# Math 5050 – Special Topics: Manifolds– Spring 2025

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Section 1 problems 1, 3, 4, 5, 8.

**1.1. A function that is $C^2$ but not $C^3$.**

Let $g : \mathbb{R} \to \mathbb{R}$ be the function in example 1.2(iii). Show that the function $h(x) = \int_0^x g(t)dt$ is $C^2$ but not $C^3$ at $x = 0$.

$$h(x) = \int_0^x g(t)dt$$
$$h'(x) = g(x) = \frac{3}{4}x^{\frac{4}{3}}$$
$$h''(x) = g'(x) = x^{\frac{1}{3}}$$
$$h'''(x) = \frac{1}{3}x^{\frac{-2}{3}}$$

which is NOT continuous at $x = 0$.

**1.3. A diffeomorphism of open interval in $\mathbb{R}$**

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$ be open subsets. A $C^\infty$ map $F : U \to V$ is called a *differomorphism* if it is bijective and has $C^\infty$ inverse $F^{-1} : V \to U$.

(a) Show that the function $f : ]-\pi/2, /pi/2[ \to \mathbb{R}, f(x) = \tan x$, is a diffeomorphism.

$$f(x) = \tan x$$
$$f^{-1}(x) = \tan^{-1} x$$
$$f^{-1}(x)' = \frac{1}{1+x^2}$$
$$f^{-1}(x)'' = \frac{-2x}{(1+x^2)^2}$$

It is clear that further differentiaion will increase the power of the denominator and the number of terms indefinitely. Thus $f^{-1} \in C^\infty$.

(b) Let $a, b$ be real numbers of $a < b$. Find a linear function $h : (a, b) \to (-1, 1)$, thus proving that any two finite open intervals are diffeomorphic.

The composite $f \circ h : (a, b) \to \mathbb{R}$ is then a diffeomorphism of an open interval with $\mathbb{R}$.

We must a function of the form $h(x) = mx + c$ and find both the slope $m$ and the $y$-intercept in terms of $a$ and $b$.

$$h(a) = -1 \text{ and } h(b) = a$$
$$m = \frac{b-a}{1-(-1)} = \frac{b-a}{2}$$
$$-1 = ma + c$$
$$1 = mb + c$$
$$0 = m(a+b) + c = \frac{b-a}{2}(b+a) + c$$
$$c = -\frac{b^2 - a^2}{2}$$
$$h(x) = \frac{b-a}{2}x - \frac{b^2-a^2}{2}$$

(c) The exponent function exp: $\mathbb{R} \to ]0, \infty[$ is a differomorphism. Use it to show that for any real numbers $a$ and $b$, the intervals $\mathbb{R}, ]a, \infty[$, and $]-\infty, b[$ are diffeomorphic.

Goal: find a map $f : ]-\infty, b[ \to ]a, \infty[$ which has the form $f(x) = ce^{-x}$ try

$$f(a) = b \implies ce^{-a} = b,\ c = be^a$$
$$f(x) = be^{x-a}$$

$f \in C^\infty$ as is $f^{-1}$. This also maps onto $\mathbb{R}$ quite well.

**1.4. A differomorphism of an open cube with $\mathbb{R}^n$**

Show that the map

$$f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^n \to \mathbb{R}^n,\ f(x_1, \ldots, x_n) = (\tan x_1, \ldots, \tan x_n),$$

is a differomorphism.

From 1.3. we can see that $f(x^i) = \tan x^i$ and $\frac{\partial f}{\partial x^i} = \frac{d \tan x^i}{dx^i}$. Thus, each $f(x^i) \in C^\infty$ for all $i = 1, \ldots, n$ and $f \in C^\infty$. The same can be said for $f^{-1}$, thus $f$ is a diffeomorphism.

**1.5. A diffeomorphism of an open ball with $\mathbb{R}^n$**

Let $\mathbf{0} = (0, 0)$ be the origin and $B(\mathbf{0}, 1)$ the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(\mathbf{0}, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the $xy$-plane in $R^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z - 1)^2 = 1,\ z < 1$$

in $\mathbb{R}^3$ as an intermediate space (Figure 1.4). First note that the map

$$f : B(\mathbf{0}, 1) \to S,\ (a, b) \mapsto (a, b, 1 - \sqrt{1 - a^2 - b^2}),$$

is a bijection.

(a) The *stereographic projection* $g : S \to \mathbb{R}^2$ from $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the intersection of the line through $(0, 0, 1)$ and $(a, b, c)$ with the $xy$-plane.

Show that it is given by

$$(a, b, c) \mapsto (u, v) = \left(\frac{a}{1 - c}, \frac{b}{1 - c}\right),\ c = 1 - \sqrt{1 - a^2 - b^2},$$

with inverse

$$(u, v) \mapsto \left(\frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, 1 - \frac{1}{\sqrt{1 + u^2 + v^2}}\right).$$

We have a line in space with two points $(0, 0, 1)$ and $(a, b, c)$ and we want to find the point where on the $xy$-plane or where $z = 0$.

$$a^2 + b^2 + (c - 1)^2 = 1 \implies c = 1 - \sqrt{1 - a^2 - b^2}$$

remembering the symmetric form for the equation of a line

$$\frac{x}{a} = \frac{y}{b} = \frac{z + 1}{1 - c}$$

When this line intersects with the $xy$-plane at $(u, v, 0)$ we have

$$\frac{u}{a} = \frac{1}{1 - c} \implies u = \frac{a}{1 - c}$$
$$\frac{v}{b} = \frac{1}{1 - c} \implies v = \frac{b}{1 - c}$$
$$(a, b, c) \mapsto \left(\frac{a}{1 - c}, \frac{b}{1 - c}\right)$$

the inverse would be a line from $(0, 0, 1)$ to $(u, v, 0)$ through a point on the hemisphere $(p, q, r)$ by using similar triangles and the radius of the sphere $R$

$$\frac{p}{R} = \frac{u}{\sqrt{1 + u^2 + v^2}},\ \frac{q}{R} = \frac{v}{\sqrt{1 + u^2 + v^2}}$$
$$\frac{1 - r}{R} = \frac{1}{\sqrt{1 + u^2 + v^2}}$$
$$R = 1 \implies (p, q, r) = \left(\frac{u}{\sqrt{1 + u^2 + v^2}}, \frac{v}{\sqrt{1 + u^2 + v^2}}, 1 - \frac{1}{\sqrt{1 + u^2 + v^2}}\right)$$

(b) Composing the two maps $f$ and $g$ gives the map

$$h = g \circ f : B(\mathbf{0}, 1) \to \mathbb{R}^2 : h(a, b) = \left(\frac{a}{\sqrt{1 - a^2 - b^2}}, \frac{b}{\sqrt{1 - a^2 - b^2}}\right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that $h$ is a differomorphism of the open disk $B(\mathbf{0},1)$ with $\mathbb{R}^2$.

$f^{-1}$ accepts a point on the hemisphere $S$ and projects it down to a point on the disc. $f^{-1}(x,y,z) = (x,y)$ thus

$$\begin{aligned} f^{-1} \circ g^{-1}(u,v) &= f^{-1}\left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}, 1 - \frac{1}{\sqrt{1+u^2+v^2}}\right) \\ &= \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right) \end{aligned}$$

(c) Generalize part (b) to $\mathbb{R}^n$.

Changing $S$ from a hemisphere to a half-hypersphere with radius=1 in $\mathbb{R}^n$ and still moving its center up one axis, $x^i$, by 1, all other parameters will be equally effected as the $x$ and $y$ coordinates where $x^i$ will respond like the $z$. Thus,

$$\begin{aligned} h(x) &= (h^1(x), h^2(x), \dots, h^k(x), \dots, h^n(x)) \\ h^k(x) &= \begin{cases} \frac{x^k}{1+\sqrt{\sum_{j=1}^n (x^j)^2}} & k \neq i \\ 1 - \frac{1}{1+\sqrt{\sum_{j=1}^n (x^j)^2}} \end{cases} \end{aligned}$$

**1.8. Bijective $C^\infty$ maps.**
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that $f$ is a bijective $C^\infty$ map, but that $f^{-1}$ is not $C^\infty$. (This example shows that a bijective $C^\infty$ map need not have a $C^\infty$ inverse. In complex analysis, the situation is quite different: a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

$$\begin{aligned} f^{-1}(x) &= x^{1/3} \\ (f^{-1}(x))' &= \frac{1}{3}x^{-2/3} \end{aligned}$$

which is not continuous at zero and therefore $f^{-1} \notin C^\infty$.