Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Notes – January – May, 2025

Definitions

1. ***Diffeomorphism***: If $f \in C^\infty$ and $f^{-1} \in C^\infty$ then f is said to be a ***diffeomorphism***. Similarly, if there exists a mapping between two sets that is a diffeomorphism, the sets are said to be ***diffeomorphic*** to each other.
2. ***Tangent Space*** at a point p. The set of all vectors rooted at p, written as $T_p(\mathbb{R}^n)$.
   Let $p = (x^1, \dots, x^n)$. The directional deriviative for each component would be described as
   $$\left.\frac{\partial}{\partial x^1}\right|_p$$
   $$\text{notice } \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \delta^i_j, \forall p.$$
   that is perpendicular and form a orthogonal basis. Thus, a Tangent Vector is also called a "Derivation".
3. ***Derivations***: any operation that supports the Liebniz Rule $D(fg) = (Df)g + fDg$.
4. ***Derivation Space***. $\mathcal{D}_p(\mathbb{R}^n)$ is the set of all derivations at p. This constitutes a vector space. There exists an isomorphism $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ defined as
   $$\begin{aligned} \phi &: T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p. \end{aligned}$$
5. ***Germ***: equivalence class of functions whose derivatives around a point are the same.
6. Vector Field vs Vector Space.
   - ***A Vector Field*** a function that assigns a vector to every point in the subset U.
     $$\begin{aligned} f &: (U \subset \mathbb{R}^m) \to T_p(\mathbb{R}^n) \\ X &\mapsto X_p = \sum a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p. \end{aligned}$$
     consider a^i as coefficent functions. We say that X is C^∞ on U if $a^i \in C^\infty$, $\forall i = 1, \dots, n$.
   - ***A Vector Space*** is any abstraciton that is closed under addition and scalar multiplication.
7. ***Dual Basis and Dual Space***. The ***Dual Basis*** is a set of functions $\alpha^i : V \to \mathbb{R}$
   $$\begin{aligned} \alpha^i &: V \to \mathbb{R} \\ \alpha^i(e_j) &= \delta^i_j \end{aligned}$$
   the ***Dual Space*** $V^\vee$ is the space of functions spanned by the Dual Basis. Elements of the Dual Space are called ***Functionals (Analysis)/1-Covectors (Differential Geometry)***.
8. ***Multi-Linear Functions and Vector Space of k-tensors*** $L_k(V)$ Let V be a vector space and V^k be k-tuples of vectors in V. A K***-linear map or*** k***-tensor*** $f : V^k \to \mathbb{R}$ such that each i^{th} component is linear. The vector space of all k-tensors on V is denoted $L_k(V)$.
   Permuting Mult-linear Functions. Given any permutation $\sigma \in S_k$
   $$f(v_1, \dots, v_k) = f(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$
   e.g., $f(x, y, z) = xyz \to f(z, x, y) = zxy$. FYI: if x, y, z are from non-commutative rings (i.e., matrices) then we must be aware of the $\text{sgn}(\sigma)$.
9. Left R-***Module***: An Abelian group R with a scalar multiplication map:
   $$\mu : R \times A \to A$$
   usually written as $\mu(r, a)$, such that $r, s \in \mathbb{R}$ and $a, b \in A$ a
   (i) (associative) $(rs)a = r(sa)$.

(ii) (identity) $1a = a$ (1 is a multiplicative identity).

(iii) (distributivity) $(r+s)a = ra + sa$ and $r(a+b) = ra + rb$.

If R is a field then R-module is precisely a vector space over R.

A ***K-Algebra over a field*** K is also a ring A that is also a vector space over K such that the ring multiplication satisfies homogeniety (scalar distributes over vector multipliation to only one of the operators).

A ***graded Algebra*** is an algebra A over a field K if it can be writte as the direct sum

$$A = \bigoplus_{i=0}^{\infty} A^i$$

of vector spaces over K such that the mupltiplication map sends $A^k \times A^l \to A^{k+l}$

10. The set of all C^∞-vector fields on U, denoted by $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the $C^\infty(U)$ ring.

$$\mathfrak{X}(U) = \{\, X : V \to V \mid X \in C^\infty(U) \,\} \text{ where } V = (\mathbb{R} \text{ or } \mathbb{C})^n$$

11. ***Derivation:*** A ***derivation*** on an algebra A is a K-multilinear function $D : A \to A$ such that

$$D(ab) = (Da)b + aDb,\ \forall a, b \in A$$

known as the ***Liebniz Rule***.

The set of all derivations on A forms a vector space, $\mathrm{Der}(C^\infty(U))$. Thus a $C^\infty(U)$ vector field gives rise to a derivation of the algebra $C^\infty(U)$. Thus the mapping

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \mathrm{Der}(C^\infty(U)) \\ X &\mapsto (f \mapsto Xf) \end{aligned}$$

this map is an isomorphism of vector spaces.

12. **Exterior Algebras $\Lambda(V)$.** The exterior algebra $\Lambda(V)$ is obtained by imposing an **anti-commutative** relation:

$$v \otimes w + w \otimes v = 0,\ \forall v, w \in V$$

this means that the quotient algebra is:

$$\Lambda(V) = T(V)/\left\langle\, v \otimes w + w \otimes v \,\right\rangle.$$

Where $T(V)$ is the **tensor algebra**

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

13. **Symmetric Algebras $S(V)$.** The symmetric algebra $S(V)$ is obtained by imposing an **commutative** relation:

$$v \otimes w - w \otimes v = 0,\ \forall v, w \in V$$

this means that the quotient algebra is:

$$S(V) = T(V)/\left\langle\, v \otimes w - w \otimes v \,\right\rangle.$$

14. ***Tensor Product*** The tensor product between two 1-covectors, $f, g : V \to \mathbb{R}$ is the 2-covector $f \otimes g$.

$$(f \otimes g)(u, v) = f(u)g(v)$$

. In general, the tensor product of a k-covector $p : V^k \to \mathbb{R}$ with a l-covector $q : v^l \to \mathbb{R}$ is the $(k+l)$-covector $p \otimes q : V^{k+l} \to \mathbb{R}$.

$$(p \otimes q)(u, v) = p(u)q(v),\ \forall u \in V^k,\ v \in V^l$$

15. ***Tensor Product(?)*** is an operator on $v \in V$ and $u \in U$ where

$$\begin{gathered} v \otimes u : V \times U \to V \oplus U \\ (v \otimes u)_{i \cdot j} = v_i \cdot u_j,\ \forall i = 1, \ldots, \dim(V),\ j = 1, \ldots, \dim(U) \end{gathered}$$

Given two vector spaces V, W with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ then the Tensor Product space $V \otimes W$ has a basis referred to as $v_i \otimes w_j$ such that given any vector $\alpha = \sum \alpha_i v_i \in V$ and $\beta = \sum \beta_j w_j \in W$ the vector $\alpha \otimes \beta$ will have $n \times m$ components and each $(\alpha \otimes \beta)_{i \times j} = \alpha_i \times \beta_j$.

α_i, β_j are all scalars. The real issue is the behavior of unit basis vectors v_i, w_j and how they are effected by the operator and the basis vectors $v_i \otimes w_j$. Thus, scalar multiplication works on either (but not both) operands and distribution over addition works over both the left and the right.

16. ***Wedge Product***

Between two covectors Let $f, g \in L_1(V)$ then for all $u, v \in V$

$$(f \wedge g)(u, v) = (f \otimes g)(u, v) - (g \otimes f)(u, v) = f(u)g(v) - f(v)g(u)$$

Between mulitple 1-covectors.

$$\begin{aligned}(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \ldots, v_k) &= \det[\alpha^1(v_j)] \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\alpha_1(v_{\sigma(1)}) \cdots \alpha_k(v_{\sigma(k)})\end{aligned}$$

Between k-covector and l-covector. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g) \in A_{k+l}(V)$$

or explicitly

$$(f \wedge g)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} f\left(v_{\sigma 1}, \ldots, v_{\sigma_k}\right) g\left(v_{\sigma_{k+1}}, \ldots, v_{\sigma k+l}\right)$$

Anticommutative. Let $f \in A_k(V)$, $g \in A_l(V)$ then

$$(f \wedge g) = (-1)^{kl} g \wedge f$$

17. **Differential k-Forms**

1-forms, covectors

$$\begin{aligned}(dx^i)\left(\frac{\partial}{\partial x^j}\bigg|_p\right) &= \frac{\partial}{\partial x^j}\bigg|_p x^i = \delta^i_j \\ (df)_p(X_p) = X_p f &= \sum a^i(p) \frac{\partial f}{\partial x^i}\bigg|_p = \sum \frac{\partial}{\partial x^i} dx^i\end{aligned}$$

18. $\Omega^k(U)$**, Vector space of C^∞ k-forms on U**.

$\Omega^0 = A_0(T_p(\mathbb{R}^n)) = C^\infty(U)$, e.g., $f \in \Omega^0$ then $f : V \to \mathbb{R}$ is a functional/covector/1-tensor.
Elements of 1-form $\Omega^1 = A_1(T_p(\mathbb{R}^n))$. For example, when $n = 3$

$$f dx + g dy + h dz, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

Elements of 2-form $\Omega^2 = A_2(T_p(\mathbb{R}^n))$. For example, when $n = 3$[1]

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy, \text{ where } f, g, h \in C^\infty(\mathbb{R}^3)$$

if $n = 4$, that is coordinates for u, v, w, x. Each form is derived from these bases

0-form $\Omega^0(\mathbb{R}^4) \in \mathbb{R}$
1-forms $\Omega^1(\mathbb{R}^4)$ summing du, dv, dw, dx,
2-forms $\Omega^2(\mathbb{R}^4)$ summing $du \wedge dv$, $du \wedge dw$, $du \wedge dx$, $dv \wedge dw$, $dv \wedge dx$, $dw \wedge dx$,
3-forms $\Omega^3(\mathbb{R}^4)$ summing $du \wedge dv \wedge dw \mid du \wedge dw \wedge dx \mid du \wedge dv \wedge dx \mid dv \wedge dw \wedge dx$
4-form $\Omega^4(\mathbb{R}^4)$ $du \wedge dv \wedge dw \wedge dx$.

Also, $U \subseteq \mathbb{R}^n$ **then** $k < n$. k-forms for $k > n$ are zero. Further $|\Omega^k(\mathbb{R}^n)| = \binom{k}{n}$ and $\left|\bigcup_k \Omega^k(\mathbb{R}^n)\right| = 2^n$ and think of $\Omega^*(U) = \bigcup_k \Omega^k(\mathbb{R}^n)$

Direct Sum. $\Omega^*(U) = \bigoplus_k \Omega^k(U)$ is an anti-commutative graded algebra over $\mathbb{R}$.
Since one can multiply C^∞ k-forms by C^∞ functions, the set $\Omega^k(U)$ of C^∞ k-forms is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$ and $\Omega^*(U)$ is also a module over C^∞ of C^∞ functions.

19. **Wedge Product of k-form.** Recall: $dx^i \wedge dx^i = 0$ for all $i = 1, \ldots, n$. Therefore, $\wedge$ only makes sense to be defined on *disjoint indice-lists*, that is, $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$ such that $I \cap J = \emptyset$. Then,

$$\begin{aligned}\wedge : \Omega^k(U) \times \Omega^l(U) &\to \Omega^{k+l}(U) \\ (\omega, \tau) &\mapsto (\omega \wedge \tau) = \sum_{I,J} a_I b_J dx^I \wedge dx^J.\end{aligned}$$

where $\omega = \sum_I a_I dx^I, \tau = \sum_J b_J, dx^J$.

[1]NOTE the cyclic order of the indices x, y, z. Switching any one of these will flip the sign.

20. **the Exterior Derivative**. If $k \geq 1$ and if $\omega = \sum_I a_i dx^I \in \Omega^k(U)$, then $d\omega \in \Omega^{k+1}(U)$ and

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_J \frac{\partial a_I}{\partial x_J} dx^J \right) \wedge dx^I$$

Example: Let $\omega \in \Omega^1(\mathbb{R}^2)$ and $\omega = f dx + g dy$, $f, g \in C^\infty(\mathbb{R}^2)$.

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \right) \wedge dy \\ &= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\ &= (g_x - f_y)\, dx \wedge dy \end{aligned}$$

Definition: Let $\bigoplus_{k=0}^{\infty} A^k$ be a graded algebra over a field K. An ***anti-derivation*** of the graded algebra A is a K-linear map $D : A \to A$ such that $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a Db$$

Proposition 4.7: Three Criterion for an Exterior Derivation

i) The ***exterior derivation*** $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

ii) $d^2 = 0$.

iii) If $f \in \mathbb{C}^\infty$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

NOTE:"In a typical school, there would be graduate level courses on Smooth Manifods and another on Remannian Manifolds."

Q: What is the difference between $\mathfrak{X}(U)$ and $C^\infty(U)$?

The difference between $\mathfrak{X}(U)$ and $C^\infty(U)$ lies in the types of objects they contain:
1. **$C^\infty(U)$: The Space of Smooth Functions** - $C^\infty(U)$ consists of all smooth (infinitely differentiable) real-valued functions defined on an open subset U of a manifold M. - Elements of $C^\infty(U)$ are scalar functions $f : U \to \mathbb{R}$. - These functions can be added and multiplied pointwise, forming an algebra over $\mathbb{R}$.

2. **$\mathfrak{X}(U)$: The Space of Smooth Vector Fields** - $\mathfrak{X}(U)$ consists of all smooth vector fields on U. - A vector field X assigns to each point $p \in U$ a tangent vector $X_p \in T_pM$, smoothly varying with p. - Vector fields act as derivations on smooth functions, meaning they satisfy the Leibniz rule:

$$X(fg) = X(f)g + fX(g), \quad \forall f, g \in C^\infty(U).$$

- The space $\mathfrak{X}(U)$ forms a module over $C^\infty(U)$, meaning smooth functions can scale vector fields: if $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then fX is also a vector field.

Key Differences

Feature	$C^\infty(U)$	$\mathfrak{X}(U)$
Elements	Smooth scalar functions $f : U \to \mathbb{R}$	Smooth vector fields $X : U \to TM$
Algebraic Structure	Commutative algebra (pointwise multiplication)	Module over $C^\infty(U)$, noncommutative under Lie bracket
Operations	Addition, multiplication	Addition, scalar multiplication by $C^\infty(U)$, Lie bracket $[X, Y]$

In summary, $C^\infty(U)$ consists of smooth functions, while $\mathfrak{X}(U)$ consists of smooth vector fields, which act as differential operators on $C^\infty(U)$.

Compare and contrast.

Set	Dim	index	basis	Delta
$L_1(U)$	n	$i = 1, \dots, n$	α^i	$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
$L_k(U)$	n^k	$I, J \in \{\underbrace{i_i, \dots, i_k}_{k \text{ times}}\}$, $i_k \in [1, \dots, n]$	$\alpha^I = \alpha^{i_1} \otimes \alpha^{i_2} \otimes \cdots \otimes \alpha^k$	
$A_k(U)$	$\binom{n}{k}$	$I, J \in \{\underbrace{i_i, \dots, i_k}_{k \text{ times}}\}$, $i_1 < i_2 < \dots i_k \in [1, n]$	$\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^k$	$\delta_I^J = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$

Supersets

Symbol	Name (set of)	Definition	Example
$\Omega^0(U)$	0-forms	{ scalar fields }	$f : V \to \mathbb{R}$ $f(x, y, z)$
$\Omega^1(U)$	1-forms	{ 1-forms, vector fields }	$d\omega(v) = A(v)dx + B(v)dy + C(v)dx$ $A, B, C : V \to \mathbb{R}$
$\Omega^k(U)$	k-forms	{ k-forms }	$\cdots + dx^1 \wedge \cdots \wedge dx^k + \cdots$
$\Omega^*(U)$	sum of k-forms	$\{ x = \sum y \mid y \in \oplus_k \Omega^k(U) \}$	$Adx + Bdx \wedge dy + Cdx \wedge dy \wedge dz,\ A, B, C : V \to \mathbb{R}$
$\mathfrak{X}(U)$	vector fields on U	$\{X \to \exists f : U \to U\}$	
$C^\infty(U)$	smooth functions on U		
$X_p = T_p(U)$	a vector field at p	$\{v \in U \mid v = p + x$ for some $x \in U\}$	

Map of $\Omega^k(\mathbb{R}^3)$

$$\begin{tikzcd}
\Omega^0(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^1(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^2(U) \arrow[r, "\mathrm{d}"] \arrow[d, "\cong"'] & \Omega^3(U) \arrow[d, "\cong"'] \\
C^\infty(U) \arrow[r, "\mathrm{grad}"'] & \mathfrak{X}(U) \arrow[r, "\mathrm{curl}"'] & \mathfrak{X}(U) \arrow[r, "\mathrm{div}"'] & C^\infty(U).
\end{tikzcd}$$

Shorthand

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_{i,j} a_i b_j = \sum_i a_i \sum_j b_j$$

$$\sum_I a_I = \sum_{n=1}^k a_{i_n}$$

$$\sum_{I,J} a_I b_J = \sum_{n=1}^k a_{i_n} \sum_{m=1}^k b_{j_m}$$

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\delta_I^J = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \cdots \delta_{i_k}^{j_k} = \begin{cases} 1 & i_n = j_n, \forall n \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Definition 0.0.1 (Exact and Closed k-forms)**.** A k-form ω on U is ***closed*** if $d\omega = 0$; it is ***exact*** if there is a $(k-1)$-form τ such that $\omega = d\tau$ on U. Since $d(d\tau) = 0$, every exact form is closed.

Definition 0.0.2 (de Rham Cohomology)**.** .

The ***k^{th}-cohomology of*** U is defined as the quotient vector space

$$H^k(U) = \frac{\{\text{closed k-forms}\}}{\{\text{exact k-forms}\}}$$

That is, each element is a vector space forming an equivalence class of k-forms.

Examples of de Rham Cohomology

De Rham cohomology provides a way to study the topology of smooth manifolds using differential forms. Below are some key examples illustrating how to compute and interpret de Rham cohomology groups.

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Example 1: Euclidean Space $\mathbb{R}^n$

For $M = \mathbb{R}^n$, we claim that the de Rham cohomology is:

$$H^k_{\mathrm{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(\mathbb{R}^n)$:**
   - The 0-forms are just smooth functions f.
   - A function is closed if $df = 0$, meaning f is constant.
   - Every constant function is not only closed but also exact since $f = d(fx)$.
   - The space of closed 0-forms is $\mathbb{R}$ (constant functions), and there are no exact forms to quotient out.
   - So, $H^0_{\mathrm{dR}}(\mathbb{R}^n) = \mathbb{R}$.
2. **$H^k_{\mathrm{dR}}(\mathbb{R}^n)$ for $k > 0$:**
   - Any closed k-form ω is locally exact due to **Poincaré's lemma**.
   - That is, every closed form is of the form $\omega = d\eta$, meaning it contributes nothing to cohomology.
   - Thus, $H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ for $k > 0$.
     This result reflects the fact that $\mathbb{R}^n$ is **contractible**, so it has trivial topology.

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Example 2: The Circle S^1

For $M = S^1$, we find:

$$H^k_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(S^1) = \mathbb{R}$**
   - Smooth functions f that satisfy $df = 0$ are constant.
   - Thus, $H^0_{\mathrm{dR}}(S^1) = \mathbb{R}$.
2. **$H^1_{\mathrm{dR}}(S^1) = \mathbb{R}$**
   - Consider the 1-form $\omega = d\theta$, where θ is the angular coordinate.
   - $d\omega = 0$, so ω is closed.
   - Is ω exact? If $\omega = d\eta$ for some η, then $d\eta = d\theta$, but no globally defined function η exists on S^1 satisfying this.
   - So ω represents a **nontrivial cohomology class**, giving $H^1_{\mathrm{dR}}(S^1) = \mathbb{R}$.
3. **$H^k_{\mathrm{dR}}(S^1) = 0$ for $k \geq 2$**
   - There are no nontrivial 2-forms on a 1-dimensional manifold.
   Interpretation

- The nontrivial $H^1_{\mathrm{dR}}(S^1)$ reflects the existence of a **loop** in S^1.
- This cohomology detects the ability to define a **non-exact closed form**, related to the winding number.

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****Example 3: The 2-Sphere S^2** For $M = S^2$:**

$$H^k_{\mathrm{dR}}(S^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ 0, & k = 1. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(S^2) = \mathbb{R}$**
   - As always, closed 0-forms are constant functions, so $H^0_{\mathrm{dR}}(S^2) = \mathbb{R}$
2. **$H^1_{\mathrm{dR}}(S^2) = 0$**
   - Any closed 1-form is exact by a higher-dimensional **Poincaré lemma**, so $H^1_{\mathrm{dR}}(S^2) = 0$.
3. **$H^2_{\mathrm{dR}}(S^2) = \mathbb{R}$**
   - The standard volume form $\omega = \sin\theta \, d\theta \wedge d\phi$ is closed.
   - It is not exact, because there is no 1-form η such that $d\eta = \omega$ (this follows from **Stokes' theorem**).
   - So ω represents a generator of $H^2_{\mathrm{dR}}(S^2)$.

Interpretation

- $H^1_{\mathrm{dR}}(S^2) = 0$ reflects that there are no **nontrivial loops** (all loops contract).
- $H^2_{\mathrm{dR}}(S^2) = \mathbb{R}$ corresponds to the existence of a volume form, a global topological feature.

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****Example 4: The Torus $T^2 = S^1 \times S^1$****

For T^2, the de Rham cohomology groups are:

$$H^k_{\mathrm{dR}}(T^2) = \begin{cases} \mathbb{R}, & k = 0, 2, \\ \mathbb{R} \oplus \mathbb{R}, & k = 1, \\ 0, & k > 2. \end{cases}$$

Computation

1. **$H^0_{\mathrm{dR}}(T^2) = \mathbb{R}$** (constant functions).
2. **$H^1_{\mathrm{dR}}(T^2) = \mathbb{R} \oplus \mathbb{R}$**
   - The torus has two independent 1-forms: $d\theta_1$ and $d\theta_2$, corresponding to the two loops in T^2.
3. **$H^2_{\mathrm{dR}}(T^2) = \mathbb{R}$**
4. The volume form $d\theta_1 \wedge d\theta_2$ represents a nontrivial 2-class.

Interpretation

- The rank of $H^1_{\mathrm{dR}}(T^2)$ reflects the **two independent loops** in the torus.
- The nontrivial $H^2_{\mathrm{dR}}(T^2)$ corresponds to the existence of a **volume form**.

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Summary Table

Manifold M	$H^0_{dR}(M)$	$H^2_{dR}(M)$	$H^1_{dR}(M)$
$\mathbb{R}^n$	$\mathbb{R}$	0	0
S^1	$\mathbb{R}$	$\mathbb{R}$	0
S^2	$\mathbb{R}$	0	$\mathbb{R}$
T^2	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$

§8 Tangent Space.

Definition 8.1.3 (Tangent vector)**.** A ***tangent vector*** at a point p is a derivation at p.

Remark 8.1.4. In general, x^i, y^j are coordinates for manifolds N, M and r^i are coordinates associated with charts. Thus $p = (x^1, \ldots, x^n)$ and $\phi(p) = (r^1, \ldots, r^k)$ where $k \leq n$.

Definition 8.1.5 (Push-back and Push-forward)**.** .

Given the mapping between manifolds N, M as $\varphi : N \to M$ with charts $\phi : N \to \mathbb{R}$ and $f' : M \to \mathbb{R}$. We define the ***push-back*** $\varphi^* f'(p)$ as [2]

$$\begin{aligned} \varphi^* f'(p) &: N \to \mathbb{R} \\ \varphi^* f'(p) &= f' \circ \varphi(p) \end{aligned}$$

Note: this is a functional from the original manifold N to the reals. Let's think of it as a short-cut from $N \to \mathbb{R}$ through M via φ. Note that the push-back operates on a chart.

The ***push-forward*** is NOT the inverse of the push-back. φ^{-1} may be undefined. Further, the push-forward does NOT operate on a chart $\psi(\varphi(p))$. The push-foward operates on the push-back $f^*\psi$ and the tangent space $T_p(N)$. The vectors in the tangent space at p, i.e., $T_p(N)$.[3] We define the push-forward f_* as

$$\begin{aligned} f_*(p) &: T_p(N) \to T_{\varphi(p)}(M) \\ (f_* V)_{\varphi(p)}(\psi) &= V|_p(f^*\psi) \end{aligned}$$

[2] beware the f-prime, f', is NOT the derivative

[3] Here we observe the behavior of functions in N by the effect from their vector changes in the tangent space.