Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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Homework #5 – December 11, 2024

I. Section 7.4, Problem 10. *Exercise 7.4.10.* Let $A \subseteq B \subseteq \mathbb{R}^n$. Show that if B is Lebesque measurable with measure zero, then A is also Legesque measurable with measure zero.

Assuming that $\mathbb{R}^n$ forms a complete measure space, $A \subseteq B$ implies that A is measurable
Recall the measure of a set is always non-negative

$$A \subseteq B \to m(A) \leq m(B) = 0$$
$$\therefore m(A) = 0$$

II. Section 7.5, problem 5

Exercise 7.5.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Legesque measurable, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a function which agrees with f outside of a set of measure zero, thus there exists a set $A \subseteq \mathbb{R}^n$ of measure zero that $f(x) = g(x)$ for all $x \in \mathbb{R}^n \backslash A$. Show that g is also Lebesque measurable. (Hint: Use Exercse 7.4.10)

Let $B \subseteq \mathbb{R}$. Then,

$$g^{-1}(B) = (g^{-1}(B) \cap \mathbb{R}^n \backslash A) \cup (g^{-1}(B) \cap A)$$

we know that $f(x) = g(x), x \in \mathbb{R}^n \backslash A$ so

$$g^{-1}(B) \cap \mathbb{R}^n \backslash A = f^{-1}(B) \cap \mathbb{R}^n \backslash A$$

and $f^{-1}(B) \cap \mathbb{R}^n \backslash A$ is measurable. Also, $g^{-1}(B) \cap A$ is also measurable because A is measurable. Therefore, $g^{-1}(B)$ is measurable as it is the union of two measurable sets. Hence g is a measurable function.

III. Let $f : \Omega \to [0, \infty)$ be measurable, $\Omega = \bigcup_{k=1}^{\infty} \Omega_k \in \mathcal{M}, \Omega_k \subseteq \Omega_{k+1}$ for all k. Then

$$\int_{\Omega} f = \lim_{k \to \infty} \int_{\Omega_k} f$$

Remark: If f is simple, then the result is precisely Lemma 4.27.

IV. Show that

$$\lim_{n \to \infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx = \int_{[0,\infty]} e^{-x} dx$$

.

$$\begin{aligned}
\lim_{n \to \infty} \int_{[0,n]} \left(1 + \frac{x}{n}\right)^n e^{-2x} dx &= \lim_{n \to \infty} \int_{[0,n]} \lim_{k \to \infty} \left(1 + \frac{x}{k}\right)^k e^{-2x} dx \\
&= \lim_{n \to \infty} \int_{[0,n]} e^{-x} dx \\
&= \int_{[0,\infty]} e^{-x} dx
\end{aligned}$$

V. If $f \in L(\Omega)$, then

$$\lim_{r \to \infty} \int_{\Omega \backslash B_r} f = 0$$

Note: Recall B_r is a the r-ball at the origin. If Ω is bounded then eventually $\Omega \backslash B_r = \emptyset$ (in this case integral is regarded to be zero) but our Ω maybe unbounded.

Choose n such that $\{g > n\} \subseteq \Omega \backslash B_r$. Then, we have

$$m(\{g > n\}) \leq m(\Omega \backslash B_r)$$

As $r \to \infty$ continue to choose higher values of n, eventually $n \to \infty$ and we know that $\lim\limits_{n \to \infty} m(\{g > k\}) = 0$ thus $m(\Omega \backslash B_r) \to 0$.

VI. skip

VII. (a) Let $f \geq 0$ be integrable on $[a, b]$. Prove that the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$. (Hint: for fixed x, use the Dominated Convergence Theorem to show that $F(x+1/n) = F(x) \to 0$ and $F(x - 1/n) - F(x) \to 0$ as $n \to \infty$. Then use this to prove continuity of F at x.)

(b) Assume f is Riemann integrable on $[a, b]$, and let F be defined as in (a). Show that F is differentiable almost everywhere, and the equality $F'(x) = f(x)$ is true almost everywhere. (The same is true for any (Lebesque) integrable function f, but this is harder to prove.)

VIII. Find an example of a uniformly bounded sequence of functions $f_n : \mathbb{R} \to [0, \infty)$ so that each f_n is Riemann integrable, but f_n converges pointwise to a function that is not Riemann integrable.

(We know this problem can't occur with the Lebesque integral, because a pointwise limit of measurable functions is measuable.)

Step 1: divide $[0, 1]$ into thirds then $f_1 = 1$ for the middle third and zero otherwise.
Step 2: divide each subinterval of Step 1 into thirds then $f_2 = 1$ for the middle of each of these subintervals
Step 3: repeat step 2 using in the subintervals of Step 2 defining f_3 accordingly
Repeat: Each f_n is Rieman Integral. $f_n \to f$ is not.

IX. Suppose $\rho : [0, \infty) \to \mathbb{R}$ is decreasing and continuous, $m(E) = m(B_r)$, where E is a measurable subset of $\mathbb{R}^n$ and $B_r \subset \mathbb{R}^n$ is the r-ball at the origin. Show that

$$\int_E \rho(|x|)dx \leq \int_{B_R} \rho(|x|)dx.$$