Math 725 – Advanced Linear Algebra

Paul Carmody

Assignment #5 – Due 9/27/23

1. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{C}^3$ where $v_1 = (1, 0, -1)$, $v_2 = (1, 1, 1)$, and $v_3 = (2, 2, 0)$. Find the dual basis.

$$
\begin{aligned}
\varphi_1(x, y, z) &= ax + by + cz \\
\varphi_1(1, 0, -1) &= a - c = 1 \\
\varphi_1(1, 1, 1) &= a + b + c = 0 \\
\varphi_1(2, 2, 0) &= 2a + 2b = 0 \\
a &= -b \\
a &= 1, b = -1, c = 0 \\
\varphi_1(x, y, z) &= x - y
\end{aligned}
$$

$$
\begin{aligned}
\varphi_2(x, y, z) &= ax + by + cz \\
\varphi_2(1, 0, -1) &= a - c = 0 \\
\varphi_2(1, 1, 1) &= a + b + c = 1 \\
\varphi_2(2, 2, 0) &= 2a + 2b = 0 \\
a &= 1, b = -1, c = 1 \\
\varphi_2(x, y, z) &= x - y + x
\end{aligned}
$$

$$
\begin{aligned}
\varphi_3(x, y, z) &= ax + by + cz \\
\varphi_3(1, 0, -1) &= a - c = 0 \\
\varphi_3(1, 1, 1) &= a + b + c = 0 \\
\varphi_3(2, 2, 0) &= 2a + 2b = 1 \\
a &= -2b \implies b = \frac{1}{2} \\
a &= -1, c = -1 \\
\varphi_3(x, y, z) &= -x + \frac{1}{2}y - z
\end{aligned}
$$

2. Let $f_1, \dots, f_m$ be linear functionals on F^n. For any $v \in F^n$ define $Tv = (f_1(v), \dots, f_m(v))$. Show that T is a linear transformation from F^n to F^m. Prove also that every linear transformation from F^n to F^m is of this form, for some $f_1, \dots, f_m$.

$$\begin{aligned}
T(cx + y) &= (f_1(cx + y), \dots, f_m(cx + y)) \\
&= (cf_1(x) + f_1(y), \dots, cf_m(x) + f_m(y)) \\
&= c(f_1(x), \dots, f_m(x)) + (f_1(y), \dots, f_m(y)) \\
&= cT(x) + T(y)
\end{aligned}$$

therefore T is a linear transformation.

Given a basis B on F^n and F^m every transformation $T \in \mathcal{L}(F^n, F^m)$ has a matrix $[T]_B = A = [a_{i,j}]$. Given any $v = (x_1, \dots, x_n)$ we know that each element $T_j(x) = \sum_{i=1}^n a_{i,j} v_j \in F$ for $j = 1, \dots, m$. These are clearly linear functionals, $T_j \in \mathcal{L}(F^n, F)$ and applies to all transformations in $\mathcal{L}(F^n, F^m)$.

3. Recall that the trace function is a linear functional on the vector space $\mathcal{M}_{n\times n}(F)$. Now prove that $\text{tr}(AB) = \text{tr}(BA)$ for any $n \times n$ matrices A and B. Conclude that similar matrices have the same trace, and hence the trace of *any* linear operator $T : V \mapsto V$ on a finite dimensional vector space V is well-defined. Conclude further that if $F = \mathbb{C}$ then it is not possible $AB - BA = I$. Why is this not true over an arbitrary field?

$$\begin{aligned}(AB)_{i,j} &= \sum_{k=1}^{n} A_{i,k}B_{k,j}\\ \text{tr}(AB) &= \sum_{l=1}^{n}(AB)_{l,l}\\ &= \sum_{l=1}^{n}\sum_{k=1}^{n} A_{l,k}B_{k,l}\\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} B_{k,l}A_{l,k}\\ &= \sum_{k=1}(BA)_{k,k}\\ &= \text{tr}(BA)\end{aligned}$$

If A, B are similar, then there exists and invertible matrix P such that $A = PBP^{-1}$

$$\begin{aligned}A &= PBP^{-1}\\ \text{tr}(A) &= \text{tr}(PBP^{-1})\\ &= \text{tr}(P)\text{tr}(B)\text{tr}(P^{-1})\\ &= \text{tr}(P)\text{tr}(P^{-1})\text{tr}(B)\\ &= \text{tr}(PP^{-1})\text{tr}(B)\\ &= \text{tr}(I)\text{tr}(B)\\ &= \text{tr}(B)\end{aligned}$$

If $F = \mathbb{C}$ then $\text{tr}(A) = u + iv$ and $\text{tr}(b) = x + iy$ then

$$\begin{aligned}AB - BA &= I\\ \text{tr}(AB - BA) &= 1\\ \text{tr}(A)\text{tr}(B) - \text{tr}(B)\text{tr}(A) &= 1\end{aligned}$$

which cannot possibly be true as the left side of this equation is zero. Over an arbitrary field, the commutativity of both multiplication causes every $xy - yx = 0$.

4. Let V be a vector space and S any subset of V. The *annihilator* of S, denoted by S° is the set of all linear functionals $f \in V^*$ with $f(v) = 0$ for all $v \in S$. Show that S° is a subspace of V^*.

Let $f, g \in S^\circ$ then for every $v \in S, (cf+g)(v) = cf(v)+g(v) = c\cdot 0+0 = 0$ therefore S° is a subspace of V^*.

a) Now suppose V is finite dimensional. Show that $\dim W + \dim W^\circ = \dim V$.

Let $k = \dim(W)$ and $B = \{v_1, \ldots, v_k\}$ be a basis for W. Then, expand B to fit a basis on V, namely $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$. And let $\{f_1, \ldots, f_n\}$ be a dual basis, hence a basis for V^*. Notice that for all $f_i, i = k+1, \ldots, n$ we have $f_i(v_j) = 0$ because $i > k$, thus given any $w \in W$ any linear combination of these functionals will be zero, hence $f_i \in W^\circ$ for all $i = k+1, \ldots, n$. These are also linearly independent as they are taken from a basis. Hence, $\dim(W^\circ) = n - k$ or $\dim W + \dim W^\circ = \dim V$

b) In an n-dimensional vector space, a subspace of dimension $n-1$ is called a hyperplane. Show that any hyperplane is the nullspace of a nonzero functional.

Given any $v \in V$ and given a functional $f : \mathcal{L}(V, F)$ that is not the zero function. Then, given a set of basis functionals, $f_1, \ldots, f_n$ we have $f(v) = a_1 f_1(v) + \cdots + a_n f_n(v)$. When $f(v) = 0$ then we have $a_1 f_1(v) + \cdots + a_n f_n(v) = 0$ or $a_1 f_1(v) + \cdots + a_{n-1} f_{n-1}(v) = -a_n f_n(v)$. Let's define new functionals, $g_i = \frac{f_i}{f_n}$. Then, $g_1(v) + \cdots + g_{n-1}(v) = -a_n$. $\dim \operatorname{span}\{g_1, \ldots, g_{n-1}\} = n - 1$ which is a hyper-plan on V^*.

c) Let W be a k-dimensional subspace of the n-dimensional vector space V. Prove that W is the intersection of $n-k$ hyperplanes.

When $k = n - 1$ we know from 4c) that there is $n - (n-1) = 1$ hyperplane, and designate it h_{n-1} and $W = h_{n-1}$.
When $k = n-2$ we know that there is one hyperplane in h_{n-1}, designated h_{n-2} which intersects with thus $W = h_{n-1} \cap h_{n-2}$.
And again at $n-3$ we have a hyperplane h_{n-3} within h_{n-2} and $W = h_{n-3} \cap h_{n-2} \cap h_{n-1}$.
Thus, when we have k dimensions we'll have $W = h_k \cap \cdots h_{n-1}$ or $n - k$ intersections.

5. Let $V = \mathcal{P}(\mathbb{R})$. Let a and b be fixed real numbers and let f be the linear functional on V defined by

$$f(p) = \int_a^b p(x)\,dx.$$

If D is the differentiation operator on V, what is $D^t f$?

$D \in \mathcal{L}(V, V)$ is the differential operator then, given the standard basis B on $\mathcal{P}(\mathbb{R})$

$$[D]_B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Thus, the matrix of D transpose is

$$[D^t]_B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & n & 0 \end{pmatrix}$$

$f \in \mathcal{L}(\mathrm{P}(\mathbb{R}), \mathbb{R})$ and given any $p \in \mathrm{P}(\mathbb{R})$ let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ then for some constant c

$$\begin{aligned}
\int p(x)dx &= c + a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \cdots + \frac{1}{n+1}a_n x^{n+1} \\
\int_a^b p(x)dx &= a_0(b-a) + \frac{1}{2}a_1(b^2 - a^2) + \frac{1}{3}a_2(b^3 - a^3) + \cdots + \frac{1}{n+1}a_n(b^{n+1} - a^{n+1}) \\
D^t f &= f \circ D \\
(D^t f)(p) &= f(Dp) \\
&= f(D(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)) \\
&= f(a_1 + 2a_2 x + 3a_3 x^2 \cdots + n a_n x^{n-1}) \\
&= a_1(b-a) + a_2(b^2 - a^2) + \cdots + a_n(b^n - a^n) \\
&= p(b) - p(a)
\end{aligned}$$

6. Let $V = \mathcal{M}_{n\times n}(F)$ and let $B \in V$ be a fixed matrix. Let $T : V \mapsto V$ be the linear transformation defined by $T(A) = AB - BA$. What is then $T^t(\mathrm{tr})$?

$$\begin{aligned}
T^t(\mathrm{tr})(A) &= \mathrm{tr} \circ T(A) \\
&= \mathrm{tr}(T(A)) \\
&= \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(B)\mathrm{tr}(A) \\
&= 0
\end{aligned}$$

Extra Questions

1. Let V be a vector space over the field F, and let $v \in V$ be a fixed vector. We define the map $L_v : V^* \mapsto F$ where $L_v(f) = f(v)$. Show that L_v is a linear functional on V^*, i.e., $L_v \in V^{**}$, the double dual of V.

2. Now let V be a finite dimensional vector space and consider the map $v \mapsto L_v$ which is a map from V to V^{**}. Show that this map is a linear isomorphism of V onto V^{**}. Conclude that if V is a finite dimensional vector space then for every linear functional L on V^* there is a unique $v \in V$ such that $L(f) = f(v)$ for every $f \in V^*$. [This is really a restatement of the isomorphism you have proved]

3. Using the above result prove that if V is a finite dimensional vector space then every basis of V^* is the dual basis to some basis of V.