Math 725 – Advanced Linear Algebra

Paul Carmody

Assignment #10 – Due 12/1/23

1. Let T be a normal operator on a finite dimensional inner product space. Prove that T is self-adjoint, positive definite, or unitary if the eigenvalues of T are real, positive, or absolute value one, respectively.

Self-adjoint implies real eigenvalues.

$T = T^*$ thus $\langle Tv, v \rangle = \langle v, T^*v \rangle = \left\langle \overline{Tv}, v \right\rangle$ which implies that $Tv = \overline{Tv}$ for all $v \in V$ and hence $Tv \in \mathbb{R}$. Given any eigenvalue λ and eigvenvector v, $Tv = \lambda v = \overline{\lambda v}$ which implies that $\lambda = \overline{\lambda}$. Hence $\lambda \in \mathbb{R}$.

Positive-definite implies positive eigenvalues.

Positive-definite means that $\langle Tv, v \rangle > 0$ for all $v \in V$. Given an eigenvalue λ and associated eigenvector v we have $\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle > 0$. Since $\langle v, v \rangle > 0$ for all $v \in V$, $\lambda > 0$.

Unitary implies eigenvalues are one.

T is unitary means that there exists an orthornormal basis such that the matrix A associated with T has the property that $A^*A = I$. The eigenvalues for A^* and the eigenvalues for A are the same any eigenvector v with eigenvalue λ will allow the following equation: $v^Tv = v^TIv = v^TA^*Av = (\lambda v^T)(\lambda v) = \lambda^2 v^Tv$ which implies that $\lambda^2 = 1$ and, being real, it must be one.

2. Let T be an operator on a finite dimensional inner product space that is both positive definite and unitary. Prove that $T = I$.

T is positive definite implies that T is diagonalizable. T is unitary implies that its eigenvalue are 1. Hence, a diagonal matrix with 1 on the diagonal is the identity.

3. Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Show that

$$\lambda_1 = \max\{x^TAx : ||x||_2 = 1\} \quad \text{and} \quad \lambda_n = \min\{x^TAx : ||x||_2 = 1\}.$$

Prove that the maximum value is achieved when $x = \pm u_1$ where u_1 is a unit eigenvector associated to λ_1 and the minimum value is achieved when $x = \pm u_n$ where u_n is a unit eigenvector associated to λ_n.

4. The Hilbert matrix H_{n+1} is an $(n+1)\times(n+1)$ matrix whose (i,j) entry is $\frac{1}{i+j-1}$.

a) Write down H_4.

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

b) Prove that if $v_1, \ldots, v_n$ are linearly independent vectors in a real inner product space V, then the matrix K whose (i,j) entry is $\langle v_i, v_j\rangle$ is a symmetric positive definite matrix. Also prove that such a matrix is invertible.

Since all $v_i, v_j \in \mathbb{R}$ then $\langle\, v_i, v_j\,\rangle = \langle\, v_j, v_i\,\rangle$. Hence K is symmetric. All entries for K are greater than zero. Thus, Ku will only increase the elements in u and $\langle\, Ku, u\,\rangle > 0$. K is both positive and symmetric.

c) Show that H_{n+1} is a symmetric positive definite matrix. [Hint: Consider $V = \mathcal{P}^{(n)}$ and the inner product $\langle f,\, g\rangle = \int_0^1 f(t)g(t)dt$.]

Allowing for the standard orthonormal basis for $\mathcal{P}^{(n)}$, $1, x, x^2, \ldots, x^n$. We can see that defining the inner product as $\langle f,\, g\rangle = \int_0^1 f(t)g(t)dt$ we'll get a transformation matrix taking the inner product of corresponding basis vectors

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & x & x^2 & \cdots & x^n \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{array} & \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{pmatrix} \end{array}$$

And since $\int_0^1 f(t)g(t)dt > 0$ for all $f, g \in \mathcal{P}^{(n)}$ and f and g commutes we see that it is symmetric as well.

d) Conclude the nontrivial fact that H_{n+1} is invertible.

5. Let $\mathcal{S}_n$ be the vector spaceof all $n \times n$ symmetric real matrices and let $PSD_n \subset \mathcal{S}_n$ be the set of all $n \times n$ positive semidefinite matrices. Prove that PSD_n is a convex cone in $\mathcal{S}_n$, i.e., show that

i) if $A \in PSD_n$ and $\lambda \geq 0$ then $\lambda A \in PSD_n$, and

$A \in PSD_n$ then $\langle Av, v \rangle > 0$ for all $v \in V$. then $\langle \lambda Av, v \rangle = \lambda \langle Av, v \rangle$ which must be non-negative when λ is non-negative.

ii) If $A, B \in PSD_n$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$ then $\lambda A + \mu B \in PSD_n$.

$\langle (\lambda A + \mu B)v, v \rangle = \langle \lambda Av + \mu Bv, v \rangle = \langle \lambda Av, v \rangle + \langle \mu Bv, v \rangle = \lambda \langle Av, v \rangle + \mu \langle Bv, v \rangle$. All terms are non-negative so the whole expression must be non-negative.

6. If A and B are two real $n \times n$ symmetric matrices, we write $A \succeq B$ if $A - B$ is positive semidefinite. Prove

i) Additivity: if $A_1 \succeq B_1$ and $A_2 \succeq B_2$ then $A_1 + A_2 \succeq B_1 + B_2$,
We know from 5 ii) above that

$$(A_1 - B_1) + (A_2 - B_2) \in PSD_n$$
$$(A_1 + A_2) - (B_1 + B_2) \in PSD_n$$
$$A_1 + A_2 \succeq B_1 + B_2$$

ii) Transitivity: if $A \succeq B$ and $B \succeq C$ then $A \succeq C$.

$$A - B \in PSD_n$$
$$B - C \in PSD_n$$
$$(A - B) + (B - C) \in PSD_n \text{ from 5 ii above}$$
$$A - C \in PSD_n$$
$$A \succeq C$$

iii) Multiplicativity: if $A \succeq B$ and Q is an invertible matrix then $QAQ^t \succeq QBQ^t$.

$$Q(A - B)Q^t = QAQ^t - QBA^t \implies QAQ^t \succeq QBQ^t$$

Extra Questions

1. Let A be $n \times n$ real symmetric positive matrix. Show that
$$\frac{\pi^{\frac{n}{2}}}{\sqrt{\det A}} = \int_{\mathbb{R}^n} e^{-\langle x, Ax \rangle}\, dx.$$