Math 5301 – Numerical Analysis– Spring 2025
w/Professor Du

Paul Carmody
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A-Conjugage Gradient Method

By Theorem 4.1 of *Finite Difference Methods for Ordinary Differential Equations* page 88. We have that ... The vectors generated in the CG Algorithm have properties provided $r_k \neq 0$ (if $r_k = 0$ then we have converged).

1. p_k is A-conjugate to all previous search directions, i.e., $p_k^T A p_j = 0$ for all $j = 1, \ldots, k-1$.
2. The residual r_k is orthogonal to all previous residuals $r_k^T r_j = 0$ for $j = 0, \ldots, k-1$.
3. The following three subspaces are identical

   $\text{span}(p_0, p_1, p_2, \ldots, p_{k-1})$,
   $\text{span}(r_0, Ar_0, A^2 r_0, \ldots, A^{k-1} r_0)$.
   $\text{span}(Ae_0, A^2 e_0, \ldots, A^k e_0)$.

Vectors from this/these subspaces take on the linear combination of basis vectors that extrapolates to a polynomial.

$$\mathcal{K}_k = \text{span}(r_0, Ar_0, A^2 r_0, \ldots, A^{k-1} r_0)$$

and is referred to as the ***Krylov Space.*** These take the form of a polynomiald

$$P_k(A) = a_0 r_0 + a_1 A r_0 + a_2 A^2 r_0 + \cdots + a_{k-1} A^{k-1} r_0$$
$$P_k(\lambda_{\max}) = a_0 r_0 + a_1 \lambda_{\max} r_0 + a_2 \lambda_{\max}^2 r_0 + \cdots + a_{k-1} \lambda_{\max}^{k-1} r_0$$

Where $\lambda_{\max}$ is the largest eigenvalue of A.

the CG algorithm converges to at most n iterations. Keep in mind that n can be very large ($>> 1000$).