Math 5111 – Real Analysis II– Sprint 2025

w/Professor Perera

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Extra Credit #2– April 11, 2025

Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if $f : X \to [0, \infty]$ is a measurable function. Calculate, justify all of the steps, the limit

$$\lim_{n\to\infty} \int_X |\sin(f(x))|^n \, d\mu.$$

Show that, in general, this result is false if $\mu(X) = \infty$.

Let $g(x) = 1$ and $h_n(x) = |\,sin(f(x))\,|^n$. Clearly, $h_n(x) \le g(x)$ for all $x \in X$ and measurable, i.e., composition of a continuous function with a measurable function is measurable.

$$\text{Let } N = \inf_{k\in\mathbb{N}} \{\, k\pi/2 > |f(x)|, \forall x \in X \,\}$$

$$\text{and } E = \{x \in X : |f(x)| = \frac{k\pi}{2}, \text{ for all } k = 1, 2, \dots, N\}$$

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} |\sin(f(x))|^n = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases}$$

$$\text{Let } h(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases} = \chi_E$$

By Lebesgue Dominated Convergence Theorem, $h \in L^1(X)$ and

$$\lim_{n\to\infty} \int_X |\sin(f(x))|^n \, d\mu = \int_X h(x) d\mu = \int_X \chi_E d\mu = \mu(E)$$

Notice first that $E \subset X$ therefore $\mu(E) < \mu(X) < \infty$. Thus,

$$\begin{aligned} E &= \bigcup_{k=1}^{N} \left\{ x : |f(x)| = \frac{(2k-1)\pi}{2} \right\} \\ \mu(E) &= \mu\left(\bigcup_{k=1}^{N} \left\{ x : |f(x)| = \frac{(2k-1)\pi}{2} \right\} \right) \\ &= \sum_{k=1}^{N} \mu\left(\left\{ x : |f(x)| = \frac{(2k-1)\pi}{2} \right\} \right) \\ &= \sum_{k=1}^{N} 0 \\ &= 0 \end{aligned}$$

For $\mu(X) = \infty$ no such N exists. However, we can still test for convergence.

$$\begin{aligned} S_n &= \int_X |\sin(f(x))|^n \, d\mu \\ S_n &= \infty, \forall n = 1, 2, \dots \end{aligned}$$

using the Ratio Test to determine convergence we have

$$\begin{aligned} S_{n+1}/S_n &= \frac{\int_X |\sin(f(x))|^{n+1} \, d\mu}{\int_X |\sin(f(x))|^n \, d\mu} \\ &= \int_X \frac{|\sin(f(x))|^{n+1}}{|\sin(f(x))|^n} d\mu \\ &= \int_X |\sin(f(x))| \, d\mu \\ &\to \infty \end{aligned}$$