Math 5102 – Real Analysis I– Fall 2024
w/Professor Liu

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7. Let $S = \{0, 1\}$ and $F = R$. If $\mathcal{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1, g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.
   - $f = g$
     $f(0) = 2(0) + 1 = 1$ and $g(0)1 + 4(0) - 2(0)^2 = 1 \to f(0) = g(0)$
     $f(1) = 2(1) + 1 = 3$ and $g(1)1 + 4(1) - 2(1)^2 = 1 + 4 - 2 = 3 \to f(1) = g(1)$
     $f = g$ since $f(s) = g(s)$ for all $s \in S$
   - $f + g = h$

$$\begin{aligned}(f+g)(t) &= f(t) + g(t)\\ &= 2t + 1 + 1 + 4t - 2t^2\\ &= 2 + 6t - 2t^2\\ (f+g)(0) &= 2 + 6(0) - 2(0)^2 = 2\\ (f+g)(1) &= 2 + 6(1) - 2(1)^2 = 6\end{aligned}$$

$$\begin{aligned}h(0) &= 5^0 + 1 = 2\\ h(1) &= 5^1 + 1 = 6\end{aligned}$$

$$(f+g)(0) = h(0) \text{ and } (f+g)(1) = h(1)$$

$$\therefore f + g = h \text{ since } (f+g)(t) = h(t), \forall t \in S$$

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of additons and scalar multiplication defined in Example 3.

    From Example 3 we have

$$(f+g)(s) = f(s) + g(s) \text{ and } (cf)(s) = c\,[\,f(s)\,]\,, \forall s \in \mathbb{R}$$

    From the additive rule of differentiation $(f+g)'(s) = f'(s) + g'(s)$ which implies that $(f+g) \in V$ and from the multiplicative rule of differentiation we know that $(cf)'(s) = c[f'(s)]$ which implies that $cf \in V$. Hence V is a vector space.

11. Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F. Prove that V is a vector space over F. (V is called the **zero vector space**).

    Given any $v, w \in V$ we have $v + w = 0 + 0 = 0 \in V$ and for all $c \in F, cv = c(0) = 0 \in V$. Therefore V is a vector space.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. (See Example 5 for the definition of a sequence.) For $\{a_n\}, \{b_n\} \in V$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \text{ and } t\{a_n\} = \{ta_n\}.$$

    Prove that, with these operations, V is a vector space. over $\mathbb{R}$.

    Given any two real sequences $\{a_n\}, \{b_n\} \in V$, we can see that the sequence $\{a_n + b_n\}$ is made up of real numbers, namely $a_i + b_i$ for all $i \in \mathbb{N}$, making it a real sequence. We can also see that $\{ca_n\}$ is made up of real numbers, namely ca_i for all $i \in \mathbb{N}$, making it a real sequence. Hence, V is a vector space.

21. Let V and W be vector spaces over a field F. Let

$$Z = \{\,(\,v, w\,) : v \in V \text{ and } w \in W\,\}$$

    Prove that Z is a vector space over F with the operations

$$(\,v_1, w_1\,) + (\,v_2, w_2\,) = (\,v_1 + v_2, w_1 + w_2\,) \text{ and } c\,(\,v_1, w_1\,) = (\,cv_1.cw_1\,)$$

Given $v_1, v_2 \in V$ and $w_1, w_2 \in W$ we can see that by definition $z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$. We want to show that $z_1 + z_2 \in Z$ and $cz_1 \in Z$ for all $c \in F$.
Well, $z_1 + z_2 = (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and since V, W are vector spaces we know that $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$. Therefore, $z_1 + z_2 \in Z$.
Additionally, $cz_1 = c(v_1, w_1) = (cv_1, cw_1)$ and we know that $cv_1 \in V$ and $cw_1 \in W$ therefore $cz_1 \in Z$.
Hence, Z is a vector space.

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3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$.

$$\begin{aligned} A^t &\implies [A_{ij}]^t = [A_{ji}] \\ (aA+bB)_t &= [\,(aA+bB)_{ji}\,] \\ &= [\,aA_{ji} + bB_{ji}\,] \\ &= a\,[\,A_{ji}\,] + b\,[\,B_{ji}\,] \\ &= aA^t + bB^t \end{aligned}$$

4. Prove that $(A^t)^t = A$ for each $A \in M_{m\times n}(F)$.

   Let $B = A^t = [A_{ji}]$ and $B^t = [A_{ji}]^t = [A_{ij}] = A$.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

   $A = [A_{ij}]$ and $A^t = [A_{ji}]$. Therefore $A + A^t = [A_{ij}] + [A_{ji}] = [A_{ij} + A_{ji}$. Given any $i, jin[1, \ldots, n]$ we can see that $(A + A^t)_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = (A + A^t)_{ji}$, hence symmetric.

10. Prove that $W_1 = \{\,(\,a_1, a_2, \ldots, a_n\,) \in F^n : a_1 + a_2 + \cdots + a_n = 0\,\}$ is a subspace of F^n, but $W_2 = \{\,(\,a_1, a_2, \ldots, a_n\,) \in F^n : a_1$ is not.

    Given $a = (\,a_1, a_2, \ldots, a_n\,), b = (\,b_1, b_2, \ldots, b_n\,) \in W_1$ we can see that $a + b = (\,a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\,)$ and $a_1 + b_1 + a_2 + b_2 + \cdots{}_n + b_n = a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n = 0 + 0 = 0$ therefore $a + b \in W_1$. Similarly, $ca = c\,(\,a_1, a_2, \ldots, a_n\,) = (\,ca_1, ca_2, \ldots, ca_n\,)$ and $ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c(0) = 0$. Therefore W_1 is a vector space.
    However, let $a = (\,a_1, a_2, \ldots, a_n\,), b = (\,b_1, b_2, \ldots, b_n\,) \in W_2$ we can see that $a + b = (\,a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\,)$ and $a_1 + b_1 + a_2 + b_2 + \cdots{}_n + b_n = a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n = 1 + 1 = 2$ which means that $a + b \notin W_2$.

15. Is the set of all differentiable real-valued functions defined on $\mathbb{R}$ a subspace of $C(\mathbb{R})$? Justify your answer.

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \bigcup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

    Let $a \in W_1 \backslash W_1 \cap W_2$ and $b \in W_1 \backslash W_1 \cap W_2$. $a \notin W_2 \implies a + b \notin W_2$ and $b \notin W_1 \implies a + b \notin W_1$. That is, unless either $a = 0$ and/or $b = 0$.