Math 5050 – Special Topics: Manifolds– Fall 2025
w/Professor Berchenko-Kogan

Paul Carmody

Section 10: Categories and Functors – June 18, 2025

Problems

10.1. **Differential of the inverse map**

If $F : N \to M$ is a differomorphism of manifolds and $p \in N$, prove that $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$.

If we define our category of objects as manifolds **Man** with morphisms as smooth maps. Then we can define a functor T :**Man**$\to$**Vect**, i.e., the category of vectors. T sends the smooth map F into a Tangent Bundle via the push forward. That is, $F_{*,p} : T_p(N) \to T_{f(p)}(M)$. Thus, concatenation is the operator of functors and

$$(F \circ F^{-1})_* = F_* \circ F_*^{-1}$$
$$(F_*)^{-1} = (F^{-1})_*$$

10.2. **Isomorphism under a functor**

Prove proposition 10.3

Proposition 10.3. Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a functor from category $\mathscr{C}$ to category $\mathscr{D}$. If $f : A \to B$ is an isomorphism in $\mathscr{C}$ then $\mathscr{F}(f) : \mathscr{F}(A) \to \mathscr{F}(B)$ is an isomorphism in $\mathscr{D}$.
Proof: Since f is isomorphic, there exists a $g \in \mathscr{C}$ such that $f \circ g = \mathbb{I}_A$ then

$$\mathscr{F}(\mathbb{I}_A) = \mathbb{I}_{\mathscr{F}}$$
$$\mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g)$$
$$\mathscr{F}(g) = \mathscr{F}(f)^{-1} \qquad \square$$

10.3. **Functorial properties of the dual**

Prove Proposition 10.5.

Proposition 10.5 (Functorial properies of the dual). Suppose V, W and S are real vector spaces.

(i) $\mathbb{I}_V : V \to V$ is the identity map on V, then $\mathbb{I}_V^\vee : V^\vee \to V^\vee$ is the identity map on $V^\vee$.
Let $f, g : V \to V$ such that $f \circ g = \mathbb{I}_V$

(ii) If $f : V \to W$ and $g : W \to S$ are linear maps, then $(g \circ f)^\vee = f^\vee \circ g^\vee$.

$$(g \circ f)^\vee \stackrel{?}{=} f^\vee \circ g^\vee$$
$$\text{Let } \phi \in S^\vee$$
$$(g \circ f)^\vee(\phi) = \phi \circ (g \circ f) \qquad \text{(LHS)}$$
$$f^\vee \circ g^\vee = f^\vee(g^\vee(\phi)) \qquad \text{(RHS)}$$
$$= f^\vee(\phi \circ g)$$
$$= \phi \circ (g \circ f)$$

10.4. **Matrix of the dual map**

Suppose a linear transformation $L : V \to \bar{V}$ is represested by the martrix $A = [a^i_j]$ relative to the basis $e_1, \ldots, e_n$ for V and $\bar{e}_1, \ldots, \bar{e}_m$ for $\bar{V}$:

$$L(e_j) = \sum_i a^i_j \bar{e}_i.$$

Let $\alpha^1, \ldots, \alpha^n$ and $\alpha^1, \ldots, \alpha^m$ be the dual bases for $V^\vee$ and $\bar{V}^\vee$, respectively. Prove that if $L^\vee(\bar{\alpha}^i) = \sum_j b^i_j \alpha^j$, then $b^i_j = a^i_j$.

10.5. **Injectivity of the dual map**

(a) Suppose V and W are vector spaces of prossibly infinite dimension over a field K. Show that if a linear map $L : V \to W$ is surjective, then its dual $L^\vee : W^\vee \to V^\vee$ is injective.

For any $v, w \in V$, where $L(v) = L(w)$

$$\begin{aligned} L^\vee(\alpha(v - w)) &= \alpha(L(v - w)) \\ &= \alpha(L(v) - L(w)) \\ &= \alpha(L(v)) - \alpha(L(w)) \\ &= L^\vee(\alpha(v)) - L^\vee(\alpha(w)) \\ &= 0 \end{aligned}$$

therefore $L^\vee$ is injective.

(b) Suppose V and W are finite-dimensional vector spaces over a field K. Prove the converse of the implication in (a).

10.6. **Functorial properties of the pullback**

Prove Proposition 10.6.

10.7. **Pullback in the top dimension**

Show that if $L : V \to V$ is a linear opeartor on a vector space V of dimension n, then the pullback $L^* : A_n(V) \to A_n(V)$ is mulitiplication by the determinant of L.