

Math 5102 – Linear Algebra– Fall 2024  
w/Professor Penner

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Homework #5 – NONE

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4. Define

$$T : M_{2 \times 1}(R) \rightarrow P_2(R) \text{ by } T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Compute  $[T]_{\beta}^{\gamma}$ .

8. Let  $V$  be an  $n$ -dimensional vector space with an ordered basis  $\beta$ . Define  $T : V \rightarrow F^n$  by  $T(x) = [x]_{\beta}$ . Prove that  $T$  is linear.
9. Let  $V$  be the vector space of complex numbers over the field  $R$ . Define  $T : V \rightarrow V$  by  $T(z) = \bar{z}$ , where  $\bar{z}$  is the complex conjugate of  $z$ . Prove that  $T$  is linear, and compute  $[T]_{\beta}$ , where  $\beta = \{1, i\}$ . (Recall by Exercise 38 of Section 2.1 that  $T$  is not linear if  $V$  is regarded as a vector space over the field  $C$ .)
13. Let  $V$  and  $W$  be vector spaces, and let  $T$  and  $U$  be nonzero linear transformations from  $V$  into  $W$ . If  $R(T) \cap R(U) = \{0\}$ , prove that  $\{T, U\}$  is a linearly independent subset of  $\mathcal{L}(V, W)$ .
16. Let  $V$  and  $W$  be vector spaces such that  $\dim(V) = \dim(W)$ , and let  $T : V \rightarrow W$  be linear. Show that there exist ordered bases  $\beta$  and  $\gamma$  for  $V$  and  $W$ , respectively, such that  $[T]_{\beta}^{\gamma}$  is a diagonal matrix.

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3. Let  $g(x) = 3 + x$ . Let  $T : P_2(R) \rightarrow P_2(R)$  and  $U : P_2(R) \rightarrow \mathbb{R}^3$  be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b).$$

Let  $\beta$  and  $\gamma$  be the standard ordered bases of  $P_2(R)$  and  $\mathbb{R}^3$ , respectively

- (a) Compute  $[U]_{\beta}^{\gamma}, [T]_{\beta}^{\gamma}$  directly. Then use Theorem 2.11 to verify your results.
  - (b) Let  $h(x) = 3 - 2x + x^2$ . Compute  $[h(x)]_{\beta}$  and  $[U(h(x))]_{\gamma}$ . Then use  $[U]_{\beta}^{\gamma}$  from (a) and Theorem 2.14 to verify your results.
9. Find linear transformations  $U, T : F^2 \rightarrow F^2$  such that  $UT = T_0$  (the zero transformation) but  $TU \neq T_0$ . Use your answer to find matrices  $A$  and  $B$  such that  $AB = 0$  but  $BA \neq 0$ .
11. Let  $V$  be a vector space, and let  $T : V \rightarrow V$  be linear. Prove that  $T^2 = T_0$  if and only if  $R(T) \subseteq N(T)$ .
12. Let  $V, W$  and  $Z$  be vector spaces, and let  $T : V \rightarrow W$  and  $U : W \rightarrow Z$  be linear.
- (a) Prove that if  $UT$  is one-to-one, then  $T$  is one-to-one. Must  $U$  also be one-to-one?
  - (b) Prove that if  $UT$  is onto, then  $U$  is onto. Must  $T$  also be onto?
  - (c) Prove that if  $U$  and  $T$  are one-to-one and onto, then  $UT$  is also.
13. Let  $A$  and  $B$  be  $n \times n$  matrices. Recall that the trace of  $A$  is defined by

$$\text{tr}(A) = \sum_{i=1}^n A_{ii}$$

Prove that  $\text{tr}(AB) = \text{tr}(BA)$  and  $\text{tr}(A) = \text{tr}(A^t)$