Math 5111 – Real Analysis II– Sprint 2025
w/Professor Liu

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Semester Notes – May 2025

January 18^{th}, 2025

Continuity is a property unlike most that we encounter in understanding functions. Typically, we start with the domain and apply it to the function and get a result in the range. A function is continuous, however, when the area around the range has the same properties as the area around the domain.

1. **continuous functions** defined using epsilon/delta. Which becomes increasingly difficult when our domain and range go beyond the Real numbers (i.e., multiple dimensions, complex numbers, sets that aren't compact).

2. **Topological definition of continuous functions** which is that an open set in the range comes from an open set in the domain. This skips over the concept of measure by simply redefining "open sets" in more abstract terms (an open set is a member of a topology). A metric topology has a way of measuring distance but it isn't necessary for this definition.

3. Analogously speaking, **a function is measurable** (a.k.a., continuous) when a measurable (a.k.a., open) set in the range comes from a measurable (a.k.a., open) set in the domain. The sets are defined by the Lebesgue Outer Measure (basically, 'not a point', which is much like but not the same as an open set in topology).

4. Definitions of "superset" topology and σ-algebra. These two concepts are so similar that they might be the same thing.

   (a) A **topology** is defined as the collection where at least the set and the empty set as members, as well as subsets where all intersections (countable) and all unions (even uncountable) are also members. "Open sets" are simply the members of the topology.

   (b) A σ-algebra is defined as having the set itself, all compliments of its members (which are subsets), and countable unions of its members.[1] The set and σ-algebra combine to make a **Measurable Space** and its members are called **Measurable Sets** (analogous to "open sets").

   Definition 0.0.1 (Defining properties of a σ-algebra)**.** .

   i. $X \in \mathfrak{M}$ which implies $\emptyset \in \mathfrak{M}$.
   ii. $E \in \mathfrak{M} \implies E^c \in \mathfrak{M}$.
   iii. Countably Additive: if E_i is a partition (or at least disjoint) then $\mu(\cup E_i) = \sum \mu(E_i)$.

   (c) If f maps from a measurable space to a topological space and each open set in the range comes from a measurable set in the domain (i.e., $f^{-1}(V)$ maps open set V from a measurable set through f) then f is said to be **Measurable Function**.

   (d) These two, σ-algebra and topology, appear to me to be logically equivalent.

   (e) Put more simply: topology is defined by arbitrary unions and countable intersections of *open sets* and a σ-algebra is defined by countable unions of *measurable sets*.

5. Further, a **Borel Set** is an element of the smallest possible σ-algebra generated by the set. It is, consequently, measurable (in the Lebesgue sense, i.e., 'not a point'). And a function is **Borel Measurable** when Borel Sets are mapped from Borel Sets.

Rules for Composing different types of these functions.

	Nomen.[2]	$f : X \to Y$	$g : Y \to Z$	$g \circ f(x) = g(f(x)) = z$
standard	open inteval	continuous	continuous	continuous
topology[3]	open	continuous	continuous	continuous
measurable	measurable	measurable	continuous	measurable
measurable	measurable	measurable	measurable	measurable
Borel Measurable	Borel	Borel measurable	continuous	???
Borel Measurable	Borel	continuous	Borel measurable	???
Borel Measurable	Borel	measurable	Borel mapping	measurable
Borel Measurable	Borel	Boreal measurable	Borel mapping	Borel measurable

[1] σ signifies infinite unions and δ (Not shown) would signify infinite intersectios

February 7, 2025

Definition 0.0.2 (Simple Integration). Where s is a simple function, that is, $\exists\{s_i\}, i = 1, \ldots, n$ and E_i such that $x \in E_i \implies s(x) = s_i$.

$$\int_X s d\mu = \sum_{i=1}^{n} s_i \mu(X \cap E_i)$$

(think E_i forms a partition on X.)

Theorem 0.0.3. *Given any positive measurable function f there exists a sequence of simple measurable functions $\{s_i\}$ such that $s_i \to f$.*

Definition 0.0.4 (Upper/Lower Semicontinuous)**.** .

A function $f : X \to \mathbb{R}$ is said to be ***lower semicontinuous, lsc,*** if $\{x \in X \mid f(x) > \alpha\}$ is $\underline{\text{open}}$ / ***upper semicontinuous, usc,*** if $\{x \in X \mid f(x) < \alpha\}$ is $\underline{\text{open}}$ **for all** $\alpha \in \mathbb{R}$.

A function $f : X \to \mathbb{R}$ is said to be f
lower semicontinous at $x_0 \iff \liminf_{x \to x_0} f(x) \geq f(x_0)$
upper semicontinuous at $x_0 \iff \limsup_{x \to x_0} f(x) \leq f(x_0)$

Note: "l.s.c. can jump down not up"
"u.s.c. can jump up not down"

Definition 0.0.5 (Integration of Positive Function). Given $f : X \to [0, \infty] \in \mathfrak{M}, E \in \mathfrak{M}(X)$

$$\int_E f d\mu = \sup \int_E s d\mu$$

supremum over all simple functions $0 \leq s < f$.

Theorem 0.0.6. *Let E_i be a partition on X then*

$$\int_X f d\mu = \int_X \sum_i f \chi_{E_i} d\mu$$

Theorem 0.0.7 (Lebesque Monotone Convergence Theorem)**.** *Given an increasing sequence of measurable functions $f_n : X \to [0, \infty] \in \mathfrak{M}$ where $f_n \to f$. Then,*

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu$$

Theorem 0.0.8 (Fatou's Lemma)**.** *Given an increasing sequence of measurable functions $f_n : X \to [0, \infty] \in \mathfrak{M}$ where $f_n \to f$. Then,*

$$\int_X \liminf_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int_X f_n d\mu$$

Theorem 0.0.9 (Lebesque Dominated Convergence Theorem)**.** *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

$$f(x) = \lim_{n \to \infty} f_n(x) \tag{1}$$

exists for evey $x \in X$. If there is a function $g \in L^1(X)$ (Lebesque Measurable) such that

$$|f_n(x)| \leq g(x) \ (n = 1, 2, \ldots, ; x \in X) \tag{2}$$

then $f \in L^1(X)$ (Lebesque Measurable),

$$\lim_{n \to \infty} \int_X |\, f_n - f \,| \, d\mu = 0, \tag{3}$$

and

$$\lim_{n \to \infty} \int_x f_n(x) d\mu = \int_x f d\mu. \tag{4}$$

Understanding 'almost everywhere', a.e. $[\mu]$

Remark 0.0.10 (Measure Zero). .

This term is almost euphemistically used and might be thought of as 'countably infinite set of distinct points'. For example, the set of rational numbers has a measure zero, is countably infinite and even dense, as are many other sets (Cantor, set of algebraic numbers). Thus giving rise to the real value of *measure* as indicating sets that actually contain real numbers (that is distiinct from rationals, integers and so on) and their neighborhoods. I have come to think of a 'measurable set' as 'not a point'.

Definition 0.0.11 (almost everywhere)**.** .

If μ is a measure on a σ-algebra $\mathfrak{M}$ and if $E \in \mathfrak{M}$, the statement 'property P holds *almost everywhere* on E' means that $\exists N \in \mathfrak{M} \to N \subset E, \mu(N) = 0$ and P holds for every $x \in E \backslash N$.

Theorem 0.0.12. *Suppose* $\{f_n\}$ *is a sequence of complex measuable functions defined* **a.e. on** X *such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for **almost all** $x, f \in L^1(\mu)$*, and*

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

Remark 0.0.13 (Local Compact Hausdorff Space). .

See page 35. The primary property that allows for Λ to be a measure on $\mathbb{R}^n$ is independent of measure and independent of geometry (i.e., inner product, orientation, etc.). In fact, the primary property is ***local compactnesss*** which is that every open set contains a neighborhood whose closer is compact (think, closed and bounded, finite subcover).

A ***Hausdorff Space, (T-4 spaces),*** means that given any two points, A and B, there exists neighborhoods around each that are disjoint. That is there exists $r, s \in \mathbb{R}$ such that $B_r(A) \cap B_s(B) = \emptyset$. (Incidentally, you could just choose $t = \min\{r, s\}$ and have n-balls of the same size).

All metric spaces are Hausdorff Spaces. Thus, the primary property for Lebesque Measure Theory is the separation of points and that they are surround by compact sets.

Properties of compactness.

- Closed sets within compact sets are compact.
- $A \subset B$ and B has a compact closure than so does A.
- Compact subsets of Hausdorff spaces are closed.
- If F is closed and K is compact in a Hausdorff space then $F \cap K$ is compact.
- If $\{K_\alpha\}$ is a collection of disjoint compact subsets in a Hausdorff space, the some finite subcollection of $\{K_\alpha\}$ are disjoint.
- U is an open set in a locally compact Hausdorff Space. $K \subset U$ is compact. Then there exists V "between" K and U (that is, $K \subset V \subset \overline{V} \subset U$).
- Compactness is invariant through continuity.
- range(f) is compact (for $f \in C_c(X)$).
- **Notation** $K \prec f$ means K is a compact set of X, $f \in C_c(X)$ and $0 \leq f \leq 1, \forall x \in X$ and $f(x) = 1, \forall x \in K$. Further,

  $$f \prec V$$

  V is open $0 \leq f(x) \leq 1, \forall x \in X$ and $\operatorname{spt}(f) \subset V$.

Definition 0.0.14 (Support of a function)**.** .
The ***support*** of a complex function on a topological space X is the closure of set

$$\operatorname{spt}(f) = \overline{\{x : f(x) \neq 0\}}.$$

[4] The collection of all continuous, complex functions with compact supports is denoted $C_c(X)$ (a vector space).

[4]Consider this in contrast to the kernel of a linear function and in contrast to the domain.

The Riesz Representation Theorem.

Theorem 0.0.15. *Let X be a locally compact Hausdorff space, and let Λ be a positive linear function on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains **all** Borel Sets in X, and there exists a **positive unique measure** μ in $\mathfrak{M}$ which represents Λ in the sense that*

(a) $\Lambda f = \int_x f d\mu$ *for every* $f \in C_c(X)$.

and which has the following additional properties;

(b) Compact implies finite: $\mu(K) < \infty$ *for every compact set* $K \subset X$.

(c) Outer Regular: For every $E \in \mathfrak{M}$, *we have*

$$\mu(E) = \inf \{\, \mu(V) : E \subset V \text{ open} \}$$

(d) Inner Regular: The relation

$$\mu(E) = \sup \{\, \mu(K) : K \subset E, K \text{ compact} \}$$

holds for every open set E *and every* $E \in \mathfrak{M}$ *with* $\mu(E) < \infty$.

(e) If $E \in \mathfrak{M}, A \subset E$, *and* $\mu(E) = 0$, *then* $A \in \mathfrak{M}$.

For the sake of clarity, let us be more explicit about the meaning of the word "positive" in the hypothesis: Λ *is assumed to be a linear functional on the complex vector space with the additional property that* Λf *is a nonnegative real number for every* f *whose range consists of nonnegative real numbers.*

Extensions

(f) If $E \in \mathfrak{M}$ *and* $\epsilon > 0$ *then there is a closed set* F *and an open set* V *such that* $F \subset E \subset V$ *and* $\mu(V - F) < \epsilon$.

(g) μ *is a regular Borel Measure on* X.

(h) and $\exists A \in F_\sigma$ *and* $B \in G_\delta$ *such that* $A \subset E \subset B$, *and* $\mu(B - A) = 0$.

Theorem 0.0.16 (Lusin's Theorem)**.** .
Suppose that X is a complex measureable function on $X, \mu(A) < \infty, f(x) = 0$ if $x \notin A$, and $\epsilon > 0$. Then there exists $g \in C_c(X)$ such that

$$\mu\left(\{x \in X : f(x) \neq g(x)\} \right) < \epsilon$$

(i.e., those x that define how f and g are different is miniscule or form an infinitely small set). Furthermore, we may arrange it so that

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|.$$

Proof. .

Step 1: First consider the case where X is compact and $0 < f < 1$.

Step 2: Next consider the case where X is compact and $f : X \to [0, \infty)$ is a bounded measurable function.

Step 3: Next consider the case where X is compact and $f : X \to (-\infty, \infty)$ is a bounded measurable function.

□

The abuse of terminology.

What I have found most frustrating is understanding fundamental concepts in Measure Theory. What I've come to realize is the misuse or abuse of terms that mathematicians have, up until now, have all accepted but are now being pushed to their limits. Typically, when you are talking about sets in the context of functions we think of them as domain, most of them are considered to be open, smooth, continuous or connected. They don't have to be but there is an assumed behavior that isn't assumed in Measure Theory. In truth, they should make that very clear and specify what they mean from the start but they never do. I've listed the terms whose meaning in context in Measure Theory has broader and more specific implications.

Set: What is meant here should be termed 'collections' which really is the same thing as the definition of 'set' but meant to regard each of its elements as either sigletons or collections of some kind be they open, closed or neither or both. Topologies are collections of sets and so are σ-algebras, further they they don't somehow connect these two elements seems remarkable strange.

Function: Up unto this point, functions have been regarded mostly as expressions that are operated on and some emphasis has been on domain and range. However, what is really considered here is not how a function operates on a single value but how it operates on sets of values. Thus, they define a function $f : X \to \mathbb{R}$ but constantly improperly refer to $f(A)$ where $A \subseteq X$. Now I refer to something that I call a "punction". A ***punction,*** p_f is defined as, given a function $f : X \to \mathbb{R}$, $p_f : D \subseteq 2^X \to R \subseteq 2^{\mathbb{R}}$. Further, $p_f^{-1} : R \to D$ and we'll see that if f is continuous/meaqsurable then p_f is 1-to-1 and onto. I believe that this is the more natural way to interpret much of Measure Theory.

Measurable Sets ¿ 0. These are best thought of as 'not a points'. Further, open sets are 'not a points' as they have this cloud-like feel about them.