Functional Analysis– Spring 2024

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p. 81 #7. If $\dim Y < \infty$ in Riesz's lemma 2.5-4, show that one can even choose $\theta = 1$.

To restate the theorem: **F. Riesz's Lemma**. Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval (0,1) there is a $z \in Z$ such that $\|z\| = 1$, $\|z - y\| < \theta$ for all $y \in Y$.

Suppose that we had a sequence θ_m such that $\lim_{m->\infty} \theta_m = 1$ and there is a corresponding sequence z_m such that $\|z_m\| = 1$ and $\|z_m - y\| < \theta_m$ for all $y \in Y$ and $m \in \mathbb{N}$. Since, $\dim Y < \infty$, Y is closed and bounded, hence every sequence converges. Thus, the sequence $\|z_m - y\|$ converges to 1 and can include 1.

p. 101 #3, 5, 6, 7, 8, 9.

3. If $T \neq 0$ is a bounded linear operator, show that for any $x \in \mathcal{D}(T)$ such that $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

   Since T is bounded, $\exists c \rightarrow \|Tx\| \leq c\|x\|$ and $c = \|T\|$. When, $\|x\| = 1$ we have $\|Tx\| \leq \|T\|$ and otherwise

$$\begin{aligned} \|T\| &= \sup_{x\in\mathcal{D}(T),x\neq 0} \frac{\|Tx\|}{\|x\|} \\ \|Tx\| &\leq \sup_{x\in\mathcal{D}(T),x\neq 0} \frac{\|Tx\|}{\|x\|} \\ 1 &\leq \sup_{x\in\mathcal{D}(T),x\neq 0} \frac{1}{\|x\|} \end{aligned}$$

   which is a strict inequality for $\|x\| < 1$.

5. Show that the operator $T : \ell^\infty \rightarrow \ell^\infty$ defined by $y = (\eta_i) = Tx, \eta_j = \xi_j/j, x = (\xi_j)$, is linear and bounded.

$$\begin{aligned} T &: \ell^\infty \rightarrow \ell^\infty \\ x = (\xi_j) &\mapsto y = (\xi_j/j) \end{aligned}$$

$$\begin{aligned} \text{let } u &= (\zeta_j) \\ T(\alpha x + \beta u) &= (\,(\,\alpha(\xi_j) + \beta(\zeta_j)\,)\,/j\,) \\ &= (\,\alpha(\xi_j)/j + \beta(\zeta_j)/j\,) \\ &= (\,\alpha(\xi_j/j) + \beta(\zeta_j/j)\,) \\ &= (\,\alpha(\xi_j/j)\,) + (\,\beta(\zeta_j/j)\,) \\ &= \alpha\,(\,(\xi_j/j)\,) + \beta\,(\,(\zeta_j/j)\,) \\ &= \alpha Tx + \beta Tu \end{aligned}$$

$$\exists c > 0 \rightarrow \|Tx\| \leq c\,\|x\|\ ,\ \forall x \in \ell^\infty$$

   Let $j \in \mathbb{N}$ be such that $\|x\| = \xi_j$. Even if $j = 1$ we can see that $\|Tx\| \leq \xi_j$ because $\xi_j \geq \xi_j/j$ for all j. Thus, $\|Tx\| \leq \|x\|$, hence T is bounded.

6. **(Range)** Show that the range $\mathcal{R}(T)$ of a bounded linear operator $T : X \rightarrow Y$ need not be closed in Y. *Hint.* Use T in Prob 5.

   Let $x = (\xi_m) \in \ell^\infty$ and $\lim_{m\rightarrow\infty} \xi_m = 0$. Then, from Prob 5,

$$\begin{aligned} T &: \ell^\infty \rightarrow \ell^\infty \\ x = (\xi_j) &\mapsto y = (\xi_j/j) \\ Tx &= (\xi_j/j) \end{aligned}$$

   Notice that if $1/j < \xi_j$ then $\xi_j/j > 1$, that is, if there exists N such that $n > N$ implies that $1/n < \xi_n$ then Tx does not converge to zero. Hence, the range of Tx is open.

7. **(Inverse operator)** Let T be a bounded linear operator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \geq b\,\|x\| \quad \text{for all } x \in X$$

   show that then $T^{-1} : Y \rightarrow X$ exists and is bounded.

   Notice that $\|Tx\| \geq b\,\|x\|$ implies that $T0 = 0$ which makes it injective, hence an inverse exists. Then,

$$\left\|T^{-1}x\right\| \leq b\,\left\|T^{-1}Tx\right\| \leq b\,\|x\|$$

   means that T is bounded.

8. Show that the inverse $T^{-1} : \mathcal{R}(T) \to X$ of a bounded linear operator $T : X \to Y$ need not be bounded. *Hint.* Use T in Prob. 5.

   Using T as defined in Prob. 5, $T^{-1}y = (\eta_j j)$. Clearly there is no c such that $\left\|T^{-1}y\right\| \leq c\,\|y\|$ for all y, therefore T^{-1} can be unbounded.

9. Let $T : C[0,1] \to C[0,1]$ be defined by

$$y(t) = \int_0^t x(\tau) d\tau.$$

   Find $\mathcal{R}(T)$ and $T^{-1} : \mathcal{R}(T) \to C[0,1]$. Is T^{-1} linear and bounded?

   After integration, each $y(t)$ will be the anti-derivative of $x(\tau)$, which is a differentiable function. That is $\mathcal{R}(T)$ will be the set of differentiable functions on $[0,1]$. $T^{-1}(z) = z'$. $\left\|T^{-1}z\right\| = \sup\limits_{t\in[0,1]} z'(t)$. However, notice that given a polynomial z^n then $T^{-1}(z^n) = nz^{n-1}$ and $\left\|T^{-1}(z^n)\right\| = n$. n is arbitrary, therefore, T^{-1} is not bounded.

p. 109 #2, 3, 4.

2. Show that the functionals defined on $C[a,b]$ by

$$f_1(x) = \int_a^b x(t)y_0(t)dt \qquad (y_o \in C[a,b])$$

$$f_2(x) = \alpha x(a) + \beta x(b) \qquad (\alpha, \beta \text{ fixed})$$

are linear and bounded.
Let $p, q \in C[a,b]$

$$f_1(\alpha p + q) = \int_a^b (\alpha p + q)(t)y_0(t)dt = \alpha \int_a^b p(t)y_0(t)dt + \int_a^b q(t)y_0(t)dt = \alpha f_1(p) + f_1(q)$$

$$\begin{aligned} f_2(\gamma p + q) &= \alpha(\gamma p + q)(a) + \beta(\gamma p + q)(b) \\ &= \alpha\gamma p(a) + \alpha q(a) + \beta\gamma p(b) + \beta q(b) \\ &= \gamma f_2(p) + f_2(q) \end{aligned}$$

$$\begin{aligned} \|f_1(x)\| &\leq \max_{t\in[a,b]} (|x(t)y_0(t)|) \\ &\leq \max_{t\in[a,b]} (|x(t)||y_0(t)|) \\ &\leq \|y_0\| \ \|x\| \end{aligned}$$

$\|y_0\|$ is a constant, hence f_1 is bounded. Then, with the extreme value theorem, there exists $s \in [a,b]$ such that $x(s) \geq x(t), \forall t \in [a,b]$

$$\begin{aligned} \|f_2(x)\| &\leq |\alpha x(s) + \beta x(s)| \\ &\leq |x(s)(\alpha + \beta)| \\ &\leq |\alpha + \beta||x(s)| \\ &\leq |\alpha + \beta| \ \|x(s)\| \end{aligned}$$

hence f_2 is bounded.

3. Find the norm of the linear functional f defined on $C[-1,1]$ by

$$f(x) = \int_{-1}^0 x(t)dt - \int_0^1 x(t)dt.$$

Let g, h linear functionals on $C[-1,1]$ and $g(x) = \int_{-1}^0 x(t)dt$, $h(x) = -\int_0^1 x(t)dt$ then

$$\begin{aligned} f(x) &= (g+h)(x) \\ \|f\| &\leq \|g\| + \|h\| \\ \|g(x)\| &= \left| \int_{-1}^0 x(t)dt \right| \leq \max_{t\in[-1,0]} |\, x(t)\,| \\ \|h(x)\| &= \left| -\int_0^1 x(t)dt \right| \leq \max_{t\in[0,1]} |\, x(t)\,| \\ \|f\| &\leq \max(\ \|g\| \ , \ \|h\| \ ) = \max_{t\in[-1,1]} |\, x(t)\,| \end{aligned}$$

4. Show that

$$f_1(x) = \max_{t\in J} x(t)$$

$$J = [a,b]$$

$$f(2) = \min_{t\in J} x(t)$$

define functionals on $C[a,b]$. Are they linear? Bounded?

It seems pretty clear that

$$\max_{t\in J}(x+y)(t) \neq \max_{t\in J} x(t) + \max_{t\in J} y(t)$$
$$\text{and } \min_{t\in J}(x+y)(t) \neq \min_{t\in J} x(t) + \min_{t\in J} y(t)$$

so, they cannot be linear. They are, however, limited by that fact that being continuous functions each $x \in C[a,b]$ must have finite values on the entire interval (no points go to infinity). Hence, bounded.