Math 725 – Advanced Linear Algebra
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Assignment #1 – Due 8/30/23

1. Show that the *positive* quadrant
$$Q = \{ (x, y) : x, y > 0\}$$
forms a vector space over $\mathbb{R}$ if we define addition by $(x_1, y_1) + (x_2, y_2) = (x_1x_2, y_1y_2)$ and scalar multiplication by $c(x, y) = (x^c, y^c)$.

Is it closed under addition? Given any two points $(x_1, y_1), (x_2, y_2)$ we have $(x_1, y_1)+(x_2, y_2) = (x_1x_2, y_1y_2)$ is clearly in Q (the only way that it could be in another quadrant would be if one of the elements x_1, x_2, y_1, y_2 is less than zero which isn't possible). Yes, it is closed under addition.

Is it closed under scalar multiplication? Given any $(x, y) \in Q$ and $c \in \mathbb{R}$ we can see $c(x, y) = (x^c, y^c)$. We know that $f(x) = c^x$ is positive definite for all $c > 0$, therefore it is closed under scalar multplication.

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2. Let E be a field and F be a subfield of E (this means that F is a field on its own right where the addition and multiplication operations of F are inherited from those of E). Explain the following: E is a vector space over F. Also, give an example with concrete fields E and F.

Looking closely at the Definition 1.19 on page 12 of the text, please note that the field F only applies to the scalars in scalar multiplciation. Thus, when we say "a set V is vector space over a field W", the scalars come from W. Thus, "E is a vector space over F" means to use any elements of E as vectors and limit the scalar values to elements of F.

An example is $\mathbb{Q} \subset \mathbb{R}$. That is, the set of real numbers is a subspace over the set of rational numbers. Given any $x, y \in \mathbb{R}$ and $q, r \in \mathbb{Q}$ we know that $qx + ry \in \mathbb{R}$.

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3. Let V and W be two vector spaces over the same field F. Explain how you can make the cartesian product $V \times W = \{(v, w) : v \in V,\ w \in W\}$ a vector space over F.

Define addition of $V \times W$ as follows. Let (v_1, w_1) and (v_2, w_2) members of $V \times W$ with $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Then $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$. Clearly, $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$ as they are both vector spaces over F. Therefore $V \times W$ is closed under addition.

Define scalar multiplication as $c(v, w) = (cv, cw)$ where $c \in F$. Clearly $cv \in V$ and $cw \in W$ because both of these are closed under scalar multiplication. Hence, $V \times W$ is closed under scalar multiplication.

Defined in this way, $V \times W$ is a vector space over F.

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4. Let $\mathbb{H}^n(\mathbb{C}) \subset M_{n\times n}(\mathbb{C})$ be the subset of *Hermitian* matrices: a square matrix A with complex coefficients is Hermitian if $A_{ij} = \overline{A_{ji}}$ for all $1 \leq i, j \leq n$ where $\overline{z}$ is the complex conjugate of z. Is $\mathbb{H}^n(\mathbb{C})$ a $\mathbb{C}$-subspace of $M_{n\times n}(\mathbb{C})$. Give a proof or a counterexample. Is it an $\mathbb{R}$-subspace of $M_{n\times n}(\mathbb{C})$?

First, No, not a $\mathbb{C}$-subspace. Quite simply.

$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{H}^n(\mathbb{C})$$
$$\text{then } (1+i)A = (1+i)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix} \notin \mathbb{H}^n(\mathbb{C})$$

However, as an $\mathbb{R}$-subspace we can see that complex conjugation is closed under addition. That is

$$\begin{aligned}\overline{(a+ib)+(c+id)} &= \overline{(a+c)+i(b+d)} \\ &= (a+c) - i(b+d) \\ &= (a-ib)+(c-id) \\ &= \overline{a+ib} + \overline{c+id}\end{aligned}$$

. Therefore, given any $A, B \in \mathbb{H}^n(\mathbb{C})$ then $\overline{(A+B)_{ij}} = \overline{A_{ij}+B_{ij}} = \overline{A_{ij}} + \overline{B_{ij}} = A_{ji} + B_{ji} = (A+B)_{ji}$ hence $A + B \in \mathbb{H}^n(\mathbb{C})$

When we limit $c \in \mathbb{R}$, we can see that complex conjugation is closed under scalar multiplication by real values. That is $\overline{c(a+ib)} = \overline{ca+icb} = ca - icb = c\overline{(a-ib)}$. Consequently, given any $A \in \mathbb{H}^n(\mathbb{C}), \overline{cA_{ij}} = c\overline{A_{ij}} = cA_{ji}$ hence $cA \in \mathbb{H}^n(\mathbb{C})$ and therefore closed.

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5. Let F be a field. Show that in F^2 there are only three types of subspaces: $\{0\}$, line generated by a nonzero vector $v \in F^2$, and F^2.

Let V be a set of vectors in $\mathbb{R}^2$ that forms a vector space and such that $V \neq \{0\}$ and $V \neq \mathbb{R}^2$. WLOC, let $x, y \in V$ and that they are linearly independent, i.e., not colinear. Thus, when $\theta x + \phi y = 0$ then $\theta = \phi = 0$ for scalars θ and ϕ. However, given any $r \in \mathbb{R}^2$ we can find scalars θ, ϕ such that $\theta x + \phi y = r$. That is, any $r \in \mathbb{R}^2$ can be represented by a linear combination of x and y. Hence, $\text{span}\{x, y\} = \mathbb{R}^2$ and since $V \subseteq \mathbb{R}^2$ then $V = \mathbb{R}^2$ which is a contradiction. Therefore, x, y must be linearly dependent implying that alll points in V are linearly dependent and hence a line through the origin.

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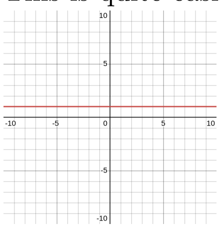
6. Let V be a vector space, and let W be a subspace of V. For a fixed vector $v \in V$, the set $v + W := \{v + w \,:\, w \in W\}$ is known as an *affine subspace* of V.

a) Under what condition(s) is an affine subspace of V a subspace of V ?

When the zero vector is a member of $v + W$ namely when $-v \in W$.

b) Draw the affine subspace of $\mathbb{R}^2$ when W is the x-axis and $v = (2, 1)$.

This is quite easily the horizontal line that goes through $(2, 1)$ or $y = 1$



c) Argue that the plane $x - 2y + 3z = 1$ is an affine subspace of $\mathbb{R}^3$.

We know that when $y = z = 0$ then $x = 1$ which is a point in the plane. Thus, given any point in this plane if we subtract the point $(1, 0, 0)$, i.e., let w be a solution to $x - 2y + 3z = 1$ subtracting $(1, 0, 0)$ from this solution will bring a point in the plane $-2y + 3z = 0$ passing through the origin which is a subspace of $\mathbb{R}^3$. Hence, $x - 2y + 3z = 1$ is an affine subspace of $\mathbb{R}^3$.

d) Show that any two affine subspaces of the form $v + W$ and $u + W$ are either equal or disjoint.

If $a \in u+W$ then there exists $x \in W$ such that $a = u+x$. Similarly, if $a \in v+W$ then there exists $y \in W$ such that $a = v+y$. Thus if $a \in (v+W) \cap (u+W)$ then $u+x = v+y$ and hence $u = v+y-x \in v+W$. Thus $u + W \subseteq v + W$ and visa versa. This also shows that if $(v + W) \cap (u + W) \neq \emptyset$ then they must be equal hence if $(v + W) \cap (u + W) = \emptyset$ they are by definition disjoint.

e) Let $v_1, \ldots, v_m$ be vectors in V. An *affine linear combination* of these vectors is a linear combination of them where the coefficients of the linear combination add up to 1: $c_1v_1 + \ldots + c_mv_m$ where $c_1 + \ldots + c_m = 1$. Let affine$\{v_1, \ldots, v_m\}$ be the set of all affine linear combinations of $v_1, \ldots, v_m$. Show that affine$\{v_1, \ldots, v_m\}$ is an affine subspace of V.

Let n be the highest number of linearly dependent vectors from $\{v_1, \ldots, v_m\}$ and reorder this list so that these vectors are listed first. We know that $c_1 + \cdots + c_m = 1$. We can make a new list $d_1, \ldots, d_{m-1}$ with one less element such that $d_i = c_i + \frac{c_m}{m-1}$ whose sum is still 1. Now we have a linear combination of $m - 1$ elements. We repeat this process, generating a new set of elements with each iteration with one less element until we have n elements. Let's call these elements $e_1, \ldots, e_n$. Let $W = \text{span}\{v_{n+1}, \ldots, v_m\}$ which we know is a vector space as they are linearly independent. Let $u = e_1v_1 + \cdots + e_nv_n$, thus affine$\{v_1, \ldots, v_m\} = u + W$.