Math 5050 – Special Topics: Manifolds– Spring 2025

w/Professor Berchenko-Kogan

Paul Carmody

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Section 4 problems:

Within the section: 4.3 (p.37): **(A basis for 3-covectors).** Let x^1, x^2, x^3, x^4 be the coordinates in $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3(T_p(\mathbb{R}^4))$.

$$\begin{aligned}
\Phi = \{ \; & dx_p^i \wedge dx_p^j \wedge dx_p^k \;\; : i < j < k \leq 4 \; \} \\
& \{ \; dx_p^1 \wedge dx_p^2 \wedge dx_p^3, \\
& \; dx_p^1 \wedge dx_p^2 \wedge dx_p^4, \\
& \; dx_p^1 \wedge dx_p^3 \wedge dx_p^4, \\
& \; dx_p^2 \wedge dx_p^3 \wedge dx_p^4 \; \} \\
|\, \Phi \,| = & \binom{4}{3} = 4
\end{aligned}$$

Within the section: 4.4 (p.38), **Wedge product of a 2-form with a 1-form.** Let ω be a 2-form and τ be a 1-form on on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z

$$\begin{aligned}
(\omega \wedge \tau)(X, Y, Z) &= (\omega \otimes \tau)(X, Y, Z) - (\tau \otimes \omega)(X, Y, Z) \\
&= \omega(X)\tau(Y, Z) - \tau(X, Y)\omega(Z) \\
(\omega \wedge \tau)(X, Y, Z) &= \frac{1}{1!2!} A(\omega \otimes \tau)(X, Y, Z)) \\
&= \frac{1}{2} \left(\omega(X, Y)\tau(Z) + \omega(Y, Z)\tau(X) + \omega(Z, X)\tau(Y) - \omega(Z, Y)\tau(X) - \omega(Y, X)\tau(Z) - \omega(X, Z)\tau(Y) \right) \\
&= \omega(X, Y)\tau(Z) + \omega(Y, Z)\tau(X) + \omega(Z, X)\tau(Y)
\end{aligned}$$

Within the section: 4.9 (p.40) **A closed 1-form on the punctured plane.** Define a 1-form on ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx - xdy).$$

Show that ω is closed.

$$\begin{aligned}
d\omega =& \frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dy \\
=& \left(\frac{-2x}{(x^2 + y^2)^2}(-ydx - xdy) + \frac{1}{x^2 + y^2}(-yd^2x - dydx) \right) dx + \\
& \left(\frac{-2y}{(x^2 + y^2)^2}(-ydx - xdy) + \frac{1}{x^2 + y^2}(-dxdy - xd^2y) \right) dyx`x
\end{aligned}$$

End of the section: 1 through 6.

4.1 **A 1-form on** $\mathbb{R}^3$.

Let ω be the 1-form $zdx - dz$ and let X be the vector $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Computer $\omega(X)$ and $d(\omega)$.

$$\begin{aligned}
\omega(X) &= (zdx - dz) \left(y\partial/\partial x + x\partial/\partial y \right) \\
&= (zdx - dz) \left(y\partial/\partial x \right) + (zdx - dz) \left(x\partial/\partial y \right) \\
&= zy\frac{\partial}{\partial x} dx - y\frac{\partial}{\partial x} dz + zx\frac{\partial}{\partial y} dx - x\frac{\partial}{\partial y} dz \qquad \text{recall } \frac{\partial}{\partial x^i} dx^j = \delta_i^j \\
&= zy
\end{aligned}$$

$$d(\omega) = d(zdx - dz) = d(zdx) - d^2z = dz \wedge dx + z \wedge d^2x = dz \wedge dx$$

4.2 **A 2-form on $\mathbb{R}^3$** At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\mathbf{a}, \mathbf{b}) = \omega_p\left(\left(\begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array}\right), \left(\begin{array}{c} b^1 \\ b^2 \\ b^3 \end{array}\right)\right) = p^3 \det\left(\begin{array}{cc} a^1 & b^1 \\ a^2 & b^2 \end{array}\right),$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternaing bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

$$\begin{aligned}
\omega(p) &= c_{xy}(p)(dx \wedge dy) + c_{yz}(p)(dy \wedge dz) + c_x z(p)(dx \wedge dz) \\
c_{xy}(p) = \omega_p(e_x, e_y) &= p^3 \left(\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{array}\right) = p^3\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 0\right) \\
c_{yz}(p) = \omega_p(e_y, e_z) &= p^3 \left(\begin{array}{cc} 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{array}\right) = 0 \\
c_{xz}(p) = \omega_p(e_x, e_z) &= p^3 \left(\begin{array}{cc} \frac{\partial}{\partial x} & 0 \\ 0 & 0 \end{array}\right) = 0
\end{aligned}$$

notice that $(dx \wedge dy)(a, b) = dx(a)dy(b) - dy(a)dx(b) = a^1b^2 - a^2b^1 = \det\left(\begin{array}{cc} a^1 & b^1 \\ a^2 & b^2 \end{array}\right)$ Thus.

$$\omega = p^3\, dx \wedge dy$$

4.3 **Exterior Calculus**.

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in of dr and $d\theta$.

$$\begin{aligned}
dx &= \cos\theta dr - r\sin\theta d\theta \\
dy &= \sin\theta dr + r\cos\theta d\theta \\
dx \wedge dy &= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= (\cos\theta dr) \wedge (\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= (\cos\theta dr) \wedge (\sin\theta dr) + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + (r\sin\theta d\theta) \wedge (r\cos\theta d\theta) \\
&= 0 + (\cos\theta dr) \wedge (r\cos\theta d\theta) - (r\sin\theta d\theta) \wedge (\sin\theta dr) + 0 \\
&= (\cos\theta dr) \wedge (r\cos\theta d\theta) + (\sin\theta dr) \wedge (r\sin\theta d\theta) \\
&= (r\cos^2\theta)(dr \wedge d\theta) + (r\sin^2\theta)(dr \wedge d\theta) \\
&= r(dr \wedge d\theta)
\end{aligned}$$

4.4 **Exterior Calculus**.

Suppose the standard coordiantes on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

$$\begin{aligned}
dx &= \sin\phi\cos\theta\, d\rho + \rho\cos\phi\cos\theta\, d\phi - \rho\sin\phi\sin\theta\, d\theta \\
dy &= \sin\phi\sin\theta\, d\rho + \rho\cos\phi\sin\theta\, d\phi + \rho\sin\phi\cos\theta\, d\theta \\
dz &= \cos\theta\, d\rho - \rho\sin\phi\, d\phi
\end{aligned}$$

We will attempt to cancel out any terms which have a $dx^i \wedge d^\wedge$ by simplifying dx, dy, and dz in the following manner

$$\begin{aligned}
dx \wedge dy \wedge dz &= (x_1\, d\rho + x_2\, d\phi + x_3\, d\theta) \wedge (y_1\, d\rho + y_2\, d\phi + y_3\, d\theta) \wedge (z_1\, d\rho + z_2\, d\phi + z_3\, d\theta) \\
&= (x_1\, d\rho \wedge y_2\, d\phi \wedge z_3\, d\theta) + (x_1\, d\rho \wedge y_3\, d\theta \wedge z_2\, d\phi) \\
&+ (x_2\, d\phi \wedge y_1\, d\rho \wedge z_3\, d\theta) + (x_2\, d\phi \wedge y_3\, d\theta \wedge z_2\, d\phi) \\
&+ (x_3\, d\theta \wedge y_1\, d\rho \wedge z_2\, d\phi) + (x_3\, d\theta \wedge y_2\, d\phi \wedge z_1\, d\rho) \\
&= (x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 + x_2y_3z_2 + x_3y_1z_2 + x_3y_2z_1)(d\rho \wedge d\phi \wedge d\theta) \\
&= \left|\begin{array}{ccc} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{array}\right| (d\rho \wedge d\phi \wedge d\theta)
\end{aligned}$$

Solving for the determinant by expanding the bottom row

$$
\begin{aligned}
\begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & \end{vmatrix} &= \rho^2 \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \sin\phi\cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix} \\
&= \rho^2\sin\phi \begin{vmatrix} \sin\phi\cos\theta & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \\ \cos\phi & -\sin\phi & \end{vmatrix} \\
&= \rho^2\sin\phi \left(\cos\phi \begin{vmatrix} \cos\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta & \cos\theta \end{vmatrix} + \sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \cos\theta \end{vmatrix} \right) \\
&= \rho^2\sin\phi \left(\cos^2\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + \sin^2\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \right) \\
&= \rho^2\sin\phi \left(\cos^2\phi + \sin^2\phi \right) \\
&= \rho^2\sin\phi
\end{aligned}
$$

That is

$$dx \wedge dy \wedge dx = (\rho^2\sin\phi)\, dr \wedge d\phi \wedge d\theta$$

4.5 **Wedge Product**. Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

$$
\begin{aligned}
\alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\
\beta &= b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2
\end{aligned}
$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

The resulting expression $\alpha \wedge \beta \in \Omega^3(\mathbb{R}^3)$. The $\dim(\Omega^3(\mathbb{R}^3)) = 1$. Thus, there will be one term of the form $dx^1 \wedge dx^2 \wedge dx^3$. Further by distributing the terms of α across the terms of β and ignoring any terms where any two elements are equal, i.e., $dx^i \wedge dx^i = 0$. We will then have

$$
\begin{aligned}
\alpha \wedge \beta &= a_1\, dx^1 \wedge (b_1\, dx^2 \wedge dx^3) + a_2\, dx^2 \wedge (b_2\, dx^3 \wedge dx^1) + a_3\, dx^3 (b_3\, dx^1 \wedge dx^2) \\
&= (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3
\end{aligned}
$$

4.6 **Wedge product and cross product**

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $v_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under the correspodence, the wedge product of 1-covectors correponds to the cross product of vectors $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.