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Michel Chipot

Elliptic Equations: An Introductory Course

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Basel · Boston · Berlin

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Preface

The goal of this book is to introduce the reader to different topics of the theory of elliptic partial differential equations avoiding technicalities and refinements. The material of the first part is written in such a way it could be taught as an introductory course. Most of the chapters – except the four first ones – are independent and some material can be dropped in a short course. The four first chapters are fundamental, the next ones are devoted to teach or present a larger spectrum of the techniques of this topics showing some qualitative properties of the solutions to these problems. Everywhere just a minimum on Sobolev spaces has been introduced, work or integration on the boundary has been carefully avoided in order not to crowd the mind of the reader with technicalities but to attract his attention to the beauty and variety of these issues. Also very often the ideas in mathematics are very simple and the discovery of them is a powerful engine to learn quickly and get further involved with a theory. We have kept this in mind all along Part 1.

Part II contains more advanced material like nonlinear problems, systems, regularity... Again each chapter is relatively independent of the others and can be read or taught separately.

We would also like to point that numerous results presented here are original and have not been published elsewhere.

This book grew out of lectures given at the summer school of Druskininkai (Lithuania), in Tokyo (Waseda University) and in Rome (La Sapienza). It is my pleasure to acknowledge the rôle of these different places and to thank K. Pileckas, Y. Yamada, D. Giachetti for inviting me to deliver these courses.

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Zürich, December 2008

Part I

Basic Techniques

Chapter 1

Hilbert Space Techniques

The goal of this chapter is to collect the main features of the Hilbert spaces and to introduce the Lax–Milgram theorem which is a key tool for solving elliptic partial differential equations.

1.1 The projection on a closed convex set

Definition 1.1. A Hilbert space H over $\mathbb{R}$ is a vector space equipped with a scalar product $(\cdot,\cdot)$ which is complete for the associated norm

$$|u| = (u,u)^{\frac{1}{2}}, \tag{1.1}$$

i.e., such that every Cauchy sequence admits a limit in H.

Examples.

1. $\mathbb{R}^n$ equipped with the Euclidean scalar product

$$(x,y) = x \cdot y = \sum_{i=1}^{n} x_i y_i \quad \forall\, x = (x_1,\dots,x_n),\; y = (y_1,\dots,y_n) \in \mathbb{R}^n.$$

   (We will prefer the notation with a dot for this scalar product.)

2. $L^2(A) = \{\, v : A \to \mathbb{R}, v \text{ measurable} \mid \int_A v^2(x)\,dx < +\infty \,\}$ with A a measurable subset of $\mathbb{R}^n$. Recall that $L^2(A)$ is in fact a set of "class" of functions. $L^2(A)$ is a Hilbert space when equipped with the scalar product

$$(u,v) = \int_A u(x)v(x)\,dx. \tag{1.2}$$

Remark 1.1. Let us recall the important "Cauchy–Schwarz inequality" which asserts that

$$|(u,v)| \le |u|\,|v| \quad \forall\, u,v \in H. \tag{1.3}$$

One of the important results is the following theorem.

Theorem 1.1 (Projection on a convex set). *Let $K \neq \emptyset$ be a closed convex subset of a Hilbert space H. For every $h \in H$ there exists a unique u such that*

$$\begin{cases} u \in K, \\ |h-u| \leq |h-v|, \quad \forall\, v \in K, \end{cases} \tag{1.4}$$

(i.e., u realizes the minimum of the distance between h and K). Moreover u is the unique point satisfying (see Figure 1.1)

$$\begin{cases} u \in K, \\ (u-h, v-u) \geq 0, \quad \forall\, v \in K. \end{cases} \tag{1.5}$$

u is called the orthogonal projection of h on K and will be denoted by $P_K(h)$.

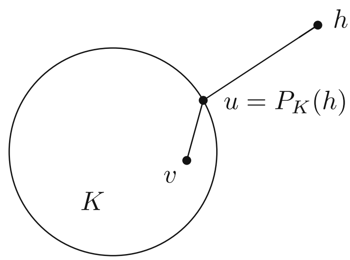


Figure 1.1: Projection on a convex set

Proof. Consider a sequence $u_n \in K$ such that when $n \to +\infty$

$$|h-u_n| \to \operatorname*{Inf}_{v \in K} |h-v| := d.$$

The infimum above clearly exists and thus such a minimizing sequence also. Note now the identities

$$\begin{aligned} |u_n-u_m|^2 &= |u_n-h+h-u_m|^2 = |h-u_n|^2+|h-u_m|^2+2(u_n-h, h-u_m) \\ |2h-u_n-u_m|^2 &= |h-u_n+h-u_m|^2 = |h-u_n|^2+|h-u_m|^2+2(h-u_n, h-u_m). \end{aligned}$$

Adding up we get the so-called parallelogram identity

$$|u_n-u_m|^2 + |2h-u_n-u_m|^2 = 2|h-u_n|^2 + 2|h-u_m|^2. \tag{1.6}$$

Recall now that a convex set is a subset of H such that

$$\alpha u + (1-\alpha)v \in K \quad \forall\, u, v \in K, \quad \forall\, \alpha \in [0,1]. \tag{1.7}$$

From (1.6) we derive

$$|u_n - u_m|^2 = 2|h - u_n|^2 + 2|h - u_m|^2 - 4\left|h - \frac{u_n + u_m}{2}\right|^2$$
$$\leq 2|h - u_n|^2 + 2|h - u_m|^2 - 4d^2 \tag{1.8}$$

since $\frac{u_n+u_m}{2} \in K$ (take $\alpha = \frac{1}{2}$ in (1.7)). Since the right-hand side of (1.8) goes to 0 when $n, m \to +\infty$, u_n is a Cauchy sequence. It converges toward a point $u \in K$ – since K is closed – such that

$$|h - u| = \operatorname*{Inf}_{v \in K} |h - v|. \tag{1.9}$$

This shows the existence of u satisfying (1.4). To prove the uniqueness of such a u one goes back to (1.8) which is valid for any u_n, u_m in K. Taking u, u' two solutions of (1.4), (1.8) becomes

$$|u - u'|^2 \leq 2|h - u|^2 + 2|h - u'|^2 - 4d^2 = 0,$$

i.e., $u = u'$. This completes the proof of the existence and uniqueness of a solution to (1.4). We show now the equivalence of (1.4) and (1.5). Suppose first that u is solution to (1.5). Then we have

$$|h-v|^2 = |h-u+u-v|^2 = |h-u|^2 + |u-v|^2 + 2(h-u, u-v) \geq |h-u|^2 \quad \forall\, v \in K.$$

Conversely suppose that (1.4) holds. Then for any $\alpha \in (0, 1)$ – see (1.7) – we have for $v \in K$

$$|h - u|^2 \leq |h - [\alpha v + (1 - \alpha)u]|^2 = |h - u - \alpha(v - u)|^2$$
$$= |h - u|^2 + 2\alpha(u - h, v - u) + \alpha^2 |v - u|^2.$$

This implies clearly

$$2\alpha(u - h, v - u) + \alpha^2 |v - u|^2 \geq 0. \tag{1.10}$$

Dividing by α and letting $\alpha \to 0$ we derive that (1.5) holds. This completes the proof of the theorem. □

Remark 1.2. If $h \in K$, $P_K(h) = h$. (1.5) is an example of variational inequality.

In the case where K is a closed subspace of H (this is a special convex set) Theorem 1.1 takes a special form.

Corollary 1.2. *Let V be a closed subspace of H. Then for every $h \in H$ there exists a unique u such that*

$$\begin{cases} u \in V, \\ |h - u| \leq |h - v|, \quad \forall\, v \in V. \end{cases} \tag{1.11}$$

Moreover u is the unique solution to

$$\begin{cases} u \in V, \\ (h - u, v) = 0, \quad \forall\, v \in V. \end{cases} \tag{1.12}$$

Proof. It is enough to show that (1.5) is equivalent to (1.12). Note that if (1.12) holds then (1.5) holds. Conversely if (1.5) holds then for any $w \in V$ one has

$$v = u \pm w \in V$$

since V is a vector space. One deduces

$$\pm(u - h, w) \geq 0 \quad \forall\, w \in V$$

which is precisely (1.12). $u = P_V(h)$ is described in Figure 1.2 below. It is the unique vector of V such that $h - u$ is orthogonal to V. □

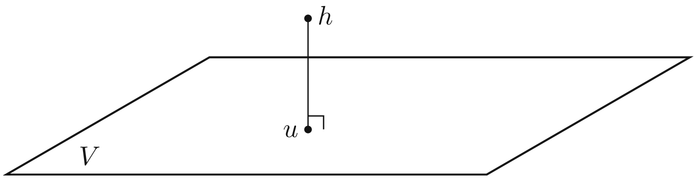


Figure 1.2: Projection on a vector space

1.2 The Riesz representation theorem

If H is a real Hilbert space we denote by H^* its dual – i.e., H^* is the set of continuous linear forms on H. If $h \in H$ then the mapping

$$v \mapsto (h, v) \tag{1.13}$$

is an element of H^*. Indeed this is a linear form that is to say a linear mapping from H into $\mathbb{R}$ and the continuity follows from the Cauchy–Schwarz inequality

$$|(h, v)| \leq |h|\,|v|. \tag{1.14}$$

The Riesz representation theorem states that all the elements of H^* are of the type (1.13) which means can be represented by a scalar product. This fact is easy to see on $\mathbb{R}^n$. We will see that it extends to infinite-dimensional Hilbert spaces. First let us analyze the structure of the kernel of the elements of H^*.

Proposition 1.3. *Let $h^* \in H^*$. If $h^* \neq 0$ the set*

$$V = \{\, v \in H \mid \langle h^*, v\rangle = 0 \,\} \tag{1.15}$$

is a closed subspace of H of codimension 1, i.e., a hyperplane of H. (We denote with brackets the duality writing $\langle h^*, v\rangle = h^*(v)$.)

Proof. Since h^* is continuous V is a closed subspace of H. Let $h \notin V$. Such an h exists since $h^* \neq 0$. Then set

$$v_0 = h - P_V(h) \neq 0. \tag{1.16}$$

Any element of $v \in H$ can be decomposed in a unique way as

$$v = \lambda v_0 + w \tag{1.17}$$

where $w \in V$. Indeed if $w \in V$ one has necessarily

$$\langle h^*, v\rangle = \lambda \langle h^*, v_0\rangle,$$

i.e., $\lambda = \langle h^*, v\rangle / \langle h^*, v_0\rangle$ and then

$$v = \frac{\langle h^*, v\rangle}{\langle h^*, v_0\rangle} v_0 + v - \frac{\langle h^*, v\rangle}{\langle h^*, v_0\rangle} v_0.$$

This completes the proof of the proposition. □

We can now show

Theorem 1.4 (Riesz representation theorem). *For any $h^* \in H^*$ there exists a unique $h \in H$ such that*

$$(h, v) = \langle h^*, v\rangle \quad \forall\, v \in H. \tag{1.18}$$

Moreover

$$|h| = |h^*|_* = \operatorname*{Sup}_{\substack{v \in H \\ v \neq 0}} \frac{\langle h^*, v\rangle}{|v|}. \tag{1.19}$$

(*This last quantity is called the strong dual norm of h^*.*)

Proof. If $h^* = 0$, $h = 0$ is the only solution of (1.18). We can assume then that $h^* \neq 0$. Let $v_0 \neq 0$ be a vector orthogonal to the hyperplane

$$V = \{\, v \in H \mid \langle h^*, v\rangle = 0 \,\},$$

(see (1.16), (1.17)). We set

$$h = \frac{\langle h^*, v_0\rangle}{|v_0|^2} v_0. \tag{1.20}$$

Due to the decomposition (1.17) we have

$$(h, v) = (h, \lambda v_0 + w) = \lambda (h, v_0) = \lambda \langle h^*, v_0\rangle = \langle h^*, \lambda v_0 + w\rangle = \langle h^*, v\rangle$$

for every $v \in H$. Thus h satisfies (1.18). The uniqueness of h is clear since

$$(h - h', v) = 0 \quad \forall\, v \in H \Longrightarrow h = h'$$

(take $v = h - h'$).

Now from (1.20) we have

$$|h| = \frac{|\langle h^*, v_0\rangle|}{|v_0|} \le |h^*|_* \tag{1.21}$$

and from (1.18)

$$|h^*|_* = \operatorname*{Sup}_{v\neq 0} \frac{(h,v)}{|v|} \le \operatorname*{Sup}_{v\neq 0} \frac{|h|\,|v|}{|v|} = |h|.$$

This completes the proof of the theorem. □

1.3 The Lax–Milgram theorem

Instead of a scalar product one can consider more generally a continuous bilinear form. That is to say if $a(u,v)$ is a continuous bilinear form on H, then for every $u \in H$

$$v \mapsto a(u,v) \tag{1.22}$$

is an element of H^*. As for the Riesz representation theorem one can ask if every element of H^* is of this type. This can be achieved with some assumptions on a which reproduce the properties of the scalar product, namely:

Theorem 1.5 (Lax–Milgram). *Let $a(u,v)$ be a bilinear form on H such that*

- *a is continuous, i.e., $\exists \Lambda > 0$ such that $|a(u,v)| \le \Lambda |u|\,|v| \quad \forall\, u, v \in H$,* (1.23)
- *a is coercive, i.e., $\exists \lambda > 0$ such that $a(u,u) \ge \lambda |u|^2 \quad \forall\, u \in H$.* (1.24)

Then for every $f \in H^*$ there exists a unique $u \in H$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall\, v \in H. \tag{1.25}$$

In the case where a is symmetric that is to say

$$a(u,v) = a(v,u) \quad \forall\, u, v \in H \tag{1.26}$$

then u is the unique minimizer on H of the functional

$$J(v) = \frac{1}{2} a(v,v) - \langle f, v\rangle. \tag{1.27}$$

Proof. For every $u \in H$, by (1.23), $v \mapsto a(u,v)$ is an element of H^*. By the Riesz representation theorem there exists a unique element in H that will be denoted by Au such that

$$a(u,v) = (Au, v) \quad \forall\, v \in H.$$

We will be done if we can show that A is a bijective mapping from H into H. (Indeed one will then have $\langle f, v\rangle = (h,v) = (Au,v) = a(u,v)\ \forall\, v \in H$ for a unique u in H.)

- A is linear.

  By definition of A for any $u_1, u_2 \in H$, $\alpha_1, \alpha_2 \in \mathbb{R}$ one has

$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = (A(\alpha_1 u_1 + \alpha_2 u_2), v) \quad \forall\, v \in H.$$

  By the bilinearity of a and the definition of A one has also

$$\begin{aligned} a(\alpha_1 u_1 + \alpha_2 u_2, v) &= \alpha_1 a(u_1, v) + \alpha_2 a(u_2, v) = \alpha_1 (Au_1, v) + \alpha_2 (Au_2, v) \\ &= (\alpha_1 Au_1 + \alpha_2 Au_2, v) \quad \forall\, v \in H. \end{aligned}$$

  Then we have

$$(A(\alpha_1 u_1 + \alpha_2 u_2), v) = (\alpha_1 Au_1 + \alpha_2 Au_2, v) \quad \forall\, v \in H$$

  and thus

$$A(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 Au_1 + \alpha_2 Au_2 \quad \forall\, u_1, u_2 \in H \quad \forall\, \alpha_1, \alpha_2 \in \mathbb{R},$$

  hence the linearity of A.

- A is injective.

  Due to the linearity of A it is enough to show that

$$Au = 0 \quad \Longrightarrow \quad u = 0.$$

  If $Au = 0$, by (1.24) and the definition of A one has

$$0 = (Au, u) = a(u, u) \geq \lambda |u|^2$$

  and our claim is proved.

- AH the image of H by A is a closed subspace of H.

  Indeed consider Au_n a sequence such that

$$Au_n \to y \quad \text{in } H.$$

  We want to show that $y \in AH$. For that note that by (1.24) one has

$$\begin{aligned} \lambda |u_n - u_m|^2 &\leq a(u_n - u_m, u_n - u_m) = (Au_n - Au_m, u_n - u_m) \\ &\leq |Au_n - Au_m|\,|u_n - u_m|, \end{aligned}$$

$$\text{i.e.,} \qquad |u_n - u_m| \leq \frac{1}{\lambda} |Au_n - Au_m|.$$

  Since Au_n converges it is a Cauchy sequence and so is u_n. Then there exists $u \in H$ such that

$$u_n \to u \quad \text{in } H$$

when $n \to +\infty$. By definition of A we have

$$a(u_n, v) = (Au_n, v) \quad \forall\, v \in H.$$

Passing to the limit in n we get

$$a(u, v) = (y, v) \quad \forall\, v \in H$$

which is also $y = Au \in H$. This completes the proof of the claim.

- A is surjective.

  If not $V = AH$ is a closed proper subspace of H. Then – see Corollary 1.2 – there exists a vector $v_0 \neq 0$ such that v_0 is orthogonal to V. Then

$$\lambda |v_0|^2 \leq a(v_0, v_0) = (Av_0, v_0) = 0$$

  which contradicts $v_0 \neq 0$. This completes the first part of the theorem.

  If we assume now that a is symmetric and u solution to (1.25) one has

$$\begin{aligned} J(v) &= J(u + v - u) \\ &= \frac{1}{2} a(u + (v-u), u + (v-u)) - \langle f, u + (v-u) \rangle \\ &= \frac{1}{2} a(u,u) - \langle f, u \rangle + a(u, v-u) - \langle f, v-u \rangle + \frac{1}{2} a(v-u, v-u) \end{aligned}$$

  (by linearity and since a is symmetric). Using (1.25) we get

$$J(v) = J(u) + \frac{1}{2} a(v-u, v-u) \geq J(u) + \frac{\lambda}{2} |v-u|^2 \geq J(u) \quad \forall\, v \in H,$$

  the last inequality being strict for $v \neq u$. Hence u is the unique minimizer of J on H. This completes the proof of the theorem. □

1.4 Convergence techniques

We recall that if h_n is a sequence in H then

$$h_n \rightharpoonup h_\infty \quad \text{in } H \quad \text{as } n \to +\infty,$$

(i.e., h_n converges towards h weakly) iff

$$\lim_{n \to +\infty} (h_n, h) = (h_\infty, h) \quad \forall\, h \in H.$$

Due to the Cauchy–Schwarz inequality it is easy to show that

$$h_n \longrightarrow h \quad \text{in } H \quad \Longrightarrow \quad h_n \rightharpoonup h \quad \text{in } H.$$

The converse is not true in general.

We also have

Theorem 1.6 (Weak compactness of balls). *If h_n is a bounded sequence in H, there exists a subsequence h_{n_k} of h_n and $h_\infty \in H$ such that*

$$h_{n_k} \rightharpoonup h_\infty.$$

Proof. See [96] □

Finally we will find useful the following theorem.

Theorem 1.7. *Let x_n, y_n be two sequences in H such that*

$$x_n \longrightarrow x_\infty, \qquad y_n \rightharpoonup y_\infty.$$

Then we have

$$(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$$

in $\mathbb{R}$.

Proof. One has

$$\begin{aligned} |(x_n, y_n) - (x_\infty, y_\infty)| &= |(x_n - x_\infty, y_n) - (x_\infty, y_\infty - y_n)| \\ &\le |x_n - x_\infty|\,|y_n| + |(x_\infty, y_\infty - y_n)|. \end{aligned}$$

Since $y_n \rightharpoonup y_\infty$, y_n is bounded (Theorem of Banach–Steinhaus), then the result follows easily. □

Exercises

1. Show (1.3). (Hint: use the fact that $(u + \lambda v,\ u + \lambda v) \ge 0\ \forall\, \lambda \in \mathbb{R}$.)
2. In $\mathbb{R}^n$ consider
$$K = \{\, x \mid x_i \ge 0\ \ \forall\, i = 1, \dots, n \,\}.$$
Show that K is a closed convex set of $\mathbb{R}^n$.

    Show that
$$P_K(y) = (y_1 \vee 0, y_2 \vee 0, \dots, y_n \vee 0) \quad \forall\, y \in \mathbb{R}^n$$
where $\vee$ denotes the maximum of two numbers.
3. Set
$$\ell^2 = \left\{ x = (x_i) \;\middle|\; \sum_{i=1}^{+\infty} x_i^2 < +\infty \right\},$$
i.e., ℓ^2 is the set of sequences square summable.

(a) Show that ℓ^2 is a Hilbert space for the scalar product

$$(x, y) = \sum_{i=1}^{+\infty} x_i y_i \quad \forall\, x, y \in \ell^2.$$

(b) One sets

$$e_n = (0, 0, \ldots, 1, 0, \ldots)$$

where 1 is located at the n^{th} slot. Show that $e_n \rightharpoonup 0$ in ℓ^2 but $e_n \not\to 0$.

4. Let $g \in L^2(A)$ where A is a measurable subset of $\mathbb{R}^n$. Set

$$K = \{\, v \in L^2(A) \mid v(x) \le g(x) \ \text{ a.e. } x \in A \,\}.$$

Show that K is a nonempty closed convex set of $L^2(A)$. Show that

$$P_K(v) = v \wedge g \quad \forall\, v \in L^2(A).$$

($\wedge$ denotes the minimum of two numbers – i.e., $(v \wedge g)(x) = v(x) \wedge g(x)$ a.e. x.)

5. If P_K denotes the projection on a nonempty closed convex set K of a Hilbert space H show that

$$|P_K(h) - P_K(h')| \le |h - h'| \quad \forall\, h, h' \in H.$$

6. Let a be a continuous, coercive, bilinear form on a real Hilbert space H. Let $f, \ell \in H^*$, $\ell \neq 0$. Set

$$V = \{\, h \in H \mid \langle \ell, h \rangle = 0 \,\}.$$

(a) Show that there exists a unique u solution to

$$\begin{cases} u \in V, \\ a(u, v) = \langle f, v \rangle \quad \forall\, v \in V. \end{cases}$$

(b) Show that there exists a unique $k \in \mathbb{R}$ such that u satisfies

$$a(u, v) = \langle f + k\ell, v \rangle \quad \forall\, v \in H.$$

(Hint: If $h \in H$, $\ell(h) = 1$, $v - \ell(v)h \in V$, $\forall\, v \in H$.)

(c) Show that there exists a unique $u_\ell \in H$ solution to

$$\begin{cases} u_\ell \in H, \\ a(v, u_\ell) = \langle \ell, v \rangle \quad \forall\, v \in H, \end{cases}$$

and that $k = -\frac{\langle f, u_\ell \rangle}{\langle \ell, u_\ell \rangle}$.

(d) If $\ell_1, \ell_2, \ldots, \ell_p \in H^*$

$$V = \{\, h \in H \mid \ell_i(h) = 0, \ \forall\, i = 1, \ldots, p \,\}$$

what can be said about u the solution to

$$u \in V, \quad a(u, v) = \langle f, v \rangle \quad \forall\, v \in V?$$

Chapter 2

A Survey of Essential Analysis

2.1 L^p-techniques

We recall here some basic techniques regarding L^p-spaces in particular those involving approximation by mollifiers.

Definition 2.1. Let $p \geq 1$ be a real number, Ω an open subset of $\mathbb{R}^n$, $n \geq 1$

$$L^p(\Omega) = \left\{ \text{“Class of functions” } v : \Omega \to \mathbb{R} \text{ s.t. } \int_\Omega |v(x)|^p \, dx < +\infty \right\}$$
$$L^\infty(\Omega) = \{ \text{ “Class of functions” } v : \Omega \to \mathbb{R} \text{ s.t. } \exists C \text{ s.t. } |v(x)| \leq C \text{ a.e. } x \in \Omega \}.$$

Equipped with the norms

$$|v|_{p,\Omega} = \left\{ \int_\Omega |v(x)|^p \, dx \right\}^{\frac{1}{p}}$$
$$|v|_{\infty,\Omega} = \text{Inf}\{ C \text{ s.t. } |v(x)| \leq C \text{ a.e. } x \in \Omega \}.$$

$L^p(\Omega)$, $L^\infty(\Omega)$ are Banach spaces (see [85]). Moreover for any $1 \leq p < +\infty$ the dual of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$ where p' is the conjugate number of p which is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1, \tag{2.1}$$

i.e., $p' = \frac{p}{p-1}$ with the convention that $p' = +\infty$ when $p = 1$. (We refer the reader to [85] for these notions.)

Definition 2.2. We denote by $L^p_{\text{loc}}(\Omega)$ the set of functions v defined on Ω and such that for any Ω' bounded such that $\overline{\Omega}' \subset \Omega$ one has

$$v \in L^p(\Omega').$$

We will note $\Omega' \subset\subset \Omega$ when $\overline{\Omega}'$ – the closure of Ω' in $\mathbb{R}^n$ – is included in Ω and compact.

We denote by $\mathcal{D}(\Omega)$ the space of functions infinitely differentiable in Ω with compact support in Ω. Recall that the support of a function ρ is defined as

$$\operatorname{Supp}\rho = \text{ the closure of the set } \{\, x \in \Omega \mid \rho(x) \neq 0 \,\} = \overline{\{\, x \in \Omega \mid \rho(x) \neq 0 \,\}}. \quad (2.2)$$

Examples 2.1.

1. Let us denote by $|x|$ the Euclidean norm of the vector $x = (x_1, \ldots, x_n)$ defined as

$$|x| = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}. \quad (2.3)$$

The function defined, for c a constant, by

$$\rho(x) = \begin{cases} c \exp\left\{ -\dfrac{1}{1-|x|^2} \right\} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \quad (2.4)$$

is a function of $\mathcal{D}(\mathbb{R}^n)$ with support

$$B_1 = \{\, x \in \mathbb{R}^n \mid |x| \leq 1 \,\}.$$

2. If $x_0 \in \Omega$ and ε positive is such that

$$B_\varepsilon(x_0) = \{\, x \in \mathbb{R}^n \mid |x - x_0| \leq \varepsilon \,\} \subset \Omega \quad (2.5)$$

then the function

$$r_\varepsilon(x) = \rho\left(\frac{x - x_0}{\varepsilon} \right) \quad (2.6)$$

is a function of $\mathcal{D}(\Omega)$ with support $B_\varepsilon(x_0)$.

Considering linear combinations of functions of the type (2.6) it is easy to construct many other functions of $\mathcal{D}(\Omega)$ and to see that $\mathcal{D}(\Omega)$ is an infinite-dimensional space on $\mathbb{R}$. Suppose that in (2.4) we choose c such that

$$\int_{\mathbb{R}^n} \rho(x)\, dx = 1 \quad (2.7)$$

and set

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon} \right). \quad (2.8)$$

We then have

$$\rho_\varepsilon \in \mathcal{D}(\Omega), \quad \operatorname{Supp}\rho_\varepsilon = B_\varepsilon(0), \quad \int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1. \quad (2.9)$$

For $u \in L^1_{\mathrm{loc}}(\Omega)$ we define the "mollifier of u" as

$$u_\varepsilon(x) = \int_\Omega u(y) \rho_\varepsilon(x - y)\, dy = (\rho_\varepsilon * u)(x). \quad (2.10)$$

It is clear that if $x \in \Omega$, $u_\varepsilon(x)$ is defined as soon as

$$\varepsilon < \operatorname{dist}(x, \partial\Omega) \tag{2.11}$$

where $\operatorname{dist}(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. Recall that $\operatorname{dist}(x, \partial\Omega) = \operatorname*{Inf}_{y \in \partial\Omega} |x - y|$ where $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^n$. Then we have

Theorem 2.1. *Let $\Omega' \subset\subset \Omega$.*

1. *For $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$, $u_\varepsilon \in C^\infty(\Omega')$.*
2. *If $u \in C(\Omega)$ – i.e., is a continuous function in Ω – then when $\varepsilon \to 0$*

$$u_\varepsilon \longrightarrow u \quad \textit{uniformly in } \Omega'. \tag{2.12}$$

3. *If $u \in L^p_{\text{loc}}(\Omega)$, $1 \le p < +\infty$ then when $\varepsilon \to 0$*

$$u_\varepsilon \longrightarrow u \quad \textit{in } L^p(\Omega'). \tag{2.13}$$

Proof. 1. Let $x \in \Omega'$. Since for $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega)$ one has $\overline{B_\varepsilon(x)} \subset \Omega$ it is clear that (2.10) is well defined. Now from the theorem of derivation under an integral one has if ∂_{x_i} denotes the partial derivative in the direction x_i

$$\partial_{x_i} u_\varepsilon(x) = \int_\Omega u(y) \partial_{x_i} \rho_\varepsilon(x - y)\, dy.$$

Repeating this process in the other directions and for any order derivative it is clear that $u_\varepsilon \in C^\infty(\Omega')$.

2. Since $u \in C(\Omega)$, u is uniformly continuous on any compact subset of Ω and for any $\delta > 0$ there exists an $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega) = \operatorname*{Inf}_{x \in \Omega', y \in \partial\Omega} |x - y|$ such that

$$x \in \Omega', \qquad |x - y| \le \varepsilon \quad \Longrightarrow \quad |u(x) - u(y)| \le \delta. \tag{2.14}$$

Let $x \in \Omega'$ one has, see (2.9):

$$u(x) - u_\varepsilon(x) = \int_\Omega \{u(x) - u(y)\} \rho_\varepsilon(x - y)\, dy$$

(by (2.10)). Thus

$$\begin{aligned} |u(x) - u_\varepsilon(x)| &= \left| \int_\Omega \{u(x) - u(y)\} \rho_\varepsilon(x - y)\, dy \right| \\ &\le \int_\Omega |u(x) - u(y)| \rho_\varepsilon(x - y)\, dy. \end{aligned}$$

In the integral above one integrates only on $B_\varepsilon(x)$ and by (2.14) it comes

$$|u(x) - u_\varepsilon(x)| \le \delta \int_{B_\varepsilon(x)} \rho_\varepsilon(x - y)\, dy = \delta \int_{B_\varepsilon} \rho_\varepsilon(z)\, dz = \delta$$

which completes the proof of assertion 2.

3. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. $u \in L^p(\Omega'')$ and thus there exists $\hat{u} \in C(\overline{\Omega}'')$ such that

$$|u - \hat{u}|_{p,\Omega''} \le \frac{\delta}{3}, \tag{2.15}$$

since the continuous functions are dense in $L^p(\Omega'')$ – see [85]. Then we have if $\hat{u}_\varepsilon = \rho_\varepsilon * \hat{u}$

$$\begin{aligned} |u - u_\varepsilon|_{p,\Omega'} &= |u - \hat{u} + \hat{u} - \hat{u}_\varepsilon + \hat{u}_\varepsilon - u_\varepsilon|_{p,\Omega'} \\ &\le |u - \hat{u}|_{p,\Omega'} + |\hat{u} - \hat{u}_\varepsilon|_{p,\Omega'} + |\hat{u}_\varepsilon - u_\varepsilon|_{p,\Omega'}. \end{aligned} \tag{2.16}$$

Now let us notice that

$$\begin{aligned} |\hat{u}_\varepsilon - u_\varepsilon|^p_{p,\Omega'} &= \int_{\Omega'} |\hat{u}_\varepsilon(x) - u_\varepsilon(x)|^p \, dx \\ &= \int_{\Omega'} \left| \int_{B_\varepsilon(x)} \{\hat{u}(y) - u(y)\} \rho_\varepsilon(x-y) \, dy \right|^p dx \\ &= \int_{\Omega'} \left| \int_{B_\varepsilon(x)} |\hat{u}(y) - u(y)| \rho_\varepsilon^{\frac{1}{p}}(x-y) \rho_\varepsilon^{\frac{1}{p'}}(x-y) \, dy \right|^p dx \\ &\le \int_{\Omega'} \int_{B_\varepsilon(x)} |\hat{u}(y) - u(y)|^p \rho_\varepsilon(x-y) \, dy \, dx \quad \text{(by Hölder's inequality)} \\ &= \int_{\Omega'} \int_{B_\varepsilon} |\hat{u}(x-z) - u(x-z)|^p \rho_\varepsilon(z) \, dz \, dx \end{aligned}$$

(we made the change of variable $z = x - y$, $B_\varepsilon = B_\varepsilon(0)$). Thus, by the Fubini theorem, we obtain

$$\begin{aligned} |\hat{u}_\varepsilon - u_\varepsilon|^p_{p,\Omega'} &\le \int_{B_\varepsilon} \rho_\varepsilon(z) \int_{\Omega'} |\hat{u}(x-z) - u(x-z)|^p \, dx \, dz \\ &\le \int_{B_\varepsilon} \rho_\varepsilon(z) \, dz \int_{\Omega''} |\hat{u}(\xi) - u(\xi)|^p \, d\xi \\ &= \int_{\Omega''} |\hat{u}(\xi) - u(\xi)|^p \, d\xi, \end{aligned}$$

for $\varepsilon < \operatorname{dist}(\Omega', \partial\Omega'')$. Thus from (2.15), (2.16) we derive

$$\begin{aligned} |u - u_\varepsilon|_{p,\Omega'} &\le \frac{2\delta}{3} + |\hat{u} - \hat{u}_\varepsilon|_{p,\Omega'} \\ &\le \delta \end{aligned}$$

for ε small enough by the point 2 since the uniform convergence implies the L^p convergence in the bounded set Ω'. This completes the proof. $\square$

Remark. If $u \in L^p(\Omega)$, Ω bounded, and if we denote by $\bar{u}$ the extension of u by 0 outside of Ω, then as a corollary of Theorem 2.1 we have

$$\bar{u}_\varepsilon = \rho_\varepsilon * \bar{u} \longrightarrow u \quad \text{in } L^p(\Omega).$$

We show now the following compactness result which is a L^p-generalization of the Arzelà–Ascoli theorem.

Theorem 2.2 (Fréchet–Kolmogorov). *Let $\Omega' \subset\subset \Omega$ and $\mathcal{L}$ a subset of $L^p(\Omega)$ $1 \leq p < +\infty$ such that*

- *$\mathcal{L}$ is bounded in $L^p(\Omega)$*
- *$\mathcal{L}$ is equicontinuous in $L^p(\Omega')$ – that is to say*

$$\begin{aligned}&\forall\, \eta > 0\ \exists\, \delta > 0, \quad \delta < \operatorname{dist}(\Omega', \partial\Omega) \text{ such that}\\ &\forall\, h \in \mathbb{R}^n, \quad |h| < \delta \Longrightarrow |\sigma_h f - f|_{p,\Omega'} < \eta \quad \forall\, f \in \mathcal{L}.\end{aligned}$$

Then $\mathcal{L}\big|_{\Omega'} = \{f|_{\Omega'} \mid f \in \mathcal{L}\}$ is relatively compact in $L^p(\Omega')$. ($\sigma_h f$ is the translated of f defined as $\sigma_h f(x) = f(x+h)$ $\forall\, h \in \mathbb{R}^n$.)

Proof. We can without loss of generality assume that Ω is bounded. If $f \in \mathcal{L}$ we denote by $\bar{f}$ its extension by 0 outside Ω and set

$$\mathcal{L}_0 = \{\, \bar{f} \mid f \in \mathcal{L}\,\}.$$

It is clear then that $\mathcal{L}_0$ is a bounded set of $L^p(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$. First we claim that for $\varepsilon < \delta$ we have

$$|\rho_\varepsilon * \bar{f} - \bar{f}|_{p,\Omega'} < \eta \quad \forall\, \bar{f} \in \mathcal{L}_0.$$

Indeed if $\varepsilon < \delta < \operatorname{dist}(\Omega', \partial\Omega)$

$$\begin{aligned}|\rho_\varepsilon * \bar{f} - \bar{f}|^p_{p,\Omega'} &= \int_{\Omega'} |\rho_\varepsilon * \bar{f}(x) - \bar{f}(x)|^p\, dx\\ &= \int_{\Omega'} \left| \int_{B_\varepsilon(x)} \{\bar{f}(y) - \bar{f}(x)\}\rho_\varepsilon(x-y)\, dy \right|^p dx\\ &= \int_{\Omega'} \left| \int_{B_\varepsilon} \{\bar{f}(x-z) - \bar{f}(x)\}\rho_\varepsilon(z)\, dz \right|^p dx\\ &\leq \int_{\Omega'} \left(\int_{B_\varepsilon} |\bar{f}(x-z) - \bar{f}(x)| \rho_\varepsilon^{\frac{1}{p}}(z) \rho_\varepsilon^{\frac{1}{p'}}(z)\, dz \right)^p dx\\ &\leq \int_{\Omega'} \int_{B_\varepsilon} |\bar{f}(x-z) - \bar{f}(x)|^p \rho_\varepsilon(z)\, dz\, dx, \quad \text{by Hölder's inequality}\\ &\leq |\sigma_z f - f|^p_{p,\Omega'} < \eta^p.\end{aligned} \tag{2.17}$$

(Recall that $\int_{B_\varepsilon} \rho_\varepsilon\, dz = 1$.) Consider then

$$\mathcal{L}_\varepsilon = \{\, \rho_\varepsilon * \bar{f} \mid \bar{f} \in \mathcal{L}_0\,\}.$$

We claim that $\mathcal{L}_\varepsilon$ satisfies the assumptions of the Arzelà–Ascoli theorem which are

- $\mathcal{L}_\varepsilon$ is bounded in $C(\overline{\Omega}')$.
  Indeed this follows from

$$|\rho_\varepsilon * \bar{f}|_\infty = \left| \int_{B_\varepsilon(x)} \bar{f}(y)\rho_\varepsilon(x-y)\,dy \right|_\infty \leq |\rho_\varepsilon|_\infty |\bar{f}|_{1,\mathbb{R}^n}$$

  ($|\cdot|_\infty$ is the L^∞-norm in $C(\overline{\Omega}')$).
- $\mathcal{L}_\varepsilon$ is equicontinuous
  Indeed if $x_1, x_2 \in \mathbb{R}^n$ we have

$$\begin{aligned} |\rho_\varepsilon * \bar{f}(x_1) - \rho_\varepsilon * \bar{f}(x_2)| &\leq \int_{\mathbb{R}^n} |\bar{f}(y)|\,|\rho_\varepsilon(x_1 - y) - \rho_\varepsilon(x_2 - y)|\,dy \\ &\leq C_\varepsilon |x_1 - x_2|\,|\bar{f}|_{1,\mathbb{R}^n} \quad \forall\, \bar{f} \in \mathcal{L}_0 \end{aligned}$$

  where C_ε is the Lipschitz constant of ρ_ε. The claim follows then from the fact that $\mathcal{L}_0$ is bounded in $L^1(\mathbb{R}^n)$.

  From above it follows then that $\mathcal{L}_\varepsilon$ is relatively compact in $C(\overline{\Omega}')$ and thus also in $L^p(\Omega')$. Given η we fix then ε such that

$$|\rho_\varepsilon * \bar{f} - \bar{f}|_{p,\Omega'} < \eta \quad \forall\, \bar{f} \in \mathcal{L}_0.$$

  Then we cover $\mathcal{L}_\varepsilon$ by a finite number of balls of radius η. It is then clear that the same balls of radius 2η cover $\mathcal{L}_0$ – which completes the proof. □

It is very useful to consider derivatives of objects having no derivative in the usual sense. This is done through duality and it is one of the great achievement of the distributions theory which emerged in the beginning of the fifties (see [86]). This is what we would like to consider next.

2.2 Introduction to distributions

For any multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote by D^α the partial derivative given by

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}. \tag{2.18}$$

We define first a notion of convergence in the space of functions $\mathcal{D}(\Omega)$.

Definition 2.3. Let φ_i, $i \in \mathbb{N}$ be a sequence of functions in $\mathcal{D}(\Omega)$. We say that

$$\varphi_i \longrightarrow \varphi \quad \text{in } \mathcal{D}(\Omega) \text{ when } i \longrightarrow +\infty \tag{2.19}$$

if the φ_i's and φ have all their supports contained in a compact subset K of Ω and if

$$D^\alpha \varphi_i \longrightarrow D^\alpha \varphi \quad \text{uniformly in } K, \quad \forall\, \alpha \in \mathbb{N}^n,$$

i.e.,

$$\lim_{i \to +\infty} \operatorname*{Sup}_{x \in K} |D^\alpha \varphi_i(x) - D^\alpha \varphi(x)| = 0 \quad \forall\, \alpha \in \mathbb{N}^n. \tag{2.20}$$

Remark 2.1. One can show that this notion of convergence can be defined by a topology on $\mathcal{D}(\Omega)$. We refer the reader to [86], [94].

We can now introduce the notion of distribution.

Definition 2.4. A distribution T on Ω is a continuous linear form on $\mathcal{D}(\Omega)$ – i.e., a linear form on $\mathcal{D}(\Omega)$ such that

$$\lim_{i \to +\infty} T(\varphi_i) = T(\varphi) \tag{2.21}$$

for any sequence φ_i such that $\varphi_i \to \varphi$ in $\mathcal{D}(\Omega)$. $T(\varphi)$ will be denoted by $\langle T, \varphi \rangle$ and the space of distributions on Ω by $\mathcal{D}'(\Omega)$.

Examples.

1. Let $T \in L^1_{\text{loc}}(\Omega)$. Then

$$\langle T, \varphi \rangle = \int_\Omega T(x)\varphi(x)\, dx \quad \forall\, \varphi \in \mathcal{D}(\Omega) \tag{2.22}$$

defines a distribution. Indeed this follows from

$$|\langle T, \varphi_i - \varphi \rangle| = \left| \int_\Omega T(x)(\varphi_i(x) - \varphi(x))\, dx \right| \le \operatorname*{Sup}_{x \in K} |\varphi_i(x) - \varphi(x)| \int_K |T|\, dx.$$

2. If T_1, T_2 are distributions, $\alpha_1, \alpha_2 \in \mathbb{R}$ then $\alpha_1 T_1 + \alpha_2 T_2$ defined as

$$\langle \alpha_1 T_1 + \alpha_2 T_2, \varphi \rangle = \alpha_1 \langle T_1, \varphi \rangle + \alpha_2 \langle T_2, \varphi \rangle \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

is a distribution. This shows that $\mathcal{D}'(\Omega)$ is a vector space on $\mathbb{R}$.

3. The Dirac mass. Let $x_0 \in \Omega$. Then

$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0) \quad \forall\, \varphi \in \mathcal{D}(\Omega) \tag{2.23}$$

defines a distribution called the Dirac mass at x_0. This is an example of a distribution which is not a function, i.e., one cannot find $T \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega T(x)\varphi(x)\, dx = \varphi(x_0) \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.24}$$

δ_{x_0} is a "measure".

To see the impossibility of having (2.24) one needs the following theorem. It establishes the consistency between equality in the distributional sense and the equality in the L^1-sense for functions. It is clear that if $T_1, T_2 \in L^1_{\text{loc}}(\Omega)$ and

$$T_1 = T_2 \quad \text{a.e. in } \Omega \tag{2.25}$$

then T_1, T_2 define the same distribution through the formula (2.22). Conversely we have

Theorem 2.3. *Suppose that $T_1, T_2 \in L^1_{\text{loc}}(\Omega)$ and*

$$\langle T_1, \varphi\rangle = \langle T_2, \varphi\rangle \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.26}$$

Then $T_1 = T_2$ a.e. in Ω.

Proof. Consider $\Omega' \subset\subset \Omega$ – i.e., Ω' is an open set included in a compact subset of Ω. If $\partial\Omega$ denotes the boundary of Ω then

$$d = \operatorname{dist}(\Omega', \partial\Omega) = \operatorname*{Inf}_{x\in\Omega', y\in\partial\Omega} |x-y| > 0. \tag{2.27}$$

If ρ_ε is defined by (2.8) for $\varepsilon < d$ one has for every $x \in \Omega'$

$$\begin{aligned}(T_1 * \rho_\varepsilon)(x) &= \int_\Omega T_1(y)\rho_\varepsilon(x-y)\,dy = \langle T_1, \rho_\varepsilon(x-\cdot)\rangle \\ &= \langle T_2, \rho_\varepsilon(x-\cdot)\rangle = (T_2 * \rho_\varepsilon)(x).\end{aligned} \tag{2.28}$$

Now when $\varepsilon \to 0$ we have

$$T_i * \rho_\varepsilon \longrightarrow T_i \quad \text{in } L^1(\Omega') \quad i = 1, 2, \tag{2.29}$$

(see Theorem 2.1). The result follows then from (2.28). □

Remark 2.2. To show that (2.24) cannot hold for $T \in L^1_{\text{loc}}(\Omega)$ it is enough to notice that (2.24) implies that $\langle T,\varphi\rangle = 0\ \forall\, \varphi \in \mathcal{D}(\Omega \setminus \{x_0\})$. Then, by Theorem 2.3, $T = 0$ a.e. in $\Omega \setminus \{x_0\}$, i.e., $T = 0$ a.e. in Ω and T is the zero distribution which is not the case of the Dirac mass.

As we fixed us as goal at the beginning of this section we can now differentiate any distribution. Indeed we have:

Definition 2.5. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then for $T \in \mathcal{D}'(\Omega)$

$$\varphi \mapsto (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle$$

defines a distribution on Ω which we denote by $D^\alpha T$. Thus we have

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.30}$$

Example. Suppose that T is a function on Ω, k times differentiable. For $|\alpha| \leq k$ denote provisionally by ∂^α the derivative of T in the usual sense – i.e.,

$$\partial^\alpha T(x) = \frac{\partial^{|\alpha|} T(x)}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} \quad \forall\, x \in \Omega. \tag{2.31}$$

Then we have

$$D^\alpha T = \partial^\alpha T, \tag{2.32}$$

i.e., the derivative in the distributional sense of T coincides with the distribution defined by the function $\partial^\alpha T$. To see this consider $\alpha = (0, \dots, 1, \dots 0)$ the 1 being at the i^{th}-slot. One has

$$\begin{aligned} \langle D^\alpha T, \varphi \rangle = -\langle T, \partial_{x_i} \varphi \rangle &= -\int_\Omega T(x) \partial_{x_i} \varphi(x)\, dx \\ &= -\int_\Omega (\partial_{x_i}(T\varphi) - \partial_{x_i} T \varphi)\, dx. \end{aligned} \tag{2.33}$$

For simplicity we set $\partial_{x_i} = \frac{\partial}{\partial_{x_i}}$ and will do that also in what follows. Now clearly by the Fubini theorem

$$\int_\Omega \partial_{x_i}(T\varphi)\, dx = \int_{x'} \int_{x_i} \partial_{x_i}(T\varphi)\, dx = 0.$$

(In the integral above we integrate first in x_i then in the other variables that we denote by x'. Note that $T\varphi$ vanishes on the boundary of Ω.) Then (2.33) becomes

$$\langle D^\alpha T, \varphi \rangle = \int_\Omega \partial_{x_i} T \varphi\, dx = \langle \partial^\alpha T, \varphi \rangle.$$

(2.32) follows then since for any $\alpha \in \mathbb{N}^n$, D^α can be obtained by iterating operators of the type ∂_{x_i}. From now on we will use the same notation for the derivative in the usual sense and in the distributional sense for C^k-functions.

Like we defined a notion of convergence in $\mathcal{D}(\Omega)$, we define now convergence in $\mathcal{D}'(\Omega)$.

Definition 2.6. Let T_i be a sequence of distributions on Ω. We say that

$$\lim_{i \to +\infty} T_i = T \quad \text{in } \mathcal{D}'(\Omega) \tag{2.34}$$

iff

$$\lim_{i \to +\infty} \langle T_i, \varphi \rangle = \langle T, \varphi \rangle \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.35}$$

One can show that the convergence above defines a topology – see [86], [94].

Example. Suppose that x_i is a sequence of points in Ω such that x_i converges toward $x_\infty \in \Omega$. Then one has

$$\delta_{x_i} \longrightarrow \delta_{x_\infty} \quad \text{in } \mathcal{D}'(\Omega).$$

Since

$$L^p(\Omega) \subset L^1_{\text{loc}}(\Omega) \quad \forall\, 1 \le p \le +\infty \tag{2.36}$$

the functions of $L^p(\Omega)$ are distributions on Ω.

Moreover we have

Proposition 2.4. *Let $T_i, T \in L^p(\Omega)$. Suppose that when $i \to +\infty$*

$$T_i \to T \quad \textit{in } L^p(\Omega), \quad (\textit{resp: } T_i \rightharpoonup T \ \textit{in } L^p(\Omega)) \quad 1 \le p < \infty \tag{2.37}$$

then one has

$$T_i \longrightarrow T \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.38}$$

Proof. We have denoted by $\rightharpoonup$ the weak convergence in $L^p(\Omega)$. Note that the strong convergence in $L^p(\Omega)$ implies the weak one and this one can be expressed as

$$\int_\Omega T_i \varphi \, dx \longrightarrow \int_\Omega T\varphi \, dx \quad \forall\, \varphi \in L^{p'}(\Omega).$$

The result follows then from the fact that $\mathcal{D}(\Omega) \subset L^{p'}(\Omega)$. □

We also have the following

Proposition 2.5. *The operator D^α, $\alpha \in \mathbb{N}^n$ is continuous on $\mathcal{D}'(\Omega)$, i.e.,*

$$T_i \longrightarrow T \quad \textit{in } \mathcal{D}'(\Omega) \quad \Longrightarrow \quad D^\alpha T_i \longrightarrow D^\alpha T \quad \textit{in } \mathcal{D}'(\Omega)$$

for $i \to +\infty$.

Proof. This follows immediately from

$$\langle D^\alpha T_i, \varphi \rangle = (-1)^{|\alpha|} \langle T_i, D^\alpha \varphi \rangle \longrightarrow (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle = \langle D^\alpha T, \varphi \rangle$$

since $D^\alpha \varphi \in \mathcal{D}(\Omega)$ $\forall\, \alpha \in \mathbb{N}^n$, $\forall\, \varphi \in \mathcal{D}(\Omega)$. □

2.3 Sobolev Spaces

The Sobolev Spaces are useful tools to solve partial differential equations. In this chapter we concentrate ourselves to the most simple ones.

Definition 2.7. Let Ω be an open subset of $\mathbb{R}^n$. We denote by $H^1(\Omega)$ the subset of $L^2(\Omega)$ defined by

$$H^1(\Omega) = \{\, v \in L^2(\Omega) \mid \partial_{x_i} v \in L^2(\Omega) \ \forall\, i = 1, \dots, n \,\}. \tag{2.39}$$

In this definition $\partial_{x_i} v$ denotes the derivative of v in the distributional sense.

Remark 2.3. In other words $v \in H^1(\Omega)$ iff $v \in L^2(\Omega)$ and for every i there exists a function v_i in $L^2(\Omega)$ – that we denote $\partial_{x_i} v$ – such that

$$-\int_\Omega v \partial_{x_i} \varphi \, dx = \int_\Omega v_i \varphi \, dx \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Example. If Ω is a bounded open set the restriction to Ω of every continuously differentiable function of $\mathbb{R}^n$ belongs to $H^1(\Omega)$.

For $u, v \in H^1(\Omega)$ one can define the scalar product

$$(u, v)_{1,2} = \int_\Omega uv + \nabla u \cdot \nabla v \, dx. \tag{2.40}$$

In the formula above ∇u denotes the gradient of u – i.e., the vector

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u) \tag{2.41}$$

and $\nabla u \cdot \nabla v$ the Euclidean scalar product of ∇u, ∇v that is to say

$$\nabla u \cdot \nabla v = \sum_{i=1}^n \partial_{x_i} u \partial_{x_i} v.$$

One has:

Theorem 2.6. *Equipped with the scalar product* (2.40) $H^1(\Omega)$ *is a Hilbert space. The associated norm will be denoted*

$$|v|_{1,2} = \left\{ \int_\Omega v^2 + |\nabla v|^2 \, dx \right\}^{\frac{1}{2}} \tag{2.42}$$

where $|\cdot|$ *stands for the Euclidean norm in* $\mathbb{R}^n$.

Proof. It is easy to see that (2.40) defines a scalar product. So, we will be done provided we show that $H^1(\Omega)$ equipped with (2.42) is complete. Let us denote by v_n a Cauchy sequence in $H^1(\Omega)$ which is a sequence such that for any ε

$$\left\{ \int_\Omega (v_n - v_m)^2 + \sum_{i=1}^n (\partial_{x_i} v_n - \partial_{x_i} v_m)^2 \, dx \right\}^{\frac{1}{2}} = |v_n - v_m|_{1,2} \le \varepsilon \tag{2.43}$$

for n, m large enough. It follows that

$$v_n, \quad \partial_{x_i} v_n \quad i = 1, \dots, n$$

are Cauchy sequences in $L^2(\Omega)$. Since $L^2(\Omega)$ is a Hilbert space there exist functions

$$u \in L^2(\Omega), \qquad u_i \in L^2(\Omega) \quad i = 1, \dots, n$$

such that

$$v_n \longrightarrow u, \qquad \partial_{x_i} v_n \longrightarrow u_i \quad \text{in } L^2(\Omega).$$

Due to Propositions 2.4, 2.5 one has also

$$\partial_{x_i} v_n \longrightarrow \partial_{x_i} u, \qquad \partial_{x_i} v_n \longrightarrow u_i \quad \text{in } \mathcal{D}'(\Omega).$$

It follows that

$$\partial_{x_i} u = u_i \in L^2(\Omega) \quad \forall\, i = 1, \ldots, n,$$

i.e., $u \in H^1(\Omega)$. Moreover letting $m \to +\infty$ in (2.43) we get

$$|v_n - u|_{1,2} \leq \varepsilon$$

for n large enough that is to say $v_n \to u$ in $H^1(\Omega)$. This completes the proof. □

In what follows we will be in need of functions vanishing on the boundary $\partial\Omega$ of Ω. However for a class of functions in $L^2(\Omega)$ the meaning of its value on $\partial\Omega$ is not clear. So we will overcome this problem by introducing $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ defined as

$$H_0^1(\Omega) = \text{ the closure of } \mathcal{D}(\Omega) \text{ in } H^1(\Omega) \tag{2.44}$$

the closure being understood for the norm (2.42). $H_0^1(\Omega)$ will play the rôle of the functions of $H^1(\Omega)$ which vanish on $\partial\Omega$. We have

Theorem 2.7. *Equipped with the scalar product* (2.40) *and the norm* (2.42) $H_0^1(\Omega)$ *is a Hilbert space.*

Proof. This follows immediately from the fact that $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$. □

Since the functions of $H_0^1(\Omega)$ are vanishing – in a certain sense – one does not need the $L^2(\Omega)$-norm to control their convergence in $H^1(\Omega)$. The $L^2(\Omega)$-norm of the derivatives will be enough. This will be a consequence of the following theorem. Before to state it let us introduce briefly the notion of directional derivative. Let ν be a unit vector in $\mathbb{R}^n$. If v is a differentiable function then the limit

$$\lim_{h \to 0} \frac{v(x + h\nu) - v(x)}{h} \tag{2.45}$$

exists and it is called the derivative of v in the direction ν. For instance $\partial_{x_1} v$ is the derivative in the direction $e_1 = (1, \ldots, 0)$, where e_1 is the first vector of the canonical basis in $\mathbb{R}^n$. To see that the limit of (2.45) exists for v smooth it is enough to note that by the mean value theorem and the chain rule one has

$$\begin{aligned} v(x + h\nu) - v(x) &= \frac{d}{dt} v(x + th\nu)\big|_\theta \quad (\theta \in (0,1)) \\ &= \nabla v(x + \theta h\nu) \cdot h\nu \end{aligned} \tag{2.46}$$

(recall that the $\cdot$ denotes here the Euclidean scalar product).

Dividing by h and letting $h \to 0$ it follows that

$$\lim_{h\to 0} \frac{v(x+h\nu)-v(x)}{h} = \nabla v(x)\cdot\nu = \partial_\nu v(x). \tag{2.47}$$

This derivative in the ν-direction will be denoted in the following as above as $\partial_\nu v$ or also $\frac{\partial v}{\partial \nu}$. Note that if $v \in H^1(\Omega)$ then for a fixed vector ν, the last equality defines a function $\partial_\nu v$ which is in $L^2(\Omega)$.

Then we can now state

Theorem 2.8 (Poincaré Inequality). *Let ν be a unit vector in $\mathbb{R}^n$, $a > 0$. Suppose that Ω is bounded in one direction more precisely*

$$\Omega \subset \{\, x \in \mathbb{R}^n \mid |x\cdot\nu| \le a \,\}. \tag{2.48}$$

Then we have

$$|v|_{2,\Omega} \le \sqrt{2}a\left|\frac{\partial v}{\partial \nu}\right|_{2,\Omega} \quad \forall\, v \in H^1_0(\Omega). \tag{2.49}$$

(*Recall that* $\frac{\partial v}{\partial \nu}$ *is defined by* (2.47). $|\cdot|_{2,\Omega}$ *was defined in Section* 2.1.)

Proof. By definition of $H^1_0(\Omega)$ and $\frac{\partial v}{\partial \nu}$ it is enough to show (2.49) for any $v \in \mathcal{D}(\Omega)$. Consider then $v \in \mathcal{D}(\Omega)$ and suppose it extended to all $\mathbb{R}^n$ by 0 outside Ω. Without loss of generality we can assume the coordinate system chosen such that $\nu = e_1$. Then for $x \in \Omega$ we have

$$v(x) = v(x) - v(-a, x_2, \ldots, x_n) = \int_{-a}^{x_1} \partial_{x_1} v(s, x_2, \ldots, x_n)\, ds.$$

Squaring this inequality and using the inequality of Cauchy–Schwarz we get

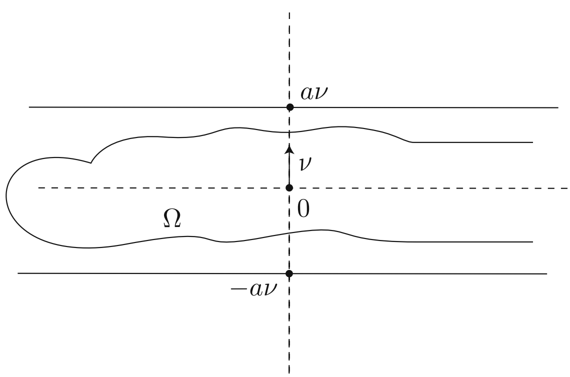


Figure 2.1: Bounded open set in one direction

$$\begin{aligned}
v^2(x) &= \left(\int_{-a}^{x_1} \partial_{x_1} v(s, x_2, \ldots, x_n)\, ds\right)^2 \le \left(\int_{-a}^{x_1} |\partial_{x_1} v(s, x_2, \ldots, x_n)|\, ds\right)^2 \\
&\le |x_1 + a| \int_{-a}^{x_1} \partial_{x_1} v(s, x_2, \ldots, x_n)^2\, ds \\
&\le |x_1 + a| \int_{-a}^{a} \partial_{x_1} v(s, x_2, \ldots, x_n)^2\, ds.
\end{aligned}$$

Integrating this inequality in x_1, we derive

$$\begin{aligned}
\int_{-a}^{a} v^2(x_1, x_2, \ldots, x_n)\, dx_1 &\le \frac{(x_1 + a)^2}{2}\bigg|_{-a}^{a} \int_{-a}^{a} \partial_{x_1} v(x_1, \ldots, x_n)^2\, dx_1 \\
&= 2a^2 \int_{-a}^{a} \partial_{x_1} v(x_1, \ldots, x_n)^2\, dx_1.
\end{aligned} \tag{2.50}$$

Integrating in the other directions leads to (2.49). □

Then we can show

Theorem 2.9. *Suppose that Ω is bounded in one direction, i.e., satisfies* (2.48) *then on $H_0^1(\Omega)$ the norms*

$$|v|_{1,2}, \qquad \big||\nabla v|\big|_{2,\Omega} \tag{2.51}$$

are equivalent.

Proof. $\big||\nabla v|\big|_{2,\Omega}$ is the $L^2(\Omega)$-norm of the gradient defined as

$$\big||\nabla v|\big|_{2,\Omega} = \left(\int_\Omega |\nabla v(x)|^2\, dx\right)^{\frac{1}{2}}.$$

Thus one has

$$\big||\nabla v|\big|_{2,\Omega} \le |v|_{1,2} \quad \forall\, v \in H_0^1(\Omega).$$

Next due to Theorem 2.8 one has for $v \in \mathcal{D}(\Omega)$

$$\int_\Omega v^2\, dx \le 2a^2 \int_\Omega \left(\frac{\partial v}{\partial \nu}\right)^2 dx \le 2a^2 \int_\Omega |\nabla v|^2\, dx \tag{2.52}$$

(see (2.47) and recall that ν is a unit vector). It follows that

$$\int_\Omega v^2 + |\nabla v|^2\, dx \le (1 + 2a^2) \int_\Omega |\nabla v|^2\, dx$$

or also

$$|v|_{1,2} \le (1 + 2a^2)^{\frac{1}{2}} \big||\nabla v|\big|_{2,\Omega}.$$

This completes the proof of the theorem since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$. □

We introduce now the dual space of $H_0^1(\Omega)$. It is denoted by $H^{-1}(\Omega)$, i.e.,

$$H^{-1}(\Omega) = (H_0^1(\Omega))^*. \tag{2.53}$$

The notation is justified by the following result:

Theorem 2.10. *$T \in H^{-1}(\Omega)$ iff there exist $T_0, T_1, \dots, T_n \in L^2(\Omega)$ such that*

$$T = T_0 + \sum_{i=1}^{n} \partial_{x_i} T_i. \tag{2.54}$$

(*The derivative in* (2.54) *is understood in the distributional sense.*)

Proof. First consider a distribution of the type (2.54). For $\varphi \in \mathcal{D}(\Omega)$ one has

$$\begin{aligned}
|\langle T, \varphi\rangle| &= |\langle T_0, \varphi\rangle - \sum_{i=1}^{n} \langle T_i, \partial_{x_i}\varphi\rangle| \\
&= \left| \int_\Omega \left(T_0\varphi - \sum_{i=1}^{n} T_i \partial_{x_i}\varphi \right) dx \right| \\
&\leq \int_\Omega \left(|T_0|\,|\varphi| + \sum_{i=1}^{n} |T_i|\,|\partial_{x_i}\varphi| \right) dx \\
&\leq \int_\Omega \left(\sum_{i=0}^{n} T_i^2 \right)^{\frac12} (|\varphi|^2 + |\nabla\varphi|^2)^{\frac12}\, dx.
\end{aligned}$$

(We used here the Cauchy–Schwarz inequality in $\mathbb{R}^{n+1}$.) By the Cauchy–Schwarz inequality again we get

$$|\langle T, \varphi\rangle| \leq \left(\int_\Omega \sum_{i=0}^{n} T_i^2\, dx \right)^{\frac12} |\varphi|_{1,2} \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{2.55}$$

By density (2.55) holds for any $\varphi \in H_0^1(\Omega)$ and $T \in (H_0^1(\Omega))^*$.

Conversely let T be in $H^{-1}(\Omega)$. By the Riesz representation theorem there exists $u \in H_0^1(\Omega)$ such that

$$\int_\Omega uv + \nabla u \cdot \nabla v\, dx = \langle T, v\rangle \quad \forall\, v \in H_0^1(\Omega).$$

Setting $T_0 = u$, $T_i = -\partial_{x_i} u$ it is clear from the equality above that

$$T = T_0 + \sum_{i=1}^{n} \partial_{x_i} T_i$$

in $\mathcal{D}'(\Omega)$. This completes the proof of the theorem. □

Remark 2.4. Note that T_0 can be chosen equal to 0 if we use the Riesz representation theorem with the scalar product $\int \nabla u \cdot \nabla v\, dx$. The strong dual norm on $H^{-1}(\Omega)$ can be defined by

$$|T|_* = \operatorname*{Sup}_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle T, v\rangle|}{|v|_{1,2}}. \tag{2.56}$$

From Theorem 2.10 one deduces easily that

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \tag{2.57}$$

$\hookrightarrow$ meaning that $L^2(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$ namely the identity is a continuous mapping. Indeed note that if $T = T_0 \in L^2(\Omega)$ then

$$|\langle T_0, v\rangle| \leq |T_0|_{2,\Omega}|v|_{2,\Omega} \leq |T_0|_{2,\Omega}|v|_{1,2} \quad \forall\, v \in H_0^1(\Omega)$$

(by the Cauchy–Schwarz inequality and (2.42)). It follows that

$$|T|_* \leq |T|_{2,\Omega}$$

which proves our assertion. One should remark that Theorem 2.10 is valid for an arbitrary domain Ω of $\mathbb{R}^n$.

Regarding the derivatives of mollifiers we have the following:

Proposition 2.11. *Suppose that $u \in L^1_{\text{loc}}(\Omega)$ is such that $\partial_{x_i} u \in L^1_{\text{loc}}(\Omega)$ (we mean here the derivative in the distributional sense). Then if $d(x, \partial\Omega) > \varepsilon$, the usual derivative in the direction x_i of $u_\varepsilon = \rho_\varepsilon * u(x)$ exists at x and is given by*

$$\partial_{x_i} u_\varepsilon(x) = \rho_\varepsilon * \partial_{x_i} u(x).$$

Proof. We recall the formula (2.10). Then differentiating under the integral sign we have

$$\begin{aligned}
\partial_{x_i} u_\varepsilon &= \int_\Omega u(y)\partial_{x_i}\rho_\varepsilon(x-y)\,dy \\
&= -\int_\Omega u(y)\partial_{y_i}(\rho_\varepsilon(x-y))\,dy \\
&= \langle \partial_{y_i} u, \rho_\varepsilon(x-\cdot)\rangle \\
&= \int_\Omega \partial_{y_i} u(y)\rho_\varepsilon(x-y)\,dy
\end{aligned}$$

which completes the proof. □

Then we can show

Proposition 2.12. *Suppose that f is a function of $\mathbb{R}$ into $\mathbb{R}$ such that*

$$f \in C^1(\mathbb{R}), \qquad f' \in L^\infty(\mathbb{R}).$$

Then if $u \in L^1_{\text{loc}}(\Omega)$, $\partial_{x_i} u \in L^1_{\text{loc}}(\Omega)$ we have $f(u) \in L^1_{\text{loc}}(\Omega)$ and the derivative in the distributional sense is given by

$$\partial_{x_i} f(u) = f'(u)\partial_{x_i} u \in L^1_{\text{loc}}(\Omega).$$

In particular if $f(0) = 0$, $u \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$) implies $f(u) \in H^1(\Omega)$ (resp. $H_0^1(\Omega)$).

Proof. Consider $u_\varepsilon = \rho_\varepsilon * u$ as in the previous proposition. By the chain rule for $\varepsilon < \operatorname{dist}(x, \partial\Omega)$, $f(u_\varepsilon)$ has a derivative in the usual sense given by

$$\partial_{x_i}(f(u_\varepsilon)) = f'(u_\varepsilon)\partial_{x_i} u_\varepsilon.$$

Let $\varphi \in \mathcal{D}(\Omega)$, $\varepsilon < \operatorname{dist}(\operatorname{Supp}\varphi, \partial\Omega)$. We have

$$-\langle f(u_\varepsilon), \partial_{x_i}\varphi\rangle = \langle f'(u_\varepsilon)\partial_{x_i} u_\varepsilon, \varphi\rangle. \tag{2.58}$$

When $\varepsilon \to 0$ we have also for any compact subset of Ω

$$\begin{aligned}
&\int_K |f(u_\varepsilon) - f(u)|\,dx \le \operatorname{Sup}|f'| \int_K |u_\varepsilon - u|\,dx \to 0,\\
&\int_K |f'(u_\varepsilon)\partial_{x_i} u_\varepsilon - f'(u)\partial_{x_i} u|\,dx\\
&\quad = \int_K |f'(u_\varepsilon)(\partial_{x_i} u_\varepsilon - \partial_{x_i} u) + \partial_{x_i} u (f'(u_\varepsilon) - f'(u))|\,dx\\
&\quad \le \operatorname{Sup}|f'| \int_K |\partial_{x_i} u_\varepsilon - \partial_{x_i} u|\,dx + \int_K |\partial_{x_i} u|\,|f'(u_\varepsilon) - f'(u)|\,dx \to 0
\end{aligned}$$

by the Lebesgue theorem and Theorem 2.1. This shows the first part of the theorem by passing to the limit in (2.58). If $u \in H^1(\Omega)$, by the formula giving the weak derivative we have of course $f(u) \in H^1(\Omega)$. If $u \in H_0^1(\Omega)$ and if $\varphi_n \in \mathcal{D}(\Omega)$ is a sequence such that $\varphi_n \to u$ in $H_0^1(\Omega)$, $\varphi_n \to u$ a.e., we have $f(\varphi_n) \to f(u)$ in $H_0^1(\Omega)$ (just replace u_ε by φ_n in the inequalities above). Now $f(\varphi_n) \in H_0^1(\Omega)$ as a C^1-function with compact support which can be approximated by mollifyers. This completes the proof. □

Remark 2.5. If $u \in H^1\Omega)$ one can drop the assumption $f(0) = 0$ when Ω is bounded. Indeed, in this case $f(u) \in H^1(\Omega)$ since $f(u) \in L^2(\Omega)$ due to the inequality

$$|f(u)| \le |f(u) - f(0)| + |f(0)| \le \operatorname{Sup}|f'||u| + |f(0)|.$$

We denote by u^+, u^- the positive and negative part of a function. Recall that

$$\begin{aligned}
u^+ &= \operatorname{Max}(u, 0) = u \vee 0, \qquad & u^- &= \operatorname{Max}(-u, 0) = -u \vee 0,\\
u &= u^+ - u^-, & |u| &= u^+ + u^-.
\end{aligned}$$

Then we have

Theorem 2.13. *Suppose that $u \in L^1_{\mathrm{loc}}(\Omega)$, $\partial_{x_i} u \in L^1_{\mathrm{loc}}(\Omega)$ (we mean here the derivative in the distributional sense) then in the distributional sense we have*

$$\partial_{x_i} u^+ = \chi_{\{u>0\}}\partial_{x_i} u$$

where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$.

Proof. Let h_ε be a continuous function such that

$$h_\varepsilon(x) = 0 \quad x \le 0, \qquad h_\varepsilon(x) = 1 \quad x \ge \varepsilon, \qquad 0 \le h_\varepsilon \le 1. \tag{2.59}$$

Define

$$H_\varepsilon(x) = \int_{-\infty}^{x} h_\varepsilon(s)\, ds.$$

Clearly $H_\varepsilon \in C^1(\mathbb{R})$ and $H'_\varepsilon = h_\varepsilon \in L^\infty(\Omega)$. Thus in the distributional sense we have (see Proposition 2.12)

$$\partial_{x_i} H_\varepsilon(u) = h_\varepsilon(u)\partial_{x_i} u.$$

This can be written as

$$-\int_\Omega H_\varepsilon(u)\partial_{x_i}\varphi\, dx = \int_\Omega h_\varepsilon(u)\partial_{x_i} u\varphi\, dx \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

Passing to the limit we obtain

$$-\int_\Omega u^+\partial_{x_i}\varphi\, dx = \int_\Omega \chi_{\{u>0\}}\partial_{x_i} u\varphi\, dx \quad \forall\, \varphi \in \mathcal{D}(\Omega)$$

and the result follows. □

Remark 2.6. One has of course

$$\partial_{x_i} u^- = -\chi_{\{u<0\}}\partial_{x_i} u$$

with an obvious notation for $\chi_{\{u<0\}}$. As a consequence we also have

$$(1 - \chi_{\{u\neq 0\}})\partial_{x_i} u = 0$$

namely

$$\partial_{x_i} u = 0 \text{ a.e. on } \{u = 0\} = \{\, x \in \Omega \mid u(x) = 0\,\}.$$

As a corollary we have

Corollary 2.14. *Suppose that $u \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) then $u^+ \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$) and we have in the distributional sense*

$$\partial_{x_i} u^+ = \chi_{\{u>0\}}\partial_{x_i} u. \tag{2.60}$$

Proof. The formula (2.60) is an immediate consequence of the previous proposition. If now $\varphi_n \in \mathcal{D}(\Omega)$ is such that

$$\varphi_n \to u \quad \text{in } H^1(\Omega),$$

choosing in (2.59) $h_\varepsilon \in C^\infty(\mathbb{R})$ – which is always possible – we see that

$$H_\varepsilon(\varphi_n) \in \mathcal{D}(\Omega) \longrightarrow H_\varepsilon(u) \in H^1_0(\Omega).$$

When $\varepsilon \to 0$, $H_\varepsilon(u) \to u^+$ in $H^1(\Omega)$ and thus $u^+ \in H^1_0(\Omega)$. This completes the proof of the theorem. □

We will need the following corollary.

Corollary 2.15. *Let f be a continuous piecewise C^1-function such that $f' \in L^\infty(\mathbb{R})$, $f(0) = 0$. If $u \in H^1(\Omega)$ then $f(u) \in H^1(\Omega)$ and*

$$\partial_{x_i} f(u) = f'(u)\partial_{x_i} u.$$

If $u \in H_0^1(\Omega)$, $f(u) \in H_0^1(\Omega)$. (See also Remark 2.5.)

Proof. Without loss of generality we can assume that f' has only one jump at 0. Then one can write

$$f(u) = f_1(u^+) + f_2(u^-)$$

where $f_i \in C^1(\mathbb{R})$. The result follows then from the previous results. $\square$

Definition 2.8 (Compact mapping). A mapping $T : A \to B$, A, B Banach spaces is said to be compact iff the image of a ball of A is relatively compact in B.

Then we have:

Theorem 2.16. *Suppose that Ω is a bounded open set of $\mathbb{R}^n$, then the canonical embedding (i.e., the identity map) is compact from $H_0^1(\Omega)$ into $L^2(\Omega)$.*

Proof. Consider

$$\mathcal{L} = \{\, v \in H_0^1(\Omega) \mid |v|_{1,2} \le R \,\}$$

($\mathcal{L}$ is the ball of radius R of $H_0^1(\Omega)$). Denote by Ω' a bounded open set such that

$$\Omega \subset\subset \Omega'.$$

(Careful, this is Ω which is compactly embedded in Ω'!) Suppose the functions of $\mathcal{L}$ extended by 0 outside Ω. We have

- $\mathcal{L}$ is bounded in $L^2(\Omega')$.
  This follows from
  $$|v|_{2,\Omega'} = |v|_{2,\Omega} \le |v|_{1,2} \le R.$$
- The extension v of a function of $\mathcal{L}$ belongs to $H_0^1(\Omega')$ – see the definition of H_0^1. Moreover for any h such that
  $$h \in \mathbb{R}^n, \qquad |h| \le \operatorname{dist}(\Omega, \partial\Omega')$$
  we have
  $$\begin{aligned} v(x+h) - v(x) &= \int_0^1 \frac{d}{dt} v(x+th)\,dt = \int_0^1 \nabla v(x+th)\cdot h\,dt \\ &\le \left\{ \int_0^1 |\nabla v(x+th)\cdot h|^2\,dt \right\}^{\frac{1}{2}}. \end{aligned}$$

Squaring and integrating the resulting inequality on Ω we get

$$\begin{aligned}\int_\Omega |v(x+h)-v(x)|^2\,dx &\leq \int_\Omega\int_0^1 |\nabla v(x+th)|^2|h|^2\,dt\,dx\\ &= |h|^2\int_0^1\int_\Omega |\nabla v(x+th)|^2\,dx\,dt\\ &\leq |h|^2\int_0^1\int_{\Omega'} |\nabla v(y)|^2\,dy\,dt\\ &\leq |h|^2R^2.\end{aligned}$$

The result follows then from Theorem 2.2. Note that the inequality above can be first established for functions in $\mathcal{D}(\Omega)$ and then holds by density. □

The theorem is used in the following and in partial differential equations in general under the following form:

Theorem 2.17. *Let u_n be a bounded sequence of $H_0^1(\Omega)$that is to say such that*

$$|u_n|_{1,2} \leq R \quad \forall\, n \in \mathbb{N}$$

for some $R > 0$. Then there exists a subsequence of n, n_k and $u \in H_0^1(\Omega)$ such that

$$u_{n_k} \longrightarrow u \quad \text{in } L^2(\Omega), \qquad u_{n_k} \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

Proof. There exists a subsequence n'_k and $u' \in H_0^1(\Omega)$ such that

$$u_{n'_k} \rightharpoonup u' \quad \text{in } H_0^1(\Omega).$$

From Theorem 2.16 there exists a subsequence of n'_k, say n_k, such that

$$u_{n_k} \longrightarrow u \quad \text{in } L^2(\Omega).$$

Of course u, u' have to be the same and this completes the proof of the theorem. □

Remark 2.7. Theorem 2.16 can be extended to $H^1(\Omega)$ and to more general spaces – provided we have a suitable extension of the functions of $H^1(\Omega)$ in a neighbourhood of Ω. This is the case for instance if we assume Ω to be piecewise C^1 – see [21], [44].

Exercises

1. (Generalized Hölder's Inequality)
   Let f_i, $i = 1, \ldots, k$ be functions in $L^{p_i}(\Omega)$ with

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = 1.$$

Show that $\prod_{i=1}^k f_i \in L^1(\Omega)$ and that

$$\left|\prod_{i=1}^k f_i\right|_{1,\Omega} \le \prod_{i=1}^k |f_i|_{p_i,\Omega}.$$

2. We set
$$\eta(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0, \\ 0 & t \le 0. \end{cases}$$
Show that η is infinitely differentiable in $\mathbb{R}$. Deduce that the function ρ defined by (2.4) belongs to $\mathcal{D}(\mathbb{R}^n)$.
3. Show that $\mathcal{D}(\Omega)$ is an infinite-dimensional vector space on $\mathbb{R}$.
4. Let $H \in L^1_{\text{loc}}(\mathbb{R})$ be the function defined by
$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$
Show that the derivative H' of H in $\mathcal{D}'(\mathbb{R})$ is given by
$$H' = \delta_0$$
where δ_0 denotes the Dirac mass at 0.
5. Show that (2.36) holds.
6. Let $T \in \mathcal{D}'(\mathbb{R})$ such that $T' = 0$. Show that T is constant.
7. Show – for instance for Ω bounded – that the function constant equal to 1 belongs to $H^1(\Omega)$ but not in $H^1_0(\Omega)$ (Hint: one can use (2.49).)
8. Show that the two norms
$$|u|_{2,\Omega} + |\partial_{x_1} u|_{2,\Omega}, \qquad |u|_{1,2}$$
are not equivalent on $H^1(\Omega)$.
9. Let $\Omega = (0,1)$. Show that
$$u \mapsto u(1)$$
is a continuous linear form on $H^1(\Omega)$ – i.e., belongs to $H^1(\Omega)^*$ – but is not a distribution (otherwise it would be the 0 one).
10. Let u be a Lipschitz continuous function in Ω. Show, when Ω is bounded, that $u \in H^1(\Omega)$ and
$$\partial_{x_i} u \in L^\infty(\Omega) \quad \forall\, i = 1, \dots, n.$$
11. Let u be a Lipschitz continuous function in Ω vanishing on $\partial\Omega$. Show that $u \in H^1_0(\Omega)$. Show that
$$H^1(\Omega) \cap C_0(\overline{\Omega}) \subset H^1_0(\Omega).$$
($C_0(\overline{\Omega})$ denotes the space of continuous functions vanishing on $\partial\Omega$.)

12. Let $u \in H^1(\Omega)$ where Ω is a domain in $\mathbb{R}^n$, i.e., a connected open subset. Show that
$$\nabla u = 0 \quad \text{in } \Omega \quad \Longrightarrow \quad u = \text{Cst}\,.$$

13. Let Ω be a bounded open set in $\mathbb{R}$. Let $p \in L^2(\Omega)$ and
$$T = p' \in H^{-1}(\Omega).$$
Show that (see (2.56))
$$|T|_* = |p|_{2,\Omega}.$$

14. Show that on $H^1(\mathbb{R})$ the two norms
$$|u|_{1,2} \quad , \quad |u'|_{2,\mathbb{R}}$$
are not equivalent.

Chapter 3

Weak Formulation of Elliptic Problems

Solving a partial differential equation is not an easy task. We explain here the weak formulation method which allows to obtain existence and uniqueness of a solution "in a certain sense" which coincides necessarily with the solution in the usual sense if it exists. First let us explain briefly in which context second-order elliptic partial differential equations do arise.

3.1 Motivation

Let Ω be a domain of $\mathbb{R}^3$. Suppose that u denotes a density of some quantity (population, heat, fluid...) which diffuses in Ω at a constant rate, i.e., the velocity of the diffusion is independent of the time. It is well known in such a situation that the velocity of the diffusion of such quantity is proportional to the gradient of u that is to say one has if $\vec{v}$ is the diffusion velocity

$$\vec{v} = -a\nabla u \tag{3.1}$$

where a is some constant depending on the problem at hand. To understand the idea behind the formula, suppose that u is a temperature. Consider two neighbouring points

$$x, \quad x + h\nu$$

where ν is a unit vector. The flow of temperature between these two points has a velocity proportional to the difference of temperature and inversely proportional to the distance between these points. In other words the component of $\vec{v}$ in the direction ν is given by

$$\vec{v} \cdot \nu = -a\frac{u(x + h\nu) - u(x)}{h}, \tag{3.2}$$

where a is some positive constant. The sign "$-$" takes into account the fact that if $u(x + h\nu) > u(x)$ then the flow of heat will go in the opposite direction to ν. (The heat, the population... diffuses from high concentrations to low ones.) Passing to the limit in h we get

$$\vec{v} \cdot \nu = -a\nabla u(x) \cdot \nu. \tag{3.3}$$

This equality holding for every ν, one has clearly (3.1). (We refer the reader to [29] for a more substantial analysis.)

Assuming (3.1) and – to simplify $a = 1$ – consider then a cube Q contained in Ω. If ∂Q denotes the different faces of Q the quantity diffusing outside of Q is given by

$$\int_{\partial Q} \vec{v} \cdot \vec{n}\, d\sigma(x)$$

where $\vec{n}$ denotes the outward normal to ∂Q – $d\sigma$ the local element of surface on ∂Q. Since we suppose that the situation is steady, that is to say does not change with time this flow through ∂Q has to be compensated by an input

$$\int_Q f\, dx$$

where $f(x)$ is the local input – of population, heat – brought into the system and we have

$$\int_{\partial Q} \vec{v} \cdot \vec{n}\, d\sigma(x) = \int_Q f\, dx. \tag{3.4}$$

Taking into account (3.1) with $a = 1$ we have

$$-\int_{\partial Q} \nabla u \cdot \vec{n}\, d\sigma(x) = \int_Q f\, dx. \tag{3.5}$$

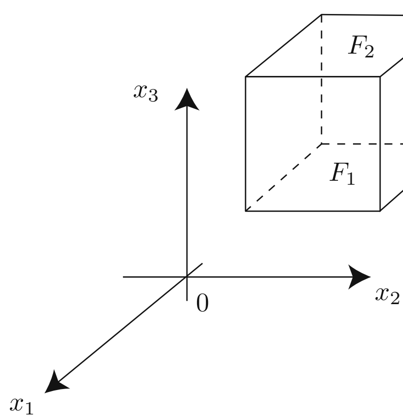


Figure 3.1.

Let us for instance evaluate the first integral above on the faces F_1, F_2 given by

$$F_i = \{\, x = (x_1, x_2, x_3) \in \partial Q \mid x_1 = \alpha_i \,\} \quad i = 1, 2.$$

One has

$$\begin{aligned} &- \int_{F_1} \nabla u \cdot \vec{n}\, d\sigma(x) - \int_{F_2} \nabla u \cdot \vec{n}\, d\sigma(x) \\ &\qquad = - \int_{\Pi(F_1)} \left(\frac{\partial u}{\partial x_1}(\alpha_1, x_2, x_3) - \frac{\partial u}{\partial x_1}(\alpha_2, x_2, x_3) \right) dx_2\, dx_3 \end{aligned}$$

where $\Pi(F_1)$ denotes the orthogonal projection of F_1 on the plane x_2, x_3. But the quantity above can also be written

$$- \int_{\Pi(F_1)} \int_{\alpha_1}^{\alpha_2} \frac{\partial^2 u}{(\partial x_1)^2}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = - \int_Q \partial^2_{x_1} u(x)\, dx.$$

Arguing for the other faces similarly we obtain from (3.5)

$$- \int_Q (\partial^2_{x_1} u + \partial^2_{x_2} u + \partial^2_{x_3} u)\, dx = \int_Q f\, dx,$$

i.e.,

$$\int_Q -\Delta u\, dx = \int_Q f\, dx \tag{3.6}$$

where Δ is the usual second-order operator defined by

$$\Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial {x_i}^2}.$$

Now the equality (3.6) is true for any Q and thus if for instance Δu and f are continuous this will force

$$-\Delta u(x) = f(x) \quad \forall\, x \in \Omega. \tag{3.7}$$

This equation is called the Laplace equation. To this equation is generally added a boundary condition. For instance

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.8}$$

(in the case of diffusion of temperature, the temperature is prescribed on the boundary of the body – this is for instance the room temperature that we can normalize to 0) or

$$\frac{\partial u}{\partial n} = \vec{v} \cdot \vec{n} = 0 \quad \text{on } \partial\Omega \tag{3.9}$$

($\vec{n}$ is the outward unit normal to $\partial\Omega$ – the body is isolated and there is no flux of temperature through its boundary).

Thus we are lead to the problem of finding u such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

This problem is called the "Dirichlet problem". Or we could be lead to look for u solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

This problem is a "Neumann problem".

3.2 The weak formulation

Suppose that we want to solve the Dirichlet problem. That is to say we give us an open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ and we would like to find a function u which satisfies

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Δ is the Laplace operator in $\mathbb{R}^n$ given by

$$\Delta = \sum_{i=1}^{n} \partial_{x_i}^2, \tag{3.13}$$

f is a function that, for the time being, we can assume continuous on $\bar{\Omega}$.

If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (3.12) pointwise one says that u is a "strong solution" of the problem (3.12) ($C^k(\Omega)$ denotes the space of functions k times differentiable in Ω, $C^0(\bar{\Omega})$ the space of continuous functions on $\bar{\Omega}$). Such a solution is not easy to find. One way to overcome the problem is to weaken our assumption on u. For instance one could only require the first equation of (3.12) to hold in the distributional sense. To proceed in this direction let us assume that u is a strong solution. Let $\varphi \in \mathcal{D}(\Omega)$. Multiplying both sides of (3.12) by φ and integrating on Ω we get

$$\int_\Omega -\Delta u \varphi \, dx = \int_\Omega f \varphi \, dx.$$

This can also be written

$$\sum_{i=1}^{n} \langle -\partial_{x_i}^2 u, \varphi \rangle = \int_\Omega f \varphi \, dx. \tag{3.14}$$

(We have seen that for a C^2-function the usual derivatives and the derivatives in the distributional sense coincide – see (2.32).) Due to the definition of the derivative in the sense of distributions this can also be written

$$\sum_{i=1}^{n} \langle \partial_{x_i} u, \partial_{x_i} \varphi \rangle = \int_\Omega f\varphi \, dx$$

or since $\partial_{x_i} u$ is a function

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.15}$$

Thus a strong solution satisfies (3.15). Now one could replace the pointwise condition

$$u = 0 \quad \text{on } \partial\Omega$$

by $u \in H_0^1(\Omega)$ then one would be lead to find u such that

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{3.16}$$

(if $u \in H_0^1(\Omega)$ then by density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, (3.15) holds for every function φ in $H_0^1(\Omega)$). (3.16) is called the weak formulation of (3.12). If u is a strong solution and if $u = 0$ on $\partial\Omega$ pointwise implies that $u \in H_0^1(\Omega)$ then u will be solution to (3.16). If on the other hand the solution to (3.16) is unique it will be the strong solution we are looking for. Thus in case of uniqueness, (3.16) will deliver the strong solution to (3.12). Now it will also deliver a "generalized solution" when a strong solution does not exist. To see that we have reached this we can state:

Theorem 3.1. *Let $f \in H^{-1}(\Omega)$. If Ω is bounded in one direction there exists a unique u solution to*

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{3.17}$$

Proof. By Theorem 2.9, the scalar product on $H_0^1(\Omega)$ can be chosen to be

$$\int_\Omega \nabla u \cdot \nabla v \, dx.$$

Then (3.17) follows immediately from the Riesz representation theorem. □

Remark 3.1. If for instance $f \in L^2(\Omega)$ – see (2.54) – then (3.17) is uniquely solvable. Now the pointwise equality (3.12) can fail – imagine a discontinuous f. Nevertheless we have found a good ersatz for the solution: we have obtained a solution in a generalized sense.

Remark 3.2. One should note that u minimizes also the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \langle f, v\rangle$$

(see Theorem 1.5). The integral

$$\int_\Omega |\nabla v|^2\,dx$$

is sometimes called Dirichlet integral.

In the same spirit as above we have

Theorem 3.2. *Let Ω be an arbitrary domain of $\mathbb{R}^n$ and $f \in L^2(\Omega)$. There exists a unique u solution to*

$$\begin{cases} u \in H^1(\Omega), \\ \displaystyle\int_\Omega \nabla u\cdot\nabla v + uv\,dx = \int_\Omega fv\,dx \quad \forall\, v \in H^1(\Omega). \end{cases} \tag{3.18}$$

Proof. One has

$$\left|\int_\Omega fv\,dx\right| \le \int_\Omega |f|\,|v|\,dx \le |f|_{2,\Omega}|v|_{2,\Omega} \le |f|_{2,\Omega}|v|_{1,2},$$

i.e.,

$$v \mapsto \int_\Omega fv\,dx$$

is a linear continuous form on $H^1(\Omega)$. The result follows then from the Riesz representation theorem. $\square$

Remark 3.3. We do not have of course $u \in H_0^1(\Omega)$. Indeed supposing Ω bounded $f = 1$ the solution is given by $u = 1 \notin H_0^1(\Omega)$ – see exercises.

Now since $\mathcal{D}(\Omega) \subset H^1(\Omega)$ we have from (3.18)

$$\begin{aligned} &\sum_{i=1}^n \langle \partial_{x_i} u, \partial_{x_i}\varphi\rangle + \langle u, \varphi\rangle = \langle f, \varphi\rangle \quad \forall\,\varphi \in \mathcal{D}(\Omega) \\ \iff \quad &\sum_{i=1}^n \langle -\partial_{x_i}^2 u, \varphi\rangle + \langle u, \varphi\rangle = \langle f, \varphi\rangle \quad \forall\,\varphi \in \mathcal{D}(\Omega), \end{aligned}$$

i.e., u is solution of

$$-\Delta u + u = f \quad \text{in } \mathcal{D}'(\Omega). \tag{3.19}$$

The boundary condition is here hidden. Suppose that $\partial\Omega$, u are "smooth" then one has for v smooth

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega \operatorname{div}(v\nabla u) - v\Delta u\,dx$$

(recall that $\operatorname{div} w = \sum_{i=1}^{n} \partial_{x_i} w_i$ for every vector $w = (w_1, \ldots, w_n)$). By the divergence formula – see [29], [44] – one has then

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma(x) + \int_\Omega -v \Delta u \, dx.$$

Going back to (3.18) we have obtained

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma(x) + \int_\Omega (-\Delta u + u) v \, dx = \int_\Omega f v \, dx,$$

i.e., by (3.19), if (3.19) holds in the usual sense

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma(x) = 0 \quad \forall\, v \text{ smooth.} \tag{3.20}$$

In a weak sense we have

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

and (3.18) is the weak formulation of the "Neumann problem"

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

Remark 3.4. In (3.18) one can replace $H^1(\Omega)$ by any closed subspace V of $H^1(\Omega)$ to get existence and uniqueness of a u such that

$$\begin{cases} u \in V, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v + uv \, dx = \int_\Omega f v \, dx \quad \forall\, v \in V. \end{cases} \tag{3.22}$$

For instance if $V = H_0^1(\Omega)$, (3.22) is the weak formulation of the problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Exercises

1. Let h, f be two continuous functions in Ω. If

$$\int_Q h \, dx = \int_Q f \, dx \quad \forall\, Q \text{ cube included in } \Omega$$

show that $h = f$ (this shows (3.7)).

2. Let Ω be a bounded open set of $\mathbb{R}^n$ and

$$V = \left\{ v \in H^1(\Omega) \,\middle|\, \int_\Omega v\,dx = 0 \right\}.$$

Show that V is a closed subspace of $H^1(\Omega)$. Show that the unique solution u to (3.22) satisfies in the distributional sense

$$-\Delta u + u = f - ⨍_\Omega f\,dx$$

where $⨍_\Omega f\,dx$ is the average of f on Ω defined as

$$⨍_\Omega f\,dx = \frac{1}{|\Omega|} \int_\Omega f(x)\,dx$$

($|\Omega|$ denotes the measure of Ω).

What is the boundary condition satisfied by u?

3. Let $f \in L^2(\Omega)$. Consider u the solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

Let $k \in \mathbb{R}$. Show that in the distributional sense

$$-\Delta(u \vee k) \le f\chi_{\{u>k\}} \quad \text{in } \Omega \qquad \vee \text{ maximum of two numbers.}$$

(Consider as test function $\left(\frac{u\vee k - k}{\varepsilon}\right)^+ \wedge v$, $\varepsilon > 0$, $v \in H_0^1(\Omega)$, $v \ge 0$, $\wedge$ minimum of two numbers.)

Chapter 4

Elliptic Problems in Divergence Form

The goal of this chapter is to introduce general elliptic problems extending those already addressed in the preceding chapter and putting them all in the same framework.

4.1 Weak formulation

We suppose here that Ω is an open subset of $\mathbb{R}^n$, $n \geq 1$. Denote by $A = A(x)$ a $n \times n$ matrix with entries a_{ij}, i.e.,

$$A(x) = (a_{ij}(x))_{i,j=1,\ldots,n}. \tag{4.1}$$

Suppose that $a_{ij} \in L^\infty(\Omega)$ $\forall\, i, j = 1, \ldots, n$ and A uniformly positive definite. In other words suppose that for some positive constants λ and Λ we have

$$|A(x)\xi| \leq \Lambda|\xi| \qquad \forall\, \xi \in \mathbb{R}^n, \qquad \text{a.e. } x \in \Omega, \tag{4.2}$$

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \qquad \forall\, \xi \in \mathbb{R}^n, \qquad \text{a.e. } x \in \Omega. \tag{4.3}$$

(In these formulas "$\cdot$" denotes the canonical scalar product in $\mathbb{R}^n$, $|\cdot|$ the associated norm and $A(x)\xi$ denotes the vector of $\mathbb{R}^n$ obtained by multiplying $A(x)$ by the column vector ξ.)

Remark 4.1. The assumption (4.2) is equivalent to assume that the a_{ij} are uniformly bounded in Ω. Indeed

$$\|A\| = \operatorname*{Sup}_{\xi \neq 0} \frac{|A\xi|}{|\xi|} \tag{4.4}$$

is a norm on the space of matrices which is equivalent to any other norm since the space of matrices is finite dimensional. In particular it is equivalent to

$$\|A\|_\infty = \operatorname*{Max}_{i,j} |a_{ij}| \tag{4.5}$$

which establishes our claim.

In addition we denote by a a function satisfying

$$a \in L^\infty(\Omega), \quad a(x) \geq 0 \quad \text{a.e. } x \in \Omega, \tag{4.6}$$

(this last assumption could be relaxed slightly – see Remark 4.8).

Then we have

Theorem 4.1. *Let $f \in H^{-1}(\Omega)$. If*

$$\text{(i)} \quad \Omega \textit{ is bounded in one direction,} \tag{4.7}$$

$$\textit{or} \quad \text{(ii)} \quad a(x) \geq a > 0 \quad \textit{a.e. } x \in \Omega, \tag{4.8}$$

for a positive constant a, then there exists a unique weak solution to

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u \cdot \nabla v + a(x)uv\,dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{4.9}$$

Proof. The proof is an easy consequence of the Lax–Milgram theorem. Indeed consider $H_0^1(\Omega)$ equipped with the scalar product defined by (2.40)

$$(u,v)_{1,2} = \int_\Omega \nabla u \cdot \nabla v + uv\,dx.$$

We have seen that $H_0^1(\Omega)$ is a Hilbert space. Consider then

$$a(u,v) = \int_\Omega A(x)\nabla u \cdot \nabla v + a(x)uv\,dx. \tag{4.10}$$

- $a(u,v)$ is a continuous bilinear form on $H_0^1(\Omega)$.

  The bilinearity is clear. For the continuity one remarks that

$$\begin{aligned} |a(u,v)| &\leq \int_\Omega |A(x)\nabla u \cdot \nabla v| + a(x)|u|\,|v|\,dx \\ &\leq \int_\Omega |A(x)\nabla u|\,|\nabla v| + a(x)|u|\,|v|\,dx \quad \text{(Cauchy–Schwarz)} \\ &\leq \int_\Omega \Lambda|\nabla u|\,|\nabla v| + |a|_\infty|u|\,|v|\,dx \end{aligned} \tag{4.11}$$

  where $|a|_\infty$ denotes the $L^\infty(\Omega)$-norm of a. It follows that

$$\begin{aligned} |a(u,v)| &\leq \operatorname{Max}\{\Lambda, |a|_\infty\} \int_\Omega |\nabla u|\,|\nabla v| + |u|\,|v|\,dx \\ &\leq \operatorname{Max}\{\Lambda, |a|_\infty\}\big\{\big||\nabla u|\big|_2\big||\nabla v|\big|_2 + |u|_2|v|_2\big\} \text{ by the Cauchy–Schwarz inequality,} \\ &\leq \operatorname{Max}\{\Lambda, |a|_\infty\}|u|_{1,2}|v|_{1,2}, \text{ by the Cauchy–Schwarz inequality in } \mathbb{R}^2. \end{aligned}$$

This completes the proof of our claim. Note that (4.11) shows at the same time that the function under the integral sign of (4.10) is integrable and thus $a(u,v)$ is well defined.

- $a(u,v)$ is coercive on $H_0^1(\Omega)$.

  Indeed one has

$$
\begin{aligned}
a(u,v) &= \int_\Omega A\nabla u \cdot \nabla v + au^2\,dx \\
&\geq \int_\Omega \lambda|\nabla u|^2 + au^2\,dx \geq \begin{matrix} \lambda\big||\nabla u|\big|_{2,\Omega}^2 & \text{in case (i)}, \\ \operatorname{Min}\{\lambda,a\}|u|_{1,2}^2 & \text{in case (ii)}. \end{matrix}
\end{aligned}
$$

  The coerciveness of $a(u,v)$ follows since in case (i) one has for some constant c

$$\big||\nabla u|\big|_{2,\Omega} \geq c|u|_{1,2}$$

  (see Theorem 2.9). Since $f \in H^{-1}(\Omega)$ the result follows then by the Lax–Milgram theorem (Theorem 1.5). □

In the case of (ii), i.e., if

$$a(x) \geq a > 0 \tag{4.12}$$

then one can replace in (4.9) $H_0^1(\Omega)$ by any closed subspace of $H^1(\Omega)$. More precisely we have

Theorem 4.2. *Assume that* (4.2), (4.3), (4.6) *and* (4.12) *hold. Let V be a closed subspace of $H^1(\Omega)$ and $f \in V^*$ where V^* denotes the dual space of V. Then there exists a unique u solution to*

$$
\begin{cases} u \in V, \\ \displaystyle\int_\Omega A\nabla u \cdot \nabla v + auv\,dx = \langle f, v\rangle \quad \forall\, v \in V. \end{cases} \tag{4.13}
$$

Proof. This is a consequence of the Lax–Milgram theorem since, as we have seen in the proof of Theorem 4.1, the form $a(u,v)$ is bilinear, continuous and coercive on V when equipped with the $|\cdot|_{1,2}$-norm. □

Remark 4.2. If $f \in L^2(\Omega)$ then

$$\langle f, v\rangle = \int_\Omega fv\,dx$$

defines a continuous linear form on any V as in Theorem 4.2 since by the Cauchy–Schwarz inequality we have

$$|\langle f, v\rangle| \leq |f|_2|v|_2 \leq |f|_2|v|_{1,2} \quad \forall\, v \in V.$$

Remark 4.3. One could replace (ii) or (4.12) by

$$a(x) \geq 0, \quad a \not\equiv 0$$

(see [29]).

Remark 4.4. In the case where A is symmetric, i.e.,

$$A(x) = A(x)^T$$

then for any $\xi, \eta \in \mathbb{R}^n$ one has

$$A\xi \cdot \eta = \xi \cdot A^T \eta = \xi \cdot A\eta$$

and $a(u, v)$ is symmetric. It follows (see Theorem 1.5) that in Theorems 4.1, 4.2 u is the unique minimizer of the functional

$$J(v) = \frac{1}{2} \int_\Omega A\nabla v \cdot \nabla v + av^2 \, dx - \langle f, v \rangle$$

on $H_0^1(\Omega)$ and V respectively. Note also that the two theorems above extend results that we had already established in the previous chapter in the case $A = \mathrm{id}$. The terminology "elliptic" comes of course from (4.3) since for a matrix A satisfying (4.3) the set

$$E = \{\, \xi \in \mathbb{R}^n \mid A\xi \cdot \xi = 1 \,\}$$

is an ellipsoid – an ellipse in the case $n = 2$.

Remark 4.5. As we could allow a to degenerate we can do the same for $A(x)$ – i.e., we can relax (4.3). Indeed in the case of Theorem 4.1, (i) for instance, what we only need is the existence of a constant $\alpha > 0$ such that

$$\int_\Omega A(x)\nabla u \cdot \nabla u + a(x)u^2 \, dx \geq \alpha |u|_{1,2}^2 \quad \forall\, u \in H_0^1(\Omega). \tag{4.14}$$

As in the case of the Laplace operator – i.e., when $A = \mathrm{id}$ – the problems above are weak formulations of elliptic partial differential equations in the usual or strong sense. This is what we would like to see now. Let us consider first the case of Theorem 4.1. Taking $v = \varphi \in \mathcal{D}(\Omega)$ we obtain

$$\sum_{i=1}^n \int_\Omega (A(x)\nabla u)_i \partial_{x_i} \varphi \, dx + \langle a(x)u, \varphi \rangle = \langle f, \varphi \rangle \tag{4.15}$$

where $(A(x)\nabla u)_i$ denotes the i^{th} component of the vector $A(x)\nabla u$. This can be written as

$$-\sum_{i=1}^n \langle \partial_{x_i} (A(x)\nabla u)_i, \varphi \rangle + \langle a(x)u, \varphi \rangle = \langle f, \varphi \rangle, \tag{4.16}$$

i.e., u is solution in the distributional sense of

$$-\sum_{i=1}^{n} \partial_{x_i}(A(x)\nabla u)_i + a(x)u = f \tag{4.17}$$

or if we use the divergence notation

$$-\operatorname{div}(A(x)\nabla u) + a(x)u = f \quad \text{in } \Omega.$$

(Recall that for a vector $\vec{\omega} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $\operatorname{div} \vec{\omega} = \sum_{i=1}^{n} \partial_{x_i}\omega_i$.) Thus since $H_0^1(\Omega)$ is the space of functions vanishing on $\partial\Omega$ the solution u of (4.9) is the weak solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

To have something which looks more like an equation – i.e., which does not involve vectors one can note that

$$(A(x)\nabla u)_i = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j} u$$

and thus the first equation of (4.18) can be written

$$\begin{aligned} & -\sum_{i=1}^{n} \partial_{x_i}\Big(\sum_{j=1}^{n} a_{ij}(x)\partial_{x_j} u\Big) + a(x)u = f \\ \iff \qquad & -\sum_{i=1}^{n}\sum_{j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + a(x)u = f. \end{aligned}$$

With the Einstein convention of repeated indices – i.e., one drops the sign Σ when two indices repeat each others, the system (4.18) has the form:

$$\begin{cases} -\partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + a(x)u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.19}$$

We now turn to the interpretation of Theorem 4.2. For that we will need the divergence formula that we recall here.

Theorem 4.3 (Divergence formula). *Suppose that Ω is a "smooth" open subset of $\mathbb{R}^n$ with outward unit normal $\vec{n}$. For any "smooth" vector field $\vec{\omega}$ in Ω we have*

$$\int_\Omega \operatorname{div} \vec{\omega}\, dx = \int_{\partial\Omega} \vec{\omega} \cdot \vec{n}\, d\sigma(x) \tag{4.20}$$

($d\sigma(x)$ is the measure area on $\partial\Omega$).

Note that in this theorem we do not precise what "smooth" means. Basically (4.20) holds when one can make sense of the different quantities $\vec{n}$, $d\sigma(x)$, the integrals, $\operatorname{div}\vec{\omega}$ occurring – see [11], [29], [44]. Note also that in the case of a simple domain as a cube the formula (4.20) can be easily established by arguing as we did between (3.5) and (3.6) with $\vec{\omega} = \nabla u$. To interpret Theorem 4.2 in a strong form we consider only the case where $V = H^1(\Omega)$, $f \in L^2(\Omega)$. Taking then $v = \varphi \in \mathcal{D}(\Omega)$ we derive as above that

$$-\operatorname{div}(A(x)\nabla u) + au = f \quad \text{in } \mathcal{D}'(\Omega). \tag{4.21}$$

Now, of course, more test-functions v are available in (4.13) and assuming everything "smooth" the second equation of (4.13) can be written as

$$\int_\Omega \operatorname{div}(vA(x)\nabla u) - \operatorname{div}(A(x)\nabla u)v + auv\,dx = \int_\Omega fv\,dx.$$

Taking into account (4.21) we derive

$$\int_\Omega \operatorname{div}(vA(x)\nabla u)\,dx = 0$$

and by the divergence formula

$$\int_{\partial\Omega} A(x)\nabla u \cdot n\,v\,d\sigma(x) = 0 \quad \forall\, v \text{ "smooth"}.$$

Thus if every computation can be performed we are ending up with

$$A(x)\nabla u \cdot n = 0 \quad \text{on } \partial\Omega.$$

Thus (4.13) is a weak formulation of the so-called Neumann problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f & \text{in } \Omega, \\ A(x)\nabla u \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.22}$$

Recall that the first equation of (4.22) can be also written as in (4.19).

Remark 4.6. In the case where $A(x) = \mathrm{Id}$ then the second equation of (4.22) is

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega$$

namely the normal derivative of u vanishes on $\partial\Omega$. We already encountered this boundary condition in the preceding chapter. In the general case and with the summation convention the second equation above can be written

$$a_{ij}(x)\partial_{x_j} u n_i = 0 \quad \text{on } \partial\Omega$$

if $n = (n_1, \dots, n_n)$. This expression is called the "conormal derivative of u".

Remark 4.7. In (4.13) if V is a finite-dimensional subspace of $H^1(\Omega)$, f a linear form on V then there exists a unique solution u. This follows from the fact that a finite-dimensional subspace is automatically closed and a linear form on it automatically continuous. Similarly in (4.9) one can replace $H_0^1(\Omega)$ by any closed subspace V and get existence and uniqueness for any $f \in V^*$. The case of finite-dimensional subspaces is especially important in the so-called finite elements method to compute approximations of the solutions to problems (4.9) and (4.13). We refer the reader to Chapter 10.

4.2 The weak maximum principle

The maximum principle has several aspects. One of it is that if $u = u(f)$ is solution to (4.9) for instance then, if f increases, so does u. First we have to make precise what do we mean by f increase. For that we have

Definition 4.1. Let $T_1, T_2 \in \mathcal{D}'(\Omega)$. We say that

$$T_1 \leq T_2 \tag{4.23}$$

iff

$$\langle T_1, \varphi\rangle \leq \langle T_2, \varphi\rangle \quad \forall\, \varphi \in \mathcal{D}(\Omega),\ \varphi \geq 0. \tag{4.24}$$

As we gave a meaning for a function of $H^1(\Omega)$ to vanish on $\partial\Omega$ we can give a meaning for such a function to be negative on $\partial\Omega$. We have

Definition 4.2. Let $u_1, u_2 \in H^1(\Omega)$ we say that

$$u_1 \leq u_2 \quad \text{on } \partial\Omega$$

iff

$$(u_1 - u_2)^+ \in H_0^1(\Omega). \tag{4.25}$$

We can now state

Theorem 4.4. *Let $A = A(x)$ be a matrix satisfying* (4.2), (4.3)*. Let $a \in L^\infty(\Omega)$ be a nonnegative function. For $u \in H^1(\Omega)$ we define $-L$ as*

$$-Lu = -\operatorname{div}(A(x)\nabla u) + a(x)u. \tag{4.26}$$

Then we have: let $u_1, u_2 \in H^1(\Omega)$ such that

$$-Lu_1 \leq -Lu_2 \quad \textit{in } \mathcal{D}'(\Omega) \tag{4.27}$$

$$u_1 \leq u_2 \qquad \textit{on } \partial\Omega \tag{4.28}$$

then

$$u_1 \leq u_2 \quad \textit{in } \Omega,$$

that is to say u_1 is a function a.e. smaller than u_2.

Proof. By (4.27), (4.24) we have

$$\int_\Omega A(x)\nabla u_1 \cdot \nabla\varphi + au_1\varphi\,dx \le \int_\Omega A(x)\nabla u_2 \cdot \nabla\varphi + au_2\varphi\,dx$$

for any $\varphi \in \mathcal{D}(\Omega)$, $\varphi \ge 0$ which is also

$$\int_\Omega A(x)\nabla(u_1 - u_2)\cdot\nabla\varphi + a(u_1 - u_2)\varphi\,dx \le 0 \quad \forall\,\varphi \in \mathcal{D}(\Omega),\ \varphi \ge 0. \tag{4.29}$$

Then we have the following claim.

Claim. Let $u \in H_0^1(\Omega)$, $u \ge 0$ then there exists $\varphi_n \in \mathcal{D}(\Omega)$, $\varphi_n \ge 0$ such that $\varphi_n \to u$ in $H^1(\Omega)$.

Proof of the claim: Since $u \in H_0^1(\Omega)$ there exists $\varphi_n \in \mathcal{D}(\Omega)$ such that

$$\varphi_n \to u \quad \text{in } H_0^1(\Omega).$$

We also have

$$\begin{aligned}\int_\Omega |\nabla(\varphi_n^+ - u)|^2\,dx &= \int_{\{\varphi_n>0\}} |\nabla(\varphi_n - u)|^2\,dx + \int_{\{\varphi_n<0\}} |\nabla u|^2 \\ &\le \int_\Omega |\nabla(\varphi_n - u)|^2 + \int_{\{u>0\}} \chi_{\{\varphi_n<0\}}|\nabla u|^2,\end{aligned} \tag{4.30}$$

see Remark 2.6. Up to a subsequence we have

$$\varphi_n \to u \ge 0 \quad \text{a.e.},$$

i.e., $\chi_{\{\varphi_n<0\}} \to 0$ a.e. on $\{u > 0\}$. It follows from (4.30)

$$\varphi_n^+ \longrightarrow u.$$

(The whole sequence converges since the limit is unique.) Then it is enough to show that $(\varphi_n)^+$ can be approximated by a nonnegative function of $\mathcal{D}(\Omega)$.

$$\rho_\varepsilon * (\varphi_n)^+$$

for ε small enough is such a function. This completes the proof of the claim. □

Returning to (4.29) we then have

$$\int_\Omega A\nabla(u_1 - u_2)\cdot\nabla v + a(u_1 - u_2)v\,dx \le 0 \quad \forall\,v \in H_0^1(\Omega),\ v \ge 0.$$

Taking $v = (u_1 - u_2)^+$ – see (4.25), (4.28) we derive

$$\begin{aligned} &\int_\Omega A\nabla(u_1 - u_2)\cdot\nabla(u_1 - u_2)^+ + a\big((u_1 - u_2)^+\big)^2\,dx \le 0 \\ \iff \quad &\int_\Omega A\nabla(u_1 - u_2)^+\cdot\nabla(u_1 - u_2)^+ + a\big((u_1 - u_2)^+\big)^2\,dx \le 0 \end{aligned} \tag{4.31}$$

(see (2.60)). It follows that $(u_1 - u_2)^+ = 0$ which completes the proof (see Exercise 12, Chapter 2). □

Remark 4.8. One can weaken slightly the assumption on a. Indeed if λ_1 is the first eigenvalue for the Dirichlet problem associated with A namely if

$$\lambda_1 = \operatorname*{Inf}_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \left\{ \int_\Omega A\nabla u \cdot \nabla u \, dx \Big/ \int_\Omega u^2 \, dx \right\} \tag{4.32}$$

(see also Chapter 9) then one can assume only $a(x) > -\lambda_1$ a.e. x (cf. (4.31)).

For a fixed a the remark above shows that the weak maximum principle holds provided Ω is located in a strip of width small enough or (cf. Paragraph 12.4) has a Lebesgue measure small enough – see (2.49). We can state that as the following corollary.

Corollary 4.5. *Let $A(x)$ be a matrix satisfying* (4.2), (4.3) *and* $a \in L^\infty(\Omega)$. *Suppose that Ω is located in a strip of width less or equal to*

$$\left\{ \frac{2\lambda}{|a^-|_{\infty,\Omega}} \right\}^{\frac{1}{2}}$$

then with the notation of Theorem 4.4 *we have*

$$-Lu_1 \leq -Lu_2 \quad \textit{in } \mathcal{D}'(\Omega), \qquad u_1 \leq u_2 \quad \textit{on } \partial\Omega$$

implies

$$u_1 \leq u_2 \quad \textit{in } \Omega.$$

Proof. Going back to (4.31) we have

$$\int_\Omega A\nabla(u_1 - u_2)^+ \cdot \nabla(u_1 - u_2)^+ + (a^+ - a^-)((u_1 - u_2)^+)^2 \, dx \leq 0.$$

Thus it comes

$$\lambda \int_\Omega |\nabla(u_1 - u_2)^+|^2 \, dx - \int_\Omega a^-((u_1 - u_2)^+)^2 \, dx \leq 0,$$

and since $a^- \leq |a^-|_{\infty,\Omega}$

$$\lambda \int_\Omega |\nabla(u_1 - u_2)^+|^2 \, dx - |a^-|_{\infty,\Omega} \int_\Omega ((u_1 - u_2)^+)^2 \, dx \leq 0.$$

Using (2.49) we deduce (a is here the half-width of the strip containing Ω!)

$$\left(\frac{\lambda}{2a^2} - |a^-|_{\infty,\Omega} \right) \int_\Omega ((u_1 - u_2)^+)^2 \, dx \leq 0$$

which implies $(u_1 - u_2)^+ = 0$ when

$$(2a)^2 < \frac{2\lambda}{|a^-|_{\infty,\Omega}}.$$

$(2a)$ is the width of the strip containing Ω. This completes the proof of the corollary. $\square$

Another corollary is

Corollary 4.6. *Under the assumptions of Theorem* 4.4 *let* $u \in H^1(\Omega)$ *satisfy*

$$-Lu \leq 0 \quad \text{in } \mathcal{D}'(\Omega), \qquad u \leq 0 \quad \text{on } \partial\Omega, \tag{4.33}$$

then we have

$$u \leq 0 \quad \text{on } \Omega. \tag{4.34}$$

Proof. It is enough to apply Theorem 4.4 with $u_1 = u$, $u_2 = 0$. □

Another consequence of Theorem 4.4 is that if u is a weak solution of a homogeneous elliptic equation in divergence form – i.e., if

$$-\operatorname{div}(A(x)\nabla u) + a(x)u = 0 \tag{4.35}$$

then a positive maximum of u cannot exceed what it is on the boundary. More precisely

Corollary 4.7. *Under the assumptions of Theorem* 4.4 *let* $u \in H^1(\Omega)$ *be such that*

$$-Lu \leq 0 \quad \text{in } \mathcal{D}'(\Omega), \qquad u \leq M \quad \text{on } \partial\Omega \tag{4.36}$$

where M *is a nonnegative constant* (*an arbitrary constant if* $a \equiv 0$). *Then*

$$u \leq M. \tag{4.37}$$

Proof. Just remark that

$$-L(u - M) = -Lu - aM \leq 0$$

and apply Corollary 4.6 to $u - M$. □

Remark 4.9. One should note that when the extension of Theorem 4.4 mentioned in Remark 4.8 holds, i.e., when one only assumes $a(x) > -\lambda_1$, a.e. x, then the two corollaries above extend as well.

Finally one should notice that the weak maximum principle also holds for the Neumann problem and we have

Theorem 4.8. *Let* $A = A(x)$ *satisfying* (4.2), (4.3) *and* $a = a(x) \in L^\infty(\Omega)$ *satisfying* (4.12). *For* $f_1, f_2 \in L^2(\Omega)$ *let* u_i, $i = 1, 2$ *be the solution to*

$$\begin{cases} u_i \in H^1(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u_i \cdot \nabla v + a(x)u_i v \, dx = \int_\Omega f_i v \, dx \quad \forall\, v \in H^1(\Omega) \end{cases} \tag{4.38}$$

(*cf. Theorem and Remark* 4.2). *Then if*

$$f_1 \leq f_2 \quad \text{in } \Omega \tag{4.39}$$

we have

$$u_1 \leq u_2 \quad \text{in } \Omega. \tag{4.40}$$

Proof. Taking $v = (u_1 - u_2)^+$ in (4.38) for $i = 1, 2$ we get by subtraction

$$\int_\Omega A(x)\nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2)^+ + a(x)(u_1 - u_2)^{+2}\, dx$$
$$= \int (f_1 - f_2)(u_1 - u_2)^+\, dx \le 0 \quad \text{(by (4.39))}.$$

This is also

$$\int_\Omega A(x)\nabla(u_1 - u_2)^+ \cdot \nabla(u_1 - u_2)^+ + a(x)(u_1 - u_2)^{+2}\, dx \le 0$$

which leads to (4.40). This completes the proof of the theorem. □

Remark 4.10. The result of Theorem 4.8 can be extended to problems of the type (4.13). The only assumptions needed being

$$(u_1 - u_2)^+ \in V, \qquad \langle f_1 - f_2, (u_1 - u_2)^+ \rangle \le 0.$$

4.3 Inhomogeneous problems

By inhomogeneous we mean that the boundary condition is nonhomogeneous that is to say is not identically equal to 0. Suppose for instance that one wants to solve the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.41}$$

($f \in H^{-1}(\Omega)$ and g is some function.) As we found a way to give a sense to $u = 0$ on $\partial\Omega$ we need to give a sense to the second equation to (4.41). The best way is to assume

$$u - g \in H_0^1(\Omega).$$

Thus let us also suppose

$$g \in H^1(\Omega).$$

Setting

$$U = u - g$$

we see that if u is solution to (4.41) – in a weak sense – then U is solution to

$$-\operatorname{div}(A(x)\nabla(U + g)) + a(x)(U + g) = f$$

in a weak sense. This can also be written as

$$-\operatorname{div}(A(x)\nabla U) + a(x)U = f - a(x)g + \operatorname{div}(A(x)\nabla g). \tag{4.42}$$

Clearly due to our assumptions the right-hand side of (4.42) belongs to $H^{-1}(\Omega)$ and thus there exists a unique weak solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla U) + a(x)U = f - ag + \operatorname{div}(A(x)\nabla g) & \text{in } \Omega, \\ U \in H_0^1(\Omega). \end{cases}$$

Then the weak solution to (4.41) is given by

$$u = U + g.$$

More precisely $u = U + g$ is the unique function satisfying

$$\begin{cases} \int_\Omega A(x)\nabla u \cdot \nabla v + a(x)uv \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \\ u - g \in H_0^1(\Omega). \end{cases} \tag{4.43}$$

Exercises

1. Establish the divergence formula for a triangle.
2. Consider for Γ_0 closed subset of $\Gamma = \partial\Omega$

   $$W = \{\, v \in C^1(\bar{\Omega}) \mid v = 0 \text{ on } \Gamma_0 \,\}$$

   ($C^1(\overline{\Omega})$ denotes the space of the restrictions of $C^1(\mathbb{R}^n)$-functions on $\overline{\Omega}$),

   $$V = \bar{W} \text{ where the closure is taken in the } H^1(\Omega)\text{-sense.}$$

   Show that the solution to (4.13) for $f \in L^2(\Omega)$ is a weak solution to

   $$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_0, \qquad A(x)\nabla u \cdot n = 0 \quad \text{on } \Gamma \setminus \Gamma_0. \end{cases}$$

   Such a problem is called a mixed (Dirichlet–Neumann) problem.
3. Let $T_1, T_2 \in \mathcal{D}'(\Omega)$. Show that

   $$\left.\begin{aligned} T_1 \le T_2 \\ T_2 \le T_1 \end{aligned}\right\} \quad \Longrightarrow \quad T_1 = T_2.$$

4. Let $T \in \mathcal{D}'(\Omega)$, $T \ge 0$. Show that for every compact $K \subset \Omega$ there exists a constant C_K such that

   $$|\langle T, \varphi\rangle| \le C_K \operatorname*{Sup}_K |\varphi| \quad \forall \varphi \in \mathcal{D}(\Omega), \operatorname{Sup}(\varphi) \subset K.$$

5. Define L_i by
$$-L_i u = -\operatorname{div}(A(x)\nabla u) + a_i u \quad i = 1, 2.$$
Show that if $u_1, u_2 \in H^1(\Omega)$
$$\begin{aligned} -L_1 u_1 &\le -L_2 u_2 && \text{in } \mathcal{D}'(\Omega),\\ 0 \le a_2 &\le a_1 && \text{in } \Omega,\\ 0_\vee u_1 &\le u_2 && \text{on } \partial\Omega, \end{aligned}$$
then $0_\vee u_1 \le u_2$ in Ω.
6. Show that for a constant symmetric matrix A
$$\lambda_1 = \operatorname*{Inf}_{\xi \ne 0} \frac{A\xi \cdot \xi}{|\xi|^2}$$
is the smallest eigenvalue of A.
7. Let $A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{pmatrix}$. Show that
$$\frac{7}{12}|\xi|^2 \le A\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^2.$$
8. Let $\Omega_1 \subset \Omega_2$ be two bounded subsets of $\mathbb{R}^n$. Let u_1, u_2 be the solutions to
$$\begin{cases} \displaystyle\int_{\Omega_i} A\nabla u_i \cdot \nabla v \, dx = \int_{\Omega_i} f_i v \, dx, \\ u_i \in H_0^1(\Omega_i), \end{cases}$$
for $i = 1, 2$ where $f_i \in L^2(\Omega_i)$. Show that
$$f_2 \ge f_1 \ge 0 \quad \Longrightarrow \quad u_2 \ge u_1 \ge 0,$$
i.e., for nonnegative f the solution of the Dirichlet problem increases with the size of the domain.
9. Under the assumptions of Theorem 4.1 show that the mapping
$$f \mapsto u$$
where u is the solution to (4.9) is continuous from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$.
10. We suppose to simplify that Ω is bounded. Let $(A_n) = (A_n(x))$ be a sequence of matrices satisfying (4.2), (4.3) for λ, Λ independent of n. For $f \in H^{-1}(\Omega)$ we denote by u_n the solution to
$$\begin{cases} u_n \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A_n(x)\nabla u_n \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

We suppose that

$$A_n \rightharpoonup A \quad \text{a.e. in } \Omega$$

(i.e., the convergence holds entry to entry).

Show that $u_n \to u$ in $H_0^1(\Omega)$ where u is the solution to

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A(x)\nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

11. Show that a subspace of $H^1(\Omega)$ of finite dimensions is closed in $H^1(\Omega)$ and that any linear form on it is automatically continuous.

Chapter 5

Singular Perturbation Problems

This theory is devoted to study problems with small diffusion velocity. One is mainly concerned with the asymptotic behaviour of the solution of such problems when the diffusion velocity approaches 0. Let us explain the situation on a classical example (see [68], [69]).

5.1 A prototype of a singular perturbation problem

Let Ω be a bounded open subset of $\mathbb{R}^n$. For $f \in H^{-1}(\Omega)$, $\varepsilon > 0$ consider u_ε the weak solution to

$$\begin{cases} -\varepsilon \Delta u_\varepsilon + u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \tag{5.1}$$

It follows from Theorem 4.1 with $A = \varepsilon \,\mathrm{Id}$ that a solution to this problem exists and is unique. The question is then to determine what happens when $\varepsilon \to 0$. Passing to the limit formally one expects that

$$u_\varepsilon \longrightarrow u_0 = f. \tag{5.2}$$

Of course this cannot be for an arbitrary norm. For instance if $f \notin H_0^1(\Omega)$ one can never have $u_\varepsilon \to f$ in $H_0^1(\Omega)$. Let us first prove:

Theorem 5.1. *Let u_ε be the solution to* (5.1). *Then we have*

$$u_\varepsilon \longrightarrow f \quad \text{in } H^{-1}(\Omega). \tag{5.3}$$

Proof. We suppose $H_0^1(\Omega)$ equipped with the norm

$$\big|\big|\nabla v\big|\big|_{2,\Omega}. \tag{5.4}$$

If $g \in H^{-1}(\Omega)$ then, by the Riesz representation theorem, there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \langle g, v \rangle \quad \forall\, v \in H_0^1(\Omega)$$

and

$$|g|_{H^{-1}(\Omega)} = \big|\big|\nabla u\big|\big|_{2,\Omega}$$

(see (1.19)) – we denote by $|g|_{H^{-1}(\Omega)}$ the strong dual norm of g in $H^{-1}(\Omega)$ defined as

$$|g|_{H^{-1}(\Omega)} = \operatorname*{Sup}_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{|\langle g, v\rangle|}{\big|\big|\nabla v\big|\big|_{2,\Omega}}.$$

Since (5.1) is meant in a weak sense we have

$$\langle f - u_\varepsilon, v\rangle = \int_\Omega \nabla(\varepsilon u_\varepsilon) \cdot \nabla v \, dx \quad \forall\, v \in H_0^1(\Omega)$$

and thus

$$|f - u_\varepsilon|_{H^{-1}(\Omega)} = \big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega}. \tag{5.5}$$

As we just saw:

$$\varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla w + u_\varepsilon w \, dx = \langle f, w\rangle \quad \forall\, w \in H_0^1(\Omega). \tag{5.6}$$

Choosing $w = \varepsilon u_\varepsilon$ we get

$$\big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega}^2 + \varepsilon |u_\varepsilon|_{2,\Omega}^2 = \langle f, \varepsilon u_\varepsilon\rangle \le |f|_{H^{-1}(\Omega)} \big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega}. \tag{5.7}$$

From this we derive that

$$\big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega}^2 \le |f|_{H^{-1}(\Omega)} \big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega}$$

and thus

$$\big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_{2,\Omega} \le |f|_{H^{-1}(\Omega)}.$$

This shows that $\varepsilon u_\varepsilon$ is bounded in $H_0^1(\Omega)$ independently of ε. So, up to a subsequence there exists $v_0 \in H_0^1(\Omega)$ such that

$$\varepsilon u_\varepsilon \rightharpoonup v_0 \quad \text{in } H_0^1(\Omega). \tag{5.8}$$

From (5.7) we derive then that

$$\varepsilon |u_\varepsilon|_{2,\Omega}^2 \le |f|_{H^{-1}(\Omega)}^2.$$

Thus

$$|\varepsilon u_\varepsilon|_{2,\Omega}^2 = \varepsilon \cdot \varepsilon |u_\varepsilon|_{2,\Omega}^2 \longrightarrow 0.$$

This means that

$$\varepsilon u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(\Omega) \tag{5.9}$$

and also in $\mathcal{D}'(\Omega)$. Since the convergence (5.8) implies convergence in $\mathcal{D}'(\Omega)$ we have $v_0 = 0$ and by uniqueness of the limit it follows that $\varepsilon u_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$. From (5.7) we derive then

$$\big|\big|\nabla(\varepsilon u_\varepsilon)\big|\big|_2^2 \le \langle f, \varepsilon u_\varepsilon\rangle \to 0.$$

By (5.5) this completes the proof of the theorem. □

The convergence we just obtained is a very weak convergence, however it will always hold. If we wish to have a convergence of u_ε in $L^2(\Omega)$ one needs to assume $f \in L^2(\Omega)$, a convergence in $H_0^1(\Omega)$ one needs $f \in H_0^1(\Omega)$ and so on. Let us examine the convergence in $L^2(\Omega)$.

Theorem 5.2. *Suppose that $f \in L^2(\Omega)$ then if u_ε is the solution to (5.1) we have*

$$u_\varepsilon \longrightarrow f \quad \text{in } L^2(\Omega). \tag{5.10}$$

Proof. The weak formulation of (5.1) is

$$\int_\Omega \varepsilon \nabla u_\varepsilon \cdot \nabla v + u_\varepsilon v \, dx = \int_\Omega f v \, dx \quad \forall \, v \in H_0^1(\Omega). \tag{5.11}$$

Taking $v = u_\varepsilon$ it comes

$$\varepsilon \big| |\nabla u_\varepsilon| \big|_{2,\Omega}^2 + |u_\varepsilon|_{2,\Omega}^2 = \int_\Omega f u_\varepsilon \, dx \le |f|_{2,\Omega} |u_\varepsilon|_{2,\Omega}$$

by the Cauchy–Schwarz inequality. Thus it follows that

$$|u_\varepsilon|_{2,\Omega} \le |f|_{2,\Omega}, \qquad \varepsilon \big| |\nabla u_\varepsilon| \big|_{2,\Omega}^2 \le |f|_{2,\Omega}^2. \tag{5.12}$$

Since u_ε is bounded in $L^2(\Omega)$ independently of ε there exists $u_0 \in L^2(\Omega)$ such that up to a subsequence

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } L^2(\Omega). \tag{5.13}$$

(In fact at this stage by Theorem 5.1 we already know that $u_\varepsilon \rightharpoonup f$ in $L^2(\Omega)$, but we give here a proof of Theorem 5.2 independent of Theorem 5.1.) To identify u_0 we go back to (5.11) that we write

$$\sqrt{\varepsilon} \int_\Omega \nabla(\sqrt{\varepsilon} u_\varepsilon) \cdot \nabla v \, dx + \int_\Omega u_\varepsilon v \, dx = \int_\Omega f v \, dx \quad \forall \, v \in H_0^1(\Omega).$$

Passing to the limit – using (5.12) we derive

$$\int_\Omega u_0 v \, dx = \int_\Omega f v \, dx \quad \forall \, v \in H_0^1(\Omega).$$

By density of $H_0^1(\Omega)$ (or $\mathcal{D}(\Omega)$) in $L^2(\Omega)$ we get

$$\int_\Omega u_0 v \, dx = \int_\Omega f v \, dx \quad \forall \, v \in L^2(\Omega)$$

that is to say $u_0 = f$ and by uniqueness of the possible limit the whole sequence u_ε satisfies (5.13), which we recall, could be derived from Theorem 5.1.

Next we remark that

$$\begin{aligned} |u_\varepsilon - f|_{2,\Omega}^2 &= |u_\varepsilon|_{2,\Omega}^2 - 2(u_\varepsilon, f)_{2,\Omega} + |f|_{2,\Omega}^2 \\ &\le 2|f|_{2,\Omega}^2 - 2(u_\varepsilon, f)_{2,\Omega} \quad \text{by (5.12).} \end{aligned} \tag{5.14}$$

$(\cdot,\cdot)_{2,\Omega}$ denotes the scalar product in $L^2(\Omega)$. Letting $\varepsilon \to 0$ we see that the right-hand side of (5.14) goes to 0 and we get $u_\varepsilon \to f$ in $L^2(\Omega)$. This completes the proof of the theorem. □

To end this section we consider the case where $f \in L^p(\Omega)$ which leads to interesting techniques since we are no more in a Hilbert space framework. So let us show:

Theorem 5.3. *Suppose that $f \in L^p(\Omega)$, $2 \le p < +\infty$. Then if u_ε is the solution to* (5.1)

$$u_\varepsilon \in L^p(\Omega), \qquad u_\varepsilon \longrightarrow f \quad \text{in } L^p(\Omega). \tag{5.15}$$

Proof. As before we have

$$\varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx + \int_\Omega u_\varepsilon v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \tag{5.16}$$

The idea is to choose

$$v = |u_\varepsilon|^{p-2} u_\varepsilon$$

to get for instance the $L^p(\Omega)$-norm of u_ε in the second integral. However one cannot do that directly since this function is not a priori in $H_0^1(\Omega)$ and one has to work a little bit. One introduces the approximation of the function $|x|^{p-2}x$ given by

$$\gamma_n(x) = \begin{cases} n^{p-1} & \text{if } x \ge n, \\ |x|^{p-2}x & \text{if } |x| \le n, \\ -n^{p-1} & \text{if } x \le -n. \end{cases} \tag{5.17}$$

Then – see Corollary 2.15 – one has for every $n > 0$

$$\gamma_n(u_\varepsilon) \in H_0^1(\Omega).$$

Letting this function in (5.16) we get

$$\varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla u_\varepsilon \gamma_n'(u_\varepsilon)\, dx + \int_\Omega u_\varepsilon \gamma_n(u_\varepsilon)\, dx = \int_\Omega f \gamma_n(u_\varepsilon)\, dx.$$

Since $\gamma_n' \ge 0$ we obtain

$$\int_\Omega u_\varepsilon \gamma_n(u_\varepsilon)\, dx \le \int_\Omega f \gamma_n(u_\varepsilon)\, dx \le |f|_{p,\Omega} |\gamma_n(u_\varepsilon)|_{p',\Omega}$$

(by Hölder's inequality). Let us set

$$\delta_n(x) = |x| \wedge n$$

where $\wedge$ denotes the minimum of two numbers. The inequality above implies that

$$|\delta_n(u_\varepsilon)|_{p,\Omega}^p = \int_\Omega \delta_n(u_\varepsilon)^p\, dx \le \int_\Omega u_\varepsilon \gamma_n(u_\varepsilon)\, dx \le |f|_{p,\Omega} |\gamma_n(u_\varepsilon)|_{p',\Omega} \le |f|_{p,\Omega} |\delta_n(u_\varepsilon)|_{p,\Omega}^{p-1}.$$

It follows that

$$\int_\Omega \delta_n(u_\varepsilon)^p \, dx \leq \int_\Omega |f|^p \, dx.$$

Letting $n \to +\infty$ by the monotone convergence theorem we get

$$\int_\Omega |u_\varepsilon|^p \, dx \leq \int_\Omega |f|^p \, dx$$

and thus $u_\varepsilon \in L^p(\Omega)$ with

$$|u_\varepsilon|_{p,\Omega} \leq |f|_{p,\Omega}. \tag{5.18}$$

We already know that $u_\varepsilon \to f$ in $H^{-1}(\Omega)$, in $L^2(\Omega)$ thus we have

$$u_\varepsilon \rightharpoonup f \quad \text{in } L^p(\Omega)\text{-weak}. \tag{5.19}$$

Now from (5.18) we deduce

$$\limsup_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^p \, dx \leq \int_\Omega |f|^p \, dx \tag{5.20}$$

and from the weak lower semi-continuity of the norm in $L^p(\Omega)$

$$\liminf_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^p \, dx \geq \int_\Omega |f|^p \, dx. \tag{5.21}$$

(A norm is a continuous, convex function and is thus also weakly lower semi-continuous – see [21].) (5.20) and (5.21) imply that

$$\lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^p \, dx = \int_\Omega |f|^p \, dx.$$

Together with (5.19) this leads to (5.15) (see exercises) and completes the proof of the theorem. □

5.2 Anisotropic singular perturbation problems

Let Ω be a bounded domain in $\mathbb{R}^2$ and f a function such that for instance

$$f = f(x_1, x_2) \in L^2(\Omega). \tag{5.22}$$

Then for every $\varepsilon > 0$ there exists a unique u_ε solution to

$$\begin{cases} -\varepsilon^2 \partial_{x_1}^2 u_\varepsilon - \partial_{x_2}^2 u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.23}$$

This is what we call an anisotropic singular perturbation problem since the diffusion velocity (see Chapter 3) is very small in the x_1 direction. Of course (5.23) is understood in a weak sense, namely u_ε satisfies

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \varepsilon^2 \partial_{x_1} u_\varepsilon \partial_{x_1} v + \partial_{x_2} u_\varepsilon \partial_{x_2} v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{5.24}$$

We would like to study the behaviour of u_ε when ε goes to 0. Formally, at the limit, if u_0 denotes the limit of u_ε when $\varepsilon \to 0$, we have

$$\int_\Omega \partial_{x_2} u_0 \partial_{x_2} v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \tag{5.25}$$

Even though we can show that $u_0 \in H_0^1(\Omega)$ the bilinear form on the left-hand side of (5.25) is not coercive on $H_0^1(\Omega)$ and the problem (5.25) is not well posed.

Let us denote by $\Pi_{x_1}(\Omega)$ the projection of Ω on the x_1-axis defined by

$$\Pi_{x_1}(\Omega) = \{\, (x_1, 0) \mid \exists\, x_2 \text{ s.t. } (x_1, x_2) \in \Omega \,\}. \tag{5.26}$$

Then for every $x_1 \in \Pi_{x_1}(\Omega) = \Pi_\Omega$ – we drop the x_1 index for simplicity – we denote by Ω_{x_1} the section of Ω above x_1 – i.e.,

$$\Omega_{x_1} = \{\, x_2 \mid (x_1, x_2) \in \Omega \,\}. \tag{5.27}$$

The closest problem to (5.25) to be well defined and suitable for the limit is the following.

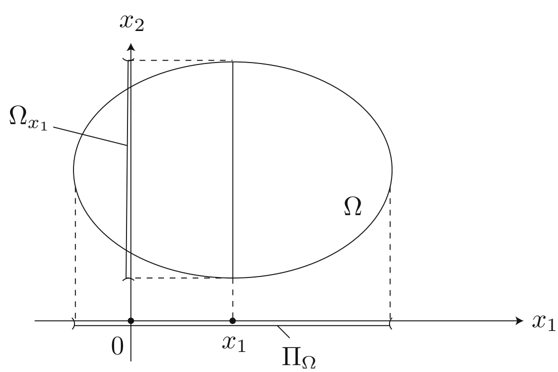


Figure 5.1.

For almost all x_1 – or for every $x_1 \in \Pi_\Omega$ if f is a smooth function – we have

$$f(x_1, \cdot) \in L^2(\Omega_{x_1}), \tag{5.28}$$

(this is a well-known result of integration).

Then there exists a unique $u_0 = u_0(x_1, \cdot)$ solution to

$$\begin{cases} u_0 \in H_0^1(\Omega_{x_1}), \\ \displaystyle\int_{\Omega_{x_1}} \partial_{x_2} u_0 \partial_{x_2} v \, dx = \int_{\Omega_{x_1}} f(x_1, \cdot) v \, dx_2 \quad \forall\, v \in H_0^1(\Omega_{x_1}). \end{cases} \tag{5.29}$$

Note that the existence and uniqueness of a solution to (5.29) is an easy consequence of the Lax–Milgram theorem since

$$\left\{ \int_{\Omega_{x_1}} (\partial_{x_2} v)^2 \, dx \right\}$$

is a norm on $H_0^1(\Omega_{x_1})$. What we would like to show now is the convergence of u_ε toward u_0 solution of (5.29) for instance in $L^2(\Omega)$. Instead of doing that on this simple problem we show how it could be embedded relatively simply in a more general class of problems. However in a first lecture the reader can just apply the technique below to this simple example. It will help him to catch the main ideas.

Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $x = (x_1, \ldots, x_n)$ the points in $\mathbb{R}^n$ and use the decomposition

$$x = (X_1, X_2), \quad X_1 = x_1, \ldots, x_p, \quad X_2 = x_{p+1}, \ldots, x_n. \tag{5.30}$$

With this notation we set, with T denoting the transposition

$$\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix} \tag{5.31}$$

with

$$\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T, \quad \nabla_{X_2} u = (\partial_{x_{p+1}} u, \ldots, \partial_{x_n} u)^T. \tag{5.32}$$

We denote by $A = (a_{ij}(x))$ a $n \times n$ matrix such that

$$a_{ij} \in L^\infty(\Omega) \quad \forall\, i, j = 1, \ldots, n, \tag{5.33}$$

and such that for some $\lambda > 0$ we have

$$\lambda |\xi|^2 \le A\xi \cdot \xi \quad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega. \tag{5.34}$$

We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{5.35}$$

where A_{11}, A_{22} are respectively $p \times p$ and $(n-p) \times (n-p)$ matrices. We then set for $\varepsilon > 0$

$$A_\varepsilon = A_\varepsilon(x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \tag{5.36}$$

For $\xi \in \mathbb{R}^n$ if we set $\xi = \binom{\bar{\xi}_1}{\bar{\xi}_2}$ where $\bar{\xi}_1 = (\xi_1, \dots, \xi_p)^T$, $\bar{\xi}_2 = (\xi_{p+1}, \dots, \xi_n)^T$ we have for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$

$$A_\varepsilon \xi \cdot \xi = A\xi_\varepsilon \cdot \xi_\varepsilon \geq \lambda |\xi_\varepsilon|^2 = \lambda\{\varepsilon^2 |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2\} \tag{5.37}$$

where we have set $\xi_\varepsilon = (\varepsilon \bar{\xi}_1, \bar{\xi}_2)$. Thus A_ε satisfies

$$A_\varepsilon \xi \cdot \xi \geq \lambda(\varepsilon^2 \wedge 1)|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega. \tag{5.38}$$

($\wedge$ denotes the minimum of two numbers.) It follows that A_ε is positive definite and for $f \in H^{-1}(\Omega)$ there exists a unique u_ε solution to

$$\begin{cases} \displaystyle\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega), \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \tag{5.39}$$

Clearly u_ε is the solution of a problem generalizing (5.24). We would like then to study the behavior of u_ε when $\varepsilon \to 0$.

Let us denote by Π_{X_1} the orthogonal projection from $\mathbb{R}^n$ onto the space $X_2 = 0$. For any $X_1 \in \Pi_{X_1}(\Omega) := \Pi_\Omega$ we denote by Ω_{X_1} the section of Ω above X_1 defined as

$$\Omega_{X_1} = \{\, X_2 \mid (X_1, X_2) \in \Omega \,\}.$$

To avoid unnecessary complications we will suppose that

$$f \in L^2(\Omega). \tag{5.40}$$

Then for a.e. $X_1 \in \Pi_\Omega$ we have $f(X_1, \cdot) \in L^2(\Omega_{X_1})$ and there exists a unique $u_0 = u_0(X_1, \cdot)$ solution to

$$\begin{cases} \displaystyle\int_{\Omega_{X_1}} A_{22} \nabla_{X_2} u_0(X_1, X_2) \cdot \nabla_{X_2} v(X_2)\, dX_2 \\ \qquad\qquad \displaystyle = \int_{\Omega_{X_1}} f(X_1, X_2) v(X_2)\, dX_2 \quad \forall\, v \in H_0^1(\Omega_{X_1}), \\ u_0(X_1, \cdot) \in H_0^1(\Omega_{X_1}). \end{cases} \tag{5.41}$$

Clearly u_0 is the natural candidate for the limit of u_ε. Indeed we have

Theorem 5.4. *Under the assumptions above if u_ε is the solution to* (5.39) *we have*

$$u_\varepsilon \longrightarrow u_0, \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(\Omega) \tag{5.42}$$

where u_0 is the solution to (5.41).

(In these convergences the vectorial convergence in $L^2(\Omega)$ means the convergence component by component.)

Proof. Let us take $v = u_\varepsilon$ in (5.39). By (5.37) we derive

$$\lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \le \langle f, u_\varepsilon \rangle \le |f|_{2,\Omega} |u_\varepsilon|_{2,\Omega}. \tag{5.43}$$

Since Ω is bounded, by the Poincaré inequality we have for some constant C independent of ε

$$|v|_{2,\Omega} \le C \big| |\nabla_{X_2} v| \big|_{2,\Omega} \quad \forall\, v \in H_0^1(\Omega). \tag{5.44}$$

From (5.43) we then obtain

$$\lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \le C |f|_{2,\Omega} \big| |\nabla_{X_2} u_\varepsilon| \big|_{2,\Omega}. \tag{5.45}$$

Dropping in this inequality the first term we get

$$\lambda \big| |\nabla_{X_2} u_\varepsilon| \big|_{2,\Omega}^2 \le C |f|_{2,\Omega} \big| |\nabla_{X_2} u_\varepsilon| \big|_{2,\Omega}$$

and thus

$$\big| |\nabla_{X_2} u_\varepsilon| \big|_{2,\Omega} \le C \frac{|f|_{2,\Omega}}{\lambda}.$$

Using this in (5.45) we are ending up with

$$\int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2} u_\varepsilon|^2 \, dx \le C^2 \frac{|f|_{2,\Omega}^2}{\lambda^2}. \tag{5.46}$$

Thus – due to (5.44) – we deduce that

$$u_\varepsilon, \quad |\varepsilon \nabla_{X_1} u_\varepsilon|, \quad |\nabla_{X_2} u_\varepsilon| \quad \text{are bounded in } L^2(\Omega). \tag{5.47}$$

(This of course independently of ε.) It follows that there exist $u_0 \in L^2(\Omega)$, $u_1 \in (L^2(\Omega))^p$, $u_2 \in (L^2(\Omega))^{n-p}$ such that – up to a subsequence

$$u_\varepsilon \rightharpoonup u_0, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup u_1, \qquad \nabla_{X_2} u_\varepsilon \rightharpoonup u_2 \quad \text{in “}L^2(\Omega)\text{”}.$$

(u_1, u_2 are vectors with components in $L^2(\Omega)$. The convergence is meant component by component.) Of course the convergence in $L^2(\Omega)$ – weak – implies the convergence in $\mathcal{D}'(\Omega)$ and by the continuity of the derivation in $\mathcal{D}'(\Omega)$ we deduce that

$$u_\varepsilon \rightharpoonup u_0, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0, \qquad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \quad \text{in } L^2(\Omega). \tag{5.48}$$

We then go back to the equation satisfied by u_ε that we expand using the different blocks of A. This gives

$$\int_\Omega \varepsilon^2 A_{11}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_1}v\,dx+\int_\Omega \varepsilon A_{12}\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_1}v\,dx+\int_\Omega \varepsilon A_{21}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}v\,dx \\ +\int_\Omega A_{22}\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_2}v\,dx=\int_\Omega fv\,dx\quad\forall\, v\in H_0^1(\Omega).$$

Passing to the limit in each term using (5.48) we get

$$\int_\Omega A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}v\,dx=\int_\Omega fv\,dx\quad\forall\, v\in H_0^1(\Omega). \tag{5.49}$$

At this point we do not know yet if for a.e. $X_1\in\Pi_\Omega$ we have

$$u_0(X_1,\cdot)\in H_0^1(\Omega_{X_1}). \tag{5.50}$$

To see this – and more – one remarks first that taking $v=u_\varepsilon$ in (5.49) and passing to the limit we get

$$\int_\Omega A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}u_0\,dx=\int_\Omega fu_0\,dx. \tag{5.51}$$

Next we compute

$$I_\varepsilon=\int_\Omega A_\varepsilon\begin{pmatrix}\nabla_{X_1}u_\varepsilon\\ \nabla_{X_2}(u_\varepsilon-u_0)\end{pmatrix}\cdot\begin{pmatrix}\nabla_{X_1}u_\varepsilon\\ \nabla_{X_2}(u_\varepsilon-u_0)\end{pmatrix}dx. \tag{5.52}$$

We get

$$I_\varepsilon=\int_\Omega \varepsilon^2 A_{11}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_1}u_\varepsilon\,dx+\int_\Omega \varepsilon A_{12}\nabla_{X_2}(u_\varepsilon-u_0)\cdot\nabla_{X_1}u_\varepsilon\,dx \\ +\int_\Omega \varepsilon A_{21}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}(u_\varepsilon-u_0)\,dx \\ +\int_\Omega A_{22}\nabla_{X_2}(u_\varepsilon-u_0)\cdot\nabla_{X_2}(u_\varepsilon-u_0)\,dx.$$

Using (5.39) with $v=u_\varepsilon$, we obtain

$$I_\varepsilon=\int_\Omega fu_\varepsilon\,dx-\int_\Omega \varepsilon A_{12}\nabla_{X_2}u_0\cdot\nabla_{X_1}u_\varepsilon\,dx-\int_\Omega \varepsilon A_{21}\nabla_{X_1}u_\varepsilon\cdot\nabla_{X_2}u_0 \\ -\int_\Omega A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}u_\varepsilon\,dx-\int_\Omega A_{22}\nabla_{X_2}u_\varepsilon\cdot\nabla_{X_2}u_0\,dx \\ +\int_\Omega A_{22}\nabla_{X_2}u_0\nabla_{X_2}u_0\,dx.$$

Passing to the limit in ε we get

$$\lim_{\varepsilon\to 0} I_\varepsilon = \int_\Omega f u_0 \, dx - \int_\Omega A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \, dx = 0.$$

Using the coerciveness assumption we have (see (5.52))

$$\lambda \int_\Omega \varepsilon^2 |\nabla_{X_1} u_\varepsilon|^2 + |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \, dx \leq I_\varepsilon.$$

It follows that

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0, \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{in } L^2(\Omega).$$

Now we also have

$$\int_{\Pi_\Omega} \int_{\Omega_{X_1}} |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \, dX_2 \, dX_1 \longrightarrow 0. \tag{5.53}$$

It follows that for almost every X_1

$$\int_{\Omega_{X_1}} |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \, dX_2 \longrightarrow 0.$$

Since

$$\left\{ \int_{\Omega_{X_1}} |\nabla_{X_2} v|^2 \, dX_2 \right\}^{\frac{1}{2}}$$

is a norm on $H_0^1(\Omega_{X_1})$ and $u_\varepsilon(X_1, \cdot) \in H_0^1(\Omega_{X_1})$ we have

$$u_0(X_1, \cdot) \in H_0^1(\Omega_{X_1})$$

and this for almost every X_1. Using then the Poincaré inequality implies

$$\begin{aligned} & \int_{\Omega_{X_1}} |u_\varepsilon - u_0|^2 \, dX_2 \leq C \int_{\Omega_{X_1}} |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \, dX_2 \\ \Longrightarrow \quad & \int_\Omega |u_\varepsilon - u_0|^2 \, dx \leq C \int_\Omega |\nabla_{X_2}(u_\varepsilon - u_0)|^2 \, dx \longrightarrow 0 \end{aligned}$$

(by (5.53)) and thus

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } L^2(\Omega). \tag{5.54}$$

All this is up to a subsequence. If we can identify u_0 uniquely then all the convergences above will hold for the whole sequence. For this purpose recall first that

$$u_0(X_1, \cdot) \in H_0^1(\Omega_{X_1}). \tag{5.55}$$

One can cover Ω by a countable family of open sets of the form

$$U_i \times V_i \subset \Omega, \quad i \in \mathbb{N}$$

where U_i, V_i are open subsets of $\mathbb{R}^p$, $\mathbb{R}^{n-p}$ respectively. One can even choose U_i, V_i hypercubes. Then choosing $\varphi \in H_0^1(V_i)$ we derive from (5.49)

$$\begin{aligned}\int_{U_i} \eta(X_1) \int_{V_i} A_{22}(X_1, X_2)\nabla_{X_2} u_0(X_1, X_2) \cdot \nabla_{X_2}\varphi(X_2)\, dX_2\, dX_1 \\ = \int_{U_i} \eta(X_1) \int_{V_i} f(X_1, X_2)\varphi(X_2)\, dX_2\, dX_1 \quad \forall\, \eta \in \mathcal{D}(U_i),\end{aligned}$$

since $\eta\varphi \in H_0^1(\Omega)$. Thus there exists a set of measure zero, $N(\varphi)$, such that

$$\int_{V_i} A_{22}(X_1, X_2)\nabla_{X_2} u_0(X_1, X_2) \cdot \nabla_{X_2}\varphi(X_2)\, dX_2 = \int_{V_i} f(X_1, X_2)\varphi(X_2)\, dX_2 \tag{5.56}$$

for all $X_1 \in U_i \setminus N(\varphi)$. Denote by φ_n a Hilbert basis of $H_0^1(V_i)$. Then (5.56) holds (replacing φ by φ_n) for all X_1 such that

$$X_1 \in U_i \setminus N_i(\varphi_n)$$

where $N_i(\varphi_n)$ is a set of measure 0. Thus for

$$X_1 \in U_i \setminus \cup_n N_i(\varphi_n)$$

we have (5.56) for any $\varphi \in H_0^1(V_i)$. This follows easily from the density in $H_0^1(V_i)$ of the linear combinations of the φ_n. Let us then choose

$$X_1 \in \Pi_\Omega \setminus \cup_i \cup_n N_i(\varphi_n)$$

(note that $\cup_i \cup_n N_i(\varphi_n)$ is a set of measure 0). Let

$$\varphi \in \mathcal{D}(\Omega_{X_1}).$$

If K denotes the support of φ we have clearly

$$K \subset \cup_i V_i$$

and thus K can be covered by a finite number of V_i that for simplicity we will denote by $V_1, \dots, V_k$. Using a partition of unity (see [86], [96] and the exercises) there exists $\psi_i \in \mathcal{D}(V_i)$ such that

$$\sum_{i=1}^{k} \psi_i = 1 \quad \text{on } K.$$

By (5.56) we derive

$$\begin{aligned}\int_K A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}\varphi\,dX_2 &= \int_K A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}\sum_i(\psi_i\varphi)\,dX_2\\ &= \sum_i\int_{V_i} A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}(\psi_i\varphi)\,dX_2\\ &= \sum_i\int_{V_i} f\psi_i\varphi\,dX_2\\ &= \int_K f\varphi\,dX_2.\end{aligned}$$

This is also

$$\int_{\Omega_{X_1}} A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}\varphi\,dX_2 = \int_{\Omega_{X_1}} f\varphi\,dX_2 \quad \forall\,\varphi\in\mathcal{D}(\Omega_{X_1})$$

and thus u_0 is the unique solution to (5.41) for a.e. $X_1\in\Pi_\Omega$. This completes the proof of the theorem. □

Exercises

1. Show that if $f\in H_0^1(\Omega)$ and if u_ε is the solution to (5.1) one has

$$u_\varepsilon \longrightarrow f \quad \text{in } H_0^1(\Omega).$$

Show that if $f\in\mathcal{D}(\Omega)$ then for every k one has

$$\Delta^k u_\varepsilon \longrightarrow \Delta^k f \quad \forall\,k\geq 1 \quad \text{in } H_0^1(\Omega),$$

(Δ^k denotes the operator Δ iterated k times).

2. One denotes by $A(x)$ a uniformly positive definite matrix. Study the singular perturbation problem

$$\begin{cases} u_\varepsilon\in H_0^1(\Omega),\\ \varepsilon\displaystyle\int_\Omega A(x)\nabla u_\varepsilon\cdot\nabla v\,dx + \int_\Omega a(x)u_\varepsilon v\,dx = \langle f,v\rangle \quad \forall\,v\in H_0^1(\Omega)\end{cases}$$

when $a(x)$ is a function satisfying

$$a(x)\geq a>0 \quad \text{a.e. } x\in\Omega,$$

and $f\in H^{-1}(\Omega)$.

3. Compute the solution to

$$\begin{cases} -\varepsilon u_\varepsilon'' + u_\varepsilon = 1 & \text{in } \Omega = (0,1), \\ u_\varepsilon \in H_0^1(\Omega). \end{cases}$$

Examine its behaviour when $\varepsilon \to 0$.

4. For $f \in L^p(\Omega)$, $p \geq 2$ let u be the solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

where A is a $n \times n$ matrix satisfying (4.2), (4.3). Show that $u \in L^p(\Omega)$ and that

$$|u|_{p,\Omega} \leq |f|_{p,\Omega}.$$

5. Suppose that $p \geq 2$.
   (i) Show that there exists a constant $c > 0$ independent of z such that

$$|1+z|^p \geq 1 - pz + c|z|^p \quad \forall\, z \in \mathbb{R}. \tag{*}$$

   (ii) Let $u_\varepsilon \in L^p(\Omega)$ be a sequence such that when $\varepsilon \to 0$

$$u_\varepsilon \rightharpoonup f \quad \text{in } L^p(\Omega),$$
$$|u_\varepsilon|_p \to |f|_p.$$

   Show that

$$u_\varepsilon \to f \quad \text{in } L^p(\Omega) \text{ strong.}$$

   (Hint: take $z = \frac{u_\varepsilon - f}{f}$ in $(*)$.)

   Show the same result for $1 < p < 2$.

6. Let Ω be a bounded domain in $\mathbb{R}^n$. For $\sigma > 0$ one considers u_σ the solution to

$$\begin{cases} u_\sigma \in H^1(\Omega), \\ \sigma \displaystyle\int_\Omega \nabla u_\sigma \cdot \nabla v \, dx + \int_\Omega u_\sigma v \, dx = \int_\Omega f v \, dx. \end{cases}$$

   (i) Show that for $f \in L^2(\Omega)$ there exists a unique solution to the problem above.
   (ii) Show that when $\sigma \to +\infty$

$$u_\sigma \to \frac{1}{|\Omega|} \int_\Omega f(x)\, dx \quad \text{in } H^1(\Omega).$$

7. (Partition of unity.)
   Let K be a compact subset of $\mathbb{R}^p$ such that

$$K \subset \cup_{i=1}^{k} V_i$$

   where V_i are open subsets.

   (i) Show that one can find open subsets V_i', V_i'' such that

$$\begin{aligned} &V_i'' \subset \overline{V}_i'' \subset V_i' \subset \overline{V}_i' \subset V_i \quad \forall\, i = 1, \dots, k, \\ &K \subset \cup_{i=1}^{k} V_i'' \end{aligned}$$

   (ii) Show that there exists a continuous function α_i such that

$$\alpha_i = 1 \text{ on } V_i'', \qquad \alpha_i = 0 \text{ outside } V_i'.$$

   (iii) Show that for ε_i small enough

$$\gamma_i = \rho_{\varepsilon_i} * \alpha_i \in \mathcal{D}(V_i).$$

   (iv) One sets

$$\psi_i = \frac{\gamma_i}{\sum_{i=1}^{k} \gamma_i}.$$

   Show that $\psi_i \in \mathcal{D}(V_i)$ and $\sum_{i=1}^{k} \psi_i = 1$ on K.

8. Let $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are bounded domains in $\mathbb{R}^p$, $\mathbb{R}^{n-p}$ respectively $(0 < p < n)$. Under the assumptions of Theorem 5.4 show that, in general, for $f \neq 0$ one cannot have

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } H^1(\Omega).$$

Chapter 6

Asymptotic Analysis for Problems in Large Cylinders

6.1 A model problem

Let us denote by Ω_ℓ the rectangle in $\mathbb{R}^2$ defined by

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1). \tag{6.1}$$

For convenience we will also set $\omega = (-1, 1)$. If $f = f(x_2)$ is a regular function –

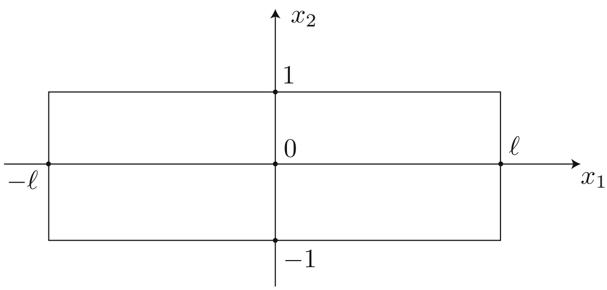


Figure 6.1.

as we will see it can be more general – there exists a unique u_ℓ solution to

$$\begin{cases} -\Delta u_\ell = f(x_2) & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell. \end{cases} \tag{6.2}$$

Of course as usual this solution is understood in the weak sense as explained in Chapters 3 and 4. Since u_ℓ vanishes at both ends of Ω_ℓ, if $f \not\equiv 0$ it is clear that u_ℓ has to depend on x_1. However due to our choice of f – independent of x_1 – u_ℓ has to experience some special properties and if ℓ is large one expects u_ℓ to be fast independent of x_1 far away from the end sides of the rectangle. In other words one expects u_ℓ to converge locally toward a function independent of x_1 when $\ell \to +\infty$. A natural candidate for the limit is of course u_∞ the weak solution to

$$\begin{cases} -\partial_{x_2}^2 u_\infty = f & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega. \end{cases} \tag{6.3}$$

We recall that $\omega = (-1, 1)$, $\partial\omega = \{-1, 1\}$. Indeed we are going to show that if $\ell_0 > 0$ is fixed then

$$u_\ell \longrightarrow u_\infty \quad \text{in } \Omega_{\ell_0} \tag{6.4}$$

with an exponential rate of convergence, that is to say the norm of $u_\ell - u_\infty$ in Ω_{ℓ_0} is a $O(e^{-\alpha\ell})$ for some positive constant α. At this point we do not precise in what norm this convergence takes place since several choices are possible.

Somehow f is just having an auxiliary rôle in our analysis and we will take it the most general possible assuming

$$f \in H^{-1}(\omega). \tag{6.5}$$

Then for $v \in H_0^1(\Omega_\ell)$ for almost every x_1

$$v(x_1, \cdot) \in H_0^1(\omega) \tag{6.6}$$

(see [30] or the exercises) and the linear form

$$\langle f, v \rangle = \int_{-\ell}^{\ell} \langle f, v(x_1, \cdot) \rangle \, dx_1 \tag{6.7}$$

defines clearly an element of $H^{-1}(\Omega_\ell)$ that we will yet denote by f. Note that for a smooth function f one has as usual

$$\langle f, v \rangle = \int_{\Omega_\ell} f v \, dx.$$

Then, with these definitions, there exist weak solutions to (6.2), (6.3) respectively that is to say u_ℓ, u_∞ such that

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), \\ \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega_\ell), \end{cases} \tag{6.8}$$

$$\begin{cases} u_\infty \in H_0^1(\omega), \\ \displaystyle\int_{\omega} \partial_{x_2} u_\infty \partial_{x_2} v \, dx_2 = \langle f, v \rangle \quad \forall\, v \in H_0^1(\omega). \end{cases} \tag{6.9}$$

We would like to show now that in this general framework – in terms of f – one has $u_\ell \to u_\infty$ when $\ell \to +\infty$.

As we have seen in Chapter 2 by the Poincaré inequality there exists a constant $c > 0$ such that

$$c \int_\omega u^2\,dx \le \int_\omega (\partial_{x_2} u)^2\,dx \quad \forall\, u \in H_0^1(\omega). \tag{6.10}$$

We will set

$$0 < \lambda_1 = \operatorname*{Inf}_{\substack{u \in H_0^1(\omega) \\ u \neq 0}} \frac{\int_\omega (\partial_{x_2} u)^2\,dx}{\int_\omega u^2\,dx}. \tag{6.11}$$

(As we will see later λ_1 is the first eigenvalue of the operator $-\partial_{x_2}^2$ for the Dirichlet problem. We will not use this here but the introduction of λ_1 is useful in order to get sharp estimates.)

Let us start with the following fundamental estimate.

Theorem 6.1. *Suppose that $\ell_2 < \ell_1 \le \ell$ then we have*

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_{\ell_2}} \le e^{-\sqrt{\lambda_1}(\ell_1-\ell_2)} \big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_{\ell_1}}. \tag{6.12}$$

Remark 6.1. This estimate shows in particular that the function

$$\ell' \mapsto e^{-\sqrt{\lambda_1}\ell'} \big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_{\ell'}}$$

is nondecreasing.

Proof of Theorem 6.1. Let $v \in H_0^1(\Omega_\ell)$. By (6.6) and (6.9) we have for almost every x_1

$$\int_\omega \partial_{x_2} u_\infty \partial_{x_2} v(x_1,\cdot)\,dx_2 = \langle f, v(x_1,\cdot)\rangle.$$

Integrating this equality on $(-\ell,\ell)$ we deduce, since u_∞ is independent of x_1,

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v\,dx = \langle f, v\rangle.$$

Combining this with (6.8) it follows that

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty)\cdot\nabla v\,dx = 0 \quad \forall\, v \in H_0^1(\Omega_\ell). \tag{6.13}$$

(Note the “ghost” rôle of f in these estimates.)

We introduce then the function ρ whose graph is depicted in Figure 6.2 on top of the next page. In particular one has

$$0 \le \rho \le 1, \quad \rho = 1 \text{ on } (-\ell_2,\ell_2), \quad \rho = 0 \text{ on } \mathbb{R}\backslash(-\ell_1,\ell_1), \quad |\rho'| \le \frac{1}{\ell_1-\ell_2}. \tag{6.14}$$

Then

$$v = (u_\ell - u_\infty)\rho(x_1) \in H_0^1(\Omega_\ell)$$

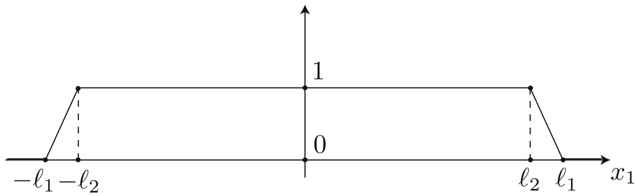


Figure 6.2.

and from (6.13) we derive

$$\int_{\Omega_{\ell_1}} \nabla(u_\ell - u_\infty) \cdot \nabla\{(u_\ell - u_\infty)\rho\}\, dx = 0. \tag{6.15}$$

It follows that we have

$$\begin{aligned}\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho\, dx &= -\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} (\nabla(u_\ell - u_\infty)\cdot\nabla\rho)(u_\ell - u_\infty)\, dx \\ &= -\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} \partial_{x_1}(u_\ell - u_\infty)\partial_{x_1}\rho(u_\ell - u_\infty)\, dx\end{aligned}$$

since ρ is independent of x_2 and $\partial_{x_1}\rho$ vanishes everywhere except on $\Omega_{\ell_1}\setminus\Omega_{\ell_2}$.

Then – see (6.14):

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho\, dx \le \frac{1}{\ell_1 - \ell_2}\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} |\partial_{x_1}(u_\ell - u_\infty)|\,|u_\ell - u_\infty|\, dx.$$

We then use the following Young inequality

$$ab \le \frac{1}{2}\left\{\frac{1}{\sqrt{\lambda_1}}a^2 + \sqrt{\lambda_1}b^2\right\}$$

to get

$$\begin{aligned}&\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho\, dx \\ &\le \frac{1}{2(\ell_1 - \ell_2)}\left\{\frac{1}{\sqrt{\lambda_1}}\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} (\partial_{x_1}(u_\ell - u_\infty))^2\, dx + \sqrt{\lambda_1}\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} (u_\ell - u_\infty)^2\, dx\right\}.\end{aligned} \tag{6.16}$$

Now by the definition of λ_1 – see also (6.6) – we have for a.e. x_1

$$\int_\omega (u_\ell - u_\infty)^2(x_1,\cdot)\, dx_2 \le \frac{1}{\lambda_1}\int_\omega (\partial_{x_2}(u_\ell - u_\infty))^2(x_1,\cdot)\, dx_2.$$

Integrating for $|x_1| \in (\ell_2, \ell_1)$ we derive

$$\int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} (u_\ell - u_\infty)^2\, dx \le \frac{1}{\lambda_1} \int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} (\partial_{x_2}(u_\ell - u_\infty))^2\, dx.$$

Going back to (6.16) we get

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 \rho\, dx \le \frac{1}{2\sqrt{\lambda_1}(\ell_1 - \ell_2)} \int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} |\nabla(u_\ell - u_\infty)|^2\, dx.$$

Since $\rho = 1$ on $(-\ell_2, \ell_2)$ – see (6.14) – this implies that

$$\int_{\Omega_{\ell_2}} |\nabla(u_\ell - u_\infty)|^2\, dx \le \frac{1}{2\sqrt{\lambda_1}(\ell_1 - \ell_2)} \int_{\Omega_{\ell_1}\setminus\Omega_{\ell_2}} |\nabla(u_\ell - u_\infty)|^2\, dx. \tag{6.17}$$

Let us set

$$F(\ell') = \int_{\Omega_{\ell'}} |\nabla(u_\ell - u_\infty)|^2\, dx. \tag{6.18}$$

We can then write (6.17)

$$F(\ell_2) \le \frac{1}{2\sqrt{\lambda_1}} \frac{F(\ell_1) - F(\ell_2)}{\ell_1 - \ell_2}. \tag{6.19}$$

Letting $\ell_1 \to \ell_2$ we obtain

$$F(\ell_2) \le \frac{1}{2\sqrt{\lambda_1}} F'(\ell_2) \quad \text{for a.e. } \ell_2. \tag{6.20}$$

(Note that $F(\ell')$ is almost everywhere derivable.) The formula above can also be written

$$(e^{-2\sqrt{\lambda_1}\ell_2} F(\ell_2))' \ge 0$$

which implies

$$e^{-2\sqrt{\lambda_1}\ell_2} F(\ell_2) \le e^{-2\sqrt{\lambda_1}\ell_1} F(\ell_1)$$

for any $\ell_2 \le \ell_1$. By (6.18) this implies (6.12) and completes the proof of the theorem. □

To complete our convergence result we will need the following proposition.

Proposition 6.2. *We have*

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_\ell} \le \sqrt{2}|u_\infty|_{1,2}, \tag{6.21}$$

where $|v|_{1,2}^2 = \int_\omega v^2 + \partial_{x_2} v^2\, dx.$

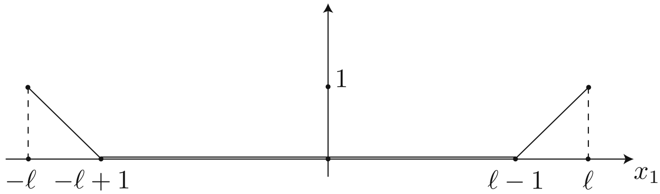


Figure 6.3: Graph of η

Proof. Note that the left-hand side of (6.21) cannot a priori go to 0, but the estimate shows already that u_ℓ is close to u_∞.

We denote by η the function whose graph is depicted in Figure 6.3. Clearly $u_\ell - u_\infty + \eta u_\infty \in H_0^1(\Omega_\ell)$. From (6.13) we derive

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla(u_\ell - u_\infty + \eta u_\infty)\, dx = 0$$

hence

$$\begin{aligned}\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 &= -\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla\{\eta u_\infty\}\, dx \\ &\leq \left\{\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\, dx\right\}^{\frac{1}{2}} \left\{\int_{\Omega_\ell} |\nabla\{\eta u_\infty\}|^2\, dx\right\}^{\frac{1}{2}}\end{aligned}$$

by the Cauchy–Schwarz inequality. Since η vanishes on $(-\ell+1, \ell-1)$ we then derive

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\, dx \leq \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |\nabla(\eta u_\infty)|^2\, dx.$$

This last integral can be evaluated to give

$$\begin{aligned}\int_{\Omega_\ell \setminus \Omega_{\ell-1}} |\nabla(\eta u_\infty)|^2\, dx &= \int_{\Omega_\ell \setminus \Omega_{\ell-1}} ((\partial_{x_1}\eta)u_\infty)^2 + (\eta \partial_{x_2} u_\infty)^2\, dx \\ &\leq \int_{\Omega_\ell \setminus \Omega_{\ell-1}} u_\infty^2 + (\partial_{x_2} u_\infty)^2\, dx = 2|u_\infty|_{1,2}^2.\end{aligned}$$

The result follows. $\square$

We can now state our exponential convergence result, namely

Theorem 6.3. *For any $\ell_0 > 0$ we have*

$$\big||\nabla(u_\ell - u_\infty)|\big|_{2,\Omega_{\ell_0}} \leq \sqrt{2} e^{\sqrt{\lambda_1}\ell_0} |u_\infty|_{1,2} e^{-\sqrt{\lambda_1}\ell}, \tag{6.22}$$

i.e., $u_\ell \to u_\infty$ in Ω_{ℓ_0} with an exponential rate of convergence.

Proof. We just apply (6.12) with $\ell_2 = \ell_0$, $\ell_1 = \ell$. The result follows then from (6.21). □

Remark 6.2. Taking $\ell_2 = \frac{\ell}{2}$, $\ell_1 = \ell$ we also have from (6.12)

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_{\ell/2}} \leq \sqrt{2}|u_\infty|_{1,2}e^{-\sqrt{\lambda_1}\frac{\ell}{2}}. \tag{6.23}$$

However we do not have

$$\big|\big|\nabla(u_\ell - u_\infty)\big|\big|_{2,\Omega_\ell} \longrightarrow 0 \tag{6.24}$$

(see Exercise 1).

6.2 Another type of convergence

Somehow the convergence in $H^1(\Omega)$ is not appealing. Pointwise convergence is more satisfactory. This can be achieved very simply by a careful use of the maximum principle. We will even extend this property to very general domains.

Let us consider the function

$$v_1(x_2) = \cos\left(\frac{\pi}{2}x_2\right) \tag{6.25}$$

the first eigenfunction of the one-dimensional Dirichlet problem on $(-1,1)$ (we refer to Chapter 9). Note that we will just be using the expression of v_1. Then we have:

Theorem 6.4. *Let u_ℓ, u_∞ be the solutions to* (6.2), (6.3). *Suppose that for some constant C we have*

$$|u_\infty(x_2)| \leq Cv_1(x_2) \quad \forall x_2. \tag{6.26}$$

Then

$$|u_\ell - u_\infty| \leq C\left\{\frac{\cosh\left(\frac{\pi}{2}x_1\right)}{\cosh\left(\frac{\pi}{2}\ell\right)}\right\}v_1(x_2). \tag{6.27}$$

Proof. Let us remark that if u denotes the function in the right-hand side of (6.27) one has

$$-\Delta u = 0$$

in the usual sense and also in the weak sense. It follows that

$$-\Delta\{u_\ell - u_\infty - u\} = 0$$

in a weak sense in Ω_ℓ. Moreover on $\partial\Omega_\ell$

$$u_\ell - u_\infty - u = -u_\infty - u = -u_\infty - Cv_1 \leq 0.$$

By the maximum principle it follows that

$$u_\ell - u_\infty \le u.$$

Arguing similarly with $u_\ell - u_\infty + u$ one derives

$$-u \le u_\ell - u_\infty \le u$$

which completes the proof of (6.27). □

Remark 6.3. The assumption (6.26) holds in many cases, for instance if $\partial_{x_2} u_\infty \in L^\infty(\omega)$ which is the case for $f \in L^2(\omega)$. As a consequence of (6.27) we have

$$|u_\ell - u_\infty|_{\infty,\Omega_{\ell_0}} \le C \cosh\left(\frac{\pi}{2}\ell_0\right) e^{-\frac{\pi}{2}\ell} \tag{6.28}$$

which gives an exponential rate of convergence for the uniform norm locally.

We can now address the case of more general domains. Suppose that Ω'_ℓ is a domain of the type of Figure 6.4. Our unique assumption being that

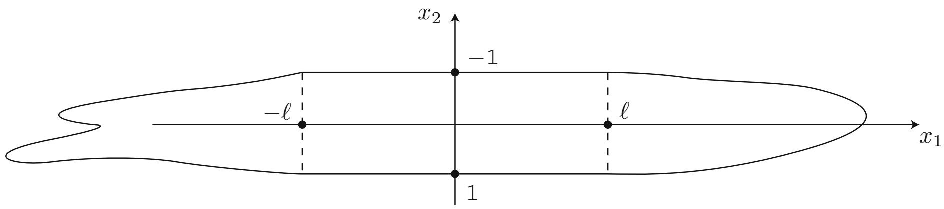


Figure 6.4.

$$\Omega'_\ell \subset \mathbb{R} \times \omega, \qquad \Omega'_\ell \cap \big((-\ell, \ell) \times \mathbb{R}\big) = \Omega_\ell. \tag{6.29}$$

Let f be a function such that

$$f \in L^2(\omega).$$

Then there exists a unique u'_ℓ solution to

$$\begin{cases} u'_\ell \in H^1_0(\Omega'_\ell), \\ \displaystyle\int_{\Omega'_\ell} \nabla u'_\ell \cdot \nabla v \, dx = \int_{\Omega'_\ell} f v \, dx \quad \forall\, v \in H^1_0(\Omega'_\ell). \end{cases} \tag{6.30}$$

Then we have

Theorem 6.5. *Let u_∞ be the solution to (6.3). For any ℓ_0 there exists a constant $C = C(\ell_0, f)$ such that*

$$|u'_\ell - u_\infty|_{\infty,\Omega_{\ell_0}} \le C e^{-\frac{\pi}{2}\ell}. \tag{6.31}$$

Proof. We suppose first that $f \geq 0$. Let u_ℓ be the solution to (6.2). By the maximum principle we have

$$0 \leq u'_\ell \quad \text{in } \Omega'_\ell, \qquad 0 \leq u_\infty \quad \text{on } \omega.$$

Thus it follows that

$$\begin{aligned} u_\ell - u'_\ell &\leq 0 \quad \text{on} \quad \partial\Omega_\ell, & -\Delta(u_\ell - u'_\ell) &= 0 \quad \text{in} \quad \Omega_\ell, \\ u'_\ell - u_\infty &\leq 0 \quad \text{on} \quad \partial\Omega'_\ell, & -\Delta(u'_\ell - u_\infty) &= 0 \quad \text{in} \quad \Omega'_\ell. \end{aligned}$$

By the maximum principle we deduce that

$$0 \leq u_\ell \leq u'_\ell \leq u_\infty \quad \text{in } \Omega_\ell.$$

By the estimate (6.28) it follows for $\ell_0 < \ell$

$$|u'_\ell - u_\infty|_{\infty,\Omega_{\ell_0}} \leq |u_\ell - u_\infty|_{\infty,\Omega_{\ell_0}} \leq Ce^{-\frac{\pi}{2}\ell}$$

(see also Remark 6.3). Thus the result follows in this case.

In the general case one writes

$$f = f^+ - f^-$$

where f^+ and f^- are the positive and negative parts of f. Then introducing $u_\ell^\pm$, $u_\infty^\pm$ the solutions to

$$\begin{aligned} u_\ell^\pm &\in H_0^1(\Omega'_\ell), & -\Delta u_\ell^\pm &= f^\pm \quad \text{in} \quad \Omega'_\ell, \\ u_\infty^\pm &\in H_0^1(\omega), & -\Delta u_\infty^\pm &= f^\pm \quad \text{in} \quad \omega, \end{aligned}$$

we have by the results above

$$|u_\ell^\pm - u_\infty^\pm|_{\infty,\Omega_{\ell_0}} \leq Ce^{-\frac{\pi}{2}\ell}.$$

Due to the linearity of the problems and uniqueness of the solution we have

$$u'_\ell = u_\ell^+ - u_\ell^-, \qquad u_\infty = u_\infty^+ - u_\infty^-.$$

The result follows then easily. □

Remark 6.4. Assuming f extended by 0 outside ω one can drop the left-hand side assumption of (6.29).

6.3 The general case

We denote a point $x \in \mathbb{R}^n$ also as $x = (X_1, X_2)$ where

$$X_1 = (x_1, \ldots, x_p), \quad X_2 = (x_{p+1}, \ldots, x_n) \tag{6.32}$$

in other words we split the components of a point in $\mathbb{R}^n$ into the p first components and the $n-p$ last ones.
Let ω_1 be an open subset of $\mathbb{R}^p$ that we suppose to satisfy

$$\omega_1 \text{ is a bounded convex domain containing } 0. \tag{6.33}$$

Let ω_2 be a bounded open subset of $\mathbb{R}^{n-p}$, then we set

$$\Omega_\ell = \ell\omega_1 \times \omega_2. \tag{6.34}$$

Note that by (6.33) we have $\Omega_\ell \subset \Omega_{\ell'}$ for $\ell < \ell'$.
We denote by

$$A(x) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{21}(X_1, X_2) & A_{22}(X_2) \end{pmatrix} = (a_{ij}(x)) \tag{6.35}$$

a $n \times n$-matrix divided into four blocks such that

$$A_{11} \text{ is a } p \times p\text{-matrix, } A_{22} \text{ is a } (n-p) \times (n-p)\text{-matrix.} \tag{6.36}$$

We assume that

$$a_{ij} \in L^\infty(\mathbb{R}^p \times \omega_2) \tag{6.37}$$

and that for some constants λ, Λ we have

$$\lambda|\xi|^2 \le A(x)\xi \cdot \xi \qquad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^p \times \omega_2 \tag{6.38}$$

$$|A(x)\xi| \le \Lambda|\xi| \qquad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^p \times \omega_2. \tag{6.39}$$

Then by the Lax–Milgram theorem for

$$f \in L^2(\omega_2) \tag{6.40}$$

there exists a unique u_∞ solution to

$$u_\infty \in H_0^1(\omega_2), \qquad \int_{\omega_2} A_{22}\nabla_{X_2}u_\infty \cdot \nabla_{X_2} v dX_2 = \int_{\omega_2} fv dX_2 \quad \forall v \in H_0^1(\omega_2). \tag{6.41}$$

(In this system ∇_{X_2} stands for the gradient in X_2, that is $(\partial_{x_{p+1}}, \ldots, \partial_{x_n})$, $dX_2 = dx_{p+1} \cdots dx_n$.)

By the Lax–Milgram theorem there exists also a unique u_ℓ solution to

$$u_\ell \in H_0^1(\Omega_\ell), \qquad \int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v dx = \int_{\Omega_\ell} fv dx \quad \forall v \in H_0^1(\Omega_\ell). \tag{6.42}$$

Moreover we have

Theorem 6.6. *There exist two constants* $c, \alpha > 0$ *independent of* ℓ *such that*

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \le ce^{-\alpha\ell}|f|^2_{2,\omega_2} \tag{6.43}$$

Proof. The proof is divided into three steps.

- **Step 1.** The equation satisfied by $u_\ell - u_\infty$.

If $v \in H^1_0(\Omega_\ell)$ then for almost every X_1 in $\ell\omega_1$ we have

$$v(X_1, \cdot) \in H^1_0(\omega_2)$$

and thus by (6.41)

$$\int_{\omega_2} A_{22}\nabla_{X_2}u_\infty \cdot \nabla_{X_2}v(X_1,\cdot)dX_2 = \int_{\omega_2} fv(X_1,\cdot)dX_2.$$

Integrating in X_1 we get

$$\int_{\Omega_\ell} A_{22}\nabla_{X_2}u_\infty \cdot \nabla_{X_2}vdx = \int_{\Omega_\ell} fvdx \quad \forall v \in H^1_0(\Omega_\ell) \tag{6.44}$$

Now for $v \in H^1_0(\Omega_\ell)$ we have

$$\begin{aligned}\int_{\Omega_\ell} A\nabla u_\infty \cdot \nabla vdx &= \int_{\Omega_\ell} A_{12}\nabla_{X_2}u_\infty \cdot \nabla_{X_1}vdx + \int_{\Omega_\ell} A_{22}\nabla_{X_2}u_\infty \cdot \nabla_{X_2}vdx \\ &= \int_{\Omega_\ell} A_{22}\nabla_{X_2}u_\infty \cdot \nabla_{X_2}vdx = \int_{\Omega_\ell} fvdx\end{aligned} \tag{6.45}$$

(since A_{12}, u_∞ are depending on X_2 only). Combining (6.42), (6.45) we get

$$\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty) \cdot \nabla vdx = 0 \quad \forall v \in H^1_0(\Omega_\ell). \tag{6.46}$$

- **Step 2.** An iteration technique.

Set $0 < \ell_0 \le \ell - 1$. There exists ρ a function of X_1 only such that

$$0 \le \rho \le 1,\ \rho = 1 \text{ on } \ell_0\omega_1,\ \rho = 0 \text{ on } \mathbb{R}^p\backslash(\ell_0+1)\omega_1,\ |\nabla_{X_1}\rho| \le c_0 \tag{6.47}$$

where c_0 is a universal constant (cf. Exercise 4). Then we have

$$(u_\ell - u_\infty)\rho^2 \in H^1_0(\Omega_\ell)$$

and from (6.46) we derive

$$\begin{aligned}
&\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty)\cdot\nabla(u_\ell - u_\infty)\rho^2 dx \\
&\qquad = -2\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty)\cdot\begin{pmatrix}\nabla_{X_1}\rho \\ 0\end{pmatrix}(u_\ell - u_\infty)\rho dx \\
&\qquad \le 2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |A\nabla(u_\ell - u_\infty)||\nabla_{X_1}\rho||u_\ell - u_\infty|\rho dx.
\end{aligned}$$

Using (6.38), (6.39), (6.47) and the Cauchy–Schwarz inequality we derive

$$\begin{aligned}
&\lambda\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\rho^2 dx \\
&\qquad \le 2c_0\Lambda\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|\rho|u_\ell - u_\infty| dx \\
&\qquad \le 2c_0\Lambda\Big(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\rho^2 dx\Big)^{\frac12}\Big(\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx\Big)^{\frac12}.
\end{aligned}$$

It follows that (recall that $\rho = 1$ on Ω_{ℓ_0})

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 dx \le \Big(2c_0\frac{\Lambda}{\lambda}\Big)^2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx. \tag{6.48}$$

Since $u_\ell - u_\infty$ vanishes on the lateral boundary of Ω_ℓ there exists a constant c_p independent of ℓ such that (see Theorem 2.8)

$$\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} (u_\ell - u_\infty)^2 dx \le c_p^2\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |\nabla_{X_2}(u_\ell - u_\infty)|^2 dx. \tag{6.49}$$

Combining this Poincaré inequality with (6.48) we get

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 dx \le C\int_{\Omega_{\ell_0+1}\setminus\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 dx \tag{6.50}$$

which is also

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 dx \le \frac{C}{1+C}\int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where $C = (2c_0c_p\frac{\Lambda}{\lambda})^2$. Iterating this formula starting from $\frac{\ell}{2}$ we obtain

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \le \big(\frac{C}{1+C}\big)^{[\frac{\ell}{2}]}\int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where $[\frac{\ell}{2}]$ denotes the integer part of $\frac{\ell}{2}$. Since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$, it comes

$$\begin{aligned}\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx &\leq e^{(-\frac{\ell}{2}+1)\ln(\frac{1+C}{C})} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \\ &= c_0 e^{-\alpha_0 \ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx\end{aligned} \tag{6.51}$$

where $c_0 = \frac{1+C}{C}$ and $\alpha_0 = \frac{1}{2}\ln(\frac{1+C}{C})$.

- **Step 3.** Evaluation of the last integral.

In (6.42) we take $v = u_\ell$. We get

$$\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla u_\ell \, dx = \int_{\Omega_\ell} f u_\ell \, dx \leq |f|_{2,\Omega_\ell} |u_\ell|_{2,\Omega_\ell}.$$

Of course by the Poincaré inequality we have

$$|u_\ell|_{2,\Omega_\ell} \leq c_p \big||\nabla u_\ell|\big|_{2,\Omega_\ell}$$

where c_p is independent of ℓ. Using the ellipticity condition we derive

$$\lambda \int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx \leq c_p |f|_{2,\Omega_\ell} \big||\nabla u_\ell|\big|_{2,\Omega_\ell}$$

and thus

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx \leq \left(\frac{c_p |f|_{2,\Omega_\ell}}{\lambda}\right)^2.$$

By a simple computation

$$\begin{aligned}|f|^2_{2,\Omega_\ell} &= \int_{\ell\omega_1} \int_{\omega_2} f^2(X_2) \, dX_2 \, dX_1 \\ &= |\ell\omega_1| \, |f|^2_{2,\omega_1} = \ell^p |\omega_1| \, |f|^2_{2,\omega_2}\end{aligned}$$

where $|\cdot|$ denotes the measure of sets. It follows that

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 \, dx \leq \frac{c_p^2 |\omega_1|}{\lambda^2} \ell^p |f|^2_{2,\omega_2}. \tag{6.52}$$

Similarly choosing $v = u_\infty$ in (6.41) we get

$$\begin{aligned}\lambda \int_{\omega_2} |\nabla u_\infty|^2 \, dX_2 &\leq \int_{\omega_2} A_{22} \nabla_{X_2} u_\infty \nabla_{X_2} u_\infty \, dX_2 = \int_{\omega_2} f u_\infty \, dX_2 \\ &\leq |f|_{2,\omega_2} |u_\infty|_{2,\Omega_2} \\ &\leq |f|_{2,\omega_2} c_p \big||\nabla u_\infty|\big|_{2,\Omega_2}.\end{aligned}$$

From this it follows that we have

$$\int_{\omega_2} |\nabla u_\infty|^2 \, dX_2 \le \frac{c_p^2}{\lambda^2} \int_{\omega_2} f^2 \, dX_2.$$

Integrating this in X_1 we derive

$$\int_{\Omega_\ell} |\nabla u_\infty|^2 \, dx \le \frac{c_p^2 |\omega_1|}{\lambda^2} \ell^p |f|^2_{2,\omega_2} \tag{6.53}$$

which is the same estimate as in (6.52). Going back to (6.51) we obtain

$$\begin{aligned}\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 \, dx &\le 2c_0 e^{-\alpha_0 \ell} \int_{\Omega_\ell} |\nabla u_\ell|^2 + |\nabla u_\infty|^2 \, dx \\ &\le 4c_0 c_p^2 \frac{|\omega_1|}{\lambda^2} \ell^p e^{-\alpha_0 \ell} |f|^2_{2,\omega_2}.\end{aligned}$$

The estimate (6.43) follows by taking α any constant smaller than α_0. $\square$

Remark 6.5. In (6.43) one can replace $\Omega_{\frac{\ell}{2}}$ by $\Omega_{a\ell}$ for any $a \in (0,1)$ at the expense of lowering α. The proof is exactly the same. However we recall that one cannot choose $a = 1$ (see the exercises).

Remark 6.6. One cannot allow A_{12} to depend on X_1. One can allow A_{11}, A_{21} to depend on ℓ provided (6.38), (6.39) are still valid.

Remark 6.7. The technique above follows [37]. It can be applied to a wide class of problems (see for instance [32] for the Stokes problem).

6.4 An application

We would like to apply the result of the previous section to the anisotropic singular perturbation problem. Indeed the two problems can be connected to each other via a scaling. Let us explain this.

With the notation of the previous section consider

$$\Omega_1 = \omega_1 \times \omega_2 \tag{6.54}$$

where ω_1 satisfies (6.33). Let us denote by $A = A(x)$ a matrix like in (6.35), i.e.,

$$A(x) = \begin{pmatrix} A_{11}(x) & A_{12}(X_2) \\ A_{21}(x) & A_{22}(X_2) \end{pmatrix}.$$

Assume that

$$\lambda |\xi|^2 \le A(x)\xi \cdot \xi \qquad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_1, \tag{6.55}$$

$$|A(x)\xi| \le \Lambda |\xi| \qquad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_1, \tag{6.56}$$

holds for λ, Λ some positive constants. Then like in (5.36) one can define for $\varepsilon > 0$

$$A_\varepsilon = A_\varepsilon(x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \tag{6.57}$$

Due to (5.37) and (5.38) for

$$f = f(X_2) \in L^2(\omega_2), \tag{6.58}$$

there exists a unique u_ε solution to

$$\begin{cases} \displaystyle\int_{\Omega_1} A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Omega_1} f v \, dx \quad \forall\, v \in H_0^1(\Omega_1), \\ u_\varepsilon \in H_0^1(\Omega_1). \end{cases} \tag{6.59}$$

Due to Theorem 5.4 we know that when $\varepsilon \to 0$, $u_\varepsilon \to u_0$ in $L^2(\Omega_1)$ where u_0 is the solution to

$$\begin{cases} \displaystyle\int_{\omega_2} A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} v \, dX_2 = \int_{\omega_2} f(X_2) v \, dX_2 \quad \forall\, v \in H_0^1(\omega_2), \\ u_0 \in H_0^1(\omega_2). \end{cases} \tag{6.60}$$

Note that here since A_{22}, f are independent of X_1 so is u_0. Due to a special situation we can have more information on the convergence of u_ε. We have

Theorem 6.7. *If $\Omega_a = a\omega_1 \times \omega_2$, $a \in (0,1)$ there exist two positive constants c, α such that*

$$\int_{\Omega_a} |\nabla(u_\varepsilon - u_0)|^2 \, dx \le c e^{-\frac{\alpha}{\varepsilon}} |f|_{2,\omega_2}^2. \tag{6.61}$$

Proof. To prove that, we rely on a scaling argument. Indeed we set

$$\varepsilon = \frac{1}{\ell}, \qquad u_\ell(X_1, X_2) = u_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big). \tag{6.62}$$

With this definition it is clear that

$$u_\ell \in H_0^1(\Omega_\ell). \tag{6.63}$$

Moreover

$$\nabla_{X_1} u_\ell(X_1, X_2) = \frac{1}{\ell} \nabla_{X_1} u_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big) \tag{6.64}$$

$$\nabla_{X_2} u_\ell(X_1, X_2) = \nabla_{X_2} u_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big). \tag{6.65}$$

Making the change of variable $X_1 \to \frac{X_1}{\ell}$ in the equation of (6.59) we obtain

$$\int_{\Omega_\ell} A_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big)\nabla u_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big) \cdot \nabla v\Big(\frac{X_1}{\ell}, X_2\Big)\, dx$$
$$= \int_{\Omega_\ell} f(X_2) v\Big(\frac{X_1}{\ell}, X_2\Big)\, dx \quad \forall\, v \in H_0^1(\Omega_1).$$

For $w \in H_0^1(\Omega_\ell)$ we have $v(X_1, X_2) = w(\ell X_1, X_2) \in H_0^1(\Omega_1)$ and

$$(\nabla v)\Big(\frac{X_1}{\ell}, X_2\Big) = (\ell \nabla_{X_1} w(X_1, X_2), \nabla_{X_2} w(X_1, X_2))^T.$$

Combining this with (6.64), (6.65) – see also (6.62) we obtain

$$\begin{aligned}
&A_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big)\nabla u_\varepsilon\Big(\frac{X_1}{\ell}, X_2\Big) \cdot \nabla v\Big(\frac{X_1}{\ell}, X_2\Big)\\
&= \begin{pmatrix} \frac{1}{\ell^2} A_{11}\left(\frac{X_1}{\ell}, X_2\right) & \frac{1}{\ell} A_{12}(X_2) \\ \frac{1}{\ell} A_{21}\left(\frac{X_2}{\ell}, X_2\right) & A_{22}(X_2) \end{pmatrix} \begin{pmatrix} \ell \nabla_{X_1} u_\ell \\ \nabla_{X_2} u_\ell \end{pmatrix} \cdot \begin{pmatrix} \ell \nabla_{X_1} w \\ \nabla_{X_2} w \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{\ell} A_{11}\left(\frac{X_1}{\ell}, X_2\right) \nabla_{X_1} u_\ell + \frac{1}{\ell} A_{12} \nabla_{X_2} u_\ell \\ A_{21}\left(\frac{X_1}{\ell}, X_2\right) \nabla_{X_1} u_\ell + A_{22} \nabla_{X_2} u_\ell \end{pmatrix} \cdot \begin{pmatrix} \ell \nabla_{X_1} w \\ \nabla_{X_2} w \end{pmatrix}\\
&= \begin{pmatrix} A_{11}\left(\frac{X_1}{\ell}, X_2\right) A_{12}(X_2) \\ A_{12}\left(\frac{X_1}{\ell}, X_2\right) A_{22}(X_2) \end{pmatrix} \begin{pmatrix} \nabla_{X_1} u_\ell \\ \nabla_{X_2} u_\ell \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} w \\ \nabla_{X_2} w \end{pmatrix}.
\end{aligned}$$

Thus setting

$$\widetilde{A}_\ell(X_1, X_2) = \begin{pmatrix} A_{11}\left(\frac{X_1}{\ell}, X_2\right) & A_{12}(X_2) \\ A_{12}\left(\frac{X_1}{\ell}, X_2\right) & A_{22}(X_2) \end{pmatrix}$$

we see that u_ℓ is solution to

$$\begin{cases} \displaystyle\int_{\Omega_\ell} \widetilde{A}_\ell \nabla u_\ell \cdot \nabla w\, dx = \int_{\Omega_\ell} f w\, dx \quad \forall\, w \in H_0^1(\Omega_\ell), \\ u_\ell \in H_0^1(\Omega_\ell). \end{cases}$$

Applying Theorem 6.6 taking into account Remarks 6.5, 6.6 we obtain the existence of some constants c_0, α_0 such that

$$\int_{\Omega_{a\ell}} |\nabla(u_\ell - u_0)|^2\, dx \le c_0 e^{-\alpha_0 \ell} |f|^2_{2,\omega_2}.$$

(Note that $u_0 = u_\infty$.) Changing $X_1 \to \ell X_1$ in the integral above we obtain

$$\ell^p \int_{\Omega_a} |\nabla(u_\ell - u_0)|^2(\ell X_1, X_2)\, dx \le c_0 e^{-\alpha_0 \ell} |f|^2_{2,\omega_2}$$

and by (6.64) for $\ell > 1$, i.e., $\varepsilon < 1$

$$
\begin{aligned}
& \frac{\ell^p}{\ell^2}\int_{\Omega_a} |\nabla(u_\varepsilon - u_0)|^2\,dx \le c_0 e^{-\alpha_0 \ell} |f|^2_{2,\omega_2} \\
\iff \quad & \int_{\Omega_a} |\nabla(u_\varepsilon - u_0)|^2\,dx \le c_0 \ell^{2-p} e^{-\alpha_0 \ell} |f|^2_{2,\omega_2} \\
& \qquad\qquad \le c e^{-\frac{\alpha}{\varepsilon}} |f|^2_{2,\omega_2}
\end{aligned}
$$

for some constant c and any α smaller than α_0. This completes the proof of the theorem. □

Exercises

1. Let λ_1 be the first eigenvalue of the problem

$$
\begin{cases}
-\partial^2_{x_2} u_\infty = \lambda_1 u_\infty & \text{in } \omega = (-1,1) \\
u_\infty = 0 & \text{on } \{-1,1\}
\end{cases}
$$

(i.e., $\lambda_1 = \left(\frac{\pi}{2}\right)^2$, $u_\infty = v_1$ given by (6.25)). Show that u_ℓ the solution to

$$
\begin{cases}
-\Delta u_\ell = \lambda_1 u_\infty & \text{in } \Omega_\ell = (-\ell,\ell)\times\omega, \\
u_\ell = 0 & \text{on } \partial\Omega_\ell
\end{cases}
$$

is given by

$$
u_\ell = \left(1 - \frac{\cosh\sqrt{\lambda_1}x_1}{\cosh\sqrt{\lambda_1}\ell}\right) u_\infty(x_2).
$$

Show then that

$$
\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\,dx \not\to 0
$$

when $\ell \to +\infty$.

2. Let u_∞, u_ℓ be the solutions to (6.41), (6.42). Show that when $p \ge 3$ we have

$$
\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2\,dx \to +\infty
$$

when $\ell \to +\infty$, $\Omega_\ell = (-\ell,\ell)^p \times \omega_2$.

3. Show that (6.26) holds when $f \in L^2(\omega)$ (see Remark 6.3).
4. Let ω_1 be a bounded convex domain containing 0. Prove that there exists a function ρ satisfying (6.47).

5. Let $\Omega = \omega_1 \times \omega_2$ where ω_1 is an open subset of $\mathbb{R}^p$ and ω_2 an open subset of $\mathbb{R}^{n-p}$. The notation is the one of Section 6.3.

   Let $u \in H_0^1(\Omega)$. Let $\varphi_n \in \mathcal{D}(\Omega)$ such that

$$\varphi_n \to u \quad \text{in } H_0^1(\Omega).$$

   (i) Show that up to a subsequence

$$\int_{\omega_2} |\nabla_{X_2}(u - \varphi_n)(X_1, X_2)| + (u - \varphi_n)^2(X_1, X_2)\, dX_2 \to 0$$

   for a.e. $X_1 \in \omega_1$.

   (ii) For a.e. $X_1 \in \omega_1$, $\nabla_{X_2} u(X_1, \cdot)$ is a function in $(L^2(\Omega))^{n-p}$. Show that this function is the gradient of $u(X_1, \cdot)$ in ω_2 in the distributional sense.

   (iii) Conclude that $u(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$.

6. We would like to show that in Theorem 6.6 one cannot relax the assumption

$$A_{12} \text{ independent of } X_1.$$

   Suppose indeed that we are under the assumptions of Theorem 6.6 with

$$A_{12} = A_{12}(X_1, X_2).$$

   Suppose that u_ℓ solution of (6.42) is converging toward u_∞ the solution to (6.41) in $H^1(\Omega_{\ell_0})$ weak.

   (i) Show that

$$\int_{\Omega_{\ell_0}} A\nabla u_\ell \cdot \nabla v\, dx = \int_{\Omega_{\ell_0}} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v\, dx \quad \forall\, v \in H_0^1(\Omega_{\ell_0}).$$

   (ii) Show that

$$\int_{\Omega_{\ell_0}} A_{12} \nabla_{X_2} u_\infty \cdot \nabla_{X_1} v = 0 \quad \forall\, v \in H_0^1(\Omega_{\ell_0}).$$

   (iii) Show that the equality above is impossible in general.

7. Let $\Omega_\ell = (-\ell, \ell) \times (-1, 1)$, $\omega = (-1, 1)$. We denote by $C_0^1(\Omega_\ell)$ the space of functions in $C^1(\overline{\Omega}_\ell)$ vanishing on $\{-\ell, \ell\} \times \omega$. We set

$$V_\ell = \text{the closure in } H^1(\Omega_\ell) \text{ of } C_0^1(\Omega_\ell).$$

   1. Show that on V_ℓ the two norms

$$\big|\big|\nabla v\big|\big|_{2,\Omega_\ell}, \qquad \big\{|v|_{2,\Omega_\ell}^2 + \big|\big|\nabla v\big|\big|_{2,\Omega_\ell}^2\big\}^{\frac{1}{2}}$$

   are equivalent.

2. Let $f \in L^2(\omega)$. Show that there exists a unique u_ℓ solution to

$$\begin{cases} u_\ell \in V_\ell, \\ \displaystyle\int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall\, v \in V_\ell. \end{cases} \tag{1}$$

3. Show that the solution to (1) is a weak solution to the problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega_\ell, \\ u_\ell = 0 \text{ on } \{-\ell, \ell\} \times \omega, \quad \dfrac{\partial u_\ell}{\partial \nu} = 0 \text{ on } (-\ell, \ell) \times \partial\omega. \end{cases}$$

4. Let us assume from now on that

$$\int_\omega f(x_2)\, dx_2 = 0.$$

Set

$$W = \left\{ v \in H^1(\omega) \;\middle|\; \int_\omega v \, dx = 0 \right\}.$$

Show that there exists a unique u_∞ solution to

$$\begin{cases} u_\infty \in W, \\ \displaystyle\int_\omega \partial_{x_2} u_\infty \partial_{x_2} v \, dx = \int_\omega f v \, dx \quad \forall\, v \in W. \end{cases} \tag{2}$$

5. Let $v \in V_\ell$. Show that $v(x_1, \cdot) \in H^1(\omega)$ for almost every x_1 and that we have

$$\int_{\Omega_\ell} \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx.$$

6. Show that u_ℓ converges towards u_∞ in $H^1(\Omega_{\ell/2})$ with an exponential rate of convergence.

8. Show that when $\int_\omega f(x_2)\, dx_2 \neq 0$ u_ℓ might be unbounded (cf. [20]).

Chapter 7

Periodic Problems

In the previous chapter we have addressed the issue of the convergence of problems set in cylinders becoming large in some directions and for data independent of these directions. To be independent of one direction could also be interpreted as to be periodic in this direction for any period. Then the question arises to see if periodic data can force at the limit the solution of a periodic problem to be periodic. This is the kind of issue that we would like to address now.

7.1 A general theory

For the sake of simplicity we consider here only the case of periodic functions with period

$$T = \prod_{i=1}^{n} (0, T_i) \tag{7.1}$$

where T_i are for $i = 1, \ldots, n$ positive numbers. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we will say that f is periodic of period T iff

$$f(x + T_i e_i) = f(x) \quad \text{a.e. } x \in \mathbb{R}^n, \ \forall\, i = 1, \ldots, n. \tag{7.2}$$

e_i denotes here the canonical basis in $\mathbb{R}^n$.

Let us then consider a matrix A and a function a such that

$$A(x), \ a(x) \text{ are } T - \text{periodic} \tag{7.3}$$

(this means for the matrix $A = (a_{ij})$ that all the entries are T periodic). We also assume that $a_{ij}, a \in L^\infty(T)$ – which implies by (7.3) that a_{ij}, a are uniformly bounded on $\mathbb{R}^n$ – and that for some positive constants λ, Λ we have

$$\lambda|\xi|^2 \le A(x)\xi \cdot \xi \qquad \text{a.e. } x \in T, \quad \forall\, \xi \in \mathbb{R}^n, \tag{7.4}$$

$$|A(x)\xi| \le \Lambda|\xi| \qquad \text{a.e. } x \in T, \quad \forall\, \xi \in \mathbb{R}^n, \tag{7.5}$$

$$\lambda \le a(x) \le \Lambda \qquad \text{a.e. } x \in T. \tag{7.6}$$

Note that in (7.4) and (7.6) we can assume the constants λ to be the same at the expense of taking the minimum of the two. The same remark applies to Λ in (7.5), (7.6). We define for $\ell \in \mathbb{R}$

$$\Omega_\ell = \prod_{i=1}^{n} (-\ell T_i, \ell T_i), \tag{7.7}$$

and consider a bounded open set Ω'_ℓ such that

$$\Omega_\ell \subset \Omega'_\ell. \tag{7.8}$$

Let us denote by V_ℓ a subspace of $H^1(\Omega'_\ell)$ such that

$$V_\ell \text{ is closed in } H^1(\Omega'_\ell), \qquad H_0^1(\Omega_\ell) \subset V_\ell \subset H^1(\Omega'_\ell). \tag{7.9}$$

(We suppose that the elements of $H_0^1(\Omega_\ell)$ are extended by 0 outside of Ω_ℓ.)

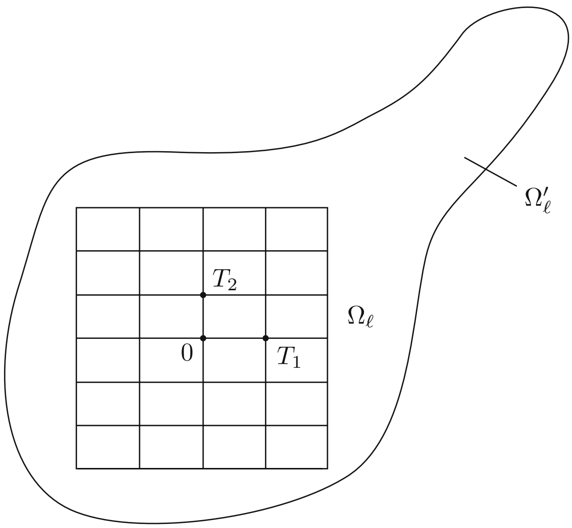


Figure 7.1.

Let us also denote by f a T-periodic function such that

$$f \in L^2(T). \tag{7.10}$$

Then by the Lax–Milgram theorem there exists a unique u_ℓ such that

$$\begin{cases} u_\ell \in V_\ell, \\ \displaystyle\int_{\Omega'_\ell} A(x)\nabla u_\ell \cdot \nabla v + a(x) u_\ell v \, dx = \int_{\Omega'_\ell} f v \, dx \quad \forall\, v \in V_\ell. \end{cases} \tag{7.11}$$

We denote by

$$H^1_{\text{per}}(T) = \text{ the closure of } \{\, v \in C^\infty(\overline{T}) \mid v \quad T\text{-periodic} \,\} \tag{7.12}$$

where $C^\infty(\overline{T})$ denotes the space of restrictions of $C^\infty(\mathbb{R}^n)$-functions to $\overline{T}$. ($v|_T$ denotes the restriction of v to the cell T, the closure is understood in the $H^1(T)$ sense.) As before $H_0^1(\Omega)$ stand for the set of $H^1(\Omega)$ functions vanishing on the boundary of some domain Ω, $H^1_{\text{per}}(T)$ stands here for the set of functions which are periodic on T – i.e., such that

$$v(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = v(x_1, \dots, x_{i-1}, T_i, x_{i+1}, \dots, x_n)$$

for almost every $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \in (0, T_1) \times \cdots \times (0, T_{i-1}) \times \{0\} \times (0, T_{i+1}) \times \cdots \times (0, T_n)$. Note that by definition $H^1_{\text{per}}(T)$ is a closed subspace of $H^1(T)$. Thus, there exists a unique u_∞ solution to

$$\begin{cases} u_\infty \in H^1_{\text{per}}(T), \\ \displaystyle\int_T A(x)\nabla u_\infty \cdot \nabla v + a(x) u_\infty v \, dx = \int_T f v \, dx \quad \forall\, v \in H^1_{\text{per}}(T). \end{cases} \tag{7.13}$$

Moreover we have:

Theorem 7.1. *For any $\ell > 0$ there exist constants $c, \sigma > 0$ independent of ℓ such that*

$$|u_\ell - u_\infty|^2_{H^1(\Omega_{\ell/2})} \le c e^{-\sigma \ell} \{ |f|^2_{2,\Omega'_\ell} + |u_\infty|^2_{H^1(\Omega_\ell)} \}. \tag{7.14}$$

Let us first prove the following lemma.

Lemma 7.2. *Let us assume that u_∞ is extended by periodicity to the whole $\mathbb{R}^n$. Then we have for every $\Omega \subset \mathbb{R}^n$, Ω bounded*

$$\int_\Omega A(x)\nabla u_\infty \cdot \nabla v + a(x) u_\infty v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \tag{7.15}$$

Proof. In other words this theorem says that if a function is T-periodic and satisfies an elliptic equation in the period cell it satisfies this equation everywhere – of course for periodic data.

Let $v \in \mathcal{D}(\Omega)$ and suppose that v is extended by 0 outside Ω. One has if we denote by τ the vector of the period – i.e.,

$$\tau = (T_1, T_2, \dots, T_n)$$

$$\begin{aligned} \int_\Omega A(x)\nabla u_\infty \cdot \nabla v + a(x) u_\infty v \, dx &= \int_{\mathbb{R}^n} A(x)\nabla u_\infty \cdot \nabla v + a(x) u_\infty v \, dx \\ &= \sum_{z \in \mathbb{Z}^n} \int_{T_z} A(x)\nabla u_\infty \cdot \nabla v + a(x) u_\infty v \, dx \end{aligned} \tag{7.16}$$

where $T_z = T + z\tau$, $z\tau$ being a shortcut for $(z_1 T_1, \dots, z_n T_n)$. Note that in this sum only a finite number of terms are not equal to 0 since v is assumed to have a compact support.

Making the change of variable $x + z\tau \to x$ in T_z and using the T-periodicity of A, a, u_∞, that is to say each of them satisfies like the function a

$$a(x + z\tau) = a(x) \quad \text{a.e. } x \in T \tag{7.17}$$

(see (7.2)) – we obtain

$$\begin{aligned} &\int_\Omega A(x)\nabla u_\infty \cdot \nabla v + a(x)u_\infty v \, dx \\ &\qquad = \sum_{z\in\mathbb{Z}^n} \int_T A(x)\nabla u_\infty \cdot \nabla v(x + z\tau) + a(x)u_\infty v(x + z\tau) \, dx \\ &\qquad = \int_T A(x)\nabla u_\infty \cdot \nabla \tilde{v} + a(x)u_\infty \tilde{v} \, dx \end{aligned} \tag{7.18}$$

where we have set

$$\tilde{v} = \sum_{z\in\mathbb{Z}^n} v(x + z\tau).$$

Now clearly this function is indefinitely differentiable and T periodic. Thus from (7.13), (7.18) we derive

$$\begin{aligned} \int_\Omega A(x)\nabla u_\infty \cdot \nabla v + a(x)u_\infty v \, dx &= \int_T f\tilde{v} \, dx \\ &= \sum_{z\in\mathbb{Z}^n} \int_T f(x)v(x + z\tau) \, dx \\ &= \sum_{z\in\mathbb{Z}^n} \int_{T_z} f(x - z\tau)v(x) \, dx \\ &= \int_\Omega fv \, dx \end{aligned} \tag{7.19}$$

by the periodicity of f. This completes the proof since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$. $\square$

Remark 7.1. One should note that the periodic extension of u_∞ to Ω is such that

$$u_\infty \in H^1(\Omega). \tag{7.20}$$

This is true for smooth functions of $H^1_{\text{per}}(T)$ and by density for any function of $H^1_{\text{per}}(T)$ – see (7.12).

End of the proof of Theorem 7.1. We proceed basically like in the proof of Theorem 6.6 – (see starting from Step 2). Let ℓ_0 be such that $0 < \ell_0 \le \ell - 1$. There exists a Lipschitz continuous function ρ such that

$$0 \le \rho \le 1, \quad \rho = 1 \text{ on } \Omega_{\ell_0}, \quad \rho = 0 \text{ on } \mathbb{R}^n \backslash \Omega_{\ell_0+1}, \quad |\nabla\rho| \le C_T. \tag{7.21}$$

(Note that this function belongs to $H^1_0(\Omega_{\ell_0+1})$.) Then we have

$$(u_\ell - u_\infty)\rho \in H^1_0(\Omega_{\ell_0+1}). \tag{7.22}$$

Since V_ℓ is supposed to contain $H^1_0(\Omega_\ell)$ – see (7.9) – combining (7.11), (7.15) we have

$$\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty)\cdot\nabla v + a(u_\ell - u_\infty)v\,dx = 0 \quad \forall\, v \in H^1_0(\Omega_\ell). \tag{7.23}$$

From (7.22) we derive

$$\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty)\cdot\nabla\{(u_\ell - u_\infty)\rho\} + a(u_\ell - u_\infty)^2\rho\,dx = 0.$$

This implies

$$\begin{aligned}\int_{\Omega_\ell} A\nabla(u_\ell - u_\infty)\cdot\nabla(u_\ell - u_\infty)\rho + a(u_\ell - u_\infty)^2\rho\,dx \\ = -\int_{\Omega_\ell}(A\nabla(u_\ell - u_\infty)\cdot\nabla\rho)(u_\ell - u_\infty)\,dx.\end{aligned}$$

In the first integral above it is enough to integrate on Ω_{ℓ_0+1} and in the second one on $\Omega_{\ell_0+1}\backslash\Omega_{\ell_0}$. Moreover using the ellipticity condition (7.4), (7.5) and (7.6) we derive easily

$$\lambda\int_{\Omega_{\ell_0+1}}\{|\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\}\rho\,dx \le \int_{\Omega_{\ell_0+1}\backslash\Omega_{\ell_0}} \Lambda C_T|\nabla(u_\ell - u_\infty)|\,|u_\ell - u_\infty|\,dx$$

(we used (7.21)).

Since $\rho = 1$ on Ω_{ℓ_0} by the usual Young inequality

$$ab \le \frac{1}{2}(a^2 + b^2)$$

we obtain

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \le \frac{\Lambda C_T}{2\lambda}\int_{\Omega_{\ell_0+1}\backslash\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx.$$

Setting $c = \frac{\Lambda C_T}{2\lambda}$ and noticing that

$$\begin{aligned}&\int_{\Omega_{\ell_0+1}\backslash\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \\ &\quad = \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx - \int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx,\end{aligned}$$

we derive then like in the proof of Theorem 6.6 that

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \le \left(\frac{c}{1+c}\right) \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx.$$

Iterating this formula from $\frac{\ell}{2}$, $\left[\frac{\ell}{2}\right]$ times ($\left[\frac{\ell}{2}\right]$ denotes the integer part of $\frac{\ell}{2}$) we obtain

$$\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \le \left(\frac{c}{1+c}\right)^{\left[\frac{\ell}{2}\right]} \int_{\Omega_{\frac{\ell}{2}+\left[\frac{\ell}{2}\right]}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx.$$

Noticing as in Theorem 6.6 that

$$\frac{\ell}{2} - 1 \le \left[\frac{\ell}{2}\right] \le \frac{\ell}{2}$$

we get

$$\begin{aligned} &\int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \\ &\qquad \le c_0 e^{-\sigma\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2\,dx \end{aligned} \tag{7.24}$$

where $c_0 = \frac{1+c}{c}$, $\sigma = \frac{1}{2}\ln\frac{1+c}{c}$.

Next we need to estimate the integral on the right-hand side of (7.24). For this choosing $v = u_\ell$ in (7.11) we get

$$\int_{\Omega'_\ell} A\nabla u_\ell \cdot \nabla u_\ell + a u_\ell^2\,dx = \int_{\Omega'_\ell} f u_\ell\,dx.$$

Using the ellipticity condition and the Cauchy–Schwarz inequality we have

$$\lambda \int_{\Omega'_\ell} |\nabla u_\ell|^2 + u_\ell^2\,dx \le |f|_{2,\Omega'_\ell} \left(\int_{\Omega'_\ell} u_\ell^2\,dx\right)^{\frac{1}{2}} \le |f|_{2,\Omega'_\ell} \left\{\int_{\Omega'_\ell} |\nabla u_\ell|^2 + u_\ell^2\,dx\right\}^{\frac{1}{2}}.$$

From this it follows that

$$\int_{\Omega'_\ell} |\nabla u_\ell|^2 + u_\ell^2\,dx \le \frac{|f|^2_{2,\Omega'_\ell}}{\lambda^2}. \tag{7.25}$$

Going back to (7.24) we have

$$\begin{aligned} |u_\ell - u_\infty|^2_{H^1(\Omega_{\ell/2})} &\le 2c_0 e^{-\sigma\ell} \int_{\Omega_\ell} |\nabla u_\ell|^2 + |\nabla u_\infty|^2 + u_\ell^2 + u_\infty^2\,dx \\ &= 2c_0 e^{-\sigma\ell} \left\{ \int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2\,dx + \int_{\Omega_\ell} |\nabla u_\infty|^2 + u_\infty^2\,dx \right\} \\ &\le 2c_0 e^{-\sigma\ell} \left\{ \frac{|f|^2_{2,\Omega'_\ell}}{\lambda^2} + |u_\infty|^2_{H^1(\Omega_\ell)} \right\}. \end{aligned}$$

The result follows by choosing $c = 2c_0(\frac{1}{\lambda^2} \vee 1)$ where $\vee$ denotes the maximum of two numbers. This completes the proof of the theorem. $\square$

Remark 7.2. One should notice that the constants c, σ are depending only on λ, Λ and T. They are independent of the choice of V_ℓ – that is to say on the boundary conditions that we choose on Ω'_ℓ and on Ω'_ℓ itself.

For the estimate (7.24) we only used (7.23) which holds in fact if we only suppose $f = f_\ell$ periodic on Ω_ℓ or $\Omega_{[\ell]+1}$ (see the proof of (7.15)).

In the case where $V_\ell = H^1_0(\Omega'_\ell)$ one can use the same technique than in Chapter 6 to show convergence of u_ℓ toward u_∞ (see the exercises).

As a corollary of formula (7.14) we have

Theorem 7.3. *Suppose that $\Omega'_\ell = \Omega_\ell$. Let u_ℓ, u_∞ be the solutions to* (7.11), (7.13)*. Then there exist two constants c_1, σ_1, depending on λ, Λ, n, T only such that*

$$|u_\ell - u_\infty|_{H^1(\Omega_{\ell/2})} \le c_1 e^{-\sigma_1 \ell} |f|^2_{2,T}. \tag{7.26}$$

Proof. Of course we suppose that we are under the assumptions of Theorem 7.1. Then by (7.24) we have

$$|u_\ell - u_\infty|^2_{H^1(\Omega_{\ell/2})} \le c_0 e^{-\sigma \ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 + (u_\ell - u_\infty)^2 \, dx \tag{7.27}$$

with $c_0 = \frac{c+1}{c}$, $\sigma = \frac{1}{2} \ln \frac{c+1}{c}$, $c = \frac{\Lambda C_T}{2\lambda}$. By (7.25) we have

$$\begin{aligned} \int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2 \, dx &\le \frac{|f|^2_{2,\Omega_\ell}}{\lambda^2} \le \frac{|f|^2_{2,\Omega_{[\ell]+1}}}{\lambda^2} \\ &\le \frac{2^n([\ell]+1)^n |f|^2_{2,T}}{\lambda^2} \le \frac{2^n(\ell+1)^n |f|^2_{2,T}}{\lambda^2}. \end{aligned} \tag{7.28}$$

We used the periodicity of f. That is to say if $N \in \mathbb{N}$

$$\int_{\Omega_N} f^2 \, dx = \sum_z \int_{T_z} f^2 \, dx = \sum_z \int_T f^2 \, dx = (2N)^n |f|^2_{2,T}. \tag{7.29}$$

(T_z are the T_z's covering Ω_N.)

Taking now $v = u_\infty$ in (7.13) we have

$$\int_T A(x) \nabla u_\infty \cdot \nabla u_\infty + a u_\infty^2 \, dx = \int_T f u_\infty \, dx.$$

Using the ellipticity condition and Cauchy–Schwarz inequality we obtain

$$\lambda \int_T |\nabla u_\infty|^2 + u_\infty^2 \, dx \le |f|_{2,T} |u_\infty|_{2,T}$$

which implies
$$\int_T |\nabla u_\infty|^2 + u_\infty^2\, dx \leq \frac{|f|_{2,T}^2}{\lambda^2}.$$
Thus since u_∞ is periodic it comes
$$\begin{aligned}\int_{\Omega_\ell} |\nabla u_\infty|^2 + u_\infty^2\, dx &\leq \int_{\Omega_{[\ell]+1}} |\nabla u_\infty|^2 + u_\infty^2\, dx \\ &= \{2([\ell]+1)\}^n \int_T |\nabla u_\infty|^2 + u_\infty^2\, dx \qquad (7.30)\\ &\leq \frac{2^n(\ell+1)^n}{\lambda^2}|f|_{2,T}^2.\end{aligned}$$
Combining then (7.27), (7.28), (7.30) we get
$$\begin{aligned}|u_\ell - u_\infty|_{H^1(\Omega_{\ell/2})}^2 &\leq 2c_0 e^{-\sigma\ell}\left\{\int_{\Omega_\ell} |\nabla u_\ell|^2 + u_\ell^2\, dx + \int_{\Omega_\ell} |\nabla u_\infty|^2 + u_\infty^2\, dx\right\}\\ &\leq \frac{4c_0 2^n(\ell+1)^n}{\lambda^2} e^{-\sigma\ell}|f|_{2,T}^2.\end{aligned}$$
The result follows with σ_1 taken as any constant less than σ. □

For Ω'_ℓ arbitrary larger than Ω_ℓ we have

Theorem 7.4. *Under the assumptions of Theorem* 7.1 *suppose that*
$$|f|_{2,\Omega'_\ell}^2 \leq Ce^{(\sigma-\delta)\ell} \qquad (7.31)$$
for some $0 < \delta < \sigma$ *and some positive constant* C. *Then we have for some constant* c'
$$|u_\ell - u_\infty|_{H^1(\Omega_{\ell/2})}^2 \leq c' e^{-\delta\ell}, \qquad (7.32)$$
i.e., u_ℓ *converges toward* u_∞ *at an exponential rate on* $\Omega_{\ell/2}$.

Proof. Combining (7.14), (7.30) and (7.31) we get
$$|u_\ell - u_\infty|_{H^1(\Omega_{\ell/2})}^2 \leq ce^{-\sigma\ell}\left\{Ce^{(\sigma-\delta)\ell} + \frac{2^n(\ell+1)^n}{\lambda^2}|f|_{2,T}^2\right\} \leq c'e^{-\delta\ell}$$
for some constant c'. □

Remark 7.3. In the case where f is periodic (7.32) holds for instance when
$$\Omega'_\ell \subset \Omega_{\ell'} \quad \text{with} \quad \ell' = O(e^{\frac{(\sigma-\delta)}{n}\ell}).$$
Note that (7.32) holds when (7.31) holds and thus f can be chosen arbitrary outside of Ω_ℓ (see also Remark 7.2) and V_ℓ can also be arbitrary provided it satisfies the assumptions of Theorem 7.1. Note that the Theorems 7.3, 7.4 show the convergence of u_ℓ toward u_∞ in $H^1(\Omega)$ for any bounded open subset of $\mathbb{R}^n$ and this at an exponential rate of convergence. This is clear since for ℓ large enough we have
$$\Omega \subset \Omega_{\ell/2}.$$

7.2 Some additional remarks

We have addressed a non-degenerate case that is to say a case where $a(x) \geq \lambda > 0$. The results developed in the preceding section can be extended easily in the case where

$$0 \leq a(x), \qquad a(x) \not\equiv 0, \tag{7.33}$$

we refer to [36] for details. In the case where

$$a \equiv 0 \tag{7.34}$$

the situation is quite different. Indeed for instance a solution to (7.13) cannot be ensured by the Lax–Milgram theorem. Let us examine what can happen in dimension 1. For that let us consider

$$\Omega_\ell = (-\ell T_1, \ell T_1), \quad \ell \in \mathbb{N} \tag{7.35}$$

and u_ℓ the weak solution to

$$\begin{cases} -(A(x)u_\ell')' = f & \text{in } \Omega_\ell, \\ u_\ell(-\ell T_1) = u_\ell(\ell T_1) = 0. \end{cases} \tag{7.36}$$

In (7.36) we suppose for instance that

$$f \in L^2(T) \tag{7.37}$$

where $T = (0, T_1)$ and A, f are T-periodic measurable functions such that for A we have

$$0 < \lambda \leq A(x) \leq \Lambda \quad \text{a.e. } x \in \mathbb{R}. \tag{7.38}$$

Then we have

Theorem 7.5. *If*

$$\int_T f(x)\, dx \neq 0, \tag{7.39}$$

the solution to (7.36) *is unbounded when* $\ell \to +\infty$.

Proof. Before going into the details of the proof one has to note that the theorem above claims that in order to obtain convergence of u_ℓ some condition on f is necessary.

Let us set

$$F(x) = \int_0^x f(s)\, ds, \qquad c_0 = \int_T f(s)\, ds. \tag{7.40}$$

We have

$$F(x + T_1) = \int_0^{x+T_1} f(s)\, ds = \int_0^x f(s)\, ds + \int_x^{x+T_1} f(s)\, ds = F(x) + c_0 \tag{7.41}$$

(cf. Theorem 8.1).

Integrating the first equation of (7.36) we have

$$-A(x)u_\ell' = F(x) + c_1$$

for some constant c_1 and thus

$$-u_\ell' = \frac{F(x)}{A(x)} + \frac{c_1}{A(x)}. \tag{7.42}$$

The value of c_1 is easily determined since

$$0 = \int_{-\ell T_1}^{\ell T_1} -u_\ell'(x)\,dx = \int_{-\ell T_1}^{\ell T_1} \frac{F(x)}{A(x)}\,dx + c_1 \int_{-\ell T_1}^{\ell T_1} \frac{dx}{A(x)}. \tag{7.43}$$

By periodicity of A we easily get

$$\int_{-\ell T_1}^{\ell T_1} \frac{dx}{A(x)} = 2\ell \int_T \frac{dx}{A(x)}. \tag{7.44}$$

Moreover by (7.41) for any integer k we have

$$\begin{aligned}\int_{kT_1}^{(k+1)T_1} \frac{F(x)}{A(x)}\,dx &= \int_{kT_1}^{(k+1)T_1} \frac{F(x+T_1) - c_0}{A(x)}\,dx \\ &= \int_{(k+1)T_1}^{(k+2)T_1} \frac{F(x)}{A(x)}\,dx - c_0 \int_T \frac{dx}{A(x)}. \end{aligned} \tag{7.45}$$

Setting

$$m_1 = \int_T \frac{dx}{A(x)}, \qquad m_2 = \int_T \frac{F(x)}{A(x)}\,dx$$

we derive – starting from $k = 0$ and iterating – that

$$\int_{kT_1}^{(k+1)T_1} \frac{F(x)}{A(x)}\,dx = \int_T \frac{F(x)}{A(x)}\,dx + c_0 k m_1 = m_2 + c_0 k m_1.$$

It follows that

$$\int_{-\ell T_1}^{\ell T_1} \frac{F(x)}{A(x)} = \sum_{k=-\ell}^{\ell-1} \int_{kT_1}^{(k+1)T_1} \frac{F(x)}{A(x)}\,dx = 2\ell m_2 - \ell c_0 m_1.$$

Going back to (7.42), (7.43) this implies hat

$$\begin{aligned} & 2\ell m_1 c_1 + 2\ell m_2 - \ell c_0 m_1 = 0 \\ \iff \quad & c_1 = \frac{c_0}{2} - \frac{m_2}{m_1}. \end{aligned}$$

Then from (7.42) integrating between 0 and ℓT_1 we get

$$\begin{aligned} u_\ell(0) &= \int_0^{\ell T_1} \frac{F(x)}{A(x)}\,dx + c_1 \int_0^{\ell T_1} \frac{dx}{A(x)} \\ &= \sum_{k=0}^{\ell-1} (m_2 + c_0 k m_1) + c_1 \ell m_1 \\ &= \ell m_2 + c_0 m_1 \frac{(\ell-1)\ell}{2} + \Big(\frac{c_0}{2} - \frac{m_2}{m_1}\Big)\ell m_1 \\ &= c_0 m_1 \frac{\ell^2}{2} \quad \longrightarrow \quad \infty \text{ when } c_0 \neq 0. \end{aligned}$$

This completes the proof of the theorem. $\square$

The interested reader will find in [36] complements on these issues, i.e., when (7.34) holds.

Exercises

1. One considers the problem

$$\begin{cases} u \in H^1_{\text{per}}(T), \\ \displaystyle\int_T A\nabla u \cdot \nabla v\,dx = \int_T fv\,dx \quad \forall\, v \in H^1_{\text{per}}(T), \end{cases} \tag{P}$$

i.e., with respect to (7.13) we choose $a = 0$.

(a) Show that the existence of a solution to (P) imposes

$$\int_T f\,dx = 0. \tag{*}$$

(b) Show that if u is a solution then $u + c$ is also solution for any constant.

(c) Under the assumption (*) show that the problem

$$\begin{cases} u \in H^1_{\text{per}}(T), \displaystyle\int_T u\,dx = 0, \\ \displaystyle\int_T A\nabla u \cdot \nabla v\,dx = \int_T fv\,dx \quad \forall\, v \in H^1_{\text{per}}(T) \end{cases}$$

admits a unique solution.

2. Under the conditions of Theorem 7.5 suppose that

$$\int_T f(x)\,dx = 0.$$

Show that u_ℓ solution to (7.36) is T-periodic.

3. Suppose that (7.1)–(7.8) hold. For any $f \in L^2(T)$ – extended by periodicity – let u_∞ be the solution to (7.13) and u_ℓ, u'_ℓ the solutions to

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), \\ \displaystyle\int_{\Omega_\ell} A(x)\nabla u_\ell \cdot \nabla v + a u_\ell v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall \, v \in H_0^1(\Omega_\ell), \end{cases}$$

$$\begin{cases} u'_\ell \in H_0^1(\Omega'_\ell), \\ \displaystyle\int_{\Omega'_\ell} A(x)\nabla u'_\ell \cdot \nabla v + a u'_\ell v \, dx = \int_{\Omega'_\ell} f v \, dx \quad \forall \, v \in H_0^1(\Omega'_\ell). \end{cases}$$

(i) Suppose $f \geq 0$. Show that

$$0 \leq u_\ell \leq u'_\ell \leq u_\infty.$$

Hint: For the last inequality notice $(u'_\ell - u_\infty)^+ \in H_0^1(\Omega'_\ell)$ and use (7.15). Show that for $\ell' > \ell$ one has

$$0 \leq u_\ell \leq u_{\ell'}.$$

Show that there exist positive constants c, σ such that

$$|u'_\ell - u_\infty|_{2,\Omega_{\ell/2}} \leq c e^{-\sigma \ell}.$$

(ii) Show the same convergence when f is arbitrary.

Chapter 8

Homogenization

The theory of homogenization is a theory which was developed in the last forty years. Its success lies in the fact that practically every partial differential equation can be homogenized. Also during the last decades composite materials have invaded our world. To explain the principle of this theory the composite materials are indeed very well adapted. Suppose that we built a "composite" i.e., a material made of different materials by juxtaposing small identical cells containing the different type of materials for instance a three material composite – see Figure 8.1 below. Assuming that we make the cells smaller and smaller at the limit we get

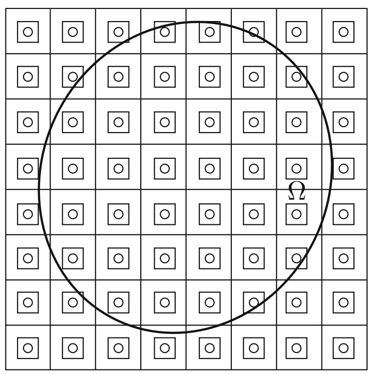


Figure 8.1.

a new material – a composite – which inherits some properties which can be very different from the ones of the materials composing it. For instance mixing three materials with different heat conductivity in the way above leads at the limit to a new material for which we can study the conductivity by cutting a piece of it – say Ω. This is such an issue which is addressed by homogenization techniques (see [14], [40], [41], [72], [46], [47], [61], [87], [92]).

8.1 More on periodic functions

As in the previous chapter we denote by

$$T = \prod_{i=1}^{n} (0, T_i) \tag{8.1}$$

the "period" of a periodic function. Let f be a T-periodic function that we suppose to be defined on the whole $\mathbb{R}^n$. Then for any $\varepsilon > 0$ the function defined by

$$f_\varepsilon(x) = f\Big(\frac{x}{\varepsilon}\Big)$$

is a periodic function with period εT – cf. Figure 8.2.

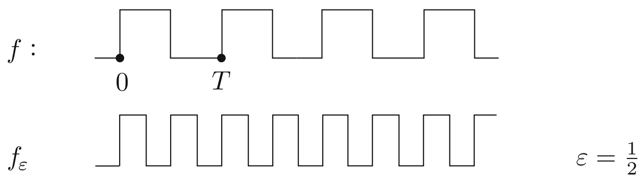


Figure 8.2.

Indeed we have for every $i = 1, \dots, n$

$$f_\varepsilon(x + \varepsilon T_i e_i) = f\Big(\frac{x + \varepsilon T_i e_i}{\varepsilon}\Big) = f\Big(\frac{x}{\varepsilon} + T_i e_i\Big) = f\Big(\frac{x}{\varepsilon}\Big) = f_\varepsilon(x). \tag{8.2}$$

Then we would like to study the behaviour of f_ε when $\varepsilon \to 0$. Of course it is clear that a pointwise convergence is out of scope as one can see in Figure 8.2 above.

Let us first remark:

Theorem 8.1 (Invariance of the integral by translation). *Let $f \in L^1(T)$ extended periodically on $\mathbb{R}^n$. For every $t_0 \in \mathbb{R}^n$ we have*

$$\int_{t_0+T} f(x)\,dx = \int_T f(x)\,dx \tag{8.3}$$

namely the integral of f on any domain obtained by translation of T is the same. ($t_0 + T = \{\, t_0 + t \mid t \in T \,\}$.)

Proof. Suppose that we are in dimension 1 and denote also T_1 by T, $t_0 + T = (t_0, t_0 + T)$ and thus

$$\int_{t_0+T} f(x)\,dx = \int_{t_0}^{t_0+T} f(x)\,dx = \int_{t_0}^{0} f(x)\,dx + \int_0^T f(x)\,dx + \int_T^{t_0+T} f(x)\,dx. \tag{8.4}$$

Making the change of variable $x \to x + T$ in the last integral leads to

$$\int_{t_0+T} f(x)\,dx = \int_0^T f(x)\,dx = \int_T f(x)\,dx. \tag{8.5}$$

In higher dimensions one remarks that

$$\int_{t_0+T} f(x)\,dx = \int_{t_0^1}^{t_0^1+T_1} \int_{t_0^2}^{t_0^1+T_2} \cdots \int_{t_0^n}^{t_0^n+T_n} f(x)\,dx$$

where $t_0 = (t_0^1, t_0^2, \dots, t_0^n)$. The result is then a consequence of (8.5) and the theorem of Fubini. □

Regarding the behaviour of f_ε we have:

Theorem 8.2 (Weak convergence of f_ε). *Let $f \in L^p(T)$, $1 < p < +\infty$, T-periodic. For every bounded domain Ω we have when $\varepsilon \to 0$*

$$f_\varepsilon \rightharpoonup \frac{1}{|T|}\int_T f(x)\,dx := \fint_T f(x)\,dx \quad \textit{in } L^p(\Omega). \tag{8.6}$$

($|T|$ *denotes the measure of* T, *thus* f_ε *converges toward its average weakly.*)

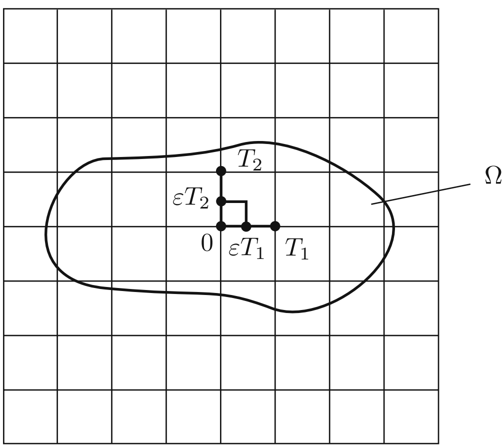


Figure 8.3.

Proof. Since Ω is bounded it can be included – see Figure 8.3 – in some big "rectangle"

$$\Omega_N = \prod_{i=1}^{n} (-NT_i, NT_i). \tag{8.7}$$

For $\varepsilon < 1$ let us denote by $\boldsymbol{\tau}_\varepsilon$ the set of the shifted copies T_ε of εT covering Ω, here by a shifted copy of εT we mean a set $\varepsilon z\tau + \varepsilon T$ (see (7.16) for the definition of τ, recall that $z\tau$ is the vector $(z_1T_1, \ldots, z_nT_n)$ with $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$). Then we have

$$\int_\Omega |f_\varepsilon|^p\,dx \le \sum_{\boldsymbol{\tau}_\varepsilon} \int_{T_\varepsilon} |f_\varepsilon|^p\,dx = (\#\boldsymbol{\tau}_\varepsilon) \int_{\varepsilon T} |f_\varepsilon|^p\,dx, \tag{8.8}$$

by Theorem 8.1 – $\#\boldsymbol{\tau}_\varepsilon$ denoting the number of elements of $\boldsymbol{\tau}_\varepsilon$. Thus we have

$$\int_\Omega |f_\varepsilon|^p\,dx \le (\#\boldsymbol{\tau}_\varepsilon) \int_{\varepsilon T} \Big| f\Big(\frac{x}{\varepsilon}\Big)\Big|^p\,dx = (\#\boldsymbol{\tau}_\varepsilon)\varepsilon^n \int_T |f(y)|^p\,dy$$

by setting $y = \frac{x}{\varepsilon}$. This implies

$$\int_\Omega |f_\varepsilon|^p\,dx \le (\#\boldsymbol{\tau}_\varepsilon)\varepsilon^n|T| \cdot \frac{1}{|T|} \int_T |f(y)|^p\,dy.$$

Now $(\#\boldsymbol{\tau}_\varepsilon)\varepsilon^n|T|$ is the volume of the T_ε covering Ω. These sets T_ε are certainly all contained in Ω_{N+1} and thus we have

$$\int_\Omega |f_\varepsilon|^p\,dx \le |\Omega_{N+1}| \frac{1}{|T|} \int_T |f(y)|^p\,dy \tag{8.9}$$

which means that f_ε is bounded in $L^p(\Omega)$ independently of ε. Thus for any $p > 1$ and up to a subsequence we have for some g

$$f_\varepsilon \rightharpoonup g \quad \text{in } L^p(\Omega). \tag{8.10}$$

We need to identify g. For that let $\varphi \in \mathcal{D}(\Omega)$ and suppose that φ is extended by 0 outside of Ω. Then we have

$$\int_\Omega f_\varepsilon(x)\varphi(x)\,dx = \int_{\mathbb{R}^n} f_\varepsilon(x)\varphi(x)\,dx = \sum_{z\in\mathbb{Z}^n} \int_{T_z(\varepsilon)} f_\varepsilon(x)\varphi(x)\,dx$$

where $T_z(\varepsilon)$ is the cell $\varepsilon T + \varepsilon z\tau$. Setting $x = \varepsilon y + \varepsilon z\tau$ we get by periodicity of f

$$\begin{aligned}\int_\Omega f_\varepsilon\varphi\,dx &= \sum_{z\in\mathbb{Z}^n} \int_T f(y)\varphi(\varepsilon y + z\tau\varepsilon)\varepsilon^n\,dy \\ &= \frac{1}{|T|} \int_T f(y) \sum_{z\in\mathbb{Z}^n} \varphi(\varepsilon y + z\tau\varepsilon)\varepsilon^n|T|\,dy.\end{aligned}$$

Now for any y

$$\sum_{z\in\mathbb{Z}^n} \varphi(\varepsilon y + \varepsilon z\tau)\varepsilon^n|T| \longrightarrow \int_\Omega \varphi(x)\,dx \tag{8.11}$$

(this is a Riemann sum for φ – note that due to the compact support of φ only a finite number of terms are not zero). Since of course this Riemann sum is bounded, by the Lebesgue theorem we get

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon\varphi\,dx = \frac{1}{|T|}\int_T f(y)\,dy\cdot\int_\Omega \varphi(x)\,dx. \tag{8.12}$$

This identifies g with $\frac{1}{|T|}\int_T f(y)\,dy$ and by the uniqueness of the limit completes the proof of the theorem. Note that since the weak limit is uniquely determined it is indeed the whole "sequence" f_ε which satisfies (8.10). □

8.2 Homogenization of elliptic equations

Let us denote by A a T-periodic matrix, namely a matrix such that each entry is T-periodic. Furthermore assume that

$$\lambda|\xi|^2 \le A(x)\xi\cdot\xi \qquad \text{a.e. } x\in\mathbb{R},\ \forall\,\xi\in\mathbb{R}^n, \tag{8.13}$$

$$|A(x)\xi| \le \Lambda|\xi| \qquad \text{a.e. } x\in\mathbb{R},\ \forall\,\xi\in\mathbb{R}^n. \tag{8.14}$$

Let $f\in H^{-1}(\Omega)$. Then for every $\varepsilon>0$ by the Lax–Milgram theorem there exists a unique u_ε solution to

$$\begin{cases} u_\varepsilon\in H_0^1(\Omega), \\ \displaystyle\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)\nabla u_\varepsilon\cdot\nabla v\,dx = \langle f,v\rangle \quad \forall\, v\in H_0^1(\Omega). \end{cases} \tag{8.15}$$

We are interested in finding the limit of u_ε when $\varepsilon\to 0$. Note that $A\left(\frac{x}{\varepsilon}\right)$ is now εT-periodic – i.e., oscillates very fast and so could suit for representing a material whose diffusion matrix is periodic with period approaching 0, that is to say a material of the type of those introduced above.

8.2.1 The one-dimensional case

In this case one can obtain relatively simply the convergence of the solution. Thus let us denote by Ω an interval, $\Omega=(\alpha,\beta)$ and by u_ε the solution to

$$\begin{cases} u_\varepsilon\in H_0^1(\Omega), \\ \displaystyle\int_\Omega A\Big(\frac{x}{\varepsilon}\Big)u_\varepsilon' v'\,dx = \langle f,v\rangle \quad \forall\, v\in H_0^1(\Omega). \end{cases} \tag{8.16}$$

In this system we assume that for some positive constants λ, Λ we have

$$\lambda\le A(x)\le\Lambda \quad \text{a.e. } x\in\Omega \tag{8.17}$$

$$A\in L^\infty(\Omega), \quad A \text{ is } T_1\text{-periodic.} \tag{8.18}$$

f is a distribution such that

$$f\in H^{-1}(\Omega). \tag{8.19}$$

Then we have

Theorem 8.3. *Under the assumption above we have*

$$u_\varepsilon \longrightarrow u_0 \quad in\ L^2(\Omega), \qquad u_\varepsilon \rightharpoonup u_0 \quad in\ H_0^1(\Omega)$$

where u_0 is the weak solution to

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A_0 u_0' v'\,dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \end{cases} \tag{8.20}$$

with A_0 given by

$$A_0 = 1 \Big/ \fint \frac{1}{A}\,dx. \tag{8.21}$$

(*See* (8.6) *for the definition of* $\fint$.)

Proof. Taking $v = u_\varepsilon$ in (8.16) we derive easily

$$\lambda |u_\varepsilon'|_{2,\Omega}^2 \le \langle f, u_\varepsilon\rangle \le |f|_* |u_\varepsilon'|_{2,\Omega} \tag{8.22}$$

where $|f|_*$ denotes the strong dual norm on $H^{-1}(\Omega)$ when $H_0^1(\Omega)$ is equipped with the norm $|u'|_{2,\Omega}$. From (8.22) it follows that

$$|u_\varepsilon'|_{2,\Omega} \le \frac{|f|_*}{\lambda}$$

that is to say u_ε is bounded in $H_0^1(\Omega)$ independently of ε. Thus, up to a subsequence, for $u_0 \in H_0^1(\Omega)$ we have

$$u_\varepsilon \longrightarrow u_0, \qquad u_\varepsilon' \rightharpoonup u_0' \quad \text{in } L^2(\Omega). \tag{8.23}$$

We have used here the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$. This also implies that

$$\Big| A\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon' \Big|_{2,\Omega} \le \Lambda |u_\varepsilon'|_{2,\Omega} \le \frac{\Lambda |f|_*}{\lambda} \tag{8.24}$$

and this quantity is also bounded independently of ε. We would like to identify now u_0. First remark that by the Riesz representation theorem there exists a function u such that

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega u' v'\,dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

Then if we set $g = u'$ we clearly have

$$f = -g'$$

where $g \in L^2(\Omega)$. It follows that

$$\left(-A\left(\frac{x}{\varepsilon}\right)u_\varepsilon'\right)' = -g' \tag{8.25}$$

and thus for some constant c_ε we have

$$A\left(\frac{x}{\varepsilon}\right)u_\varepsilon' = g + c_\varepsilon. \tag{8.26}$$

Since $A\left(\frac{x}{\varepsilon}\right)u_\varepsilon'$ is bounded in $L^2(\Omega)$, the constant c_ε is bounded and up to a subsequence one has

$$c_\varepsilon \longrightarrow c_0. \tag{8.27}$$

Writing (8.26) as

$$u_\varepsilon' = (g + c_\varepsilon) \cdot \frac{1}{A\left(\frac{x}{\varepsilon}\right)},$$

since – up to the subsequence above –

$$g + c_\varepsilon \longrightarrow g + c_0, \qquad \frac{1}{A\left(\frac{x}{\varepsilon}\right)} \rightharpoonup \fint_\Omega \frac{1}{A}\,dx \quad \text{in } L^2(\Omega) \tag{8.28}$$

we have

$$u_\varepsilon' \rightharpoonup (g + c_0) \fint_\Omega \frac{1}{A}\,dx \quad \text{in } L^2(\Omega).$$

From (8.23) we deduce that

$$u_0' = (g + c_0) \fint_\Omega \frac{1}{A}\,dx$$

and thus u_0 is solution to

$$\begin{cases} -\left(\dfrac{1}{\fint_\Omega \frac{1}{A}\,dx} u_0'\right)' = -g' = f \quad \text{in } \Omega, \\ u_0 \in H_0^1(\Omega). \end{cases} \tag{8.29}$$

Since the possible limit is uniquely determined, the whole sequence u_ε converges toward u_0 solution of (8.29) in $L^2(\Omega)$ and u_ε' toward u_0' weakly in $L^2(\Omega)$. This completes the proof of the theorem. □

Remark 8.1. Without using the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ the proof above shows that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega).$$

Note that the result is a little bit surprising. Indeed, passing formally to the limit in (8.16) one would expect u_0 to satisfy

$$\int_\Omega \left(\fint A\,dx\right) u_0' v'\,dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega)!$$

8.2.2 The n-dimensional case

We denote by $\widetilde{H}^1_{\rm per}(T)$ the space

$$\widetilde{H}^1_{\rm per}(T) = \left\{ v \in H^1_{\rm per}(T) \;\middle|\; \fint_T v\,dx = 0 \right\}. \tag{8.30}$$

(Recall the definition (7.12) of $H^1_{\rm per}(T)$.)

Then we have

Theorem 8.4. *On $\widetilde{H}^1_{\rm per}(T)$ the norms*

$$\big||\nabla v|\big|_{2,T} \quad \text{and} \quad \{\big||\nabla v|\big|^2_{2,T} + |v|^2_{2,T}\}^{\frac12} \tag{8.31}$$

are equivalent. Note that the second norm in (8.31) *is the usual H^1-norm.*

Proof. Clearly the norm of the left-hand side of (8.31) is smaller than the one on the right-hand side. We only need to prove that

$$\Big(|v|^2_{2,T} + \big||\nabla v|\big|^2_{2,T}\Big)^{\frac12} \le C\big||\nabla v|\big|_{2,T}.$$

By density it is enough to show this for a smooth function v. For $x, y \in T$ we have

$$v(x) - v(y) = -\int_0^{|x-y|} \frac{d}{dr} v(x + r\omega)\,dr, \qquad \omega = \frac{y-x}{|y-x|}.$$

Integrating with respect to y on T we get

$$|T|\left\{ v(x) - \fint_T v(y)\,dy \right\} = -\int_T \int_0^{|x-y|} \frac{d}{dr} v(x + r\omega)\,dr\,dy.$$

Since $\fint_T v(y)\,dy = 0$ we deduce

$$\begin{aligned}
|v(x)| &= \frac{1}{|T|}\left|\int_T \int_0^{|x-y|} \frac{d}{dr} v(x + r\omega)\,dr\,dy\right| \\
&\le \frac{1}{|T|}\int_T \int_0^{|x-y|} \left|\frac{d}{dr} v(x + r\omega)\right| dr\,dy \\
&= \frac{1}{|T|}\int_T \int_0^{|x-y|} |\nabla v(x + r\omega)\cdot\omega|\,dr\,dy \\
&\le \frac{1}{|T|}\int_T \int_0^{|x-y|} |\nabla v(x + r\omega)|\,dr\,dy.
\end{aligned}$$

(In the computation above we used the chain rule and the fact that $|\omega| = 1$.) Let us denote by d_T the diameter of T and suppose that the function $|\nabla v(z)|$ is extended by 0 for $z \notin T$. Then we derive from above for $x \in T$

$$
\begin{aligned}
|v(x)| &\leq \frac{1}{|T|} \int_{|x-y|<d_T} \int_0^{+\infty} |\nabla v(x+r\omega)|\, dr\, dy \\
&= \frac{1}{|T|} \int_0^{d_T} \int_{|\omega|=1} \int_0^{+\infty} |\nabla v(x+r\omega)|\, dr \rho^{n-1}\, d\omega\, d\rho
\end{aligned}
$$

(since $|y-x|\omega = \rho\omega = y-x$ and $dy = \rho^{n-1}\, d\rho\, d\omega$, $d\omega$ is the superficial measure on the unit sphere). It follows that

$$
|v(x)| \leq \frac{1}{|T|} \frac{{d_T}^n}{n} \int_0^{+\infty} \int_{|\omega|=1} |\nabla v(x+r\omega)|\, d\omega\, dr.
$$

Taking as new variable $y = x + r\omega$, we derive – since $r = |y-x|$ –

$$
|v(x)| \leq \frac{1}{|T|} \frac{{d_T}^n}{n} \int_T |x-y|^{1-n} |\nabla v(y)|\, dy \quad \forall\, x \in T.
$$

Using now the Cauchy–Schwarz inequality it becomes

$$
\begin{aligned}
|v(x)| &\leq \frac{1}{|T|} \frac{{d_T}^n}{n} \int_T |x-y|^{\frac{1-n}{2}} |x-y|^{\frac{1-n}{2}} |\nabla v(y)|\, dy \\
&\leq \frac{1}{|T|} \frac{{d_T}^n}{n} \left(\int_T |x-y|^{1-n}\, dy \right)^{\frac{1}{2}} \left(\int_T |x-y|^{1-n} |\nabla v(y)|^2\, dy \right)^{\frac{1}{2}}.
\end{aligned}
$$

Squaring and integrating on T in x we deduce that

$$
\begin{aligned}
\int_T v(x)^2\, dx &\leq \frac{1}{|T|^2} \frac{{d_T}^{2n}}{n^2} \int_T \left(\int_T |x-y|^{1-n}\, dy \int_T |x-y|^{1-n} |\nabla v(y)|^2\, dy \right) dx \\
&\leq \frac{1}{|T|^2} \frac{{d_T}^{2n}}{n^2} \left\{ \operatorname*{Sup}_{x\in T} \int_T |x-y|^{1-n}\, dy \right\} \int_T \int_T |x-y|^{1-n} |\nabla v(y)|^2\, dy\, dx \\
&= \frac{1}{|T|^2} \frac{{d_T}^{2n}}{n^2} \left\{ \operatorname*{Sup}_{y\in T} \int_T |x-y|^{1-n}\, dx \right\} \int_T \left(\int_T |x-y|^{1-n}\, dx \right) |\nabla v(y)|^2\, dy \\
&\leq \frac{1}{|T|^2} \frac{{d_T}^{2n}}{n^2} \left\{ \operatorname*{Sup}_{y\in T} \int_T |x-y|^{1-n}\, dx \right\}^2 \int_T |\nabla v(y)|^2\, dy. \qquad (8.32)
\end{aligned}
$$

We just need yet to compute the supremum appearing above. For that let us denote by $B_R(y)$ the ball of center y such that

$$
|T| = |B_R(y)|.
$$

We have

$$
\int_T |x-y|^{1-n}\, dx \leq \int_{B_R(x)} |x-y|^{1-n}\, dx.
$$

Indeed this follows from

$$\begin{aligned}\int_T |x-y|^{1-n}\,dx &= \int_{T\cap B_R(y)} |x-y|^{1-n}\,dx + \int_{T\setminus B_R(y)} |x-y|^{1-n}\,dx\\ &\le \int_{T\cap B_R(y)} |x-y|^{1-n}\,dx + R^{1-n}|T\setminus B_R(y)|\\ &= \int_{T\cap B_R(y)} |x-y|^{1-n}\,dx + R^{1-n}|B_R(y)\setminus T|\\ &\le \int_{T\cap B_R(y)} |x-y|^{1-n}\,dx + \int_{B_R(y)\setminus T} |x-y|^{1-n}\,dx\\ &= \int_{B_R(y)} |x-y|^{1-n}\,dx.\end{aligned}$$

It follows that for every $y \in T$

$$\int_T |x-y|^{1-n}\,dx \le \int_0^R \int_{|\omega|=1} d\omega\,d\rho = Rn\omega_n$$

where ω_n is the volume of the unit ball. Since $|T| = \omega_n R^n$ we have for any $y \in T$

$$\int_T |x-y|^{1-n}\,dx \le n\omega_n^{1-\frac{1}{n}}|T|^{\frac{1}{n}}.$$

Combining this with (8.32) we get

$$|v|_{2,T} \le {d_T}^n\Big(\frac{\omega_n}{|T|}\Big)^{1-\frac{1}{n}}\big|\nabla v\big|_{2,T}.$$

It follows that

$$|v|_{2,T}^2 + \big|\nabla v\big|_{2,T}^2 \le \bigg(1+\Big\{{d_T}^n\Big(\frac{\omega_n}{|T|}\Big)^{1-\frac{1}{n}}\Big\}^2\bigg)\big|\nabla v\big|_{2,T}^2.$$

This completes the proof of the theorem. □

By the Lax–Milgram theorem and the result above it follows then that for every $\ell = 1,\dots,n$ there exists a unique w^ℓ solution to

$$\begin{cases} w^\ell \in \widetilde{H}^1_{\mathrm{per}}(T),\\ \displaystyle\int_T A^T\nabla w^\ell\cdot\nabla v\,dx = -\int_T a_{\ell i}\partial_{x_i} v\,dx \quad \forall\, v \in \widetilde{H}^1_{\mathrm{per}}(T),\end{cases} \tag{8.33}$$

where A^T denotes the transposed matrix of A. Moreover if we extend w^ℓ by periodicity to the whole $\mathbb{R}^n$ we have for every bounded open set Ω in $\mathbb{R}^n$

$$\int_\Omega A^T\nabla w^\ell\cdot\nabla v\,dx = -\int_\Omega a_{\ell i}\partial_{x_i} v\,dx \quad \forall\, v \in H^1_0(\Omega). \tag{8.34}$$

(The proof is almost identical to the one of Lemma 7.2.)

Then we can state our main result.

Theorem 8.5. *Under the assumptions* (8.13), (8.14) *let* u_ε *be the solution to* (8.15). *Set for* $i, \ell = 1, \ldots, n$

$$\tilde{a}_{\ell i} = \fint_T a_{ji}(y)\{\delta_{j\ell} + \partial_{y_j} w^\ell(y)\}\, dy. \tag{8.35}$$

Set $\widetilde{A} = (\tilde{a}_{ij})$. *Then when* $\varepsilon \to 0$ *we have*

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{in } H_0^1(\Omega) \tag{8.36}$$

where u_0 *is the solution to*

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \widetilde{A}\nabla u_0 \cdot \nabla v\, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{8.37}$$

Proof. Taking $v = u_\varepsilon$ in (8.15) we derive

$$\lambda \int_\Omega |\nabla u_\varepsilon|^2\, dx \le \int_\Omega A\nabla u_\varepsilon \cdot \nabla u_\varepsilon\, dx = \langle f, u_\varepsilon\rangle \le |f|_* \big||\nabla u_\varepsilon|\big|_{2,\Omega} \tag{8.38}$$

where $|f|_*$ denotes the strong dual norm of f in $H^{-1}(\Omega)$ when $H_0^1(\Omega)$ is equipped with the norm $\big||\nabla v|\big|_{2,\Omega}$. It follows that we have

$$\big||\nabla u_\varepsilon|\big|_{2,\Omega} \le \frac{|f|_*}{\lambda}. \tag{8.39}$$

We also have

$$\big||A\nabla u_\varepsilon|\big|_{2,\Omega}^2 = \int_\Omega |A\nabla u_\varepsilon|^2\, dx \le \Lambda^2 \int_\Omega |\nabla u_\varepsilon|^2\, dx$$

and by (8.39)

$$\big||A\nabla u_\varepsilon|\big|_{2,\Omega} \le \frac{\Lambda}{\lambda}|f|_*. \tag{8.40}$$

Since $A\nabla u_\varepsilon$, ∇u_ε are bounded in $L^2(\Omega)$ independently of ε – up to a subsequence – there exist $A_0 \in (L^2(\Omega))^n$, $u_0 \in H_0^1(\Omega)$ such that

$$A\nabla u_\varepsilon \rightharpoonup A_0, \qquad u_\varepsilon \to u_0 \quad \text{in } L^2(\Omega), \quad u_\varepsilon \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega) \tag{8.41}$$

(by the last convergence we have $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ in $L^2(\Omega)$). From the equation satisfied by u_ε, that is to say

$$\int_\Omega A\nabla u_\varepsilon \cdot \nabla v\, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega)$$

we derive by taking the limit in ε

$$\int_\Omega A_0 \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega) \tag{8.42}$$

(recall that A_0 is a vector with components in $L^2(\Omega)$). The whole point is now to identify A_0. For that we consider

$$v_\varepsilon(x) = x_\ell + \varepsilon w^\ell\Big(\frac{x}{\varepsilon}\Big). \tag{8.43}$$

(We drop the index ℓ in v_ε for the sake of simplicity.) For $\zeta \in \mathcal{D}(\Omega)$ take

$$v = v_\varepsilon \zeta \tag{8.44}$$

as test function in (8.15) we get writing A for $A(\frac{x}{\varepsilon})$:

$$\int_\Omega A\nabla u_\varepsilon \cdot \nabla\{v_\varepsilon \zeta\}\, dx = \langle f, v_\varepsilon \zeta \rangle.$$

This also reads

$$\int_\Omega (A\nabla u_\varepsilon \cdot \nabla v_\varepsilon)\zeta \, dx + \int_\Omega (A\nabla u_\varepsilon \cdot \nabla \zeta) v_\varepsilon \, dx = \langle f, v_\varepsilon \zeta \rangle.$$

Using the transpose of A this can be written

$$\begin{aligned} &\int_\Omega (A^T\nabla v_\varepsilon \cdot \nabla u_\varepsilon)\zeta \, dx + \int_\Omega (A\nabla u_\varepsilon \cdot \nabla \zeta) v_\varepsilon \, dx = \langle f, v_\varepsilon \zeta \rangle \\ \iff \quad &\int_\Omega A^T\nabla v_\varepsilon \cdot \nabla\{u_\varepsilon \zeta\}\, dx - \int_\Omega (A^T\nabla v_\varepsilon \cdot \nabla \zeta) u_\varepsilon \, dx \\ &\qquad + \int_\Omega (A\nabla u_\varepsilon \cdot \nabla \zeta) v_\varepsilon \, dx = \langle f, v_\varepsilon \zeta \rangle. \end{aligned} \tag{8.45}$$

Let us examine the first integral above. From the definition of v_ε we have

$$\int_\Omega A^T \nabla v_\varepsilon \nabla\{u_\varepsilon\zeta\}\, dx = \int_\Omega A^T\Big(\frac{x}{\varepsilon}\Big)\Big\{e_\ell + \nabla w^\ell\Big(\frac{x}{\varepsilon}\Big)\Big\} \cdot \nabla\{u_\varepsilon \zeta\}\, dx.$$

By a change of variable $y = \frac{x}{\varepsilon}$ in the second integral above we get if $\tilde{u}_\varepsilon(y) = \{u_\varepsilon \zeta\}(\varepsilon y)$

$$\int_\Omega A^T\Big(\frac{x}{\varepsilon}\Big)\Big\{e_\ell + \nabla w^\ell\Big(\frac{x}{\varepsilon}\Big)\Big\}\nabla\{u_\varepsilon\zeta\} = \int_{\frac{1}{\varepsilon}\Omega} A^T(y)\{e_\ell + \nabla w^\ell(y)\}\nabla \tilde{u}_\varepsilon(y)\, dy = 0.$$

This is indeed a consequence of (8.34) since $a_{\ell i}\partial_{x_i} v = A^T e_\ell \cdot \nabla v$. Thus we arrive now to

$$-\int_\Omega (A^T\nabla v_\varepsilon \cdot \nabla \zeta) u_\varepsilon \, dx + \int_\Omega (A\nabla u_\varepsilon \cdot \nabla \zeta) v_\varepsilon \, dx = \langle f, v_\varepsilon \zeta \rangle. \tag{8.46}$$

Recall that

$$v_\varepsilon = x_\ell + \varepsilon w^\ell\Big(\frac{x}{\varepsilon}\Big).$$

Hence

$$v_\varepsilon \longrightarrow x_\ell \quad \text{in } L^2(\Omega). \tag{8.47}$$

(Since $w^\ell\left(\frac{x}{\varepsilon}\right)$ is bounded in $L^2(\Omega)$ – see Theorem 8.2.) Moreover

$$\nabla v_\varepsilon = e_\ell + \nabla w^\ell\Big(\frac{x}{\varepsilon}\Big) \rightharpoonup e_\ell + \fint_T \nabla w^\ell \, dx = e_\ell \quad \text{in } L^2(\Omega). \tag{8.48}$$

and thus $v_\varepsilon\zeta \rightharpoonup x_\ell\zeta$ in $H_0^1(\Omega)$. In addition we can write now (8.46) as

$$-\int_\Omega \Big\{a_{ji}\Big(\frac{x}{\varepsilon}\Big)\Big\{\delta_{j\ell} + \partial_{y_j} w^\ell\Big(\frac{x}{\varepsilon}\Big)\Big\}(\partial_{x_i}\zeta)u_\varepsilon \, dx + \int_\Omega (A\nabla u_\varepsilon \cdot \nabla\zeta)v_\varepsilon \, dx = \langle f, v_\varepsilon\zeta\rangle.$$

Passing to the limit in ε we get

$$-\int_\Omega (\tilde a_{\ell i}\partial_{x_i}\zeta)u_0 \, dx + \int_\Omega (A_0 \cdot \nabla\zeta)x_\ell \, dx = \langle f, x_\ell\zeta\rangle.$$

This is also since $\tilde a_{i\ell}$ is constant

$$\int_\Omega (\tilde a_{\ell i}\partial_{x_i}u_0)\zeta \, dx + \int_\Omega (A_0 \cdot \nabla\zeta)x_\ell \, dx = \langle f, x_\ell\zeta\rangle. \tag{8.49}$$

Taking now $v = x_\ell\zeta$ in (8.42) we obtain

$$\int_\Omega (A_0 \cdot e_\ell)\zeta \, dx + \int_\Omega (A_0 \cdot \nabla\zeta)x_\ell \, dx = \langle f, x_\ell\zeta\rangle. \tag{8.50}$$

From (8.49), (8.50) we derive

$$\int_\Omega (\tilde a_{\ell i}\partial_{x_i}u_0)\zeta \, dx = \int_\Omega (A_0 \cdot e_\ell)\zeta \, dx \quad \forall\, \zeta \in \mathcal{D}(\Omega).$$

It follows that for every ℓ one has

$$A_0 \cdot e_\ell = \tilde a_{\ell i}\partial_{x_i}u_0.$$

From (8.42) we see then that u_0 is solution to

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \tilde a_{\ell i}\partial_{x_i}u_0\partial_{x_\ell}v \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{8.51}$$

If the matrix $\tilde a$ is positive definite we arrive to a unique u_0 solution to (8.37) or (8.51) and the whole sequence converges toward u_0 in $H_0^1(\Omega)$-weak. This completes the proof of the theorem since we have: □

Lemma 8.6. *The matrix $\widetilde{A} = (\tilde{a}_{ij})$ defined by* (8.35) *is positive definite.*

Proof. By the definition of w^ℓ we have

$$\int_T a_{ji}\partial_{y_j}(y_\ell + w^\ell)\partial_{y_i} v \, dy = 0 \quad \forall\, v \in \widetilde{H}^1_{\text{per}}(T).$$

It follows that for every k we have (taking $v = w_k$ in the equation above)

$$\int_T a_{ji}\partial_{y_j}(y_\ell + w^\ell)\partial_{y_i}(y_k + w^k)\, dy = \int_T a_{jk}\partial_{y_j}(y_\ell + w^\ell)\, dy = |T|\tilde{a}_{\ell k}.$$

It follows – with the summation convention

$$\begin{aligned}\tilde{a}_{\ell k}\xi_k\xi_\ell &= \frac{1}{|T|}\int_T a_{ji}\partial_{y_j}(\xi_\ell(y_\ell + w^\ell))\partial_{y_i}(\xi_k(y_k + w^k))\, dy \\ &\geq \frac{1}{|T|}\int_T \lambda \left|\nabla\left(\sum_\ell \xi_\ell(y_\ell + w^\ell)\right)\right|^2 dy.\end{aligned}$$

This last integral is nonnegative and vanishes iff

$$\nabla\left\{\sum_\ell \xi_\ell(y_\ell + w^\ell)\right\} \equiv 0 \quad \text{on } T.$$

But

$$\partial_{y_i}\left\{\sum_\ell \xi_\ell(y_\ell + w^\ell)\right\} = \xi_i + \sum_\ell \xi_\ell \partial_{y_i} w^\ell.$$

If this vanishes we have

$$\int_T \left\{\xi_i + \sum_\ell \xi_\ell\partial_{y_i} w^\ell\right\} dy = \xi_i |T| = 0,$$

i.e., $\xi = 0$. This shows that $\widetilde{A}$ is positive definite. This completes the proof of the lemma. $\square$

Remark 8.2. (8.35) can be written in a synthetic way as

$$\widetilde{A}^T e_\ell = \fint_T A^T \nabla(x_\ell + w^\ell)\, dX. \tag{8.52}$$

Note that one also has $u_\varepsilon \to u_0$ in $L^2(\Omega)$.

Exercises

1. Show that the convergence (8.6) also holds in $L^1(\Omega)$ weak and $L^\infty(\Omega)$-weak*.
2. For $f \in L^2(\Omega)$, $\Omega = (\alpha, \beta)$, one considers u_ε the solution to

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \int_\Omega A\Big(\dfrac{x}{\varepsilon}\Big) u_\varepsilon' v' \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \end{cases}$$

   (a) Show that $\xi_\varepsilon = A\left(\frac{x}{\varepsilon}\right) u_\varepsilon' \in H^1(\Omega)$ and is bounded in $H^1(\Omega)$ independently of ε.

   (b) Show that up to a subsequence there exist $\xi_0 \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ such that

$$\xi_\varepsilon \longrightarrow \xi_0, \qquad u_\varepsilon' \rightharpoonup u_0' \quad \text{in } L^2(\Omega).$$

   (c) Rediscover that u_0 is the solution to (8.29).
3. Show that if A is positive definite so is A^T (cf. (8.33))
4. Recover Theorem 8.3 using (8.35).
5. Let a be a T-periodic function satisfying for $\lambda, \Lambda > 0$

$$0 < \lambda \le a(x) \le \Lambda \quad \text{a.e. } x \in T.$$

   If Ω is a bounded open set of $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$, let u_ε be the weak solution to

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ -\Delta u_\varepsilon + a\Big(\dfrac{x}{\varepsilon}\Big) u_\varepsilon = f \quad \text{in } \Omega. \end{cases}$$

   Show that

$$u_\varepsilon \longrightarrow u_0 \quad \text{in } H_0^1(\Omega)$$

   where u_0 is the solution to

$$\begin{cases} u_0 \in H_0^1(\Omega), \\ -\Delta u_0 + \overline{a} u_0 = f \quad \text{in } \Omega \end{cases}$$

   with $\overline{a} = ⨍_T a(x)\, dx$.
6. Assuming that $H^1_{\text{per}}(T)$ is compactly embedded in $L^2(T)$ give an alternative proof to Theorem 8.4.
7. (i) Show that

$$u(x) = \frac{1}{n\omega_n} \int_\Omega \frac{\nabla u(y) \cdot (x - y)}{|x - y|^n} \, dy \quad \forall\, u \in \mathcal{D}(\Omega).$$

   (Hint: see the proof of Theorem 8.4.)

(ii) Deduce from (i) that for $u \in \mathcal{D}(\Omega)$

$$|u(x)| \le \frac{1}{n\omega_n}\left\{\int_\Omega |x-y|^{1-n}\,dy\right\}^{\frac{1}{2}}\left\{\int_\Omega |x-y|^{1-n}|\nabla u|^2\,dy\right\}^{\frac{1}{2}}.$$

(iii) If Ω is of bounded measure deduce that

$$|u|_{2,\Omega} \le \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}\big|\,|\nabla u|\,\big|_{2,\Omega} \quad \forall\, u \in H_0^1(\Omega).$$

(Hint: see the proof of Theorem 8.4.)

8. (i) For $f \in L^1(\Omega)$, $\alpha \in [0,1)$ we set

$$V_\alpha(f)(x) = \int_\Omega |x-y|^{-n\alpha} f(y)\,dy.$$

We suppose that Ω is an open subset with finite measure. Show that

$$V_\alpha(1) \le \frac{1}{1-\alpha}\omega_n^\alpha|\Omega|^{1-\alpha}.$$

(ii) Show that for any $1 \le p < +\infty$

$$f \in L^p(\Omega) \Longrightarrow V_\alpha(f) \in L^p(\Omega)$$

and one has

$$|V_\alpha(f)|_{p,\Omega} \le \frac{1}{1-\alpha}\omega_n^\alpha|\Omega|^{1-\alpha}|f|_{p,\Omega}.$$

(iii) Deduce that

$$|u|_{p,\Omega} \le \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}\big|\,|\nabla u|\,\big|_{p,\Omega} \quad \forall\, u \in \mathcal{D}(\Omega).$$

9. Show that

$$\left\{ v \in C^\infty(\overline{T}) \;\middle|\; v \text{ } T\text{-periodic}, \ \fint_T v(x)\,dx = 0 \right\}$$

is dense in $\widetilde{H}^1_{\mathrm{per}}(T)$.

Chapter 9

Eigenvalues

9.1 The one-dimensional case

Suppose to simplify that Ω is the interval $(-a, a)$, $a > 0$.

Definition 9.1. λ is called an eigenvalue for the Dirichlet problem if there exists a function $u \neq 0$ weak solution to the problem

$$\begin{cases} -u'' = \lambda u & \text{in } \Omega, \\ u(-a) = u(a) = 0. \end{cases} \tag{9.1}$$

u is then called an eigenfunction corresponding to the eigenvalue λ.

One should remark that if u is an eigenfunction of the problem (9.1) so is μu for every real μ, i.e., the solution of the problem (9.1) is not unique.

Note that λ is necessarily positive since a weak solution to (9.1) is a function satisfying

$$u \in H_0^1(\Omega), \qquad \int_\Omega u'v'\,dx = \lambda \int_\Omega uv\,dx \quad \forall\, v \in H_0^1(\Omega), \quad u \neq 0. \tag{9.2}$$

Taking $v = u$ it comes

$$\int_\Omega u'^2\,dx = \lambda \int_\Omega u^2\,dx \tag{9.3}$$

hence $\lambda > 0$ since $u' = 0$ would imply $u = 0$.

The first equation of (9.1) is a second-order linear differential equation which possesses a subspace of solutions of dimension 2. It is then easy to check that the general solution of this equation is given by

$$u = A\sin\big(\sqrt{\lambda}(x+a)\big) + B\cos\big(\sqrt{\lambda}(x+a)\big), \quad A, B \in \mathbb{R}. \tag{9.4}$$

(Since $\sin(\sqrt{\lambda}(x+a))$, $\cos(\sqrt{\lambda}(x+a))$ are independent solutions.)

In order to match the boundary conditions of (9.1) we must have

$$B = 0, \qquad \sin\big(\sqrt{\lambda}(2a)\big) = 0.$$

Thus, the possible eigenvalues of the problem are given by λ_k such that

$$2a\sqrt{\lambda_k} = k\pi \quad \Longleftrightarrow \quad \lambda_k = \left(\frac{k\pi}{2a}\right)^2 \quad k = 1, 2, \ldots. \tag{9.5}$$

They form a discrete countable sequence. The corresponding eigenfunctions are then given by

$$u_k = A_k \sin\frac{k\pi}{2a}(x+a). \tag{9.6}$$

They are simple in the sense that the eigenspaces – the spaces of eigenfunctions – are of dimension 1. If one wants to choose as basis of eigenfunctions the one which has L^2-norm equal to 1 – one says that this is a normalized eigenfunction – one has to choose A_k such that

$$A_k^2 \int_{-a}^{a} \sin^2\frac{k\pi}{2a}(x+a)\,dx = 1$$

which leads to

$$u_k = \frac{1}{\sqrt{a}}\sin\frac{k\pi}{2a}(x+a). \tag{9.7}$$

It is clear from (9.5) that the smallest eigenvalue is given by

$$\lambda_1 = \left(\frac{\pi}{2a}\right)^2, \tag{9.8}$$

and the so-called first eigenfunction by

$$u_1 = \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}. \tag{9.9}$$

On this simple example we already see some properties emerging that one could try to generalize in higher dimensions for an elliptic operator. These are

- λ_1 is decreasing with the size of the domain
- λ_1 is simple – namely the dimension of the associated eigenspace is 1
- u_1 does not change sign
- the u_k's are building a Hilbert basis of $L^2(\Omega)$ – see below for this concept.

9.2 The higher-dimensional case

Let Ω be a bounded open subset of $\mathbb{R}^n$. For the sake of simplicity we restrict ourselves to the Laplace operator for the Dirichlet problem that is to say we are interested in finding the λ_k's such that for some $u \neq 0$ we have in a weak sense

$$\begin{cases} -\Delta u = \lambda_k u & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{9.10}$$

Then we have

Theorem 9.1. *The first eigenvalue* λ_1 *for the Dirichlet problem* (9.10) *is given by*

$$\lambda_1 = \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx}. \tag{9.11}$$

(*By the first eigenvalue we mean the smallest one. The quotient above is called a Rayleigh quotient.*)

Proof. First we remark that if λ_k is eigenvalue for (9.10) we have

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda_k \int_\Omega uv \, dx \quad \forall\, v \in H_0^1(\Omega),$$

for some $u \neq 0$, $u \in H_0^1(\Omega)$. Taking $v = u$ we get

$$\lambda_k = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \tag{9.12}$$

and λ_k is certainly bigger than the infimum in (9.11).

Next we remark that by the Poincaré inequality (see Theorem 2.8 or (2.52)) the infimum (9.11) is positive. Let us show that this infimum is indeed achieved. Let u_n be a minimizing sequence, that is to say a sequence such that

$$u_n \in H_0^1(\Omega) \qquad \frac{\int_\Omega |\nabla u_n|^2}{\int_\Omega u_n^2} \longrightarrow \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega v^2 \, dx} := \operatorname{Inf}. \tag{9.13}$$

By replacing eventually u_n by $u_n/|u_n|_{2,\Omega}$ one can assume that

$$|u_n|_{2,\Omega} = 1. \tag{9.14}$$

Then, clearly, from (9.13) we have for n large

$$\int_\Omega |\nabla u_n|^2 \, dx \leq 1 + \operatorname{Inf}.$$

This together with (9.14) implies that u_n is bounded in $H^1(\Omega)$. Up to a subsequence there is a $u \in H_0^1(\Omega)$ such that

$$u_n \longrightarrow u \quad \text{in } L^2(\Omega), \qquad u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega). \tag{9.15}$$

By (9.14), (9.15) it is clear that

$$|u|_{2,\Omega} = 1.$$

Moreover by the weak lower semi-continuity of the norm

$$\mathrm{Inf} = \underline{\lim} \int_\Omega |\nabla u_n|^2 \, dx \geq \int_\Omega |\nabla u|^2 \, dx = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \geq \mathrm{Inf}\,.$$

It follows that u is a function $\neq 0$ for which the infimum (9.11) is achieved. Note that the infimum is then achieved for any multiple of u. We claim that u is an eigenfunction. Indeed let $u \neq 0$ be a point where the infimum (9.11) is achieved. Let $v \in H_0^1(\Omega)$. Consider

$$\frac{\int_\Omega |\nabla(u + \lambda v)|^2 \, dx}{\int_\Omega (u + \lambda v)^2 \, dx} = R(\lambda). \tag{9.16}$$

For λ small enough this quotient is well defined and 0 is an absolute minimum of $R(\lambda)$. It follows that

$$\frac{d}{d\lambda} R(0) = \frac{d}{d\lambda} \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx + 2\lambda \int_\Omega \nabla u \cdot \nabla v \, dx + \lambda^2 \int_\Omega |\nabla v|^2 \, dx}{\int_\Omega u^2 \, dx + 2\lambda \int_\Omega uv \, dx + \lambda^2 \int_\Omega v^2 \, dx} \right\}(0) = 0.$$

We have to derive second-order polynomials in λ and a rapid computation shows that the equation above is equivalent to

$$\int_\Omega \nabla u \cdot \nabla v \, dx \cdot \int_\Omega u^2 \, dx = \int_\Omega |\nabla u|^2 \, dx \int_\Omega uv \, dx$$

and we get

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \int_\Omega uv \, dx \quad \forall\, v \in H_0^1(\Omega).$$

This shows that u is an eigenfunction for the value λ_1 given by (9.11). This completes the proof of the theorem. $\square$

Remark 9.1. λ_1 is thus the best constant in the Poincaré inequality – i.e., the largest constant c such that

$$c \int_\Omega v^2 \, dx \leq \int_\Omega |\nabla v|^2 \, dx \quad \forall v \in H_0^1(\Omega).$$

On the way to establish our assertions from the end of Section 9.1 we have:

Theorem 9.2. *Let $\lambda_1 = \lambda_1(\Omega)$ be the first eigenvalue for the Dirichlet problem* (9.10) *on Ω. If $\Omega \subset \Omega'$ we have $\lambda_1(\Omega) \geq \lambda_1(\Omega')$.*

Proof. A function in $H_0^1(\Omega)$ extended by 0 outside Ω is in $H_0^1(\Omega')$. The result follows then directly from Property (9.11). □

We also have

Theorem 9.3. *Let $\lambda_1 = \lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet problem* (9.10) *and u a corresponding eigenfunction. Then we have, if Ω is supposed to be connected,*

- *λ_1 is simple*
- *u does not change of sign in Ω.*

Proof. We know that u realizes the infimum (9.11). It is clear (see Corollary 2.14) that $|u|$ also realizes the infimum (9.11). Thus both $|u|$ and u are eigenfunctions of (9.10) and also

$$u^+ = \frac{|u| + u}{2} \quad \text{and} \quad u^- = \frac{|u| - u}{2}.$$

That is to say we have

$$-\Delta u^+ = \lambda_1 u^+ \geq 0. \tag{9.17}$$

If $f \geq 0, \not\equiv 0$ one can show that the weak solution to

$$-\Delta u = f, \quad u \in H_0^1(\Omega)$$

is positive – i.e., $u > 0$ (see [17], [31]). (This property which arises for elliptic differential equations is very clear in dimension 1. Indeed if

$$-u'' \geq 0 \quad \text{on} \quad \Omega = (\alpha, \beta), \qquad u(\alpha) = u(\beta) = 0,$$

then u is a concave function vanishing at end points. If it vanishes in between at some point the concavity imposes that it vanishes identically. But then $-u'' = f$ has to vanish identically which is impossible for $f \not\equiv 0$.)

Going back to (9.17) we see that if u^+ is not identically equal to 0 then u^+ is positive in the whole Ω – i.e., $u^- = 0$ and thus u cannot change of sign since if $u^+ \equiv 0$ then $u = -u^-$.

To see now that λ_1 is simple let u_1, u_2 be two corresponding eigenfunctions. If u_1, u_2 are linearly independent replacing u_2 by $u_1 + \mu u_2$ one can assume that u_1, u_2 are orthogonal in $L^2(\Omega)$. But this is impossible: since u_1, u_2 have constant sign one cannot have

$$\int_\Omega u_1 u_2 \, dx = 0.$$

This completes the proof of the theorem. □

Definition 9.2. Let H be a real separable Hilbert space. A Hilbert basis or an orthonormal Hilbert basis of H is a family of linear independent vectors $(e_n)_{n\geq 1}$ such that

- $|e_n| = 1$, $(e_n, e_m) = 0 \;\forall\, n, m \geq 1$, $n \neq m$,
- the linear combinations of the e_i's are dense in H.

Definition 9.3. Let H be a real Hilbert space and L a continuous linear operator from H into itself. One says that L is

- positive iff $(Lx, x) \geq 0 \;\forall\, x \in H$
- self-adjoint iff $(Lx, y) = (x, Ly) \;\forall\, x, y \in H$.

Then we have

Theorem 9.4. *Let H be a separable real Hilbert space of infinite dimension. Let L be a self-adjoint positive compact operator. Then there exists a sequence of positive eigenvalues (μ_n) converging toward 0 and a sequence of corresponding eigenvectors (e_n) such that (e_n) is an orthonormal Hilbert basis of H.*

Proof. See [21]. □

As an application we have

Theorem 9.5. *Let Ω be a bounded open set of $\mathbb{R}^n$. There exists an orthonormal Hilbert basis $(e_n)_{n\geq 1}$ of $L^2(\Omega)$ and a sequence of positive numbers $(\lambda_n)_{n\geq 1}$ such that $\lambda_n \to +\infty$ when $n \to +\infty$ and*

$$\begin{cases} e_n \in H_0^1(\Omega), \\ -\Delta e_n = \lambda_n e_n \quad \text{in } \Omega. \end{cases} \tag{9.18}$$

The λ_n's are called the eigenvalues of the Dirichlet problem in Ω, the e_n's are the corresponding eigenvectors or eigenfunctions.

Proof. Let us define in the Hilbert space $L^2(\Omega)$ the operator L by $L(f) = u$ where u is the solution to

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^1(\Omega). \end{cases} \tag{9.19}$$

- L is compact. This is due to the compactness of the canonical embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$.
- L is self-adjoint. Indeed if $u = L(f)$, $v = L(g)$ we have

$$\left.\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f L(g) \, dx \\ \int_\Omega \nabla v \cdot \nabla u \, dx &= \int_\Omega g L(f) \, dx \end{aligned}\right\} \implies \int_\Omega f L(g) \, dx = \int_\Omega L(f) g \, dx \tag{9.20}$$

for every $f, g \in L^2(\Omega)$.

- L is positive. Indeed, taking $v = u$ in (9.19) we have

$$\int_\Omega f L(f)\,dx = \int_\Omega \nabla u \cdot \nabla u\,dx \geq 0 \quad \forall f \in L^2(\Omega). \tag{9.21}$$

According to Theorem 9.4 there exists a sequence of eigenvalues $(\mu_n)_{n\geq 1}$ converging toward 0 and a basis of orthonormal eigenvectors $(e_n)_{n\geq 1}$ such that

$$Le_n = \mu_n e_n, \tag{9.22}$$

i.e.,

$$\begin{cases} e_n \in H_0^1(\Omega), \\ \mu_n \displaystyle\int_\Omega \nabla e_n \cdot \nabla v\,dx = \int_\Omega e_n v\,dx \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{9.23}$$

Setting $\lambda_n = \frac{1}{\mu_n}$ this completes the proof of the theorem. □

Remark 9.2. One should note that two eigenfunctions orthogonal in $L^2(\Omega)$ are also orthogonal in $H_0^1(\Omega)$. One can also show that $e_n \in C^\infty(\Omega)\ \forall n \geq 1$.

9.3 An application

In $\Omega_\ell = (-\ell, \ell) \times \omega$, $\omega = (-1, 1)$ we consider again the problem

$$\begin{cases} -\Delta u_\ell = f & \text{in } \Omega_\ell, \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases} \tag{9.24}$$

where $f = f(x_2) \in L^2(\omega)$ and the associated limit u_∞ defined as the solution of the problem

$$\begin{cases} -\partial_{x_2}^2 u_\infty = f & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega. \end{cases} \tag{9.25}$$

We have seen in Chapter 6 that, when ℓ goes to $+\infty$, u_ℓ converges to u_∞ with an exponential rate of convergence. We would like to show that by another argument. Let us denote by $(\varphi_n)_{n\geq 1}$ the orthonormal basis of eigenfunctions associated to the Dirichlet problem (9.25) – i.e.,

$$\begin{cases} -\partial_{x_2}^2 \varphi_n = \lambda_n \varphi_n & \text{in } \omega, \\ \varphi_n \in H_0^1(\omega). \end{cases}$$

Since $(\varphi_n)_{n\geq 1}$ is an orthonormal basis of $L^2(\omega)$ we have

$$u_\infty = \sum_{n=1}^{+\infty} a_n \varphi_n, \qquad a_n = (u_\infty, \varphi_n),$$

$(\cdot,\cdot)$ denotes the scalar product in $L^2(\omega)$. It is clear that for any n

$$h_n = \frac{\cosh\sqrt{\lambda_n}x_1}{\cosh\sqrt{\lambda_n}\ell}\varphi_n(x_2)$$

is harmonic – i.e., $\Delta h_n = 0$. Thus by the uniqueness of the solution to (9.24) we have

$$u_\ell - u_\infty = -\sum_{n=1}^{+\infty} a_n \frac{\cosh\sqrt{\lambda_n}x_1}{\cosh\sqrt{\lambda_n}\ell}\varphi_n(x_2), \tag{9.26}$$

i.e., we have an explicit formula for u_ℓ. Then with a simple computation (see [35]) one can show that the estimate (6.22) is sharp – see the exercises.

Exercises

1. Let $\Omega_\ell = (-\ell,\ell)\times(-1,1)$. Find all the eigenvalues and eigenfunctions of the problem
$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega_\ell, \\ u \in H_0^1(\Omega_\ell). \end{cases}$$
2. Let Ω be a bounded open set of $\mathbb{R}^n$ and $(e_n)_{n\geq 1}$ be the orthonormal Hilbert basis of $L^2(\Omega)$ given by Theorem 9.5. Prove that $(\frac{1}{\sqrt{\lambda_n}}e_n)$ is an orthonormal Hilbert basis of $H_0^1(\Omega)$.
3. Show Theorems 9.3, 9.5 for a general symmetric elliptic problem, i.e.,
$$\begin{cases} \nabla\cdot(A\nabla u) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
where $A = A^T$.
4. Using (9.26) show that the estimate (6.22) is sharp.

Chapter 10

Numerical Computations

Suppose that Ω is a bounded open subset of $\mathbb{R}^2$. We would like to compute numerically the solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.1}$$

For simplicity we restrict ourselves to the two-dimensional case but the methods extend easily to higher dimensions (see [38]).

10.1 The finite difference method

This is perhaps the oldest method in numerical analysis and also the simplest. It consists in replacing the derivative by its difference quotients – i.e., $\partial_{x_i} u(x)$ by

$$\frac{u(x + he_i) - u(x)}{h} \tag{10.2}$$

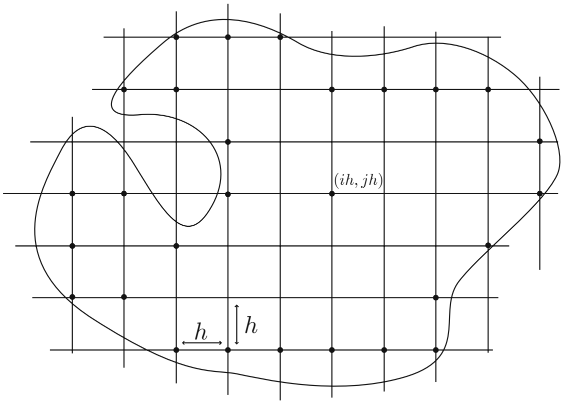
(h is some positive number taken smaller and smaller). Let us make things more precise. First we cover the domain Ω by a grid of size h – see Figure 10.1. If (ih, jh) is a point of the grid we call neighbours of (ih, jh) the four points

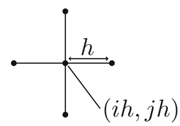
$$((i-1)h, jh), ((i+1)h, jh), (ih, (j-1)h), (ih, (j+1)h). \tag{10.3}$$

We denote by Ω_h, Γ_h the sets of grid points defined by

$$\Omega_h = \{(ih, jh) \in \Omega \text{ having their four neighbours in } \Omega\}, \tag{10.4}$$
$$\Gamma_h = \{(ih, jh) \in \Omega\} \setminus \Omega_h. \tag{10.5}$$

(The set Γ_h is depicted in Figure 10.1 on top of the next page.)





the four neighbours of (ih, jh)

Figure 10.1.

Iterating (10.2), i.e., writing

$$\begin{aligned}
\partial^2_{x_i} u(x) &\backsimeq \frac{\partial_{x_i} u(x) - \partial_{x_i} u(x - he_i)}{h} \\
&\backsimeq \frac{\{u(x + he_i) - u(x)\} - \{u(x) - u(x - he_i)\}}{h^2} \\
&\backsimeq -\frac{2u(x) - u(x + he_i) - u(x - he_i)}{h^2}
\end{aligned}$$

we deduce that

$$\Delta u(ih, jh) \backsimeq -\frac{1}{h^2}\Big\{ 4u(ih, jh) - \sum_{z \in N(ih,jh)} u(z) \Big\} \tag{10.6}$$

where $N(ih, jh)$ denotes the set of neighbours of (ih, jh), i.e., the points given by (10.3). Thus the expression on the right-hand side of (10.6) is defined for any $(ih, jh) \in \Omega_h$. Let us denote by u_y the approximation of $u(y)$ for $y \in \Omega_h$. According to (10.1), (10.6) it is reasonable to assume that $(u_y)_{y \in \Omega_h}$ satisfies

$$\begin{cases} -\Delta_h u_y := -\dfrac{1}{h^2}\Big\{ \displaystyle\sum_{z \in N(y)} u_z - 4u_y \Big\} = f(y) & \forall\, y \in \Omega_h, \\ u_y = 0 & \forall\, y \in \Gamma_h. \end{cases} \tag{10.7}$$

The system (10.7) is just a linear system with unknown u_y, $y \in \Omega_h$. To show that it possesses a unique solution we will need the following lemma

Lemma 10.1 (Discrete Maximum Principle). *Let u_y, $y \in \Omega_h$ satisfying*

$$\begin{cases} -\Delta_h u_y = -\dfrac{1}{h^2}\left\{ \displaystyle\sum_{z \in N(y)} u_z - 4u_y \right\} \leq 0 \quad \forall\, y \in \Omega_h, \\ u_y \text{ given on } \Gamma_h. \end{cases} \tag{10.8}$$

Then we have

$$u_y \leq \operatorname*{Max}_{\Gamma_h} u_z \quad \forall\, y \in \Omega_h, \tag{10.9}$$

i.e., u_y, $y \in \Omega_h \cup \Gamma_h$ achieves its maximum on Γ_h.

Proof. Suppose that the maximum of u_y, $y \in \Omega_h \cup \Gamma_h$ is achieved for some point $y_0 \in \Omega_h$. Then from the inequality (10.8) we have

$$4u_{y_0} \leq \sum_{z \in N(y_0)} u_z \leq 4u_{y_0}$$

since each u_z is less or equal to u_{y_0}. This forces

$$u_z = u_{y_0} \quad \forall\, z \in N(y_0).$$

Repeating the operation if $z \in N(y_0) \cap \Omega_h$ we will have

$$u_t = u_{y_0} \quad \forall\, t \in N(z)$$

else $z \in \Gamma_h$ and we have a point of Γ_h reaching the maximum, i.e.,

$$u_{y_0} \leq \operatorname*{Max}_{\Gamma_h} u_z \leq u_{y_0}.$$

This implies (10.9). Note that one will always reach such a point z since the grid points are finite. □

Remark 10.1. If the maximum of u_y is achieved in Ω_h it does not imply that u_y is constant in Ω_h. Some connectedness of this set is necessary – see Figure 10.1 above.

Remark 10.2. Similarly if u_y satisfies

$$-\Delta_h u_y \geq 0 \quad \forall\, y \in \Omega_h \tag{10.10}$$

then one has

$$\min_{\Gamma_h} u_z \leq u_y \quad \forall\, y \in \Omega_h. \tag{10.11}$$

Indeed it is enough to apply Lemma 10.1 to $-u_y$. It follows that when

$$-\Delta_h u_y = 0 \quad \forall\, y \in \Omega_h \tag{10.12}$$

one has

$$\min_{\Gamma_h} u_z \leq u_y \leq \operatorname*{Max}_{\Gamma_h} u_z \quad \forall\, y \in \Omega_h. \tag{10.13}$$

Then one can prove:

Theorem 10.2. *The system* (10.7) *possesses a unique solution.*

Proof. This is a linear system. It is enough to show that the corresponding homogeneous system

$$\begin{cases} -\dfrac{1}{h^2}\Big\{ \displaystyle\sum_{z\in N(y)} u_z - 4u_y \Big\} = 0 & \forall\, y \in \Omega_h, \\ u_y = 0 & \forall\, y \in \Gamma_h \end{cases} \tag{10.14}$$

has only 0 as solution. This is an obvious consequence of (10.13). □

Then we would like to show that

$$u_z - u(z) \longrightarrow 0$$

when $h \to 0$ – i.e., the different values of u_z are approximate values of $u(z)$. We have indeed

Theorem 10.3. *Suppose that $u \in C^2(\overline{\Omega})$ is solution to* (10.1)*. Suppose that Ω is such that for h small enough one has*

$$\forall\, y \in \Omega_h, \quad \textit{the segments joining } y \textit{ to its neighbours are in } \Omega. \tag{10.15}$$

Then for any $\varepsilon > 0$ we have

$$|u_z - u(z)| \le \varepsilon \quad \forall\, z \in \Omega_h, \tag{10.16}$$

for h small enough.

Proof.
• 1. We first check the "consistency" of the system (10.7). By consistency we mean how accurate $u(z)$, $z \in \Omega_h$ is the solution of a linear system close to (10.7). For a C^2-function we have

$$\begin{aligned} u(y + he_i) - u(y) &= \int_0^1 \frac{d}{dt} u(y + the_i)\, dt \\ &= (t-1)\frac{d}{dt}u(y + the_i)\Big|_0^1 - \int_0^1 (t-1)\frac{d^2}{dt^2} u(y + the_i)\, dt \\ &= \partial_{x_i} u(y) h - \int_0^1 (t-1)\partial_{x_i}^2 u(y + the_i) h^2\, dt. \end{aligned}$$

(Note that this computation assumes that the segment $(y, y + he_i)$ belongs to Ω.) Similarly if the segment $(y - he_i, y)$ is contained in Ω we get

$$u(y - he_i) - u(y) = -\partial_{x_i} u(y) h - \int_0^1 (t-1)\partial_{x_i}^2 u(y - the_i) h^2\, dt.$$

It follows under the conditions above that

$$\begin{aligned} &u(y+he_i) - 2u(y) + u(y-he_i) \\ &\qquad = h^2 \int_0^1 (1-t)\{\partial_{x_i}^2 u(y+the_i) + \partial_{x_i}^2 u(y-the_i)\}\,dt. \end{aligned} \tag{10.17}$$

This implies clearly that

$$\begin{aligned} \frac{u(y+he_i) - 2u(y) + u(y-he_i)}{h^2} &= \partial_{x_i}^2 u(y) \\ + \int_0^1 (1-t)\{\partial_{x_i}^2 u(y+the_i) &+ \partial_{x_i}^2 u(y-the_i) - 2\partial_{x_i}^2 u(y)\}\,dt. \end{aligned} \tag{10.18}$$

We then set

$$\varepsilon_i(y,h) = \int_0^1 (1-t)\{\partial_{x_i}^2 u(y+the_i) + \partial_{x_i}^2 u(y-the_i) - 2\partial_{x_i}^2 u(y)\}\,dt, \tag{10.19}$$

to get

$$\frac{u(y+he_i) - 2u(y) + u(y-he_i)}{h^2} = \partial_{x_i}^2 u(y) + \varepsilon_i(y,h). \tag{10.20}$$

Remark 10.3. If u belongs to $C^2(\overline{\Omega})$, $\partial_{x_i}^2 u$ is in particular uniformly continuous in Ω and one has

$$|\varepsilon_i(y,h)| \le \varepsilon(h) \quad i = 1,2, \quad \forall\, y, \tag{10.21}$$

where $\varepsilon(h) \to 0$ when $h \to 0$. In particular if $\partial_{x_i}^2 u$ is uniformly Lipschitz continuous it follows from (10.19) that

$$|\varepsilon_i(y,h)| \le Ch \quad i = 1,2, \quad \forall\, y, \tag{10.22}$$

for some constant C.

Due to our assumption (10.15), (10.20) holds for any $y \in \Omega_h$ and for $i = 1,2$. It follows then that $u(y)$, $y \in \Omega_h$ satisfies

$$\begin{aligned} -\Delta_h u(y) &= -\frac{1}{h^2}\Big\{\sum_{z \in N(y)} u(z) - 4u(y)\Big\} \\ &= f(y) - \varepsilon_1(y,h) - \varepsilon_2(y,h) \quad \forall\, y \in \Omega_h. \end{aligned} \tag{10.23}$$

• 2. *The convergence.*
We set

$$U_y = u_y - u(y).$$

It follows from (10.7), (10.23) by subtraction that U_y satisfies

$$\begin{cases} -\Delta_h U_y = \varepsilon_1(y,h) + \varepsilon_2(y,h) & \forall\, y \in \Omega_h, \\ U_y = -u(y) & \forall\, y \in \Gamma_h. \end{cases} \tag{10.24}$$

We then decompose U_y in two parts. More precisely we introduce U_y^1, U_y^2 the solution to

$$\begin{cases} -\Delta_h U_y^1 = \varepsilon_1(y,h) + \varepsilon_2(y,h) & \forall\, y \in \Omega_h, \\ U_y^1 = 0 & \forall\, y \in \Gamma_h, \end{cases} \tag{10.25}$$

$$\begin{cases} -\Delta_h U_y^2 = 0 & \forall\, y \in \Omega_h, \\ U_y^2 = -u(y) & \forall\, y \in \Gamma_h. \end{cases} \tag{10.26}$$

Clearly both systems admit a unique solution. This is clear by Theorem 10.2 for the first one. For (10.26) this follows from the fact that $U_y^2 + u(y)$ is solution of a system of the type (10.7). By uniqueness of the solution of such systems we have

$$U_y = U_y^1 + U_y^2 \quad \forall\, y \in \Omega_h \cup \Gamma_h. \tag{10.27}$$

We would like to estimate U_y^1, U_y^2.

For $y_0 \in \Omega$ set

$$\delta(y) = M - \frac{|y - y_0|^2}{4} \tag{10.28}$$

where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^2$ – i.e.,

$$\delta(y) = M - \frac{1}{4}\{(y_1 - y_{0,1})^2 + (y_2 - y_{0,2})^2\}$$

if $y = (y_1, y_2)$, $y_0 = (y_{0,1}, y_{0,2})$, and $M = \mathrm{Max}_{y \in \overline{\Omega}} \frac{|y-y_0|^2}{4}$. Note that – see for instance (10.18)

$$-\Delta\delta(y) = -\Delta_h \delta(y) = 1. \tag{10.29}$$

By (10.21), (10.25) one has then

$$\begin{aligned} -\Delta_h\{U_y^1 - 2\varepsilon(h)\delta(y)\} &= \varepsilon_1(y,h) + \varepsilon_2(y,h) - 2\varepsilon(h) \le 0 \quad \forall\, y \in \Omega_h, \\ U_y^1 - 2\varepsilon(h)\delta(y) &\le 0 \quad \forall\, y \in \Gamma_h, \end{aligned}$$

and

$$\begin{aligned} -\Delta_h\{-U_y^1 - 2\varepsilon(h)\delta(y)\} &= -\varepsilon_1(y,h) - \varepsilon_2(y,h) - 2\varepsilon(h) \le 0 \quad \forall\, y \in \Omega_h, \\ -U_y^1 - 2\varepsilon(h)\delta(y) &\le 0 \quad \forall\, y \in \Gamma_h. \end{aligned}$$

From the discrete maximum principle we derive that

$$|U_y^1| \le 2\varepsilon(h)\delta(y) \quad \forall\, y \in \Omega_h \cup \Gamma_h.$$

If d_Ω denotes the diameter of Ω we obtain (since $0 \le \delta \le M \le \frac{d_\Omega^2}{4}$)

$$|U_y^1| \le \frac{d_\Omega^2}{2}\varepsilon(h) \quad \forall\, y \in \Omega_h \cup \Gamma_h. \tag{10.30}$$

Since u is in $C^2(\overline{\Omega})$ and thus in particular Lipschitz continuous we have for some constant C

$$|u(y)| \leq Ch \quad \forall\, y \in \Gamma_h. \tag{10.31}$$

(Note that $\operatorname{dist}(y, \partial\Omega) \leq h$ for $y \in \Gamma_h$.) It follows from the discrete maximum principle that

$$|U_y^2| \leq Ch \quad \forall\, y \in \Omega_h \cup \Gamma_h. \tag{10.32}$$

Combining (10.30), (10.32) we derive

$$|U_y| = |u_y - u(y)| \leq Ch + \frac{d_\Omega^2}{2}\varepsilon(h) \quad \forall\, y \in \Omega_h \cup \Gamma_h. \tag{10.33}$$

This completes the proof of the convergence result. □

Remark 10.4. In the case where in addition $\partial_{x_i}^2 u$ are Lipschitz continuous we have (see (10.22)) for some constant C

$$|u_y - u(y)| \leq Ch \quad \forall\, y \in \Omega_h \cup \Gamma_h. \tag{10.34}$$

Remark 10.5. The assumption (10.15) holds for every domain such that its sections (in the directions x_1 and x_2) are convex.

10.2 The finite element method

The method has been introduced in the middle of the last century and developed together with the functional analysis approach for solving elliptic equations. The idea – for instance for solving the Dirichlet problem – is to replace $H_0^1(\Omega)$ by a finite-dimensional space, that is to say, to apply the so-called Galerkin method. Now the possible choices for such a finite-dimensional subspace are infinite. One can build it by splitting the domain Ω in small elements and choosing a basis of functions with support in these elements. This introduces a "mesh" on the domain, but one can also develop "mesh free" techniques by choosing the basis of the finite-dimensional space without any decomposition of the domain Ω in different subsets.

Let us first consider the abstract Lax–Milgram setting of the problem. That is to say let $a = a(u, v)$ be a bilinear, continuous, coercive form on H – see (1.23), (1.24). For $f \in H^*$ the dual of H there exists a unique u solution to

$$\begin{cases} u \in H, \\ a(u, v) = \langle f, v\rangle \quad \forall\, v \in H, \end{cases} \tag{10.35}$$

(see Theorem 1.5). Let us consider then $\widehat{H}$ a closed subspace of H (at this point $\widehat{H}$ could be finite dimensional or not). Then by the Lax–Milgram theorem there exists a unique $\hat{u}$ solution to

$$\begin{cases} \hat{u} \in \widehat{H}, \\ a(\hat{u}, v) = \langle f, v \rangle \quad \forall\, v \in \widehat{H}. \end{cases} \tag{10.36}$$

Then one can ask oneself how close $\hat{u}$ is to u, in particular when $\widehat{H}$ is "approximating" H in a sense to be precised. To answer this question we have the following simple lemma.

Theorem 10.4 (Céa's Lemma). *Let a be a bilinear form on H satisfying* (1.23), (1.24). *Let u, $\hat{u}$ be the solutions to* (10.35), (10.36) *respectively. Then we have*

$$|u - \hat{u}| \le \frac{\Lambda}{\lambda} \operatorname*{Inf}_{v \in \widehat{H}} |u - v|. \tag{10.37}$$

Proof. From (10.35), (10.36) we derive by subtraction since $\widehat{H} \subset H$

$$a(u - \hat{u}, v) = 0 \quad \forall\, v \in \widehat{H}.$$

Replacing v by $v - \hat{u}$ which also belongs to $\widehat{H}$ we obtain

$$\begin{aligned} & a(u - \hat{u}, v - \hat{u}) = 0 \quad \forall\, v \in \widehat{H} \\ \iff \qquad & a(u - \hat{u}, v - u + u - \hat{u}) = 0 \quad \forall\, v \in \widehat{H}. \end{aligned}$$

Using the bilinearity of a again we get

$$a(u - \hat{u}, u - \hat{u}) = a(u - \hat{u}, u - v).$$

From the continuity and the coerciveness of a this implies

$$\begin{aligned} \lambda |u - \hat{u}|^2 &\le \Lambda |u - \hat{u}|\, |u - v| \qquad && \forall\, v \in \widehat{H}, \\ \text{i.e.,} \qquad |u - \hat{u}| &\le \frac{\Lambda}{\lambda} |u - v| && \forall\, v \in \widehat{H} \end{aligned}$$

and (10.37) follows. This completes the proof of the theorem. □

Remark 10.6. One can write (10.37) as

$$|u - \hat{u}| \le \frac{\Lambda}{\lambda} |u - P_{\widehat{H}}(u)| \tag{10.38}$$

where $P_{\widehat{H}} u$ denotes the orthogonal projection of u on $\widehat{H}$ – see Theorem 1.1.

Let us now consider a "sequence" of such subspaces $\widehat{H}$. More precisely for $h \in \mathbb{R}$ denote by $\widehat{H}_h$ a closed subspace of H and suppose that

$$\widehat{H}_h \text{ is dense in } H \text{ when } h \to 0 \tag{10.39}$$

that is to say

$$\forall\, v \in H, \quad \exists\, \hat{v}_h \in \widehat{H}_h \text{ such that } \hat{v}_h \to v \text{ when } h \to 0. \tag{10.40}$$

Then we have

Theorem 10.5. *Under the assumptions of Theorem* 10.4 *and* (10.40), *if* $\hat{u}_h$ *denotes the solution to*

$$\begin{cases} \hat{u}_h \in \widehat{H}_h, \\ a(\hat{u}_h, v) = \langle f, v \rangle \quad \forall\, v \in \widehat{H}_h, \end{cases} \tag{10.41}$$

we have

$$\hat{u}_h \longrightarrow u \quad \text{in } H \tag{10.42}$$

where u *is the solution to* (10.35).

Proof. Let $\widehat{U}_h$ be the sequence of $\widehat{H}_h$ such that

$$\widehat{U}_h \longrightarrow u \quad \text{in } H.$$

By (10.37) we have

$$|u - \hat{u}_h| \leq \frac{\Lambda}{\lambda} |u - \widehat{U}_h|$$

and the result follows. □

Perhaps it is time to show how our abstract results above are fitting concrete situations. We will argue for the sake of simplicity in $\mathbb{R}^2$, i.e., Ω will be a bounded domain in $\mathbb{R}^2$ – see Figure 10.2. Let us denote by $\widehat{\Omega}$ a polygonal subdomain of

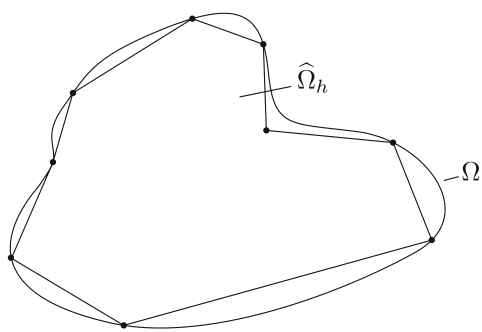


Figure 10.2.

Ω (the "^" reminds of the polygonal character of its boundary). This domain is supposed to approximate Ω and to measure the degree of approximation we introduce h defined as

$$h = \operatorname*{Sup}_{x \in \partial\widehat{\Omega}} \operatorname{dist}(x, \partial\Omega) \tag{10.43}$$

and we denote then $\widehat{\Omega}$ by $\widehat{\Omega}_h$. To fit our abstract framework we set $\widehat{H}_h = H_0^1(\widehat{\Omega}_h)$.

Let us then denote by u, $\hat{u}_h$ the weak solution of the Dirichlet problems

$$\begin{cases} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega), \end{cases} \tag{10.44}$$

$$\begin{cases} \hat{u}_h \in H_0^1(\widehat{\Omega}_h), \\ \displaystyle\int_{\widehat{\Omega}_h} \nabla \hat{u}_h \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\widehat{\Omega}_h). \end{cases} \tag{10.45}$$

Then as a consequence of our abstract analysis we have

Theorem 10.6. *For $f \in H^{-1}(\Omega)$ let u, $\hat{u}_h$ be the solutions to* (10.44), (10.45). *If $\widehat{\Omega}_h$ is such that $h \to 0$ (see* (10.43)) *then we have*

$$u_h \longrightarrow u \quad \textit{in } H_0^1(\Omega). \tag{10.46}$$

(*We assume $\hat{u}_h$ extended by 0 outside $\widehat{\Omega}_h$.*)

Proof. By Theorem 10.5 it is enough to show that $\widehat{H}_h = H_0^1(\widehat{\Omega}_h)$ is dense in $H_0^1(\Omega)$ when $h \to 0$, i.e., we have (10.40). Let $v \in H_0^1(\Omega)$ and denote by $\hat{v}_h \in \widehat{H}_h$ the function defined as

$$\hat{v}_h = P_{\widehat{H}_h}(v). \tag{10.47}$$

We mean here the projection when $H_0^1(\Omega)$ is equipped with the scalar product

$$(u, v) \mapsto \int_\Omega \nabla u \cdot \nabla v \, dx.$$

By Corollary 1.2 we then have

$$\int_\Omega \nabla (v - \hat{v}_h) \cdot \nabla w \, dx = 0 \quad \forall\, w \in H_0^1(\widehat{\Omega}_h). \tag{10.48}$$

Taking $w = \hat{v}_h$ we derive

$$\big|\big|\nabla \hat{v}_h\big|\big|_{2,\Omega}^2 = \int_\Omega \nabla \hat{v}_h \cdot \nabla \hat{v}_h \, dx = \int_\Omega \nabla v \cdot \nabla \hat{v}_h \le \big|\big|\nabla v\big|\big|_{2,\Omega} \big|\big|\nabla \hat{v}_h\big|\big|_{2,\Omega},$$

i.e.,

$$\big|\big|\nabla \hat{v}_h\big|\big|_{2,\Omega} \le \big|\big|\nabla v\big|\big|_{2,\Omega} \tag{10.49}$$

and $\hat{v}_h$ is bounded in $H_0^1(\Omega)$. Thus, up to a subsequence, when $h \to 0$ we have for some $v_0 \in H_0^1(\Omega)$

$$\hat{v}_h \rightharpoonup v_0 \quad \text{in } H_0^1(\Omega). \tag{10.50}$$

Let us choose $w \in \mathcal{D}(\Omega)$. Then for h small enough we clearly have $w \in H_0^1(\widehat{\Omega}_h)$ and from (10.48)

$$\int_\Omega \nabla (v - \hat{v}_h) \cdot \nabla w = 0.$$

Passing to the limit when $h \to 0$ we derive

$$\int_\Omega \nabla(v - v_0) \cdot \nabla w = 0 \quad \forall\, w \in \mathcal{D}(\Omega).$$

By density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$, the equality above holds for any $w \in H_0^1(\Omega)$ and thus

$$v_0 = v.$$

Since the possible limit of $\hat{v}_h$ is unique the whole sequence $\hat{v}_h$ satisfies (10.50) with $v_0 = v$. Finally from (10.48) we derive

$$\int_\Omega |\nabla(v - \hat{v}_h)|^2\, dx = \int_\Omega \nabla(v - \hat{v}_h) \cdot \nabla v\, dx \to 0 \quad \text{when } h \to 0.$$

This completes the proof of the density of $H_0^1(\widehat{\Omega}_h)$ in $H_0^1(\Omega)$ when $h \to 0$ and the proof of the theorem. □

Remark 10.7. Due to Theorem 10.6 it is clear that when we wish to compute the solution u of (10.44) we can replace Ω by a polygonal domain included in Ω which is close to Ω. – see also Exercise 4.

Taking into account the remark above we suppose now that Ω is a polygonal domain of $\mathbb{R}^2$. We are going to present the simplest finite element method – the so-called P_1-method.

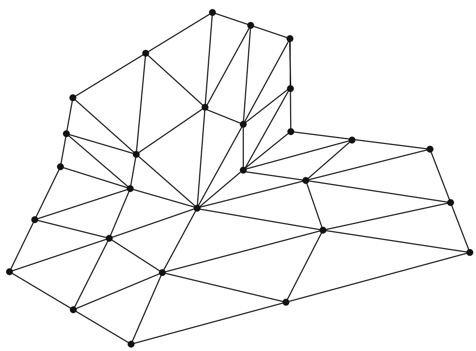


Figure 10.3.

Definition 10.1. A triangulation τ of Ω is a set of triangles T_i, $i = 1, \dots, n$ such that

$$\overline{\Omega} = \bigcup_{i=1}^{n} T_i \tag{10.51}$$

$$T_i \cap T_j = \emptyset, \text{ one point or one common side } \forall\, i \neq j. \tag{10.52}$$

(We suppose that the triangles are closed.)

Definition 10.2. For a triangulation τ of Ω, the mesh size h is the parameter defined by

$$h = \operatorname*{Sup}_{T\in\tau} \operatorname{diam} T \tag{10.53}$$

where $\operatorname{diam} T$ denotes the diameter of T – i.e., its largest side. One denotes then by τ_h a triangulation τ of Ω of mesh size h.

Remark 10.8. By (10.52) the following situation is excluded for the triangles of a triangulation: i.e., two neighbours cannot have in common only part of a side.

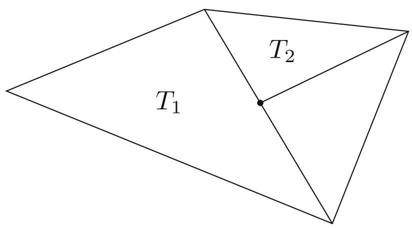


Figure 10.4.

If τ_h is a triangulation of Ω we define V_h as

$$V_h = \{\, v : \Omega \to \mathbb{R} \mid v \text{ is continuous}, v|_T \in P_1 \ \forall T \in \tau_h,\ \ v = 0 \text{ on } \partial\Omega \,\}. \tag{10.54}$$

In this definition $v|_T$ denotes the restriction of v to T, P_1 is the set of polynomials of degree less or equal to 1 – i.e., the set of affine functions on $\mathbb{R}^2$.

Let us denote by

$$x^i = (x_1^i, x_2^i), \quad i = 1, \dots, N \tag{10.55}$$

the vertices of the triangles of τ_h which are inside Ω. Then we have

Theorem 10.7. *V_h is a finite-dimensional subspace of $H_0^1(\Omega)$ and*

$$\dim V_h = N. \tag{10.56}$$

Proof. 1. A function $v \in V_h$ is uniquely determined by $v(x^i)$, $i = 1, \dots, N$. Indeed consider a triangle T with for instance vertices

$$x^1 = (x_1^1, x_2^1), \qquad x^2 = (x_1^2, x_2^2), \qquad x^3 = (x_1^3, x_2^3). \tag{10.57}$$

We claim that if v is affine on T it is uniquely determined by $v(x^i)$, $i = 1, 2, 3$. On T the function v is given by

$$v(x_1, x_2) = ax_1 + bx_2 + c \tag{10.58}$$

where a, b, c are such that

$$\begin{cases} ax_1^1 + bx_2^1 + c = v(x^1), \\ ax_1^2 + bx_2^2 + c = v(x^2), \\ ax_1^3 + bx_2^3 + c = v(x^3). \end{cases} \tag{10.59}$$

Clearly a, b, c are uniquely determined provided that the determinant

$$\begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 1 \end{vmatrix} = \begin{vmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 - x_1^1 & x_2^2 - x_2^1 & 0 \\ x_1^3 - x_1^1 & x_2^3 - x_2^1 & 0 \end{vmatrix} = \begin{vmatrix} x_1^2 - x_1^1 & x_2^2 - x_2^1 \\ x_1^3 - x_1^1 & x_2^3 - x_2^1 \end{vmatrix}$$

is different of 0. This is the case iff the vectors

$$x^2 - x^1, \qquad x^3 - x^1$$

are linearly independent. Since we do not allow flat triangles in our triangulation this is of course always the case for every triangle. This shows that $v(x^i)$ $i = 1, \ldots, N$ determines uniquely v. Note that the event mentioned in Remark 10.8 would create problems at this stage.

2. *Building of a basis on V_h.*

A function $v \in V_h$, vanishing on the boundary is uniquely determined by its values on the inside nodes of the triangulation. In particular there exists a unique function λ_i such that

$$\lambda_i \in V_h, \qquad \lambda_i(x^j) = \delta_{ij} \quad \forall\, i, j = 1, \ldots, N. \tag{10.60}$$

We claim that $(\lambda_i)_{i=1,\ldots,N}$ is a basis of V_h. These functions are indeed linearly independent since

$$\sum_{i=1}^{N} \alpha_i \lambda_i = 0 \quad \Longrightarrow \quad \sum_{i=1}^{N} \alpha_i \lambda_i(x^j) = \alpha_j = 0 \quad \forall\, j = 1, \ldots, N.$$

Moreover if $v \in V_h$ then the function

$$\sum_{i=1}^{N} v(x^i)\lambda_i$$

is a function of V_h which coincides with v at the inside nodes of the triangulation and thus everywhere which shows that

$$v = \sum_{i=1}^{N} v(x^i)\lambda_i$$

and $(\lambda_i)_{i=1,\ldots,N}$ is a basis of V_h. This established (10.56).

3. $V_h \subset H_0^1(\Omega)$.

First remark that if $v \in V_h$, then v is C^1 inside each triangle T of τ_h. Let us denote by

$$\partial_{x_i} v|_T \quad i = 1, 2$$

the usual derivative (in the direction x_i) of v in T. Then we set

$$v_i = \sum_{T \in \tau_h} (\partial_{x_i} v|_T) \chi_T \tag{10.61}$$

where χ_T denotes the characteristic function of T. v_i is piecewise constant and thus

$$v, v_1, v_2 \in L^2(\Omega).$$

Let us show that

$$\partial_{x_i} v = v_i \quad i = 1, 2 \tag{10.62}$$

in the distributional sense. Let $\varphi \in \mathcal{D}(\Omega)$. We have

$$\langle \partial_{x_i} v, \varphi \rangle = -\langle v, \partial_{x_i} \varphi \rangle = -\int_\Omega v \partial_{x_i} \varphi \, dx = -\sum_{T \in \tau_h} \int_T v \partial_{x_i} \varphi \, dx. \tag{10.63}$$

Now on each triangle T by the Green formula – or a simple integration we have

$$\begin{aligned} \int_T v \partial_{x_i} \varphi \, dx &= \int_T \partial_{x_i}(v\varphi) - (\partial_{x_i} v)\varphi \, dx \\ &= -\int_T (\partial_{x_i} v)\varphi + \int_{\partial T} v\varphi\nu_i \, d\sigma(x) \end{aligned} \tag{10.64}$$

where $\nu = (\nu_1, \nu_2)$ denotes the outward unit normal to ∂T. If one integrates on a side S of T which belongs to $\partial\Omega$ one has

$$\int_S v\varphi\nu_i \, d\sigma(x) = 0$$

(since $\varphi \in \mathcal{D}(\Omega)$ vanishes on S). If one integrates on a side S belonging to T_1 and T_2 one has

$$\int_{S \cap \partial T_1} v\varphi\nu_i \, d\sigma(x) + \int_{S \cap \partial T_2} v\varphi\nu_i \, d\sigma(x) = 0$$

(since the outward normals to T_1 and T_2 have opposite directions). Thus if we are summing up (10.64) the boundary integrals disappear and we get (see (10.63))

$$\langle \partial_{x_i} v, \varphi \rangle = \sum_{T \in \tau_h} \int_T (\partial_{x_i} v)\varphi = \langle v_i, \varphi \rangle \tag{10.65}$$

which establishes (10.62). Note that since $v \in H^1(\Omega) \cap C(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$ one has $v \in H_0^1(\Omega)$ (cf. Exercise 11, Chapter 2). This completes the proof of the theorem. $\square$

Since V_h is finite dimensional, it is closed in $H_0^1(\Omega)$ and by the Lax–Milgram theorem there is a unique u_h solution to

$$\begin{cases} u_h \in V_h, \\ \displaystyle\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in V_h. \end{cases} \tag{10.66}$$

One expects now that

- u_h will converge toward u, the weak solution to (10.1) when $h \to 0$. (10.67)
- u_h is computable – since only a finite number of values are needed. (10.68)

Regarding the first point one has to avoid the situation where some of the triangles of τ_h get flat when $h \to 0$. This will be done through the following assumption. First, for T a triangle, denote by h_T the diameter of T – i.e., the length of the largest side of T and by ρ_T the diameter of the largest circle which can fit in T (see the Figure 10.5 below).

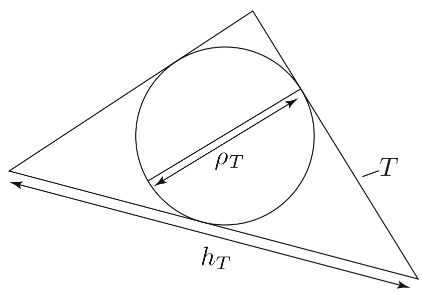


Figure 10.5.

Then we have

Definition 10.3. A family of triangulations (τ_h) is said regular or non-degenerate when $h \to 0$ if there is a constant $\gamma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \gamma \quad \forall\, T \in (\tau_h)\ \forall\, h, \tag{10.69}$$

namely the triangles cannot get too flat when $h \to 0$.

Then we have

Theorem 10.8. *Under the assumptions above, if u_h denotes the solution to* (10.66) *and if (τ_h) is regular we have*

$$u_h \longrightarrow u \quad \textit{in } H_0^1(\Omega) \tag{10.70}$$

where u is the solution to (10.1).

Proof. By Theorem 10.5 applied for

$$H = H_0^1(\Omega), \qquad a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \widehat{H}_h = V_h$$

it is enough to show that V_h is dense in $H_0^1(\Omega)$ when $h \to 0$. Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$ it is enough to show that for every $v \in \mathcal{D}(\Omega)$ there exists a sequence of functions v_h such that $v_h \in V_h$ and $v_h \to v$ in $H_0^1(\Omega)$. Of course a natural candidate for v is the interpolant $\hat{v}_h$ of v given by

$$\hat{v}_h = \sum_{i=1}^N v(x^i)\lambda_i, \tag{10.71}$$

i.e., the function of V_h which coincides with v at the inside nodes of the triangulation. We have indeed

Lemma 10.9. *Let $v \in C^2(\overline{\Omega})$, $v = 0$ on $\partial\Omega$. Let $\hat{v}_h$ be the interpolant of v given by* (10.71)*. If* (10.69) *holds we have for some constant C independent of h*

$$\big|\big|\nabla(v - \hat{v}_h)\big|\big|_{2,\Omega} \le Ch \tag{10.72}$$

(*recall the definition of h given by* (10.53)).

Assuming the lemma proved the proof of the theorem is complete. □

Proof of the lemma.

1. Let T be an arbitrary triangle and v a continuous function of class C^2 on each triangle vanishing at the vertices of all T. Without loss of generality we can assume that 0 is a vertex of T. Let ν_1, ν_2 be the unit vectors on the sides of T – see Figure 10.6. Suppose $a_i = \alpha_i \nu_i$. Then we have

$$\begin{aligned} 0 = v(a_i) - v(0) &= \int_0^{\alpha_i} \frac{d}{dt} v(t\nu_i)\, dt \\ &= \int_0^{\alpha_i} \nabla v(t\nu_i) \cdot \nu_i \, dt \\ &= \int_0^{\alpha_i} \partial_{\nu_i} v(t\nu_i)\, dt. \end{aligned}$$

By the mean value theorem, there is a point p_i on the segment $(0, a_i)$ such that

$$\partial_{\nu_i} v(p_i) = 0.$$

Let x be an arbitrary point of T. We have

$$\begin{aligned} \partial_{\nu_i} v(x) &= \partial_{\nu_i} v(x) - \partial_{\nu_i} v(p_i) \\ &= \int_0^1 \frac{d}{dt} \partial_{\nu_i} v(p_i + t(x - p_i))\, dt \\ &= \int_0^1 \nabla \partial_{\nu_i} v(p_i + t(x - p_i)) \cdot (x - p_i)\, dt. \end{aligned} \tag{10.73}$$

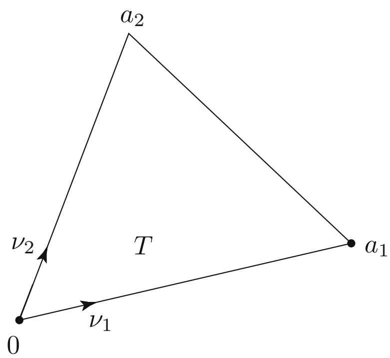


Figure 10.6.

We denote by M a constant such that

$$\operatorname*{Sup}_T |\partial^2_{x_i x_j} v| \leq M \quad \forall T \quad \forall i, j = 1, 2. \tag{10.74}$$

Then from (10.73) we easily derive

$$|\partial_{\nu_i} v(x)| \leq 2Mh_T \quad i = 1, 2. \tag{10.75}$$

Without loss of generality we can assume $\nu_1 = e_1$. Then if θ denotes the angle of the vectors ν_1, ν_2 we have

$$|\partial_{x_1} v(x)| \leq 2Mh_T, \qquad |\cos\theta \partial_{x_1} v(x) + \sin\theta \partial_{x_2} v(x)| \leq 2Mh_T.$$

We then derive

$$|\partial_{x_2} v(x)| \leq \frac{4Mh_T}{|\sin\theta|}.$$

Now it is easy to see that

$$|\sin\theta| \geq \frac{\rho_T}{h_T}$$

and we derive

$$|\partial_{x_2} v(x)| \leq 4M\frac{h_T^2}{\rho_T}, \qquad |\partial_{x_1} v(x)| \leq 2Mh_T \leq 2M\frac{h_T^2}{\rho_T}$$

and thus for some constant $K = K(v)$ we obtain

$$|\nabla v(x)| \leq K\frac{h_T^2}{\rho_T} \quad \forall x \in T, \ \forall T. \tag{10.76}$$

2. Applying step 1 to $v - \hat{v}_h$ we get

$$|\nabla(v - \hat{v}_h)(x)| \le K \frac{h_T^2}{\rho_T} \quad \forall\, x \in T,\ \forall\, T. \tag{10.77}$$

(Note that in (10.74) the constant M depends on v only since the second derivatives of $\hat{v}$ are vanishing on each triangle.)

From (10.77) we derive

$$\begin{aligned} \big|\big|\nabla(v - \hat{v}_h)\big|\big|_{2,\Omega}^2 &= \sum_{T\in\tau_h} \int_T |\nabla(v - \hat{v}_h)(x)|^2\, dx \\ &\le K^2 \sum_{T\in\tau_h} \left(\frac{h_T}{\rho_T}\right)^2 h_T^2 |T| \\ &\le K^2\gamma^2 h^2 \sum_{T\in\tau_h} |T| = K^2\gamma^2h^2|\Omega| \end{aligned}$$

where $|T|$ denotes the measure of T (see (10.69), (10.53)). It follows then that

$$\big|\big|\nabla(v - \hat{v})\big|\big|_{2,\Omega} \le K\gamma|\Omega|^{\frac{1}{2}} h \tag{10.78}$$

which completes the proof of the lemma. $\square$

To compute u_h (see (10.68)) one should notice that

$$u_h = \sum_{i=1}^N \xi_i \lambda_i \tag{10.79}$$

for some uniquely determined ξ_i (see Theorem 10.7). Moreover by (10.66) u_h is uniquely determined by

$$\int_\Omega \nabla u_h \cdot \nabla v\, dx = \int_\Omega f v\, dx \quad \forall\, v \in V_h. \tag{10.80}$$

Of course since the λ_i's built a basis of V_h, (10.80) will be satisfied if and only if

$$\int_\Omega \nabla u_h \cdot \nabla \lambda_j\, dx = \int_\Omega f \lambda_j\, dx \quad \forall\, j = 1, \ldots, N.$$

Thus, by (10.79), we see that the ξ_i are uniquely determined as the solution of the $N \times N$ linear system

$$\sum_{i=1}^N \xi_i \int_\Omega \nabla\lambda_i \cdot \nabla\lambda_j\, dx = \int_\Omega f\lambda_j\, dx \quad \forall\, j = 1, \ldots, N. \tag{10.81}$$

Since we know that u_h exists and is unique the system above is uniquely solvable and in particular the matrix

$$\left(\int_\Omega \nabla\lambda_i \cdot \nabla\lambda_j \, dx\right)_{i,j}$$

is invertible. Note that this matrix is sparse since

$$\int_\Omega \nabla\lambda_i \cdot \nabla\lambda_j \, dx = 0$$

as soon as the support of λ_i, λ_j are disjoint. (By (10.60) the support of λ_i is the union of the triangles having x^i for vertex.) Thus at the end – like for the finite difference method – the approximate solution of our problem is found by solving a linear system of equations.

Exercises

1. One considers the problem

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Set $h = \frac{1}{N}$. One denotes by u_i, $i = 1, \dots, N-1$ the approximation of $u(ih)$. Find the system satisfied by u_i using the finite difference method.

One introduces

$$\begin{aligned} V_h = \{v \in H_0^1(\Omega) \mid\ & v \text{ is continuous}, \\ & v \text{ is affine on each interval } (ih, (i+1)h)\,\} \end{aligned}$$

and λ_i the function of V_h satisfying

$$\lambda_i(jh) = \delta_{ij} \quad \forall\, i, j = 1, \dots, N-1.$$

Show that there exists a unique $u_h \in V_h$ solution to

$$\begin{cases} u_h \in V_h, \\ \displaystyle\int_\Omega u_h' v' \, dx = \int_\Omega f v \, dx \quad \forall\, v \in V_h. \end{cases}$$

Show that

$$u_h = \sum_{i=1}^{N-1} \xi_i \lambda_i.$$

Find the linear system satisfied by $\xi = (\xi_i)$.

2. Let T be a triangle with vertices K_1, K_2, K_3 and such that the angles θ_i satisfy for some positive δ
$$0 < \delta \leq \theta_i \leq \frac{\pi}{2} - \delta.$$
Denote by λ_i the affine functions such that
$$\lambda_i(K_j) = \delta_{ij} \quad \forall\, i, j = 1, 2, 3.$$
Show that for $i, j = 1, 2, 3$ there exist constants C_1, C_2 such that
$$\nabla\lambda_i \cdot \nabla\lambda_j \leq -\frac{C_1}{h_T^2} < 0 \quad \forall\, i \neq j,$$
$$\frac{1}{h_T} \leq |\nabla\lambda_i| \leq \frac{C_2}{h_T} \quad \forall\, i$$
(h_T is the diameter of T).
3. Discretize with finite differences the Dirichlet problem
$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$
Show the convergence of this discretization. One will assume all the data smooth.
4. Let $\widehat{\Omega}_h$ be a polygonal domain which is not necessarily included in Ω. Suppose that for every $\varepsilon > 0$ one has for h small enough
$$\{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon \,\} \subset \widehat{\Omega}_h \subset \{\, x \in \mathbb{R}^2 \mid \operatorname{dist}(x, \Omega) < \varepsilon \,\}.$$
If u, $\hat{u}_h$ are respectively the solutions to (10.44), (10.45) show that
$$\hat{u}_h \longrightarrow u \quad \text{in } H^1(\Omega).$$
Extend this result in $\mathbb{R}^n$.
5. Prove Lemma 10.9 in higher dimensions.
6. Extend the P_1-method in $\mathbb{R}^3$ and then in $\mathbb{R}^n$.
7. Let $\Omega = (\alpha, \beta)$ be an interval
$$\alpha = t_0 < t_1 < \cdots < t_n = \beta$$
a subdivision of Ω. Set $h = \underset{i=0,\dots,n-1}{\operatorname{Max}} (t_{i+1} - t_i)$. If $u \in C^2(\overline{\Omega})$ and $\hat{u}$ denotes the interpolant of u – i.e., the piecewise affine function such that
$$\hat{u}(t_i) = u(t_i) \quad \forall i = 0, \dots, n$$
show that
$$|\hat{u}' - u'|_{2,\Omega} \leq \frac{h}{\sqrt{2}} |u''|_{2,\Omega}.$$

8. A rectangle in $\mathbb{R}^2$ is a set of the type

$$Q = [a_1, b_1] \times [a_2, b_2].$$

A "triangulation" τ_h of Ω is a decomposition of $\overline{\Omega}$ in rectangles such that

$$\overline{\Omega} = \bigcup_{Q \in \tau_h} Q$$

and for any Q_1, $Q_2 \in \tau = \tau_h$, $Q_1 \cap Q_2 = \emptyset$, one point or a common side. Set

$$\mathbb{Q}_1 = \{P \mid P(x) = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2,\ a_i \in \mathbb{R}\}$$
$$V_h = \{v : \Omega \to \mathbb{R} \mid v \text{ is continuous, } v|_Q \in \mathbb{Q}_1 \forall\, Q \in \tau_h,\ v = 0 \text{ on } \partial\Omega\}.$$

(i) Show that a function of $\mathbb{Q}_1$ is uniquely determined by its value on the vertices of Q.

(ii) Find a basis of V_h.

(iii) Develop a $\mathbb{Q}_1$-finite element method ($h = \operatorname*{Sup}_{Q \in \tau} \operatorname{diam} Q$).

Part II

More Advanced Theory

Chapter 11

Nonlinear Problems

The goal of this chapter is to introduce nonlinear problems – i.e., problems whose solution does not depend linearly on the data. We will restrict ourselves to simple techniques of existence or uniqueness.

11.1 Monotone methods

To simplify our exposition we will assume that Ω is a bounded domain in $\mathbb{R}^n$. Suppose that for $f \in H^{-1}(\Omega)$ we want to solve

$$\begin{cases} -\Delta u + F(u) = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{11.1}$$

In order for the first equation (11.1) to make sense we need to have $F(u) \in H^{-1}(\Omega)$ and thus some hypothesis on F. Let us assume that F is uniformly Lipschitz continuous that is to say for some constant $L > 0$ we have

$$|F(z) - F(z')| \leq L|z - z'| \quad \forall\, z, z' \in \mathbb{R}. \tag{11.2}$$

Then, clearly, if $v \in L^2(\Omega)$ we have $F(v) \in L^2(\Omega)$. This follows from the continuity of F and from the inequality

$$\begin{aligned} |F(v)| &= |F(v) - F(0) + F(0)| \\ &\leq |F(v) - F(0)| + |F(0)| \leq L|v| + |F(0)|. \end{aligned} \tag{11.3}$$

The last function above belongs to $L^2(\Omega)$ since we have assumed Ω bounded. Thus the first equation of (11.1) makes sense for F satisfying (11.2). Let us then introduce the following definition:

Definition 11.1 (super- and sub-solutions). We say that $\overline{u}$ (respectively $\underline{u}$) is a supersolution (respectively subsolution) to (11.1) iff

$$\begin{aligned}
&\overline{u} \in H^1(\Omega), \quad \overline{u} \geq 0 \text{ on } \partial\Omega \text{ (in the sense of Definition 4.2)},\\
&-\Delta\overline{u} + F(\overline{u}) \geq f \quad \text{in a weak sense – i.e.,}\\
&\int_\Omega \nabla\overline{u}\cdot\nabla v + F(\overline{u})v\,dx \geq \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega),\ v \geq 0,\\
\text{(resp.}\quad &\underline{u} \in H^1(\Omega), \quad \underline{u} \leq 0 \quad \text{on } \partial\Omega,\\
&\int_\Omega \nabla\underline{u}\cdot\nabla v + F(\underline{u})v\,dx \leq \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega),\ v \geq 0).
\end{aligned} \tag{11.4}$$

Then we have (see [3]–[5])

Theorem 11.1. *Let $f \in H^{-1}(\Omega)$. Suppose that there exist $\underline{u}$, $\overline{u}$ respectively sub- and supersolutions to* (11.1) *such that*

$$\underline{u} \leq \overline{u}. \tag{11.5}$$

Then there exist u_m, u_M solutions to (11.1) *such that*

$$\underline{u} \leq u_m \leq u_M \leq \overline{u} \tag{11.6}$$

and such that for any solution u of (11.1) *satisfying $\underline{u} \leq u \leq \overline{u}$ one has*

$$\underline{u} \leq u_m \leq u \leq u_M \leq \overline{u}. \tag{11.7}$$

(*u_m, u_M are respectively the minimal and the maximal solution to* (11.1) *between $\underline{u}$ and $\overline{u}$.*)

Proof. We consider for $\lambda \geq L$ the mapping S which assigns for every $v \in L^2(\Omega)$, $u = S(v) \in L^2(\Omega)$ weak solution to

$$\begin{cases} -\Delta u + \lambda u = \lambda v - F(v) + f & \text{in } \Omega,\\ u \in H_0^1(\Omega). \end{cases} \tag{11.8}$$

(This mapping goes in fact from $L^2(\Omega)$ into $H_0^1(\Omega)$.) A solution to (11.1) is a fixed point for S. As seen in (11.3) for $v \in L^2(\Omega)$, $\lambda v - F(v) + f \in H^{-1}(\Omega)$ and the existence of $u = S(v)$ follows from the Lax–Milgram theorem.

1. S is continuous from $L^2(\Omega)$ into itself.

   Indeed if $u_i = S(v_i)$ one derives easily from (11.8) that

$$-\Delta(u_1 - u_2) + \lambda(u_1 - u_2) = (\lambda v_1 - F(v_1)) - (\lambda v_2 - F(v_2)).$$

Taking in the weak form of the equation above $u_1 - u_2$ as test function we get

$$\begin{aligned}\lambda|u_1 - u_2|_{2,\Omega}^2 &\leq \int_\Omega \lambda(v_1 - v_2) - (F(v_1) - F(v_2))(u_1 - u_2)\,dx \\ &\leq \{\lambda|v_1 - v_2|_{2,\Omega} + |F(v_1) - F(v_2)|_{2,\Omega}\}|u_1 - u_2|_{2,\Omega} \\ &\leq \{\lambda|v_1 - v_2|_{2,\Omega} + L|v_1 - v_2|_{2,\Omega}\}|u_1 - u_2|_{2,\Omega}\end{aligned}$$

(we used here the Cauchy–Schwarz inequality and (11.2)).

It follows that

$$|u_1 - u_2|_{2,\Omega} \leq \left(\frac{\lambda + L}{\lambda}\right)|v_1 - v_2|_{2,\Omega}$$

and the continuity of S follows.

2. S is monotone – i.e., $v_1 \geq v_2 \Rightarrow S(v_1) \geq S(v_2)$

Indeed for $v_1 \geq v_2$ one has by (11.2)

$$\begin{aligned}(\lambda v_1 - F(v_1)) - (\lambda v_2 - F(v_2)) &= \lambda(v_1 - v_2) - (F(v_1) - F(v_2)) \\ &\geq \lambda(v_1 - v_2) - L(v_1 - v_2) \geq 0.\end{aligned} \tag{11.9}$$

It follows from (11.8) that

$$-\Delta u_1 + \lambda u_1 \geq -\Delta u_2 + \lambda u_2$$

in a weak sense and by the weak maximum principle that $u_1 \geq u_2$.

3. We consider then the following sequences

$$\begin{cases} \underline{u}_0 = \underline{u}, \\ \underline{u}_k = S(\underline{u}_{k-1}) \quad k \geq 1 \end{cases} \qquad \begin{cases} \overline{u}_0 = \overline{u}, \\ \overline{u}_k = S(\overline{u}_{k-1}) \quad k \geq 1. \end{cases}$$

We claim that the following inequalities hold for every $k \geq 1$

$$\underline{u} \leq \underline{u}_{k-1} \leq \underline{u}_k \leq \overline{u}_k \leq \overline{u}_{k-1} \leq \overline{u}. \tag{11.10}$$

Indeed by definition of $\underline{u}_1$ we have

$$-\Delta\underline{u}_1 + \lambda\underline{u}_1 = \lambda\underline{u}_0 - F(\underline{u}_0) + f \geq -\Delta\underline{u}_0 + \lambda\underline{u}_0$$

(since $\underline{u}_0$ is subsolution). By the maximum principle it follows that

$$\underline{u}_0 \leq \underline{u}_1. \tag{11.11}$$

Applying S^{k-1} to both sides of (11.11) and using the monotonicity of S we get

$$\underline{u}_{k-1} \leq \underline{u}_k.$$

One can show the same way that

$$\overline{u}_k \le \overline{u}_{k-1} \le \cdots \le \overline{u}_1 \le \overline{u}_0 = \overline{u}.$$

Finally the inequality

$$\underline{u}_k \le \overline{u}_k$$

follows from

$$\underline{u}_0 \le \overline{u}_0$$

where we apply S^k. This completes the proof of (11.10).

4. We pass to the limit

Since $\underline{u}_k$ and $\overline{u}_k$ are monotone sequences between $\underline{u}$ and $\overline{u}$ by the Lebesgue theorem there exist two functions u_m and u_M such that

$$\underline{u}_k \longrightarrow u_m, \qquad \overline{u}_k \longrightarrow u_M \quad \text{in } L^2(\Omega). \tag{11.12}$$

It is clear by the definitions of $\underline{u}_k$, $\overline{u}_k$ and the continuity of S in $L^2(\Omega)$ that u_m and u_M are fixed points for S and thus solutions to (11.1). Moreover if u is a solution to (11.1) such that

$$\underline{u}_0 = \underline{u} \le u \le \overline{u} = \overline{u}_0$$

applying S^k to these inequalities one gets

$$\underline{u}_k \le u = S^k(u) \le \overline{u}_k$$

and passing to the limit

$$u_m \le u \le u_M$$

which completes the proof of the theorem. □

Example 11.1. There exists a weak solution to

$$\begin{cases} -\Delta u = 1 - |u| & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{11.13}$$

This follows from the fact that $-1, 1$ are respectively weak subsolution and supersolution to (11.13).

Remark 11.1. We used only the Lipschitz continuity of F on $[\underline{u}, \overline{u}]$. Thus there also exists a weak solution to

$$\begin{cases} -\Delta u = 1 - |u|^p & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

which lies between -1 and 1 for every $p > 1$.

As a corollary of Theorem 11.1 we have

Theorem 11.2. *Let $f \in L^2(\Omega)$ and F a Lipschitz continuous monotone function such that*

$$F(0) = 0. \tag{11.14}$$

Then there exists a unique solution to

$$\begin{cases} -\Delta u + F(u) = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{11.15}$$

Proof. We consider $\underline{u}$ and $\overline{u}$ the weak solutions to

$$\begin{cases} -\Delta \underline{u} = -f^- & \text{in } \Omega, \\ \underline{u} \in H_0^1(\Omega), \end{cases} \qquad \begin{matrix} -\Delta \overline{u} = f^+ & \text{in } \Omega, \\ \overline{u} \in H_0^1(\Omega), \end{matrix}$$

where f^- and f^+ are respectively the negative and positive part of f. By the weak maximum principle we have

$$\underline{u} \leq 0 \leq \overline{u}.$$

It follows, by the monotonicity of F, that

$$-\Delta \underline{u} + F(\underline{u}) \leq -\Delta \underline{u} = -f^- \leq f \leq f^+ = -\Delta \overline{u} \leq -\Delta \overline{u} + F(\overline{u}).$$

Since in addition $\underline{u} = 0 = \overline{u}$ on $\partial\Omega$ the existence of a solution to (11.15) follows from Theorem 11.1. If now u_1, u_2 are two solutions to (11.15) by subtraction we get

$$-\Delta(u_1 - u_2) = -(F(u_1) - F(u_2)) \quad \text{in } \Omega,$$

i.e.,

$$\int_\Omega \nabla(u_1 - u_2) \cdot \nabla v \, dx = -\int_\Omega (F(u_1) - F(u_2)) v \, dx \quad \forall\, v \in H_0^1(\Omega).$$

Taking $v = u_1 - u_2$, by the monotonicity of F we get

$$\int_\Omega |\nabla(u_1 - u_2)|^2 \, dx \leq 0$$

that is to say $u_1 = u_2$. This completes the proof of the theorem. □

Remark 11.2. One can easily remove the assumption (11.14) by writing the first equation of (11.15) as

$$-\Delta u + F(u) - F(0) = f - F(0). \tag{11.16}$$

Remark 11.3. Except in very peculiar situations – like for instance in Theorem 11.2 – uniqueness is lost for nonlinear problems. For instance if $\Omega = (0,\pi)^2$ the problem

$$\begin{cases} -\Delta u + 2|u| = 0 \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{11.17}$$

admits

$$u = k \sin x \sin y$$

as solution for any $k \leq 0$.

To end this section we would like to consider an interesting example – the so-called logistic equation (see [16]). If a is a positive constant we are looking for u solution to

$$\begin{cases} -\Delta u = u(a-u) & \text{in } \Omega, \\ u \in H_0^1(\Omega), & u > 0, \ u \text{ bounded}. \end{cases} \tag{11.18}$$

Note that $u = 0$ is solution to (11.18) if we do not require $u > 0$. Then we have

Theorem 11.3. *Let us denote by λ_1 the first eigenvalue for the Dirichlet problem in Ω. The problem* (11.18) *admits a solution iff $a > \lambda_1$. Moreover this solution is unique.*

Proof. Suppose first that $a \leq \lambda_1$. Under its weak form the problem (11.18) can be written as

$$\int_\Omega \nabla u \cdot \nabla v - auv \, dx = -\int_\Omega u^2 v \, dx \quad \forall\, v \in H_0^1(\Omega).$$

Taking $v = u$ we obtain

$$\int_\Omega |\nabla u|^2 \, dx - a \int_\Omega u^2 \, dx = -\int_\Omega u^3 \, dx.$$

Since $a \leq \lambda_1$ we have

$$\int_\Omega |\nabla u|^2 \, dx - a \int_\Omega u^2 \, dx \geq \int_\Omega |\nabla u|^2 \, dx - \lambda_1 \int_\Omega u^2 \, dx \geq 0.$$

It follows that

$$\int_\Omega u^3 \, dx \leq 0$$

and u is necessarily equal to 0.

Suppose now that $\lambda_1 < a$. First we claim that if u is bounded then

$$u \leq a \quad \text{in } \Omega.$$

Indeed taking $v = (u-a)^+$ in the weak formulation of (11.18) we get

$$\int_\Omega \nabla u \cdot \nabla (u-a)^+ \, dx = \int_\Omega u(a-u)(u-a)^+ \, dx$$

$$\iff \qquad \int_\Omega \nabla (u-a) \cdot \nabla (u-a)^+ \, dx = \int_{\{u>a\}} u(a-u)(u-a) \, dx \leq 0,$$

$\{u > a\}$ denoting the set defined as $\{ x \mid u(x) > a \}$. It follows that

$$\int_\Omega |\nabla (u-a)^+|^2 \, dx \leq 0,$$

i.e., $(u-a)^+ = 0$ and this shows that $u \leq a$.

Next we remark that $\overline{u} = a$ is a supersolution for (11.18). Then consider φ_1 a positive first eigenfunction corresponding to λ_1. For any $t > 0$ we have

$$-\Delta(t\varphi_1) = \lambda_1 t\varphi_1 \leq (t\varphi_1)(a - t\varphi_1)$$

provided that

$$\lambda_1 \leq a - t\varphi_1 \quad \iff \quad t\varphi_1 \leq a - \lambda_1.$$

(We remind that φ_1 can always be chosen positive. Moreover it is bounded.) Since $a - \lambda_1 > 0$ for t small enough we have

$$-\Delta(t\varphi_1) \leq (t\varphi_1)(a - t\varphi_1)$$

and $\underline{u} = t\varphi_1$ is a subsolution to (11.18). It follows then from Theorem 11.1 that there exists a maximal solution u to (11.18) such that

$$t\varphi_1 \leq u \leq a.$$

Of course u is everywhere positive.

Let us now turn to the issue of uniqueness. Suppose that u_1, u_2 are two solutions to (11.18). From the weak formulation we have

$$\int_\Omega \nabla u_1 \cdot \nabla v \, dx = \int_\Omega u_1(a-u_1)v \, dx \quad \forall\, v \in H_0^1(\Omega),$$

$$\int_\Omega \nabla u_2 \cdot \nabla v \, dx = \int_\Omega u_2(a-u_2)v \, dx \quad \forall\, v \in H_0^1(\Omega).$$

Taking $v = u_2$ in the first equation, $v = u_1$ in the second and subtracting the two equalities we get

$$0 = \int_\Omega u_1 u_2 (u_2 - u_1) \, dx.$$

Suppose then that u_2 is the maximal solution built between 0 and a. One has of course

$$u_2 \geq u_1 > 0$$

and thus by the integral equality above $u_1 = u_2$. This completes the proof of the theorem. $\square$

Remark 11.4. One can replace u bounded by

$$u(a-u)v \in L^1(\Omega)$$

for any $v \in H_0^1(\Omega)$ since we only use this property for all our computations to make sense. In fact one could also replace $u > 0$ in (11.18) by $u \geq 0$, $u \not\equiv 0$ since one can show that a nonnegative solution to (11.18) is automatically positive.

11.2 Quasilinear equations

A quasilinear elliptic equation in divergence form is an equation of the type

$$-\operatorname{div}(A(x,u)\nabla u) + a(x,u)u = f, \tag{11.19}$$

i.e., the coefficients are depending on u – or eventually on its derivatives. The motivation for introducing such an equation and nonlinear equations in general is to take into account the reaction of a system to its own state. For instance the diffusion of the temperature in a material has a velocity given by the Fourier law

$$\vec{v} = -A\nabla u.$$

A is a constant proper to each material. However it is easy to imagine that this constant for a piece of metal is not the same at 0^{o} and 200^{o}: it depends on the temperature of the material itself and a more realistic Fourier law can be written as

$$\vec{v} = -A(u)\nabla u. \tag{11.20}$$

This would lead to an equation of the type (11.19). In the preceding section we addressed the case of a nonlinear lower order term. Here for the sake of simplicity we will restrict ourselves to the case where

$$A(x,u) = A(x,u)\,\mathrm{Id}, \qquad a(x,u) = 0.$$

$A(x,u)$ is a real number.

Thus let us suppose that Ω is a bounded subset of $\mathbb{R}^n$. Consider the following quasilinear Dirichlet problem. One wants to find u solution to

$$\begin{cases} \displaystyle\int_\Omega A(x,u)\nabla u \cdot \nabla v\, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{11.21}$$

In this equation $f \in H^{-1}(\Omega)$. First let us assume that

$$A(x,u) = A(u) \tag{11.22}$$

where A is a function satisfying

$$A \text{ is continuous}, \qquad 0 < \lambda \leq A(u) \leq \Lambda \quad u \in \mathbb{R} \tag{11.23}$$

for some constant λ, Λ.

Then under the assumptions above we have

Theorem 11.4. *Let $f \in H^{-1}(\Omega)$ and A satisfying* (11.23). *Then there exists a unique solution u of the problem*

$$\begin{cases} \displaystyle\int_\Omega A(u)\nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{11.24}$$

Proof. For $u \in H_0^1(\Omega)$ let us introduce

$$U = U(x) = \int_0^{u(x)} A(s)\, ds. \tag{11.25}$$

By Proposition 2.12 we have

$$U \in H_0^1(\Omega), \qquad \partial_{x_i} U = A(u) \partial_{x_i} u$$

and thus U given by (11.25) is solution to

$$\begin{cases} \displaystyle\int_\Omega \nabla U \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega), \\ U \in H_0^1(\Omega). \end{cases} \tag{11.26}$$

Of course (11.26) admits a unique solution. By (11.23) it is clear that the function

$$\mathcal{A}(t) = \int_0^t A(s)\, ds$$

is a C^1, monotone increasing function which admits a C^1 monotone increasing inverse $\mathcal{A}^{-1}$. It follows that if U is solution to (11.26) then $\mathcal{A}^{-1}(U)$ is solution to (11.24). This completes the proof of the theorem since $\mathcal{A}$ realizes a one-to-one mapping between the solution to (11.24) and (11.26). $\square$

The case where A is depending on x is a little bit more involved. We will need the following definition.

Definition 11.2. A function $A = A(x, u)$ is a Carathéodory function provided

$$\text{(i)} \quad u \mapsto A(x,u) \quad \text{is continuous for a.e. } x \in \Omega, \tag{11.27}$$

$$\text{(ii)} \quad x \mapsto A(x,u) \quad \text{is measurable for every } u \in \mathbb{R}. \tag{11.28}$$

Example 11.2. If $A : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is continuous then A is a Carathéodory function. In a first reading one can only consider this case which is already general enough!

In order to be able to solve (11.21) we will assume that

$$A \text{ is a Carathéodory function} \tag{11.29}$$

$$\exists\, \lambda, \Lambda \text{ such that } 0 < \lambda \le A(x,u) \le \Lambda \quad \text{a.e. } x \in \Omega \quad \forall\, u \in \mathbb{R}. \tag{11.30}$$

Under the assumptions above we have

Theorem 11.5. *Let A satisfy* (11.29), (11.30). *Then for every $f \in H^{-1}(\Omega)$ there exists a solution to* (11.21).

Proof. We use the Schauder fixed point theorem (see the appendix). For $w \in L^2(\Omega)$ we introduce $u = S(w)$ solution to

$$\begin{cases} \displaystyle\int_\Omega A(x,w)\nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{11.31}$$

The existence of u just follows by the Lax–Milgram theorem. Note that by our assumptions on A we have $A(x,w) \in L^\infty(\Omega)$. We will be done if we can show that the mapping S considered as a mapping from $L^2(\Omega)$ into itself has a fixed point.

1. **A priori estimates**

   We take as test function in (11.31) $v = u$. It follows that

$$\int_\Omega A(x,w)|\nabla u|^2 \, dx = \langle f, u \rangle \le |f|_* \big||\nabla u|\big|_{2,\Omega} \tag{11.32}$$

   if we denote by $|f|_*$ the strong dual norm of f when $H_0^1(\Omega)$ is equipped with the norm $\big||\nabla u|\big|_{2,\Omega}$. From (11.32) we deduce by (11.30)

$$\lambda \big||\nabla u|\big|_{2,\Omega}^2 \le |f|_* \big||\nabla u|\big|_{2,\Omega}.$$

   Thus if c_p denotes the constant of the Poincaré inequality we have

$$|u|_{2,\Omega} \le c_p \big||\nabla u|\big|_{2,\Omega} \le \frac{c_p |f|_*}{\lambda}. \tag{11.33}$$

2. **Fixed point argument**

   We introduce B the ball defined by

$$B = \left\{ v \in L^2(\Omega) \;\middle|\; |v|_{2,\Omega} \le \frac{c_p|f|_*}{\lambda} \right\}. \tag{11.34}$$

   B is a closed convex set of $L^2(\Omega)$. Moreover, by (11.33), S is a mapping from B into B and $S(B)$ is relatively compact in B (by the compactness of the embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$). If in addition we can show that S is continuous we will be done – i.e., S will have a fixed point – thanks to the Schauder fixed point theorem (see the appendix).

3. **Continuity of S**

   Let $w_n \in L^2(\Omega)$ be a sequence such that

$$w_n \longrightarrow w \quad \text{in } L^2(\Omega).$$

Set $u_n = S(w_n)$. The estimate (11.33) applies to u_n – since it is independent of w and we have

$$\big|\big|\nabla u_n\big|\big|_{2,\Omega} \le \frac{|f|_*}{\lambda},$$

i.e., u_n is bounded in $H_0^1(\Omega)$ independently of n. Then, up to a subsequence we have

$$u_n \rightharpoonup u_\infty \quad \text{in } H_0^1(\Omega), \qquad u_n \longrightarrow u_\infty \quad \text{in } L^2(\Omega)$$

for some $u_\infty \in H_0^1(\Omega)$. Moreover – up to a subsequence – we can assume that

$$w_n \longrightarrow w \quad \text{a.e. in } \Omega.$$

Due to the assumptions on A this will imply that

$$A(x, w_n) \longrightarrow A(x, w) \quad \text{a.e. in } \Omega.$$

We are then in the position to pass to the limit in the equation

$$\int_\Omega A(x, w_n)\nabla u_n \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega). \tag{11.35}$$

Indeed by the Lebesgue theorem

$$A(x, w_n)\nabla v \longrightarrow A(x, w)\nabla v \quad \text{in } L^2(\Omega).$$

Since $\nabla u_n \rightharpoonup \nabla u_\infty$ in $L^2(\Omega)$ we obtain from (11.35)

$$\int_\Omega A(x, w)\nabla u_\infty \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega),$$

i.e., $u_\infty = u = S(w)$. Since the possible limit of the sequence u_n is uniquely determined, the whole sequence u_n converges toward u in $L^2(\Omega)$. This shows the continuity of S and completes the proof of the theorem. □

Let us address now the issue of uniqueness (see also [6], [7]). For that let us suppose that there exists a function ω such that

$$\omega(t) > 0 \quad \forall\, t > 0, \quad \omega \text{ is continuous, nondecreasing} \tag{11.36}$$

and

$$|A(x, u) - A(x, v)| \le \omega(|u - v|) \quad \text{a.e. } x \in \Omega, \quad \forall\, u \ne v \in \mathbb{R}. \tag{11.37}$$

Moreover let us assume that

$$\int_{0^+} \frac{dt}{\omega(t)} = +\infty. \tag{11.38}$$

Remark 11.5. The condition (11.38) implies that necessarily

$$\omega(t) \to 0 \quad \text{when} \quad t \to 0.$$

If for instance we suppose that $u \mapsto A(x,u)$ is uniformly Lipschitz continuous which means that for some positive constant L we have

$$|A(x,u) - A(x,v)| \le L|u-v| \quad \text{a.e. } x \in \Omega, \quad \forall\, u,v \in \mathbb{R}, \tag{11.39}$$

then clearly

$$\omega(t) = Lt$$

fulfills the conditions (11.36)–(11.38).

Under these conditions we have

Theorem 11.6. *Under the assumptions of Theorem* 11.5 *and if the "modulus of continuity" of* A, ω *satisfies* (11.38) *then the problem* (11.21) *admits a unique solution.*

Proof. Consider u_1, u_2 two solutions to (11.21). For $\varepsilon > 0$ let us define F_ε as

$$F_\varepsilon(x) = \begin{cases} \displaystyle\int_\varepsilon^x \frac{dt}{\omega^2(t)} & \text{if } x \ge \varepsilon, \\ 0 & \text{if } x \le \varepsilon. \end{cases} \tag{11.40}$$

Clearly F_ε is a continuous piecewise C^1-function such that

$$|F'_\varepsilon(x)| \le \frac{1}{\omega^2(\varepsilon)} \quad \forall\, x.$$

Thus it follows from Corollary 2.15 that

$$F_\varepsilon(u_1 - u_2) \in H^1_0(\Omega). \tag{11.41}$$

From (11.21) we have for $i = 1, 2$

$$\int_\Omega A(x,u_i)\nabla u_i \cdot \nabla v\, dx = \langle f, v\rangle \quad \forall\, v \in H^1_0(\Omega).$$

We deduce then that

$$\begin{aligned} &\int_\Omega A(x,u_1)\nabla u_1 \cdot \nabla v\, dx = \int_\Omega A(x,u_2)\nabla u_2 \cdot \nabla v\, dx \\ \iff &\int_\Omega A(x,u_1)\nabla (u_1 - u_2) \cdot \nabla v\, dx = \int_\Omega (A(x,u_2) - A(x,u_1))\nabla u_2 \cdot \nabla v\, dx \\ &\qquad\qquad\qquad\qquad \forall\, v \in H^1_0(\Omega). \end{aligned} \tag{11.42}$$

Taking $v = F_\varepsilon(u_1 - u_2)$ we derive that

$$\begin{aligned}\int_{\{u_1-u_2>\varepsilon\}} & A(x,u_1)\frac{|\nabla(u_1-u_2)|^2}{\omega^2(u_1-u_2)}\,dx \\ &= \int_{\{u_1-u_2>\varepsilon\}} \frac{(A(x,u_2)-A(x,u_1))\nabla u_2\cdot\nabla(u_1-u_2)}{\omega^2(u_1-u_2)}\,dx\end{aligned} \tag{11.43}$$

where $\{u_1-u_2>\varepsilon\} = \{\,x\in\Omega \mid (u_1-u_2)(x)>\varepsilon\,\}$.

From (11.43) we obtain by (11.30), (11.37) and using the Cauchy–Schwarz inequality

$$\begin{aligned}\lambda\int_{\{u_1-u_2>\varepsilon\}} & |\nabla(u_1-u_2)|^2/\omega^2(u_1-u_2)\,dx \\ &\le \int_{\{u_1-u_2>\varepsilon\}} \frac{|A(x,u_2)-A(x,u_1)||\nabla u_2||\nabla(u_1-u_2)|}{\omega^2(u_1-u_2)}\,dx \\ &\le \int_{\{u_1-u_2>\varepsilon\}} \frac{|\nabla u_2||\nabla(u_1-u_2)|}{\omega(u_1-u_2)} \\ &\le \left\{\int_{\{u_1-u_2>\varepsilon\}} |\nabla(u_1-u_2)|^2/\omega^2(u_1-u_2)\,dx\right\}^{\frac12}\left\{\int_{\{u_1-u_2>\varepsilon\}}|\nabla u_2|^2\,dx\right\}^{\frac12}.\end{aligned}$$

From this it follows that

$$\begin{aligned}\int_{\{u_1-u_2>\varepsilon\}} |\nabla(u_1-u_2)|^2/\omega^2(u_1-u_2)\,dx &\le \frac{1}{\lambda^2}\int_{\{u_1-u_2>\varepsilon\}}|\nabla u_2|^2\,dx \\ &\le \frac{1}{\lambda^2}\int_\Omega |\nabla u_2|^2\,dx.\end{aligned} \tag{11.44}$$

Let us then set

$$G_\varepsilon(x) = \begin{cases} \displaystyle\int_\varepsilon^x \frac{dt}{\omega(t)} & \text{if } x\ge\varepsilon, \\ 0 & \text{if } x\le\varepsilon.\end{cases} \tag{11.45}$$

With this definition (11.44) becomes

$$\int_\Omega |\nabla G_\varepsilon(u_1-u_2)|^2\,dx \le \frac{1}{\lambda^2}\int_\Omega |\nabla u_2|^2\,dx. \tag{11.46}$$

Clearly G_ε is a continuous piecewise C^1-function satisfying the assumptions of Corollary 2.15 and we have

$$G_\varepsilon(u_1-u_2)\in H_0^1(\Omega).$$

Using the Poincaré inequality in (11.46) we derive that for some constant c_p we have

$$\int_\Omega |G_\varepsilon(u_1-u_2)|^2\,dx \le \frac{c_p}{\lambda^2}\int_\Omega |\nabla u_2|^2\,dx.$$

Passing to the limit in ε and by Fatou's lemma we get

$$\int_\Omega \underline{\lim} |G_\varepsilon(u_1 - u_2)|^2 \, dx \le \frac{c_p}{\lambda^2} \int_\Omega |\nabla u_2|^2 \, dx.$$

This implies that

$$\underline{\lim}_{\varepsilon \to 0} |G_\varepsilon(u_1 - u_2)|^2 < +\infty \quad \text{a.e. } x.$$

By the definition of G_ε and (11.38) this implies that

$$u_1 - u_2 \le 0 \quad \text{a.e. } x.$$

Reversing the rôle of u_1 and u_2 we conclude that

$$u_1 = u_2.$$

This completes the proof of the theorem. □

The problems that we just addressed are of the so-called local type. That is to say the nonlinear character depends on u at each point. We just saw that somehow uniqueness can be preserved in this case. This is no more the case in the so-called nonlocal case. Now, by Remark 11.5, it is clear that uniqueness holds in the case where A is Lipschitz continuous in u.

11.3 Nonlocal problems

Let us consider the following model problem. We are looking for u solution to

$$\begin{cases} \displaystyle\int_\Omega A\Big(\int_\Omega u(x)\,dx\Big) \nabla u \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall\, v \in H_0^1(\Omega) \\ u \in H_0^1(\Omega). \end{cases} \tag{11.47}$$

f is here a distribution in $H^{-1}(\Omega)$ and A a function which is assumed to satisfy only

$$0 \ne A(z) \quad \forall\, z \in \mathbb{R}. \tag{11.48}$$

Let us introduce φ the solution of the Dirichlet problem

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega, \\ \varphi \in H_0^1(\Omega). \end{cases} \tag{11.49}$$

Moreover let us denote by E the set defined as

$$E = \Big\{ \mu \in \mathbb{R} \;\Big|\; A(\mu)\mu = \int_\Omega \varphi(x)\,dx \Big\}. \tag{11.50}$$

Finally for $\mu \in \mathbb{R}$ denote by u_μ the solution to

$$\begin{cases} \displaystyle\int_\Omega A(\mu)\nabla u_\mu \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \\ u_\mu \in H_0^1(\Omega). \end{cases} \tag{11.51}$$

Then we have

Theorem 11.7. *The mapping*

$$\mu \mapsto u_\mu \tag{11.52}$$

is a one-to-one mapping from the set E onto the set S of the solutions to (11.47).

Proof.

1. Let us show first that $u_\mu \in S$ for every $\mu \in E$.
   Indeed from (11.51) and by the uniqueness of the solution of the Dirichlet problem we have
   $$A(\mu)u_\mu = \varphi,$$
   since $A(\mu)u_\mu$ is a weak solution to (11.49) ($A(\mu)$ is just a constant). Integrating over Ω we derive that
   $$A(\mu)\int_\Omega u_\mu \, dx = \int_\Omega \varphi(x)\, dx = A(\mu)\mu$$
   since $\mu \in E$. It follows that
   $$\mu = \int_\Omega u_\mu \, dx \tag{11.53}$$
   and $u_\mu \in S$, since u_μ satisfies (11.47).
2. Let us show that for any $u \in S$ there exists a unique $\mu \in E$ such that $u = u_\mu$.
   Indeed let $u \in S$. Then from (11.47) and the uniqueness of the solution to (11.49) we have
   $$A\Big(\int_\Omega u \, dx\Big)u = \varphi.$$
   Integrating on Ω we derive
   $$A\Big(\int_\Omega u \, dx\Big)\int_\Omega u \, dx = \int_\Omega \varphi \, dx, \tag{11.54}$$
   i.e., $\mu = \int_\Omega u \, dx \in E$ and $u = u_\mu$. To show that μ is unique suppose that $u_{\mu_1} = u_{\mu_2}$ for two μ's. Then by (11.53), $\mu_1 = \mu_2$. This completes the proof of the theorem. $\square$

As a corollary we have

Theorem 11.8. *Suppose that A is a continuous function such that*

$$\lambda \leq A(z) \leq \Lambda \quad \forall\, z \in \mathbb{R} \tag{11.55}$$

where λ, Λ are positive constants. Then there exists a solution to (11.47). *In addition* (11.47) *might have several solutions – up to a continuum.*

Proof. The solutions of (11.47) are determined by solving the equation

$$A(\mu)\mu = \int_\Omega \varphi\, dx. \tag{11.56}$$

If

$$\int_\Omega \varphi\, dx = 0$$

then $\mu = 0$ is the only solution to (11.56) and thus (11.47) possesses a unique solution. Else μ is necessarily different of 0 and the equation (11.56) is equivalent to

$$A(\mu) = \frac{\int_\Omega \varphi\, dx}{\mu}. \tag{11.57}$$

One has if $\int_\Omega \varphi\, dx > 0$

$$\lim_{\mu \to +\infty} \frac{\int \varphi\, dx}{\mu} = 0, \qquad \lim_{\mu \to 0} \frac{\int \varphi\, dx}{\mu} = +\infty$$

and thus there always exists a μ satisfying (11.57). The case where $\int_\Omega \varphi\, dx < 0$ can be treated the same way.

We represent in the figures below the different possibilities for existence of solutions to (11.57), i.e., to (11.47). For simplicity we consider only the case where

$$\int_\Omega \varphi\, dx > 0.$$

The solution to (11.57) are obtained at the intersection of the graph of A and the hyperbola

$$\mu \longrightarrow \int \varphi\, dx/\mu = h(\mu).$$

□

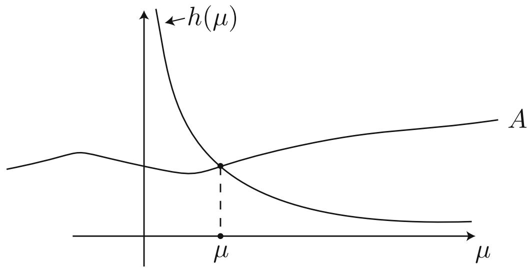


Figure 11.1: A case with unique solution

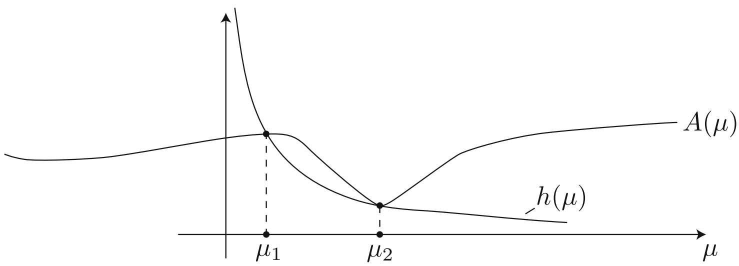


Figure 11.2: A case with two solutions

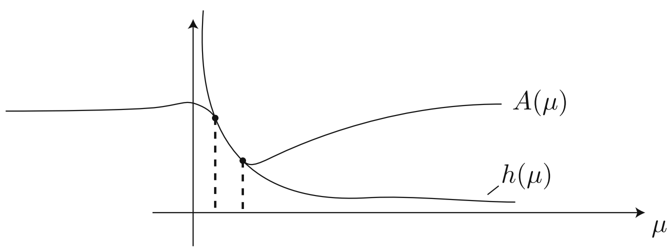


Figure 11.3: A case with a continuum of solutions

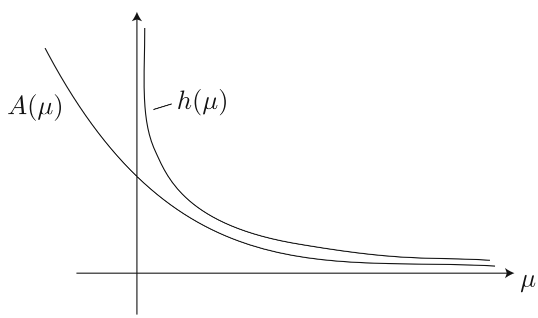


Figure 11.4: A case with no solution

11.4 Variational inequalities

In this section we are going to consider a class of nonlinear problems associated to linear operators. We have encountered our first variational inequality in Chapter 1 when dealing with the projection on a convex set. Thus our first step in the theory is a generalization of the Lax–Milgram theorem when H – the whole space – is replaced by a closed convex set. The notation being the one of Chapter 1 we have

Theorem 11.9. *Let $K \neq \emptyset$ be a closed convex set of a real Hilbert space H. Let $a(u,v)$ be a bilinear form on H satisfying the assumptions of Theorem 1.5 that is to say (1.23), (1.24). Then for every $f \in H^*$ the dual of H there exists a unique u solution to*

$$\begin{cases} u \in K, \\ a(u, v-u) \geq \langle f, v-u \rangle \quad \forall\, v \in K. \end{cases} \tag{11.58}$$

Moreover if a is symmetric u is the unique minimizer over K of the functional

$$J(v) = \frac{1}{2} a(u,v) - \langle f, v \rangle.$$

Proof. 1. *An equivalent formulation.*

As in Chapter 1 we denote by $(\cdot,\cdot)$ the scalar product in H. Then by the Riesz representation theorem (see also Theorem 1.5) for every $u \in H$ there exists a unique $Au \in H$ such that

$$a(u,v) = (Au, v) \quad \forall\, v \in H. \tag{11.59}$$

Similarly the linear form f can be written for some $\tilde{f} \in H$ as

$$\langle f, v \rangle = (\tilde{f}, v) \quad \forall\, v \in H.$$

Then the second equation of (11.58) becomes

$$\begin{aligned} &(Au - \tilde{f}, v-u) \geq 0 \quad \forall\, v \in K \\ \iff \quad &(u, v-u) \geq (u - \varepsilon(Au - \tilde{f}), v-u) \quad \forall\, v \in K \end{aligned} \tag{11.60}$$

for any $\varepsilon > 0$. We will fix ε later on. It is clear that $u \in K$ solves (11.60) iff

$$u = P_K(u - \varepsilon(Au - \tilde{f}))$$

where P_K denotes the projection from H onto K, that is to say if u is a fixed point for the mapping from K into itself defined by

$$u \mapsto P_K(u - \varepsilon(Au - \tilde{f})). \tag{11.61}$$

2. *A fixed point argument.*

First we claim that P_K is a contraction on H that is to say it holds

$$|P_K(u_1) - P_K(u_2)| \leq |u_1 - u_2| \quad \forall\, u_1, u_2 \in H \tag{11.62}$$

if $|\cdot|$ denotes the norm in H. Indeed from the definition of P_K – see Theorem 1.1 – we have

$$\begin{aligned}(P_K(u_1), v - P_K(u_1)) &\geq (u_1, v - P_K(u_1)) \quad \forall\, v \in K, &(11.63)\\ (P_K(u_2), v - P_K(u_2)) &\geq (u_2, v - P_K(u_2)) \quad \forall\, v \in K. &(11.64)\end{aligned}$$

Taking $v = P_K(u_2)$ in (11.63) and $v = P_K(u_1)$ in (11.64) we obtain by adding the two inequalities

$$|P_K(u_1) - P_K(u_2)|^2 \leq (P_K(u_1) - P_K(u_2), u_1 - u_2) \leq |P_K(u_1) - P_K(u_2)|\,|u_1 - u_2|$$

which gives (11.62).

If we denote $T_\varepsilon(u)$ the mapping defined by (11.61) we then have

$$\begin{aligned}|T_\varepsilon(u_1) - T_\varepsilon(u_2)|^2 &\leq |(u_1 - \varepsilon A u_1) - (u_2 - \varepsilon A u_2)|^2 &(11.65)\\ &= |u_1 - u_2|^2 - 2\varepsilon(A(u_1 - u_2), u_1 - u_2) + \varepsilon^2 |A(u_1 - u_2)|^2.\end{aligned}$$

(We recall that A is linear – see the proof of Theorem 1.5.)

Taking $v = u$ in (11.59) we get

$$(Au, u) = a(u, u) \geq \lambda |u|^2 \quad \forall\, u \in H. \tag{11.66}$$

Then taking $v = Au$ we obtain

$$|Au|^2 = a(u, Au) \leq \Lambda |u|\,|Au|,$$

i.e.,

$$|Au| \leq \Lambda |u|. \tag{11.67}$$

Going back to (11.65) and using (11.66), (11.67) with $u = u_1 - u_2$ we derive that

$$\begin{aligned}|T_\varepsilon(u_1) - T_\varepsilon(u_2)|^2 &\leq |u_1 - u_2|^2 - 2\varepsilon\lambda |u_1 - u_2|^2 + \varepsilon^2 \Lambda^2 |u_1 - u_2|^2\\ &= \{1 - \varepsilon(2\lambda - \varepsilon\Lambda^2)\} |u_1 - u_2|^2.\end{aligned}$$

Choosing $\varepsilon < \frac{2\lambda}{\Lambda^2}$ it is clear that T_ε is a contraction and by the Banach fixed point theorem (applied on K) possesses a unique fixed point which is the solution to (11.58).

3. *The symmetric case.*

In this case one proceeds as in Theorem 1.5 that is to say one computes

$$\begin{aligned}
J(v) &= \frac{1}{2}a(v,v) - \langle f, v\rangle = J(v-u+u) \\
&= \frac{1}{2}a(v-u+u, v-u+u) - \langle f, v-u+u\rangle \\
&= \frac{1}{2}a(u,u) - \langle f, u\rangle + a(u, v-u) - \langle f, v-u\rangle + \frac{1}{2}a(v-u, v-u)
\end{aligned}$$

since a is symmetric.

We then derive – see (11.58) –

$$J(v) \geq J(u) + \frac{1}{2}a(v-u, v-u) \geq J(u) + \frac{\lambda}{2}|v-u|^2 \quad \forall\, v \in K.$$

It follows that u is the unique minimizer of J on K. This completes the proof of the theorem. □

Remark 11.6. In the case where $K = V$ is a closed vector subspace of H one recovers the Lax–Milgram theorem. Indeed in (11.58) one can choose $v = u \pm w$ where $w \in V$ which leads to

$$a(u, w) = \langle f, w\rangle \quad \forall\, w \in V.$$

To have applications of the result above one can consider again the framework of Chapter 4 that is to say consider

$$a(u,v) = \int_\Omega A(x)\nabla u \cdot \nabla v + a(x) uv\, dx \quad \forall\, u, v \in H^1(\Omega) \tag{11.68}$$

where A and a satisfy (4.2), (4.3) and (4.6).

If φ is a measurable function in Ω and if

$$K_\varphi = \{\, v \in H_0^1(\Omega) \mid v(x) \geq \varphi(x) \text{ a.e. } x \in \Omega \,\} \tag{11.69}$$

we have

Theorem 11.10. *Let Ω be a bounded open set of $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$ and $a(u,v)$ given by* (11.68)*. If* (4.2), (4.3), (4.6) *hold and if $K_\varphi \neq \emptyset$ then there exists a unique u solution to*

$$\begin{cases} u \in K_\varphi, \\ a(u, v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K_\varphi. \end{cases} \tag{11.70}$$

u is called the solution of a one obstacle problem.

Proof. It is enough to show that K_φ is a closed convex set of $H_0^1(\Omega)$. We leave the details of the proof to the reader. □

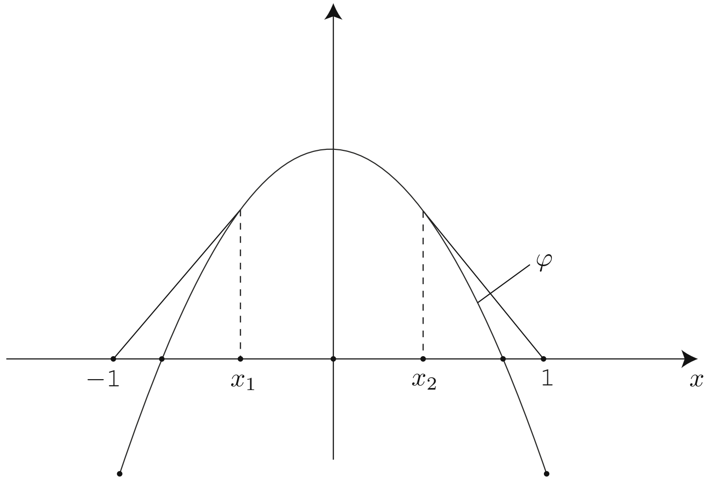


Figure 11.5.

Example. Consider $\Omega = (-1, 1)$, $f = 0$, φ given by Figure 11.5 above, i.e.,

$$\varphi \in C^2(\Omega), \quad \varphi'' \leq 0 \text{ in } \Omega, \quad \varphi(-1), \varphi(1) \leq 0, \ \varphi \text{ non-identically negative.} \tag{11.71}$$

Then the solution to

$$\begin{cases} u \in K_\varphi, \\ \displaystyle\int_\Omega u'(v-u)'\,dx \geq 0 \quad \forall\, v \in K_\varphi, \end{cases} \tag{11.72}$$

is the function u obtained by drawing from the end points of Ω the tangents to the graph of φ. If x_1, x_2 are the tangent points

$$u(x) = \begin{cases} \varphi'(x_1)(x+1) & \text{on } (-1, x_1), \\ \varphi(x) & \text{on } (x_1, x_2), \\ \varphi'(x_2)(x-1) & \text{on } (x_2, 1). \end{cases} \tag{11.73}$$

To see that let $v \in K_\varphi$ and $v_n \in \mathcal{D}(\Omega)$ such that

$$v_n \to v \quad \text{in } H_0^1(\Omega). \tag{11.74}$$

It is clear that u is a C^1-function on Ω. Thus we have

$$\begin{aligned} \int_\Omega u'(v_n-u)'\,dx &= \int_{-1}^{x_1} u'(v_n-u)'\,dx + \int_{x_1}^{x_2} u'(v_n-u)'\,dx + \int_{x_2}^{1} u'(v_n-u)'\,dx \\ &= \varphi'(x_1)(v_n-u)\Big|_{-1}^{x_1} + \int_{x_1}^{x_2} \varphi'(v_n-\varphi)'\,dx + \varphi'(x_2)(v_n-u)\Big|_{x_2}^{1} \end{aligned}$$

$$\begin{aligned}
&= \varphi'(x_1)(v_n - u)(x_1) \\
&\quad + \int_{x_1}^{x_2} \{(\varphi'(v_n - \varphi))' - \varphi''(v_n - \varphi)\}\, dx - \varphi'(x_2)(v_n - u)(x_2) \\
&= - \int_{x_1}^{x_2} \varphi''(v_n - \varphi)\, dx
\end{aligned} \tag{11.75}$$

since $u(x_i) = \varphi(x_i)$, $i = 1, 2$.

Letting $n \to +\infty$ we obtain by (11.71)

$$\int_\Omega u'(v-u)'\, dx \geq - \int_{x_1}^{x_2} \varphi''(v - \varphi)\, dx \geq 0 \quad \forall\, v \in K_\varphi$$

which implies that u is the solution to (11.72) since $u \in K_\varphi$. Note that if $\varphi \leq 0$ then $u = 0$ solves (11.72). For more results on variational inequalities we refer the interested reader to [10], [13], [18]–[20], [23], [24], [26]–[28], [62]–[66], [68], [84], [90], [95].

Exercises

1. Let $f \in H^{-1}(\Omega)$.
   a) Show that there exists $f_n \in L^2(\Omega)$ such that
   $$f_n \longrightarrow f \quad \text{in } H^{-1}(\Omega).$$
   b) Under the assumptions of Theorem 11.2 let u_n be the solution to
   $$\begin{cases} -\Delta u_n + F(u_n) = f_n & \text{in } \Omega, \\ u_n \in H_0^1(\Omega). \end{cases}$$
   Show that u_n is bounded independently of n in $H_0^1(\Omega)$. Deduce that (11.15) admits a unique solution for any $f \in H^{-1}(\Omega)$.
2. Let A be a matrix whose coefficients are Carathéodory functions satisfying the usual ellipticity condition. Namely we suppose that
   $$\begin{aligned} &\lambda|\xi|^2 \leq A(x,u)\xi \cdot \xi \qquad && \text{a.e. } x \in \Omega, \quad \forall\, u \in \mathbb{R}, \quad \forall\, \xi \in \mathbb{R}^n, \\ &|A(x,u)\xi| \leq \Lambda|\xi| && \text{a.e. } x \in \Omega, \quad \forall\, u \in \mathbb{R}, \quad \forall\, \xi \in \mathbb{R}^n, \end{aligned}$$
   Show the existence of a function u solution to
   $$\begin{cases} \displaystyle\int_\Omega A(x,u)\nabla u \cdot \nabla v\, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$
   (Ω is supposed to be bounded, $f \in H^{-1}(\Omega)$.) Give conditions on A in order to have uniqueness.

3. Show that under the conditions of Theorem 11.6 the weak maximum principle holds for u solution to (11.21)
4. Let $g \in L^2(\Omega)$. In the spirit of Theorem 11.7 solve the problem

$$\begin{cases} \int_\Omega A\Big(\int_\Omega gu\,dx\Big)\nabla u \cdot \nabla v\,dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases}$$

5. Prove that if $f \not\equiv 0$ then we can drop the assumption (11.48) in Theorem 11.7 by replacing E by the set

$$E = \Big\{\mu \in \mathbb{R} \;\Big|\; A(\mu) \neq 0,\; A(\mu)\mu = \int_\Omega \varphi(x)\,dx\Big\}.$$

6. Show that when A is a function satisfying

$$0 < \lambda \leq A(z) \leq \Lambda \quad \forall\, z \in \mathbb{R}$$

then existence of a solution might fail in Theorem 11.8 when A fails to be continuous.

7. Under the assumptions of Theorem 11.9 show that (11.58) is equivalent to

$$\begin{cases} u \in K, \\ a(v, v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \end{cases}$$

8. Suppose that $\varphi \in H_0^1(\Omega)$. Under the assumptions of Theorem 11.10 and if u denotes the solution to (11.70) show that the mapping

$$\varphi \mapsto u$$

is continuous from $H_0^1(\Omega)$ into itself.

9. Suppose that we are in the conditions of Theorem 11.10 and let u be the solution to (11.70). Suppose in addition that $u - \varphi$ is continuous. Show that

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) + a(x)u &\geq f \quad \text{in } \Omega, \\ -\operatorname{div}(A(x)\nabla u) + a(x)u &= f \quad \text{in } \{u - \varphi > 0\} \end{aligned}$$

($\{u - \varphi > 0\} = \{\, x \in \Omega \mid u - \varphi(x) > 0\,\}$).

10. Let A be a Carathéodory function satisfying (11.30). Let φ be a measurable function such that

$$K_\varphi = \{\, v \in H_0^1(\Omega) \mid v(x) \geq \varphi(x) \text{ a.e. in } \Omega\,\} \neq \emptyset.$$

Show that there exists a u solution to

$$\begin{cases} u \in K_\varphi, \\ \displaystyle\int_\Omega A(x,u)\nabla u \cdot \nabla(v-u)\,dx \geq \langle f, v-u\rangle \quad \forall\, v \in K_\varphi. \end{cases}$$

(Ω is supposed to be bounded, $f \in H^{-1}(\Omega)$.) Show that when A satisfies (11.36)–(11.38) the solution u is unique.

11. Let Ω be a bounded open set of $\mathbb{R}^n$, α a positive constant. One sets

$$K_1 = \{\, v \in H_0^1(\Omega) \mid |\nabla v(x)| \leq \alpha \text{ a.e. } x \in \Omega \,\},$$
$$K_2 = \left\{ v \in H_0^1(\Omega) \;\middle|\; \int_\Omega |v(x)|\,dx \leq \alpha \right\},$$
$$K_3 = \{\, v \in H_0^1(\Omega) \mid |v(x)| \leq \alpha \text{ a.e. } x \in \Omega \,\}.$$

Show that K_i is a closed convex set of $H_0^1(\Omega)$ for $i = 1, 2, 3$ and that for $f \in H^{-1}(\Omega)$ there exists a unique u_i solution to

$$\begin{cases} u_i \in K_i, \\ \displaystyle\int_\Omega \nabla u_i \cdot \nabla(v-u_i)\,dx \geq \langle f, v-u_i\rangle \quad \forall\, v \in K_i. \end{cases}$$

Let u be the solution to

$$\begin{cases} u \in H_0^1(\Omega), \\ -\Delta u = f \qquad \text{in } \Omega. \end{cases}$$

Show that when $|f|_{H^{-1}}$ is small enough one has $u_2 = u$.

Chapter 12

L^∞-estimates

In general any quantity appearing in nature is finite. Thus a natural question is to derive some conditions on our data that will assure the solution of our elliptic equation to be bounded.

12.1 Some simple cases

First in dimension 1 let us consider u solution to

$$\begin{cases} -(A(x)u')' = f_0 - f_1' & \text{in } \Omega = (\alpha, \beta), \\ u \in H_0^1(\Omega), \end{cases} \tag{12.1}$$

where $f_0, f_1 \in L^2(\Omega)$, $A \in L^\infty(\Omega)$, $0 < \lambda \le A(x) \le \Lambda$ a.e. $x \in \Omega$. It is clear that (12.1) possesses a unique weak solution u. Moreover, integrating the first equation in (12.1) we have

$$-A(x)u' = \int_\alpha^x f_0(s)\,ds - f_1 + c_0$$

where c_0 is some constant. The constant is easily determined by noting that

$$\int_\alpha^\beta u'(s)\,ds = 0.$$

Note that this equality holds for every $u \in \mathcal{D}(\Omega)$ and by density for any $u \in H_0^1(\Omega)$. From

$$-u' = \frac{1}{A(x)} \int_\alpha^x f_0(s)\,ds - \frac{f_1}{A(x)} + \frac{c_0}{A(x)} \tag{12.2}$$

we derive

$$c_0 \int_\alpha^\beta \frac{dx}{A(x)} = \int_\alpha^\beta \frac{f_1(x)}{A(x)}\,dx - \int_\alpha^\beta \frac{1}{A(x)} \int_\alpha^x f_0(s)\,ds\,dx.$$

From this it follows easily if $|\Omega| = \beta - \alpha$ that we have

$$\frac{|\Omega|}{\Lambda}|c_0| \le |c_0| \int_\alpha^\beta \frac{dx}{A(x)} \le \frac{1}{\lambda}|f_1|_{1,\Omega} + \frac{|\Omega|}{\lambda}|f_0|_{1,\Omega}.$$

Thus we have

$$|c_0| \le \frac{\Lambda}{\lambda}\left\{\frac{|f_1|_{1,\Omega}}{|\Omega|} + |f_0|_{1,\Omega}\right\}. \tag{12.3}$$

From (12.2) we derive easily

$$|u'| \le \frac{1}{\lambda}|f_0|_{1,\Omega} + \frac{1}{\lambda}|f_1(x)| + \frac{1}{\lambda}|c_0|$$

and finally

$$\begin{aligned} |u(x)| = \left|\int_\alpha^x u'(s)\,dx\right| &\le \int_\alpha^x |u'(s)|\,dx \\ &\le \frac{|\Omega|}{\lambda}|f_0|_{1,\Omega} + \frac{1}{\lambda}|f_1|_{1,\Omega} + \frac{1}{\lambda}|\Omega||c_0| \\ &\le \frac{|\Omega|}{\lambda}|f_0|_{1,\Omega} + \frac{1}{\lambda}|f_1|_{1,\Omega} + \frac{\Lambda}{\lambda^2}\{|\Omega||f_0|_{1,\Omega} + |f_1|_{1,\Omega}\} \\ &\le \frac{\lambda+\Lambda}{\lambda^2}\{|\Omega||f_0|_{1,\Omega} + |f_1|_{1,\Omega}\}. \end{aligned}$$

We have thus proved

Theorem 12.1. *Under the assumptions above the unique solution to* (12.1) *satisfies*

$$|u|_{\infty,\Omega} \le \frac{\lambda+\Lambda}{\lambda^2}\{|\Omega||f_0|_{1,\Omega} + |f_1|_{1,\Omega}\}. \tag{12.4}$$

Let us now examine some simple cases in higher dimensions. Suppose that

$$\Omega \subset S_\nu = \{\, x \mid (x - x_0)\cdot\nu \in (-a,a)\,\}$$

ν being a unit vector. In other words Ω is included in a strip of size $2a$ and thus is bounded in one direction. For $f \in L^\infty(\Omega)\cap H^{-1}(\Omega)$ there exists a unique solution u to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{12.5}$$

Moreover we have

Theorem 12.2. *Under the assumptions above the solution u to* (12.5) *satisfies*

$$|u|_{\infty,\Omega} \le \frac{a^2}{2}|f|_{\infty,\Omega}. \tag{12.6}$$

Proof. Let us denote by u_1 the solution to

$$\begin{cases} -\Delta u_1 = 1 & \text{in } \Omega, \\ u_1 \in H_0^1(\Omega). \end{cases}$$

We clearly have

$$-\Delta(u_1|f|_{\infty,\Omega}) = |f|_{\infty,\Omega} \geq f = -\Delta u$$

and by the weak maximum principle

$$u \leq u_1|f|_{\infty,\Omega}.$$

Since $-u$ is the solution to (12.5) corresponding to $-f$ it also follows that

$$-u \leq u_1|f|_{\infty,\Omega}$$

and thus

$$|u| \leq u_1|f|_{\infty,\Omega}. \tag{12.7}$$

If $g = \frac{1}{2}\{a^2 - ((x - x_0) \cdot \nu)^2\}$ we have

$$g \geq 0 \text{ on } \partial\Omega \qquad -\Delta g = |\nu|^2 = 1 = -\Delta u_1 \geq -\Delta 0.$$

By the weak maximum principle again it follows that

$$0 \leq u_1 \leq g \leq \frac{a^2}{2}. \tag{12.8}$$

The proof of the theorem follows by combining (12.7) and (12.8). □

We consider now the problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{12.9}$$

Ω is an arbitrary domain in $\mathbb{R}^n$, A is an elliptic matrix satisfying (4.2), (4.3) and a is a bounded function such that

$$0 < \lambda \leq a(x) \quad \text{a.e. } x \in \Omega. \tag{12.10}$$

For

$$f \in L^\infty(\Omega) \cap H^{-1}(\Omega)$$

by Theorem 4.1 there exists a unique solution u to (12.9). Moreover we have

Theorem 12.3. *Under the assumptions above $u \in L^\infty(\Omega)$ and*

$$|u|_\infty \leq \frac{|f|_\infty}{\lambda}. \tag{12.11}$$

Proof. One remarks setting

$$L(u) = -\operatorname{div}(A(x)\nabla u) + a(x)u$$

that

$$\begin{aligned} L\Big(\frac{|f|_\infty}{\lambda}\Big) &= a\frac{|f|_\infty}{\lambda} \geq f = Lu \quad \text{in } \Omega, & \frac{|f|_\infty}{\lambda} &\geq u \quad \text{on } \partial\Omega \\ L\Big(\frac{-|f|_\infty}{\lambda}\Big) &= -a\frac{|f|_\infty}{\lambda} \leq f = Lu \quad \text{in } \Omega, & \frac{-|f|_\infty}{\lambda} &\leq u \quad \text{on } \partial\Omega. \end{aligned}$$

Thus by the weak maximum principle we derive

$$-\frac{|f|_\infty}{\lambda} \leq u \leq \frac{|f|_\infty}{\lambda} \quad \text{in } \Omega$$

which leads to (12.11). This completes the proof of the theorem. □

12.2 A more involved estimate

We would like to present in this section a L^∞-estimate due to Stampacchia ([64]). We consider u a weak solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u \leq f_0 + \sum_{i=1}^n \partial_{x_i} f_i & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega, \end{cases} \tag{12.12}$$

where Ω is a domain of finite measure in $\mathbb{R}^n$, $n \geq 2$ (see also Definition 4.2). We will suppose that A satisfies (4.2), (4.3) and that

$$0 \leq a(x) \leq \Lambda \quad \text{a.e. in } \Omega. \tag{12.13}$$

Moreover we will assume that

$$f_0 \in L^q(\Omega), \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{n}, \quad f_i \in L^p(\Omega) \quad \forall\, i = 1, \dots, n \text{ for some } p > n \geq 2. \tag{12.14}$$

Then we have

Theorem 12.4 (Stampacchia). *Under the assumption above $u^+ \in L^\infty(\Omega)$ and*

$$u^+ \leq C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}|\Omega|^{\frac{1}{n}-\frac{1}{p}} \tag{12.15}$$

where $C = C(\lambda, n, p)$ is a constant independent of u, $|f|^2 = \sum_{i=1}^n f_i^2$, $|\Omega|$ is the measure of Ω.

Proof. For $k > 0$ we denote by F_k the function

$$F_k(z) = (z - k)^+. \tag{12.16}$$

One should remark that $F_k(z) = 0$ on $z \leq k$ and is equal to $z - k$ for $z > k$. Note also that $F_k(u) = F_k(u^+)$ and therefore $F_k(u) \in H_0^1(\Omega)$. Thus taking as test function in the weak formulation of (12.12) $F_k(u)$ we get

$$\int_\Omega A(x)\nabla F_k(u) \cdot \nabla F_k(u) + a(x)uF_k(u) \leq \int_\Omega f_0 F_k(u) - f_i \partial_{x_i} F_k(u)\, dx. \tag{12.17}$$

Using the ellipticity of the matrix A and the Hölder inequality it comes by (12.13)

$$\lambda \int_\Omega |\nabla F_k(u)|^2\, dx \leq |f_0|_{q,\Omega} |F_k(u)|_{q',\Omega} + \big||f|\big|_{p,\Omega} \big||\nabla F_k(u)|\big|_{p',\Omega}$$

where q', p' denote the conjugate exponents of q, p respectively. Recall that $p' = \frac{p}{p-1}$ and $\frac{1}{q'} = 1 - \frac{1}{q} = 1 - \frac{1}{p} - \frac{1}{n} = \frac{1}{p'} - \frac{1}{n} = \frac{1}{p'^*}$. For a real number $0 < r < n$ the Sobolev exponent for r, r^* is defined as

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{n}. \tag{12.18}$$

Thus we obtain

$$\lambda \big||\nabla F_k(u)|\big|_{2,\Omega}^2 \leq |f_0|_{q,\Omega} |F_k(u)|_{p'^*,\Omega} + \big||f|\big|_{p,\Omega} \big||\nabla F_k(u)|\big|_{p',\Omega}$$

where p'^* is the Sobolev exponent of $p' < 2 \leq n$. It follows – for a constant $C = C(\lambda, n, p)$ independent of Ω – that we have by the Sobolev–Gagliardo–Nirenberg inequality – (see below the next section)

$$\big||\nabla F_k(u)|\big|_{2,\Omega}^2 \leq C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\} \big||\nabla F_k(u)|\big|_{p',\Omega}. \tag{12.19}$$

Let us denote by $A(k)$ the set defined as

$$A(k) = \{\, x \in \Omega \mid u(x) > k \,\}. \tag{12.20}$$

By Hölder's inequality we have

$$\big||\nabla F_k(u)|\big|_{p',\Omega}^{p'} = \int_{A(k)} |\nabla u|^{p'}\, dx \leq \left(\int_{A(k)} |\nabla u|^2\, dx \right)^{\frac{p'}{2}} |A(k)|^{1-\frac{p'}{2}},$$

where $|\cdot|$ denotes the measure of sets. It follows that

$$\big||\nabla F_k(u)|\big|_{p',\Omega}^2 \leq \big||\nabla F_k(u)|\big|_{2,\Omega}^2 |A(k)|^{\frac{2}{p'}-1}$$

and from (12.19)

$$\big||\nabla F_k(u)|\big|_{p',\Omega} \leq C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\} |A(k)|^{\frac{2}{p'}-1}.$$

Using again the Sobolev embedding theorem we get

$$|F_k(u)|_{p'^*,\Omega} \le C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}|A(k)|^{\frac{2}{p'}-1}, \tag{12.21}$$

where $C = C(\lambda, p, n)$. For $h > k$ we clearly have

$$A(h) \subset A(k) \tag{12.22}$$

and from (12.21) we derive

$$\begin{aligned}(h-k)|A(h)|^{\frac{1}{p'^*}} &\le \left(\int_{A(h)} \{(u-k)^+\}^{p'^*}\,dx\right)^{\frac{1}{p'^*}} \\ &\le |F_k(u)|_{p'^*,\Omega} \le C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}|A(k)|^{\frac{2}{p'}-1}\end{aligned}$$

which can be written as

$$|A(h)| \le \{C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}/(h-k)\}^{p'^*}|A(k)|^{p'^*\left(\frac{2}{p'}-1\right)}. \tag{12.23}$$

Then the result will follow from the following lemma.

Lemma 12.5. *Let $\Phi : [0, +\infty) \to \mathbb{R}^+$ be a nonincreasing function such that*

$$\Phi(h) \le \left(\frac{C}{h-k}\right)^{\alpha} \Phi(k)^{\beta} \quad h > k \tag{12.24}$$

where C, α, β are positive constants. Then if $\beta > 1$

$$\Phi(d) = 0 \quad \textit{for } d = C\Phi(0)^{\frac{\beta-1}{\alpha}} \cdot 2^{\frac{\beta}{\beta-1}}.$$

Proof. Let $h_n = d - \frac{d}{2^n}$, $n \in \mathbb{N}$. By (12.24) we have

$$\Phi(h_{n+1}) \le \left(\frac{C}{h_{n+1}-h_n}\right)^{\alpha} \Phi(h_n)^{\beta} = \left(\frac{C}{d}\right)^{\alpha} 2^{(n+1)\alpha}\Phi(h_n)^{\beta}. \tag{12.25}$$

By induction we claim that

$$\Phi(h_n) \le \Phi(0)2^{\frac{n\alpha}{1-\beta}}. \tag{12.26}$$

If $n = 0$ (12.26) is clear. Assuming it holds for n we get from (12.25)

$$\begin{aligned}\Phi(h_{n+1}) &\le \left(\frac{C}{d}\right)^{\alpha} 2^{(n+1)\alpha}\Phi(0)^{\beta}2^{\frac{n\alpha\beta}{1-\beta}} \\ &= (\Phi(0)^{\frac{1-\beta}{\alpha}}2^{\frac{\beta}{1-\beta}})^{\alpha}2^{(n+1)\alpha}\Phi(0)^{\beta}2^{\frac{n\alpha\beta}{1-\beta}} \quad \text{(by definition of } d\text{)} \\ &= \Phi(0)2^{\frac{(n+1)\alpha}{1-\beta}}.\end{aligned}$$

From (12.25) we then derive

$$0 \le \Phi(d) \le \Phi(h_n) \le \Phi(0)2^{\frac{n\alpha}{1-\beta}} \to 0 \quad \text{when } n \to 0.$$

This completes the proof of the lemma. $\square$

End of the proof of Theorem 12.4. We have $\alpha = p'^*$, $\beta = p'^*\left(\frac{2}{p'} - 1\right)$, i.e., since $\frac{1}{p'^*} = \frac{1}{p'} - \frac{1}{n} = 1 - \frac{1}{p} - \frac{1}{n}$

$$\beta = \left(2 - \frac{2}{p} - 1\right) \Big/ \left(1 - \frac{1}{p} - \frac{1}{n}\right) = \left(1 - \frac{2}{p}\right) \Big/ \left(1 - \frac{1}{p} - \frac{1}{n}\right) > 1$$

since $\frac{1}{p} < \frac{1}{n}$. Moreover

$$\frac{\beta - 1}{\alpha} = \frac{\beta}{\alpha} - \frac{1}{\alpha} = \frac{2}{p'} - 1 - \frac{1}{p'^*} = \frac{1}{n} - \frac{1}{p},$$

$$\frac{\beta - 1}{\beta} = 1 - \frac{1}{\beta} = 1 - \frac{1 - \frac{1}{p} - \frac{1}{n}}{1 - \frac{2}{p}} = \frac{\frac{1}{n} - \frac{1}{p}}{1 - \frac{2}{p}}.$$

From (12.23) and Lemma 12.5 we then derive that

$$\begin{aligned} u \le d &\le C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}\Phi(0)^{\frac{1}{n} - \frac{1}{p}} \\ &\le C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}|\Omega|^{\frac{1}{n} - \frac{1}{p}}. \end{aligned}$$

This completes the proof of the theorem. □

Remark 12.1. In the estimate (12.15) one can replace f_0 by f_0^+ since in (12.17) the term $-\int_\Omega f_0^- F_k(u)\,dx$ is nonpositive.

Under the assumptions of Theorem 12.4 as a corollary we have

Corollary 12.6. *Under the assumptions* (12.13) *and* (12.14) *if u is the weak solution to*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u = f_0 + \displaystyle\sum_{i=1}^{n} \partial_{x_i} f_i \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{12.27}$$

one has $u \in L^\infty(\Omega)$ and an estimate

$$|u|_{\infty,\Omega} \le C\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\}|\Omega|^{\frac{1}{n} - \frac{1}{p}} \tag{12.28}$$

for some constant $C = C(\lambda, n, p)$.

Proof. It is enough to apply Theorem 12.4 to u and $-u$. □

12.3 The Sobolev–Gagliardo–Nirenberg inequality

If $p < n$ and if p^* denotes the Sobolev exponent that we encountered earlier given by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \tag{12.29}$$

we have

Theorem 12.7 (Sobolev–Gagliardo–Nirenberg). *Let $1 \leq p < n$. There exists a constant $C = C(p,n)$ such that*

$$|v|_{p^*,\mathbb{R}^n} \leq C\big|\,|\nabla v|\,\big|_{p,\mathbb{R}^n} \quad \forall\, v \in C_c^1(\mathbb{R}^n). \tag{12.30}$$

($C_c^1(\mathbb{R}^n)$ denotes the space of C^1-functions with compact support.)

Proof. 1. *The case $p = 1$.*

In this case the Sobolev exponent 1^* is given by

$$1^* = \frac{n}{n-1}.$$

Let $v \in C_c^1(\mathbb{R}^n)$. Since v has compact support we have

$$v(x) = \int_{-\infty}^{x_i} \partial_{x_i} v(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)\,dt$$

and thus

$$|v(x)| \leq \int_{-\infty}^{+\infty} |\partial_{x_i} v(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_n)|\,dt := f_i(\hat{x}_i). \tag{12.31}$$

We have denoted by $\hat{x}_i$ the point $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ and f_i the function above. From (12.31) we then derive

$$\int_{\mathbb{R}^n} |v(x)|^{\frac{n}{n-1}}\,dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^n f_i(\hat{x}_i)^{\frac{1}{n-1}}\,dx. \tag{12.32}$$

Let us suppose that we have established the following lemma:

Lemma 12.8. *Let $f_i(\hat{x}_i) \in L^{n-1}(\mathbb{R}^{n-1})$ for $i = 1,\dots,n$ then $\prod_{i=1}^n f_i(\hat{x}_i) \in L^1(\mathbb{R}^n)$ and we have*

$$\Big|\prod_{i=1}^n f_i(\hat{x}_i)\Big|_{1,\mathbb{R}^n} \leq \prod_{i=1}^n |f_i(\hat{x}_i)|_{n-1,\mathbb{R}^{n-1}}.$$

($|f_i(\hat{x}_i)|_{n-1,\mathbb{R}^{n-1}}$ denotes the $L^{n-1}(\mathbb{R}^{n-1})$-norm of f_i, the measure of integration being $dx_1 \dots \widehat{dx_i} \dots dx_n$ where $\widehat{dx_i}$ is dropped in the product of measures.)

From (12.32) and this lemma we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |v(x)|^{\frac{n}{n-1}}\,dx &\leq \prod_{i=1}^n \Big\{\int_{\mathbb{R}^{n-1}} f_i(\hat{x}_i)\,dx_1 \dots \widehat{dx_i} \dots dx_n\Big\}^{\frac{1}{n-1}} \\
&= \prod_{i=1}^n \Big\{\int_{\mathbb{R}^n} |\partial_{x_i} v|\,dx\Big\}^{\frac{1}{n-1}} \\
&\leq \Big(\int_{\mathbb{R}^n} |\nabla v|\,dx\Big)^{\frac{n}{n-1}},
\end{aligned}$$

which is exactly (12.30) for $p = 1$. The constant C being here 1.

2. *The case of a general* $1 < p < n$.

If $p > 1$ we have

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} < 1 - \frac{1}{n} = \frac{1}{1^*},$$

i.e., $p^* > 1^*$. It follows that for $v \in C_c^1(\mathbb{R}^n)$, $|v|^{\frac{p^*}{1^*}}$ belongs to $C_c^1(\mathbb{R}^n)$ and from Part 1 above we have

$$\left(\int_{\mathbb{R}^n} |v|^{p^*}\,dx\right)^{\frac{1}{1^*}} \le \int_{\mathbb{R}^n} |\nabla\{|v|^{\frac{p^*}{1^*}}\}|\,dx. \tag{12.33}$$

An easy computation gives

$$\nabla|v|^{\frac{p^*}{1^*}} = \frac{p^*}{1^*}|v|^{\frac{p^*}{1^*}-1}\operatorname{sign} v\nabla v$$

and thus from (12.33) and the Hölder inequality we derive

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |v|^{p^*}\,dx\right)^{\frac{1}{1^*}} &\le \frac{p^*}{1^*}\int_{\mathbb{R}^n} |v|^{\frac{p^*}{1^*}-1}|\nabla v|\,dx\\ &\le \frac{p^*}{1^*}\left(\int_{\mathbb{R}^n} |v|^{\left(\frac{p^*}{1^*}-1\right)p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n} |\nabla v|^p\,dx\right)^{\frac{1}{p}}.\end{aligned} \tag{12.34}$$

By an easy computation we obtain

$$\frac{p^*}{1^*} - 1 = \frac{(1-\frac{1}{n}) - \left(\frac{1}{p}-\frac{1}{n}\right)}{\frac{1}{p}-\frac{1}{n}} = \frac{p^*}{p'}, \qquad \frac{1}{1^*} - \frac{1}{p'} = \left(1-\frac{1}{n}\right) - \left(1-\frac{1}{p}\right) = \frac{1}{p^*}.$$

Thus (12.34) becomes

$$\left(\int_{\mathbb{R}^n} |v|^{p^*}\,dx\right)^{\frac{1}{1^*}} \le \frac{p^*}{1^*}\left(\int_{\mathbb{R}^n} |v|^{p^*}\,dx\right)^{\frac{1}{p'}}\left(\int_{\mathbb{R}^n} |\nabla v|^p\,dx\right)^{\frac{1}{p}}$$

and thus

$$|v|_{p^*,\mathbb{R}^n} \le \frac{p^*}{1^*}\big||\nabla v|\big|_{p,\mathbb{R}^n}.$$

This completes the proof of (12.30) with $C = \frac{p^*}{1^*} = \frac{p(n-1)}{n-p}$. □

3. *The proof of Lemma* 12.8.

We proceed by induction on n. For $n = 2$, $n - 1 = 1$ the inequality reduces to

$$|f_1(\hat{x}_1)f_2(\hat{x}_2)|_{1,\mathbb{R}^2} = \int_{\mathbb{R}^2} |f_1(x_2)||f_2(x_1)|\,dx_1\,dx_2 = |f_1|_{1,\mathbb{R}}|f_2|_{1,\mathbb{R}}.$$

Thus let us assume that the lemma holds for $n-1$. We denote by x_i in the integrals an integration in x_i. Thus we have

$$\int_{\mathbb{R}^n} |f_1(\hat{x}_1)| \dots |f_n(\hat{x}_n)|\, dx$$
$$= \int_{\mathbb{R}^{n-1}} |f_n(\hat{x}_n)| \int_{x_n} |f_1(\hat{x}_1)| \dots |f_{n-1}(\hat{x}_{n-1})|\, dx_n\, dx_1 \dots dx_{n-1}.$$

Using the generalized Hölder inequality (see the exercises in Chapter 2) we get

$$\int_{\mathbb{R}^n} |f_1(\hat{x}_1)| \dots |f_n(\hat{x}_n)|\, dx$$
$$\leq \int_{\mathbb{R}^{n-1}} |f_n(\hat{x}_n)| \prod_{i=1}^{n-1} \Big(\int_{x_n} |f_i(\hat{x}_i)|^{n-1}\, dx_n \Big)^{\frac{1}{n-1}} dx_1 \dots\, dx_{n-1}.$$

Using in this last integral the Hölder inequality with power $n-1$ and $\frac{n-1}{n-2}$ we obtain

$$\int_{\mathbb{R}^n} |f_1(\hat{x}_1)| \dots |f_n(\hat{x}_n)|\, dx$$
$$\leq \Big(\int_{\mathbb{R}^{n-1}} |f_n(\hat{x}_n)|^{n-1}\, dx_1 \dots dx_{n-1} \Big)^{\frac{1}{n-1}}$$
$$\times \Big(\int_{\mathbb{R}^{n-1}} \prod_{i=1}^{n-1} \Big(\int_{x_n} |f_i(\hat{x}_i)|^{n-1}\, dx_n \Big)^{\frac{1}{n-2}} dx_1 \dots dx_{n-1} \Big)^{\frac{n-2}{n-1}}.$$

Applying the induction assumption in this last integral we derive

$$\int_{\mathbb{R}^n} |f_1(\hat{x}_1)| \dots |f_n(x_n)|\, dx$$
$$\leq \Big(\int_{\mathbb{R}^{n-1}} |f_n(\hat{x}_n)|^{n-1}\, dx_1 \dots\, dx_{n-1} \Big)^{\frac{1}{n-1}}$$
$$\times \prod_{i=1}^{n-1} \Big(\int_{\mathbb{R}^{n-1}} |f_i(\hat{x}_i)|^{n-1}\, dx_1, \dots, \widehat{dx_i} \dots dx_n \Big)^{\frac{1}{n-1}}$$

which completes the proof of the lemma. □

Remark 12.2. The constant $C = C(p,n) = \frac{p(n-1)}{n-p}$ that we obtained is not the sharpest one. We refer to [91] for this kind of issue.

Remark 12.3. In Theorem 12.4 we used (12.30) with p given by $p' < 2 \leq n$ and for a function $v \in H_0^1(\Omega)$. It is clear that $C_c^1(\Omega)$ as $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$. For $p < 2$ the inequality (12.30) holds for $v \in H_0^1(\Omega)$ since for $p < 2$ we have $L^2(\Omega) \hookrightarrow L^p(\Omega)$ (recall that the measure of Ω is finite).

12.4 The maximum principle on small domains

First let us see how the estimates of Paragraph 12.2 can be combined with the maximum principle.

Definition 12.1. For $u \in H^1(\Omega)$ we denote by $\operatorname*{Max}_{\partial\Omega} u$ the smallest constant k such that

$$u \leq k \quad \text{on } \partial\Omega \tag{12.35}$$

in the sense of Definition 4.2. (In the case where no such constant exists $\operatorname*{Max}_{\partial\Omega} u = +\infty$.)

Remark 12.4. One can have $k = -\infty$. If $k := \inf\{\tilde{k};\ \tilde{k} \text{ satisfies (12.35)}\} \in \mathbb{R}$, then k also satisfies (12.35), see Exercise 4.

Then we have

Theorem 12.9. *Under the assumptions of Theorem* 12.4 *let* $u \in H^1(\Omega)$ *satisfying*

$$-\operatorname{div}(A(x)\nabla u) + a(x)u \leq f_0 + \sum_{i=1}^{n} \partial_{x_i} f_i \quad \text{in } \Omega \tag{12.36}$$

then for the constant $C = C(\lambda, n, p)$ *of Theorem* 12.4 *we have*

$$u^+ \leq \operatorname*{Max}_{\partial\Omega} u^+ + C\big\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\big\}|\Omega|^{\frac{1}{n}-\frac{1}{p}}. \tag{12.37}$$

Proof. If $k = \operatorname{Max}_{\partial\Omega} u^+ = +\infty$ there is nothing to prove. Else if $k = \operatorname{Max}_{\partial\Omega} u^+ < +\infty$ from (12.35) we derive

$$-\operatorname{div}(A(x)\nabla(u-k)) + a(x)(u-k) \leq f_0 + \sum_{i=1}^{n} \partial_{x_i} f_i - a(x)k \leq f_0 + \sum_{i=1}^{n} \partial_{x_i} f_i$$

(note that $a(x)$, $k \geq 0$). Moreover

$$(u-k)^+ = (u^+ - k)^+ \in H_0^1(\Omega)$$

(by definition of k). Applying Theorem 12.4 we derive

$$u - k \leq C\big\{|f_0|_{q,\Omega} + \big||f|\big|_{p,\Omega}\big\}|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

which gives (12.35). $\square$

We also have

Theorem 12.10 (Maximum Principle for small domains). *Suppose that* $A = A(x)$ *satisfies* (4.2) *and* (4.3). *Suppose that* $a \in L^\infty(\Omega)$ *satisfies for some positive constant* a_0

$$-a_0 \leq a \quad \text{a.e. in } \Omega. \tag{12.38}$$

Let $u \in H^1(\Omega) \cap L^q(\Omega)$, $q > \frac{n}{2}$ solution to

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + a(x)u \le 0 & \text{in } \Omega, \\ u \le 0 & \text{on } \partial\Omega. \end{cases} \tag{12.39}$$

Denote by $C = C(\lambda, n, p)$, $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ the constant of Theorem 12.4. *Then if*

$$Ca_0|\Omega|^{\frac{2}{n}} < 1 \tag{12.40}$$

one has

$$u \le 0 \quad \text{in } \Omega.$$

(*Note that a is not supposed to be nonnegative anymore.*)

Proof. If $a = a^+ - a^-$ since $a^- = \operatorname{Max}\{0, -a\}$ one has $0 \le a^- \le a_0$. Moreover from (12.39) we derive

$$-\operatorname{div}(A(x)\nabla u) + a^+u \le a^-u = a^-u^+ - a^-u^- \le a^-u^+.$$

It follows from Theorem 12.9 applied with $f_0 = a^-u^+$ that

$$|u^+|_{\infty,\Omega} \le C|a^-u^+|_{q,\Omega}|\Omega|^{\frac{1}{n}-\frac{1}{p}}$$

($|\cdot|_{\infty,\Omega}$ denotes the $L^\infty(\Omega)$-norm). We then obtain

$$\begin{aligned} |u^+|_{\infty,\Omega} &\le Ca_0|u^+|_{\infty,\Omega}|\Omega|^{\frac{1}{q}}|\Omega|^{\frac{1}{n}-\frac{1}{p}} \\ &= Ca_0|\Omega|^{\frac{2}{n}}|u^+|_{\infty,\Omega}. \end{aligned}$$

From (12.40) it follows that $u^+ = 0$. This completes the proof of the theorem. □

Remark 12.5. By Theorem 12.7 we have $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ with

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}.$$

Thus when $2^* > \frac{n}{2}$ we just need to assume $u^+ \in H_0^1(\Omega)$ in Theorem 12.10. This is the case when $n \le 5$. Also in the case of equality in the equations (12.39) u belongs to $L^q(\Omega)$ – see [89]. To remove the assumption in $u \in L^q(\Omega)$ we refer the reader to the exercises.

Exercises

1. Reproducing the proof of Theorem 12.1 estimate the L^∞-norm of the solution to

$$\begin{cases} -(A(x)u')' + a(x)u = f_0 - f_1' & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

2. Let $u \in H^1(\Omega)$ solution to (12.39).
   (i) Show that
   $$-\operatorname{div}(A(x)\nabla u) + a^+ u \le a_0 u^+ \quad \text{in } \Omega.$$
   (ii) Deduce that
   $$\big||\nabla u^+|\big|^2_{2,\Omega} \le \frac{a_0}{\lambda}|u^+|^2_{2,\Omega}.$$
   (iii) Using Exercise 7(iii) in Chapter 8 show that when
   $$\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{2}{n}}\frac{a_0}{\lambda} < 1$$
   then the weak maximum principle holds – i.e., one has $u \le 0$ in Ω.
3. Let $u \in H^1(\Omega)$ such that $u \ge 0$ a.e. in Ω and $k > 0$ a constant. Prove that $u + k \notin H^1_0(\Omega)$, hence $\operatorname{Max}_{\partial\Omega} u \ge 0$.
4. Let $u \in H^1(\Omega)$. Assume that $k := \inf\{\tilde{k};\ \tilde{k} \text{ satisfies (12.36)}\} \in \mathbb{R}$. Prove that k also satisfies (12.35) (then k is a minimum).
5. Prove Lemma 12.5 without assuming that the function ϕ is nonincreasing.

Chapter 13

Linear Elliptic Systems

13.1 The general framework

Sometimes, in many physical situations, one has not only to look for a scalar u but for a vector $\mathbf{u} = (u^1, \dots, u^m)$. (In what follows a bold letter will always denote a vector.) This could be a position in space, a displacement, a velocity… So one is in need to introduce systems of equations. The simplest one is of course the one consisting in m copies of the Dirichlet problem, that is to say

$$\begin{cases} -\Delta u^1 = f^1 & \text{in } \Omega, \\ \cdots\cdots\cdots\cdots & \\ -\Delta u^m = f^m & \text{in } \Omega, \\ \mathbf{u} = (u^1, \dots, u^m) = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.1}$$

This is a system of m equations with m unknowns. If for instance $f \in \mathbb{L}^2(\Omega) := (L^2(\Omega))^m$ and since we have here a system of m Dirichlet problems it is clear that it possesses a unique solution (just apply the theorem of Lax–Milgram to each equation). However it is possible to write it down in a more condensed form. Indeed choosing $\mathbf{v} = (v^1, \dots, v^m) \in \mathbb{H}_0^1(\Omega) := (H_0^1(\Omega))^m$ multiplying each equation by v^i, integrating in Ω (i.e., considering the weak form of each equation) and summing up in i we see that $\mathbf{u}$ satisfies

$$\begin{cases} \displaystyle\sum_{i=1}^m \int_\Omega \nabla u^i \cdot \nabla v^i \, dx = \sum_{i=1}^m \int_\Omega f^i v^i \, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\Omega), \\ \mathbf{u} \in \mathbb{H}_0^1(\Omega). \end{cases} \tag{13.2}$$

With the convention of repeated indices and writing with a dot the scalar product we are ending up with

$$\begin{cases} \displaystyle\int_\Omega \nabla u^i \cdot \nabla v^i \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\Omega), \\ \mathbf{u} \in \mathbb{H}_0^1(\Omega). \end{cases} \tag{13.3}$$

We have in fact

$$\nabla u^i \cdot \nabla v^i = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \partial_{x_\alpha} u^i \partial_{x_\alpha} v^i \tag{13.4}$$

and if $\nabla \mathbf{u}$ denotes the Jacobian matrix of u, i.e., the $m \times n$ matrix

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_{x_1} u^1 & ,\dots, & \partial_{x_n} u^1 \\ \vdots & & \vdots \\ \partial_{x_1} u^m & ,\dots, & \partial_{x_n} u^m \end{pmatrix} \tag{13.5}$$

then (13.4) is nothing but the Euclidean scalar product of the matrices $\nabla \mathbf{u}$ and $\nabla \mathbf{v}$ considered as vectors in $\mathbb{R}^{m\times n}$. Thus our initial system takes finally the following condensed form:

$$\begin{cases} \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\Omega), \\ \mathbf{u} \in \mathbb{H}_0^1(\Omega). \end{cases} \tag{13.6}$$

We just have shown that the solution of (13.1) – that is to say $\mathbf{u} = (u^1, \dots, u^m)$ – satisfies (13.6). Conversely if $\mathbf{u} = (u^1, \dots, u^m)$ is solution to (13.6) choosing $\mathbf{v} = (0, \dots, v^i, 0, \dots, 0)$ in (13.6) or (13.3) we obtain for every i and with no summation in i

$$\begin{cases} \displaystyle\int_\Omega \nabla u^i \cdot \nabla v^i\, dx = \int_\Omega f^i v^i\, dx \quad \forall\, v^i \in H_0^1(\Omega) \ \ \forall\, i = 1, \dots, m, \\ u^i \in H_0^1(\Omega) \qquad\qquad\qquad\qquad\quad \forall\, i = 1, \dots, m. \end{cases} \tag{13.7}$$

We see then that the different components of $\mathbf{u}$ are solution to (13.1) and thus (13.1) and (13.6) are perfectly equivalent. If we introduce the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx, \tag{13.8}$$

we see that we are again in the framework of the Lax–Milgram theorem but with this time $\mathbf{u}$, $\mathbf{v}$ vectors, i.e., belonging to a product of Hilbert spaces.

One should now notice that (13.1) is a very peculiar system in the sense that the equations are uncoupled – i.e., the k^{th} one only depends on the k^{th} component of $\mathbf{u}$.

Some more complicated systems can be investigated. We copy here our analysis of Chapter 4. Namely we replace $A(x)$, a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$, ($\mathbb{R}^n$ was the space where ∇u was living) by $\mathcal{A}(x)$ a linear mapping from $\mathbb{R}^{m\times n}$ into $\mathbb{R}^{m\times n}$ where we denote by $\mathbb{R}^{m\times n}$ the space of $m \times n$ matrices with real coefficients ($\nabla \mathbf{u} \in \mathbb{R}^{m\times n}$). Similarly we can replace $a(x)uv$, the lower order term of Chapter 4, namely a bilinear form on $\mathbb{R}$ by a bilinear form on $\mathbb{R}^m$ defined as

$$A(x)\mathbf{u} \cdot \mathbf{v} \tag{13.9}$$

(here $A(x)$ is a $m \times m$ matrix, "$\cdot$" denotes the scalar product in $\mathbb{R}^m$). Thus the bilinear form extending the one considered in Chapter 4 to the case of systems can be defined as

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathcal{A}(x)\nabla \mathbf{u} \cdot \nabla \mathbf{v} + A(x)\mathbf{u} \cdot \mathbf{v}\, dx \tag{13.10}$$

where here for a.e. x

$$\mathcal{A}(x) \in \mathcal{L}(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}), \qquad A(x) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), \tag{13.11}$$

$$\nabla \mathbf{u}(x) \in \mathbb{R}^{m\times n}, \qquad \mathbf{u}(x) \in \mathbb{R}^m. \tag{13.12}$$

$\mathcal{L}(V, V)$ denotes the space of linear mappings from V into itself. Now if V is of dimension k then $\mathcal{L}(V, V)$ is of dimension k^2 – i.e., is determined by k^2 coefficients. Thus it might be useful to introduce these coefficients. For instance to write

$$\mathcal{A}(x)\nabla \mathbf{u} = (A_{ij}^{\alpha\beta}(x)\partial_{x_\beta} u^j)_{\substack{\alpha=1,\dots,n,\\ i=1,\dots,m}} \tag{13.13}$$

$$A(x)\mathbf{u} = (a_{ij}(x)u^j)_{i=1,\dots,m}. \tag{13.14}$$

We see that $\mathcal{A}(x)\nabla \mathbf{u}$ is a $m \times n$ matrix and $\mathcal{A}$ is equivalent to the data of $(m\times n)^2$ coefficients. To distinguish those running from 1 to n from these running from 1 to m we denote the first ones by Greek letters and the second ones by Latin letters. Note also that in (13.13) we assume a summation in β, j and in (13.14) a summation in j. Thus with the summation convention and with these coefficients explicitly written our bilinear form (13.10) becomes

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega A_{ij}^{\alpha\beta}(x)\partial_{x_\beta} u^j \partial_{x_\alpha} v^i + a_{ij}(x)u^i v^j\, dx. \tag{13.15}$$

As before if $\partial_{x_\alpha} v^i$, $\partial_{x_\beta} u^j$, u^i, $v^j \in L^2(\Omega)$ the integral (13.15) is perfectly defined as soon as $A_{ij}^{\alpha\beta}$, $a_{ij} \in L^\infty(\Omega)$. We can formulate these later assumptions as

$$|\mathcal{A}(x)M| \le \Lambda |M| \quad \forall\, M \in \mathbb{R}^{m\times n}, \tag{13.16}$$

$$|A(x)\xi| \le \Lambda |\xi| \qquad \forall\, \xi \in \mathbb{R}^m, \tag{13.17}$$

for some positive constant Λ, $|\cdot|$ denoting the Euclidean norm of vectors or matrices (for instance $|M| = (\sum_{ij} m_{ij}^2)^{\frac{1}{2}}\ \forall\, M = (m_{ij}) \in \mathbb{R}^{m\times n}$).

Let us introduce more precisely the spaces that we will use in this chapter. As already mentioned we will set

$$\mathbb{L}^2(\Omega) = (L^2(\Omega))^m. \tag{13.18}$$

This space is a Hilbert space when equipped with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx.$$

(This follows from the fact that $\mathbf{u}_n$ is a Cauchy sequence in $\mathbb{L}^2(\Omega)$ iff each component of $\mathbf{u}_n$ is a Cauchy sequence.)

Similarly we will set

$$\mathbb{H}^1(\Omega) = (H^1(\Omega))^m, \qquad \mathbb{H}^1_0(\Omega) = (H^1_0(\Omega))^m \tag{13.19}$$

and we will consider on these spaces the scalar products

$$\int_\Omega (\nabla \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{u} \cdot \mathbf{v})\, dx, \qquad \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx, \tag{13.20}$$

where $\nabla \mathbf{u}$ denotes the Jacobian matrix of $\mathbf{u}$. Equipped with the first scalar product of (13.20) both $\mathbb{H}^1(\Omega)$, $\mathbb{H}^1_0(\Omega)$ are Hilbert spaces. In the case where Ω is bounded in one direction then the two scalar products of (13.20) are equivalent on $\mathbb{H}^1_0(\Omega)$. With this definition if $A^{\alpha\beta}_{ij}$, $a_{ij} \in L^\infty(\Omega)$ and satisfy (13.16), (13.17) the bilinear form $a(\mathbf{u}, \mathbf{v})$ is continuous on $\mathbb{H}^1(\Omega)$. Indeed this follows from

$$\begin{aligned}
|a(\mathbf{u},\mathbf{v})| &= \left| \int_\Omega \mathcal{A}(x)\nabla\mathbf{u}\cdot\nabla\mathbf{v} + A(x)\mathbf{u}\cdot\mathbf{v}\, dx \right| \\
&\le \int_\Omega |\mathcal{A}(x)\nabla\mathbf{u}\cdot\nabla\mathbf{v}| + |A(x)\mathbf{u}\cdot\mathbf{v}|\, dx \\
&\le \int_\Omega |\mathcal{A}(x)\nabla\mathbf{u}|\,|\nabla\mathbf{v}| + |A(x)\mathbf{u}|\,|\mathbf{v}|\, dx \\
&\le \Lambda \int_\Omega |\nabla\mathbf{u}|\,|\nabla\mathbf{v}| + |\mathbf{u}|\,|\mathbf{v}|\, dx \\
&\le \Lambda \big\{ \big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} \big|\big|\nabla\mathbf{v}\big|\big|_{2.\Omega} + \big|\big|\mathbf{u}\big|\big|_{2,\Omega} \big|\big|\mathbf{v}\big|\big|_{2,\Omega} \big\} \\
&\le \Lambda |\mathbf{u}|_{1,2} |\mathbf{v}|_{1,2}
\end{aligned} \tag{13.21}$$

where we have set

$$|\mathbf{u}|_{1,2} = \left\{ \int_\Omega |\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\, dx \right\}^{\frac12}. \tag{13.22}$$

Recall that $|\cdot|$ denotes the Euclidean norm of vectors or matrices considered as elements of $\mathbb{R}^{m\times n}$.

The coerciveness of a requires some assumptions.

Definition 13.1 (Legendre condition). We say that $\mathcal{A}$ satisfies the Legendre condition if there exists a $\lambda > 0$ such that

$$\mathcal{A}(x)M \cdot M \ge \lambda |M|^2 \quad \forall\, M \in \mathbb{R}^{m\times n}, \text{ a.e. } x \in \Omega. \tag{13.23}$$

Definition 13.2 (Legendre–Hadamard). We say that $\mathcal{A}$ satisfies the Legendre–Hadamard condition if

$$\mathcal{A}(x)M \cdot M \ge \lambda |M|^2 \quad \forall\, M \in \mathbb{R}^{m\times n},\ \text{Rank } M = 1, \text{ a.e. } x \in \Omega, \tag{13.24}$$

i.e., the inequality (13.23) holds only for rank one matrices.

Example. Suppose that for $n = m = 2$

$$\mathcal{A}(x)M \cdot M = |M|^2 + k \det M \tag{13.25}$$

where k is some constant, det denotes the determinant of a matrix, $|\cdot|$ its Euclidean norm. Clearly if $\operatorname{Rank} M = 1$, one has $\det M = 0$ and thus

$$\mathcal{A}(x)M \cdot M = |M|^2$$

and (13.24) holds with $\lambda = 1$. Now if M is the matrix

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{one has} \quad \mathcal{A}(x)M \cdot M = 2 - k < 0$$

for k large enough in such a way that (13.23) cannot hold.

Even so the Legendre–Hadamard condition is weaker than the Legendre one it is enough to get coerciveness – at least on $\mathbb{H}_0^1(\Omega)$ – for the bilinear form a when $\mathcal{A}$ is independent of x – i.e., in the case of constant coefficients. More precisely we have

Proposition 13.1. *Suppose that $\mathcal{A}$ independent of x satisfies* (13.24) *then we have*

$$\int_\Omega \mathcal{A}\nabla \mathbf{u} \cdot \nabla \mathbf{u}\, dx \geq \lambda \int_\Omega |\nabla \mathbf{u}|^2\, dx \quad \forall\, \mathbf{u} \in \mathbb{H}_0^1(\Omega). \tag{13.26}$$

Proof. One should remark that a rank one matrix can be expressed as

$$c \otimes d = (c_i d_j)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \tag{13.27}$$

namely with $c \in \mathbb{R}^m$, $d \in \mathbb{R}^n$. It is enough to show (13.26) for $\mathbf{u} \in (\mathcal{D}(\Omega))^m$. Then we can extend v by 0 outside Ω. We denote by $\hat{f}$ the Fourier transform of f given by

$$\hat{f}(y) = \int_{\mathbb{R}^n} e^{-2\pi i y \cdot x} f(x)\, dx. \tag{13.28}$$

Since the functions we are dealing with are real valued we have

$$\begin{aligned} \int_\Omega \mathcal{A}\nabla \mathbf{u} \cdot \nabla \mathbf{u}\, dx &= \frac{1}{2} \int_{\mathbb{R}^n} \{\overline{A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j} \partial_{x_\alpha} u^i + A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \overline{\partial_{x_\alpha} u^i}\}\, dx \\ &= \frac{1}{2} \Big\{ \int_{\mathbb{R}^n} \overline{\widehat{A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j}}\, \widehat{\partial_{x_\alpha} u^i}\, dy + \int_{\mathbb{R}^n} \widehat{A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j}\, \overline{\widehat{\partial_{x_\alpha} u^i}}\, dy \Big\} \end{aligned} \tag{13.29}$$

by the Plancherel formula (see [85]).

If $\hat{u}^k = a^k + ib^k$ we have

$$Re(\overline{\hat{u}^k} \hat{u}^\ell) = \frac{1}{2} \{\overline{\hat{u}^k} \hat{u}^\ell + \hat{u}^k \overline{\hat{u}^\ell}\} = a^k a^\ell + b^k b^\ell.$$

Thus from (13.29) we derive, since $\widehat{\partial_{x_\beta} u^j}(y) = 2\pi i y_\beta \hat{u}^j$

$$\begin{aligned}\int_\Omega \mathcal{A}\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,dx &= \int_{\mathbb{R}^n} (2\pi)^2 A_{ij}^{\alpha\beta} y_\alpha y_\beta \{a^j a^i + b^j b^i\}\,dy \\ &\geq \int_{\mathbb{R}^n} (2\pi)^2 \lambda |y|^2 \{|a|^2 + |b|^2\}\,dy.\end{aligned}$$

(Note that when $c \otimes d$ is given by (13.27) one has $|c\otimes d|^2 = |c|^2|d|^2$.) Thus we have now

$$\int_\Omega \mathcal{A}\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,dx \geq \lambda \int_{\mathbb{R}^n} (2\pi)^2 |y|^2 |\hat{u}|^2\,dy = \lambda \int_{\mathbb{R}^n} \overline{\widehat{\partial_{x_\alpha} u^i}}\,\widehat{\partial_{x_\alpha} u^i}\,dy = \lambda \int_\Omega |\nabla\mathbf{u}|^2\,dy.$$

This completes the proof of the proposition. □

On the matrix A we can make the following assumptions

$$A(x)\zeta\cdot\zeta \geq 0 \qquad \forall\,\zeta\in\mathbb{R}^m,\ \text{a.e. } x\in\Omega \tag{13.30}$$

$$A(x)\zeta\cdot\zeta \geq \lambda|\zeta|^2 \qquad \forall\,\zeta\in\mathbb{R}^m,\ \text{a.e. } x\in\Omega. \tag{13.31}$$

If $A_{ij}^{\alpha\beta}$, $a_{ij} \in L^\infty(\Omega)$ then we have

Proposition 13.2. (i) *Suppose that*

$$\mathcal{A} \textit{ is independent of } x \textit{ and } \mathcal{A} \textit{ satisfies } (13.24) \textit{ or } \mathcal{A} \textit{ satisfies } (13.23) \tag{13.32}$$

then if $A(x)$ satisfies (13.30) *and Ω is bounded in one direction $a(\mathbf{u},\mathbf{v})$ is coercive on $\mathbb{H}_0^1(\Omega)$.*

(ii) *Suppose that $\mathcal{A}$ satisfies* (13.23) *and A satisfies* (13.31) *then $a(\mathbf{u},\mathbf{v})$ is coercive on $\mathbb{H}^1(\Omega)$.*

Proof. (i) By (13.23) or Proposition 13.1 we have

$$a(\mathbf{u},\mathbf{u}) = \int_\Omega \mathcal{A}\nabla\mathbf{u}\cdot\nabla\mathbf{u} + A(x)\mathbf{u}\cdot\mathbf{v}\,dx \geq \int_\Omega \mathcal{A}\nabla\mathbf{u}\cdot\nabla\mathbf{u}\,dx \geq \lambda\int_\Omega |\nabla\mathbf{u}|^2\,dx$$

and the result follows.

(ii) It follows immediately from

$$a(\mathbf{u},\mathbf{u}) = \int_\Omega \mathcal{A}(x)\nabla\mathbf{u}\cdot\nabla\mathbf{u} + A(x)\mathbf{u}\cdot\mathbf{u}\,dx \geq \lambda\int_\Omega |\nabla\mathbf{u}|^2 + |\mathbf{u}|^2\,dx.$$

This completes the proof of the proposition. □

As an obvious corollary we have

Theorem 13.3. (i) *Suppose that $\mathcal{A}$ satisfies* (13.32), *A satisfies* (13.30) *and Ω is bounded in one direction. Then for every $\mathbf{f}\in\mathbb{H}^{-1}(\Omega) = (H^{-1}(\Omega))^n$ there exists a unique $\mathbf{u}$ solution to*

$$\begin{cases}\mathbf{u}\in\mathbb{H}_0^1(\Omega), \\ \displaystyle\int_\Omega \mathcal{A}\nabla\mathbf{u}\cdot\nabla\mathbf{v} + A\mathbf{u}\cdot\mathbf{v}\,dx = \langle\mathbf{f},\mathbf{v}\rangle \quad \forall\,\mathbf{v}\in\mathbb{H}_0^1(\Omega).\end{cases} \tag{13.33}$$

(ii) *If $\mathcal{A}$ satisfies* (13.23) *and A satisfies* (13.31)*, for every $f \in \mathbb{L}^2(\Omega)$ and every $\mathbb{V}$ closed subspace of $\mathbb{H}^1(\Omega)$ there exists a unique $\mathbf{u}$ solution to*

$$\begin{cases} \mathbf{u} \in \mathbb{V}, \\ \displaystyle\int_\Omega \mathcal{A}\nabla\mathbf{u} \cdot \nabla\mathbf{v} + A\mathbf{u} \cdot \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{V}. \end{cases} \tag{13.34}$$

(*In* (13.33)*, $\langle \mathbf{f}, \mathbf{v}\rangle$ stands for $\langle f^i, v^i\rangle$ with the summation convention.*)

13.2 Some examples

• *Diffusion of population with competition.*

In this example $\mathbf{u} = (u^1, \dots, u^m)$ is the vector density of population of different species, i.e., u^i is the density of population of the species u^i. If all these species are living in an environment Ω their diffusion could be described by $\mathbf{u}$ solution to

$$\begin{cases} \mu_i \displaystyle\int_\Omega \nabla u^i \cdot \nabla v^i\, dx + \int_\Omega A(x)\mathbf{u} \cdot \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{H}^1_0(\Omega), \\ \mathbf{u} \in \mathbb{H}^1_0(\Omega). \end{cases} \tag{13.35}$$

In strong form and if $A = (a_{ij})$ this system can be written (with no summation in i) as

$$\begin{cases} -\mu_i \Delta u^i + a_{ij} u^j = f^i \text{ in } \Omega & \forall\, i = 1, \dots, m \\ u^i \in H^1_0(\Omega) & \forall\, i = 1, \dots, m. \end{cases} \tag{13.36}$$

The term in $A\mathbf{u}$ stands for the competition between the different species. A particular important case is when

$$\begin{aligned} &a_{ii} > 0 \quad \forall\, i = 1, \dots, m, \\ &a_{ij} \le 0 \quad \forall\, j \ne i, \quad \forall\, i = 1, \dots, m. \end{aligned}$$

• *The Stokes problem.*

If Ω is a bounded open set of $\mathbb{R}^n$ ($n = 2$ or 3 in the applications) the Stokes problem consists in finding the velocity $\mathbf{u}$ of a fluid and its pressure p satisfying

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.37}$$

(μ is the viscosity of the fluid, $\mathbf{f}$ the applied forces. One neglects in a first approximation the nonlinear effect. The velocity of the fluid is supposed to vanish on the boundary of the container Ω.)

A natural space adapted to the problem is

$$\widehat{\mathbb{H}}_0^1(\Omega) = \{\, \mathbf{v} \in \mathbb{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \,\}. \tag{13.38}$$

We have

Lemma 13.4. $\widehat{\mathbb{H}}_0^1(\Omega)$ *is a closed subspace of* $\mathbb{H}_0^1(\Omega)$.

Proof. Indeed if $\mathbf{v}_n$ is a sequence of $\widehat{\mathbb{H}}_0^1(\Omega)$ converging toward v one has for every i

$$\begin{aligned} & \partial_{x_i} v_n^i \longrightarrow \partial_{x_i} v && \text{in } L^2(\Omega) \\ \Longrightarrow \qquad & 0 = \operatorname{div} \mathbf{v}_n \longrightarrow \operatorname{div} \mathbf{v} = 0 && \text{in } L^2(\Omega) \end{aligned}$$

and $\mathbf{v} \in \widehat{\mathbb{H}}_0^1(\Omega)$. Note that the equality $\operatorname{div} \mathbf{v} = 0$ in (13.38) can be taken in the distributions or in the $L^2(\Omega)$- sense. This completes the proof of the Lemma. □

If we multiply the first equation of (13.37) by a smooth function of $\widehat{\mathbb{H}}_0^1(\Omega)$ and integrate on Ω the pressure disappears and we are lead to find a $\mathbf{u}$ solution to

$$\begin{cases} \mathbf{u} \in \widehat{\mathbb{H}}_0^1(\Omega), \\ \mu \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \quad \forall \, \mathbf{v} \in \widehat{\mathbb{H}}_0^1(\Omega). \end{cases} \tag{13.39}$$

Having recast our problem in term of the velocity only we have

Theorem 13.5. *For* $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$ *there exists a unique solution to* (13.39).

Proof. This is a trivial consequence of Theorem 13.3. □

Remark 13.1. If Ω is a Lipschitz domain and if $\mathbf{u}$ satisfies (13.39) one can show that there exists $p \in L^2(\Omega)$ – unique up to an additive constant – such that

$$-\mu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega$$

(we refer the reader to [56], [83], [93]).

• *The system of elasticity.*

We state here the problem in a very simple framework. First we suppose

$$m = n$$

and in the applications n will be equal to 2 or 3. The unknown of the problem is the displacement $\mathbf{u}(x)$ of a particular x of an elastic body occupying a domain Ω in space (see [29], [39]). We denote by $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x)$ coefficients in $L^\infty(\Omega)$ experiencing in addition the symmetry property

$$A_{ij}^{\alpha\beta} = A_{j\alpha}^{\beta i} = A_{\beta\alpha}^{ji} \quad \forall \, i, j, \alpha, \beta = 1, \ldots, n. \tag{13.40}$$

Moreover, we replace the Legendre or the Legendre–Hadamard condition by assuming that for some $\lambda > 0$ we have

$$\mathcal{A}(x)M \cdot M \geq \lambda |M|^2 \quad \forall\, M \in \mathbb{R}^{n\times n}, \quad M^T = M, \quad \text{a.e. } x \in \Omega. \tag{13.41}$$

(M^T denotes the transposed matrix of M – i.e., the matrix derived from M by taking for the i^{th} column of M^T the i^{th} row of M.) In other words we are assuming that the Legendre condition holds for symmetric matrices only.

Under the assumptions above we have

Theorem 13.6. *Let* $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$*. There exists a unique* u *solution to*

$$\begin{cases} \mathbf{u} \in \mathbb{H}^1_0(\Omega), \\ \displaystyle\int_\Omega \mathcal{A}(x)\nabla\mathbf{u} \cdot \nabla\mathbf{v}\, dx = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in \mathbb{H}^1_0(\Omega). \end{cases} \tag{13.42}$$

Proof. We start first with the following remark due to our symmetry conditions (13.40). Let $M = (m_{i\alpha})$ be a $n \times n$ matrix. We have with the summation convention

$$\begin{aligned} \mathcal{A}(x)&\Big(\frac{M+M^T}{2}\Big) \cdot \Big(\frac{M+M^T}{2}\Big) \\ &= A^{\alpha\beta}_{ij}\Big(\frac{m_{i\alpha}+m_{\alpha i}}{2}\Big)\Big(\frac{m_{j\beta}+m_{\beta j}}{2}\Big) \\ &= \frac{1}{4}A^{\alpha\beta}_{ij} m_{i\alpha}m_{j\beta} + \frac{1}{4}A^{\alpha\beta}_{ij} m_{i\alpha}m_{\beta j} + \frac{1}{4}A^{\alpha\beta}_{ij} m_{\alpha i}m_{j\beta} + \frac{1}{4}A^{\alpha\beta}_{ij} m_{\alpha i}m_{\beta j} \\ &= A^{\alpha\beta}_{ij} m_{i\alpha}m_{j\beta} = \mathcal{A}(x)M \cdot M. \end{aligned} \tag{13.43}$$

Next, due to the Lax–Milgram theorem, we remark that the existence and uniqueness of a solution to (13.42) will follow if the left-hand side of the second equality of (13.42) is coercive. Using (13.43) we have by (13.41)

$$\begin{aligned} \int_\Omega \mathcal{A}(x)\nabla\mathbf{u} \cdot \nabla\mathbf{u}\, dx &= \int_\Omega \mathcal{A}(x)\frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2} \cdot \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2} \\ &\geq \lambda \Big|\Big|\frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2}\Big|\Big|^2_{2,\Omega} \quad \forall\, \mathbf{u} \in \mathbb{H}^1_0(\Omega). \end{aligned} \tag{13.44}$$

Recall that $|\cdot|$ denotes the Euclidean norm of the matrices considered as vectors in $\mathbb{R}^{n\times n}$.

We adopt here the classical notation

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\{\partial_{x_i} u^j + \partial_{x_j} u^i\}, \tag{13.45}$$

i.e., we denote by $\varepsilon_{ij}(\mathbf{u})$ the entries of the matrix $\frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. Then Theorem 13.6 is a consequence of the following lemma.

Lemma 13.7 (Korn's inequality). *We have*

$$\Big|\Big|\frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2}\Big|\Big|_{2,\Omega}^2 \geq \frac{1}{2}\big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}^2 \quad \forall\, \mathbf{u} \in \mathbb{H}_0^1(\Omega). \tag{13.46}$$

Proof of the lemma. By density of $(\mathcal{D}(\Omega))^n$ in $\mathbb{H}_0^1(\Omega)$ it is enough to show (13.46) for $\mathbf{u} \in (\mathcal{D}(\Omega))^n$. Then we have by (13.45)

$$|\varepsilon_{ij}(\mathbf{u})|_{2,\Omega}^2 = \frac{1}{4}\int_\Omega (\partial_{x_i}u^j + \partial_{x_j}u^i)^2\,dx = \frac{1}{4}\int_\Omega (\partial_{x_i}u^j)^2 + 2\partial_{x_i}u^j\partial_{x_j}u^i + (\partial_{x_j}u^i)^2\,dx. \tag{13.47}$$

An integration by parts shows that

$$\int_\Omega \partial_{x_i}u^j\partial_{x_j}u^i\,dx = \int_\Omega \partial_{x_i}u^i\partial_{x_j}u^j\,dx.$$

Thus summing up the inequalities (13.47) – recall that $\varepsilon_{ii}(\mathbf{u}) = \partial_{x_i}u^i$ – we obtain

$$\sum_{i,j}|\varepsilon_{ij}(\mathbf{u})|_{2,\Omega}^2 = \frac{1}{2}\sum_{i,j}|\partial_{x_i}u^j|_{2,\Omega}^2 + \frac{1}{2}|\operatorname{div} u|_{2,\Omega}^2 \geq \frac{1}{2}\sum_{i,j}|\partial_{x_i}u^j|_{2,\Omega}^2$$

which is (13.46). This completes the proof of the lemma and the theorem. □

Remark 13.2. We could have added in (13.42) some lower order terms. $\mathbf{u}$ is the displacement of the particles inside Ω under the action of forces $\mathbf{f}$ (for instance the gravity) when the boundary of the body is maintained fixed. At the expenses of a more complex Korn inequality one could consider an elastic body maintained fixed only on a part of its boundary – see Figure 13.1 below. We refer the reader to [39], [29] for details.

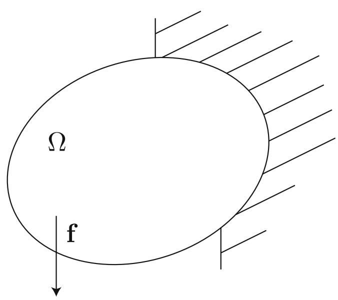


Figure 13.1.

For a material which is homogeneous and isotropic one has

$$A_{ij}^{\alpha\beta} = \lambda\delta_{i\alpha}\delta_{j\beta} + \mu\delta_{i\beta}\delta_{j\alpha} + \mu\delta_{ij}\delta_{\alpha\beta} \tag{13.48}$$

where δ_{ij} is the Kronecker symbol defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{13.49}$$

μ and λ are the so-called Lamé's constants (see [39]). It is easy to see that the symmetry relations (13.40) hold. Moreover if $M = (m_{i\alpha}), M' = (m_{j\beta}) \in \mathbb{R}^{n\times n}$ we have

$$\begin{aligned} A_{ij}^{\alpha\beta} m_{i\alpha} m'_{j\beta} &= \lambda m_{ii} m'_{jj} + \mu m_{ij} m'_{ji} + \mu m_{ij} m'_{ij} \\ &= \lambda \operatorname{tr} M \operatorname{tr} M' + \mu m_{ij}(m'_{ij} + m'_{ji}) \end{aligned} \tag{13.50}$$

with the summation convention and with tr denoting the trace of the matrices – i.e., the sum of the diagonal coefficients.

One can remark that

$$m_{ij}(m'_{ij} + m'_{ji}) = m_{ji}(m'_{ij} + m'_{ji}) \tag{13.51}$$

and (13.50) can take the more symmetric form

$$A_{ij}^{\alpha\beta} m_{i\alpha} m'_{j\beta} = \lambda \operatorname{tr} M \operatorname{tr} M' + 2\mu \frac{M + M^T}{2} \cdot \frac{M' + M'^T}{2}. \tag{13.52}$$

It is clear that for $\lambda \geq 0$, $\mu > 0$ the inequality (13.41) holds for λ in (13.41) given by 2μ and thus we have

Theorem 13.8. *Let $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$, $\lambda \geq 0$, $\mu > 0$. There exists a unique solution to*

$$\begin{cases} \mathbf{u} \in \mathbb{H}_0^1(\Omega), \\ \lambda \displaystyle\int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx + 2\mu \int_\Omega \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})\, dx = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbb{H}_0^1(\Omega). \end{cases} \tag{13.53}$$

Moreover setting

$$\sigma_{ij}(\mathbf{u}) = \lambda \operatorname{div} \mathbf{u} \delta_{ij} + 2\mu \varepsilon_{ij}(\mathbf{u}), \tag{13.54}$$

$\mathbf{u}$ *is a weak solution to the system*

$$\begin{cases} -\partial_{x_j} \sigma_{ij}(\mathbf{u}) = f^i & \text{in } \Omega, \ i = 1, \dots, n, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{13.55}$$

Proof. The existence and uniqueness of a solution to (13.53) follows from Theorem 13.6 and (13.52). To obtain (13.55) it is enough to notice that by (13.51) one has

$$\begin{aligned} &\lambda \int_\Omega \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}\, dx + 2\mu \int_\Omega \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})\, dx \\ &\qquad = \lambda \int_\Omega \operatorname{div} \mathbf{u} \partial_{x_j} v^j\, dx + 2\mu \int_\Omega \varepsilon_{ij}(\mathbf{u}) \partial_{x_j} v^i\, dx = \langle \mathbf{f}, \mathbf{v} \rangle \end{aligned}$$

for any $\mathbf{v} \in (\mathcal{D}(\Omega))^n$. □

Exercises

1. In the spirit of paragraph 13.1 introduce a general theory for higher order elliptic systems.
2. Show that if $A_{ij}^{\alpha\beta}$ are measurable (13.16) is equivalent to $A_{ij}^{\alpha\beta} \in L^\infty(\Omega)$ $\forall\, \alpha, \beta, i, j$.
3. Show that for $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$ and every $\varepsilon > 0$ there exists a unique $\mathbf{u}_\varepsilon$ solution to

$$\begin{cases} \mathbf{u} \in \mathbb{H}_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla \mathbf{u}_\varepsilon \cdot \nabla \mathbf{v}\, dx + \frac{1}{\varepsilon} \int_\Omega \operatorname{div} \mathbf{u}_\varepsilon \cdot \operatorname{div} \mathbf{v}\, dx = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall\, v \in \mathbb{H}_0^1(\Omega). \end{cases}$$

   Show that when $\varepsilon \to 0$, $\mathbf{u}_\varepsilon \to \mathbf{u}$ in $\mathbb{H}_0^1(\Omega)$ where $\mathbf{u}$ is the solution to the Stokes problem.
4. In $\mathbb{R}^n$ we consider a cylindrical domain of the form

$$\Omega_\ell = \ell\omega_1 \times \omega_2 \quad \omega_1 \subset \mathbb{R}^p, \quad \omega_2 \subset \mathbb{R}^{n-p}$$

   (ω_1 satisfies the assumptions of Paragraph 6.3. We refer to it for the notation). Let $\mathcal{A}$ be the coefficients of a linear constant elliptic system satisfying the Legendre–Hadamard condition. Consider $\mathbf{u}_\ell$ the solution to

$$\begin{cases} \mathbf{u}_\ell \in \mathbb{H}_0^1(\Omega_\ell), \\ \displaystyle\int_{\Omega_\ell} \mathcal{A} \nabla \mathbf{u}_\ell \cdot \nabla \mathbf{v}\, dx = \int_{\Omega_\ell} \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\Omega_\ell), \end{cases} \tag{P_ℓ}$$

   where $\mathbf{f} = \mathbf{f}(X_2) \in \mathbb{L}^2(\omega_2)$ – ($x = (X_1, X_2)$ where $X_1 \in \mathbb{R}^p$, $X_2 \in \omega_2$). Let $\mathbf{u}_\infty$ be the solution to

$$\begin{cases} \mathbf{u}_\infty \in \mathbb{H}_0^1(\omega_2) = (H_0^1(\omega_2))^m, \\ \displaystyle\int_{\omega_2} \mathcal{A} \nabla \mathbf{u}_\infty \cdot \nabla \mathbf{v}\, dx = \int_{\omega_2} \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\omega_2) \end{cases} \tag{P_∞}$$

   where $\mathbf{f} = (f^1, \dots, f^m)$. Note that $\mathbf{v}$ and $\mathbf{u}_\infty$ are depending on X_2 only – see Section 6.3.

   Show that there exist two constants $C, \alpha > 0$ independent of ℓ such that

$$\big|\big|\nabla(\mathbf{u}_\ell - \mathbf{u}_\infty)\big|\big|_{2,\Omega_{\ell/2}} \le ce^{-\alpha\ell}\big|\big|\mathbf{f}\big|\big|_{2,\omega_2}.$$

   Extend this to $\mathcal{A} = \mathcal{A}(x)$ where $\mathcal{A}$ satisfies the Legendre condition or is of the elasticity type.

Chapter 14

The Stationary Navier–Stokes System

14.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$. Like for the Stokes problem one is looking for a couple

$$(\mathbf{u}, p) \tag{14.1}$$

representing respectively the velocity of a fluid and its pressure such that

$$\begin{cases} -\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{14.2}$$

μ is the viscosity of the fluid. Note that here one is taking into account the nonlinear effect. The operator $\mathbf{u}\cdot\nabla$ is defined as

$$u^i\partial_{x_i} \tag{14.3}$$

with the summation convention in i. We will restrict ourselves to the physical relevant cases – i.e., $n = 2$ or 3. We refer the reader to [45], [56] for a physical background on the problem (see also [51], [52], [68]). Eliminating the pressure as we did in the preceding chapter we are reduced to find $\mathbf{u}$ such that

$$\begin{cases} \mathbf{u} \in \widehat{\mathbb{H}}_0^1(\Omega), \\ \mu\displaystyle\int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\,\mathbf{v}\in\widehat{\mathbb{H}}_0^1(\Omega), \end{cases} \tag{14.4}$$

where $\widehat{\mathbb{H}}_0^1(\Omega)$ is defined by (13.38). In order to do that we introduce the trilinear form defined as

$$t(\mathbf{w};\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega w^i\partial_{x_i}u^j v^j\,dx \tag{14.5}$$

(with the summation convention in i, j). One should notice that if

$$w^i, \partial_{x_i} u^j, v^j \in L^2(\Omega)$$

it is not clear that the product above is integrable. This will follow however for $\mathbf{w}, \mathbf{v} \in \mathbb{H}^1_0(\Omega)$ from the Sobolev embedding theorem.

Indeed we have

Proposition 14.1. *There exists a constant $C = C(\Omega)$ such that*

$$|t(\mathbf{w};\mathbf{u},\mathbf{v})| \leq C\big|\big|\nabla \mathbf{w}\big|\big|_{2,\Omega}\big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}\big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega} \quad \forall\, \mathbf{w}, \mathbf{v} \in \mathbb{H}^1_0(\Omega), \quad \forall\, \mathbf{u} \in \mathbb{H}^1(\Omega). \tag{14.6}$$

Proof. From the formula (12.30) we have for some constant $C = C(n)$

$$|v|_{2^*,\Omega} \leq C\big|\big|\nabla v\big|\big|_{2,\Omega} \quad \forall\, v \in H^1_0(\Omega),$$

where $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$, i.e., 2^* can be chosen arbitrary for $n = 2$ and equal to 6 for $n = 3$. Thus in both cases we have (for instance applying Hölder's inequality)

$$|v|_{4,\Omega} \leq C(\Omega)|v|_{6,\Omega} \leq C(\Omega)\big|\big|\nabla v\big|\big|_{2,\Omega} \quad \forall\, v \in H^1_0(\Omega).$$

From Hölder's inequality again we then have since $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$ for every i, j

$$\begin{aligned}\left|\int_\Omega w^i \partial_{x_i} u^j v^j \, dx\right| &\leq \int_\Omega |w^i|\,|\partial_{x_i} u^j|\,|v^j|\, dx \\ &\leq |w^i|_{4,\Omega} |\partial_{x_i} u^j|_{2,\Omega} |v^j|_{4,\Omega} \\ &\leq C\big|\big|\nabla w^i\big|\big|_{2,\Omega}\big|\big|\nabla u^j\big|\big|_{2,\Omega}\big|\big|\nabla v^j\big|\big|_{2,\Omega} \\ &\leq C\big|\big|\nabla \mathbf{w}\big|\big|_{2,\Omega}\big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}\big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega}\end{aligned} \tag{14.7}$$

and the result follows due to the formula (14.5) after summation in i, j. This completes the proof of the proposition. □

Proposition 14.2. *Let $\mathbf{w} \in \mathbb{H}^1(\Omega)$ with $\operatorname{div} \mathbf{w} = 0$. Then we have*

$$t(\mathbf{w};\mathbf{v},\mathbf{v}) = 0 \qquad \forall\, \mathbf{v} \in \mathbb{H}^1_0(\Omega) \tag{14.8}$$

$$t(\mathbf{w};\mathbf{v},\mathbf{u}) = -t(\mathbf{w};\mathbf{u},\mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbb{H}^1_0(\Omega). \tag{14.9}$$

Proof. Let us first choose $\mathbf{v} \in (\mathcal{D}(\Omega))^n$, $n = 2$ or 3. Then by (14.5) we have – with the summation convention

$$\begin{aligned}t(\mathbf{w};\mathbf{v},\mathbf{v}) &= \int_\Omega w^i \partial_{x_i} v^j v^j \\ &= \frac{1}{2}\int_\Omega w^i \partial_{x_i} (v^j)^2 \\ &= -\frac{1}{2}\langle \partial_{x_i} w^i, (v^j)^2\rangle = 0.\end{aligned}$$

(Note that $(v^j)^2 \in \mathcal{D}(\Omega)$.)

Thus (14.8) holds for $\mathbf{v} \in (\mathcal{D}(\Omega))^n$. By density of $(\mathcal{D}(\Omega))^n$ in $\mathbb{H}_0^1(\Omega)$ (14.8) follows since t, trilinear form, is continuous at 0 for the $\mathbb{H}^1(\Omega)$ norm (see (14.6)) and thus is continuous at any point.

To prove (14.9) it is enough to notice that

$$\begin{aligned} 0 = t(\mathbf{w}; \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) &= t(\mathbf{w}; \mathbf{u}, \mathbf{u}) + t(\mathbf{w}; \mathbf{u}, \mathbf{v}) + t(\mathbf{w}; \mathbf{v}, \mathbf{u}) + t(\mathbf{w}; \mathbf{v}, \mathbf{v}) \\ &= t(\mathbf{w}; \mathbf{u}, \mathbf{v}) + t(\mathbf{w}, \mathbf{v}, \mathbf{u}) \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbb{H}_0^1(\Omega) \end{aligned}$$

(by (14.8)). This completes the proof of the proposition. □

14.2 Existence and uniqueness result

For $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$ we will denote by $\langle \mathbf{f}, \mathbf{v} \rangle$ the continuous linear form on $\mathbb{H}_0^1(\Omega)$ defined as

$$\langle \mathbf{f}, \mathbf{v} \rangle = \langle f^i, v^i \rangle \tag{14.10}$$

with the summation convention. Clearly (14.10) defines a continuous linear form on $\mathbb{H}_0^1(\Omega)$ and we denote by $|\mathbf{f}|_*$ the strong dual norm of this linear form when $\mathbb{H}_0^1(\Omega)$ is equipped with the norm

$$\big||\nabla \mathbf{v}|\big|_{2,\Omega}. \tag{14.11}$$

Then we have

Theorem 14.3. *For $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$, $\mu > 0$ there exists a solution $\mathbf{u}$ to*

$$\begin{cases} \mathbf{u} \in \widehat{\mathbb{H}}_0^1(\Omega), \\ \mu \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, dx = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall\, \mathbf{v} \in \widehat{\mathbb{H}}_0^1(\Omega) \end{cases} \tag{14.12}$$

such that

$$\big||\nabla \mathbf{u}|\big|_{2,\Omega} \le \frac{|\mathbf{f}|_*}{\mu}. \tag{14.13}$$

In order to prove Theorem 14.3 we will use a variant of the Brouwer fixed point theorem (see the appendix). More precisely

Lemma 14.4. *Let H be a finite-dimensional Hilbert space equipped with the scalar product $(\cdot,\cdot)$. Let $T : H \to H$ be a continuous mapping such that for some $\nu > 0$ we have*

$$(T(v), v) \ge 0 \quad \forall\, v \in H, \quad |v| = \nu. \tag{14.14}$$

Then there exists an element $u \in H$ such that

$$|u| \le \nu, \qquad T(u) = 0. \tag{14.15}$$

Proof of the Lemma. If not

$$T(u) \neq 0 \quad \forall\, |u| \leq \nu.$$

Then consider the mapping

$$u \mapsto F(u) = -\frac{\nu T(u)}{|T(u)|}.$$

This mapping is a continuous mapping from $B_\nu(0)$ into itself and by the Brouwer fixed point theorem it admits a fixed point in $B_\nu(0)$, i.e., a point u such that

$$-\frac{\nu T(u)}{|T(u)|} = u.$$

Due to this equality one has

$$|u| = \nu \qquad -(T(u), u)\nu = |u|^2 |T(u)| > 0$$

which contradicts (14.14) and completes the proof of the lemma. □

Proof of the theorem. $\widehat{\mathbb{H}}^1_0(\Omega)$ is a closed subspace of $\mathbb{H}^1_0(\Omega)$ and thus separable. Let us consider $\mathbf{v}_n$ a countable basis of $\widehat{\mathbb{H}}^1_0(\Omega)$, and denote by V_k the finite-dimensional subspace of $\widehat{\mathbb{H}}^1_0(\Omega)$ generated by $\mathbf{v}_1, \ldots, \mathbf{v}_k$. One will assume V_k equipped with the scalar product of $\mathbb{H}^1_0(\Omega)$, namely

$$\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx. \tag{14.16}$$

By the Lax–Milgram theorem for $\mathbf{v} \in V_k$ there exists a unique $\mathbf{u} = T(\mathbf{v})$ solution to

$$\begin{cases} \mathbf{u} \in V_k, \\ \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{w}\, dx = \mu \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{w}\, dx + \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\, dx - \langle \mathbf{f}, \mathbf{w} \rangle \quad \forall\, \mathbf{w} \in V_k. \end{cases} \tag{14.17}$$

(Note that the right-hand side of the equation above is a continuous linear form in $\mathbf{w}$.) Taking $\mathbf{w} = \mathbf{v}$ and applying Proposition 14.2 we see that

$$\begin{aligned} \int_\Omega \nabla T(\mathbf{v}) \cdot \nabla \mathbf{v} &= \mu \big|\big|\nabla \mathbf{v}\big|\big|^2_{2,\Omega} - \langle \mathbf{f}, \mathbf{v} \rangle \\ &\geq \mu \big|\big|\nabla \mathbf{v}\big|\big|^2_{2,\Omega} - |\mathbf{f}|_* \big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega} \\ &= \big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega} \big\{\mu \big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega} - |\mathbf{f}|_*\big\} \\ &\geq 0 \quad \text{for} \quad \big|\big|\nabla \mathbf{v}\big|\big|_{2,\Omega} \leq \frac{|\mathbf{f}|_*}{\mu}. \end{aligned}$$

Applying Lemma 14.4 (note that T is continuous on V_k) we see that there exists $\mathbf{u}_k \in V_k$ such that

$$\big|\big|\nabla \mathbf{u}_k\big|\big|_{2,\Omega} \leq \frac{|\mathbf{f}|_*}{\mu} \tag{14.18}$$

and $T(\mathbf{u}_k) = 0$ which is

$$\mu \int_\Omega \nabla \mathbf{u}_k \cdot \nabla \mathbf{w}\, dx + \int_\Omega (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k \cdot \mathbf{w}\, dx = \langle \mathbf{f}, \mathbf{w}\rangle \quad \forall\, \mathbf{w} \in V_k. \tag{14.19}$$

From (14.18) it follows that the sequence $\mathbf{u}_k$ is bounded independently of k in $\widehat{\mathbb{H}}_0^1(\Omega)$. Up to a subsequence we have

$$\mathbf{u}_k \rightharpoonup \mathbf{u} \quad \text{in } \widehat{\mathbb{H}}_0^1(\Omega), \qquad \mathbf{u}_k \to \mathbf{u} \quad \text{in } \mathbb{L}^4(\Omega).$$

(We use here the fact that the injection from $\mathbb{H}_0^1(\Omega)$ into $\mathbb{L}^4(\Omega)$ is compact. This follows easily from (12.30) and arguments given in Chapter 2.) Noticing by Proposition 14.2 that

$$\int_\Omega (\mathbf{u}_k \cdot \nabla)\mathbf{u}_k \cdot \mathbf{w}\, dx = -\int_\Omega (\mathbf{u}_k \cdot \nabla)\mathbf{w} \cdot \mathbf{u}_k\, dx$$

and by (14.7) we can pass to the limit in (14.19) to get

$$\mu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{w}\, dx - \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{w} \cdot \mathbf{u}\, dx = \langle \mathbf{f}, \mathbf{w}\rangle \quad \forall\, w \in V_k \quad \forall\, k. \tag{14.20}$$

(By (14.7) we have that $t(\mathbf{w}; \mathbf{u}, \mathbf{v})$ is continuous at 0 on $\mathbb{L}^4(\Omega) \times \mathbb{H}_0^1(\Omega) \times \mathbb{L}^4(\Omega)$ and thus by trilinearity at every point, which leads to (14.20).)

Applying Proposition 14.2 again we see that u satisfies then

$$\begin{cases} \mathbf{u} \in \widehat{\mathbb{H}}_0^1(\Omega), \\ \mu \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{w}\, dx + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{w}\, dx = \langle \mathbf{f}, \mathbf{w}\rangle \quad \forall\, \mathbf{w} \in \widehat{\mathbb{H}}_0^1(\Omega) \end{cases}$$

and thus $\mathbf{u}$ is solution to (14.12). (We have also used here the density of $\bigcup_k V_k$ in $\mathbb{H}_0^1(\Omega)$.) The estimate (14.13) follows from the lower semi-continuity of the norm and (14.18). This completes the proof of the theorem. $\square$

The problem of uniqueness is a difficult issue for this nonlinear problem. However for small data $\mathbf{f}$ we can show uniqueness. More precisely let us set

$$\|t\| = \operatorname*{Sup}_{\mathbf{w},\mathbf{u},\mathbf{v} \neq 0} \frac{|t(\mathbf{w}; \mathbf{u}, \mathbf{v})|}{\big|\big|\nabla \mathbf{w}\big|\big|_2 \big|\big|\nabla \mathbf{u}\big|\big|_2 \big|\big|\nabla \mathbf{v}\big|\big|_2} \tag{14.21}$$

where for simplicity we dropped the index Ω in $|\cdot|_{2,\Omega}$. It is clear by (14.6) that the supremum (14.21) does exist.

Moreover we have

Theorem 14.5. *Suppose that* $\mathbf{f} \in \mathbb{H}^{-1}(\Omega)$ *and*

$$|\mathbf{f}|_* < \frac{\mu^2}{\|t\|}, \tag{14.22}$$

then the solution to (14.12) *is unique.*

Proof. Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two solutions. By subtraction we get

$$\mu \int_\Omega \nabla(\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla \mathbf{v}\, dx + t(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}) - t(\mathbf{u}_2; \mathbf{u}_2, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \widehat{\mathbb{H}}_0^1(\Omega).$$

Taking $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ we derive

$$\begin{aligned}
&\mu \big||\nabla(\mathbf{u}_1 - \mathbf{u}_2)|\big|_{2,\Omega}^2 \\
&\quad = t(\mathbf{u}_2; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - t(\mathbf{u}_1; \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) \\
&\quad = t(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) + t(\mathbf{u}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - t(\mathbf{u}_1; \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) \\
&\quad = t(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) - t(\mathbf{u}_1; \mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) \\
&\quad = t(\mathbf{u}_2 - \mathbf{u}_1; \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)
\end{aligned}$$

by Proposition 14.2. It follows from the definition of $\|t\|$ that

$$\mu \big||\nabla(\mathbf{u}_1 - \mathbf{u}_2)|\big|_{2,\Omega}^2 \leq \|t\| \big||\nabla \mathbf{u}_2|\big|_{2,\Omega} \big||\nabla(\mathbf{u}_1 - \mathbf{u}_2)|\big|_{2,\Omega}^2. \tag{14.23}$$

Now taking $\mathbf{v} = \mathbf{u}$ in (14.12) one easily sees that

$$\mu \big||\nabla \mathbf{u}|\big|_{2,\Omega}^2 = \langle \mathbf{f}, \mathbf{u} \rangle \leq |\mathbf{f}|_* \big||\nabla \mathbf{u}|\big|_{2,\Omega},$$

i.e., all solution to (14.12) satisfies (14.13). From (14.23) we then derive

$$\big||\nabla(\mathbf{u}_1 - \mathbf{u}_2)|\big|_{2,\Omega}^2 \leq \frac{\|t\| |\mathbf{f}|_*}{\mu^2} \big||\nabla(\mathbf{u}_1 - \mathbf{u}_2)|\big|_{2,\Omega}^2$$

which implies by (14.22) that $\mathbf{u}_1 = \mathbf{u}_2$. This completes the proof of the theorem. $\square$

Exercise

1. Let t be a k-multilinear form on N a normed space. Show that the continuity at $\mathbf{0}$ is equivalent to the continuity of t at any point and also to the existence of a constant C such that

$$|t(a_1, \dots, a_k)| \leq C|a_1| \dots |a_k| \quad \forall\, a_i \in N,\ i = 1, \dots, k.$$

($|\cdot|$ denotes the norm in N and the absolute value in $\mathbb{R}$.)

2. Show that
$$t(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0$$
for all smooth functions such that $\mathbf{w} \cdot \nu = 0$ on $\partial\Omega$. (ν is the outward unit normal to $\partial\Omega$.)
3. Let $\mathbf{u} \in \widehat{H}_0^1(\Omega)$. For $x \in \Omega$ one denotes by Ω_{x_i} the section
$$\Omega_{x_i} = \{(y_1, \ldots, x_i, \ldots, y_n) \in \Omega\}.$$
Show that
$$\int_{\Omega_{x_i}} u^i(x)\, dx_1, \ldots, \widehat{dx_i}, \ldots dx_n = 0$$
(the measure above is the Lebesgue measure where dx_i has been dropped).
4. (Poiseuille flow) Let $\Omega = \mathbb{R} \times \omega$ where ω is a bounded open set of $\mathbb{R}^2$. One considers the problem
$$\begin{cases} -\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 \text{ on } \mathbb{R}\times\partial\omega, \ \displaystyle\int_\omega u^1(x_1, x')\, dx' = F, & \end{cases}$$
($x = (x_1, x')$, F is a given constant). Show that the problem admits a solution $(\mathbf{u}, p)$ of the type
$$\mathbf{u} = (u^1, 0, 0), \qquad p = cx_1$$
where u^1 is solution of a Dirichlet problem in ω and c a constant which can be determined.

Chapter 15

Some More Spaces

15.1 Motivation

Let Ω be a bounded open set of $\mathbb{R}^n$. Suppose that we would like to solve the following nonlinear Dirichlet problem:

$$\begin{cases} -\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i}u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{15.1}$$

For $p \in (1, +\infty)$ the operator

$$\Delta_p = \partial_{x_i}(|\nabla|^{p-2}\partial_{x_i}) \tag{15.2}$$

is called the p-Laplace operator. Note that in the formula above we are making the summation convention and for $p = 2$, Δ_2 is just the usual Laplace operator. A weak formulation would lead to replace the first equation of (15.1) by

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \tag{15.3}$$

for any test function v. In the case where $p = 2$ (15.3) was making sense for

$$f, v, \partial_{x_i}u, \partial_{x_i}v \in L^2(\Omega). \tag{15.4}$$

In the case where $p \neq 2$, the integral on the right-hand side of (15.3) makes sense for (recall Hölder's inequality)

$$f \in L^{p'}(\Omega), \qquad v \in L^p(\Omega)$$

and the one on the left-hand side makes sense for

$$\partial_{x_i}u, \partial_{x_i}v \in L^p(\Omega).$$

This suggests to introduce Sobolev spaces of the model of $H^1(\Omega)$ but where $L^2(\Omega)$ is replaced by $L^p(\Omega)$.

Next suppose that we are interested in solving the problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{15.5}$$

Such a problem arises for instance in elasticity, $\vec{\nu}$ denotes the outward unit normal to the boundary $\partial\Omega$ of Ω. Then a weak formulation would lead to replace the first equation of (15.5) by

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx. \tag{15.6}$$

Here the first integral of (15.6) makes sense for

$$\Delta u, \Delta v \in L^2(\Omega).$$

Thus the $L^2(\Omega)$- framework is yet suitable in this case, however one needs more derivatives in $L^2(\Omega)$ than in the case of the Dirichlet problem.

Guided by the two examples above we introduce for $k \in \mathbb{N}$, $1 \leq p \leq +\infty$

$$W^{k,p}(\Omega) = \{\, v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega) \ \ \forall\, \alpha \in \mathbb{N}^n, |\alpha| \leq k \,\}. \tag{15.7}$$

(See Section 2.2 for the notation, the derivatives are taken in the distributional sense.) These spaces generalize the space H^1 constructed on $L^2(\Omega)$ since we have

$$W^{1,2}(\Omega) = H^1(\Omega). \tag{15.8}$$

15.2 Essential features of the Sobolev spaces $W^{k,p}$

We equip the spaces $W^{k,p}(\Omega)$ with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leq k} |D^\alpha u|_{p,\Omega}^p \right)^{\frac{1}{p}}. \tag{15.9}$$

Note that we will identify $W^{0,p}(\Omega)$ with $L^p(\Omega)$ and for $p = \infty$ the norm (15.9) is

$$\|u\|_{k,\infty} = \operatorname*{Max}_{|\alpha| \leq k} |D^\alpha u|_{\infty,\Omega},$$

(cf. the exercises).

Theorem 15.1. *Let Ω be an open subset of $\mathbb{R}^n$. Equipped with the norm* (15.9)*, $W^{k,p}(\Omega)$ is a Banach space.*

Proof. We leave to the reader to show that $W^{k,p}(\Omega)$ is a vector space and that (15.9) defines a norm. Let us show the completeness of these spaces for $p < +\infty$

the case $p = +\infty$ being analogous. Let u_n be a Cauchy sequence in $W^{k,p}(\Omega)$ – i.e., such that for n, m large enough we have

$$\sum_{|\alpha| \leq k} |D^\alpha u_n - D^\alpha u_m|_{p,\Omega}^p \leq \varepsilon^p \tag{15.10}$$

(note that for $\alpha = \mathbf{0}$, $D^\alpha u = u$). Then for any α, $|\alpha| \leq k$, $D^\alpha u_n$ is a Cauchy sequence in $L^p(\Omega)$ and there exists $u^\alpha \in L^p(\Omega)$ such that

$$D^\alpha u_n \longrightarrow u^\alpha \quad \text{in } L^p(\Omega)$$

(in particular $u_n \to u^{\mathbf{0}} = u$ in $L^p(\Omega)$). By the continuity of the derivation in $\mathcal{D}'(\Omega)$ we have

$$D^\alpha u_n \longrightarrow D^\alpha u$$

and thus $u \in W^{k,p}(\Omega)$. Moreover, letting $m \to +\infty$ in (15.10) we have for n large enough

$$\sum_{|\alpha| \leq k} |D^\alpha u_n - D^\alpha u|_{p,\Omega}^p \leq \varepsilon^p$$

that is to say $u_n \to u$ in $W^{k,p}(\Omega)$. This completes the proof of the theorem. $\square$

Copying the definition of $H_0^1(\Omega)$ we set for $1 \leq p < +\infty$

$$\begin{aligned} W_0^{k,p}(\Omega) &= \overline{\mathcal{D}(\Omega)} \\ &= \text{the closure of } \mathcal{D}(\Omega) \text{ in } W^{k,p}(\Omega). \end{aligned}$$

Theorem 15.2. *$W_0^{k,p}(\Omega)$ is a Banach space when endowed with the $\|\cdot\|_{k,p}$-norm.*

Proof. This is a trivial consequence of the definition. $\square$

Remark 15.1. In the case $p = 2$, $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$ are Hilbert spaces for the scalar product

$$\sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v),$$

$(\cdot,\cdot)$ being the canonical scalar product in $L^2(\Omega)$. In this case we will prefer the notation

$$H^k(\Omega), \qquad H_0^k(\Omega)$$

for the spaces.

We now establish a $L^p(\Omega)$-Poincaré inequality generalizing (2.49). We have

Theorem 15.3 (Poincaré Inequality). *Let Ω be an open set bounded in one direction. More precisely suppose that we are in the situation of Theorem 2.8. Then we have*

$$|v|_{p,\Omega} \leq \frac{2a}{p^{\frac{1}{p}}} \left| \frac{\partial v}{\partial \nu} \right|_{p,\Omega} \qquad \forall\, v \in W_0^{1,p}(\Omega). \tag{15.11}$$

Proof. We just sketch it since it follows the lines of the proof of Theorem 2.8. We suppose as there $\nu = e_1$. We then have for $v \in \mathcal{D}(\Omega)$

$$\begin{aligned}|v|^p(x) &= \left|\int_{-a}^{x_1} \partial_{x_1} v(s, x_2, \dots, x_n)\, ds\right|^p \leq \left(\int_{-a}^{x_1} |\partial_{x_1} v(s, x_2, \dots, x_n)|\, ds\right)^p \\ &\leq |x_1 + a|^{\frac{p}{p'}} \int_{-a}^{a} |\partial_{x_1} v(s, x_2, \dots, x_n)|^p\, ds \quad \text{(by Hölder)} \\ &= |x_1 + a|^{p-1} \int_{-a}^{a} |\partial_{x_1} v(s, x_2, \dots, x_n)|^p\, ds.\end{aligned}$$

Integrating in x_1 between $(-a, a)$ and then in the other directions the result follows. □

As a consequence we have

Theorem 15.4. *Suppose that Ω is bounded in one direction. Then on $W_0^{k,p}(\Omega)$ the norms*

$$\|u\|_{k,p} \quad \text{and} \quad |u\|_{k,p} = \left(\sum_{|\alpha|=k} |D^\alpha u|_{p,\Omega}^p\right)^{\frac{1}{p}} \tag{15.12}$$

are equivalent.

Proof. One has trivially

$$|u\|_{k,p} \leq \|u\|_{k,p} \quad \forall\, u \in W_0^{k,p}(\Omega).$$

Now since Ω is bounded in one direction by the Poincaré inequality we have for $u \in \mathcal{D}(\Omega)$ and for some constant C independent of u

$$|u|_{p,\Omega} \leq \frac{2a}{p^{\frac{1}{p}}} \left|\frac{\partial u}{\partial \nu}\right|_{p,\Omega} \leq C\left(\sum_{i=1}^{n} |\partial_{x_i} u|_{p,\Omega}^p\right)^{\frac{1}{p}}. \tag{15.13}$$

(We used here the fact that in $\mathbb{R}^n$ all the norms are equivalent.) Taking $u = \partial_{x_j} u$ in (15.13) we get summing up in j

$$\sum_{j=1}^{n} |\partial_{x_j} u|_{p,\Omega}^p \leq C \sum_{i,j} |\partial_{x_i x_j}^2 u|_{p,\Omega}^p.$$

Continuing this process the result follows, i.e., we obtain for some other constant C

$$\|u\|_{k,p} \leq C\left(\sum_{|\alpha|=k} |D^\alpha u|_{p,\Omega}^p\right)^{\frac{1}{p}} = C|u\|_{k,p} \quad \forall\, u \in \mathcal{D}(\Omega)$$

and by density (15.12) holds. This completes the proof of the theorem. □

We also have

Theorem 15.5. $W_0^{k,p}(\Omega)$, $W^{k,p}(\Omega)$ *are reflexive for any k, $1 < p < +\infty$.*

(See for instance [1].)

15.3 An application

As we were able to solve the problem of Dirichlet weakly we can now solve in a weak sense (15.5). Indeed we have

Theorem 15.6. *Suppose that $f \in L^2(\Omega)$ then if Ω is bounded in one direction there exists a unique u solution to*

$$\begin{cases} \int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx \quad \forall\, v \in H_0^2(\Omega), \\ u \in H_0^2(\Omega). \end{cases} \tag{15.14}$$

Proof. We apply the Lax–Milgram theorem. Indeed $v \mapsto \int_\Omega f v \, dx$ is a continuous linear form on $H_0^2(\Omega)$. Moreover $(u, v) \mapsto \int_\Omega \Delta u \Delta v \, dx$ is a continuous bilinear form on $H_0^2(\Omega)$. We will be done if we can show that this bilinear form is coercive. This is a consequence of the following simple remark. If $u \in \mathcal{D}(\Omega)$ then we have

$$\begin{aligned} \int_\Omega (\Delta u)^2 \, dx &= \int_\Omega \Big(\sum_i \partial_{x_i}^2 u \Big)^2 \, dx \\ &= \int_\Omega \sum_{i,j} \partial_{x_i}^2 u \partial_{x_j}^2 u \, dx \\ &= \int_\Omega \sum_{i,j} (\partial_{x_i x_j}^2 u)^2 \, dx. \end{aligned} \tag{15.15}$$

Indeed the last equality follows by integration by parts since $u \in \mathcal{D}(\Omega)$ or from the definition of distributional derivative. From (15.15) and by density of $\mathcal{D}(\Omega)$ in $H_0^2(\Omega)$ we derive

$$\int_\Omega (\Delta u)^2 \, dx \geq c \|u\|_{2,2}^2 \qquad \forall\, u \in H_0^2(\Omega)$$

for some constant c. The coerciveness of the bilinear form follows and this completes the proof of the theorem. □

Remark 15.2. $H_0^1(\Omega)$ is the space of functions of $H^1(\Omega)$ "vanishing" on $\partial\Omega$. Thus in this spirit $H_0^2(\Omega)$ is the space of functions of $H^2(\Omega)$ such that $u = 0$, $\partial_{x_i} u = 0$ on $\partial\Omega$ for any i. This last condition seems to be stronger than $u = 0$, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. This is not the case. Indeed for a smooth function, $u = 0$ on $\partial\Omega$ implies that all tangential derivatives vanish. Since the normal derivative vanishes too so does the gradient.

We introduced above the spaces built on $L^p(\Omega)$ with the idea to solve the problem (15.1). However we see that the left-hand side of the equality (15.3) is not linear in u and thus does not define a bilinear form. Some more work is then needed in order to be able to reach a solution for (15.1).

Exercises

1. Let $p, q \in [1, +\infty]$. Set

$$W^{1,p,q}(\Omega) = \{\, v \in L^p(\Omega) \mid \partial_{x_i} v \in L^q(\Omega) \;\; \forall\, i = 1, \ldots, n \,\}.$$

   Show that $W^{1,p,q}(\Omega)$ is a Banach space when equipped with the norm

$$|v|_{p,\Omega} + \big||\nabla v|\big|_{q,\Omega}.$$

2. Let Ω be a measurable subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a measurable function.
   (i) Show that

$$\operatorname{ess\,sup} |f| \leq \varliminf_{p \to +\infty} |f|_{p,\Omega}.$$

   (ii) Assume that $f \in L^\infty(\Omega) \cap L^{p_0}(\Omega)$ for some $p_0 \geq 1$. Prove that $f \in L^p(\Omega)$ $\forall\, p \geq p_0$ and that

$$\lim_{p \to +\infty} |f|_{p,\Omega} = |f|_{\infty,\Omega}.$$

3. Let T be a continuous linear form on $W_0^{1,p}(\Omega)$, $1 \leq p < +\infty$. Show that there exist $T_0, T_1, \ldots, T_n \in L^{p'}(\Omega)$ such that

$$T = T_0 + \sum_{i=1}^{n} \partial_{x_i} T_i$$

   (compare to Theorem 2.10).

   Hint: consider the map $u \mapsto (u, \nabla u)$ from $W_0^{1,p}(\Omega)$ into $\big(L^p(\Omega)\big)^{n+1}$.

Chapter 16

Regularity Theory

16.1 Introduction

We have seen in Chapter 3 that when u is a regular or strong solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{16.1}$$

then u is a weak solution of it. This was an argument to introduce weak solutions since strong solutions could not slip through our existence theory. Having obtained a weak solution to (16.1) one can then ask oneself if this solution also satisfies (16.1) in the usual sense. This requires of course some assumptions on f: recall that we know how to solve (16.1) for very general $f \in H^{-1}(\Omega)$. Let us try to get some insight through the one-dimensional problem

$$\begin{cases} -u'' = f & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{16.2}$$

When solved for $f \in L^2(\Omega)$, the solution u belongs to $H^1(\Omega)$. However since in the distributional sense

$$u'' = -f \tag{16.3}$$

we have in fact $u \in H^2(\Omega)$. Thus $f \in L^2(\Omega)$ implies $u \in H^2(\Omega)$. If we derive k times the equation (16.3) in the distributional sense we get

$$u^{(k+2)} = -f^{(k)}. \tag{16.4}$$

(k) denotes the k^{th} derivative. Thus

$$f \in H^k(\Omega) \quad \Longleftrightarrow \quad u \in H^{k+2}(\Omega). \tag{16.5}$$

A natural question is to try to establish (16.5) for u solution to (16.1) in higher dimensions. The difficulty is due to the fact that a priori $\Delta u \in L^2(\Omega)$ does not imply that all second derivatives of u belong to $L^2(\Omega)$. Saying that in a different manner $\Delta u \in H^k(\Omega)$ does not necessarily imply that $u \in H^{k+2}(\Omega)$. When it is not clear that derivatives exist one can work with difference quotients. Let us explain this briefly. For $u \in W^{1,p}(\Omega)$, $1 \leq p$, $h \in \mathbb{R}$ let us set

$$\tau_{h,s}(u) = \tau_h(u) = \frac{u(x + he_s) - u(x)}{h}. \tag{16.6}$$

In the expression above e_s denotes the s^{th} vector of the canonical basis of $\mathbb{R}^n$, i.e.,

$$e_s = (0, \dots, 1, \dots, 0)$$

where the 1 is located on the s^{th} slot. The quotient above is defined for $h < \operatorname{dist}(x, \partial\Omega)$.

Remark 16.1. It is easy to show that

$$\tau_{h,s}(u) \longrightarrow \partial_{x_s} u \quad \text{in } \mathcal{D}'(\Omega) \tag{16.7}$$

when $h \to 0$.

We have

Proposition 16.1.

(i) *Suppose that $u, v \in L^2(\Omega)$ and one of them has compact support $K \subset \Omega$. Then for $h < \operatorname{dist}(K, \partial\Omega)$ we have*

$$\int_\Omega u\tau_h(v)\,dx = -\int_\Omega \tau_{-h}(u)v\,dx. \tag{16.8}$$

(ii) *If $u \in W^{1,p}(\Omega)$ and $\Omega' \subset\subset \Omega$ for $h < \operatorname{dist}(\Omega', \partial\Omega)$ we have $\tau_h(u) \in W^{1,p}(\Omega')$ and*

$$\partial_{x_k}(\tau_h u) = \tau_h(\partial_{x_k} u). \tag{16.9}$$

(iii) *For $h < \operatorname{dist}(x, \partial\Omega)$ we have*

$$\tau_h(uv)(x) = u(x + he_s)\tau_h v(x) + \tau_h u(x)v(x). \tag{16.10}$$

Proof. (i) The functions are supposed to be extended by 0 when necessary. As above we drop when it is clear the index s for simplicity. We have

$$\begin{aligned}\int_\Omega u\tau_h(v)\,dx &= \int_\Omega u(x)\frac{v(x + he_s) - v(x)}{h}\,dx \\ &= \int_\Omega \frac{u(x)v(x + he_s)}{h}\,dx - \int_\Omega \frac{u(x)v(x)}{h}\,dx.\end{aligned}$$

Changing x into $x - he_s$ in the first integral it comes

$$\begin{aligned}\int_\Omega u\tau_h(v)\,dx &= \int_\Omega \frac{u(x-he_s)v(x)}{h}\,dx - \int_\Omega \frac{u(x)v(x)}{h}\,dx\\ &= -\int_\Omega \frac{u(x-he_s)-u(x)}{-h}v(x)\,dx = -\int_\Omega \tau_{-h}(u)v\,dx.\end{aligned}$$

This completes the proof of (i).

(ii) Let $\varphi \in \mathcal{D}(\Omega')$, $h < \operatorname{dist}(\Omega', \partial\Omega)$. We have

$$\begin{aligned}\langle \partial_{x_k}(\tau_h u), \varphi\rangle &= -\langle \tau_h u, \partial_{x_k}\varphi\rangle\\ &= -\int_\Omega \tau_h(u)\partial_{x_k}\varphi\,dx\\ &= \int_\Omega u\tau_{-h}(\partial_{x_k}\varphi)\,dx \qquad \text{by (i).}\end{aligned}$$

It follows (note that the assumptions on φ and h imply that $\tau_h\varphi \in \mathcal{D}(\Omega)$)

$$\begin{aligned}\langle \partial_{x_k}(\tau_h u), \varphi\rangle &= \int_\Omega u\partial_{x_k}(\tau_{-h}\varphi)\,dx\\ &= -\int_\Omega \partial_{x_k}u(\tau_{-h}\varphi)\,dx = \int_\Omega \tau_h(\partial_{x_k}u)\varphi\,dx\\ &= \langle \tau_h(\partial_{x_k}u), \varphi\rangle.\end{aligned}$$

This completes the proof of (ii).

(iii) One has by definition of τ_h

$$\begin{aligned}\tau_h(uv)(x) &= \frac{u(x+he_s)v(x+he_s) - u(x)v(x)}{h}\\ &= u(x+he_s)\frac{v(x+he_s)-v(x)}{h} + \frac{u(x+he_s)-u(x)}{h}v(x)\end{aligned}$$

which completes the proof of the proposition. □

The following theorem relies the difference quotients to the derivatives.

Theorem 16.2. (i) *Assume that $u \in W^{1,p}(\Omega)$, $1 \le p \le +\infty$. Let $\Omega' \subset\subset \Omega$. Then for $|h| < \operatorname{dist}(\Omega', \partial\Omega)$ we have*

$$|\tau_{h,s}u|_{p,\Omega'} \le |\partial_{x_s}u|_{p,\Omega}. \tag{16.11}$$

(ii) *Assume that $u \in L^p(\Omega)$, $1 \le p \le +\infty$. If there exists a constant C such that for all $\Omega' \subset\subset \Omega$*

$$|\tau_{h,s}u|_{p,\Omega'} \le C \quad \forall\, h,\ |h| < \operatorname{dist}(\Omega', \partial\Omega), \tag{16.12}$$

then $\partial_{x_s}u \in L^p(\Omega)$ and

$$|\partial_{x_s}u|_{p,\Omega} \le C. \tag{16.13}$$

Proof. (i) Suppose first that u is smooth. Then we have

$$\begin{aligned}\tau_{h,s}u(x) = \frac{u(x+he_s)-u(x)}{h} &= \frac{1}{h}\int_0^h \frac{d}{dt}u(x+te_s)\,dt\\ &= \fint_{[0,h]} \frac{d}{dt}u(x+te_s)\,dt\end{aligned}$$

where $\fint_A$ denotes the average on A. Thus we derive for $p<+\infty$

$$\begin{aligned}|\tau_{h,s}u|_{p,\Omega'}^p &= \int_{\Omega'}\left|\fint_{[0,h]}\frac{d}{dt}u(x+te_s)\,dt\right|^p dx\\ &\le \int_{\Omega'}\fint_{[0,h]}|\partial_{x_s}u(x+te_s)|^p\,dt\,dx \qquad \text{(by Hölder's inequality)}\\ &\le \fint_{[0,h]}\int_{\Omega'}|\partial_{x_s}u(x+te_s)|^p\,dx\,dt \qquad \text{(by Fubini's theorem)}\\ &\le \fint_{[0,h]}\int_{\Omega}|\partial_{x_s}u(x)|^p\,dx\,dt\\ &= |\partial_{x_s}u|_{p,\Omega}^p.\end{aligned}$$

If u is not smooth one replaces it by $u_\varepsilon = \rho_\varepsilon * u$ for ε small enough and passes to the limit in ε. To have the result for $p=+\infty$ one can pass to the limit in $p\to+\infty$ since one has

$$\lim_{p\to+\infty}|u|_{p,\widetilde{\Omega}} = |u|_{\infty,\widetilde{\Omega}} \quad \forall\, u\in L^\infty(\Omega)$$

(if $\widetilde{\Omega}$ has finite measure see Exercise 2 of the previous chapter).

(ii) By assumption $\tau_{h,s}u$ is bounded in $L^p(\Omega')$. So we can extract a "sequence" such that

$$\tau_{h,s}u \rightharpoonup v$$

in $L^p(\Omega')$ when $h\to 0$. We have already noticed that

$$\tau_{h,s}u \longrightarrow \partial_{x_s}u \quad \text{in } \mathcal{D}'(\Omega').$$

It follows that $\partial_{x_s}u = v\in L^p(\Omega')$. Thus $u\in W^{1,p}(\Omega')$. Moreover passing to the limit-inf in

$$|\tau_{h,s}u|_{p,\Omega'}\le C$$

we obtain by the weak lower semi-continuity of the norm

$$|\partial_{x_s}u|_{p,\Omega'}\le C.$$

This completes the proof of the theorem, since this holds for any Ω'. □

16.2 The translation method

This technique uses only the structure of ellipticity – i.e., the coerciveness and the continuity – so there is no additional difficulty to present it on systems. We thus consider a function $\mathbf{u}$ which satisfies

$$\begin{cases} \mathbf{u} \in \mathbb{H}^1(\Omega), \\ \int_\Omega \mathcal{A}(x)\nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall\, \mathbf{v} \in \mathbb{H}_0^1(\Omega) \end{cases} \tag{16.14}$$

where – to simplify we will assume

$$\mathbf{f} \in \mathbb{L}^2(\Omega). \tag{16.15}$$

(See Chapter 13 for the notation. Recall that $\mathbf{u}$ has m entries.) We assume that the operator $\mathcal{A}$ is having its coefficients in $L^\infty(\Omega)$ in such a way that it holds for some constant Λ

$$|\mathcal{A}(x)M| \le \Lambda |M| \tag{16.16}$$

(see (13.16)).

We will consider the following coerciveness assumption

$$\begin{aligned} &\exists\, \lambda_1, \lambda_2 > 0 \text{ such that} \\ &\int_\Omega \mathcal{A}(x)\nabla \mathbf{u} \cdot \nabla \mathbf{u}\, dx \ge \lambda_1 \big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}^2 - \lambda_2 \big|\big|\mathbf{u}\big|\big|_{2,\Omega}^2 \quad \forall\, \mathbf{u} \in \mathbb{H}_0^1(\Omega). \end{aligned} \tag{16.17}$$

Remark 16.2. Note that in (16.14) we did not specify any boundary condition – i.e., $\mathbf{u}$ could be in any subspace of $\mathbb{H}^1(\Omega)$. The assumption (16.17) called the Gårding inequality holds of course when $\mathcal{A}$ satisfies the Legendre condition or when $\mathcal{A}$ is an operator with constant coefficients satisfying the Legendre–Hadamard condition. Having stated our coerciveness assumption under the form (16.17) we do not need any assumption on Ω.

We have then

Theorem 16.3. *Suppose that* $\mathbf{u}$ *satisfies* (16.14)*. If* $\mathcal{A} \in C^{0,1}(\Omega)$ *– i.e., the coefficients* $A_{ij}^{\alpha\beta}$ *are uniformly Lipschitz continuous in* Ω *– and* (16.15)*–*(16.17) *hold, then for every* $\Omega' \subset\subset \Omega$*,* $\mathbf{u} \in \mathbb{H}^2(\Omega') = (H^2(\Omega'))^m$ *and we have for some constant* C *independent of* u

$$\big|\big|D^2\mathbf{u}\big|\big|_{2,\Omega'} \le C\big\{\big|\big|\mathbf{f}\big|\big|_{2,\Omega} + \big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}\big\} \tag{16.18}$$

($D^2\mathbf{u}$ *denotes the vector of all second derivatives of the components of* $\mathbf{u}$).

Proof. Let us choose $\eta \in \mathcal{D}(\Omega)$ such that

$$0 \le \eta \le 1, \qquad \eta = 1 \text{ on } \Omega'. \tag{16.19}$$

Then the function

$$\mathbf{v} = -\eta\tau_{-h,s}(\eta\tau_{h,s}\mathbf{u}) \in \mathbb{H}_0^1(\Omega) \tag{16.20}$$

for h small enough, see Proposition 16.1 (ii). Of course $\tau_{h,s}\mathbf{u}$ is the vector with components $\tau_{h,s}u^i$. We will drop the index s in what follows. Using (16.14) and with the summation convention we obtain

$$\begin{aligned} &-\int_\Omega A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j\{\eta\partial_{x_\alpha}(\tau_{-h}(\eta\tau_h u^i))\}\,dx \\ &\qquad = \int_\Omega A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j(\partial_{x_\alpha}\eta)\tau_{-h}(\eta\tau_h u^i)\,dx - \int_\Omega \mathbf{f}\cdot\eta\tau_{-h}(\eta\tau_h\mathbf{u})\,dx. \end{aligned} \tag{16.21}$$

Let us denote by I the left-hand side of the equality above. Using Proposition 16.1 we derive

$$\begin{aligned} I &= \int_\Omega \tau_h(A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j\eta)\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &= \int_\Omega \{A_{ij}^{\alpha\beta}\eta\tau_h(\partial_{x_\beta}u^j) + \partial_{x_\beta}u^j(x+he_s)\tau_h(A_{ij}^{\alpha\beta}\eta)\}\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &= \int_\Omega \{A_{ij}^{\alpha\beta}\eta\partial_{x_\beta}(\tau_h u^j) + \partial_{x_\beta}u^j(x+he_s)\tau_h(A_{ij}^{\alpha\beta}\eta)\}\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &= \int_\Omega \{A_{ij}^{\alpha\beta}\partial_{x_\beta}(\eta\tau_h u^j) - A_{ij}^{\alpha\beta}\partial_{x_\beta}\eta\tau_h u^j + \partial_{x_\beta}u^j(x+he_s)\tau_h(A_{ij}^{\alpha\beta}\eta)\}\partial_{x_\alpha}(\eta\tau_h u^i)\,dx. \end{aligned}$$

Combining this computation with (16.21) we get

$$\begin{aligned} &\int_\Omega A_{ij}^{\alpha\beta}\partial_{x_\beta}(\eta\tau_h u^j)\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &\quad = \int_\Omega A_{ij}^{\alpha\beta}\partial_{x_\beta}\eta\tau_h u^j\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &\qquad - \int_\Omega \partial_{x_\beta}u^j(x+he_s)\tau_h(A_{ij}^{\alpha\beta}\eta)\partial_{x_\alpha}(\eta\tau_h u^i)\,dx \\ &\qquad + \int_\Omega A_{ij}^{\alpha\beta}\partial_{x_\beta}u^j(\partial_{x_\alpha}\eta)\tau_{-h}(\eta\tau_h u^i)\,dx \\ &\qquad - \int_\Omega \mathbf{f}\cdot\eta\tau_{-h}(\eta\tau_h\mathbf{u})\,dx. \end{aligned} \tag{16.22}$$

Since we assumed the coefficients $A_{ij}^{\alpha\beta}$ uniformly Lipschitz continuous there exists a constant L such that

$$|\tau_h(A_{ij}^{\alpha\beta}\eta)| \le L \quad \forall\, i,j,\alpha,\beta.$$

Then from (16.22) we derive

$$\begin{aligned}\int_\Omega \mathcal{A}\nabla(\eta\tau_h\mathbf{u})\cdot\nabla(\eta\tau_h\mathbf{u})\,dx \leq \Lambda\int_\Omega &|\partial_{x_\beta}\eta(\tau_h u^j)|\,|\nabla(\eta\tau_h\mathbf{u})|\,dx\\ &+ LC\int_\Omega |\nabla\mathbf{u}(x+he_s)|\,|\nabla(\eta\tau_h\mathbf{u})|\,dx\\ &+ \Lambda C\int_\Omega |\nabla\mathbf{u}|\,|\tau_{-h}(\eta\tau_h\mathbf{u})|\,dx\\ &+ \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big|\big|\tau_{-h}(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}\end{aligned}$$

(C is a constant depending on η, $|\cdot|$ denotes the norm of matrices or vectors). Using now Theorem 16.2 (note that the first integral in the right-hand side is an integral on the support of η) we get

$$\begin{aligned}&\int_\Omega \mathcal{A}\nabla(\eta\tau_h\mathbf{u})\cdot\nabla(\eta\tau_h\mathbf{u})\,dx\\ &\qquad\leq \big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}\big\{\Lambda C\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + LC\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \Lambda C\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big\}.\end{aligned}$$

We used above the Cauchy–Schwarz inequality. Thus by (16.17) we obtain for some new constant C

$$\lambda_1\big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}^2 - \lambda_2\big|\big|\eta\tau_h\mathbf{u}\big|\big|_{2,\Omega}^2 \leq C\big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}\big\{\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big\}.$$

Using the Young inequality

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$$

we obtain

$$\lambda_1\big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}^2 \leq \lambda_2\big|\big|\eta\tau_h\mathbf{u}\big|\big|_{2,\Omega}^2 + \varepsilon\big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}^2 + \frac{C^2}{4\varepsilon}\big\{\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big\}^2.$$

Choosing $\varepsilon = \frac{\lambda_1}{2}$ and using (16.11) for the term in λ_2 we obtain for some new constant $C = C(\lambda_1, \lambda_2, \Lambda, L, \eta)$

$$\big|\big|\nabla(\eta\tau_h\mathbf{u})\big|\big|_{2,\Omega}^2 \leq C^2\big\{\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big\}^2.$$

In particular since $\eta = 1$ on Ω' we get for any $\Omega'' \subset\subset \Omega'$

$$\big|\big|\nabla(\tau_{h,s}\mathbf{u})\big|\big|_{2,\Omega''}^2 \leq C^2\big\{\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}\big\}^2$$

where C is independent of Ω''. Using Theorem 16.2, (ii) we obtain

$$\big|\big|\nabla(\partial_{x_s}\mathbf{u})\big|\big|_{2,\Omega'}^2 \leq C^2\big\{\big|\big|\nabla\mathbf{u}\big|\big|_{2,\Omega} + \big|\big|\mathbf{f}\big|\big|_{2,\Omega}^2\big\}. \tag{16.23}$$

Since this inequality holds for any s this provides an estimate for all second derivatives of $\mathbf{u}$. This completes the proof of the theorem. □

Remark 16.3. As a consequence of the theorem above we have achieved part of the programme mentioned in the introduction of this chapter namely if u is a weak solution to

$$\begin{cases} u \in H^1(\Omega), \\ -\Delta u = f \qquad \text{in } \Omega \end{cases} \tag{16.24}$$

then $f \in L^2(\Omega)$ implies that $u \in H^2(\Omega')$ for any subdomain $\Omega' \subset\subset \Omega$ and we have for some constant C

$$\|u\|_{2,2,\Omega'} \leq C\big\{|f|_{2,\Omega} + |u|_{2,\Omega} + \big||\nabla u|\big|_{2,\Omega}\big\} \tag{16.25}$$

(we denote by $\|u\|_{2,2,\Omega'}$ the norm in $H^2(\Omega')$ defined in (15.9)). Indeed Theorem 16.3 clearly applies here for $m = 1$ and the bilinear form

$$(u, v) \mapsto \int_\Omega \nabla u \cdot \nabla v \, dx.$$

This allows us to bound the second derivatives of u in $L^2(\Omega')$. The rest of the norm of $\|u\|_{2,2,\Omega'}$ is included in the right-hand side of (16.25). We would like to iterate this process but before we would like to analyze the connection between higher Sobolev spaces and regularity.

16.3 Regularity of functions in Sobolev spaces

Definition 16.1. We say that a function u is Hölder continuous in $\overline{\Omega}$ with order $\alpha \in (0, 1)$ if we have for some constant C

$$|u(x) - u(y)| \leq C|x - y|^\alpha \quad \forall\, x, y \in \Omega. \tag{16.26}$$

We will denote by $C^\alpha(\overline{\Omega})$ the set of the (uniformly) Hölder continuous functions in Ω.

Then we have

Theorem 16.4. *Let Ω be an open set in $\mathbb{R}^n$. We have for any $1 < p$*

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \quad \forall\, p < n, \tag{16.27}$$

$$W_0^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega}) \quad \forall\, p > n. \tag{16.28}$$

p^* is the Sobolev exponent given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, $\hookrightarrow$ means an algebraic inclusion together with the continuity of the canonical embedding – see below Exercise 2 for the metric of C^α, $\alpha = 1 - \frac{n}{p}$.

Proof. (16.27) follows directly from (12.30). (16.28) will be a consequence of the following estimate. Namely there exists a constant $C = C(p, n)$ such that

$$|v(x) - v(y)| \leq C\big||\nabla v|\big|_{p,\mathbb{R}^n}|x - y|^\alpha \quad \forall\, x, y \in \mathbb{R}^n, \ \forall\, v \in C_c^\infty(\mathbb{R}^n). \tag{16.29}$$

For $x, y \in \mathbb{R}^n$, set $r = |x-y|$ and denote by D the domain

$$D = B_r(x) \cap B_r(y).$$

One has

$$|v(x) - v(y)| \leq |v(x) - v(z)| + |v(z) - v(y)| \quad \forall\, z \in D.$$

By integration on D in z it follows that

$$|D|\, |v(x) - v(y)| \leq \int_D |v(x) - v(z)|\, dz + \int_D |v(z) - v(y)|\, dz. \tag{16.30}$$

(Recall that $|D|$ is the measure of D.) Let us evaluate the first integral above – the second one can be treated the same way. From

$$v(z) - v(x) = \int_0^1 \frac{d}{dt} v(x + t(z-x))\, dt = \int_0^1 \nabla v(x + t(z-x)) \cdot (z-x)\, dt$$

we derive

$$|v(x) - v(z)| \leq \int_0^1 |\nabla v(x + t(z-x))|\, |z-x|\, dt \quad \forall\, z \in D.$$

It follows then that we have

$$\begin{aligned}\int_D |v(x) - v(z)|\, dz &\leq r \int_D \int_0^1 |\nabla v(x + t(z-x))|\, dt\, dz \\ &= r \int_0^1 \int_D |\nabla v(x + t(z-x))|\, dz\, dt.\end{aligned}$$

Setting $\xi = x + t(z-x)$, $d\xi = t^n dz$ we get

$$\int_D |v(x) - v(z)|\, dz \leq r \int_0^1 \int_{B_{tr}(x)} |\nabla v(\xi)| t^{-n}\, d\xi\, dt.$$

Using Hölder's inequality it comes

$$\begin{aligned}\int_D |v(x) - v(z)|\, dz &\leq r \int_0^1 \left(\int_{B_{tr}(x)} |\nabla v(\xi)|^p\, d\xi \right)^{\frac{1}{p}} |B_{tr}(x)|^{1-\frac{1}{p}} t^{-n}\, dt \\ &\leq r \big|| \nabla v |\big|_{p,\mathbb{R}^n} \int_0^1 \{(tr)^n \omega_n\}^{1-\frac{1}{p}} t^{-n}\, dt \\ &= r^{n+1-\frac{n}{p}} \big|| \nabla v |\big|_{p,\mathbb{R}^n} \omega_n^{1-\frac{1}{p}} \int_0^1 t^{-\frac{n}{p}}\, dt \\ &= \frac{1}{1-\frac{n}{p}} \omega_n^{1-\frac{1}{p}} \big|| \nabla v |\big|_{p,\mathbb{R}^n} r^{n+1-\frac{n}{p}},\end{aligned}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$. Since the second integral of (16.30) can be estimated the same way we get

$$|D|\,|v(x)-v(y)| \le \frac{2}{1-\frac{n}{p}}\omega_n^{1-\frac{1}{p}}\big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n}|x-y|^{n+1-\frac{n}{p}}.$$

It is clear that

$$|D| = c(n)|x-y|^n$$

for some constant $c(n)$. It follows that for some constant $c = C(n,p)$ we have

$$|v(x)-v(y)| \le C\big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n}|x-y|^{1-\frac{n}{p}}. \tag{16.31}$$

This completes the proof of (16.29).

As a consequence of (16.29) we also have

$$|v(x)| \le |v(y)| + C\big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n}|x-y|^{1-\frac{n}{p}}$$

and thus integrating this in y on $B_1(x)$ we obtain

$$\begin{aligned}|v(x)| &\le \fint_{B_1(x)} |v(y)|\,dy + C\big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n} \fint_{B_1(x)} |x-y|^{1-\frac{n}{p}}\,dy\\ &\le \fint_{B_1(x)} |v(y)|\,dy + C\big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n}\end{aligned}$$

where $\fint$ denotes the average. Applying Hölder's inequality in the first integral we deduce that

$$|v(x)| \le C\big\{|v|_{p,\mathbb{R}^n} + \big|\big|\nabla v\big|\big|_{p,\mathbb{R}^n}\big\} \tag{16.32}$$

for some constant $C = C(p,n)$. If now $v_n \in \mathcal{D}(\Omega)$ is such that

$$v_n \longrightarrow v \quad \text{in } W_0^{1,p}(\Omega)$$

then by (16.32) v_n is a Cauchy sequence for the uniform convergence and thus converges toward a continuous function which also satisfies (16.29) and is a continuous representative for v. $\square$

Remark 16.4. Note that even for Ω not necessarily bounded we have obtained

$$|v|_{\infty,\Omega} \le C(p,n)\big|\big|v\big|\big|_{1,p} \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{16.33}$$

As a consequence we have

Theorem 16.5. *Let Ω be an open set in $\mathbb{R}^n$.*

$$\text{If } kp < n, \qquad W_0^{k,p}(\Omega) \hookrightarrow L^{\frac{np}{n-kp}}(\Omega) \tag{16.34}$$

$$\text{If } kp > n,\ \frac{n}{p} \notin \mathbb{N}, \qquad W_0^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\overline{\Omega}) \text{ with } m = \left[k - \frac{n}{p}\right],\ \alpha = k - \frac{n}{p} - m. \tag{16.35}$$

(*Recall that* $[\cdot]$ *denotes the integer part of a real number.*)

In (16.35) $C^{m,\alpha}(\overline{\Omega})$ *denotes the space of functions of class* C^m *in* $\overline{\Omega}$ *such that the derivatives of order* m *belong to* $C^\alpha(\overline{\Omega})$.

Proof. One proceeds by iteration.

In the case where $k < \frac{n}{p}$ – arguing eventually first for smooth functions one has $W_0^{k,p}(\Omega) \hookrightarrow W_0^{k-1,p^*}(\Omega) \hookrightarrow \cdots \hookrightarrow L^{p^{*k}}(\Omega)$ where p^{*k} stands for the fact that we have taken k times the $*$ operation, i.e., since

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

we have

$$\frac{1}{p^{*k}} = \frac{1}{p} - \frac{k}{n} = \frac{n - kp}{np}$$

and (16.34) follows.

In the case where $k > \frac{n}{p}$, $\frac{n}{p} \notin \mathbb{N}$, $u \in W_0^{k,p}(\Omega)$ we have by the first part and the preceding theorem that all the derivatives of order m of u belong to $C^\alpha(\overline{\Omega})$. Then u is derivable in the usual sense m times and its partial derivatives of order m belong to $C^\alpha(\overline{\Omega})$ – see also Exercise 3. This completes the proof of the theorem. □

Remark 16.5. In the case where $\frac{n}{p} \in \mathbb{N}$ one has of course

$$W_0^{k,p}(\Omega) \hookrightarrow C^{m-1,\alpha}(\overline{\Omega}) \tag{16.36}$$

for every $\alpha \in (0,1)$.

16.4 The bootstrap technique

We would like to show now that the solutions we have defined by a weak formulation are in fact smooth – i.e., differentiable in the usual sense – provided our data are themselves smooth. Applying successively Theorem 16.3 we will gain at each step some additional smoothness. Such reasoning – typical in partial differential equations – is called "bootstrapping".

We have

Theorem 16.6. *Under the assumptions of Theorem* 16.3 *and if*

$$A_{ij}^{\alpha\beta} \in C^{k,1}(\overline{\Omega}), \quad \mathbf{f} \in \mathbb{H}^k(\Omega) \tag{16.37}$$

then for every subdomain Ω' *we have* $\mathbf{u} \in \mathbb{H}^{k+2}(\Omega') := (H^{k+2}(\Omega'))^m$ *and for some constant* C *independent of* $\mathbf{u}$

$$\big|\big|D^{k+2}\mathbf{u}\big|\big|_{2,\Omega'} \le C\big\{\|\mathbf{f}\|_{k,2} + \big|\big|\nabla \mathbf{u}\big|\big|_{2,\Omega}\big\}. \tag{16.38}$$

(*For a function* $\mathbf{v}$, $D^k\mathbf{v}$ *denotes the vector of the* k^{th} *derivatives of each component of* $\mathbf{v}$.)

Proof. Let $\mathbf{u}$ be a solution to (16.14) that we can also write as

$$\int_\Omega A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \partial_{x_\alpha} v^i \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \quad \forall \, \mathbf{v} \in \mathbb{H}_0^1(\Omega). \tag{16.39}$$

If $\mathbf{v} \in \big(\mathcal{D}(\Omega)\big)^m$ we can use for any $s, -\partial_{x_s}\mathbf{v}$ as test function in (16.39). We get

$$\int_\Omega A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \partial_{x_\alpha}(-\partial_{x_s} v^j) \, dx = \int_\Omega \mathbf{f} \cdot (-\partial_{x_s}\mathbf{v}) \, dx. \tag{16.40}$$

We know by Theorem 16.3 that

$$\nabla \mathbf{u} \in \mathbb{H}^1(\Omega') \quad \forall \, \Omega' \subset\subset \Omega,$$

i.e., each entry of this Jacobian matrix is in $H^1(\Omega')$. If in addition $A_{ij}^{\alpha\beta} \in C^{1,1}(\overline{\Omega})$ then

$$A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j \in \mathbb{H}^1(\Omega'),$$

and permuting first ∂_{x_α} and ∂_{x_s} we get from (16.40)

$$\begin{aligned}
&\int_\Omega \partial_{x_s}(A_{ij}^{\alpha\beta} \partial_{x_\beta} u^j) \partial_{x_\alpha} v^j \, dx = \int_\Omega \partial_{x_s}\mathbf{f} \cdot \mathbf{v} \, dx \qquad (16.41)\\
\iff &\int_\Omega A_{ij}^{\alpha\beta} \partial_{x_\beta}(\partial_{x_s} u^j) \partial_{x_\alpha} v^j \, dx = \int_\Omega \partial_{x_s}\mathbf{f} \cdot \mathbf{v} \, dx - \int_\Omega \partial_{x_s}(A_{ij}^{\alpha\beta}) \partial_{x_\beta} u^j \partial_{x_\alpha} v^i \, dx\\
\iff &\int_\Omega \mathcal{A}(x) \nabla \partial_{x_s}\mathbf{u} \cdot \nabla \mathbf{v} \, dx = \int_\Omega \partial_{x_s}\mathbf{f} \cdot \mathbf{v} \, dx + \int_\Omega \partial_{x_\alpha}(\partial_{x_s}(A_{ij}^{\alpha\beta}) \partial_{x_\beta} u^j) v^i \, dx.
\end{aligned}$$

Since for every α and for $\Omega' \subset\subset \Omega'' \subset\subset \Omega$

$$\partial_{x_\alpha}(\partial_{x_s}(A_{ij}^{\alpha\beta}) \partial_{x_\beta} u^j) \in L^2(\Omega'')$$

we can apply Theorem 16.3 to $\partial_{x_s}\mathbf{u} \in \mathbb{H}^1(\Omega'')$ to get that all derivatives of order 3 of $\mathbf{u}$ are bounded in $L^2(\Omega')$ by a constant times the $L^2(\Omega'')$-norm of the first derivatives of $\mathbf{f}$ and the $L^2(\Omega'')$-norm of the second derivatives of u (note that (16.41) is satisfied for all $v \in \mathbb{H}_0^1(\Omega'')$). Since the second derivatives of u in Ω'' can be bounded by an estimate like (16.18) this provides (16.38) in the case $k = 1$ – repeating of course the estimate for every s. Continuing this process or applying what we just did to $\partial_{x_s}\mathbf{u}$ we obtain (16.38). This completes the proof of the theorem. □

As a consequence we have

Theorem 16.7 (Regularity of the solution of elliptic systems). *Under the assumptions of Theorem* 16.3 *if* $\mathbf{u}$ *is solution to* (16.14) *and if*

$$A_{ij}^{\alpha\beta} \in C^\infty(\Omega), \qquad f^i \in C^\infty(\Omega) \quad \forall \, i, j, \alpha, \beta$$

then $\mathbf{u} \in \big(C^\infty(\Omega)\big)^m$.

Proof. By the previous theorem we have

$$\mathbf{u} \in \mathbb{H}^k(\Omega') \quad \forall\, k,\ \forall\, \Omega' \subset\subset \Omega.$$

Take η a function of $\mathcal{D}(\Omega)$ which is equal to 1 on Ω'. Then we have

$$\eta\mathbf{u} \in \mathbb{H}_0^k(\Omega) \quad \forall\, k.$$

Since $k - \frac{n}{2}$ will finish by pass any integer q we get by Theorem 16.5

$$\eta\mathbf{u} \in \big(C^q(\Omega)\big)^m \quad \forall\, q$$

in particular since $\eta = 1$ on Ω' we have

$$\mathbf{u} \in \big(C^\infty(\Omega')\big)^m$$

and the result follows since Ω' is arbitrary. □

Remark 16.6. If u is the weak solution of the Dirichlet problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\Delta u = f \text{ in } \Omega \end{cases}$$

then $f \in C^\infty(\Omega)$ implies that $u \in C^\infty(\Omega)$ and the equation is satisfied in the usual sense in Ω. This is indeed a trivial consequence of the previous theorem in the case where $m = 1$, i.e., $\mathbf{u}$ has only one component.

Remark 16.7. We did not include – for the sake of simplicity lower order terms in (16.14). This can be done and we refer the reader to the exercises.

Exercises

1. Reproduce the proof of Theorem 16.3 in the simple case where u is weak solution to
$$-\Delta u = f \quad \text{in } \Omega.$$
2. Let Ω be a bounded open subset of $\mathbb{R}^n$. Show that $C^\alpha(\overline{\Omega})$ is a Banach space when equipped with the norm
$$\|u\|_\alpha = |u|_{\infty,\Omega} + [u]_\alpha$$
where $[u]_\alpha = \operatorname{Sup}_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.
3. If f is a continuous function on $\mathbb{R}$ show that
$$\int_0^x f(s)\,ds$$
is differentiable in the distributional sense and has for derivative f. Show that a distribution which admits for derivative (in the distributional sense) a continuous function is derivable in the usual sense. Extend the preceding question in $\mathbb{R}^n$.
4. State and prove Theorem 16.3 when some lower order terms are included.

Chapter 17

The p-Laplace Equation

In this chapter we would like to consider in more details the Dirichlet problem for the p-Laplace operator, namely, if Ω is an open subset of $\mathbb{R}^n$, $1 < p < +\infty$ the problem

$$\begin{cases} -\partial_{x_i}(|\nabla u|^{p-2}\partial_{x_i} u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{17.1}$$

As remarked before the problem presents two interesting features: it requires to work in L^p-spaces and one cannot rely on Lax–Milgram to solve it and thus it offers two directions of extensions for our techniques.

17.1 A minimization technique

One way to show existence of a solution to (17.1) is to use a minimization technique. In the case where $p = 2$ we have seen that the solution to (17.1) is also the unique minimizer of the functional

$$J_2(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx. \tag{17.2}$$

It is in particular a critical point of this functional. Thus a way to solve some partial differential equations is via the finding of critical points. For $f \in L^{p'}(\Omega)$ let us set

$$J_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega fv\,dx. \tag{17.3}$$

The functional J_p is well defined for $v \in W_0^{1,p}(\Omega)$. Moreover we have

Theorem 17.1. *Let Ω be an open subset of $\mathbb{R}^n$ bounded in one direction. There exists u minimizing the functional J_p on $W_0^{1,p}(\Omega)$. Moreover u is solution to (17.1).*

Proof. We use here the so-called direct method in the calculus of variations (see [43], [53]). It consists to obtain a minimizer as the limit of a minimizing sequence. Let us single out the different steps of the method. First when the infimum is 0 then 0 is a minimizer, so we will assume the infimum negative.

• 1. *The functional J_p is bounded from below.*

Indeed by the Hölder inequality we have

$$\left|\int_\Omega fv\,dx\right| \leq |f|_{p',\Omega}|v|_{p,\Omega} \leq C|f|_{p',\Omega}\big||\nabla v|\big|_{p,\Omega}$$

(by the Poincaré inequality of Theorem 15.3). It follows then that

$$\begin{aligned} J_p(v) &\geq \frac{1}{p}\big||\nabla v|\big|_{p,\Omega}^p - C|f|_{p',\Omega}\big||\nabla v|\big|_{p,\Omega} \\ &\geq \frac{1}{p}\big||\nabla v|\big|_{p,\Omega}^p - \frac{1}{p}\big||\nabla v|\big|_{p,\Omega}^p - \frac{1}{p'}\{C|f|_{p',\Omega}\}^{p'} \\ &= -\frac{1}{p'}\{C|f|_{p',\Omega}\}^{p'} \end{aligned} \tag{17.4}$$

(by the Young inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$).

• 2. *Minimizing sequences are bounded.*

Let u_n be a minimizing sequence – i.e., a sequence such that

$$J_p(u_n) \longrightarrow \operatorname*{Inf}_{W_0^{1,p}(\Omega)} J_p(v). \tag{17.5}$$

One has then for n large enough

$$\begin{aligned} &\frac{1}{p}\big||\nabla u_n|\big|_{p,\Omega}^p - \int_\Omega fu_n\,dx \leq 0 \\ \Longrightarrow \qquad &\big||\nabla u_n|\big|_{p,\Omega}^p \leq p\int_\Omega fu_n\,dx \\ &\qquad\qquad \leq p|f|_{p',\Omega}|u_n|_{p,\Omega} \\ &\qquad\qquad \leq p|f|_{p',\Omega}C\big||\nabla u_n|\big|_{p,\Omega}. \end{aligned} \tag{17.6}$$

From this we derive

$$\big||\nabla u_n|\big|_{p,\Omega} \leq \{p|f|_{p',\Omega}C\}^{\frac{1}{p-1}} \tag{17.7}$$

and thus u_n is bounded in $W_0^{1,p}(\Omega)$.

• 3. *u_n converges toward a minimizer.*

Since u_n is bounded and $W_0^{1,p}(\Omega)$ is reflexive – up to a subsequence – there exists a $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega). \tag{17.8}$$

It is clear that J_p is continuous since

$$v \mapsto \int_\Omega f v \, dx$$

is continuous and $v \mapsto \big||\nabla v|\big|_{p,\Omega}$ also. Moreover since $p \geq 1$, $J_p(v)$ is convex and thus lower semi-continuous for the weak convergence. It follows that

$$J_p(u) \leq \liminf_n J_p(u_n) = \operatorname*{Inf}_{W_0^{1,p}(\Omega)} J_p(v) \tag{17.9}$$

and thus u is a minimizer of J_p on $W_0^{1,p}(\Omega)$.

• 4. *u is a critical point for J_p or equivalently a solution to* (17.1).

Since u is a minimizer of J_p, for any $v \in W_0^{1,p}(\Omega)$ the function

$$\lambda \longrightarrow \frac{1}{p} \int_\Omega |\nabla(u + \lambda v)|^p \, dx - \int_\Omega f(u + \lambda v) \, dx = J_p(u + \lambda v)$$

possesses a minimum at 0. It follows that

$$\frac{d}{d\lambda} J_p(u + \lambda v)\Big|_0 = 0 \quad \forall \, v \in W_0^{1,p}(\Omega). \tag{17.10}$$

By a simple computation one has

$$\begin{aligned}
\frac{d}{d\lambda} |\nabla(u + \lambda v)|^p &= \frac{d}{d\lambda} \{|\nabla(u + \lambda v)|^2\}^{p/2} \\
&= \frac{p}{2} \{|\nabla(u + \lambda v)|^2\}^{\frac{p}{2} - 1} \{2\nabla u \cdot \nabla v + 2\lambda |\nabla v|^2\} \\
&= p |\nabla(u + \lambda v)|^{p-2} \{\nabla u \cdot \nabla v + \lambda |\nabla v|^2\}.
\end{aligned}$$

By the usual rule of the derivation under the integral sign (note that the last expression above is integrable), (17.10) implies then that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall \, v \in W_0^{1,p}(\Omega), \tag{17.11}$$

i.e., we have shown the existence of a solution to (17.11). This completes the proof of the theorem. □

Remark 17.1. One could assume only $f \in W_0^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$ and get the same existence result.

Remark 17.2. Since the function $x \mapsto |x|^p$ is strictly convex, so is the functional J_p and thus it possesses a unique minimizer. However the uniqueness of the minimizer does not imply necessarily the uniqueness of a solution to (17.1) – i.e., the uniqueness of a critical point for J_p. In our case the strict convexity also implies uniqueness of a solution to (17.1). We will see it below thanks to useful elementary inequalities.

First we have

Proposition 17.2. *Let $1 < p < +\infty$, then there exists a constant C_p such that*

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big| \leq C_p|\xi - \eta|\{|\xi| + |\eta|\}^{p-2} \quad \forall\, \xi, \eta \in \mathbb{R}^n. \tag{17.12}$$

Proof. It is enough to show that for any $\xi \neq \eta$

$$\Phi(\xi,\eta) = \frac{\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big|}{|\xi - \eta|\{|\xi| + |\eta|\}^{p-2}} \leq C_p.$$

For $\xi = 0$ one has

$$\Phi(0,\eta) = 1$$

and with no loss of generality we can assume $\xi \neq 0$. Then one notices that Φ is invariant by orthogonal transformations – i.e.,

$$\Phi(Q\xi, Q\eta) = \Phi(\xi,\eta)$$

for every orthogonal transformation Q. Thus performing such a transformation we can then suppose $\xi = |\xi|e_1$. Using the homogeneity of Φ, i.e., the fact that

$$\Phi(|\xi|e_1, \eta) = \Phi\Big(e_1, \frac{\eta}{|\xi|}\Big)$$

we can assume $\xi = e_1$, i.e., we are reduced to show that

$$\Phi(e_1,\eta) = \frac{|e_1 - |\eta|^{p-2}\eta|}{|e_1 - \eta|\{1 + |\eta|\}^{p-2}} \leq C_p.$$

For η close to e_1 one has – since $X \mapsto X^{p-2}$ is differentiable near 1

$$\begin{aligned} |e_1 - |\eta|^{p-2}\eta| &= |e_1 - \eta + (1 - |\eta|^{p-2})\eta| \\ &\leq |e_1 - \eta| + C\big|1 - |\eta|\big|\,|\eta| \\ &\leq |e_1 - \eta|\{1 + C|\eta|\} \end{aligned}$$

and thus

$$\Phi(e_1,\eta) \leq \frac{1 + C|\eta|}{\{1 + |\eta|\}^{p-2}} \leq C(\varepsilon)$$

for $|\eta - e_1| \leq \varepsilon$. Since clearly $\Phi(e_1,\eta)$ is continuous on $\mathbb{R}^n \setminus \{e_1\}$ by a compactness argument it is enough to show that $\Phi(e_1,\eta)$ is bounded for large $|\eta|$. This follows from the inequality

$$\Phi(e_1,\eta) \leq \frac{1 + |\eta|^{p-1}}{(|\eta| - 1)\{1 + |\eta|\}^{p-2}}.$$

This completes the proof of the proposition. □

The next inequalities could be called coerciveness inequalities for their left-hand side part. Let us set

$$N_p(\xi,\eta) = \{|\xi| + |\eta|\}^{p-2}|\xi-\eta|^2 \tag{17.13}$$

since this quantity will play an important rôle in the following. Note that when $p<2$ it can be extended by continuity at $(0,0)$. We have

Proposition 17.3. *Let $1<p<+\infty$. There exist two positive constants c_p, C_p such that for every $\xi,\eta\in\mathbb{R}^n$*

$$c_pN_p(\xi,\eta) \le (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \le C_pN_p(\xi,\eta) \tag{17.14}$$

a dot denotes here the Euclidean scalar product in $\mathbb{R}^n$.

Proof. The bound from above follows directly from the previous proposition. Next by a simple computation we have

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) = |\xi|^p + |\eta|^p - \{|\xi|^{p-2} + |\eta|^{p-2}\}(\xi\cdot\eta). \tag{17.15}$$

Noticing that

$$(\xi\cdot\eta) = \frac{1}{2}\{|\xi|^2 + |\eta|^2 - |\xi-\eta|^2\}$$

we obtain

$$\begin{aligned}
&(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta)\\
&\quad= \frac{1}{2}\{|\xi|^{p-2} + |\eta|^{p-2}\}|\xi-\eta|^2 + \frac{1}{2}|\xi|^p + \frac{1}{2}|\eta|^p - \frac{1}{2}|\eta|^{p-2}|\xi|^2 - \frac{1}{2}|\xi|^{p-2}|\eta|\\
&\quad= \frac{1}{2}\{|\xi|^{p-2} + |\eta|^{p-2}\}|\xi-\eta|^2 + \frac{1}{2}(|\xi|^{p-2} - |\eta|^{p-2})(|\xi|^2 - |\eta|^2).
\end{aligned}$$

If $p\ge 2$ the function $X\mapsto X^{p-2}$ is nondecreasing and it comes

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \ge \frac{1}{2}\{|\xi|^{p-2} + |\eta|^{p-2}\}|\xi-\eta|^2.$$

By exchanging the rôle of ξ and η one can always assume

$$|\xi| \ge |\eta|.$$

Then we have

$$|\xi| \ge \frac{|\xi|+|\eta|}{2}.$$

Using again the fact that $X\to X^{p-2}$ is nondecreasing we obtain

$$\begin{aligned}
(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) &\ge \frac{1}{2}\left\{\left(\frac{|\xi|+|\eta|}{2}\right)^{p-2}\right\}|\xi-\eta|^2\\
&= \frac{1}{2^{p-1}}N_p(\xi,\eta).
\end{aligned}$$

This is the first inequality of (17.14).

To complete the proof in the case $p < 2$ we will need the following lemma

Lemma 17.4. *Let $1 < p < +\infty$. There exist two constants $0 < c_p \leq 1 \leq K_p$ such that for every a, b such that $0 < b \leq a$*

$$c_p\{a+b\}^{p-2}(a-b)^2 \leq (a^{p-1} - b^{p-1})(a-b) \leq K_p\{a+b\}^{p-2}(a-b)^2. \quad (17.16)$$

Proof of the lemma. Taking $b = 0$ we see that $c_p \leq 1 \leq K_p$. Dividing both sides of (17.16) by a^p the inequalities reduce to

$$c_p\{1+z\}^{p-2}(1-z) \leq 1 - z^{p-1} \leq K_p\{1+z\}^{p-2}(1-z) \quad \forall\, z \in (0,1].$$

Considering the function

$$z \longrightarrow \frac{1 - z^{p-1}}{(1+z)^{p-2}(1-z)}$$

since

$$\lim_{z\to 1} \frac{1-z^{p-1}}{1-z}\frac{1}{(1+z)^{p-2}} = \frac{p-1}{2^{p-2}}$$

one sees that this continuous function is bounded from above and from below by two positive constants. This completes the proof of the lemma. □

End of the proof of the proposition. Going back to (17.15) we have

$$\begin{aligned}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) \\ = (|\xi|^{p-1} - |\eta|^{p-1})(|\xi| - |\eta|) + \{|\xi|^{p-2} + |\eta|^{p-2}\}(|\xi||\eta| - (\xi\cdot\eta)).\end{aligned}$$

If $p \leq 2$ we have since the function $X \to X^{p-2}$ is nonincreasing

$$|\xi|^{p-2} + |\eta|^{p-2} \geq (|\xi| + |\eta|)^{p-2} + (|\xi| + |\eta|)^{p-2} = 2(|\xi| + |\eta|)^{p-2}.$$

Thus it comes, using Lemma 17.4

$$\begin{aligned}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) &\geq \{c_p(|\xi| - |\eta|)^2 + 2c_p(|\xi||\eta| - (\xi\cdot\eta))\}\{|\xi| + |\eta|\}^{p-2} \\ &= c_p\{|\xi| + |\eta|\}^{p-2}|\xi - \eta|^2.\end{aligned}$$

This completes the proof of the proposition. □

Remark 17.3. If $p \geq 2$ we derive from (17.16) and since $X \to X^{p-2}$ is nondecreasing that

$$\begin{aligned}(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi-\eta) &\leq \{K_p(|\xi| - |\eta|)^2 + 2K_p(|\xi||\eta| - (\xi\cdot\eta))\}\{|\xi| + |\eta|\}^{p-2} \\ &= K_p\{|\xi| + |\eta|\}^{p-2}|\xi - \eta|^2.\end{aligned}$$

This gives us another proof of the right-hand side inequality (17.14) with a precise constant.

As a trivial consequence of the formula above we have

Theorem 17.5. *Suppose that* Ω *is an open subset of* $\mathbb{R}^n$ *bounded in one direction then problem* (17.1) *admits a unique solution.*

Proof. If u_1, u_2 are two weak solution of (17.1) then we have

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \cdot \nabla v - |\nabla u_2|^{p-2}\nabla u_2 \cdot \nabla v \, dx = 0 \quad \forall\, v \in W_0^{1,p}(\Omega).$$

Taking $v = u_1 - u_2$ the result follows from Proposition 17.3. □

17.2 A weak maximum principle and its consequences

As a generalization of the uniqueness result above we have the following

Theorem 17.6. *Under the assumptions of Theorem* 17.1*, let* u_i*,* $i = 1, 2$ *the weak solutions to*

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i \cdot \nabla v \, dx = \int_\Omega f_i v \, dx \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{17.17}$$

If we have

$$f_1 \leq f_2 \quad \textit{in } \Omega \tag{17.18}$$

and

$$u_1 \leq u_2 \quad \textit{on } \partial\Omega, \quad (\textit{i.e.}, \ (u_1 - u_2)^+ \in W_0^{1,p}(\Omega))$$

then

$$u_1 \leq u_2 \quad \textit{in } \Omega. \tag{17.19}$$

Proof. By subtraction we have

$$\int_\Omega \big\{|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\big\} \cdot \nabla v \, dx = \int_\Omega (f_1 - f_2) v \, dx$$

for every $v \in W_0^{1,p}(\Omega)$. Taking

$$v = (u_1 - u_2)^+ \in W_0^{1,p}(\Omega) \tag{17.20}$$

we derive that

$$\int_\Omega \big\{|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\big\} \cdot \nabla (u_1 - u_2)^+ \, dx \leq 0.$$

From Proposition 17.3 it follows that for some constant $c_p > 0$

$$c_p \int_{\{u_1 > u_2\}} \{|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2}\} |\nabla (u_1 - u_2)|^2 \, dx \leq 0$$

where $\{u_1 > u_2\}$ denotes the set defined by

$$\{u_1 > u_2\} = \{\, x \in \Omega \mid u_1(x) > u_2(x) \,\}.$$

It follows that

$$\nabla(u_1 - u_2)^+ = 0 \quad \text{in } \Omega$$

and by the Poincaré inequality that $(u_1 - u_2)^+ = 0$ in Ω. This completes the proof of the theorem. □

As a consequence we can show that the solution to (17.1) remains bounded when

$$f \in L^\infty(\Omega) \cap L^{p'}(\Omega). \tag{17.21}$$

We will suppose that for some $a > 0$, Ω is included in a strip of size $2a$. More precisely

$$\Omega \subset S_\nu = \{\, x \mid (x - x_0) \cdot \nu \in (-a, a) \,\} \tag{17.22}$$

where ν is a unit vector. Then we have

Theorem 17.7. *Under the assumptions* (17.21) *and* (17.22), *let u be the unique weak solution to* (17.1). *We have $u \in L^\infty(\Omega)$ and*

$$|u|_{\infty,\Omega} \le \frac{p-1}{p} a^{\frac{p}{p-1}} |f|_{\infty,\Omega}^{\frac{1}{p-1}}. \tag{17.23}$$

(*Note that this generalizes* (12.6).)

Proof. Set $\alpha = 1 + \frac{1}{p-1}$ and consider

$$g = a^\alpha - |(x - x_0) \cdot \nu|^\alpha \ge 0 \quad \text{on } \Omega.$$

(Recall that "$\cdot$" denotes the scalar product in $\mathbb{R}^n$.)

One has

$$\begin{aligned}\nabla g &= -\nabla\{((x - x_0) \cdot \nu)^2\}^{\frac{\alpha}{2}} = -\frac{\alpha}{2}\{((x - x_0) \cdot \nu)^2\}^{\frac{\alpha}{2}-1} \cdot 2((x - x_0) \cdot \nu)\nu \\ &= -\alpha|(x - x_0) \cdot \nu|^{\alpha-2}((x - x_0) \cdot \nu)\nu.\end{aligned}$$

It follows that – recall that $|\nu| = 1$:

$$|\nabla g|^{p-2} = \alpha^{p-2}|(x - x_0) \cdot \nu|^{(\alpha-1)(p-2)}.$$

Thus we deduce

$$|\nabla g|^{p-2}\nabla g = -\alpha^{p-1}|(x - x_0) \cdot \nu|^{(\alpha-2)+(\alpha-1)(p-2)}((x - x_0) \cdot \nu)\nu.$$

Now from

$$\alpha - 2 + (\alpha - 1)(p - 2) = \frac{1}{p-1} - 1 + \frac{p-2}{p-1} = 0$$

it follows that

$$|\nabla g|^{p-2}\nabla g = -\alpha^{p-1}((x-x_0)\cdot\nu)\nu$$

and thus

$$-\nabla\cdot(|\nabla g|^{p-2}\nabla g) = \alpha^{p-1}\nu_i\partial_{x_i}((x-x_0)\cdot\nu) = \alpha^{p-1}$$

(we used the summation convention and the notation $\nabla\cdot = \operatorname{div}$). It follows that we have – for instance in the $\mathcal{D}'(\Omega)$ sense – see also (15.2)

$$-\Delta_p u = f \le |f|_{\infty,\Omega} = -\Delta_p\left\{\frac{|f|_{\infty,\Omega}^{\frac{1}{p-1}}}{\alpha}g\right\}.$$

By the weak maximum principle we deduce that

$$u \le \frac{|f|_{\infty,\Omega}^{\frac{1}{p-1}}}{\alpha}g. \tag{17.24}$$

Since $-u$ is solution to (17.1) corresponding to $-f$ we derive

$$|u| \le \frac{|f|_{\infty,\Omega}^{\frac{1}{p-1}}}{\alpha}g.$$

Since

$$\frac{1}{\alpha} = \frac{p-1}{p} \quad\text{and}\quad 0 \le g \le a^\alpha = a^{\frac{p}{p-1}}$$

the result follows. □

17.3 A generalization of the Lax–Milgram theorem

In Section 17.1 we have solved the problem (17.1) by generalizing the Dirichlet principle which consists in minimizing the functional

$$\frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx$$

on $H_0^1(\Omega)$ in order to obtain the solution of the Dirichlet problem. Considering (17.2) we could as well have tried to extend the theory of existence of a solution to

$$a(u,v) = \langle f, v\rangle$$

to forms $a(u,v)$ which are linear in v but not necessarily in u. This is what we would like to do now. (Recall that in this case one can write $a(u,v) = \langle Au, v\rangle$ for some possibly nonlinear operators A.) We will do that also with the goal to extend to nonlinear operators our theory of variational inequalities (see Chapter 11).

• 1. *The finite-dimensional case.*

In this paragraph we suppose that X is a finite-dimensional space with dual X^*.

Then we have

Theorem 17.8. *Let K be a nonempty compact convex subset of X and A a continuous mapping from K into X^*. Then for every $f \in X^*$ there exists a solution u to the problem*

$$\begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \forall\, v \in K, \\ u \in K. \end{cases} \tag{17.25}$$

$\langle \cdot,\cdot\rangle$ is the pairing between X^* and X.

Proof. We can define an Euclidean structure on X – that is to say a scalar product $(\cdot,\cdot)$. Denote by j the linear mapping from X^* into X such that

$$\langle x', x\rangle = (j(x'), x) \quad \forall\, x \in X.$$

If P_K denotes the projection on K for the Euclidean structure above then the mapping

$$x \mapsto P_K(x - j(Ax - f))$$

is continuous from K into itself. Then by the Brouwer fixed point theorem (see Corollary A.6) it has a fixed point u which satisfies

$$\begin{cases} (u, v-u) \geq (u - j(Au-f), v-u) \quad \forall\, v \in K, \\ u \in K, \end{cases}$$

i.e., u is solution to (17.25). This completes the proof of the theorem. □

Remark 17.4. In the case where K is unbounded then (17.25) does not necessarily have a solution. Indeed, choose for instance $X = X^* = \mathbb{R} = K$ with the pairing

$$\langle x', x\rangle = x' \cdot x.$$

For $A : \mathbb{R} \to (0, +\infty)$ the problem

$$\langle Au, v-u\rangle \geq 0 \quad \forall\, v \in \mathbb{R}$$

is equivalent to $Au = 0$ which does not have a solution. Thus in the case where K is unbounded some additional assumptions are necessary. This is what we would like to consider now.

Definition 17.1. Let K be an unbounded convex subset of X and A a mapping from K into X^*. We say that A is coercive on K if there exists $v_0 \in K$ such that

$$\langle Av - Av_0, v - v_0\rangle / |v - v_0| \longrightarrow +\infty \quad \text{when } v \in K, \quad |v| \to +\infty \tag{17.26}$$

($|\cdot|$ is a norm in X).

Then we have

Theorem 17.9. *Let K be a closed convex subset of X and $A : K \to X^*$ be a continuous coercive map. For every $f \in X^*$ there exists u solution to*

$$\begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \forall\, v \in K, \\ u \in K. \end{cases} \tag{17.27}$$

Proof. Since the map $v \to Av - f$ is continuous and coercive if we can solve (17.27) for $f = 0$ the general case will follow. We suppose then $f = 0$ and denote by B_R the ball

$$B_R = \{\, v \in X \mid |v| \leq R \,\}.$$

From Theorem 17.8 there exists u_R solution to

$$\begin{cases} \langle Au_R, v-u_R\rangle \geq 0 \quad \forall\, v \in K \cap B_R, \\ u_R \in K \cap B_R. \end{cases} \tag{17.28}$$

If $v_0 \in K$ is a point such that (17.26) holds let us choose $R > |v_0|$. By (17.28) we have

$$\langle Au_R, v_0 - u_R\rangle \geq 0. \tag{17.29}$$

On the other hand

$$\begin{aligned} \langle Au_R, v_0 - u_R\rangle &= -\langle Au_R - Av_0, u_R - v_0\rangle + \langle Av_0, v_0 - u_R\rangle \\ &\leq -\langle Au_R - Av_0, u_R - v_0\rangle + |Av_0|_* |v_0 - u_R| \\ &= |u_R - v_0| \left\{ -\frac{\langle Au_R - Av_0, u_R - v_0\rangle}{|u_R - v_0|} + |Av_0|_* \right\} \end{aligned}$$

where $|\cdot|_*$ denotes the strong dual norm in X^*. If for every R, $|u_R| = R$ then taking R large enough in the inequality above will lead to – see (17.26)

$$\langle Au_R, v_0 - u_R\rangle < 0$$

which is in contradiction with (17.29). Thus there exists some R large enough for which

$$|u_R| < R.$$

Then for every $v \in K$ one can find $\varepsilon > 0$ small enough such that

$$\varepsilon v + (1-\varepsilon)u_R = u_R + \varepsilon(v - u_R) \in K \cap B_R.$$

It follows by (17.28) that

$$\varepsilon\langle Au_R, v - u_R\rangle \geq 0,$$

i.e.,

$$\langle Au_R, v - u_R\rangle \geq 0 \quad \forall\, v \in K$$

and u_R is the solution we were looking for. This completes the proof of the theorem. $\square$

• 2. *The infinite-dimensional case.*

We assume now that X is a reflexive Banach space with norm $|\cdot|$ and dual X^*. K is a nonempty closed convex subset of X, A a mapping from K into X^*.

Definition 17.2. We shall say that A is monotone iff

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall\, u, v \in K. \tag{17.30}$$

A is strictly monotone if the equality in (17.30) holds only for $u = v$.

Definition 17.3. We shall say that $A : K \to X^*$ is continuous on finite-dimensional subspaces of X if for every finite-dimensional subspace M of X the mapping

$$A : K \cap M \longrightarrow X^*$$

is weakly continuous, that is to say for every $v \in X$

$$u \mapsto \langle Au, v\rangle$$

is continuous on $K \cap M$.

Under the assumptions above we have:

Theorem 17.10. *Let K be a nonempty closed convex subset of a reflexive Banach space X. Let $f \in X^*$ and $A : K \to X^*$ be a monotone mapping continuous on finite-dimensional subspaces of X. Then if*

- *K is bounded, or*
- *A is coercive on K (see Definition 17.1),*

there exists u solution to the variational inequality

$$\begin{cases} \langle Au, v - u\rangle \geq \langle f, v - u\rangle \quad \forall\, v \in K, \\ u \in K. \end{cases} \tag{17.31}$$

Moreover if A is strictly monotone the solution of (17.31) *is unique.*

In the proof of Theorem 17.10 we will use the following lemma

Lemma 17.11 (Minty's Lemma). *Under the assumptions of Theorem* 17.10 *the variational inequality* (17.31) *is equivalent to*

$$\begin{cases} \langle Av, v - u\rangle \geq \langle f, v - u\rangle \quad \forall\, v \in K, \\ u \in K. \end{cases} \tag{17.32}$$

Proof of the lemma. If u is solution to (17.31) we have

$$\langle Av, v - u\rangle = \langle Au, v - u\rangle + \langle Av - Au, v - u\rangle \geq \langle f, v - u\rangle \quad \forall\, v \in K,$$

by the monotonicity of A and (17.31). Thus u solution of (17.31) implies that u is solution to (17.32). Conversely let u be solution to (17.32). Changing in (17.32) v in

$$u + t(v-u) = tv + (1-t)u \in K$$

for $t \in (0,1)$ we get

$$\begin{aligned} &\langle A(u+t(v-u)), t(v-u)\rangle \geq \langle f, t(v-u)\rangle \\ \Longrightarrow \quad &\langle A(u+t(v-u)), v-u\rangle \geq \langle f, v-u\rangle \quad \forall\, v \in K. \end{aligned}$$

Using the continuity of A on finite-dimensional subspaces we get (17.31) by letting $t \to 0$ in the inequality above. This completes the proof of the Lemma. □

Proof of the theorem. Without loss of generality we can assume $f = 0$. Suppose then first that K is bounded. Consider

$$C(v) = \{\, u \in K \mid \langle Av, v-u\rangle \geq 0 \,\}.$$

It is clear that $C(v)$ is a weakly closed subset of K which is bounded. Thus $C(v)$ is weakly compact and if

$$\cap_{v \in K} C(v) = \emptyset$$

one can find $v_1, \ldots, v_n$ in K such that

$$C(v_1) \cap \cdots \cap C(v_n) = \emptyset. \tag{17.33}$$

Consider then M the subspace of X spanned by $v_1, \ldots, v_n$. Let us denote by i the canonical injection from M into X and by r the restriction mapping defined by

$$r(x') = x'|_M \quad \forall\, x' \in X^*.$$

By applying Theorem 17.8 and Minty's Lemma to the operator

$$v \mapsto r \circ A \circ i(v)$$

we find that there exists a solution of the problem

$$\begin{cases} \langle Av, v-u\rangle \geq 0 \quad \forall\, v \in K \cap M, \\ u \in K \cap M. \end{cases}$$

Of course such a u belongs to $C(v_1) \cap \cdots \cap C(v_n)$ which gives a contradiction to (17.33) and thus $\bigcap_{v \in K} C(v) \neq \emptyset$ which proves the theorem in the case where K is bounded. One deals with the case K unbounded exactly as in Theorem 17.9. To prove uniqueness in the case where A is strictly monotone it is enough to remark that if u_1, u_2 are two solutions to (17.31) then

$$\begin{aligned} \langle Au_1, v-u_1\rangle &\geq \langle f, v-u_1\rangle \quad \forall\, v \in K, \\ \langle Au_2, v-u_2\rangle &\geq \langle f, v-u_2\rangle \quad \forall\, v \in K. \end{aligned}$$

Taking $v = u_2$ in the first inequality, $v = u_1$ in the second, adding we obtain

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \leq 0.$$

The strict monotonicity implies $u_1 = u_2$. This completes the proof of the theorem. □

As a corollary we have

Corollary 17.12. *Let A be a monotone, coercive operator from a reflexive Banach space X into its dual X^*. If A is continuous on finite-dimensional subspaces of X then A is onto, i.e., for every $f \in X^*$ there exists a solution to*

$$Au = f. \tag{17.34}$$

If in addition A is strictly monotone then the solution of (17.34) *is unique.*

Proof. Taking $K = X$ we derive from Theorem 17.10 that there exists $u \in X$ such that

$$\langle Au, v - u \rangle \geq \langle f, v - u \rangle \quad \forall\, v \in X.$$

Replacing v by $u \pm v$ in this inequality we get

$$\langle Au, v \rangle = \langle f, v \rangle \quad \forall\, v \in X.$$

This completes the proof. □

Remark 17.5. In view of this corollary the coerciveness assumption seems to be more natural. Indeed in order to a monotone mapping A from $\mathbb{R}$ to $\mathbb{R}$ to be onto one needs that

$$|Ax| \longrightarrow +\infty \quad \text{when} \quad |x| \to +\infty,$$

i.e., assuming $A(0) = 0$

$$\frac{|Ax||x|}{|x|} = \frac{Ax \cdot x}{|x|} \longrightarrow +\infty \quad \text{when} \quad |x| \to +\infty.$$

• 3. *Applications.*

We go back to (17.1) and we define $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ by setting

$$\langle Au, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \quad \forall\, v \in W_0^{1,p}(\Omega).$$

It is easy to see that A defined that way satisfies all the assumptions of Theorem 17.10 and thus there exists a unique solution to (17.1).

Exercises

1. Detail the proof of (17.11).
2. Derive the Lax–Milgram theorem from Theorem 17.10.
3. Let $u \in W_0^{1,p}(\Omega)$ and $v \in W^{1,p}(\Omega)$ such that $v \geq 0$ a.e. on Ω. Prove that $(u-v)^+ \in W_0^{1,p}(\Omega)$. Deduce rigorously (17.24).
4. Let $u, v \in W^{1,p}(\Omega)$ such that $u^+ \in W_0^{1,p}(\Omega)$, $v^- \in W_0^{1,p}(\Omega)$. Prove that $(u-v)^+ \in W_0^{1,p}(\Omega)$.
5. Let Ω be a bounded open set of $\mathbb{R}^n$, $\varepsilon > 0$, $f \in L^{p'}(\Omega)$.
   (i) Show that there exists a u_ε unique weak solution to
   $$\begin{cases} -\varepsilon \partial_{x_i}(|\nabla u_\varepsilon|^{p-2}\partial_{x_i} u_\varepsilon) + |u_\varepsilon|^{p-2}u_\varepsilon = f \quad \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,p}(\Omega). \end{cases}$$
   (ii) Show that $u_\varepsilon \to \operatorname{sign} f |f|^{\frac{1}{p-1}}$ in $L^{p'}(\Omega)$ when $\varepsilon \to 0$ ($\operatorname{sign} f$ denotes the sign of f).

Chapter 18

The Strong Maximum Principle

Since we have seen that weak solutions can be smooth (see Chapter 16) in this chapter we would like to introduce some tools and properties appropriate for smooth solutions of elliptic problems (see also [15], [17], [44], [48]–[50], [54], [55], [57], [80]–[82]).

18.1 A first version of the maximum principle

Let $\Omega = (a, b)$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$u'' > 0 \quad \text{on} \quad (a, b). \tag{18.1}$$

Then we claim that

$$u(x) < u(a) \vee u(b) \quad \forall\, x \in (a, b) \tag{18.2}$$

where $\vee$ denotes the maximum of two numbers. Indeed – if not – u achieves an interior maximum x_0. At this point we have

$$u(x) \le u(x_0) \quad \forall\, x \text{ close to } x_0. \tag{18.3}$$

Using the Taylor expansion it comes

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{u''(\xi_x)}{2}(x - x_0)^2$$

where ξ_x lies between x_0 and x. From (18.3) we derive

$$(x - x_0)\Big\{u'(x_0) + u''(\xi_x)\Big(\frac{x - x_0}{2}\Big)\Big\} \le 0$$

for x close to x_0. This forces

$$u'(x_0) = 0, \qquad u''(\xi_x) \le 0$$

for x close to x_0. Letting $x \to x_0$ we see that at a point x_0 where u reaches a local maximum we have

$$u'(x_0) = 0, \qquad u''(x_0) \leq 0.$$

By (18.1) such a point cannot exist and thus (18.2) holds.

A natural issue is to generalize this result in higher dimension replacing for instance u'' by a linear combination of second derivatives of u. To this purpose let us consider Ω a bounded open subset of $\mathbb{R}^n$ and a_{ij}, b_i, $i, j = 1, \ldots, n$ functions in Ω such that the a_{ij} satisfy

$$a_{ij}(x)\xi_i\xi_j \geq 0 \quad \forall\, x \in \Omega,\ \forall\, \xi \in \mathbb{R}^n. \tag{18.4}$$

Then we have

Theorem 18.1. *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying*

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^{n} b_i(x)\partial_{x_i} u > 0 \quad \forall\, x \in \Omega. \tag{18.5}$$

Then we have

$$u(x) < \operatorname*{Max}_{\partial\Omega} u(x) \quad \forall\, x \in \Omega. \tag{18.6}$$

($\partial\Omega$ denotes the boundary of Ω.)

Proof. Since for a C^2-function we have

$$\partial^2_{x_i x_j} u = \partial^2_{x_j x_i} u$$

there is no loss of generality in assuming that $A = (a_{ij})$ is a symmetric matrix. If not one replaces a_{ij} by $\frac{1}{2}(a_{ij} + a_{ji})$ in (18.5). Since $u \in C(\overline{\Omega})$ and $\overline{\Omega}$ is compact, u achieves its maximum at some point in $\overline{\Omega}$. If (18.6) does not hold then u achieves its maximum at a point $x_0 \in \Omega$. Then for every $\xi \in \mathbb{R}^n$ the function

$$v(t) = u(x_0 + t\xi)$$

achieves a local maximum at $t = 0$. This implies (see above) that

$$v'(0) = 0, \qquad v''(0) \leq 0,$$

i.e., by the chain rule

$$\partial_{x_i} u(x_0)\xi_i = 0, \qquad \partial^2_{x_i x_j} u(x_0)\xi_i\xi_j \leq 0 \quad \forall\, \xi \in \mathbb{R}^n. \tag{18.7}$$

Since $A = (a_{ij}(x_0))$ is symmetric there exists an orthogonal matrix Q and a diagonal matrix D such that

$$A = QDQ^T.$$

Suppose that

$$D = \begin{pmatrix} d_1(x_0) & & 0 \\ & \ddots & \\ 0 & & d_n(x_0) \end{pmatrix}.$$

If "$\cdot$" denotes the usual scalar product in $\mathbb{R}^n$ then (18.4) reads

$$A\xi \cdot \xi \geq 0 \quad \forall\, \xi \in \mathbb{R}^n$$

(for convenience we dropped the x). This implies that

$$D\xi \cdot \xi = Q^T AQ\xi \cdot \xi = AQ\xi \cdot Q\xi \geq 0 \quad \forall\, \xi \in \mathbb{R}^n$$

and in particular – for $\xi = e_i$ –

$$d_i(x_0) \geq 0 \quad \forall\, i = 1, \ldots, n. \tag{18.8}$$

If $\partial^2 u$ denotes the Hessian matrix of u at x_0 we have

$$\sum_{i,j=1}^n a_{ij}(x_0)\partial^2_{x_i x_j} u(x_0) = \operatorname{tr}(A\partial^2 u) = \operatorname{tr}(QDQ^T\partial^2 u) = \operatorname{tr}(DQ^T\partial^2 uQ), \tag{18.9}$$

(we used the fact that for two matrices M_1, M_2, $\operatorname{tr}(M_1M_2) = \operatorname{tr}(M_2M_1)$). If we set $C = Q^T\partial^2 u(x_0)Q = (c_{ij})$ by (18.7) we have

$$C\xi \cdot \xi = \partial^2 u(x_0)Q\xi \cdot Q\xi \leq 0 \quad \forall\, \xi \in \mathbb{R}^n$$

and thus as above $c_{ii}(x_0) \leq 0\ \forall\, i = 1, \ldots, n$. Combining (18.7), (18.8), (18.9) we get

$$\sum_{i,j=1}^n a_{ij}(x_0)\partial^2_{x_i x_j} u(x_0) + \sum_{i=1}^n b_i(x_0)\partial_{x_i} u(x_0) = \operatorname{tr}(DC) = \sum_{i=1}^n d_i(x_0)c_{ii}(x_0) \leq 0$$

which contradicts (18.5) and concludes the proof. □

Remark 18.1. The result above holds true for $A = (a_{ij}) \equiv 0$.

The next issue is to see what happens when the inequality (18.5) is not strict. In this case we have

Theorem 18.2. *Suppose in addition to* (18.4) *that there exists a vector* $\xi \neq 0$ *such that for some positive constants* γ, δ

$$a_{ij}(x)\xi_i\xi_j \geq \gamma > 0, \qquad b_i(x)\xi_i \geq -\delta \quad \forall\, x \in \Omega. \tag{18.10}$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfying*

$$\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(x)\partial_{x_i} u \geq 0 \quad \forall\, x \in \Omega \tag{18.11}$$

then

$$u(x) \leq \operatorname*{Max}_{\partial\Omega} u(x) \quad \forall\, x \in \overline{\Omega}. \tag{18.12}$$

Proof. Consider the function

$$u_\varepsilon(x) = u(x) + \varepsilon e^{\alpha(\xi\cdot x)}$$

(α is a positive constant to be chosen later on). One has

$$\partial_{x_i} u_\varepsilon = \partial_{x_i} u + \varepsilon\alpha e^{\alpha(\xi\cdot x)}\xi_i,$$
$$\partial^2_{x_i x_j} u_\varepsilon = \partial^2_{x_i x_j} u + \varepsilon\alpha^2 e^{\alpha(\xi\cdot x)}\xi_i\xi_j.$$

Thus it comes (we drop the x for convenience)

$$\begin{aligned}
&\sum_{i,j=1}^n a_{ij}\partial^2_{x_i x_j} u_\varepsilon + \sum_{i=1}^n b_i\partial_{x_i} u_\varepsilon \\
&\quad = \sum_{i,j=1}^n a_{ij}\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i\partial_{x_i} u + \varepsilon e^{\alpha(\xi\cdot x)}\{\alpha^2 a_{ij}\xi_i\xi_j + \alpha b_i\xi_i\} \\
&\quad \geq \varepsilon e^{\alpha(\xi\cdot x)}\{\alpha^2\gamma - \alpha\delta\} > 0
\end{aligned}$$

for α large enough. Applying Theorem 18.1 we deduce that

$$u_\varepsilon(x) < \operatorname*{Max}_{\partial\Omega} u_\varepsilon \leq \operatorname*{Max}_{\partial\Omega} u + \varepsilon \operatorname*{Max}_{\partial\Omega} e^{\alpha(\xi\cdot x)} \quad \forall\, x \in \Omega.$$

The result follows by letting $\varepsilon \to 0$. □

Remark 18.2.

1. In the case of (18.11) one cannot have (18.6) since for instance $u \equiv$ cst is a solution to (18.11).
2. Suppose that in $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, $u = u(x,t)$ satisfies

$$\Delta u \pm u_t \geq 0 \quad \text{in } \Omega$$

then Theorem 18.2 applies – take $\xi = e_1$ in $\mathbb{R}^{n+1}$ and consider t as x_{n+1}. Thus the maximum principle works the same way for elliptic problems – i.e., involving an elliptic operator as the Laplacian – and for parabolic problems involving operators of the type $\pm u_t - Au$ where A is an elliptic operator in the x variable.

As a corollary of Theorem 18.2 we have the following

Theorem 18.3. *Suppose that we are under the assumptions of Theorem* 18.2 *– namely* (18.4), (18.10) *hold. Let us denote by c a function such that*

$$c(x) \geq 0 \quad \forall\, x \in \Omega. \tag{18.13}$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(x)\partial_{x_i} u - c(x)u \geq 0 \quad \forall\, x \in \Omega \tag{18.14}$$

then

$$u(x) \le \operatorname*{Max}_{\partial\Omega} u^+(x) \quad \forall\, x \in \overline{\Omega}. \tag{18.15}$$

Proof. Consider the open set

$$\Omega^+ = \{\, x \in \Omega \mid u(x) > 0 \,\}.$$

If $\Omega^+ = \emptyset$ then (18.15) clearly holds. Else by (18.14) we have

$$\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(x)\partial_{x_i} u \ge c(x)u(x) \ge 0 \quad \text{on } \Omega^+$$

and by Theorem 18.2

$$u(x) \le \operatorname*{Max}_{\partial\Omega^+} u(x) \le \operatorname*{Max}_{\partial\Omega} u^+(x).$$

This completes the proof of the theorem. □

Remark 18.3. (Minimum principle). Under the assumptions of Theorem 18.3 if u is such that

$$Lu := \sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(x)\partial_{x_i} u - c(x)u \le 0 \quad \forall\, x \in \Omega \tag{18.16}$$

then $-u$ satisfies (18.14) and one has

$$-u(x) \le \operatorname*{Max}_{\partial\Omega}(-u)^+(x) \quad \Longleftrightarrow \quad u(x) \ge -\operatorname*{Max}_{\partial\Omega} u^-(x).$$

Remark 18.4. Condition (18.13) cannot be dropped. Indeed for instance $u = \sin nx$ satisfies

$$u'' + n^2 u = 0 \quad \text{on} \quad \Omega = (0,\pi)$$

and the maximum principle does not hold on $(0,\pi)$.

Remark 18.5. A consequence of the maximum principle is the uniqueness for the Dirichlet problem associated to the operator L defined in (18.16) or more generally if $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$\begin{cases} -Lu \le -Lv & \text{in } \Omega, \\ u \le v & \text{on } \partial\Omega, \end{cases} \tag{18.17}$$

then we have

$$u \le v \quad \text{in } \Omega. \tag{18.18}$$

18.2 The Hopf maximum principle

As a consequence of Theorem 18.2, in one dimension, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$u'' \geq 0 \quad \text{in} \quad \Omega = (a, b) \tag{18.19}$$

then we have

$$u(x) \leq M = \text{Max}\{u(a), u(b)\} \quad \forall\, x \in \Omega.$$

Suppose now that the maximum M is achieved at some point $x_0 \in \Omega$. Then we have

$$u'(x_0) = 0$$

and from (18.19), $u' \leq 0$ on the left of x_0 and $u' \geq 0$ on the right. In other words u is nonincreasing on the left of x_0 and nondecreasing on the right. But since $M = u(x_0)$ is the maximum we have

$$u \equiv M \quad \text{in } \Omega.$$

In other words when (18.19) holds then u cannot achieve its maximum inside Ω without being constant and equal to this maximum on Ω. This is what we would like to generalize now.

Theorem 18.4 (The Hopf Maximum principle). *Let us denote by c a bounded function. We assume in addition that for some positive λ*

$$\sum_{ij} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n.$$

Let $u \in C^2(\Omega)$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_i x_j} u + \sum_{i=1}^n b_i(x)\partial_{x_i} u - c(x)u \geq 0 \quad \text{in } \Omega. \tag{18.20}$$

Consider a point $x_0 \in \partial\Omega$ and suppose that

(i) *u is continuous at x_0* (18.21)

(ii) *$u(x_0) > u(x) \quad \forall\, x \in \Omega$* (18.22)

(iii) *$\partial\Omega$ satisfies an interior sphere condition at x_0 (see below).* (18.23)

Let ν be an exterior direction to the ball involved in (iii) *(see below) then if $\frac{\partial u}{\partial \nu}(x_0)$ exists, we have*

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \tag{18.24}$$

for

$$\begin{array}{ll} \bullet\ c = 0 & u(x_0) \text{ arbitrary} \\ \bullet\ c \geq 0 & u(x_0) \geq 0 \\ \bullet\ c \text{ arbitrary} & u(x_0) = 0. \end{array} \tag{18.25}$$

Proof. First let us explain what is meant by (iii). We assume that there exists a ball $B_R(y) = \{ x \mid |x - y| < R \} \subset \Omega$ such that x_0 is the only point of $\partial\Omega$ which belongs to its boundary (see Figure 18.1 below).

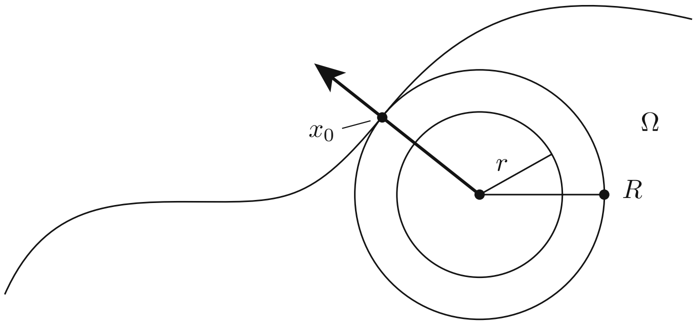


Figure 18.1.

Consider the function v defined as

$$v(x) = e^{-\alpha\rho^2} - e^{-\alpha R^2} \geq 0 \tag{18.26}$$

where $\rho = |x - y|$, $r < \rho < R$ and α is a positive constant that we will choose later. We have

$$\partial_{x_i} v = \partial_{x_i} e^{-\alpha|x-y|^2} = -2\alpha e^{-\alpha\rho^2}(x_i - y_i)$$
$$\partial^2_{x_i x_j} v = 4\alpha^2 e^{-\alpha\rho^2}(x_i - y_i)(x_j - y_j) - 2\alpha e^{-\alpha\rho^2}\delta_{ij}$$

where δ_{ij} denotes the Kronecker symbol. Thus if Lu is defined as the left-hand side of (18.20) let us denote by L^+ the operator defined as L with c replaced by c^+. We have with the summation convention

$$\begin{aligned} L^+v &= a_{ij}\{4\alpha^2 e^{-\alpha\rho^2}(x_i - y_i)(x_j - y_j) - 2\alpha e^{-\alpha\rho^2}\delta_{ij}\} + b_i\{-2\alpha e^{-\alpha\rho^2}(x_i - y_i)\} \\ &\quad - c^+\{e^{-\alpha\rho^2} - e^{-\alpha R^2}\} \\ &\geq e^{-\alpha\rho^2}\{4\alpha^2 a_{ij}(x_i - y_i)(x_j - y_j) - 2\alpha(a_{ii} + b_i(x_i - y_i)) - c^+\} \\ &\geq e^{-\alpha\rho^2}\{4\alpha^2\lambda|x - y|^2 - 2\alpha(|a_{ii}| + |b|\,|x - y|) - c^+\} \\ &\geq e^{-\alpha\rho^2}\{4\alpha^2\lambda r^2 - 2\alpha(|a_{ii}| + |b|R) - c^+\} \geq 0 \quad \text{on } B_R(y) \setminus B_r(y) \end{aligned}$$

for α large enough.

On the boundary of $B_r(y)$ we have, since u is continuous,

$$u - u(x_0) \leq -\gamma < 0$$

for some positive constant γ. Thus for ε small enough we have

$$u - u(x_0) + \varepsilon v \leq 0 \quad \text{on } \partial B_r(y). \tag{18.27}$$

Moreover on $\partial B_R(y)$ we have (see (18.22), (18.26))

$$u - u(x_0) + \varepsilon v \le 0 \quad \text{on } \partial B_R(y).$$

Finally we have

$$\begin{aligned} L^+(u - u(x_0) + \varepsilon v) &= Lu - c^- u + c^+ u(x_0) + \varepsilon L^+(v) \\ &\ge -c^-(u - u(x_0)) - c^- u(x_0) + c^+ u(x_0) \\ &\ge 0 \end{aligned}$$

in all cases (18.25). Applying Theorem 18.3 with L replaced by L^+ we deduce

$$u - u(x_0) + \varepsilon v \le 0 \quad \text{on } B_R(y) \setminus B_r(y).$$

Since v vanishes at x_0 we have for $h > 0$ small enough

$$u(x_0 - h\nu) - u(x_0) + \varepsilon(v(x_0 - h\nu) - v(x_0)) \le 0,$$

(since ν is an exterior direction to $B_R(y)$; i.e., $(x_0 - y) \cdot \nu > 0$). This implies

$$\frac{u(x_0 - h\nu) - u(x_0)}{-h} \ge -\varepsilon \frac{v(x_0 - h\nu) - v(x_0)}{-h}.$$

Letting $h \to 0$ we obtain

$$\frac{\partial u}{\partial \nu}(x_0) \ge -\varepsilon \frac{\partial v}{\partial \nu}(x_0).$$

But

$$\frac{\partial v}{\partial \nu}(x_0) = \nabla v(x_0) \cdot \nu = -2\alpha e^{-\alpha|x_0 - y|^2}(x_0 - y) \cdot \nu < 0$$

and the result follows. □

As a corollary we have

Theorem 18.5. *Suppose that we are under the assumptions of Theorem* 18.4. *Let* $u \in C^2(\Omega)$ *satisfying*

$$Lu \ge 0 \quad \textit{in } \Omega.$$

If Ω *is connected and*

$$\begin{aligned} &c = 0 \quad \textit{and } u \textit{ achieves its maximum inside } \Omega \\ &c \ge 0 \quad \textit{and } u \textit{ achieves a nonnegative maximum inside } \Omega \\ &c \textit{ arbitrary and } u \textit{ achieves a } 0 \textit{ maximum in } \Omega \end{aligned} \tag{18.28}$$

then u *is constant in* Ω.

Proof. Suppose on the contrary u nonconstant with a maximum M achieved in Ω. Set

$$\Omega^- = \{ x \mid u(x) < M \}.$$

Since u is not constant Ω^- is an open subset of Ω which is not empty and such that $\partial\Omega^- \cap \Omega$ is not empty (if not, then $\Omega \subset \Omega^- \cup (\mathbb{R}^n - \overline{\Omega^-})$ which contradicts the connectivity of Ω). Let $y \in \Omega^-$ such that y is closer from $\partial\Omega^-$ than from $\partial\Omega$. Consider then $B_R(y)$ the largest ball contained in Ω^-. There are points $x_0 \in \partial B_R(y)$ such that $u(x_0) = M$. At such a point by Theorem 18.4 one has

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

(ν is the outward normal of the set $B_R(y)$) but this contradicts the fact that $x_0 \in \Omega$ is a point where the maximum is achieved and hence where $\nabla u(x_0) = 0$. Thus u is constant. This completes the proof of the theorem. □

Remark 18.6. A consequence of Theorem 18.5 is that if Ω is bounded and M denotes the maximum of u in $\overline{\Omega}$ and if we are under the assumptions of Theorem 18.5 then

$$u \equiv M \text{ in } \Omega \quad \text{or} \quad u < M \text{ in } \Omega.$$

18.3 Application: the moving plane technique

The maximum principle that we just saw allows to obtain qualitative properties for the solution of elliptic equations. Let us see that on an example (cf. [15], [54]).

Definition 18.1. Let $\Omega \subset \mathbb{R}^n$. We say that Ω is convex in the x_1 direction if

$$(x_1, x'), (y_1, x') \in \Omega \Longrightarrow (\alpha x_1 + (1-\alpha) y_1, x') \in \Omega \quad \forall\, \alpha \in (0,1). \tag{18.29}$$

Then we have:

Theorem 18.6. *Let Ω be a bounded domain in $\mathbb{R}^n$ convex in the x_1 direction and symmetric with respect to the hyperplane $x_1 = 0$ (see Figure 18.2 on top of the next page). Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution to*

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{18.30}$$

where f is a Lipschitz continuous function. Then u is symmetric with respect to the hyperplane $x_1 = 0$ and

$$\partial_{x_1} u > 0 \quad \text{on} \quad \Omega^- = \{ x \in \Omega \mid x_1 < 0 \}. \tag{18.31}$$

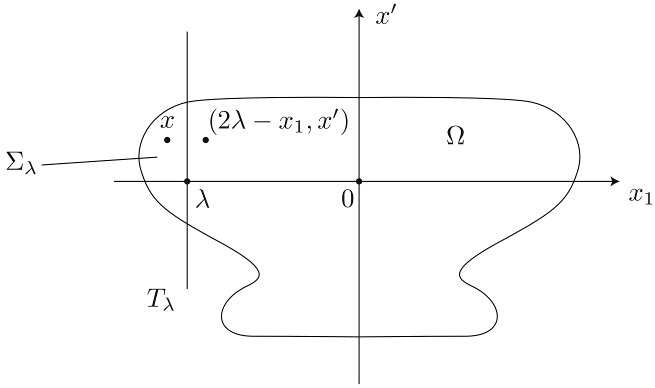


Figure 18.2.

Proof. One uses the so-called moving plane technique. Set

$$-a = \operatorname*{Inf}_{(x_1,x')\in\Omega} x_1. \tag{18.32}$$

For $-a < \lambda < 0$ we define T_λ, Σ_λ as

$$T_\lambda = \{\, x \mid x_1 = \lambda \,\}, \qquad \Sigma_\lambda = \{\, x \in \Omega \mid x_1 < \lambda \,\}. \tag{18.33}$$

Define

$$w_\lambda(x) = u(2\lambda - x_1, x') - u(x_1, x'), \qquad x = (x_1, x') \in \Sigma_\lambda. \tag{18.34}$$

We have in Σ_λ

$$\begin{aligned} -\Delta w_\lambda &= -\Delta u(2\lambda - x_1, x') + \Delta u(x_1, x') \\ &= \frac{\{f(u(2\lambda - x_1, x')) - f(u(x_1, x'))\}}{w_\lambda} w_\lambda \\ &=: c(x, \lambda) w_\lambda \end{aligned}$$

with the convention that $c(x, \lambda) = 0$ if $w_\lambda = 0$. Moreover, we have a uniform bound with respect to λ since

$$|c(x, \lambda)| \le L \quad \forall\, x \in \Sigma_\lambda \;\forall\, \lambda, \tag{18.35}$$

where L is the Lipschitz constant of f. Since u is positive we also have

$$w_\lambda \ge 0, \qquad w_\lambda \not\equiv 0 \quad \text{on } \partial\Sigma_\lambda.$$

(Note that $w_\lambda = 0$ on $T_\lambda \cap \Omega$.) For $0 < \lambda + a$ small enough, Σ_λ is narrow in the x_1 direction and from the weak maximum principle we deduce

$$w_\lambda(x) \ge 0 \quad \text{in } \Sigma_\lambda. \tag{18.36}$$

(See (18.35) and Corollary 4.5.) By the Hopf maximum principle (Theorem 18.5) we have in fact

$$w_\lambda(x) > 0 \quad \text{in } \Sigma_\lambda. \tag{18.37}$$

Let $(-a, \mu)$ be the largest interval for which

$$w_\lambda(x) > 0 \quad \text{in } \Sigma_\lambda \quad \forall\, \lambda \in (-a, \mu). \tag{18.38}$$

We claim that $\mu = 0$. If not suppose $\mu < 0$. By continuity we have

$$w_\mu(x) \geq 0 \quad \text{in } \Sigma_\mu$$

and by the Hopf maximum principle

$$w_\mu(x) > 0 \quad \text{in } \Sigma_\mu.$$

Let us denote by K a compact set in Σ_μ such that

$$|\Sigma_\mu \setminus K| \leq \frac{\delta}{2}$$

where δ is a real number that we will fix later. ($|\cdot|$ is the Lebesgue measure.) On K we have

$$w_\mu(x) \geq m > 0.$$

Thus by continuity for ε small enough we have

$$w_{\mu+\varepsilon}(x) \geq \frac{m}{2} > 0 \quad \text{on } K.$$

We can also choose ε small enough in such a way that

$$|\Sigma_{\mu+\varepsilon} \setminus K| \leq \delta.$$

Now for δ small enough (see (18.35) and Theorem 12.10) the maximum principle applies to $\widetilde{\Sigma}_{\mu+\varepsilon} = \Sigma_{\mu+\varepsilon} \setminus K$. More precisely since

$$\begin{aligned} -\Delta w_{\mu+\varepsilon} - c w_{\mu+\varepsilon} &= 0 \quad \text{in } \widetilde{\Sigma}_{\mu+\varepsilon}, \\ w_{\mu+\varepsilon} &\geq 0 \quad \text{on } \partial\widetilde{\Sigma}_{\mu+\varepsilon}, \end{aligned}$$

we derive $w_{\mu+\varepsilon} \geq 0$ in $\widetilde{\Sigma}_{\mu+\varepsilon}$ and finally by the Hopf maximum principle

$$w_{\mu+\varepsilon} > 0 \quad \text{in } \Sigma_{\mu+\varepsilon}.$$

This contradicts the maximality of μ and thus we have

$$w_\lambda(x) > 0 \quad \text{in } \Sigma_\lambda \quad \forall\, \lambda \in (-a, 0).$$

Letting $\lambda \to 0$ in the inequality above and recalling the definition of w_λ we arrive to

$$u(-x_1, x') \geq u(x_1, x') \quad \forall\, (x_1, x') \in \Omega, \quad x_1 < 0.$$

Since $u(-x_1, x')$ is also a positive solution to (18.30) we have

$$u(x_1, x') \geq u(-x_1, x') \quad \forall\, (x_1, x') \in \Omega, \quad x_1 < 0.$$

This shows that

$$u(-x_1, x') = u(x_1, x') \quad \forall\, (x_1, x') \in \Omega, \quad x_1 < 0$$

and the symmetry of u with respect to the hyperplane $x_1 = 0$ is proved. Now for any $\lambda \in (-a, 0)$ we have

$$w_\lambda(x_1, x') > 0 \quad \text{in } \Sigma_\lambda, \qquad w_\lambda(x_1, x') = 0 \quad \text{on } T_\lambda \cap \Omega.$$

By the Hopf maximum principle (see (18.25))

$$\partial_{x_1} w_\lambda(\lambda, x') < 0 \quad \forall\, x' \text{ such that } (\lambda, x') \in \Omega.$$

Going back to (18.34) we get

$$-2\partial_{x_1} u(\lambda, x') < 0 \quad \forall\, x' \text{ such that } (\lambda, x') \in \Omega.$$

This is (18.31) and this completes the proof of the theorem. □

Remark 18.7. One has of course due to the symmetry of u

$$\partial_{x_1} u < 0 \quad \text{on} \quad \Omega^+ = \{\, x \in \Omega \mid x_1 > 0 \,\}$$

and by the continuity of $\partial_{x_i} u$, $\partial_{x_i} u = 0$ on $T_0 \cap \Omega$.

As a corollary we have the following famous result due to Gidas–Ni–Nirenberg ([54]):

Corollary 18.7. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a positive solution to*

$$\begin{cases} -\Delta u = f(u) & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0), \end{cases}$$

where f is Lipschitz continuous. Then u is radially symmetric namely

$$u(x) = \tilde{u}(|x|)$$

for some function $\tilde{u}$. Moreover $\tilde{u}' < 0$ on $(0, R)$.

Proof. This follows directly from Theorem 18.6 since u is symmetric with respect to any hyperplane passing through the origin. □

Remark 18.8. $\tilde{u}$ is of course solution to

$$\begin{cases} -\tilde{u}_{rr} - \frac{n-1}{r}\tilde{u}_r = f(\tilde{u}) \quad \text{on } (0, R), \\ \tilde{u}'(0) = 0, \ u(R) = 0. \end{cases}$$

Exercises

1. Show that Theorem 18.2 holds when (18.10) is replaced by the existence of v such that

$$\sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j} v + \sum_{i=1}^n b_i(x)\partial_{x_i} v > 0 \quad \forall\, x \in \Omega.$$

2. Let Ω be a bounded domain symmetric in x_1. Let $f \in L^2(\Omega)$ be a function symmetric in x_1. Show that the solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

is symmetric in x_1.

3. (a) Justify carefully the fact that $w_\lambda \neq 0$ on $\partial\Sigma_\lambda$.
   (b) Justify carefully the use of the maximum principle in the proof of Theorem 18.6.
4. Let $u \in C^2(\mathbb{R}^n)$ satisfying

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

Let B_1 denote the unit ball of $\mathbb{R}^n$, $d\sigma$ the superficial measure on ∂B_1. Set

$$U(r) = \fint_{\partial B_1} u(r\sigma)\, d\sigma.$$

(i) Show that U is solution to

$$-(r^{n-1}U')' = r^{n-1} \fint_{\partial B_1} f(r\sigma)\, d\sigma,$$

$$\text{i.e.,} \qquad -\Delta U = \fint_{\partial B_1} f(r\sigma)\, d\sigma.$$

(ii) If u is harmonic that is to say satisfies

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^n$$

show that U is constant. Deduce that

$$u(x_0) = \fint_{\partial B_R(x_0)} u(y)\, d\sigma(y) \quad \forall\, x_0 \in \mathbb{R}^n,\ \forall\, R > 0.$$

Chapter 19

Problems in the Whole Space

We would like to consider here elliptic equations set in the whole $\mathbb{R}^n$. For instance if A is a positive definite matrix and a a nonnegative function we are going to consider the equation

$$-\operatorname{div}(A(x)\nabla u) + a(x)u = 0 \quad \text{in } \mathbb{R}^n. \tag{19.1}$$

Let us first start by considering the harmonic functions, i.e., the functions satisfying (19.1) with $A(x) \equiv \mathrm{Id}$, $a = 0$.

19.1 The harmonic functions, Liouville theorem

Definition 19.1. We say that a function is harmonic if $u \in C^2(\mathbb{R}^n)$ and

$$\Delta u(x) = \sum_{i=1}^{n} \partial_{x_i}^2 u = 0 \quad \forall\, x \in \mathbb{R}^n. \tag{19.2}$$

Then we have

Theorem 19.1. *Let u be a harmonic function and $x_0 \in \mathbb{R}^n$. We have*

$$u(x_0) = \frac{1}{|\partial B_R(x_0)|} \int_{\partial B_R(x_0)} u(y)\, d\sigma(y) \quad \forall\, x_0 \in \mathbb{R}^n,\ \forall\, R > 0. \tag{19.3}$$

($d\sigma$ denotes the superficial measure on $\partial B_R(x_0)$.)

Proof. For any r we have

$$\begin{aligned} U(r) &= \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(y)\, d\sigma(y) \\ &= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_1(x_0)} u(x_0 + r\omega) r^{n-1}\, d\sigma(\omega) \\ &= \frac{1}{n\omega_n} \int_{\partial B_1(x_0)} u(x_0 + r\omega)\, d\sigma(\omega), \end{aligned}$$

where ω_n denotes the volume of the unit ball. By differentiation it comes

$$\begin{aligned}
U'(r) &= \frac{1}{n\omega_n}\int_{\partial B_1(x_0)} \frac{d}{dr}u(x_0+r\omega)\,d\sigma(\omega)\\
&= \frac{1}{n\omega_n}\int_{\partial B_1(x_0)} \nabla u(x_0+r\omega)\cdot\omega\,d\sigma(\omega)\\
&= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B_r(x_0)} \nabla u(y)\cdot\frac{(y-x_0)}{r}\,d\sigma(y)\\
&= \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \frac{\partial u}{\partial\nu}(y)\,d\sigma(y)\\
&= \frac{1}{|\partial B_r(x_0)|}\int_{B_r(x_0)} \Delta u\,dx = 0 \quad \text{(by the Green formula).}
\end{aligned}$$

Then U is a constant function and letting $r \to 0$ we see that it is equal to $u(x_0)$. This completes the proof of the theorem. □

Theorem 19.2. *Let u be a harmonic function and $x_0 \in \mathbb{R}^n$. We have*

$$u(x_0) = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u(x)\,dx \quad \forall\, x_0 \in \mathbb{R}^n,\ \forall\, R > 0. \tag{19.4}$$

Proof. From the equality (19.3) we derive

$$n\omega_n r^{n-1}u(x_0) = \int_{\partial B_r(x_0)} u(y)\,d\sigma(y) \quad \forall\, 0 < r < R.$$

Integrating in r we get

$$u(x_0)\omega_n R^n = \int_0^R \int_{\partial B_r(x_0)} u(y)\,d\sigma(y)\,dr = \int_{B_R(x_0)} u(x)\,dx.$$

This completes the proof of the theorem. □

As a consequence we can show

Theorem 19.3. *Let u be a nonnegative harmonic function then*

$$u = \mathrm{cst}\,. \tag{19.5}$$

Proof. Let x, y be two points of $\mathbb{R}^n$. Note that

$$B_R(x) \subset B_{R+|y-x|}(y)$$

for any $R > 0$ due to the inequality

$$|z-y| \le |z-x| + |x-y| < R + |x-y| \quad \forall\, z \in B_R(x).$$

Then by Theorem 19.2 we have

$$\begin{aligned} u(x) = \fint_{B_R(x)} u(z)\,dz &= \frac{1}{|B_R(x)|}\int_{B_R(x)} u(z)\,dz \\ &\le \frac{1}{|B_R(x)|}\int_{B_{R+|y-x|}(y)} u(z)\,dz \\ &= \frac{1}{|B_R(x)|}|B_{R+|y-x|}(y)| \fint_{B_{R+|y-x|}(y)} u(z)\,dz \\ &= \left(\frac{R+|y-x|}{R}\right)^n u(y). \end{aligned}$$

Letting $R \to +\infty$ we deduce that $u(x) \le u(y)$. Since the two points are arbitrary we have similarly $u(y) \le u(x)$ and the result follows. □

More generally we have

Theorem 19.4 (Liouville). *Let u be a harmonic function, bounded from above or from below, then u is constant.*

Proof. Without loss of generality (at the expense of changing u in $-u$) one can assume that u is bounded from below, i.e.,

$$u \ge c$$

for some constant c. By Theorem 19.3, $u - c$ is constant, since this is a harmonic function, and so does u. This completes the proof of the theorem. □

Remark 19.1. The result above gives us some information about the harmonic functions: besides the constants they have to be unbounded from below and from above. Note that in one dimension the harmonic functions – i.e., the functions such that

$$u'' = 0$$

are the affine functions $x \mapsto Ax + B$. Unless they are constant they are unbounded from below and from above.

In our research of information on the harmonic functions one can try to find out if – besides the constants – there are some which are radially symmetric that is to say of the type

$$u(x) = v(r) \qquad r = |x|. \tag{19.6}$$

By a simple application of the chain rule we have for $r \neq 0$

$$\begin{aligned} \partial_{x_i} u(x) &= v'(r) \cdot \frac{x_i}{r}, \\ \partial_{x_i}^2 u(x) &= v''(r)\frac{x_i^2}{r^2} + v'(r)\left\{\frac{1}{r} - \frac{x_i^2}{r^3}\right\}. \end{aligned}$$

By summing up we see that

$$\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r) \quad (r \neq 0). \tag{19.7}$$

We already feel some problem for $r = 0\ldots$ So suppose that we want to find the harmonic functions in $\mathbb{R}^n \setminus \{0\}$ which in addition are radially symmetric. We have to solve

$$v''(r) + \frac{n-1}{r} v'(r) = 0 \quad r \neq 0$$

or equivalently

$$\begin{aligned} r^{n-1} v''(r) + (n-1) r^{n-2} v'(r) = 0 \quad r \neq 0, \\ (r^{n-1} v')' = 0 \quad r \neq 0. \end{aligned} \tag{19.8}$$

This gives

$$v' = \frac{A}{r^{n-1}}$$

for some constant A and by integration

$$v = \begin{cases} A \ln r + B & \text{if } n = 2, \\ \dfrac{A}{r^{n-2}} + B & \text{if } n \geq 2. \end{cases} \tag{19.9}$$

(The case $n = 1$ was considered in Remark 19.1.) Thus again besides the constant we cannot find a radially symmetric harmonic function. However the functions given by (19.9) are playing an important rôle.

Definition 19.2. We call "fundamental solution" of the Laplace operator the function defined by

$$F_n(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| & \text{if } n = 2, \\ \dfrac{-1}{n(n-2)\omega_n |x|^{n-2}} & \text{if } n > 2. \end{cases} \tag{19.10}$$

The theorem below justifies Definition 19.2. We have:

Theorem 19.5. *We have in the distributional sense*

$$\Delta F_n = \delta_0 \tag{19.11}$$

where δ_0 denotes the Dirac mass at 0.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We have by definition of the derivative in the distributional sense

$$\begin{aligned} \langle \Delta F_n, \varphi \rangle &= \langle F_n, \Delta \varphi \rangle \\ &= \int_{\mathbb{R}^n} F_n \Delta \varphi \, dx. \end{aligned} \tag{19.12}$$

(Note that this integral makes sense since $F_n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.) Denote by $B_R(0)$ a big ball containing the support of φ then we have

$$\int_{\mathbb{R}^n} F_n \Delta\varphi \, dx = \int_{B_R(0)\setminus B_\varepsilon(0)} F_n \Delta\varphi \, dx + \int_{B_\varepsilon(0)} F_n \Delta\varphi \, dx. \tag{19.13}$$

Let us consider the first integral of the right-hand side of the equation above. By the Green formula we have

$$\begin{aligned}\int_{B_R(0)\setminus B_\varepsilon(0)} \nabla F_n \cdot \nabla\varphi \, dx &= \int_{B_R(0)\setminus B_\varepsilon(0)} \operatorname{div}(F_n \nabla\varphi) - F_n \Delta\varphi \, dx \\ &= \int_{\partial\{B_R(0)\setminus B_\varepsilon(0)\}} F_n \frac{\partial\varphi}{\partial\nu} \, d\sigma - \int_{B_R(0)\setminus B_\varepsilon(0)} F_n \Delta\varphi \, dx \\ &= \int_{\partial\{B_R(0)\setminus B_\varepsilon(0)\}} \varphi \frac{\partial F_n}{\partial\nu} \, d\sigma - \int_{B_R(0)\setminus B_\varepsilon(0)} \varphi \Delta F_n \, dx\end{aligned}$$

(the last equality is obtained by exchanging the rôle of F_n and φ). Since F_n is harmonic and $\varphi = 0$ in a neighborhood of $\partial B_R(0)$ we obtain

$$\int_{B_R(0)\setminus B_\varepsilon(0)} F_n \Delta\varphi \, dx = -\int_{\partial B_\varepsilon(0)} \varphi \frac{\partial F_n}{\partial\nu} \, d\sigma + \int_{\partial B_\varepsilon(0)} F_n \frac{\partial\varphi}{\partial\nu} \, d\sigma$$

(in the formula above ν denotes the inward normal to $\partial B_\varepsilon(0)$). Thus from (19.12) and (19.13) we derive

$$\langle \Delta F_n, \varphi \rangle = -\int_{\partial B_\varepsilon(0)} \varphi \frac{\partial F_n}{\partial\nu} \, d\sigma + \int_{\partial B_\varepsilon(0)} F_n \frac{\partial\varphi}{\partial\nu} \, d\sigma + \int_{B_\varepsilon(0)} F_n \Delta\varphi \, dx. \tag{19.14}$$

Since $F_n \Delta\varphi \in L^1(B_1(0))$, the last integral in the right-hand side of (19.14) converges toward 0 when $\varepsilon \to 0$. Denote by M a constant such that

$$|\nabla\varphi| \le M \quad \text{on } B_1(0).$$

Then for $\varepsilon < 1$ we have

$$\left| \int_{\partial B_\varepsilon(0)} F_n \frac{\partial\varphi}{\partial\nu} \, d\sigma \right| \le F_n(\varepsilon) M |\partial B_\varepsilon(0)| = C F_n(\varepsilon) \varepsilon^{n-1},$$

for some constant C independent of ε. Then it is clear by the definition of F_n that the integral above converge toward 0 when $\varepsilon \to 0$. We have now (with the summation in i)

$$\frac{\partial F_n}{\partial\nu} = \nabla F_n \cdot \nu = F_n'(r) \cdot \frac{x_i}{r} \cdot \left(-\frac{x_i}{r} \right) = -F_n'(r) \quad \text{on } \partial B_\varepsilon(0)$$

where $r = |x|$ (we used the same notation for $F_n(x) = F_n(|x|)$). Thus we have

$$\int_{\partial B_\varepsilon(0)} \varphi \frac{\partial F_n}{\partial\nu} \, d\sigma = -F_n'(\varepsilon) \int_{\partial B_\varepsilon(0)} \varphi \, d\sigma.$$

Since by (19.10)

$$F_n'(\varepsilon) = \frac{1}{|\partial B_\varepsilon(0)|}$$

we get

$$\int_{\partial B_\varepsilon(0)} \varphi \frac{\partial F_n}{\partial \nu}\, d\sigma = -\frac{1}{|\partial B_\varepsilon(0)|} \int_{\partial B_\varepsilon(0)} \varphi\, d\sigma \to -\varphi(0)$$

when $\varepsilon \to 0$. Passing to the limit in (19.14) we obtain

$$\langle \Delta F_n, \varphi \rangle = \varphi(0) = \langle \delta_0, \varphi \rangle.$$

This completes the proof of the theorem. □

We can now prove

Theorem 19.6. *Let u be a harmonic function in $\mathbb{R}^n$. Then u is analytic and thus uniquely determined by its values on any open subset of $\mathbb{R}^n$.*

Proof. Arguing as above (see below (19.13))

$$\begin{aligned}
&\int_{B_R(y)\setminus B_\varepsilon(y)} \nabla F_n(x-y)\cdot \nabla u(x)\, dx \\
&\qquad = \int_{B_R(y)\setminus B_\varepsilon(y)} \{\operatorname{div}(F_n(x-y)\nabla u) - F_n \Delta u\}\, dx \\
&\qquad = \int_{\partial\{B_R(y)\setminus B_\varepsilon(y)\}} F_n(x-y) \frac{\partial u}{\partial \nu}(x)\, d\sigma(x) \\
&\qquad = \int_{\partial\{B_R(y)\setminus B_\varepsilon(y)\}} u(x) \frac{\partial F_n}{\partial \nu}(x-y)\, d\sigma(x)
\end{aligned}$$

(by exchanging the rôle of F_n and u). It follows that (for ν denoting the outward unit normal)

$$\begin{aligned}
&\int_{\partial B_R(y)} F_n(x-y)\frac{\partial u}{\partial \nu}(x)\, d\sigma(x) - \int_{\partial B_\varepsilon(y)} F_n(x-y)\frac{\partial u}{\partial \nu}(x)\, d\sigma(x) \\
&\qquad = \int_{\partial B_R(y)} u(x)\frac{\partial F_n}{\partial \nu}(x-y)\, d\sigma(x) - \int_{\partial B_\varepsilon(y)} u(x)\frac{\partial F_n}{\partial \nu}(x-y)\, d\sigma(y).
\end{aligned}$$

Passing to the limit in ε as in (19.14) we obtain

$$u(x) = \int_{\partial B_R(y)} u(x)\frac{\partial F_n}{\partial \nu}(x-y)\, d\sigma(x) - \int_{\partial B_R(y)} F_n(x-y)\frac{\partial u}{\partial \nu}(x)\, d\sigma(x).$$

Clearly since F_n is analytic in $\mathbb{R}^n \setminus \{0\}$ the right-hand side of the inequality above is analytic and the result follows. □

19.2 The Schrödinger equation

We would like to see now if the equation (19.1) can have bounded solutions when $a \not\equiv 0$. A particular case is the so-called stationary Schrödinger equation. It has the form

$$-\Delta u + a(x)u = 0. \tag{19.15}$$

This equation gave rise to numerous work during the last decades – see for instance [67], [88] and the references there.

In order to obtain a nontrivial bounded solution to (19.15) one needs to impose some decay on a at infinity. This will be clarified later on. So, let us assume

$$a \in L^\infty_{\text{loc}}(\mathbb{R}^n), \quad a \geq 0, \quad a \not\equiv 0, \tag{19.16}$$

and

$$\int_{\mathbb{R}^n} a(x)|x|^{2-n}\, dx < +\infty \tag{19.17}$$

with

$$n \geq 3. \tag{19.18}$$

Then we have

Theorem 19.7 (Grigor'yan [58] and [76]–[79], [8]). *Let us assume that* (19.16)–(19.18) *hold. Then there exists a function* $u \neq 0$ *such that*

$$0 \leq u \leq 1 \quad \textit{in } \mathbb{R}^n \tag{19.19}$$

which satisfies

$$-\Delta u + au = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^n). \tag{19.20}$$

Proof. Denote by u_k the solution (weak) to

$$\begin{cases} -\Delta u_k + au_k = 0 & \text{in } B_k(0), \\ u_k = 1 & \text{on } \partial B_k(0). \end{cases} \tag{19.21}$$

($B_k(0)$ denotes the ball of center 0 and radius k. We refer to Chapter 4 for the existence of a solution.) By the maximum principle we have

$$0 \leq u_k \leq 1. \tag{19.22}$$

From this we derive that

$$-\Delta u_{k+1} + au_{k+1} = -\Delta u_k + au_k \quad \text{in } B_k(0),$$

and

$$u_{k+1} \leq u_k \quad \text{on } \partial B_k(0).$$

By the maximum principle again

$$0 \le u_{k+1} \le u_k \quad \text{in } B_k(0). \tag{19.23}$$

Combining (19.22), (19.23) it follows from the Lebesgue convergence theorem that there exists u such that

$$u_k \longrightarrow u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^n).$$

Then clearly u satisfies

$$-\Delta u + au = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \tag{19.24}$$

We will be done if we can show that

$$u \not\equiv 0. \tag{19.25}$$

Let us assume by contradiction that

$$u \equiv 0. \tag{19.26}$$

Let $\zeta \in C^\infty(\mathbb{R}^n)$, $0 \le \zeta \le 1$ be a function such that

$$\zeta(x) = \begin{cases} 0 & \text{for } |x| < R, \\ 1 & \text{for } |x| > R+1. \end{cases}$$

R will be chosen later on. For $k > R+1$ if $a_k = \frac{1}{k^{n-2}}$ we have

$$\frac{\zeta}{|x|^{n-2}} - a_k \in H^1_0(B_k(0)).$$

From the weak formulation of (19.21) we get

$$\int_{B_k(0)} \nabla u_k \cdot \nabla \Big\{ \frac{\zeta}{|x|^{n-2}} - a_k \Big\} dx + \int_{B_k(0)} a(x) u_k \Big\{ \frac{\zeta}{|x|^{n-2}} - a_k \Big\} dx = 0. \tag{19.27}$$

The first integral above can also be written as

$$\begin{aligned} &\int_{B_k(0)} \nabla u_k \cdot \nabla \Big\{ \frac{\zeta}{|x|^{n-2}} - a_k \Big\} dx \\ &\quad = \int_{B_k(0)} \nabla (u_k - 1) \cdot \nabla \Big\{ \frac{\zeta}{|x|^{n-2}} - a_k \Big\} dx \\ &\quad = \int_{B_k(0)} (1 - u_k) \Delta \Big\{ \frac{\zeta}{|x|^{n-2}} - a_k \Big\} dx \\ &\quad = \int_{B_k(0)} \Delta \Big\{ \frac{\zeta}{|x|^{n-2}} \Big\} dx - \int_{B_k(0)} u_k \Delta \Big(\frac{\zeta}{|x|^{n-2}} \Big) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial B_k(0)} \frac{\partial}{\partial \nu}\left(\frac{\zeta}{|x|^{n-2}}\right) d\sigma(x) - \int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx \\
&= \int_{\partial B_k(0)} \frac{\partial}{\partial \nu}\left(\frac{1}{|x|^{n-2}}\right) d\sigma(x) - \int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx \\
&= (2-n) \int_{\partial B_k(0)} k^{1-n}\, d\sigma(x) - \int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx \\
&= (2-n)\, \sigma_n - \int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx,
\end{aligned}
$$

where σ_n is the area of the unit sphere in $\mathbb{R}^n$. From (19.27) we then deduce easily

$$
-\int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx + \int_{B_k(0)} a(x) u_k \frac{\zeta}{|x|^{n-2}}\, dx \geq (n-2)\sigma_n. \tag{19.28}
$$

Now $\Delta\left(\frac{\zeta}{|x|^{n-2}}\right)$ has compact support in $B_{R+1}(0) \setminus B_R(0)$. Letting $k \to +\infty$ if $u_k \to 0$ we derive then that

$$
\lim_{k \to +\infty} -\int_{B_k(0)} u_k \Delta\left(\frac{\zeta}{|x|^{n-2}}\right) dx = 0.
$$

Since also

$$
\int_{\{|x|>R\}} a(x)|x|^{2-n}\, dx \geq \int_{B_k(0)} a(x) u_k \frac{\zeta}{|x|^{n-2}}\, dx
$$

we get passing to the limit in (19.28)

$$
\int_{\{|x|>R\}} a(x)|x|^{2-r}\, dx \geq (n-2)\sigma_n.
$$

Since R can be arbitrary this contradicts (19.17) and completes the proof of the theorem. □

Remark 19.2. A function a with compact support provides a very simple example of a function for which (19.17) holds.

The example given in the remark above and (19.17) itself shows that some decay of a seems to be needed in order to get a nontrivial bounded solution. Also we did not consider the case of the dimension less than 3. This is what we would like to consider now.

Suppose that u is a weak solution to

$$
\begin{cases}
u \in H^1_{\mathrm{loc}}(\mathbb{R}^n), \\
-\operatorname{div}(A(x)\nabla u) + a(x)u = 0 \quad \text{in } \mathbb{R}^n,
\end{cases} \tag{19.29}
$$

where A is a positive definite matrix satisfying for some constants λ and Λ

$$\lambda|\xi|^2 \le A(x)\xi\cdot\xi \qquad \forall\,\xi\in\mathbb{R}^n,\ \text{a.e. } x\in\mathbb{R}^n, \tag{19.30}$$

$$|A(x)\xi| \le \Lambda|\xi| \qquad \forall\,\xi\in\mathbb{R}^n,\ \text{a.e. } x\in\mathbb{R}^n. \tag{19.31}$$

$a\in L^\infty_{\text{loc}}(\mathbb{R}^n)$ is a nonnegative function. Let us denote by ρ a smooth function such that

$$0\le\rho\le 1, \qquad \rho=1 \text{ on } B_{\frac{1}{2}}, \qquad \rho=0 \text{ outside } B_1, \tag{19.32}$$

$$|\nabla\rho|\le c_\rho. \tag{19.33}$$

Then we can show

Theorem 19.8. *Set $B_r=B_r(0)$. If u is a solution to (19.29) we have*

$$\int_{B_r}\{|\nabla u|^2+au^2\}\rho^2\Big(\frac{x}{r}\Big)dx \le \frac{C}{r}\Big\{\int_{B_r\setminus B_{r/2}}|\nabla u|^2\rho^2\Big(\frac{x}{r}\Big)dx\Big\}^{\frac{1}{2}}\Big\{\int_{B_r\setminus B_{r/2}}u^2\,dx\Big\}^{\frac{1}{2}} \tag{19.34}$$

where C is a constant independent of r.

Proof. Consider then

$$u\rho^2\Big(\frac{x}{r}\Big)\in H_0^1(B_r).$$

From (19.29) we derive

$$\int_{B_r}A(x)\nabla u\cdot\nabla\Big\{u\rho^2\Big(\frac{x}{r}\Big)\Big\}+a(x)u^2\rho^2\Big(\frac{x}{r}\Big)dx=0.$$

This implies

$$\int_{B_r}(A(x)\nabla u\cdot\nabla u)\rho^2\Big(\frac{x}{r}\Big)dx+a(x)u^2\rho^2\Big(\frac{x}{r}\Big)dx = -\int_{B_r}2A(x)\nabla u\cdot\nabla\rho\Big(\frac{x}{r}\Big)\frac{1}{r}u\rho\Big(\frac{x}{r}\Big)dx.$$

In this last integral one integrates in fact on $B_r\setminus B_{r/2}$. Thus it comes by (19.30), (19.31)

$$\int_{B_r}\{\lambda|\nabla u|^2+a(x)u^2\}\rho^2\Big(\frac{x}{r}\Big)dx\le 2\frac{\Lambda}{r}c_\rho\int_{B_r\setminus B_{r/2}}|\nabla u|\,|u|\rho\Big(\frac{x}{r}\Big)dx.$$

By the Cauchy–Schwarz inequality we obtain

$$(\lambda\wedge 1)\int_{B_r}\{|\nabla u|^2+au^2\}\rho^2\Big(\frac{x}{r}\Big)dx \le 2\frac{\Lambda}{r}c_\rho\Big\{\int_{B_r\setminus B_{r/2}}|\nabla u|^2\rho^2\Big(\frac{x}{r}\Big)dx\Big\}^{\frac{1}{2}}\Big\{\int_{B_r\setminus B_{r/2}}u^2\,dx\Big\}^{\frac{1}{2}}$$

where $\lambda \wedge 1$ denotes the minimum of λ and 1. This inequality is nothing but (19.34) and the proof of the theorem is complete. □

As a first application of this inequality we have

Theorem 19.9. *If $n \leq 2$ there is no nontrivial bounded solution to* (19.29)*, i.e., every bounded solution to* (19.29) *is a constant which is equal to 0 when $a \not\equiv 0$.*

Proof. From (19.34) we derive

$$\int_{B_r} |\nabla u|^2 \rho^2\Big(\frac{x}{r}\Big) dx \leq \frac{C}{r}\Big\{\int_{B_r \setminus B_{r/2}} |\nabla u|^2 \rho^2\Big(\frac{x}{r}\Big) dx\Big\}^{\frac{1}{2}} \Big\{\int_{B_r \setminus B_{r/2}} u^2\, dx\Big\}^{\frac{1}{2}} \quad (19.35)$$

from which it follows that

$$\int_{B_r} |\nabla u|^2 \rho^2\Big(\frac{x}{r}\Big) dx \leq \frac{C^2}{r^2} \int_{B_r \setminus B_{r/2}} u^2\, dx.$$

Since $\rho\left(\frac{x}{r}\right) = 1$ on $B_{r/2}$ we get

$$\int_{B_{r/2}} |\nabla u|^2\, dx \leq \frac{C^2}{r^2} \int_{B_r \setminus B_{r/2}} u^2\, dx.$$

If u is bounded, we derive when $n \leq 2$ that

$$\int_{B_{r/2}} |\nabla u|^2\, dx \leq \frac{C^2}{r^2} \int_{B_r \setminus B_{r/2}} u^2\, dx \leq C'$$

for some constant C' independent of r. Thus the mapping

$$r \mapsto \int_{B_{r/2}} |\nabla u|^2\, dx$$

is nondecreasing and bounded. It has a limit when $r \to +\infty$. From (19.35) we deduce then that

$$\int_{B_{r/2}} |\nabla u|^2\, dx \leq C'^{\frac{1}{2}} \Big\{\int_{B_r} |\nabla u|^2\, dx - \int_{B_{r/2}} |\nabla u|^2\, dx\Big\}^{\frac{1}{2}} \to 0$$

when $r \to +\infty$. This shows that $u = \text{cst}$ and completes the proof of the theorem. □

We now consider the case when a is not small at infinity. To simplify we will assume

$$a(x) \geq \lambda > 0 \quad \text{a.e. } x \qquad (19.36)$$

referring the reader to [22] for deeper results. We have

Theorem 19.10. *Under the assumptions of Theorem* 19.9, (19.36), *and n arbitrary there is no nontrivial bounded solution to* (19.29).

Proof. From (19.34) we derive easily

$$\begin{aligned}\int_{B_{r/2}} \{|\nabla u|^2 + au^2\}\, dx &\le \frac{C}{r}\left\{\int_{B_r} \{|\nabla u|^2 + au^2\}\, dx\right\}^{\frac{1}{2}} \left\{\int_{B_r\setminus B_{r/2}} u^2\, dx\right\}^{\frac{1}{2}} \\ &\le \frac{C}{r}\left\{\int_{B_r} \{|\nabla u|^2 + au^2\}\, dx\right\}^{\frac{1}{2}} \left\{\int_{B_r} \frac{a}{\lambda} u^2\, dx\right\}^{\frac{1}{2}} \\ &\le \frac{C'}{r}\int_{B_r} \{|\nabla u|^2 + au^2\}\, dx\end{aligned}$$

where $C' = \frac{C}{\sqrt{\lambda}}$ is independent of r. Iterating this formula we get

$$\int_{B_{r/2^{k+1}}} \{|\nabla u|^2 + au^2\}\, dx \le \frac{(C')^k}{r^k}\int_{B_{r/2}} \{|\nabla u|^2 + au^2\}\, dx. \tag{19.37}$$

Now from (19.34) we also have

$$\int_{B_r} \{|\nabla u|^2 + au^2\}\rho^2\Big(\frac{x}{r}\Big) dx \le \frac{C}{r}\left\{\int_{B_r} \{|\nabla u|^2 + au^2\}\rho^2\Big(\frac{x}{r}\Big) dx\right\}^{\frac{1}{2}} \left\{\int_{B_r\setminus B_{r/2}} u^2\, dx\right\}^{\frac{1}{2}}$$

which leads to

$$\int_{B_r} \{|\nabla u|^2 + au^2\}\rho^2\Big(\frac{x}{r}\Big) dx \le \frac{C^2}{r^2}\int_{B_r\setminus B_{r/2}} u^2\, dx$$

and to

$$\int_{B_{r/2}} \{|\nabla u|^2 + au^2\}\, dx \le \frac{C^2}{r^2}\int_{B_r\setminus B_{r/2}} u^2\, dx.$$

Going back to (19.37) we obtain

$$\int_{B_{r/2^{k+1}}} \{|\nabla u|^2 + au^2\}\, dx \le \frac{C'^k}{r^k}\cdot\frac{C^2}{r^2}\int_{B_r\setminus B_{r/2}} u^2\, dx.$$

If u is bounded we obtain

$$\int_{B_{r/2k+1}} \{|\nabla u|^2 + au^2\}\, dx \le \frac{C_k}{r^{k+2-n}}.$$

Choosing $k + 2 - n > 0$ and letting $r \to +\infty$ completes the proof. $\square$

Exercises

1. Using Theorem 19.2 show that a harmonic function cannot achieve a local maximum or a local minimum without being constant.
2. Show that in arbitrary dimension there is no nontrivial solution u to (19.29) such that
$$\frac{1}{r^2}\int_{B_r\setminus B_{r/2}} u^2\,dx \le C \quad \forall\, r>0.$$
3. A function is called "subharmonic" if
$$u\in C^2(\mathbb{R}^n), \qquad \Delta u \ge 0.$$
Show that for a subharmonic function we have
$$u(x_0) \le \frac{1}{|\partial B_R(x_0)|}\int_{\partial B_R(x_0)} u(y)\,d\sigma(y) \qquad \forall\, x_0\in\mathbb{R}^n, \quad \forall\, R>0,$$
$$u(x_0) \le \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u(y)\,dy \qquad \forall\, x_0\in\mathbb{R}^n, \quad \forall\, R>0.$$
4. Show that there exist nontrivial positive bounded subharmonic functions. (Hint: Use Theorem 19.7. Compare to [73].)
5. Rephrase Theorems 19.1, 19.2 for
$$u\in C^2(\Omega), \qquad \Delta u = 0 \quad \text{in } \Omega$$
where Ω is some open set of $\mathbb{R}^n$.
6. Let $b,c\in\mathbb{R}$, $b\neq 0$. Show that for $f\in L^\infty(\mathbb{R})$ the equation
$$u''+cu'+bu=f \quad \text{in } \mathbb{R}$$
admits a unique solution $u=\Lambda(f)$. Show that Λ is a bounded operator from $L^\infty(\mathbb{R})$ into itself.
7. Let $\Omega=\mathbb{R}\times(-1,1)$. Let u be a weak solution to
$$\begin{cases} -\Delta u = 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega = \mathbb{R}\times\{-1,1\}. \end{cases} \tag{1}$$
The first equation of (1) means that for any bounded set $\Omega'\subset\Omega$ one has
$$u\in H^1(\Omega') \quad \text{and} \quad \int_{\Omega'} \nabla u\cdot\nabla v\,dx = 0 \quad \forall\, v\in H_0^1(\Omega').$$
The second condition means that for any function $\rho=\rho(x_1)$ which is bounded, piecewise C^1, continuous and with support in $(-\ell,\ell)$ one has
$$\rho u\in H_0^1(\Omega_\ell)$$
where $\Omega_\ell=(-\ell,\ell)\times(-1,1)$.

(i) Using the technique of Chapter 6 show that for any $\ell_2 < \ell_1$ one has

$$\big||\nabla u|\big|_{2,\Omega_{\ell_2}} \le e^{-\sqrt{\lambda_1}(\ell_1-\ell_2)}\big||\nabla u|\big|_{2,\Omega_{\ell_1}}$$

(λ_1 is defined by (6.11)).

(ii) (Liouville type theorem). Show that there is no nontrivial solution to (1) such that for some $\alpha < \sqrt{\lambda_1}$, $C > 0$ one has

$$\big||\nabla u|\big|_{2,\Omega_{\ell_1}} \le Ce^{\alpha\ell_1}.$$

(iii) Show that this result is sharp since

$$u = e^{\frac{\pi}{2}x_1}\cos\frac{\pi}{2}x_2$$

is a nontrivial solution to (1).

8. Let u be such that

$$u \in L^2_{\text{loc}}(\Omega), \qquad \Delta u = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Show that u is harmonic.

9. We suppose that

$$\int_0^{+\infty} ra(r)\,dr = +\infty.$$

Show that there is no bounded nontrivial radially symmetric solution to

$$-\Delta u + a(r)u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

(See [22].)

10. Let u be a harmonic function. Show that

$$|\nabla u(y)| \le \frac{n}{R}\operatorname*{Sup}_{\partial B_R(y)}|u|.$$

Give another proof of the Liouville theorem for bounded harmonic functions.

Appendix

Fixed Point Theorems

We present here for the reader's convenience some fixed point theorems that we used in the text. We follow some lines of [47].

A.1 The Brouwer fixed point theorem

In this section $B_r(x)$ denotes the closed ball defined by

$$B_r(x) = \{\, y \in \mathbb{R}^n \mid |y - x| \leq r \,\}.$$

Theorem A.1. *Let $B_1(0)$ be the unit ball in $\mathbb{R}^n$. Let $\mathbf{F} : B_1(0) \to B_1(0)$ be a continuous mapping then $\mathbf{F}$ admits a fixed point that is to say a point for which*

$$\mathbf{F}(x) = x. \tag{A.1}$$

Proof. Let us give first a naive proof when $n = 1$. In this case $F : [-1, 1] \to [-1, 1]$. If we set

$$G(x) = F(x) - x$$

we have $G(-1) = F(-1)+1 \geq 0$, $G(1) = F(1)-1 \leq 0$ since $F(-1), F(1) \in [-1, 1]$. Since G is continuous there exists a point x such that $G(x) = 0$, i.e., such that (A.1) holds. $\square$

The proof in higher dimensions is more involved and requires some preliminaries.

Definition A.1. Let A be a $n \times n$ matrix. The matrix of the cofactors of A is the matrix given by

$$\mathrm{Cof}\, A = ((-1)^{i+k} \det A_{ik})_{i,k=1,\ldots,n} \tag{A.2}$$

where A_{ik} denotes the matrix deduced from A by deleting the i^{th} row and the k^{th} column.

We then have

Proposition A.2. *Let* $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^2*-mapping. Denote by* $\operatorname{Cof}(\nabla\mathbf{u})_{ik}$ *the entries of* $\operatorname{Cof}(\nabla\mathbf{u})$*. Then we have with the summation convention in* k

$$\partial_{x_k}(\operatorname{Cof}(\nabla\mathbf{u})_{ik}) = 0 \quad \forall\, i = 1, \dots, n, \tag{A.3}$$

i.e., the rows of the matrix $\operatorname{Cof}(\nabla\mathbf{u})$ *are divergence free.*

Proof. By definition of the matrix $\operatorname{Cof} A$ we have

$$(\det A)\,\mathrm{Id} = A^T \operatorname{Cof} A. \tag{A.4}$$

This matricial equality can also be written as

$$(\det A)\delta_{ij} = a_{ki}(\operatorname{Cof} A)_{kj}$$

(we set $A = (a_{ij})$). In particular since $(\operatorname{Cof} A)_{kj}$ contains no term in a_{ki} we have

$$\partial_{a_{ki}}(\det A) = (\operatorname{Cof} A)_{ki}. \tag{A.5}$$

Replacing now A by $\nabla\mathbf{u}$ in (A.4) we derive

$$(\det \nabla\mathbf{u})\delta_{ij} = \partial_{x_i} u^k (\operatorname{Cof} \nabla\mathbf{u})_{kj} \quad \forall\, i, j = 1, \dots, n,$$

where $\mathbf{u} = (u^1, \dots, u^n)$.

Taking the derivative in the direction x_j and summing up we obtain

$$\partial_{x_j}(\det \nabla\mathbf{u})\delta_{ij} = \partial^2_{x_i x_j} u^k (\operatorname{Cof} \nabla\mathbf{u})_{kj} + \partial_{x_i} u^k \partial_{x_j} (\operatorname{Cof} \nabla\mathbf{u})_{kj}. \tag{A.6}$$

Using the chain rule and (A.5) we also have

$$\partial_{x_i}(\det \nabla\mathbf{u}) = (\operatorname{Cof} \nabla\mathbf{u})_{kj} \cdot \partial_{x_i}(\partial_{x_j} u^k)$$

and (A.6) becomes

$$\partial_{x_i} u^k \partial_{x_j}(\operatorname{Cof} \nabla\mathbf{u})_{kj} = 0 \quad \forall\, i = 1, \dots, n. \tag{A.7}$$

If $\nabla\mathbf{u}(x)$ is invertible then we obtain

$$\partial_{x_j}(\operatorname{Cof} \nabla\mathbf{u}(x))_{kj} = 0.$$

If $\det \nabla\mathbf{u}(x) = 0$ then replacing $\mathbf{u}$ by $\mathbf{u} + \varepsilon\mathbf{x}$ the result follows by letting $\varepsilon \to 0$. This completes the proof of the proposition. □

As a consequence we have

Proposition A.3. *Let Ω be a smooth bounded open set of $\mathbb{R}^n$ and $\mathbf{u}, \mathbf{v} : \overline{\Omega} \to \mathbb{R}^n$ be $C^2(\overline{\Omega})$-mappings such that*

$$\mathbf{u} = \mathbf{v} \quad \text{on } \partial\Omega. \tag{A.8}$$

Then we have

$$\int_\Omega \det \nabla \mathbf{u} \, dx = \int_\Omega \det \nabla \mathbf{v} \, dx. \tag{A.9}$$

Proof. We set for $t \in [0, 1]$

$$f(t) = \int_\Omega \det \nabla \{\mathbf{u} + t(\mathbf{v} - \mathbf{u})\} \, dx.$$

By the chain rule we have (see (A.5))

$$\begin{aligned} f'(t) &= \int_\Omega \frac{d}{dt} \det\{\nabla(\mathbf{u} + t(\mathbf{v} - \mathbf{u}))\} \, dx \\ &= \int_\Omega \big(\mathrm{Cof}\{\nabla(\mathbf{u} + t(\mathbf{v} - \mathbf{u}))\}\big)_{ik} \partial_{x_k}(v^i - u^i) \, dx \\ &= \int_\Omega \partial_{x_k}\big(\mathrm{Cof}\{\nabla(\mathbf{u} + t(\mathbf{v} - \mathbf{u}))\}_{ik}(v^i - u^i)\big) \, dx \quad \text{(by (A.3))} \\ &= 0 \qquad \text{(by (A.8))}. \end{aligned}$$

It follows that f is constant on $(0, 1)$ and since it is continuous $f(0) = f(1)$. This is (A.9) and the proof of the proposition is complete. □

Proof of Brouwer's theorem. 1. One cannot find $\mathbf{w} : B_1(0) \to \partial B_1(0)$ which is C^2 and satisfies $\mathbf{w}(x) = x$ on $\partial B_1(0)$.

Indeed in case such a function exists by the proposition above we have

$$\int_{B_1(0)} \det \nabla \mathbf{w} = \int_{B_1(0)} \det \nabla \mathbf{Id}(x) = |B_1(0)| \tag{A.10}$$

where $|B_1(0)|$ denotes the measure of the unit ball. On the other hand we have

$$\mathbf{w} \cdot \mathbf{w} \equiv 1$$

and differentiating we derive

$$\nabla \mathbf{w} \cdot \mathbf{w} \equiv 0.$$

It follows that $\det \nabla \mathbf{w} \equiv 0$ and a contradiction to (A.10).

2. One cannot find $\mathbf{w} : B_1(0) \to \partial B_1(0)$ continuous and satisfying $\mathbf{w}(x) = x$ on $\partial B_1(0)$. Suppose such a function exists. We extend it by x outside $B_1(0)$. Then we consider

$$\mathbf{w}_\varepsilon(x) = \rho_\varepsilon * \mathbf{w}(x), \tag{A.11}$$

i.e., we mollify each component of $\mathbf{w}$. Then for ε small enough one has

$$\mathbf{w}_\varepsilon(x) = x \quad \text{on } \partial B_2(0). \tag{A.12}$$

Moreover – see Chapter 2 – $\mathbf{w}_\varepsilon \to \mathbf{w}$ uniformly on $B_2(0)$ and for ε small enough we have $|\mathbf{w}_\varepsilon| > \frac{1}{2}$. It follows that the mapping

$$\tilde{\mathbf{w}}_\varepsilon(\mathbf{x}) = \frac{2\mathbf{w}_\varepsilon}{|\mathbf{w}_\varepsilon|}$$

is a smooth mapping satisfying all the assumptions of part 1 with $B_1(0)$ replaced by $B_2(0)$. This completes the proof of this part.

3. Let $\mathbf{F}$ be a continuous map such that $\mathbf{F}(x) \neq x \;\forall\, x$. Let $\mathbf{w}(x)$ be the point of $\partial B_1(0)$ where the half-line $(\mathbf{F}(x), x)$ crosses $\partial B_1(0)$. Then $\mathbf{w}$ is a mapping as in Part 3 and we get a contradiction. This completes the proof of the theorem. □

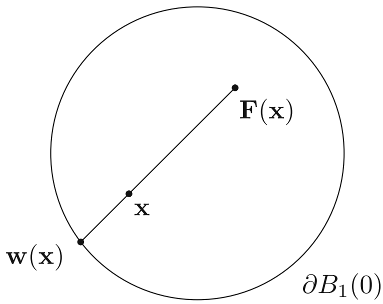


Figure A.1.

Corollary A.4. *Let K be a compact convex subset of $\mathbb{R}^n$ and $\mathbf{F} : K \to K$ be a continuous mapping. Then $\mathbf{F}$ admits a fixed point.*

Proof. Since K is compact it is bounded. Let $R > 0$ such that $K \subset B_R(0)$. Set

$$\mathbf{G} = \mathbf{F} \circ \mathbf{P}_K$$

where $\mathbf{P}_K$ denotes the projection on the convex set K (see Theorem 1.1 and (11.62)). Then $\mathbf{G}$ is a continuous mapping from $B_R(0)$ into itself and by Theorem A.1 it possesses a fixed point x, i.e., such that

$$\mathbf{F}(\mathbf{P}_K(x)) = x.$$

Since $x \in K$ this is also a fixed point for $\mathbf{F}$. This completes the proof. □

A.2 The Schauder fixed point theorem

The Schauder fixed point theorem is a generalization of the Brouwer fixed point theorem in infinite dimensions. More precisely we have

Theorem A.5. *Let K be a compact convex subset of a Banach space B. Let F be a continuous mapping from K into itself, then F admits a fixed point.*

Proof. Since K is compact, for any $\varepsilon > 0$ one can cover K with $N(\varepsilon)$ open balls

$$B_\varepsilon(x_k) \quad k = 1, \ldots, N = N(\varepsilon).$$

Denote by K_ε the convex hull of the x_k. Consider the mapping I_ε from K into K_ε defined by

$$I_\varepsilon(x) = \frac{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i)) x_i}{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i))}.$$

Note that I_ε is well defined and continuous (see Exercise 3). By a simple computation one has for every $x \in K$

$$\begin{aligned} |I_\varepsilon(x) - x| &= \left| \frac{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i))(x_i - x)}{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i))} \right| \\ &\leq \frac{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i))|x_i - x|}{\sum_i \operatorname{dist}(x, K \setminus B_\varepsilon(x_i))} \leq \varepsilon. \end{aligned} \tag{A.13}$$

Consider then the mapping

$$F_\varepsilon = I_\varepsilon \circ F.$$

As a mapping from K_ε into itself it has a fixed point x_ε. Now up to a subsequence (K is compact) one can assume that

$$x_\varepsilon \longrightarrow x_0.$$

We have then by (A.12)

$$|F(x_\varepsilon) - x_\varepsilon| = |F(x_\varepsilon) - I_\varepsilon(F(x_\varepsilon))| \leq \varepsilon.$$

Letting $\varepsilon \to 0$ we see – since F is continuous – that x_0 is a fixed point. This completes the proof of the theorem. □

As a corollary we have

Corollary A.6. *Let K be a closed convex set of a Banach space B and F a continuous mapping from K into K such that $F(K)$ is precompact. Then F admits a fixed point.*

Proof. Apply the previous theorem for the compact convex set $\widetilde{K}$ = the closure of the convex hull of $F(K)$ (see Exercise 4). □

Remark A.3. The compactness in the analysis above cannot be dropped. Consider for instance the space of real sequences

$$\ell^2 = \left\{ (x_i)_i \;\Big|\; \sum_{i=1}^{+\infty} x_i^2 < +\infty \right\}.$$

This is a Hilbert space for the norm

$$|x|_2 = \left\{ \sum_{i=1}^{+\infty} x_i^2 \right\}^{\frac{1}{2}} \quad \forall\, x = (x_i).$$

Now the application defined by

$$F(x) = (0, \sqrt{1 - |x|_2^2}, x_1, x_2, \dots,) \quad \forall\, x = (x_1, x_2, \dots)$$

maps the unit ball of ℓ^2 into the unit sphere. A fixed point x would be such that

$$F^n(x) = x \quad \forall\, n$$

$(F^n = \underbrace{F \circ F \circ \cdots \circ F}_{n\text{-times}})$ and thus

$$x_1 = x_2 = \cdots = x_n = 0 \quad \forall\, n.$$

The only possibility would be $x = 0$ which clearly is not a fixed point since it does not belong to the unit sphere.

Exercises

1. Let $\mathbf{u} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$; $\mathbf{u} \in C^1(\Omega)$. Prove that for all $x \in \Omega$, there exists $\varepsilon_x > 0$ such that $\det \nabla(\mathbf{u} + \varepsilon\, \mathbf{Id})(x) \neq 0 \;\forall\, \varepsilon \in (-\varepsilon_x, \varepsilon_x) \setminus \{0\}$.
2. Let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \rho\, dx = 1$ and $\rho(x) = \rho(-x) \;\forall\, x \in \mathbb{R}^n$. Prove that $\mathbf{Id} * \rho = \mathbf{Id}$ (we mollify each component of the identity mapping) and justify (A.12) for ρ_ε well chosen.
3. (a) Prove that the mapping $I_\varepsilon : K \to K_\varepsilon$ defined in the proof of Theorem A.5 is well defined and continuous.
   (b) Justify carefully the inequality (A.13).
4. Let A be a precompact subset of a Banach space X. Prove that the closure of the convex hull of A is a compact convex set.

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