

Math 5411 – Mathematical Statistics I– Fall 2024  
w/Nezamoddini-Kachouie

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Homework Short #3 – October 7, 2024

Chapter 4: 2, 5, 6 (a,c,d), 14

2. If  $X$  is a discrete uniform random variable – that is,  $P(X = k) = 1/n$  for  $k = 1, 2, \dots, n$  – find  $E(X)$  and  $\text{Var}(X)$ .

$$\begin{aligned} E(X) &= \sum_{k=1}^n kp(X = k) \\ &= \sum_{k=1}^n k(1/k) \\ &= \sum_{k=1}^n 1 \\ \mu &= n \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[(x - \mu)^2] \\ &= \sum_{k=1}^n (k - \mu)^2 p(X = k) \\ &= \sum_{k=1}^n (k - 1)^2 (1/k) \\ &= \sum_{k=1}^n \frac{k^2 - 2k + 1}{k} \\ &= \sum_{k=1}^n (k - 2 + 1/k) \\ &= \sum_{k=1}^n k - 2n + \sum_{k=1}^n 1/k \\ &= \frac{n(n+1)}{2} - 2n + \sum_{k=1}^n 1/k \end{aligned}$$

5. Let  $X$  have the density

$$f(x) = \frac{1 + \alpha x}{2}, \quad -1 \leq x \leq 1, \quad -1 \leq \alpha \leq 1$$

Find  $E(X)$  and  $\text{Var}(X)$ .

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-1}^1 x \frac{1 + \alpha x}{2} dx \\ &= \int_{-1}^1 \frac{x + \alpha x^2}{2} dx \\ &= \frac{1}{2} \int_{-1}^1 x + \alpha x^2 dx \\ &= \frac{1}{2} \left[ \frac{x^2}{2} + \frac{\alpha x^3}{3} \right]_{-1}^1 \\ &= \frac{1}{2} \left( \left( \frac{1}{2} + \frac{\alpha}{3} \right) - \left( \frac{1}{2} + \frac{-\alpha}{3} \right) \right) \\ &= \frac{1}{2} \left( \frac{\alpha}{3} + \frac{\alpha}{3} \right) \\ &= \frac{\alpha}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E[(x - \mu)^2] \\ &= \int_{-1}^1 (x - \mu)^2 f(x) dx \\ &= \int_{-1}^1 \left( x - \frac{\alpha}{3} \right)^2 \frac{1 + \alpha x}{2} dx \\ &= \int_{-1}^1 \left( x^2 - \frac{2\alpha x}{3} + \frac{\alpha^2}{9} \right) \frac{1 + \alpha x}{2} dx \\ &= \frac{1}{2} \int_{-1}^1 \left( x^2 - \frac{2\alpha x}{3} + \frac{\alpha^2}{9} + \alpha x^3 - \frac{2\alpha^2 x^2}{3} + \frac{\alpha^3 x}{9} \right) dx \\ &= \frac{1}{2} \int_{-1}^1 \left( x^2 - \frac{2\alpha x}{3} + \frac{\alpha^2}{9} + \alpha x^3 + \frac{\alpha^3 x}{9} \right) dx \\ &= \frac{1}{2} \left[ -\frac{2\alpha x^2}{6} + \frac{\alpha^2 x}{9} + \alpha x^4/4 + \frac{\alpha^3 x^2}{18} \right]_{-1}^1 \\ &= \frac{1}{2} \left( -\frac{4\alpha}{6} + \frac{2\alpha^2}{9} + \frac{\alpha}{2} + \frac{\alpha^3}{9} \right) \end{aligned}$$

6. Let  $X$  be a continuous random variable with probability density function  $f(x) = 2x, 0 \leq x \leq 1$ .

(a) Find  $E(X)$ .

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^1 x(2x) dx \\ &= \int_0^1 2x^2 dx \\ &= \left[ \frac{2x^3}{3} \right]_0^1 \\ &= \frac{2}{3} \end{aligned}$$

- (b) Use Theorem A in Section 4.1.1 to find  $E(X^2)$  and compare to your answer part(b).

$$\begin{aligned}
 E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^1 x^2 (2x) dx \\
 &= \int_0^1 2x^3 \\
 &= \left[ \frac{x^4}{2} \right]_0^1 \\
 &= \frac{1}{2}
 \end{aligned}$$

- (c) Find  $\text{Var}(X)$  according to the definition of variance given in Section 4.2. Also, find  $\text{Var}(X)$  by using Theorem B of Section 4.2.

$$\begin{aligned}
 \text{Var}(X) &= E(X^2) - \mu^2 \\
 &= \frac{1}{2} - \left( \frac{2}{3} \right)^2 \\
 &= \frac{1}{2} - \frac{4}{9} \\
 &= \frac{5}{18}
 \end{aligned}$$

14. Let  $X$  be a continuous random variable with the density function

$$f(x) = 2x, \quad 0 \leq x \leq 1$$

- (a) Find  $E(X)$ .

$$\begin{aligned}
 E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
 &= \int_0^1 x(2x) dx \\
 &= \int_0^1 2x^2 \\
 &= \left[ \frac{2x^3}{3} \right]_0^1 \\
 &= \frac{2}{3}
 \end{aligned}$$

(b) Find  $E(X^2)$  and  $\text{Var}(X)$ .

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^1 x^2 (2x) dx \\ &= \int_0^1 2x^3 \\ &= \left[ \frac{x^4}{2} \right]_0^1 \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= \frac{1}{2} - \left( \frac{2}{3} \right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{5}{18} \end{aligned}$$