without using any measure theory or any theorems about Lebesque integration. (This is to impress you with the power of Lebesque integral. A nice proof was given by W. F. Eberlein in *Communications of Pure and Applied Mathematics*, Vol. X, pp 3357-360, 1957.)

Page 56, 8, 14

Pg 56, 8

Construct a Borel set $E \subset \mathbb{R}^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is it possible ot have $m(E) < \infty$ for such a set? Page 57, 13
Page 59, 19
Page 70, 1, 2
Page 70, 4,5,11,12
Page 70, 7, 10

show that $1 < r < p < s \leq \infty$ and $f \in L^r(\mu) \cap L^s(\mu)$, then $f^p(\mu)$ and $||f||_p \leq \max\{||f||_r, ||f||_s\}$.