Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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Homework Exercises 1 – December 11, 2024

1. Let $f : X \to Y$
   (a) If f is continuous, $A \subset X$. Show that $f(\overline{A}) \subseteq \overline{f(A)}$.
   f is continuous means that given any open set $V \subset Y$ then $f^{-1}(V)$ is open. Let $A = f^{-1}(V)$. Then, $\overline{A} = A \cup \partial A$ and $A \cap \partial A =$, that is they are disjoint. Thus,
   $$\begin{aligned} f(\overline{A}) &= f(A \cup \partial A) \\ &= f(A) \cup f(\partial A) \end{aligned}$$
   How do I show that $f(\partial A) \subset \overline{f(A)}$?

   Let $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = x \in \partial A$. $f(x_n) \in f(A), \forall n$. Either, $\lim_{n\to\infty} f(x_n) = f(x) \in f(A)$ or for any $\epsilon > 0, \exists N > 0 \to |f(x) - f(x_n)| < \epsilon$ whenever $n > N$. That is, $f(x) \notin A$ but infinitely close to $f(A)$ hence $f(x) \in \partial A$. Thus, in general, $f(x_n) \in f(A) \cup \partial f(A) = \overline{f(A)}, \forall n$.
   (b) Suppose $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$, is f continuous? Prove your claim.
   Let $C \subset f(X)$ and let $\{y_n\} \to y \in \overline{C}$. Then $\exists \{x_n\} \in f^{-1}(\overline{C}) \to f(x_n) = y_n$. Furthermore, for every $\epsilon > 0$, $\exists N > 0 \to |y_n - y| = |f(x_n) - y| < \epsilon \implies n > N$. Actually, it says more than that. Because $\{y_n\} \to y \in \overline{C}$ we can say that it converges uniformly ($\overline{C}$ is closed and bounded), that is for every $\epsilon > 0$, $\exists N > 0 \to |y_n - y_m| = |f(x_n) - f(x_m)| < \epsilon \implies n, m > N$. We can see that $f^{-1}(\overline{C})$ is also bounded, therefore $\{x_n\}$ must converge. Let $A = f^{-1}(\overline{C})$. Clearly, all $\{x_n\} \in A$ given any $\{y_n\} \in C$. Hence, f is continuous.

2. Let X be a compact metric space, $f : X \to X$ satisfies
   $$d(f(x), f(y)) < d(x, y) \text{ for all distinct } x, y \in X.$$
   Show that there is a unique $x^* \in X$ such that $f(x^*) = x^*$.
   What its really saying is given a function that maps back onto itself and that the space between mappings is always less than the space between beginnings, then there is a point where function maps back onto itself.
   Given any $x, x_1 \in X$ then let $\epsilon_1 = d(x, x_1) - d(f(x), f(x_1))$. Pick $x_2 \in X$ such that $d(x_1, x_2) < \epsilon_1$ then $d(f(x_1), f(x_2) < d(x_1, x_2) < \epsilon_1$. Pick $\epsilon_2 = d(x_1, x_2) - d(f(x_1), f(x_2))$. Clearly, $\epsilon_2 < \epsilon_1$. Repeat this over and over again: $\epsilon_i = d(x_{i-1}, x_i) - d(f(x_{i-1}), f(x_i))$ and pick $x_i \in X$ such that $d(x_{i-1}, x_i) < \epsilon_{i-1}$. We can see that $\epsilon_i \to 0$ as $i \to \infty$.

3. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous, $\varphi_n : [a, b] \to [c, d]$ converges uniformly on $[a, b]$. Show that $F_n := f(\cdot, \varphi_n(\cdot))$ also converges uniformaly on $[a, b]$.

4. Let $D = (a, b) \times (c, d)$, $f : D \to \mathbb{R}$ satisfies the following
   (a) for $\forall y \in (c, d)$, $f(\cdot, y) \in C\,(a, b)$.
   (b) for all $x \in (a, b)$, $f(x, \cdot)$ is Lipschitz, namely there is $L > 0$ such that
   $$|\,f(x, y_1) - f(x, y_2)\,| \leq L\,|\,y_1 - y_2\,| \text{ for } y_1, y_2 \in (c, d)$$
   Show that $v \in C(D)$.

5. If $f : \mathbb{R}^2 \to \mathbb{R}$ has bounded partial derivative $\partial_x f$ and $\partial_y f$, show that $f \in C(\mathbb{R}^2)$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $\partial_x f(0, 0)$ exists and $\partial_f f$ is continuous at $(0, 0)$. Show that f is differentiable at $(0, 0)$.

7. Show that $f : B^m_r(a) \to \mathbb{R}^n$ is differentiable at a iff there is a map $A : B_r(0) \to \mathbb{R}^{n\times m}$ continuous at a such that
   $$f(a + h) - f(a) = A(h)h \text{ for } h \in B^r(0).$$

8. Let $f : B^m_r(a) \to \mathbb{R}^n$ be differentiable at a,
   $$|\,f(x) - f(a)\,| \geq |\,x - a\,| \text{ for } x \in B_r(a).$$
   Show that $\mathrm{rank} f'(a) = m$.

9. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable, $h \in \mathbb{R}^m$. If f is bounded and $h \cdot \nabla f(x) - f(x)$ for all $x \in \mathbb{R}^m$, show that $f(x) = 0$ for al l$x \in \mathbb{R}^m$.

10. Let $\Omega \subset \mathbb{R}^2$ be open and connected. If $f : \Omega \to \mathbb{R}$ be diffentiable, $\nabla f(x, y) = 0$ for all $(x, y) \in \Omega$. Show that f is a constant function.