Math 5110 – Real Analysis I– Fall 2024
w/Professor Liu

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Homework #3 – TBD: October 31, 2024

I. Let $\Omega \subset \mathbb{R}^m, a \in \Omega^o$. If $f : \Omega \to \mathbb{R}$ is continuous at $a, g : \Omega \to \mathbb{R}$ is differentiable at a and $g(a) = 0$, show that fg is differentiable at a. (Note fg is the function whose value at $x \in \Omega$ is $f(x)g(x)$.

II. skip II

III. Find the total derivative (i.e., derivative matrices) of the following functions at the given points.

(a) $f(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ x_1x_3^2 \\ x_1 + x_2 + x_3 \end{pmatrix}$ at $(x_1, x_2, x_3) = (1, 0, 1)$.

(b) $f(x) = \binom{x^2}{e^x}$ at $x = 3$.

(c) $f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2x_4 + \sin(x_3x_4)$ at $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

IV. Section 6.2 Problem 2.

Exercise 6.2.2. Prove Lemma 6.2.4. (Hint: prove by contradiction. If $L_1 \neq L_2$, then there exists a vector v such that $L_1v \neq L_2v$; this vector must be non-zero (why?). Now apply the definition of derivative, and try to specialize to the case where $x = x_0 + tv$ for some scalar t, to obtain a contradiction.)

Lemma 6.2.4 (Uniqueness of derivatives). *Let E be subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be an interior point of E, and let $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Suppose that f is differentiable at x_0 with derivatives L_1, and also diffentiable at x_0 with derivative L_2. Then $L_1 = L_2$*

V. Section 6.3, problem 3 and problem 4.

Exercise 6.3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x, y) := \frac{x^3}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) := 0$. Show that f is not differentiable at $(0, 0)$, despite being differentiable in every direction $v \in \mathbb{R}^2$ at $(0, 0)$. Explain why this does not contradict Thoerem 6.3.8.

Exercise 6.3.4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function such that $f'(x) = 0$ for all $x \in \mathbb{R}^n$. Shwo that f is constant. (Hint: you may use the mean-value theorem or fundamental theorem of calculus for one-dimensional functions, but bear in mind that there is a direct analogue to these theorems for several-variable functions. I would not advise proeeding via first principles.) For a tougher challenge, replace the domain $\mathbb{R}^n$ by an open connected subet Ω of $\mathbb{R}^n$.

VI. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable, $\alpha \in \mathbb{R}$. If $f(tx) = t^\alpha f(x)$ for $\forall x \in \mathbb{R}^m$ and $t > 0$, we say that f is homogeneous of order α. Show that f is homogeneous of order α iff $x \cdot \nabla f(x) = \alpha f(x)$, that is

$$x^1\partial_1 f(x) + \cdots + x^m\partial_m f(x) = \alpha f(x).$$

This equation is classically written as

$$x^1\frac{\partial f}{\partial x^1} + \cdots + \frac{\partial f}{\partial x^m} = \alpha f(x).$$

Hint: As in the development of the theory in the text, a basic idea to study multivariable functions is to convert them into single-variable functions by restricting the variable x in a fixed direction. For example, for this problem you may consider the function $\varphi(t) = f(t)$.

VII. (a) Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1-map,

$$|f(x) - f(y)| \geq |x - y|, \forall x, y \in \mathbb{R}^m,$$

then $\forall a \in \mathbb{R}^m, \det f'(a) \neq 0$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable, and assume $f(0, 0) = \langle 1, 2 \rangle$, and

$$Df(0, 0) = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Let $g(x, y) = \left\langle xy^2, y + 2, 2x - 3y \right\rangle$.. Find $D(g \circ f)(0, 0)$.

VIII. Let $f : E \to \mathbb{R}$ be defined on some open set $E \subset \mathbb{R}^2$, and assume the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 6.3.8 (continuity of partial derivatives implies f is differentiable) which we discussed in class.

IX. Let $F(x, y, z) = \begin{pmatrix} x + y \\ x^2 y \\ z + 2x \end{pmatrix}$.

(a) At what points $x_0, y_0, z_0)$ does F have a local inverse, i.e., a function F^{-1} defined on an open set V containing $F(x_0, y_o, z_o)$, such that $F(F^{-1}(x, y, z)) = (x, y, z)$ for all $(x, y, z) \in V$?

(b) What is $D(F^{-1})(2, 1, 3)$? (Hint: $F(1, 1, 1) = (2, 1, 3)$.)

X. When does the equation $x_1^2 + 2x_2^3 - xd^4 + \ln(1 + x_4^2 1) = 1$ define a function $x_4 = g(x_1, x_2, x_3)$ implicitly? Find $\nabla g(1, 0, -1)$.