

Math 5102 – Linear Algebra– Fall 2024
w/Professor Pendera

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Homework #7 – NONE

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Page 116: 4. Let T be the linear operator on \mathbb{R}^2 defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix}$$

let β be the standard ordered basis for \mathbb{R}^2 , and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[T]_{\beta'}$

Page 116: 11 Let V be a finite-dimensional vector space with ordered bases α, β and γ .

- (a) Prove that if Q and R are the changed of coordinate matrices that change α -coordinates in β -coordinates and β -coordinates into γ -coordinates, respectively, then RQ is the change of coordinate matrix that changes α -coordinates to γ -coordinates.
- (b) Prove that if Q changes α -coordinates into β -coordinates, then Q^{-1} changes β -coordinates into α -coordinates.

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Page 124: 3. For each of the following vector spaces V and bases β , find explicit formulas for vectors of the dual basis β^* for V^* , as in Example 4.

- (a) $V = \mathbb{R}^3; \beta = \{(1, 0, 1), (1, 2, 1), (0, 0, 1)\}$

$$\begin{array}{l} \delta_{ij} = f_i(\beta_j) \\ \begin{array}{c} i \backslash j \\ 1 \end{array} \quad \begin{array}{c} 1 \\ 1 = f_1(1, 0, 1) \\ 1 = a + c \\ f_1(x, y, z) = x - \frac{1}{2}y \end{array} \quad \begin{array}{c} 2 \\ 0 = f_1(1, 2, 1) \\ 0 = a + 2b + 1 \\ \end{array} \quad \begin{array}{c} 3 \\ 0 = f_1(0, 0, 1) \\ 0 = c \end{array} \\ \\ \begin{array}{c} 2 \\ 0 = f_2(1, 0, 1) \\ d = -f \\ f_2(x, y, z) = \frac{1}{2}y \end{array} \quad \begin{array}{c} 1 = f_2(1, 2, 1) \\ 1 = d + 2e + f \\ \end{array} \quad \begin{array}{c} 0 = f_2(0, 0, 1) \\ f = 0 \end{array} \\ \\ \begin{array}{c} 3 \\ 0 = f_3(1, 0, 1) \\ k = -m \\ f_3(x, y, z) = -x + z \end{array} \quad \begin{array}{c} 0 = f_3(1, 2, 1) \\ 0 = k + 2l + m \\ \end{array} \quad \begin{array}{c} 1 = f_3(0, 0, 1) \\ 1 = m \end{array} \end{array}$$

- (b) $V = P_2(\mathbb{R}); \beta = \{1, x, x^2\}$

$$\begin{array}{l} f_i(\beta_j) = \delta_{ij} \\ \begin{array}{c} i \backslash j \\ 1 \end{array} \quad \begin{array}{c} 1 \\ 1 = f_1(1, 0, 0) \\ f_1(x, y, z) = x \end{array} \quad \begin{array}{c} x \\ 0 = f_1(0, 1, 0) \end{array} \quad \begin{array}{c} x^2 \\ 0 = f_1(0, 0, 1) \end{array} \\ \\ \begin{array}{c} 2 \\ 0 = f_2(1, 0, 0) \\ f_2(x, y, z) = y \end{array} \quad \begin{array}{c} 1 = f_2(0, 1, 0) \end{array} \quad \begin{array}{c} 0 = f_2(0, 0, 1) \end{array} \\ \\ \begin{array}{c} 3 \\ 0 = f_3(1, 0, 0) \\ f_3(x, y, z) = z \end{array} \quad \begin{array}{c} 0 = f_3(0, 1, 0) \end{array} \quad \begin{array}{c} 1 = f_3(0, 0, 1) \end{array} \end{array}$$

Page 124: 6 Define $f \in (\mathbb{R}^2)^*$ by $f(x, y) = 2x + y$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (3x + 2y, x)$.

(a) Compute $T^t(f)$.

$$T^t(f) = (f \circ T)(x, y) = f(T(x, y)) = f(3x + 2y, x) = 2(3x + 2y) + x = 7x + 4y$$

(b) Compute $[T^t]_{\beta^*}$, where β is the standard ordered basis for \mathbb{R}^2 and $\beta^* = \{f_1, f_2\}$ is the dual basis, by finding scalars a, b, c , and d such that $T^t(f_1) = af_1 + cf_2$ and $T^t(f_2) = bf_1 + df_2$.

$$\begin{aligned}\beta^* &= \{x, y\} \\ T^t(x) &= ax + cy, \quad T^t(y) = bx + dy \\ T^t(f(x, y)) &= T^t(2x + y) \\ &= T^t(2x) + T^t(y) \\ &= af_1(T(2, 0))\end{aligned}$$

(c) Compute $[T]_{\beta}$ and $([T]_{\beta})^t$, and compare your results with (b).

$$\begin{aligned}T(1, 0) &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } T(0, 1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ [T]_{\beta} &= \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \\ ([T]_{\beta})^t &= \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}\end{aligned}$$

Page 124: 7 Let $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ with respective standard ordered bases β and γ . Define $T : V \rightarrow W$ by

$$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0)),$$

where $p'(x)$ is the derivative of $p(x)$.

1. For $f \in W^*$ defined by $f(a, b) = a - 2b$, compute $T^t(f)$.
2. Compute $[T^t]_{\gamma^*}^{\beta^*}$ without appealing to Theorem 2.25.
3. Compute $[T]_{\beta}^{\gamma}$ and its transpose, and compare your results with (b).

Page 124: 11 let V and W be infinite-dimensional vector spaces over F , and let ψ_1 and ψ_2 be the isomorphisms between V and V^{**} and W and W^{**} , respectively, as defined in Theorem 2.26. Let $T : V \rightarrow W$ be linear, and define $T^{tt} = (T^t)^t$. Prove that the diagram depicted in Figure 2.6 commutes (i.e, prove that $\psi_2 T = T^{tt} \psi_1$).

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \psi_1 \bigcap \downarrow & & \downarrow \psi_2 \\ V^{**} & \xrightarrow{T^{tt}} & W^{**} \end{array}$$