Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Notes – January – May, 2025

Definitions

1. **Diffeomorphism**: If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then f is said to be a **diffeomorphism**. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be **diffeomorphic** to each other.

2. **Tangent Space** at a point p. The set of all vectors rooted at p, written as $T_p(\mathbb{R}^n)$.

3. **Derivations**: any operation that supports the Liebniz Rule $(D(fg) = (Df)g + fDg)$.

4. **Derivation Space**. $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p. This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ defined as

$$\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$$
$$v \mapsto D_v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p .$$

5. **Germ**: equivalence class of functions whose derivatives around a point are the same.

6. Vector Field vs Vector Space.

   - **A Vector Field** a function that assigns a vector to every point in the subset U.

   $$f : (U \subset \mathbb{R}^m) \to T_p(\mathbb{R}^n)$$
   $$X \mapsto X_p = \sum a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p .$$

   consider a^i as coeffecient functions. We say that X is C^∞ on U if $a^i \in C^\infty$, $\forall i = 1, \ldots, n$.
   - **A Vector Space** is any abstraciton that is closed under addition and scalar multiplication.

7. **Dual Basis and Dual Space**. The **Dual Basis** is a set of functions $\alpha^i : V \to \mathbb{R}$

$$\alpha^i : V \to \mathbb{R}$$
$$\alpha^i(e_j) = \delta^i_j$$

   the **Dual Space** $V^\vee$ is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called **Functionals (Analysis)/1-Covectors (Differential Geometry)**.

8. **Multi-Linear Functions** Let V be a vector space and V^k be k-tuples of vectors in V. A K-**linear map or** k-**tensor** $f : V^k \to \mathbb{R}$ such that each ith component is linear. The vector space of all k-tensors on V is denoted $L_k(V)$.

   Permuting Mult-linear Functions. Given any permutation $\sigma \in S_k$

$$f(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

   e.g., $f(x, y, z) = xyz \to f(z, x, y) = zxy$. FYI: if x, y, z are from non-commutative rings (i.e., matrices) then we must be aware of the $\text{sgn}(\sigma)$.

9. Left R-**Module**: An Abelian group R with a scalar multiplication map:

$$\mu : R \times A \to A$$

   usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a

   (i)   (associative) $(rs)a = r(sa)$.
   (ii) (identity) $1a = a$ (1 is a multiplicative identity).
   (iii) (distributivity) $(r + s)a = ra + sa$ and $r(a + b) = ra + rb$.

   If R is a field then R-module is precisely a vector space over R.

   A K-**Algebra over a field** K is also a ring A that is also a vector space over K such that the ring multiplication satisfies homogeniety (scalar distributes over vector multipliation to only one of the operators).

   A **graded Algebra** is an algebra A over a field K if it can be writte as the direct sum

$$A = \bigoplus_{i=0}^\infty A^i$$

   of vector spaces over K such that the mupltiplication map sends $A^k \times A^l \to A^{k+l}$

10. The set of all C^∞-vector fields on U, denoted by $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the $C^\infty(U)$ ring.

$$\mathfrak{X}(U) = \{\, X : V \to V \mid X \in C^\infty(U) \,\} \text{ where } V = (\mathbb{R} \text{ or } \mathbb{C})^n$$

11. ***Derivation:*** A ***derivation*** on an algebra A is a K-multilinear function $D : A \to A$ such that

$$D(ab) = (Da)b + aDb, \ \forall a, b \in A$$

known as the ***Liebniz Rule***.

The set of all derivations on A forms a vector space, $\operatorname{Der}(C^\infty(U))$. Thus a $C^\infty(U)$ vector field gives rise to a derivation of the algebra $C^\infty(U)$. Thus the mapping

$$\varphi : \mathfrak{X}(U) \to \operatorname{Der}(C^\infty(U))$$
$$X \mapsto (f \mapsto Xf)$$

this map is an isomorphism of vector spaces.

12. **Exterior Algebras** $\Lambda(V)$**.** The exterior algebra $\Lambda(V)$ is obtained by imposing an **anti-commutative** relation:

$$v \otimes w + w \otimes v = 0, \ \forall v, w \in V$$

this means that the quotient algebra is:

$$\Lambda(V) = T(V)/\langle\, v \otimes w + w \otimes v \,\rangle.$$

Where $T(V)$ is the **tensor algebra**

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

13. **Symmetric Algebras** $S(V)$**.** The symmetric algebra $S(V)$ is obtained by imposing an **commutative** relation:

$$v \otimes w - w \otimes v = 0, \ \forall v, w \in V$$

this means that the quotient algebra is:

$$S(V) = T(V)/\langle\, v \otimes w - w \otimes v \,\rangle.$$

14. ***Tensor Product*** The tensor product between two 1-covectors, $f, g : V \to \mathbb{R}$ is the 2-covector $f \otimes g$.

$$(f \otimes g)(u, v) = f(u)g(v)$$

. In general, the tensor product of a k-covector $p : V^k \to \mathbb{R}$ with a l-covector $q : v^l \to \mathbb{R}$ is the $(k + l)$-covector $p \otimes q : V^{k+l} \to \mathbb{R}$.

$$(p \otimes q)(u, v) = p(u)q(v), \ \forall u \in V^k, \ v \in V^l$$

15. ***Tensor Product(?)*** is an operator on $v \in V$ and $u \in U$ where

$$v \otimes u : V \times U \to V \oplus U$$
$$(v \otimes u)_{i \cdot j} = v_i \cdot u_j, \ \forall i = 1, \ldots, \dim(V), \ j = 1, \ldots, \dim(U)$$

Given two vector spaces V, W with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

α_i, β_j are all scalars. The real issue is the behavior of unit basis vectors v_i, w_j and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

16. ***Wedge Product***

Between two covectors Let $f, g \in L_1(V)$ then for all $u, v \in V$

$$(f \wedge g)(u, v) = (f \otimes g)(u, v) - (g \otimes f)(u, v) = f(u)g(v) - f(v)g(u)$$

Between mulitple 1-covectors.

$$(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \ldots, v_k) = \det[\alpha^1(v_j)]$$
$$= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)})$$

Between k-covector and l-covector. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A_{k+l}(V)$$

or explicitly

$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f\left(v_{\sigma 1}, \ldots, v_{\sigma_k}\right) g\left(v_{\sigma_{k+1}}, \ldots, v_{\sigma k+l}\right)$$

Anticommutative. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$(f \wedge g) = (-1)^{kl} g \wedge f$$

17. **Differential k-Forms**

    1-forms, covectors

$$(dx^i)\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta^i_j$$
$$(df)_p(X_p) = X_p f = \sum a^i(p) \left.\frac{\partial f}{\partial x^i}\right|_p = \sum \frac{\partial}{\partial x^i} dx^i$$

18. $\Omega^k(U)$, **Vector space of C^∞ k-forms on U.**

    $\Omega^0 = A_0(T_p(\mathbb{R}^n)) = C^\infty(U)$, e.g., $f \in \Omega^0$ then $f : V \to \mathbb{R}$ is a functional/covector/1-tensor.

    Elements of 1-form $\Omega^1 = A_1(T_p(\mathbb{R}^n))$. For example, when $n = 3$

$$f dx + g dy + h dz, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

    Elements of 2-form $\Omega^2 = A_2(T_p(\mathbb{R}^n))$. For example, when $n = 3$[1]

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

    if $n = 4$, that is coordinates for u, v, w, x. Each form is derived from these bases

        0-form $\Omega^0(\mathbb{R}^4) \in \mathbb{R}$

        1-forms $\Omega^1(\mathbb{R}^4)$ summing du, dv, dw, dx,

        2-forms $\Omega^2(\mathbb{R}^4)$ summing $du \wedge dv, du \wedge dw, du \wedge dx, dv \wedge dw, dv \wedge dx, dw \wedge dx$,

        3-forms $\Omega^3(\mathbb{R}^4)$ summing $du \wedge dv \wedge dw \mid du \wedge dw \wedge dx \mid du \wedge dv \wedge dx \mid dv \wedge dw \wedge dx$

        4-form $\Omega^4(\mathbb{R}^4)$ $du \wedge dv \wedge dw \wedge dx$.

    Also, $U \subseteq \mathbb{R}^n$ **then** $k < n$. k-forms for $k > n$ are zero. Further $|\Omega^k(\mathbb{R}^n)| = \binom{k}{n}$ and $\left|\bigcup_k \Omega^k(\mathbb{R}^n)\right| = 2^n$ and think of $\Omega^*(U) = \bigcup_k \Omega^k(\mathbb{R}^n)$

    Direct Sum. $\Omega^*(U) = \bigoplus_k \Omega^k(U)$ is an anti-commutative graded algebra over $\mathbb{R}$.

        Since one can multiply C^∞ k-forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k-forms is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$ and $\Omega^*(U)$ is also a module over C^∞ of C^∞ functions.

19. **Wedge Product of k-form.** Recall: $dx^i \wedge dx^i = 0$ for all $i = 1, \ldots, n$. Therefore, $\wedge$ only makes sense to be defined on *disjoint indice-lists*, that is, $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$ such that $I \cap J = \emptyset$. Then,

$$\wedge : \Omega^k(U) \times \Omega^l(U) \to \Omega^{k+l}(U)$$
$$(\omega, \tau) \mapsto (\omega \wedge \tau) = \sum_{I,J} a_I b_J dx^I \wedge dx^J.$$

where $\omega = \sum_I a_I dx^I, \tau = \sum_J b_J, dx^J$.

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[1]NOTE the cyclic order of the indices x, y, z. Switching any one of these will flip the sign.

20. **the Exterior Derivative**. If $k \geq 1$ and if $\omega = \sum_I a_i dx^I \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$ and

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_J \frac{\partial a_I}{\partial x_J} dx^J \right) \wedge dx^I$$

Example: Let $\omega \in \Omega^1(\mathbb{R}^2)$ and $\omega = f dx + g dy$, $f, g \in C^\infty(\mathbb{R}^2)$.

$$d\omega = df \wedge dx + dg \wedge dy$$
$$= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy$$
$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$
$$= (g_x - f_y) \, dx \wedge dy$$

Definition: Let $\bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K. An **anti-derivation** of the graded algebra A is a K-linear map $D : A \to A$ such that $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a Db$$

Proposition 4.7: Three Criterion for an Exterior Derivation

i) The **exterior derivation** $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

ii) $d^2 = 0$.

iii) If $f \in C^\infty$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

NOTE: "In a typical school, there would be graduate level courses on Smooth Manifods and another on Remannian Manifolds."

Q: What is the difference between $\mathfrak{X}(U)$ and $C^\infty(U)$?

The difference between $\mathfrak{X}(U)$ and $C^\infty(U)$ lies in the types of objects they contain:

1. **$C^\infty(U)$: The Space of Smooth Functions** - $C^\infty(U)$ consists of all smooth (infinitely differentiable) real-valued functions defined on an open subset U of a manifold M. - Elements of $C^\infty(U)$ are scalar functions $f : U \to \mathbb{R}$. - These functions can be added and multiplied pointwise, forming an algebra over $\mathbb{R}$.

2. **$\mathfrak{X}(U)$: The Space of Smooth Vector Fields** - $\mathfrak{X}(U)$ consists of all smooth vector fields on U. - A vector field X assigns to each point $p \in U$ a tangent vector $X_p \in T_pM$, smoothly varying with p. - Vector fields act as derivations on smooth functions, meaning they satisfy the Leibniz rule:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(U).$$

- The space $\mathfrak{X}(U)$ forms a module over $C^\infty(U)$, meaning smooth functions can scale vector fields: if $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then fX is also a vector field.

Key Differences

Feature	$C^\infty(U)$	$\mathfrak{X}(U)$
Elements	Smooth scalar functions $f : U \to \mathbb{R}$	Smooth vector fields $X : U \to TM$
Algebraic Structure	Commutative algebra (pointwise multiplication)	Module over $C^\infty(U)$, noncommutative under Lie bracket
Operations	Addition, multiplication	Addition, scalar multiplication by $C^\infty(U)$, Lie bracket $[X, Y]$

In summary, $C^\infty(U)$ consists of smooth functions, while $\mathfrak{X}(U)$ consists of smooth vector fields, which act as differential operators on $C^\infty(U)$.

Compare and contrast.

Set	Dim	index	basis	Delta
$L_1(U)$	n	$i = 1, \ldots, n$	α^i	$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
$L_k(U)$	n^k	$I, J \in \underbrace{\{i_i, \ldots, i_k\}}_{k \text{ times}}, i_k \in [1, \ldots, n]$	$\alpha^I = \alpha^{i_1} \otimes \alpha^{i_2} \otimes \cdots \otimes \alpha^k$	
$A_k(U)$	$\binom{n}{k}$	$I, J \in \underbrace{\{i_i, \ldots, i_k\}}_{k \text{ times}}, i_1 < i_2 < \ldots i_k \in [1, n]$	$\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^k$	$\delta_I^J = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$

Supersets

Symbol	Name (set of)	Definition	Example
$\Omega^0(U)$	0-forms	{ scalar fields }	$f : V \to \mathbb{R}$ $f(x, y, z)$
$\Omega^1(U)$	1-forms	{ 1-forms, vector fields }	$d\omega(v) = A(v)dx + B(v)dy + C(v)dx$ $A, B, C : V \to \mathbb{R}$
$\Omega^k(U)$	k-forms	{ k-forms }	$\cdots + dx^1 \wedge \cdots \wedge dx^k + \cdots$
$\Omega^*(U)$	sum of k-forms	{ $x = \sum y \,|\, y \in \oplus_k \Omega^k(U)$ }	$Adx + Bdx \wedge dy + Cdx \wedge dy \wedge dz, \; A, B, C : V \to \mathbb{R}$
$\mathfrak{X}(U)$	vector fields on U	$\{X \to \exists f : U \to U\}$	
$C^\infty(U)$	smooth functions on U		
$X_p = T_p(U)$	a vector field at p	$\{v \in U \,|\, v = p + x \text{ for some } x \in U\}$	

$$\text{Map of } \Omega^k(\mathbb{R}^3)$$

$$
\begin{array}{ccccccc}
\Omega^0(U) & \xrightarrow{\;\mathrm{d}\;} & \Omega^1(U) & \xrightarrow{\;\mathrm{d}\;} & \Omega^2(U) & \xrightarrow{\;\mathrm{d}\;} & \Omega^3(U) \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
C^\infty(U) & \underset{\text{grad}}{\longrightarrow} & \mathfrak{X}(U) & \underset{\text{curl}}{\longrightarrow} & \mathfrak{X}(U) & \underset{\text{div}}{\longrightarrow} & C^\infty(U).
\end{array}
$$

Shorthand

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_I a_I = \sum_{n=1}^k a_{i_n}$$

$$\sum_{I,J} a_I b_J = \sum_{n=1}^k a_{i_n} \sum_{m=1}^k b_{j_m}$$

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_I^J = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_k}^{j_k} = \begin{cases} 1 & i_n = j_n, \forall n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$