

# Functional Analysis– Spring 2024

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Assignment #3– March 17, 2024

p. 126 #8 Show that the dual space of the space  $c_0$  is  $\ell^1$ . (Cf. Prob. 1 in Sec. 2.3.)

Want to show that

1. every element of  $c'_0$  is an element of  $\ell^1$

Let  $(e_k)$  be the unique Schauder basis for  $\ell^1$  where  $e_k = (\delta_{jk})$ . Let  $x = (\xi_j) \in c_0$ , that is  $\lim_{j \rightarrow \infty} \xi_j = 0$  which has the unique representation  $x = \sum_{j=1}^{\infty} \xi_j e_j$ . Let  $f \in c'_0$ , that is  $f : c_0 \rightarrow \mathbb{R}$  which is linear and bounded. Therefore,

$$\begin{aligned} f(x) &= \sum_{j=1}^{\infty} \xi_j f(e_j) \\ |f(e_j)| &\leq \|f\| \|e_j\| = \|f\| \\ \|f(x)\| &\leq \|f\| \left| \sum_{j=1}^{\infty} \xi_j \right| \leq \|f\| \sum_{j=1}^{\infty} |\xi_j| = \|f\| \|x\|_{\ell^1} \end{aligned}$$

which means that  $f \in \ell^1$ .

2. that the norm over  $c'_0$  is the norm over  $\ell^1$ . want to show that  $|f(x)| = \|x\|$ . Let  $\gamma = \sup_j f(e_j)$

$$|f(x)| = \left| \sum_{j=1}^{\infty} \xi_j f(e_j) \right| \leq \gamma \sum_{j=1}^{\infty} |\xi_j| = \gamma \|x\|$$

p. 135 #9 Prove

$$\begin{aligned} \operatorname{Re} \langle x, y \rangle &= \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 \right) \\ \operatorname{Im} \langle x, y \rangle &= \frac{1}{4} \left( \|x + iy\|^2 - \|x - iy\|^2 \right) \end{aligned}$$

Remember that

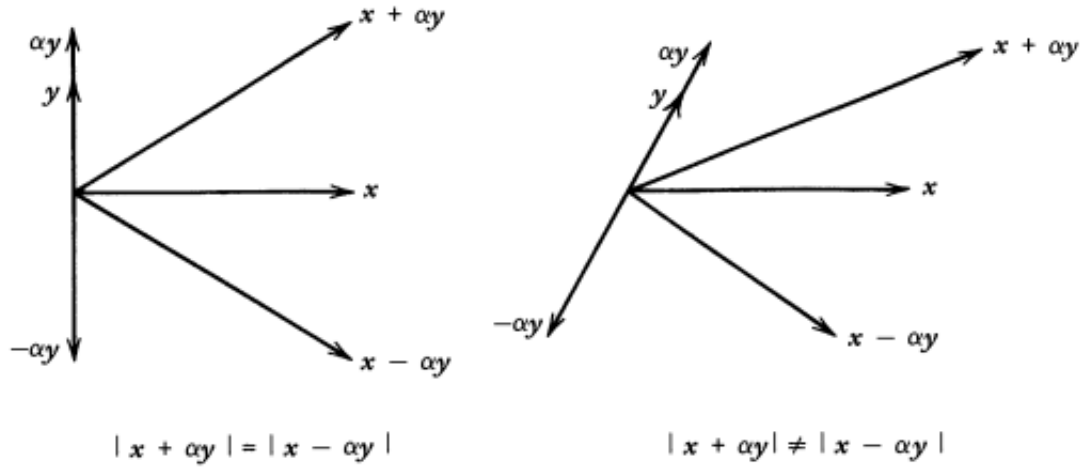
$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + 2 \operatorname{Re} \langle x, y \rangle \\ \text{and } \|x + y\|^2 + \|x - y\|^2 &= 2 \|x\|^2 + 2 \|y\|^2 \\ \text{thus } 2 \operatorname{Re} \langle x, y \rangle &= \|x + y\|^2 - \|x\|^2 - \|y\|^2 \\ &= \|x + y\|^2 - (\|x\|^2 + \|y\|^2) \\ &= \|x + y\|^2 - \frac{1}{2} (\|x + y\|^2 + \|x - y\|^2) \\ 4 \operatorname{Re} \langle x, y \rangle &= \|x + y\|^2 - \|x - y\|^2 \\ \operatorname{Re} \langle x, y \rangle &= \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \end{aligned}$$

Notice that

$$\begin{aligned} \text{Notice } \|x + iy\|^2 + \|x - iy\|^2 &= 2 \|x\|^2 + 2 \|y\|^2 \\ \|x - iy\|^2 &= \langle x - iy, x - iy \rangle \\ &= \langle x, x - iy \rangle + \langle iy, x - iy \rangle \\ &= \langle x, x \rangle + \langle x, iy \rangle - \langle iy, x \rangle + \langle iy, iy \rangle \\ &= \|x\|^2 + |i|^2 \|y\|^2 - i \langle x, y \rangle - i \overline{\langle x, y \rangle} \\ &= \|x\|^2 + \|y\|^2 + 2 \operatorname{Im} \langle x, y \rangle \\ 2 \operatorname{Im} \langle x, y \rangle &= \|x - iy\|^2 - (\|x\|^2 + \|y\|^2) \\ &= \|x - iy\|^2 - \frac{1}{2} (\|x + iy\|^2 + \|x - iy\|^2) \\ 4 \operatorname{Im} \langle x, y \rangle &= \|x - iy\|^2 - \|x + iy\|^2 \end{aligned}$$

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7. Show that in an inner product space,  $x \perp y$  if and only if  $\|x + \alpha y\| = \|x - \alpha y\|$  (see Fig. 25.)



**Fig. 25.** Illustration of Prob. 7 in the Euclidean plane  $\mathbf{R}^2$

Assuming that  $x \perp y$  then  $\langle x, y \rangle = 0$  and

$$\begin{aligned}\|x + \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \\ &= \|x\|^2 + |\alpha|^2 \|y\|^2 \\ \|x - \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \|y\|^2 - 2|\alpha| \operatorname{Re} \langle x, y \rangle \\ &= \|x\|^2 + |\alpha|^2 \|y\|^2\end{aligned}$$

Assuming that they are equal

$$\begin{aligned}\|x + \alpha y\|^2 - \|x - \alpha y\|^2 &= \left( \|x\|^2 + |\alpha|^2 \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \right) - \left( \|x\|^2 + |\alpha|^2 \|y\|^2 - 2|\alpha| \operatorname{Re} \langle x, y \rangle \right) \\ &= 4|\alpha| \operatorname{Re} \langle x, y \rangle\end{aligned}$$

which can only be zero when  $\operatorname{Re} \langle x, y \rangle = 0$  or  $x \perp y$ .

8. Show that in an inner product space,  $x \perp y$  if and only if  $\|x + \alpha y\| \geq \|x\|$  for all scalars  $\alpha$ .

Assuming that  $x \perp y$  then

$$\begin{aligned}\|x + \alpha y\|^2 &= \|x\|^2 + |\alpha|^2 \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle \\ &= \|x\|^2 + |\alpha|^2 \|y\|^2\end{aligned}$$

$|\alpha| \geq 0$  for all  $\alpha$  as well as  $\|y\| \geq 0$  for all  $y$ . Thus,  $\|x + \alpha y\| \geq \|x\|$ .

Assuming that this statement is true then for all  $\alpha \in \mathbb{C}$  and  $y \in Y$

$$\begin{aligned}\|x\|^2 + |\alpha|^2 \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle &\geq \|x\|^2 \\ |\alpha|^2 \|y\|^2 + 2|\alpha| \operatorname{Re} \langle x, y \rangle &\geq 0 \\ |\alpha|^2 \|y\|^2 &\geq -2|\alpha| \operatorname{Re} \langle x, y \rangle\end{aligned}$$

the left side is positive and the right side is negative therefore  $\operatorname{Re} \langle x, y \rangle = 0$  and  $x \perp y$ .

9. Let  $V$  be the vector space of all continuous complex-valued functions on  $J = [a, b]$ . Let  $X_1 = (V, \|\cdot\|_\infty)$ , where  $\|x\|_\infty = \max_{t \in J} |x(t)|$ ; and let  $X_2 = (V, \|\cdot\|_2)$ , where

$$\|x\|_2 = \langle x, x \rangle^{1/2}, \quad \langle x, y \rangle = \int_a^b x(t) \overline{y(t)} dt$$

Show that the identity mapping  $x \mapsto x$  of  $X_1$  onto  $X_2$  is continuous.  
(It is not a homeomorphism.  $X_2$  is not complete.)

Let  $T : X_1 \rightarrow X_2$  such that  $Tx = x$ . We want to show that given any  $\epsilon > 0$  when  $\|Tx - Ty\| < \epsilon$ , then  $\|x - y\| \leq \delta$  for some  $\delta > 0$  dependent on  $\epsilon$ .

$$\begin{aligned} \epsilon > \|Tx - Ty\|_2^2 &= \langle x - y, x - y \rangle \\ &= \int_a^b (x(t) - y(t)) \overline{(x(t) - y(t))} dt \\ &= \int_a^b \operatorname{Re} (x(t) - y(t))^2 + \operatorname{Im} (x(t) - y(t))^2 dt \\ &\geq \int_a^b \|x(t) - y(t)\|_2^2 dt \end{aligned}$$

by the Extreme Value Theorem, there exists  $p \in J$  such that  $x(p) \geq x(t) - y(t), \forall t \in J$  i.e.,  $|x(p) - y(p)| = \|x - y\|_\infty$ . Thus.

$$\int_a^b \|x(t) - y(t)\|_2^2 dt \leq (b - a)(x(p) - y(p))^2 = (b - a) \|x - y\|_\infty^2 < \delta$$

10. **(Zero Operator)** Let  $T : X \rightarrow X$  be a bounded linear operator on a complex inner product space  $X$ . If  $\langle Tx, x \rangle = 0$  for all  $x \in X$ , show that  $T = 0$ .

Show that this does not hold in the case of a real inner product space. *Hint.* Consider a rotation of the Euclidean plane.

$$\begin{aligned} 0 &= \langle T(x + y), x + y \rangle \\ &= \langle Tx + Ty, x + y \rangle \\ &= \langle Tx, x + y \rangle + \langle Ty, x + y \rangle \\ &= \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle \\ &= \langle Tx, y \rangle + \langle Ty, x \rangle \end{aligned}$$

The Inner Product is positive definite, which means that the only way that the right side can be equal to zero is if  $Tx = 0$  for  $x \in x$ .

p. 150 #2, 3a, 6,

2. Show that the subset  $M = \{y = (\eta_j) \mid \sum \eta_j = 1\}$  of complex space  $\mathbf{C}^n$  (cf 3.1-4) is complete and convex. Find the vector of minimum norm in  $M$ .

$$\begin{aligned}\langle x, y \rangle &= \sum \xi_j \bar{\eta}_j \\ \|x\|^2 &= \sum \xi_j \bar{\xi}_j\end{aligned}$$

$M$  is convex if given any two points  $x = (\xi_n), y = (\eta_n) \in M$  then  $z \in M$  where  $z = \alpha x + (1 - \alpha)y$  and any  $0 \leq \alpha \leq 1$ .

$$\begin{aligned}z &= \alpha x + (1 - \alpha)y \\ &= \alpha(\xi_n) + (1 - \alpha)(\eta_n) \\ &= (\alpha\xi_n) + (\eta_n) - (\alpha\eta_n) \\ &= (\alpha\xi_n + \eta_n - \alpha\eta_n) \\ \text{let } z &= (\zeta_n) \\ \sum \zeta_n &= \sum (\alpha\xi_n + \eta_n - \alpha\eta_n) \\ &= \sum \alpha\xi_n + \sum \eta_n - \sum \alpha\eta_n \\ &= \alpha \sum \xi_n + \sum \eta_n - \alpha \sum \eta_n \\ &= \alpha + 1 - \alpha \\ &= 1\end{aligned}$$

$M$  is complete when every sequences in  $M$  converges to an element in  $M$ .

Given a convergent sequence  $x_p = (\xi_n^p) \in M$  where  $p$  is an index then  $\sum \xi_n^p = 1$  for all  $p \in \mathbb{N}$ . If  $x_p \rightarrow x$  then  $x \in M$ .

3. (a) Show that the vector space  $X$  of all real-valued continuous functions on  $[-1, 1]$  is the direct sum of the set of all even continuous functions and the set of all odd continuous functions on  $[-1, 1]$ .

Define the inner product over the set of continuous functions on  $[-1, 1]$  as

$$\langle x, y \rangle = \int_{-1}^1 x(t)y(t)dt$$

Let  $E$  be the set of even functions. That is for all  $f \in E$  then  $f(x) = f(-x)$ . Let  $g \in E^\perp$  then

$$\begin{aligned}0 = \langle f, g \rangle &= \int_{-1}^1 f(t)g(t)dt \\ &= \int_{-1}^0 f(t)g(t)dt + \int_0^1 f(t)g(t)dt\end{aligned}$$

notice that the function  $h(t) = f(t)g(t)$  on  $t \in [-1, 1]$  is odd. Yet,  $f$  is even, thus  $g$  must be odd. Since  $g$  is arbitrary  $E^\perp$  is filled with odd functions. And we know that  $X = E \oplus E^\perp$ .

6. Show that  $Y = \{x \mid x = (\xi_j) \in \ell^2, \xi_{2n} = 0, n \in \mathbb{N}\}$  is a closed subspace of  $\ell^2$  and find  $Y^\perp$ . What is  $Y^\perp$  if  $Y = \text{span}\{e_1, \dots, e_n\} \subset \ell^2$ , where  $e_j = (\delta_{jk})$ ?

Let  $x = (\xi_j), y = (\eta_j) \in Y$ . Then,  $x + \alpha y = (\xi_j + \alpha\eta_j)$ . Whenever,  $j$  is even  $\xi_j = \eta_j = 0$  and  $\xi_j + \alpha\eta_j = 0$  thus  $x + \alpha y \in Y$ . Further, let  $x_n = (\xi_j^n) \in Y$  be a convergent sequence and  $x_n \rightarrow x$ . NOTE:  $\xi^n$  is NOT an exponent but an index. Then

$$\|x_n - x\|^2 = \sum_{i=1}^{\infty} (\xi_i^n - \xi_i)^2$$

when  $i$  is even we have zero. Each of the terms on the right must be positive. Thus,  $\lim_{n \rightarrow \infty} \|x_n - x\|^2 = 0$  which implies that  $\xi_n^m \rightarrow \xi_i$  for all  $i \in \mathbb{N}$ . Thus,  $x \in Y$  and  $Y$  is closed.