Homework 1
MTH 5230 (PDE), Fall 2025
Due Wednesday, September 3 in class

Please show your work and justify all answers.

Part I:

1. (a) Consider an initial value problem for the linear transport equation with a bounded, one-dimensional spatial domain:

$$\begin{cases} u_t + 3u_x = 0, & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = 0. \end{cases}$$

(The last line gives a *boundary condition* for u at $x = 0$, which we did not have in the lecture because we were considering the case where the spatial domain is all of $\mathbb{R}^n$, so there was no spatial boundary.)

We assume that $g(0) = 0$, so that the initial condition and boundary condition agree at the corner $(x,t) = (0,0)$.

Find a formula for $u(x,t)$ using the same method as was seen in class, i.e. use the fact that a certain directional derivative of u is zero. What do you notice about your solution for large times?

(b) Next, derive a solution formula for the same problem with a more general boundary condition and source term:

$$\begin{cases} u_t + 3u_x = f(x,t), & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = h(t). \end{cases}$$

We assume that $g(0) = h(0)$, so that the initial condition and boundary condition agree at the corner $(x,t) = (0,0)$.

(c) Why did we only specify the boundary condition at the left-hand boundary $(x = 0)$, not the right-hand boundary $(x = 1)$? In other words, what would go wrong if we specified boundary conditions on both sides, as in

$$\begin{cases} u_t + 3u_x = f(x,t), & 0 < x < 1, t > 0, \\ u(x,0) = g(x), \\ u(0,t) = h_0(t), \\ u(1,t) = h_1(t), \end{cases}$$

for some given functions $h_0(t)$ and $h_1(t)$? (We can assume $g(0) = h_0(0)$ and $g(1) = h_1(0)$, so that the initial and boundary conditions agree at the corners.)

2. In one space dimension, Laplace's equation $\Delta u = 0$ becomes an ODE:

$$u''(x) = 0.$$

Describe all solutions to this ODE posed on the real line $(-\infty, \infty)$. Next, for given constants c_1, c_2, find the unique solution to $u''(x) = 0$ on the interval $[0, 1]$ with $u(0) = c_1$ and $u(1) = c_2$.

Do these one-dimensional harmonic functions satisfy the mean value property $u(x) = ⨍_{B(x,r)} u(y)\,\mathrm{d}y$? Why or why not?

3. Let $z = x + iy$ be a complex variable (x and y are real numbers). Recall that a complex function $f(z) = u(x, y) + iv(x, y)$ is *complex-differentiable* if u and v are continuously differentiable (as functions of x and y) and the Cauchy-Riemann equations are satisfied:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

If f is complex-differentiable, and in addition the real and imaginary parts $u(x, y)$ and $v(x, y)$ are C^2 (twice continuously differentiable) functions, then show that u and v are harmonic.

In fact, the extra assumption that u and v are C^2 is not truly needed here, since complex-differentiable functions defined in any open subset of the complex plane are automatically C^∞ and analytic, but that is outside the scope of this problem.

Part II:

Evans, Chapter 2, problems 1, 2, 5(a,b,d).

Hint for problem 2: Recall that an $n \times n$ matrix O is orthogonal if $O^t O = I$, or in other words,

$$\sum_{i=1}^{n} O_{ji} O_{ki} = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Let $y = Ox$, i.e. $y_i = \sum_{k=1}^{n} O_{ik} x_k$, and apply the multivariable chain rule to find $\Delta v = \sum_{i=1}^{n} v_{x_i x_i}$ in terms of y-derivatives of u.