Math 5110 – Real Analysis I– Fall 2024
w/Professor Liu

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I. This problem reviews continuity for functions on real line.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $a \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(a) Show that $f(x) = x^2$ is continuous at $x = 2$.
Given an $\epsilon > 0$, when $|f(x) - 4| < \epsilon$, $|x^2 - 4| < \epsilon$. Let $\delta < \sqrt{\epsilon + 4}$
If

$$
\begin{aligned}
(2+\delta)^2 - 4 &< \epsilon \\
(2+\delta)^2 &< \epsilon + 4 \\
2 + \delta &< \sqrt{\epsilon+4} \\
\delta &< \sqrt{\epsilon+4} - 2
\end{aligned}
$$

(b) Suppose that f is continuous at a and $f(a) \neq 0$. Show that f is nonzero in some open interval containing a.
Since f is continuous at a and $f(a) \neq 0$ then for every $\epsilon > 0$ such that when $|f(x) - f(a)| < \epsilon$. Without loss of generality, assume $f(a)$ is positive, $f(a) > 0$. Choose $0 < \epsilon < f(a)$ then $0 < f(a) - \epsilon < f(x) < f(a) + \epsilon$. Therefore, $f(x) \neq 0$

II. This problem review derivatives.

(a) Let $f(x) = x^n$ for some positive integer n. Using the definition of the derivative, and the binomial theorem, show that $f'(x^n) = nx^{n-1}$.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$f'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$
$$f(x) = x^n$$

thus

$$f'(x^n) = \lim_{h\to 0} \frac{(x-h)^n - x^n}{h}$$
$$= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-h)^k - x^n}{h}$$

remove the first entry from the summation

$$f'(x^n) = \lim_{h\to 0} \frac{x^n + \sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$
$$= \lim_{h\to 0} \frac{\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k}{h}$$
$$= \sum_{k=1}^{n} \lim_{h\to 0} \frac{\binom{n}{k} x^{n-k} h^k}{h}$$
$$= \lim_{h\to 0} \frac{\binom{n}{1} x^{n-1} h}{h} + \lim_{h\to 0} \frac{\binom{n}{2} x^{n-2} h^2}{h} + \cdots + \lim_{h\to 0} \frac{\binom{n}{n} x^{0} h^n}{h}$$
$$= nx^{n-1} + \lim_{h\to 0} \frac{\binom{n}{2} x^{n-2} h^2}{h} + \cdots + \lim_{h\to 0} \frac{\binom{n}{n} x^{0} h^n}{h}$$

the $\lim\limits_{h\to 0} \frac{h^k}{h} = \lim\limits_{h\to 0} h^{k-1} = 0$ for all $k \geq 1$ therefore

$$f'(x^n) = nx^{n-1}$$

(b) Is the function

$$f(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x \leq 0 \end{cases}$$

differentiable at $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$f(x)$ is both continuous and differentiaable at $x = 0$.

III. This problem reviews sup and inf.

For any subset $A \subset \mathbb{R}$, we say that M is an *upper bound* for A if $x \leq M$ for all $x \in A$. If a set A has a finite upper bound, we say it is *boundared above.* It is a theorem about the set $\mathbb{R}$ that *for any set $A \subset \mathbb{R}$ that is bounded above, there exists a least (smallest) upper bound for A.* This least upper bound is called supermum of A, and denoted $\sup A$. By definition, the number $\sup A$ has two properties.

(i) $x \leq \sup A$ for all $x \in A$ (i.e., $\sup A$ is an upper bound for M).

(ii) for any M that is an upper bound for A, we have $\sup A \leq M$.

For sets that are not bounded above, we say that $\sup A = +\infty$. we often write things like

$$\sup_{x \in A} f(x),$$

to denote the supremum of the set $\{f(x) : x \in A\}$, where f is a some function.

Similarly, any set that is bounded below has a *greatest lower bound* called the *infimum*, denoted $\inf A$. It satisfies the same properties as $\sup A$ with the inequalities reversed.

(a) Find $\sup A$ and $\inf A$ for $A = (1, 2]$, $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, and $A = \{0, 1, 2, 3, \dots\}$.

- $A = (1, 2]$, $\sup A = 2$ and $\inf A = 1$
- $A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, $\sup A = 1$, and $\inf A = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.
- $A = \{0, 1, 2, 3, \dots\}$. $\sup A = \lim\limits_{n\to\infty} \quad = \infty$, and $\inf A = 0$

(b) Find $\sup\limits_{x\in(0,1)} (1 + x^2)^{-1}$

Let $f(x) = (1+x^2)^{-1}$. On the interval $(0, 1)$ we can see that it is strictly decreasing, that is $a < b \implies f(a) > f(b)$. Thus, $\sup\limits_{x\in(0,1)} f(x) = f(0) = (1 + 0^2)^{-1} = 1$.

(c) Assume that $\sup A < \infty$, and show that for every $\epsilon > 0$, there exists $x \in A$ such that $x > \sup A - \epsilon$.

Given any $\epsilon > 0$ let $x > \sup A - \epsilon$. If $x \notin A$ then x is an upper bound of A, i.e., $x \in M$ and $x < \sup A$, but that violates proper (ii). Hence, $x \in A$.

(d) For any two functions $f, g : \mathbb{R} \to \mathbb{R}$, and any set $A \subset \mathbb{R}$, show that $\sup\limits_{x\in A} (f(x) + g(x)) \leq \sup\limits_{x\in A} f(x) + \sup\limits_{x\in A} g(x)$.

$$
\begin{aligned}
f(x) \leq \sup_{x\in A} f(x) \text{ and } g(x) &\leq \sup_{x\in A} g(x), \forall x \in x \in A \\
\therefore f(x) + g(x) &\leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x), \forall x \in A \\
\text{and } \sup_{x\in A} (f(x) + g(x)) &\leq \sup_{x\in A} f(x) + \sup_{x\in A} g(x)
\end{aligned}
$$

IV. Section 1.1, Exercise 5, 6, 13.

Exercise 1.1.5. Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers verify the identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right), \tag{1.3}$$

and conclude *Cauchy-Schwarz inequality*

$$\left|\sum_{i=1}^{n} a_1 b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=j}^{n} b_j^2\right)^{1/2}$$

Let's start by expanding the center term

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}((a_i b_j)^2 + (a_j b_i)^2 - (a_i b_j a_j b_i)^2) \\
&= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{i=1}^{n} b_i^2 \sum_{j=1}^{n} a_i^2 - 2\sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j b_j \\
&= 2\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - 2\left(\sum_{i=1}^{n} a_i b_i\right)^2
\end{aligned}$$

Equation 1.3 then becomes

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

which is true. Since

$$\begin{aligned}
\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 &= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \\
\left(\sum_{i=1}^{n} a_i b_i\right)^2 &= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \\
\therefore \left|\sum_{i=1}^{n} a_i b_i\right| &\leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}
\end{aligned}$$

Let's start by taking the square of the distance from $a + b$ to zero using the ℓ^2.

$$\begin{aligned}
d_{\ell^2}(a + b, 0)^2 &= \sum_{i=1}^{n}(a_i + b_i)^2 \\
&= \sum_{i=1}^{n}(a_i^2 + b_i^2 + 2a_i b_i) \\
&= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i
\end{aligned}$$

apply Cauchy-Schwarz and factor.

$$\begin{aligned} d_{\ell^2}(a+b,0)^2 &\le d_{\ell^2}(a,0) + d_{\ell^2}(b,0) + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2} \\ &\le d_{\ell^2}(a,0) + d_{\ell^2}(b,0) + 2\left(\, d_{\ell^2}(a,0)\cdot d_{\ell^2}(b,0)\,\right)^{1/2} \\ &\le \left(\, d_{\ell^2}(a,0)^{1/2} + d_{\ell^2}(b,0)^{1/2}\,\right)^2 \end{aligned}$$

Expand the ℓ^2 metrics and take the square root of both sides and

$$\left(\sum_{i=1}^{n}(a_i+b_i)^2\right)^{1/2} \le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=j}^{n} b_j^2\right)^{1/2}$$

Exercise 1.1.6 Show that $(\mathbb{R}^n, d_{l^2})$ in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5)

Example 1.1.6 (Euclidean spaces). Let $n \geq 1$ be a natural number, and let $\mathbb{R}^n$ be the space of n-tupes of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \end{aligned}$$

We must prove that d_{ℓ^2} is symmetric, positive definite and that the triangle inequality holds.

Symmetric: show that $d_{\ell^2}(x, y) = d_{\ell^2}(y, x)$.

$$\begin{aligned} d_{l^2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2} \\ &= d_{l^2}((y_1, \ldots, y_n), (x_1, \ldots, x_n)) \end{aligned}$$

Positive Definite: show that $d_{\ell^2}(x, y) \geq 0$ and $d_{\ell^2}(x, y) = 0 \to x = y$.

The square root is taken as a positive value. $d_{l^2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = 0$ implies that each $x_i - y_i = 0$ therefore $x = y$.

Triangle Inequality: show that $d_{\ell^2}(x, z) \leq d_{\ell^2}(x, y) + d_{\ell^2}(y, z)$

Excercise 1.1.5 proves the triangulate inequality replacing $a_i = x_i$ and $b_i = y_i$.

Exercise 1.1.13 Prove Proposition 1.1.19.

Proposition 1.1.19 (Convergence in a the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^{\infty}$ convergent to x with respect to the discrete metric d_{disc} if and only if there exists $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.*

Remember that:

$$d_{\text{disc}}(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

($\Longrightarrow$) assume that $x^{(n)} \to x$ under d_{disc}. Then, for any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $x^{(n)} - x < \epsilon$. Clearly, $x^{(n)} - x$ can be equal to either 1 or 0. Thus, $x^{(n)} - x = 0$ or $x^{(n)} = x$ and hence true for all $n > N$.

($\Longleftarrow$) assume that $\exists N > m$ such that when $n > N, x^{(n)} = x$. Given any $\epsilon > 0$ and $n > N$ we can see that $x^{(n)} - x = 0 < \epsilon$. Therefore $x^{(n)} \to x$.

V. For this problem only, you do not need to give proofs. Just write the answers.

For each set, identify the boundary, interior, and closure of A, and say whether A is open , closed, both or neither. We are working in $\mathbb{R}^2$ with standard distance. Unless othewise noted, the ambient space is $\mathbb{R}^2$.

(a) $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$.
Boundary: $\partial A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$
Interior: $A^o = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1\}$
Closure: $\overline{A} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1\}$
A is open.

(b) $A = \{(1/n, 2/n) : n = 1, 2, 3, \dots\}$ (Note: $(1/n, 2/n)$ is a vector in $\mathbb{R}^2$, not an open interval in $\mathbb{R}$.)
Boundary: $\partial A = A$
Interior: $A^o = \emptyset$
Closure: $\overline{A} = A$
A is closed.

(c) $A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) \leq 1\}$, in the relative topology with respect to $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$.
Boundary: $\partial_Y A = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, d(x, 0) = 1\}$ the right semi-circle combined with the y-axis from 1 to -1.
Interior: $A^o = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, d(x, 0) < 1\}$
Closure: $\overline{A} = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, d(x, 0) \leq 1\}$
A is closed relative to Y.

VI. Let (X, d) be a metric space.

(a) For a given point $x_0 \in X$, show the singleton set $\{x_0\}$ is closed.
let $E = \{x_0\}^c = X \backslash \{x_0\}$. Given any $x \in E$ and $0 < \epsilon < |x_0 - x|$ we can easily see that there exists a ball $B = B_d(x, \epsilon)$ such that $B \cap \{x_0\} = \emptyset$. Further, $\partial E = \{x_0\}$ and $\{x_0\} \notin E$ thus E is open. This implies that $E^c = \{x_0\}$ is closed.

(b) Let $x_0 \in X$ and $r > 0$. Show that the ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open.
Let $E = B(x_0, r), x \in E$ and $0 < \epsilon_x < r - d(x_0, x)$. Then, for any $y \in B(x, \epsilon_x)$ we can see that $d(x_0, y) < d(x_0, x) + \epsilon_x < y$ therefore $y \in E$ and the open ball $B(x, \epsilon_x) \subset E$.

$$\bigcup_{x \in E} x = E$$
$$\text{then } B(x, \epsilon_x) \subset E, \forall x \in E$$
$$\bigcup_{x \in E} B(x, \epsilon_x) = E$$

thus $B(x_0, r)$ is the union of open balls.