Math 725 – Advanced Linear Algebra

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Assignment #11 – Due 12/8/23

1. Let A be an invertible square matrix. Show that $|\det A| = \sigma_1\sigma_2\cdots\sigma_n$.

$$\begin{aligned} A &= U\Sigma V^T \\ |\det A| &= |\det(U\Sigma V^T)| \\ &= |\det(U)\det(\Sigma)\det(V^T)| \\ &= |\det(\Sigma)| \\ &= |\sigma_1 \cdot \sigma_2 \cdots \sigma_n| \\ &= \sigma_1 \cdot \sigma_2 \cdots \sigma_n \end{aligned}$$

because U and V are orthonormal, their determinant is one and all $\sigma_i > 0$.

2. Let A be a nonzero $m \times n$ matrix. Prove that $\sigma_1 = \max\{||Au|| : ||u|| = 1\}$.

Suppose that this is not true. Then let $\mu = \max\{||Au|| : ||u|| = 1\}$ and let u' be such that $\mu = ||Au'||$ and $||u'|| = 1$. Then, $\sigma_1 - \mu \geq 0$ because σ_1 is the greatest eigenvalue which implies that $||Au'|| = \mu < ||\sigma_1 u'|| = \sigma_1$ hence a contradiction.

3. Let A and A' be two nonzero $m \times n$ matrices with respective largest singular values σ_1 and σ_1'. Prove that the largest singular value of $A + A'$ is bounded above by $\sigma_1 + \sigma_1'$.

$$\begin{aligned} ||(A + A')x|| &\leq ||Ax|| + ||A'x|| \\ &\leq (||A|| + ||A'||)||x|| \\ &\leq (\sigma_1 + \sigma_1')||x|| \end{aligned}$$

4. Suppose A is an $m \times n$ matrix and B is $n \times m$ matrix obtained by rotating A ninety degrees clockwise on paper (not a standard matrix operation). Do A and B have the same singular values? Prove or give a counterexample.

$$\text{Let } A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n & 0 \end{pmatrix}$$

$$\text{Then } B = \begin{pmatrix} 0 & \cdots & 0 & \lambda_1 \\ 0 & \cdots & \lambda_2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Clearly, A has distinct eigenvalues of λ_i which are not at all the same for B.

5. Let A be an $m \times n$ matrix of rank $r > 0$ with singular values $\sigma_1, \ldots, \sigma_r$. Show that $||A||_F = \sqrt{\sigma_1^2 + \ldots + \sigma_r^2}$.

Let the SVD of $A = U\Sigma V^T$ then

$$\begin{aligned}
||A||_F &= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}^2} \\
&= ||A^T|| \\
||A^T A|| &= ||(U\Sigma V^T)^T U\Sigma V^T|| \\
&= ||V\Sigma^T U^T U\Sigma V^T|| \\
&= ||V\Sigma^T \Sigma V^T|| \\
&= ||\Sigma^T \Sigma|| \\
&= \sum_{i=1}^{r} \Sigma_{ii}^2 \\
&= \sum_{i=1}^{r} \sigma_i^2
\end{aligned}$$