Math 5102 – Linear Algebra– Fall 2024

w/Professor Penera

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Homework #7 – NONE

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Page 116: 4. Let T be the linear operator on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\left(\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right)^{-1} = \left(\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right)$$

to find $[T]_{\beta'}$

Page 116: 11 Let V be a finite-dimensional vector space with ordered bases α, β and γ.

(a) Prove that if Q and R are the changed of coordinate matrices that change α-coordinates in β-coordinates and β-coordiantes into γ-coordinates, respectiely, then RQ is the change of coordinate matrix that changes α-coordinates to γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

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Page 124: 3. For each of the following vector spaces V and bases β, find explicit formulas for vectors of the dual basis β^* for V^*, as in Example 4.

(a) $V = \mathbb{R}^3; \beta = \{(1,0,1), (1,2,1), (0,0,1)\}$

$$\delta_{ij} = f_i(\beta_j)$$

$$\begin{array}{cccc}
i\backslash j & 1 & 2 & 3 \\
1 & 1 = f_1(1,0,1) & 0 = f_1(1,2,1) & 0 = f_1(0,0,1) \\
 & 1 = a + c & 0 = a + 2b + 1 & 0 = c \\
 & f_1(x,y,z) = x - \frac{1}{2}y & & \\
 & & & \\
2 & 0 = f_2(1,0,1) & 1 = f_2(1,2,1) & 0 = f_2(0,0,1) \\
 & d = -f & 1 = d + 2e + f & f = 0 \\
 & f_2(x,y,z) = \frac{1}{2}y & & \\
 & & & \\
3 & 0 = f_3(1,0,1) & 0 = f_3(1,2,1) & 1 = f_3(0,0,1) \\
 & k = -m & 0 = k + 2l + m & 1 = m \\
 & f_3(x,y,z) = -x + z & &
\end{array}$$

(b) $V = P_2(\mathbb{R}); \beta = \{1, x, x^2\}$

$$f_i(\beta_j) = \delta_{ij}$$

$$\begin{array}{cccc}
i\backslash j & 1 & x & x^2 \\
1 & 1 = f_1(1,0,0) & 0 = f_1(0,1,0) & 0 = f_1(0,0,1) \\
 & f_1(x,y,z) = x & & \\
2 & 0 = f_2(1,0,0) & 1 = f_2(0,1,0) & 0 = f_2(0,0,1) \\
 & f_2(x,y,z) = y & & \\
3 & 0 = f_3(1,0,0) & 0 = f_3(0,1,0) & 1 = f_3(0,0,1) \\
 & f_3(x,y,z) = z & &
\end{array}$$

Page 124: 6 Define $f \in (\mathbb{R}^2)^*$ by $f(x,y) = 2x + y$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (3x+2y, x)$.

(a) Compute $T^t(f)$.

$$T^t(f) = (f \circ T)(x,y) = f(T(x,y)) = f(3x+2y, x) = 2(3x+2y) + x = 7x + 4y$$

(b) Compute $[T^t]_{\beta^*}$, where β is the standard ordered basis for $\mathbb{R}^2$ and $\beta^* = \{f_1, f_2\}$ is the dual basis, by finding scalars a, b, c, and d such that $T^t(f_1) = af_1 + cf_2$ and $T^t(f_2) = bf_1 + df_2$.

$$\begin{aligned}
\beta^* &= \{x, y\} \\
T^t(x) &= ax + cy,\ T^t(y) = bx + dy \\
T^t(f(x,y)) &= T^(2x + y) \\
&= T^(2x) + T^(y) \\
&= af_1(T(2,0))
\end{aligned}$$

(c) Compute $[T]_\beta$ and $(\,[T]_\beta\,)^t$, and compare your results with (b).

$$\begin{aligned}
T(1,0) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } T(0,1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\
[T]_\beta &= \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \\
(\,[T]_\beta\,)^t &= \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}
\end{aligned}$$

Page 124: 7 Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ. Define $T : V \to W$ by

$$T(p9x)) =)p(0) - 2p(1), p(0) + p'(0)),$$

where $p'(x)$ is the derivative of $p(x)$.

1. For $f] \in W^*$ defined by $f(a,b) = a - 2b$, comptuer $T^t(f)$.
2. Compute $[T^t]_{\gamma^*}^{\beta^*}$ without appelaing to Theorem 2.25.
3. Compute $[T]_\beta^\gamma$ and its transpose, and compare your results with (b).

Page 124: 11 let V and W be infinite-dimensional vector spaces over F, and let ψ_1 and ψ_2 be the isomorphisms between V and V^{**} and W and W^{**}, respectively, as defined in Theorem 2.26. Let $T : V \to W$ be linear, and defien $T^{tt} = (T^t)^t$. Prove that the diagram depicted in Figure 2.6 commutes (i.e, prove that $\psi_2 T = T^{tt}\psi_1$).

$$\begin{array}{ccc}
V & \xrightarrow{T} & W \\
\psi_1 \bigcap \downarrow & & \downarrow \psi_2 \\
V^{**} & \xrightarrow{T^{tt}} & W^{**}
\end{array}$$