

Math 5111 – Real Analysis II– Sprint 2025

w/Professor Perera

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Homework (not graded) – May 2025

Pg 30, 1

Does there exist an infinite σ -algebra which has only countably many members?

First we establish the smallest possible measurable set containing a point, B_x . Then demonstrate that these are unique and distinct (that is, $x \neq y \implies B_x \cap B_y = \emptyset$). Then assuming that there are countably many B_x show that there are still some missing points not in the $\bigcup_{x \in \mathbb{Q}} B_x$.

Pg 30, 2

Prove an analogue of Theorem 1.8 for n functions.

Pg 30, 3

Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r , then f is measurable.

Let $E_r = \{x : f(x) \geq r\}$ each of which is measurable and $E = \bigcup_{r \in \mathbb{Q}} E_r$. E is measurable because it is the countable union of measurable sets. Given any measurable set $I \in \mathbb{R}$ we can clearly see that $f(E) \cap I \neq \emptyset$ because \mathbb{Q} is dense.

$$f(E) = f\left(\bigcup_{r \in I \cap \mathbb{Q}} E_r\right)$$

We are saying that I is measurable, therefore it can be made up of the union of disjoint measurable sets A and B where $B = I \cap f(E)$ and $A = I \setminus B$. $f^{-1}(I) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. We can see that $f^{-1}(B)$ is measurable, How do we show that $f^{-1}(A)$ is?

Notice, $E_r \in \mathfrak{M} \implies E_r^c \in \mathfrak{M}$. Now, we can try to connect $f^{-1}(A)$ to these sets.

Pg 30, 4

Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a) $\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} a_n$.

$$\begin{aligned} \sup_{n \rightarrow \infty} \{-a_n\} &= -\inf_{n \rightarrow \infty} \{a_n\} \\ \limsup_{n \rightarrow \infty} \{-a_n\} &= \lim_{n \rightarrow \infty} \sup_{k > n} \{-a_k\} = \lim_{n \rightarrow \infty} (-\inf_{k > n} \{a_k\}) = -\liminf_{n \rightarrow \infty} \{a_n\} \end{aligned}$$

(b) $\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$ provided none of the sums is $\infty - \infty$.

$$\limsup_{n \rightarrow \infty} \{a_n + b_n\} = \lim_{n \rightarrow \infty} \sup_{k > n} \{a_k + b_k\}$$

keep mind that $a_k + b_k$ could be zero, but $\sup a_k + \sup b_k$ could only be zero after evaluating both sequences. Thus,

$$\begin{aligned} \sup_{k > n} \{a_k + b_k\} &\leq \sup_{k > n} \{a_k\} + \sup_{k > n} \{b_k\}, \forall n \\ \lim_{n \rightarrow \infty} \sup_{k > n} \{a_k + b_k\} &\leq \lim_{n \rightarrow \infty} \sup_{k > n} \{a_k\} + \lim_{n \rightarrow \infty} \sup_{k > n} \{b_k\}, \forall n \\ \limsup_{n \rightarrow \infty} \{a_n + b_n\} &\leq \limsup_{n \rightarrow \infty} \{a_n\} + \limsup_{n \rightarrow \infty} \{b_n\} \end{aligned}$$

(c) if $a_n \leq b_n$, for all n , then

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} b_n$$

Show by example that strick inequality can hold in (b).

Pg 30, 5

(a) Suppose $f : X \rightarrow [-\infty, \infty]$ and $g : X \rightarrow [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

are measurable.

Given any $y \in X$ let $s_y = \{x : f(x) < g(y)\}$. f is measurable implies that s_y is measurable. Now taking the intersection over all $y \in X$, that is

$$\bigcup_{y \in X} s_y = \{x : f(x) < g(x)\} \in \mathfrak{M}$$

(b) Prove that the set of points at which a sequence of measurable real-value functions converges (to a finite limit) is measurable.

Let $f, f_n : X \rightarrow [0, \infty] \in \mathfrak{M}$ and let $f_n \rightarrow f, \forall x \in X$. Given any measurable set $p \in X$ the sequence of sets $\{f_i(p)\}$ are all measurable. Further, since $f_i(x) \rightarrow f(x), \forall x \in X$ we can say $f_i(p) \rightarrow f(p), \forall x \in p$. Thus, for any measurable set P and $\exists p \in \mathfrak{M}, n \rightarrow f_n(p) = P$.

Given any measurable set $P \subset [-\infty, \infty]$ then $f_n^{-1}(P) \in \mathfrak{M}, \forall n$. Let $p_n \subset X$ such that $p_n = f_n^{-1}(P)$. Then, $\cap_n f_n(p_n) = \cap_n f(p_n) = P$. Thus, f is measurable.

Pg 30, 6

Let X be an uncountable set, let \mathfrak{M} be the collection of all sets $E \subset X$ such that either E and E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that \mathfrak{M} is a σ -algebra in X and that μ is a measure on \mathfrak{M} . Describe the corresponding measurable functions and their integrals.

Pg 30, 7

Suppose $f_n : X \rightarrow [0, \infty]$ is measurable fo $rn = 1, 2, \dots, f_1 \geq f_2 \geq f_3 \geq \dots \geq 0, f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

and show that this conclusion does *not* follow if the condition “ $f_1 \in L^1(\mu)$ ” is omitted.

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \lim_{n \rightarrow \infty} \sup \int_X s_n d\mu$$

where each s_n are the simple function less than f_n for all x . These are all less than f and whose $\limsup_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} S_n$ of simple functions less than f . thus

$$\limsup_{n \rightarrow \infty} \int_X s_n d\mu = \limsup_{n \rightarrow \infty} \int_X S_n d\mu = \int_X f d\mu$$

Pg 30, 8

Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

Fatou's Lemma given any any $f_n : X \rightarrow [0, \infty) \in \mathfrak{M}$ then

$$\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu$$

The sequence of functions does not converge but oscilates. Thus, LHS goes to zero and the RHS goes to 1.

Pg 30, 9

Suppose μ is a positive measure on X , $f : X \rightarrow [0, \infty]$ is measurable $\int_X f d\mu = c$, where $0 < c < \infty$, and α is constant. Prove that

$$\lim_{n \rightarrow \infty} \int_X n \log[1 + (f/n)^\alpha] d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf . If $\alpha < 1$, Fatous lemma can be applied.

Pg 30, 10

Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X , and $f_n \rightarrow f$ uniformly on X . Prove that

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

and show that the hypothesis “ $\mu(X) < \infty$ ” cannot be omitted.

Pg 30, 11

Show that

$$A = \bigcap_{n,j=1}^{\infty} \bigcup_{k=n}^{\infty} E^k$$

in Theorem 1.41 and hence prove the theorem without any reference to integration.

Pg 30, 12

Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

Pg 30, 13

Show that proposition 1.24(c) is also true when $c = \infty$.