Math 5102 – Linear Algebra– Fall 2024

w/Professor Penera

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Homework #6 – NONE

Page 107: 3, 4, 5, 15, 16

Page 107:3. Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) F^3 and $P_3(F)$.
not isomorphic $\dim F^3 = 3$ and $\dim P_3(F) = 4$.

(b) F^4 and $P_3(F)$.
isomorphic $\dim F^4 = 4$ and $\dim P_3(F) = 4$.

(c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$.
isomorphic $\dim M_{2\times 2} = 4$ and $\dim P_3(F) = 4$.

(d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \text{tr}(A) = 0\}$ and $\mathbb{R}^4$.
Not isomorphic. $\dim V = 2$ and $\dim \mathbb{R}^4 = 4$

Page 107:4. Let A and B be $n \times n$ matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

$$\begin{aligned}
(AB)^{-1}(AB) &= I \\
(AB)^{-1}ABB^{-1} &= IB^{-1} \\
(AB)^{-1}AA^{-1} &= B^{-1}A^{-1} \\
(AB)^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

Page 107:5. Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

$$\begin{aligned}
AA^{-1} &= I \\
(AA^{-1})^t &= I^t \\
A^t(A^{-1})^t &= I \\
(A^t)^{-1} &= (A^{-1})^t
\end{aligned}$$

Page 107:15. Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

($\Longrightarrow$) T is an isomorphism, thus T is one-to-one and onto. Thus,

$$\begin{aligned}
&\forall w \in W, !\exists v \in V \to v = \sum_{i=1}^{n} \alpha_i \beta_i \text{ and } T(v) = w \\
&w = T(v) = T\left(\sum_{i=1}^{n} \alpha_i \beta_i\right) = \sum_{i=1}^{n} \alpha_i T(\beta_i) \\
&\text{also } \forall i, j, i \neq j, 0 = aT(\beta_i) - bT(\beta_j) = T(a\beta_i - b\beta_j) \to a = b = 0 \\
&\therefore \text{span}\,\{\,T(\beta_i)\,\} = W
\end{aligned}$$

($\Longleftarrow$) $\{\,T(\beta_i)\,\}$ is a basis for W.

$$\dim \text{span}\,\{\,T(\beta_i)\,\} = n \to \text{onto}$$

Page 107:16. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n} \to (F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Page 116: 4, 11

Page 116:4) Let T be the linear operator $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[T]_{\beta'}$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \implies [T]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\begin{aligned} [T]_{\beta'} &= Q^{-1}[T]_\beta Q \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 5 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 11 \\ -5 & -7 \end{pmatrix} \end{aligned}$$

Page 116:11) Let V be a finite-dimensional vector space with ordered bases α, β and γ.

(a) Prove that if Q and R are the changed of coordinate matrices that change α-coordinates in β-coordinates and β-coordiantes into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates to γ-coordinates.

Q and R are invertible and therefore commutative.

$$\begin{aligned} [T]_\beta &= Q^{-1}[T]_\alpha Q \text{ and } T_\gamma = R^{-1}[T]_\beta R \\ T_\gamma &= R^{-1}\left(Q^{-1}[T]_\alpha Q\right)R \\ &= (QR)^{-1}[T]_\alpha QR \end{aligned}$$

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.