Math 725 – Advanced Linear Algebra

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Assignment #9 – Due 11/15/23

1. Let T be an operator on a finite dimensional inner product space.
a) Show that range(T^*) is equal to the orthogonal complement of null(T).

For $v \in \text{null}(T)$ we have $\langle Tv, w \rangle = 0$ for all $w \in W$. And $\langle Tv, w \rangle = \langle v, T^* w \rangle = 0$. Which means that $T^* w$ must be orthogonal to v for all w, hence range(T) is orthogonal to all $v \in \text{null}(T)$.

b) Assume that T is invertible. Prove that T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$.

$$
\begin{aligned}
I &= I^* \\
TT^{-1} &= (TT^{-1})^* \\
&= (T^{-1})^* T^* \\
(TT^{-1})^{-1} &= ((T^{-1})^* T^*)^{-1} \\
I &= (T^*)^{-1}((T^{-1})^*)^{-1} \\
(T^{-1})^* &= (T^*)^{-1}
\end{aligned}
$$

2. Let $V = \mathcal{M}_{n\times n}(\mathbb{C})$ with the inner product $\langle A, B \rangle = \text{tr}(AB^*)$. Let P be a fixed invertible matrix in V, and let T_P be the linear operator on V defined by $T_P(A) = P^{-1}AP$. Find the adjoint of T_P.

$$
\begin{aligned}
\langle T_P(A), A \rangle &= \text{tr}(P^{-1}APA^*) = \text{tr}(AP^{-1}A^*P) = \langle A, T_P(A^*) \rangle \\
T_P^* A &= T_P(A^*)
\end{aligned}
$$

We can commute P and P^{-1} because they are invertible.

3. Let V be a finite dimensional inner product space and let W be a subspace of V. Then $V = W \oplus W^{\perp}$ where $W^{\perp}$ is the orthogonal complement of W in V. In this case every vector $v \in V$ can be written as $v = w + u$ where $w \in W$ and $u \in W^{\perp}$ are unique vectors. We define a linear operator $U : V \mapsto V$ by $U(v) = w - u$ where $v = w + u$ is the unique decomposition.

a) Prove that U is both self-adjoint and unitary. [Hint: diagonalize U].

Since V is a finite dimensional inner product space, there exists an orthonormal basis B. Also, W is U-invariant, i.e., $x \in W$, $\langle U(v), x \rangle = \langle w - u, x \rangle = \langle w, x \rangle - \langle u, x \rangle = \langle w, x \rangle$. Then

$$[U]_B^B = \begin{pmatrix} U|_W & 0 \\ 0 & U|_{W^\perp} \end{pmatrix}$$

We can see that $||U|_W(w)|| = ||w||$ which means that $U|_W$ is unitary and $||U|_{W^\perp}(u)|| = ||u||$ which means that $U|_{W^\perp}$ is also unitary, thus $[U]_B^B$ is also unitary. Let $x = a + b$ where $a \in W$ and $b \in W^\perp$. Then,

$$\begin{aligned} \langle U(v), x \rangle &= \langle w - u, x \rangle \\ &= \langle w, x \rangle - \langle u, x \rangle \\ &= \langle w, a \rangle + \langle w, b \rangle - \langle u, a \rangle - \langle u, b \rangle \\ \text{since } \langle u, a \rangle &= \langle w, b \rangle = 0 \\ \langle U(v), x \rangle &= \langle w, a \rangle - \langle w, b \rangle + \langle u, a \rangle - \langle u, b \rangle \\ &= \langle w + u, a - b \rangle \\ &= \langle v, U(x) \rangle \end{aligned}$$

hence self-adjoint.

b) Prove that, conversely, if an operator on V is both self-adjoint and unitary, it has to be as U induced by some subspace W. [Hint: what are the eigenvalues of this operator?].

Let λ be an eigenvalue for U and x be the eigenvector associated with λ. $U(x) = \lambda x$. Since U is unitary we have

$$\begin{aligned} U(x)|| &= ||x|| \\ &= ||\lambda x|| \\ &= |\lambda| \, ||x|| \end{aligned}$$

since $\lambda \in \mathbb{R}$, λ is 1 or -1. Since, U is self-adjoint, the eigenvectors of distinct eigenvalues are orthogonal to each other. Thus, the eigenspace for $\lambda_1 = 1$ will be orthogonal to the eigenspace for $\lambda_{-1} = -1$. Let W be the eigenspace for λ_1 then $W^\perp$ will be the eigenspace for λ_{-1}. Thus, we must have $U|_W(v) = v$ and $U|_{W^\perp}(v) = -v$ and $U = U_W \oplus U_{W^\perp}$ or $U(v) = w - u$ when $w \in W$ and $u \in W^\perp$.

4. Prove that T is normal if and only if $T = U_1 + iU_2$ where U_1 and U_2 are self-adjoint which commute.

Suppose there exists V_1, V_2 such that $T = V_1 + iV_2$ then

$$\begin{aligned} T + T &= (U_1 + iU_2) - (V_1 + iV_2) \\ &= (U_1 - V_1) + i(U_2 - V_2) \\ U_1 &= V_1 \text{ and } U_2 = V_2 \end{aligned}$$

since T is normal, U_1 and U_2 are self-adjoint. And,

$$\begin{aligned} U_1 &= \frac{1}{2}(T + T^*),\ U_1^* = U_1 \\ U_2 &= \frac{1}{2}(T - T^*),\ U_2^* = U_2 \\ U_1U_2 &= \left(\frac{1}{2}(T + T^*)\right)\left(\frac{1}{2}(T - T^*)\right) \\ &T \text{ is normal and commutes with } T^* \\ &= \left(\frac{1}{2}(T - T^*)\right)\left(\frac{1}{2}(T + T^*)\right) \\ &= U_2U_1 \end{aligned}$$

5. Let T be a normal operator on a finite dimensional complex inner product space. Show that there exists a polynomial f with complex coefficients such that $T^* = f(T)$. [Hint: diagonalize T].

Since T is normal, $T = Q^*\Lambda Q$ where Q is orthonormal and made up of column vectors of eigenvectors and Λ is diagonal filled with eigenvalues of T. Then we can see that for any term $f(x) = x^n$ then $f(T) = T^n = (Q^*\Lambda Q)(Q^*\Lambda Q)\cdots(Q^*\Lambda Q)$, n times. Since $QQ^* = I$ we can see that $T^n = Q^*\Lambda^n Q$. All polynomials are made up of these terms, and we can factor out Q, Q^* from each we have $f(T) = Q^*f(\Lambda)Q$ for any polynomials f. The adjoint, $T^* = (Q^*\Lambda Q)^* = Q^*\Lambda^*Q)$. Thus, we are now looking for a solution to $\Lambda^* = f(\Lambda)$. Both Λ and Λ^* are diagonal and filled with the same eigenvalues. It sems life $f(x) = x$.

6. Suppose T is a self-adjoint operator on a complex inner product space V of finite dimension. Let $\lambda \in \mathbb{C}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $||v|| = 1$ and $||Tv - \lambda v|| < \epsilon$. Prove that T has an eigenvalue μ such that $|\lambda - \mu| < \epsilon$.

$$\begin{aligned} T &= Q^*\Lambda Q \\ ||Tv - \lambda v|| &= ||Q^*\Lambda Qv - \lambda v|| \\ &= ||Q^*\Lambda Q - \lambda||\,|v| \\ &= ||Q^*\Lambda Q - \lambda|| \\ &< \epsilon \end{aligned}$$

there must exist an eigenvalue μ such that $|\mu - \lambda| < \epsilon|$

Extra Questions
These extra questions will help you go through the proof of the following theorem.

Theorem 1. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ be an invertible matrix. Then there exists a unique lower triangular matrix L with positive diagonal elements such that LA is unitary.

1. Let $\alpha_1, \ldots, \alpha_n$ be the rows of A and let $\beta_1, \ldots, \beta_n$ be an orthogonal basis obtained by the Gram-Schmidt procedure. Recall that this means $\text{span}(\alpha_1, \ldots, \alpha_j) = \text{span}(\beta_1, \ldots, \beta_j)$ for each $j = 1, \ldots, n$. Show that $\beta_j = \alpha_j - \sum_{i<j} c_{ij}\alpha_i$ for each $j = 1, \ldots, n$ and some scalars c_{ij}. [Hint: how does Gram-Schmidt work? Review.]

2. Let U be the matrix whose ith row is $\beta_i/||\beta_i||$. Clearly, U is unitary. Construct the matrix L as in the statement of the theorem such that $LA = U$.

3. Now you will prove the uniqueness of L. Suppose L_1 and L_2 are two lower triangular matrices with positive diagonals such that L_1A and L_2A are both unitary. First prove that $(L_1A)(L_2A)^{-1} = L_1L_2^{-1}$ is lower triangular and unitary. Conclude that $(L_1L_2^{-1})^* = (L_1L_2^{-1})^{-1}$ and hence $L_1L_2^{-1}$ is simultaneouly upper triangular and lower triangular. Hence $L_1L_2^{-1}$ is a diagonal matrix with positive diagonal entries. Finally, using the fact $L_1L_2^{-1}$ is also unitary and hence has eigenvalues with absolute value one, argue that $L_1L_2^{-1} = I$.

4. As a corollary, prove that for every complex invertible matrix A there exists a unique lower triangular matrix N with positive diagonals and a unique unitary matrix U such that $A = NU$.