# Math 5050 – Special Topics: Manifolds– Spring 2025

w/Professor Berchenko-Kogan

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### 6.1 Differential Structure on $\mathbb{R}$.

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \mathbb{I} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure give by the maximal atlas of the chart $(\mathbb{R}, \psi : \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

1. Show that these two differentiable structures are distinct.

   WTS that there is a chart in $\mathbb{R}$ that is incompatible with a chart in $\mathbb{R}'$. Let $\phi = \mathbb{I}$ and $\psi = x^{1/3}$ then

   $$\psi \circ \phi^{-1} = x^{1/3} \text{ and } \phi \circ \psi^{-1} = x^3$$

   these are compatible if they are diffeomorphic to each other. However,

   $$\begin{aligned}
   \text{Let } g(x) &= \phi \circ \psi^{-1} = x^3 \in C^\infty \\
   \text{Let } h(x) &= \psi \circ \phi^{-1} = x^{1/3} \\
   h'(x) &= \frac{1}{3x^{1/3}} \\
   h'(0) &\text{ does not exist} \\
   \therefore h &\notin C^\infty
   \end{aligned}$$

   and $\mathbb{R}$ is incompatible with $\mathbb{R}'$.

2. Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$, (*Hint:* The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired differomorphism: in fact, this map is not smooth).

   Let $f : \mathbb{R} \to \mathbb{R}'$ and set

   $$\begin{aligned}
   f(x) &= x^3 \\
   \left(\psi \circ f \circ \phi^{-1}\right)(x) &= \left(\psi \circ f \circ \mathbb{I}\right)(x) = \left(\psi \circ f\right)(x) \\
   &= \psi(x^3) = \left(x^3\right)^{1/3} = x \\
   \text{and } \left(\phi \circ f^{-1} \circ \psi^{-1}\right)(x) &= \left(\phi \circ f^{-1}\right)(x^3) \\
   &= \phi(f^{-1}(x^3)) = \mathbb{I}((x^3)^{1/3}) = x
   \end{aligned}$$

   therefore $f$ is a diffeomorphism.

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### 6.2 The smoothness of inclusion map.

Let $M$ and $N$ be manifolds and let $q_o$ be a point in $N$. Prove that the inclusion map $i_{q_o} : M \to M \times N, i_{q_0}(p) = (p, q_0)$, is $C^\infty$.

Let $(U, \phi)$ be a chart on $M$ and $(V, \psi)$ be a chart on $N$. Then $(U \times V, \phi \times \psi)$ is a chart on $M \times N$. Observe

$$\begin{aligned}
(\phi \times \psi) \circ i_{q_0} \circ \phi^{-1} &: \phi(U) \to M \times N \\
(\phi \times \psi) \circ i_{q_0} \circ \phi^{-1}(u) &= (\phi \times \psi) \circ i_{q_0}(\phi^{-1}(u)) \\
&= (\phi \times \psi)(\phi^{-1}(u), q_0) \\
&= \left(\phi(\phi^{-1}(x)),\ \psi(q_0)\right) \\
&= \left(u,\ \psi(q_0)\right)
\end{aligned}$$

which is a smooth map.

**6.4 Local coordinate systems.**

Find all points in $\mathbb{R}^3$ in a neighborhood of which the funciton $x, x^2+y^2+z^2-1, z$ can serve as a local coordinate system.

This can only happen where the Jacobian is not zero.

$$\begin{aligned} J(F) &= \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial}{\partial x}(x^2+y^2+z^2-1) & \frac{\partial z}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial|}{\partial y}(x^2+y^2+z^2-1) & \frac{\partial z}{\partial y} \\ \frac{\partial x}{\partial z} & \frac{\partial}{\partial z}(x^2+y^2+z^2-1) & \frac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2x & 0 \\ 0 & 2y & 0 \\ 0 & 2z & 1 \end{vmatrix} \\ &= 2y \end{aligned}$$

Thus, as long as $y \neq 0$ we have a local coordinate system.