# Algebraic measure theory

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# ALGEBRAIC MEASURE THEORY

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**Abstract:** The purpose of algebraic measure theory is to make algebra out of both traditional and noncommutative measure theory. This is accomplished by associating a partially ordered abelian group $G$ with a measure-carrying structure $L$ in such a way that real-valued group homomorphisms on $G$ correspond to measures on $L$.

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**1. Introduction.** The representation of integrals and measures by linear functionals on a partially ordered vector space is a commonplace feature of the theory of measure and integration. The resulting cross fertilization between functional analysis and the theory of measure and integration has been immensely beneficial to both disciplines.

In 1979, G.T. Rüttimann [59] launched a new discipline, *noncommutative measure theory*, which includes traditional measure theory as a special case, but extends the theory to measures on non-Boolean structures such as *orthomodular lattices* (OMLs) and *orthomodular posets* (OMPs) [38]. For instance, projection operators on a Hilbert space form an OML, and by a celebrated theorem of A.M. Gleason [20, 28] probability measures on this lattice correspond to von Neumann density operators on the Hilbert space. Already there has been considerable progress in extending major results from the conventional theory of measure, integration, and probability to noncommutative measure theory [10, 17, 18, 19, 20, 34].

The connection between traditional measure theory and partially ordered vector spaces does not survive the transition to noncommutative measure theory. But, divested of scalars, a partially ordered vector space becomes an additive partially ordered abelian group, and this is the key for placing noncommutative measure theory in an algebraic context. Indeed , the connection between noncommutative measure theory and partially ordered abelian groups is completely analogous to the usual measure theory/ordered-vector-space connection. This is the basis of what we propose to call *algebraic measure theory* (as opposed to traditional functional-analytic measure theory).

A noteworthy precedent for algebrizing functional analysis by "stripping partially ordered vector spaces of scalars" can be found in the work of Goodearl, Handelman, Shen, *et al.* on the application of the theory of partially ordered abelian groups with interpolation to the study of rings, C*-algebras, and Markov chains [29].

In algebraic measure theory a (possibly) non-Boolean measure-carrying structure $L$ is represented by the order interval $[0, u]$ in a partially ordered abelian group $G$ with order unit $u$. The group $G$ has a natural dual object $\Omega$ which forms a compact convex subset of the real vector space $\mathbb{R}^L$. Elements of $\Omega$ may be regarded as probability measures on $L$ and linear combinations thereof as signed measures on $L$. Contact is reestablished with functional analysis by passing to the partially ordered Banach space $Aff(\Omega)$ of affine continuous real-valued functions on $\Omega$.

The purpose of this paper is to present some of the basic ideas of algebraic measure theory. All proofs are omitted, but references to the literature indicate where most of the missing proofs can be found. The presentation is focused mainly on the classification of measure-carrying structures, the representation of these structures by groups, and the representation of measures by group homomorphisms.

**2. The Spectrum of Quantum Logics.** Noncommutative measure theory is motivated in part by probabilistic and statistical theories in physics. For instance, measures on a $\sigma$-field of subsets of phase space figure prominently in statistical mechanics, measures on an OML of projection operators arise in orthodox quantum mechanics, and measures on partial algebras of positive self-adjoint operators are employed in the contemporary theory of quantum measurement. The measure-carrying structures that arise in physical theories are known collectively as *quantum logics* (QLs), even if they are entirely classical [4, 7, 14, 15, 16, 37, 48, 52, 54, 55].

Quantum logics can be classified by the degree to which they behave like classical Boolean structures. This classification, starting with Boolean structures and progressing to more and more general structures, has been called the *spectrum of quantum logics* by R.J. Greechie. This is not the place for a detailed review of the historical development of QLs, but a brief sketch will help to persuade the reader that the introduction of successively more general structures has been dictated by considerations other than mere mathematical curiosity.

In *An Investigation of the Laws of Thought* [8], George Boole launched the study of the algebra that now bears his name. Of all Boolean algebras, the two-element algebra $\mathbf{2} = \{0, 1\}$ is not only the simplest, but it is the building block from which all the others are formed. Indeed, every Boolean algebra is a subdirect product of copies of $\mathbf{2}$. We take $\mathbf{2}$ as the starting point for our spectrum of QLs. Note that $\mathbf{2}$ can be regarded as the field of all subsets of a one-element set.

In 1933, Kolmogorov [41], guided by previous work of Fréchet, laid the formal mathematical foundations for classical probability theory. Following Fréchet, Kolmogorov took $\sigma$-fields ($\sigma$Fs) of sets as the carriers of probability measures. We take the transition $\mathbf{2} \rightarrow \sigma\mathrm{F}$ from the simple Boolean algebra $\mathbf{2}$ to the more general $\sigma$-fields as the first step in the formation of the spectrum of quantum logics.

Following the publication of Kolmogorov's *Grundbegriffe*, and heavily influenced by the publication of D. Maharam's remarkable classification of Boolean measure algebras [47], a theory of measures on general BAs began to take shape. In 1948 [42], Kolmogorov partially repudiated his own earlier work by suggesting that probability theory based on BAs eliminates some of the unsatisfactory aspects of $\sigma$F-based theory and fits well with Maharam's theory. Thus, since Boolean algebras are more general than $\sigma$-fields, we extend our QL-spectrum to $\mathbf{2}\to\sigma\mathrm{F}\to\mathrm{BA}$.

In 1936, Birkhoff and von Neumann [7] argued that propositions about a quantum-mechanical system cannot be represented by a BA. Although they realized that the projection lattice of an infinite-dimensional Hilbert space is not modular, they proposed a *modular orthocomplemented lattice* (MOL) as the logic of a quantum-mechanical system. Their insistence on modularity was in part due to von Neumann's desire for a probability measure invariant under the symmetry group of the logic—what he called an "a priori thermodynamic weight of states." As a BA is an MOL, the spectrum of quantum logics now becomes $\mathbf{2}\to\sigma\mathrm{F}\to\mathrm{BA}\to\mathrm{MOL}$. Much of von Neumann's subsequent work on continuous geometries [50] and rings of operators [51] was motivated by his search for a suitable MOL to serve as the logic of quantum mechanics and the carrier of quantum-mechanical probabilities.

In 1937, K. Husimi [36] observed that the projection lattice of a Hilbert space, finite or infinite dimensional, forms what is now called an *orthomodular lattice* (OML). By then, physicists were routinely using Hilbert-space-based quantum mechanics and, as every MOL is an OML, the quantum-logic spectrum took the form

$$\mathbf{2}\to\sigma\mathrm{F}\to\mathrm{BA}\to\mathrm{MOL}\to\mathrm{OML}.$$

In 1957, G. Mackey published an expository article on the mathematical foundations of quantum mechanics [46] based on lectures he was giving at Harvard. In this article, he referred to propositions about a physical system as *questions*. As early as 1962 [23], it was noticed that Mackey's questions naturally band together to form a structure called an *orthomodular poset* (OMP) which is more general than an OML. Thus, by 1962, the spectrum of quantum logics looked like this:

$$\mathbf{2}\to\sigma\mathrm{F}\to\mathrm{BA}\to\mathrm{MOL}\to\mathrm{OML}\to\mathrm{OMP}.$$

By the Stone representation theorem [64], there is a compact, Hausdorff, totally disconnected topological space $X$, called a *Stone space*, associated with a Boolean algebra $B$, and $B$ is isomorphic to the field of compact open subsets of $X$. If $B_1$ and $B_2$ are Boolean algebras with Stone spaces $X_1$ and $X_2$, respectively, then with the usual cartesian product topology, $X_1 \times X_2$ is again a Stone space, and the Boolean algebra of all compact open subsets of $X_1 \times X_2$ is the *tensor product* $B_1 \otimes B_2$ of $B_1$ and $B_2$.

Two measurable spaces $(X_1, \mathcal{M}_1)$ and $(X_2, \mathcal{M}_2)$ in traditional measure theory can be combined to form the measurable space $(X_1 \times X_2, \mathcal{M}_3)$, where $\mathcal{M}_3$ is the $\sigma$-field generated by all measurable rectangles $A \times B$, $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$. The Boolean algebra $\mathcal{M}_3$ is closely related to the tensor product of the Boolean algebras $\mathcal{M}_1$ and $\mathcal{M}_2$. This construction is exploited, for instance, in the usual approach to the study of joint probability distributions.

An analogous construction for the projection lattices of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ has long been used by physicists to represent coupled quantum-mechanical systems by forming the projection lattice of the Hilbert-space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ [4, 37].

In 1979, C.H. Randall and D.J. Foulis showed that it is impossible to extend the tensor product construction from Boolean algebras and Hilbert-space projection lattices to the full category of OMLs or OMPs [57]. The category of OMLs turns out to be too small to support tensor products; indeed, it is quite possible for a tensor product of OMLs to exist, but fail to be even an OMP. In [56] a larger category was proposed, the category of *orthoalgebras* (OAs), and it was shown that, at least for those OAs that admit a good supply of measures, a tensor product exists and is again an OA [5, 26, 40, 65]. In 1989, Rüttimann extended noncommutative measure theory to include the study of measures on OAs [60]. So, in the 1980s, the quantum-logic spectrum was enlarged to

$$\mathbf{2} \to \sigma\mathrm{F} \to \mathrm{BA} \to \mathrm{MOL} \to \mathrm{OML} \to \mathrm{OMP} \to \mathrm{OA}.$$

More or less in parallel with the development of noncommutative measure theory, people working on the foundations of quantum mechanics and the theory of measurement were studying a generalization of the notion of a spectral measure on a Hilbert space. Whereas a spectral measure takes its values in the OML of projection operators, the more general measure, called a *positive-operator-valued* (POV) measure, takes on values in the set $\mathbb{E}(\mathcal{H})$ of bounded, self-adjoint operators lying between the zero operator $\mathbf{0}$ and the identity operator **1** on a Hilbert space $\mathcal{H}$. Following G. Ludwig [45], the operators in $\mathbb{E}(\mathcal{H})$ are called *effect operators* on $\mathcal{H}$.

As early as 1973, A.S. Holevo used POV-measures in a paper on statistical decision theory for quantum systems [35]. Since then, POV-measures have been employed in the development of stochastic quantum mechanics[53, 63], quantum mechanics on phase space [61, 62], the theory of measurement [11], and in a host of other areas including quantum optics, quantum communication, and quantum information theory. The use of POV-measures enables the simultaneous measurement (with attendant unsharpness) of complementary quantum-mechanical observables [39].

In 1989, R. Giuntini and H. Greuling [27] abstracted some of the salient properties of the algebra $\mathbb{E}(\mathcal{H})$ of Hilbert-space effects to formulate a generalized orthoalgebra in which the classical law of noncontradiction, $p \wedge p' = 0$ could fail, thus allowing for the possibility of unsharp or fuzzy propositions. In 1994, F. Kôpka and F. Chovanec[43] and M.K. Bennett and D.J. Foulis [24] suggested alternative abstractions of $\mathbb{E}(\mathcal{H})$ that turned out to be mathematically equivalent to the generalized orthoalgebras of Giuntini and Greuling. These structures are now called *effect algebras* (EAs) [31]. Thus in the 1990s, the QL-spectrum was extended to

$$\mathbf{2} \to \sigma\text{F} \to \text{BA} \to \text{MOL} \to \text{OML} \to \text{OMP} \to \text{OA} \to \text{EA}.$$

The QL-spectrum shown above forms the "backbone" of a much richer hierarchy of quantum logics. The structural details of the entire QL-hierarchy are not fully worked out, but it certainly is not linearly ordered as is our QL-spectrum. Also, it remains to be seen whether EAs are the ultimate quantum-logical generalization of Boolean algebras. For instance, ideals in an EA are not necessarily EAs, rather they form what are called BCK-algebras [21], and it may again be necessary to extend the QL-hierarchy to include these even more general structures.

**3. Effect Algebras.** In the following definition, think of $L = \{p, q, r, ...\}$ as a collection of testable propositions about some system or situation. Think of $p \oplus q$ as the disjunction of two propositions $p$ and $q$ that are simultaneously testable but refute each other. Presumably, the unit proposition $u$ always tests true, and the zero proposition 0 always tests false.

**3.1 Definition.** An *effect algebra* is a system $(L, 0, u, \oplus)$ consisting of a set

$L$, special elements $0, u \in L$ called the *zero* and the *unit*, respectively, and a partially defined binary operation $\oplus$ on $L$, called the *orthosum*, such that, for all $p, q, r \in L$:

(i) If $p \oplus q$ and $(p \oplus q) \oplus r$ are defined, then $q \oplus r$ and $p \oplus (q \oplus r)$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

(ii) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(iii) For each $p \in L$, there is a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = u$.

(iv) If $p \oplus u$ is defined, then $p = 0$.

Axioms (i) and (ii) are the *associative* and *commutative* laws, respectively. Axiom (iii), called the *orthosupplementation law*, provides a sort of "negation" $q$ for each $p \in L$. Axiom (iv), a *weak consistency law*, is interpreted as the requirement that no proposition other than 0 refutes $u$. Following standard mathematical practice, we often refer to $L$, rather than to $(L, 0, u, \oplus)$, as an effect algebra. Also, if $p, q, r \in L$ and we write an equation such as $p \oplus q = r$, we understand it to mean that $p \oplus q$ is defined and $p \oplus q = r$.

**3.2 Definition.** Let $L$ be an effect algebra with unit $u$ and let $p, q, r \in L$.

(i) If $p \oplus q$ is defined, we say that $p$ and $q$ are *orthogonal*, in symbols $p \perp q$.

(ii) If $p \perp q$ and$(p \oplus q) \perp r$, we say that $p$, $q$, and $r$ are *jointly orthogonal* and we write $p \oplus q \oplus r := (p \oplus q) \oplus r = p \oplus (q \oplus r)$. (The notation $:=$ means *equals by definition.*)

(iii) $L$ is *orthocoherent* iff $p \perp q$, $p \perp r$, and $q \perp r \Rightarrow p$, $q$, and $r$ are jointly orthogonal.

(iv) $p$ and $q$ are *Mackey compatible* iff there exist jointly orthogonal elements $p_1, q_1, d \in L$ with $p = p_1 \oplus d$ and $q = q_1 \oplus d$.

(v) Define $p'$ to be the unique element in $L$ such that $p \perp p'$ and $p \oplus p' = u$. The mapping $' \colon L \to L$ is called the *orthosupplementation* and $p'$ is called the *orthosupplement* of $p$.

(vi) If there exists $r \in L$ with $p \perp r$ and $p \oplus r = q$, we say that $p$ is *less than or equal to* $q$, in symbols $p \leq q$.

Proofs of the following can be found in [6, 24, 32].

**3.3 Theorem.** *Let $L$ be an effect algebra with unit $u$ and let $p, q, r \in L$. Then:*

(i) *$\leq$ is a partial order on $L$ and $0 \leq p \leq u$.*

(ii) *$p \perp q \Leftrightarrow p \leq q'$.*

(iii) *$p = p''$ and $p \leq q \Leftrightarrow q' \leq p'$.*

(iv) *$p \leq q \Rightarrow q = p \oplus (p \oplus q')'$.*

(v) *If $p \perp r$ and $q \perp r$, then $p \oplus r \leq q \oplus r \Leftrightarrow p \leq q$.*

The next theorem shows how a Boolean algebra can be regarded as a special kind of effect algebra. Recall that a Boolean algebra can be defined as a complemented distributive lattice.

**3.4 Theorem.** *Let $B$ be a Boolean algebra with unit $u$. For $p, q \in B$, define $p \oplus q$ iff $p \wedge q = 0$, in which case $p \oplus q := p \vee q$. Then $(B, 0, u, \oplus)$ is an orthocoherent effect algebra in which every pair of elements is Mackey compatible. Conversely, if $B$ is an orthocoherent effect algebra in which every pair of elements is Mackey compatible, there is one and only one way to organize $B$ into a Boolean algebra such that the Boolean partial order coincides with the effect-algebra partial order.*

If we attribute an order-theoretic property to an effect algebra $L$, or to elements in $L$, we understand that it applies to the bounded partially ordered set $(L, 0, u, \leq)$. For instance, following standard order-theoretic usage, an element $a \in L$ is an *atom* iff $a \neq 0$ and, for all $b \in L$, $b \leq a \Rightarrow b = 0$ or $b = a$. Note that, if $p, q \in L$, then the infimum $p \wedge q$ and supremum $p \vee q$ may or may not exist in the partially ordered set $(L, \leq)$. If we write an equation such as $p \wedge q = r$, we mean that $p \wedge q$ exists in $(L, \leq)$ and it equals $r$. Likewise for $p \vee q = r$.

**3.5 Definition.** Let $L$ be an effect algebra.

(i) Elements $p, q \in L$ are *disjoint* iff $p \wedge q = 0$.

(ii) $L$ is *lattice ordered* iff $(L, \leq)$ is a lattice, i.e., $p \wedge q$ and $p \vee q$ exist for all $p, q \in L$.

(iii) $L$ is a *scale effect algebra*, or simply a *scale algebra*, iff it is totally ordered by $\leq$.

A Boolean algebra $B$, regarded as an effect algebra as in Theorem 3.4, is lattice ordered and two elements $p, q \in B$ are disjoint iff they are orthogonal. Evidently, all scale algebras are lattice ordered. An example of a lattice-ordered effect algebra in which disjoint elements are not necessarily orthogonal is as follows.

**3.6 Example.** Let $\mathcal{H} \neq \{0\}$ be a Hilbert space and let $\mathbb{P}(\mathcal{H})$ denote the set of all projection operators (i.e., bounded, self-adjoint, idempotent operators) on $\mathcal{H}$. Denote the zero and identity operators on $\mathcal{H}$ by $\mathbf{0}$ and $\mathbf{1}$, respectively. If $P, Q \in \mathbb{P}(\mathcal{H})$, define $P \oplus Q$ iff $PQ = \mathbf{0}$, in which case $P \oplus Q := P + Q$. Then $(\mathbb{P}(\mathcal{H}), \mathbf{0}, \mathbf{1}, \oplus)$ is a lattice-ordered effect algebra in which orthogonal elements are disjoint, but disjoint elements are not necessarily orthogonal. In fact, $\mathbb{P}(\mathcal{H})$ has the property that disjoint elements are orthogonal iff $\mathcal{H}$ is one dimensional and $\mathbb{P}(\mathcal{H})$ is isomorphic to $\mathbf{2}$. Two projections $P, Q \in \mathbb{P}(\mathcal{H})$ are Mackey compatible iff they commute as operators on $\mathcal{H}$, i.e., iff $PQ = QP$.

Here is an example of an effect algebra in which disjoint elements are orthogonal, but not vice versa.

**3.7 Example.** Let $[0, 1] \subseteq \mathbb{R}$ be the real unit interval. For $p, q \in [0, 1]$, define $p \oplus q$ iff $p + q \leq 1$, in which case $p \oplus q := p + q$. Then $([0, 1], 0, 1, \leq)$ is a scale effect algebra, called the *standard scale.* In the standard scale, two elements are disjoint iff at least one of them is zero; hence, disjoint elements are orthogonal, but not (in general) vice versa. Two elements of $[0, 1]$ are Mackey compatible iff they are orthogonal.

**3.8 Definition.** Let $L$ be an effect algebra and let $p_1, p_2, ..., p_n$ be a finite sequence of (not necessarily distinct) elements in $L$. By recursion, we say that $p_1, p_2, ..., p_n$ is a *jointly orthogonal* sequence and we define the *orthosum* $p_1 \oplus p_2 \oplus \cdots \oplus p_n \in L$ iff $p_1, p_2, ..., p_{n-1}$ is a jointly orthogonal sequence and $(p_1 \oplus p_2 \oplus \cdots \oplus p_{n-1}) \perp p_n$, in which case $p_1 \oplus p_2 \oplus \cdots \oplus p_n := (p_1 \oplus p_2 \oplus \cdots \oplus p_{n-1}) \oplus p_n$.

**3.9 Definition.** Let $L$ be an effect algebra and let $p \in L$. We define $0p := 0$ and $1p := p$. If $n$ is an integer greater than 1, let $p_i := p$ for $i = 1, 2, \ldots, n$. We say that *$np$ is defined* in $L$ iff the sequence $p_1, p_2, \ldots, p_n$ is jointly orthogonal, in which case $np := p_1 \oplus p_2 \oplus \cdots \oplus p_n$. In particular, we say that $p$ is *isotropic* iff $2p$ is defined, i.e., iff $p \perp p$. The effect algebra $L$ is *regular* iff any two isotropic elements are orthogonal.

In the standard scale algebra $[0, 1] \subseteq \mathbb{R}$ (Example 3.7), the isotropic elements are the elements $p \leq \frac{1}{2}$, whence $[0, 1]$ is regular. The system $\mathbb{E}(\mathcal{H})$ of all effect operators on the Hilbert space $\mathcal{H}$ is organized into an effect algebra—the prototypic effect algebra—as follows.

**3.10 Example.** Recall that $\mathbb{E}(\mathcal{H})$ consists of all bounded self-adjoint operators $A$ on the Hilbert space $\mathcal{H}$ such that $\mathbf{0} \leq A \leq \mathbf{1}$, where $\leq$ is the usual partial order on the set of bounded self-adjoint operators. If $A, B \in \mathbb{E}(\mathcal{H})$, define $A \oplus B$ iff $A + B \leq \mathbf{1}$, in which case $A \oplus B := A + B$. Then $(\mathbb{E}(\mathcal{H}), \mathbf{0}, \mathbf{1}, \oplus)$ is a regular effect algebra called the *standard effect algebra over $\mathcal{H}$*.

As expected, a *morphism* $\phi : L_1 \to L_2$ from the effect algebra $L_1$ to the effect algebra $L_2$ is defined by the requirements that $\phi$ maps the unit of $L_1$ to the unit of $L_2$ and preserves existing orthosums. An *isomorphism* is a bijective morphism $\phi$ such that $\phi^{-1}$ is also a morphism. An appropriate definition of a *$\sigma$-morphism*, generalizing the notion of a $\sigma$-homomorphism of Boolean algebras, can be found in [22, 34].

A subset $S \subseteq L$ of an effect algebra $L$ is a *subeffect algebra* of $L$ iff the zero and unit elements of $L$ belong to $S$, $S$ is closed under existing orthosums, and $S$ is closed under orthosupplementation. Evidently, a subeffect algebra $S$ of $L$ is again an effect algebra in its own right, and the effect-algebra partial order on $S$ is the restriction to $S$ of the effect-algebra partial order on $L$. For instance $\mathbb{P}(\mathcal{H})$ (Example 3.6) is a subeffect algebra of $\mathbb{E}(\mathcal{H})$ (Example 3.10) [32].

**3.11 Definition.** A *measure* on the effect algebra $L$ is a mapping $\mu : L \to \mathbb{R}$ such that, for all $p, q \in L$, $p \perp q \Rightarrow \mu(p \oplus q) = \mu(p) + \mu(q)$. If also $0 \leq \mu(p)$ for all $p \in L$, then $\mu$ is a *positive measure.* If $u$ is the unit of $L$, then a *probability measure* (or *state*) on $L$ is a positive measure $\omega$ on $L$ such that $\omega(u) = 1$.

Note that the measures in Definition 3.11 are *finitely additive* and $\mathbb{R}$-*valued.* Using techniques developed in [22, 34], it is possible to define and deal with $\sigma$-additive measures. Also, measures taking on values in the extended reals can be handled more or less as in traditional measure theory.

**3.12 Definition.** If $L$ is an effect algebra, then $\Omega(L)$ is the set of all probability measures on $L$ and $\Omega_\sigma(L)$ is the subset of $\Omega(L)$ consisting of the $\sigma$-additive probability measures on $L$.

If $\mu$ is a positive measure on the effect algebra $L$, and if $p, q \in L$ with $p \leq q$, then there exists $r \in L$ with $p \oplus r = q$, whence $\mu(p) + \mu(r) = \mu(q)$, and it follows that $\mu(p) \leq \mu(q)$. Thus, every positive measure is order preserving. Conversely, a requirement that positive measures determine the partial order of $L$ is the *sine qua non* of algebraic measure theory.

**3.13 Definition.** A subset $\Delta$ of $\Omega(L)$ is *full* (or *order determining*) iff, for all $p, q \in L$, the condition $\omega(p) \leq \omega(q)$ for all $\omega \in \Delta$ implies that $p \leq q$.

Any Boolean algebra carries a full set of probability measures, as does $\mathbb{P}(\mathcal{H})$ in Example 3.6, the standard scale $[0, 1] \subseteq \mathbb{R}$ in Example 3.7, and $\mathbb{E}(\mathcal{H})$ in Example 3.10.

**3.14 Lemma.** If $L$ is an effect algebra and $\Omega(L)$ is full, then $L$ is regular.

If $L$ is an effect algebra, then $\Omega(L)$ is a subset of the real vector space $\mathbb{R}^L$, and as such it is a convex set. It is usually understood that $\Omega(L)$ is topologized as a subset of $\mathbb{R}^L$ with the product topology. The set of extreme points of $\Omega(L)$, denoted by $\partial_e\Omega(L)$, plays an important role in algebraic measure theory and is understood to carry the relative topology inherited from $\Omega(L)$.

**4. Effect Algebras and the Spectrum of Quantum Logics.** In this section we characterize the effect algebras in the QL-spectrum

$$\mathbf{2}\rightarrow\sigma\text{F}\rightarrow\text{BA}\rightarrow\text{MOL}\rightarrow\text{OML}\rightarrow\ \text{OMP}\rightarrow\text{OA}\rightarrow\text{EA},$$

starting at the "top" and working downward. *For the remainder of this section, we assume that $L$ is an effect algebra with unit $u$.*

**4.1 Definition.** An element $p \in L$ is *sharp* iff $p \wedge p' = 0$.

If the elements of $L$ are regarded as propositions, the sharp elements are those that satisfy the classical *law of noncontradictiion.* (One has to be careful here as $p \wedge q$ does not always represent a logical conjunction of $p$ and $q$ [55].) In the standard effect algebra $\mathbb{E}(\mathcal{H})$ (Example 3.10), the sharp elements are the projections in the subeffect algebra $\mathbb{P}(\mathcal{H})$.

**4.2 Definition.** $L$ is an *orthoalgebra* (OA) iff every element in $L$ is sharp.

**4.3 Lemma.** The following conditions are mutually equivalent:

(i) $L$ is an OA.

(ii) There are no nonzero isotropic elements in $L$.

(iii) If $p, q \in L$ with $p \perp q$, then $p \oplus q$ is a minimal upper bound in $L$ for $p$ and $q$.

(iv) $p \in L \Rightarrow p \vee p' = u$.

Since the standard scale algebra $[0, 1] \subseteq \mathbb{R}$ and the standard effect algebra $\mathbb{E}(\mathcal{H})$ contain nonzero isotropic elements, they both fail to be OAs. Also, as the only isotropic element in an OA is 0, it follows that every OA is (trivially) regular.

**4.4 Definition.** An element $p \in L$ is *principal* iff, for all $q, r \in L$, $q \perp r$ and $q, r \leq p \Rightarrow q \oplus r \leq p$.

**4.5 Definition.** $L$ is an *orthomodular poset* (OMP) iff every element in $L$ is principal.

**4.6 Lemma.** Every principal element in $L$ is sharp; hence every OMP is an OA.

**4.7 Lemma.** The following conditions are mutually equivalent:

(i) $L$ is an OMP.

(ii) $L$ is an orthocoherent OA.

(iii) $L$ is an OA and every finite pairwise orthogonal sequence of elements in $L$ is jointly orthogonal.

(iv) If $p, q \in L$ with $p \perp q$, then $p \vee q = p \oplus q$.

**4.8 Example.** The *Wright triangle* $W_{14}$ is an effect algebra with fourteen elements, namely 0 and $u$, six atoms $a, b, c, d, e, f$, and six coatoms $a', b', c', d', e', f'$. It is determined by the conditions $a \oplus b \oplus c = c \oplus d \oplus e = e \oplus f \oplus a = u$ and their consequences, e.g., $b \oplus c = a'$. The Wright triangle is an OA and it carries a full set $\Omega(W_{14})$ of probability measures, but it is not an OMP.

**4.9 Definition.** An *orthomodular lattice* (OML) is a lattice-ordered OMP.

**4.10 Lemma.** $L$ is an OML iff $L$ is a lattice-ordered OA.

The projection lattice $\mathbb{P}(\mathcal{H})$ of a Hilbert space, and more generally the projection lattice of a von Neumann algebra, provide examples of OMLs. Here is an example of an OMP that is not an OML.

**4.11 Example.** Let $\mathcal{Q}[0,1]$ be the set of all Borel subsets $M$ of the unit interval $[0,1] \subseteq \mathbb{R}$ such that the Lebesgue measure of $M$ is a rational number. If $M, N \in \mathcal{Q}[0,1]$, define $M \oplus N$ iff $M \cap N = \emptyset$, in which case $M \oplus N := M \cup N$. Then $(\mathcal{Q}[0,1], \emptyset, [0,1], \oplus)$ is an OMP, it carries a full set $\Omega(\mathcal{Q}[0,1])$ of probability measures, but it is not an OML. Two elements $M, N \in \mathcal{Q}[0,1]$ are Mackey compatible iff $M \cap N \in \mathcal{Q}[0,1]$.

Beneath the OMLs in our QL-spectrum are the BAs. By Theorem 3.4, the BAs can be characterized as the orthocoherent effect algebras in which every pair of elements is Mackey compatible. There are OAs that are not orthocoherent, but in which every pair of elements is Mackey compatible. The set $\mathcal{B}[0,1]$ of all Borel subsets of $[0,1] \subseteq \mathbb{R}$ forms a Boolean algebra and $\mathcal{Q}[0,1]$ is a subeffect algebra of $\mathcal{B}[0,1]$, whence a subeffect algebra of a Boolean algebra need not be a Boolean algebra, nor even an OML.

The relations among the Boolean algebras $\mathbf{2}\to\sigma\mathrm{F}\to\mathrm{BA}$ at the beginning of the QL-spectrum are well known and, apart from calling attention to the theorem of Loomis and Sikorski showing that a $\sigma$-complete BA is the quotient of a $\sigma$F by a $\sigma$-ideal [44], there is no need to review them here.

**5. The Spectrum of Fuzzy Logics.** As soon as one drops beneath the "top" of the QL-spectrum, one is dealing with effect algebras that satisfy the law of noncontradiction, $p \wedge p' = 0$. Violation of this law, i.e., the existence of nonsharp elements, is the defining characteristic of *fuzzy* or *unsharp* logics, so a second "spectrum" opens up beneath the EAs—the *spectrum of* (possibly) *fuzzy logics* (FLs). Until now, the influence of this second spectrum on the progress of noncommutative measure theory has been limited, but there are indications that it will have a stronger role to play in the future. For instance, the range of a POV measure, consisting of so-called *coexistent effects* [11], appears to be an important type of fuzzy logic.

As is the case with the QLs, the FLs actually form an elaborate and incompletely understood hierarchy, so the linearly-ordered *FL-spectrum* that we present here is only intended to help fix our ideas. Also, it will be more convenient to develop our FL-spectrum from the "top down." *In what follows, we maintain our assumption that $L$ is an effect algebra with unit $u$.*

If $P$ is a lattice and $p, q, r, s \in P$ with $p, q \le r, s$ (meaning $p \le r$, $p \le s$, $q \le r$, and $q \le s$), then $p \vee q \le r \wedge s$, and any element $t \in P$ with $p \vee q \le t \le r \wedge s$ satisfies $p, q \le t \le r, s$. Thus, a lattice, and in particular any lattice-ordered effect algebra, has the property in the following definition.

**5.1 Definition.** A partially ordered set $P$ has the *interpolation property* iff, for all $p, q, r, s \in P$, $p, q \le r, s \Rightarrow$ there exists $t \in P$ with $p, q \le t \le r, s$. An effect algebra $L$ with the interpolation property is called an *interpolation effect algebra* (IEA).

If $L$ is finite and has the interpolation property, then it is lattice ordered [6].

**5.2 Example.** Denote the cardinal number of a set $X$ by $^{\#}X$. Suppose $X$ is finite and $^{\#}X$ is even. Define $\mathcal{P}_{even}(X)$ to be the set of all subsets $M \subseteq X$ such that $^{\#}M$ is even. If $M, N \in \mathcal{P}_{even}(X)$, define $M \oplus N$ iff $M \cap N = \emptyset$, in which case $M \oplus N := M \cup N$. Then $(\mathcal{P}_{even}(X), \emptyset, X, \oplus)$ is an OMP with a full set $\Omega(\mathcal{P}_{even}(X))$ of probability measures, but if $^{\#}X \geq 6$, it is not lattice ordered, hence it fails to be an IEA.

**5.3 Definition.** $L$ has the *Riesz decomposition property* (RD), or is an *RD-effect algebra* iff, for all $p, q, r \in L$ with $p \perp q$ and $r \leq p \oplus q$, there exist $p_1, q_1 \in L$ such that $p_1 \leq p$, $q_1 \leq q$, and $r = p_1 \oplus q_1$.

If $L$ is an RD-effect algebra, then it has the interpolation property [6], so our FL-spectrum takes the initial form RD$\to$IEA$\to$EA. The simplest example of an IEA that is not RD is as follows.

**5.4 Example.** The *diamond* $D$ is the four-element effect algebra $D = \{0, u, a, b\}$ satisfying the relations $a \oplus a = b \oplus b = u$ and all consequences thereof. The diamond $D$ is lattice ordered; in fact, *as a lattice* it is isomorphic to the four-element Boolean algebra $\mathbf{2}^2$. (Thus two nonisomorphic effect algebras can be isomorphic as partially ordered sets.) As $D$ is lattice ordered, it is an IEA, and $b \leq u = a \oplus a$, but $b$ cannot be written as an orthosum of subelements of $a$, whence $D$ is not RD. Note that $\Omega(D)$ is not full.

**5.5 Lemma.** Let $L$ be an OA. Then:

(i) If $L$ is an IEA, then $L$ is an OMP.

(ii) If $L$ is RD, then $L$ is a BA.

In view of the fact that a Boolean algebra is an orthoalgebra with the Riesz decomposition property, Lemma 5.5 (ii) provides an alternative characterization of a BA as an OA with the RD property.

The next example shows that an RD-effect algebra need not be lattice ordered. Recall that an *antilattice* is a partially ordered set $P$ such that, for $p, q \in P$, $p \wedge q$ exists in $P$ iff $p \leq q$ or $q \leq p$. The only antilattices that are also lattices are the totally ordered sets.

**5.6 Example.** In the euclidean plane $\mathbb{R}^2$, let $U$ be the open unit square $U := \{(x,y) \in \mathbb{R}^2 \mid 0 < x, y < 1\}$ and let $S := U \cup \{(0,0),(1,1)\}$. If $p, q \in S$, define $p \oplus q$ iff $p+q \in S$, in which case $p \oplus q := p+q$. Then $(S,(0,0),(1,1),\oplus)$ is an RD-effect algebra, $\Omega(S)$ is full, $S$ is an antilattice, and $S$ is not a scale effect algebra.

In 1957, C.C.Chang introduced *MV-algebras* to serve as algebraic models for multivalued logics [12]. With a simplified set of axioms, MV-algebras were later employed by Mundici as an aid in the classification of AF C*-algebras [49]. By translating results of Chovanec and Kôpka [13] into the language of effect algebras, it can be shown that MV-algebras, as originally defined by Chang and Mundici, are mathematically equivalent to the MV-effect algebras defined as follows.

**5.7 Definition.** An *MV-effect algebra* is a lattice-ordered RD-effect algebra.

As a lattice, an MV-effect algebra is distributive [6, 12]. The diamond $D$ (Example 5.4) is a distributive lattice-ordered effect algebra that is not MV. The finite distributive lattice-ordered effect algebras were completely classified in [32], but the infinite case is still open.

Recall that $L$ is an OA iff orthogonal elements of $L$ are disjoint. For lattice-ordered effect algebras, we have the following result involving the converse condition.

**5.8 Lemma.** If $L$ is lattice ordered, then $L$ is MV iff disjoint elements of $L$ are orthogonal.

As every MV-effect algebra has the RD property, our FL-spectrum enlarges to

$$\text{MV}\rightarrow\text{RD}\rightarrow\text{IEA}\rightarrow\text{EA}.$$

If $P$ is a partially ordered set with a smallest element 0, then a mapping ${}^{\sim}\colon P \to P$ is a *pseudocomplementation* iff, for all $p, q \in P$, $p \wedge q = 0 \Leftrightarrow p \leq q^{\sim}$. A pseudocomplementation on $P$, if it exists, is unique.

**5.9 Definition.** $L$ is a *Heyting effect algebra* (HEA) iff $L$ is an MV-effect algebra and there is a pseudocomplementation ${}^{\sim}\colon L \to L$ such that $p^{\sim\prime} \leq p^{\sim\sim}$

for all $p \in L$.

Every scale algebra, and in particular the standard scale [0,1], is an HEA. Also, any finite totally ordered set can be organized in one and only one way into a scale algebra, hence an HEA.

If $P$ is a lattice with 0, then a *Heyting implication connective* on $P$ is a mapping $\supset : P \times P \to P$ such that, for all $p, q, r \in P$, $(p \wedge q) \leq r \Leftrightarrow p \leq (q \supset r)$. A Heyting implication connective $\supset$ on $P$, if it exists, is unique, and $p \mapsto p^{\sim} := (p \supset 0)$ is a pseudocomplementation on $P$. On a Boolean algebra, the material implication connective $(p \supset q) := p' \vee q$ is a Heyting implication connective. See [31] for a proof of the following.

**5.10 Theorem.** If $L$ is an HEA, then $L$ admits a Heyting implication connective given by

$$(p \supset q) := (p' \oplus (p \wedge q))'^{\sim} \vee q$$

for all $p, q \in L$.

**5.11 Example.** If $[0,1] \subseteq \mathbb{R}$ is the real unit interval, let $F$ be the set of all continuous functions $f : [0,1] \to [0,1]$. Define $z$ and $u$ in $F$ by $z(x) := 0$ and $u(x) := 1$ for all $x \in [0,1]$. If $f, g \in F$, define $f \oplus g$ iff $f(x) + g(x) \leq 1$ for all $x \in [0,1]$, in which case $(f \oplus g)(x) := f(x) + g(x)$ for all $x \in [0,1]$. Then $(F, z, u, \oplus)$ is an MV-effect algebra, but it is not an HEA.

With the annexation of HEAs to our FL-spectrum, we arrive at

$$\text{HEA}\to\text{MV}\to\text{RD}\to\text{IEA}\to\text{EA}.$$

Evidently, every BA is an HEA. Note that the standard scale $[0,1]$ is an HEA, but not a BA, so we can complete our FL-spectrum by annexing the initial part $\mathbf{2} \to \sigma\text{F}\to\text{BA}$. Hence, for the rather course classification of FLs under consideration here, our FL-spectrum takes the form

$$\mathbf{2} \to \sigma\text{F}\to\text{BA}\to \text{MV}\to\text{RD}\to\text{IEA}\to\text{EA}.$$

If $L$ is an FL, then an $\omega \in \Omega(L)$ is more likely to be regarded as an assignment of *truth values* on the standard scale $[0,1]$ rather than a probability model for propositions $p \in L$. This is especially true for $\omega \in \partial_e\Omega(L)$. For instance, if $B$ is a BA, then $\partial_e\Omega(B)$ is the set of all $\mathbf{2}$-valued measures on $B$, and $\omega \in \partial_e\Omega(B)$ can be regarded as an assignment of truth values $0 =$ false and $1 =$ true to the propositions in $B$.

**6. Measurable Functions, Observables, and Integrals.** In this section we give a very brief sketch indicating how the notions in the heading are handled in algebraic measure theory.

Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{B}(\mathbb{R})$ be the $\sigma$-field of real Borel sets. In traditional measure theory a function $f : X \to \mathbb{R}$ is (Borel) *measurable* iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}(\mathbb{R})$. Thus a measurable function $f$ induces a Boolean $\sigma$-homomorphism $\phi : \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ given by $\phi(E) := f^{-1}(E)$ for all $E \in \mathcal{B}(\mathbb{R})$. Both $\mathcal{B}(\mathbb{R})$ and $\mathcal{M}$ are $\sigma$Fs and $\phi : \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ is an effect-algebra $\sigma$-morphism. Furthermore, if $\mu : \mathcal{M} \to \mathbb{R}$ is an $\mathbb{R}$-valued $\sigma$-additive measure on $\mathcal{M}$, then the composition $\mu \circ \phi : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a $\sigma$-additive measure on $\mathcal{B}(\mathbb{R})$.

A POV measure $\Phi$ is perfectly analogous to the mapping $\phi : \mathcal{B}(\mathbb{R}) \to \mathcal{M}$ induced by a measurable function $f$, except that $\Phi$ takes on values in the standard effect algebra $\mathbb{E}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$. Specifically, a (*normalized*) *positive-operator-valued* (POV) *measure* over the Hilbert space $\mathcal{H}$ is a mapping $\Phi : \mathcal{B}(\mathbb{R}) \to \mathbb{E}(\mathcal{H})$ such that $\Phi(\mathbb{R}) = \mathbf{1}$ and, for every countable family $(E_i)$ of pairwise disjoint sets in $\mathcal{B}(\mathbb{R})$, $\Phi(\bigcup_i(E_i)) = \sum_i \Phi(E_i)$, with convergence in the strong operator topology. Consequently, a POV measure is the same thing as an effect algebra $\sigma$-morphism $\Phi : \mathcal{B}(\mathbb{R}) \to \mathbb{E}(\mathcal{H})$. To pursue the analogy, we observe that a bounded, self-adjoint, trace-class operator (i.e., a *von Neumann density operator*) $J$ on $\mathcal{H}$ determines a $\sigma$-additive $\mathbb{R}$-valued measure $\mu$ according to $\mu(A) := \mathrm{trace}(JA)$ for all $A \in \mathbb{E}(\mathcal{H})$, and the composition $\mu \circ \Phi : \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a $\sigma$-additive measure on $\mathcal{B}(\mathbb{R})$.

In the contemporary quantum theory of measurement, a POV measure $\Phi : \mathcal{B}(\mathbb{R}) \to \mathbb{E}(\mathcal{H})$ is called a (possibly *unsharp*) *observable*. (An observable $\Phi$ taking on values in the OML $\mathbb{P}(\mathcal{H}) \subseteq \mathbb{E}(\mathcal{H})$ is a *sharp observable*.) Thus, the following definition provides a simultaneous generalization of measurable function on the one hand and Hilbert-space observable on the other.

**6.1 Definition.** An *observable* for the effect algebra $L$ is an effect-algebra morphism $\alpha : \mathcal{B}(\mathbb{R}) \to L$. A $\sigma$-*observable* is an effect-algebra $\sigma$-morphism $\alpha : \mathcal{B}(\mathbb{R}) \to L$.

Let $\alpha : \mathcal{B}(\mathbb{R}) \to L$ be a $\sigma$-observable and let $\mu : L \to \mathbb{R}$ be a $\sigma$-additive $\mathbb{R}$-valued measure, so that $\mu \circ \alpha \in \Omega_\sigma(\mathcal{B}(\mathbb{R}))$ is a $\sigma$-additive $\mathbb{R}$-valued measure on the $\sigma$-field $\mathcal{B}(\mathbb{R})$ of real Borel sets. Then one can define the *integral* $\int \alpha d\alpha$ in the usual way in terms of the measure $\mu \circ \alpha$. For instance, suppose $\alpha$ has *bounded support* in the sense that there is a closed interval $[a, b] \subseteq \mathbb{R}$ such that, for all $E \in \mathcal{B}(\mathbb{R})$, $E \cap [a, b] = \emptyset \Rightarrow \alpha(E) = 0 \in L$. Then one can define $\int \alpha d\alpha := \lim \sum_i y_i(\mu \circ \alpha)(E_i)$ where $E_1, E_2, \ldots, E_n$ is a Borel partition of $[a, b]$, $y_i \in E_i$ and the limit is taken as the partition is refined and its norm goes to zero.

Suppose $\alpha : \mathcal{B}(\mathbb{R}) \to L$ is a $\sigma$-observable and let $\Delta \subseteq \Omega_\sigma(L)$ be a full convex set of $\sigma$-additive probability measures on $L$. Then $\omega \in \Delta \Rightarrow \omega \circ \alpha \in \Omega_\sigma(\mathcal{B}(\mathbb{R}))$, so we can and do define $\mathcal{A} : \Delta \to \Omega_\sigma(\mathcal{B}(\mathbb{R}))$ by $\mathcal{A}(\omega) := \omega \circ \alpha$ for all $\omega \in \Delta$. Then both $\Delta$ and $\Omega_\sigma(\mathcal{B}(\mathbb{R}))$ are convex sets and $\mathcal{A}$ is an affine mapping from $\Delta$ to $\Omega_\sigma(\mathcal{B}(\mathbb{R}))$. This provides the motivation for the following alternative definition of a $\sigma$-observable, featured in the work of Beltrametti and Bugajski [1, 2, 3, 9].

**6.2 Definition.** If $\Delta$ is a full convex subset of $\Omega_\sigma(L)$, then a $\Delta$-*observable* is an affine mapping $\mathcal{A} : \Delta \to \Omega_\sigma(\mathcal{B}(\mathbb{R}))$.

To deal with vector-valued observables or observables taking on values in a manifold, one replaces $\mathbb{R}$ in the discussion above by a vector space $V$ or a manifold $M$ and $\mathcal{B}(\mathbb{R})$ by an appropriate $\sigma$-field of subsets of $V$ or $M$.

**7. Effect Algebras and Partially Ordered Abelian Groups.** For proofs of the results in this section, see [6, 24, 25].

Let $G$ be an additively-written abelian group. A subset $G^+$ of $G$ is called a *positive cone* iff $G^+ + G^+ \subseteq G^+$ and $G^+ \cap (-G^+) = \{0\}$. A *partially ordered abelian group* is an abelian group $G$ together with a distinguished positive cone $G^+ \subseteq G$. Such a group is in fact partially ordered by the relation $\leq$ defined for $p, q \in G$ by $p \leq q \Leftrightarrow q - p \in G^+$; furthermore, the relation $\leq$ is *translation invariant* in the sense that, for all $p, q, r \in G$, $p \leq q \Rightarrow p + r \leq q + r$. Conversely, if the abelian group $G$ is partially ordered as a set by the translation invariant relation $\leq$, then $G^+ := \{g \in G \mid 0 \leq g\}$ is a positive cone in $G$ and the partial order $\leq$ coincides with the partial order determined by $G^+$.

If $G$ is a partially ordered abelian group and $H$ is a subgroup of $G$, then $H^+ := G^+ \cap H$ is a positive cone in $H$, called the *induced positive cone*. The partial order on $H$ determined by the induced positive cone is just the restriction to $H$ of the partial order on $G$. The system $\mathbb{R}$ of real numbers, regarded as an additive abelian group, is partially (and in fact totally) ordered by the usual positive cone $\mathbb{R}^+ := \{x^2 \mid x \in \mathbb{R}\}$. The additive group $\mathbb{Z}$ of integers is a subgroup of $\mathbb{R}$ and the induced positive cone $\mathbb{Z}^+ := \mathbb{R}^+ \cap \mathbb{Z}$ yields the usual (total) order on $\mathbb{Z}$.

**7.1 Definition.** Let $G$ be a partially ordered abelian group and let $u \in G^+$. Then the *interval* $G^+[0,u]$ is defined by $G^+[0,u] := \{p \in G \mid 0 \leq p \leq u\}$. If $p, q \in G^+[0,u]$, define $p \oplus q$ iff $p + q \leq u$, in which case $p \oplus q := p + q$.

**7.2 Theorem.** If $G$ is a partially ordered abelian group and $u \in G^+$, then $(G^+[0,u], 0, u, \oplus)$ is an effect algebra.

**7.3 Definition.** The effect algebra $L$ is called an *interval effect algebra* iff there is a partially ordered abelian group $G$ and an element $u \in G^+$ such that $L$ is isomorphic to $G^+[0,u]$.

Part (ii) of the following theorem shows that the interval effect algebras are the measure-carrying structures of primary importance in algebraic measure theory.

**7.4 Theorem.** Let $L$ be an effect algebra.

(i) If $L$ is a subeffect algebra of an interval effect algebra, then $L$ is an interval effect algebra.

(ii) If $\Omega(L)$ is full, then $L$ is an interval effect algebra.

(iii) If $L$ is an interval effect algebra, then $\Omega(L) \neq \emptyset$ and $\Omega(L)$ is a compact convex set.

As the next example shows, it is possible to realize an interval effect algebra in more than one way. In the example, $\mathbb{Z}_2 := \{0, 1\}$ is the additive group of integers modulo 2 and the cartesian product of two groups is understood to be a group with coordinatewise addition.

**7.5 Example.** Define the three partially ordered abelian groups $G$, $H$, and $K$ as follows:

(i) $G := \mathbb{Z} \times \mathbb{Z}$ and $G^+ := \mathbb{Z}^+ \times \mathbb{Z}^+$.

(ii) $H := \mathbb{Z}$ and $H^+ := \{3n + 4m \mid n, m \in \mathbb{Z}^+\}$.

(iii) $K := \mathbb{Z} \times \mathbb{Z}_2$ and $K^+ := \{(n, \alpha) \mid 0 \neq n \in \mathbb{Z}^+, \alpha \in \mathbb{Z}_2\} \cup \{(0, 0)\}$.

Then $G^+[(0, 0), (1, 1)]$, $H^+[0, 7]$, and $K^+[(0, 0), (2, 1)]$ are all isomorphic to the BA $\mathbf{2}^2$.

We now outline a construction that associates a canonical partially ordered abelian group $G$ and a distinguished element $u \in G^+$ with an interval effect algebra $L$ in such a way that $L$ is isomorphic to $G^+[0, u]$. The group $G$, which is called the *universal group* (or for short, the *unigroup*) for $L$, has the following property: for every other partially ordered abelian group $H$ in which $L$ can be realized (up to isomorphism) as an interval $H^+[0, v]$, there is an order-preserving group homomorphism $\phi^* \colon G \to H$ such that the restriction $\phi$ of $\phi^*$ to $G^+[0, u]$ is an effect-algebra isomorphism $\phi \colon G^+[0, u] \to H^+[0, v]$.

Since much of the construction of $G$ can be carried out even if $L$ is not an interval effect algebra, *we begin by assuming only that $L$ is an effect algebra with unit $u$.*

**7.6 Definition.** If $p_1, p_2, \ldots, p_n \in L$, $k_1, k_2, \ldots, k_n \in \mathbb{Z}^+$, $k_i p_i$ is defined for $i = 1, 2, \ldots, n$, and $k_1 p_1, k_2 p_2, \ldots, k_n p_n$ is a jointly orthogonal sequence in $L$, we define $\bigoplus_i k_i p_i := k_1 p_1 \oplus k_2 p_2 \oplus \cdots \oplus k_n p_n$ and refer to $\bigoplus_i k_i p_i$ as an *orthocombination* of $p_1, p_2, \ldots, p_n$ with *coefficients* $k_1, k_2, \ldots, k_n$.

**7.7 Definition.** A subset $A \subseteq L$ is a set of *orthogenerators* for $L$ iff $0 \notin A$ and every element $p \in L$ can be written as an orthocombination of a finite sequence $p_1, p_2, \ldots, p_n$ of elements in $A$.

If $L$ is finite, then the set $A$ of all atoms in $L$ is a set of orthogenerators for $L$. If $L$ is infinite and no other set of orthogenerators suggests itself, one can always take the set $A := L \setminus \{0\}$ as a set of orthogenerators. It can be shown that $L$ is finite iff it admits a finite set of orthogenerators. In what follows, *we assume that $A$ is a set of orthogenerators for $L$.*

**7.8 Definition.** Organize the set $\mathbb{Z}^A$ of all functions $k : A \to \mathbb{Z}$ into a partially ordered abelian group with pointwise operations and with positive cone $(\mathbb{Z}^+)^A := \{k \in \mathbb{Z}^A \mid a \in A \Rightarrow k(a) \in \mathbb{Z}^+\}$.

(i) If $k \in \mathbb{Z}^A$, then the *support* of $k$ is defined by $\mathrm{supp}(k) := \{a \in A \mid k(a) \neq 0\}$.

(ii) $\mathbb{Z}^{[A]}$ is the subgroup of $\mathbb{Z}^A$ consisting of all $k \in \mathbb{Z}^A$ such that $\mathrm{supp}(k)$ is finite.

(iii) If $0 \neq p \in L$, $0 \leq k \in \mathbb{Z}^{[A]}$, and $S := \mathrm{supp}(k)$, we write $p = \bigoplus_{a \in A} k(a)a$ iff the orthocombination $\bigoplus_{s \in S} k(s)s$ exists in $L$ and $p = \bigoplus_{s \in S} k(s)s$. Also, by definition, if $k = 0$, then $\bigoplus_{a \in A} k(a)a := 0$.

(iv) $T := \{t \in \mathbb{Z}^{[A]} \mid u = \bigoplus_{a \in A} t(a)a\}$.

**7.9 Definition.** Choose and fix $s \in T$.

(i) $T - s := \{t - s \mid t \in T\}$.

(ii) $< T - s >$ is the subgroup of $\mathbb{Z}^{[A]}$ generated by $T - s$.

(iii) $G$ is the quotient group given by $G := \mathbb{Z}^{[A]}/ < T - s >$.

(iv) $\xi : \mathbb{Z}^{[A]} \to G$ is the canonical group epimorphism with kernel $< T - s >$.

**7.10 Lemma.** $< T - s >$ is independent of the choice of $s \in T$ and there is a uniquely determined mapping $\gamma : L \to G$ such that, for $p \in L$ and $0 \leq k \in \mathbb{Z}^{[A]}$, $\gamma(p) = \xi(k) \Leftrightarrow p = \bigoplus_{a \in A} k(a)a$.

**7.11 Definition.** If $\gamma$ is the mapping in Lemma 7.10, the pair $(G, \gamma)$ is called a *universal group* of (or for) $L$.

It turns out that, in spite of the arbitrary choice of the set $A$ and the element $s \in T$, a universal group of $L$ is uniquely determined up to a group isomorphism; hence we shall refer to it as *the* universal group of $L$. There are finite OMLs having the group $G = \{0\}$ as their universal group, but these are of little interest for algebraic measure theory as they carry only the zero measure. *In what follows, we assume that* $(G, \gamma)$ *is the universal group of* $L$.

**7.12 Definition.** If $K$ is an abelian group, then a *$K$-valued measure* on $L$ is a mapping $\phi\colon L \to K$ such that, for $p, q \in L$, $p \perp q \Rightarrow \phi(p \oplus q) = \phi(p) + \phi(q)$. If $K$ is a partially ordered abelian group, then a $K$-valued measure $\phi\colon L \to K$ is *positive* iff $\phi(L) \subseteq K^+$.

Note that a measure on $L$ (Definition 3.11) is the same thing as an $\mathbb{R}$-valued measure on $L$.

**7.13 Lemma.** $\gamma : L \to G$ is a $G$-valued measure on $L$ and $\gamma(L)$ generates the group $G$.

The next theorem shows that the $K$-valued measures on $L$ correspond uniquely to group homomorphisms from $G$ to $K$. (This is the *universal property* of $(G, \gamma)$).

**7.14 Theorem.** If $K$ is an abelian group and $\phi : L \to K$ is a $K$-valued measure on $L$, there is a uniquely determined group homomorphism $\phi^*\colon G \to K$ such that $\phi = \phi^* \circ \gamma$. Furthermore, the correspondence $\phi \leftrightarrow \phi^*$ between $K$-valued measures $\phi$ on $L$ and group homomorphisms $\phi^* : G \to K$ is a bijection.

As shown by the following theorem, the universal group of an interval effect algebra is especially well behaved.

**7.15 Theorem.** $L$ is an interval effect algebra iff $G$ can be organized into a partially ordered abelian group in such a way that $\gamma\colon L \to G^+[0, \gamma(u)]$ is an effect-algebra isomorphism. Furthermore, if $L$ is an interval effect algebra, then:

(i) $G = G^+ - G^+$, so $G^+$ generates $G$.

(ii) Every element $g \in G^+$ has the form $g = \sum_{i=1}^{n} g_i$ with $g_i \in G^+[0, \gamma(u)]$.

(iii) If $K$ is an abelian group and $\phi : G^+[0, \gamma(u)] \to K$ is a $K$-valued measure, then $\phi$ can be extended uniquely to a group homomorphism $\phi^* : G \to K$.

**8. Unigroups.** If $L$ is an interval effect algebra and $(G, \gamma)$ is its universal group, then we can use the effect-algebra isomorphism $\gamma : L \to G^+[0, \gamma(u)]$ in Theorem 7.15 to *identify $L$ with the unit interval* $G^+[0, \gamma(u)]$. In so doing, of course, we identify the unit $u$ with $\gamma(u)$. This suggests the following definition, which introduces the fundamental measure-carrying objects for algebraic measure theory.

**8.1 Definition.** A *unigroup* is a partially ordered abelian group $G$ with a distinguished element $u \in G^+$ called the *unit* such that:

(i) $G = G^+ - G^+$.

(ii) Every element $g \in G^+$ has the form $g = \sum_{i=1}^{n} g_i$ with $g_i \in G^+[0, u]$ for $i = 1, 2, \ldots, n$.

(iii) If $K$ is an abelian group and $\phi : G^+[0, u] \to K$ is a $K$-valued measure, then $\phi$ can be extended to a group homomorphism $\phi^* : G \to K$.

For the unigroup $G$ with unit $u$, the interval effect algebra $G^+[0, u]$ is called the *unit interval.*

By Theorem 7.15, the unit intervals in unigroups are, up to isomorphism, the interval effect algebras. By Theorem 7.4, the unit interval in a unigroup always carries at least one probability measure and, conversely, if $\Omega(L)$ is full, then $L$ can be realized as the unit interval in a unigroup. Owing to parts (i) and (ii) of Definition 8.1, the extension $\phi^*$ of $\phi$ in part (iii) is uniquely determined by $\phi$.

**8.2 Example.** The system $\mathbb{R}$ of real numbers, regarded as a partially ordered abelian group under addition, forms a unigroup with unit 1, and its unit interval is the standard scale $[0, 1] = \mathbb{R}^+[0, 1]$.

If $G$ and $H$ are unigroups with units $u$ and $v$, respectively, then we understand that $\lambda : G \to H$ is a *unigroup morphism* iff $\lambda$ is a group homomorphism, $\lambda(G^+) \subseteq H^+$, and $\lambda(u) = v$. Evidently, a unigroup morphism is order-preserving.

**8.3 Theorem.** Let $G$ and $H$ be unigroups with units $u$ and $v$, respectively. Then, if $\lambda : G \to H$ is a unigroup morphism, the restriction of $\lambda$ to $G^+[0, u]$ is an effect-algebra morphism of $G^+[0, u]$ onto $H^+[0, v]$. Conversely, if $\alpha : G^+[0, u] \to H^+[0, v]$ is an effect-algebra morphism, then $\alpha$ extends uniquely to a unigroup morphism $\alpha^* : G \to H$.

Let $G$ be a unigroup with unit $u$ and let $\mu$ be an $\mathbb{R}$-valued measure on the unit interval $L := G^+[0, u]$ in $G$. By Definition 8.1 (iii) with $K := \mathbb{R}$, $\mu$ can be extended uniquely to a group homomorphism $\mu^* : G \to \mathbb{R}$. Conversely, the restriction to $L$ of any group homomorphism from $G$ to $\mathbb{R}$ is a measure on $L$. Consequently, *there is a bijective correspondence* $\mu \leftrightarrow \mu^*$ *between measures on* $L$ *and group homomorphisms from* $G$ *to* $\mathbb{R}$. Also, $\mu$ is positive iff $\mu^*(G^+) \subseteq \mathbb{R}^+$. In particular, a probability measure $\omega \in \Omega(L)$ is the same thing as a unigroup morphism $\omega : L \to [0, 1] = \mathbb{R}^+[0, 1]$, whence *the probability measures* $\omega \in \Omega(L)$ *are in bijective correspondence* $\omega \leftrightarrow \omega^*$ *under extension and restriction with unigroup morphisms* $\omega^* : G \to \mathbb{R}$.

What about traditional measure theory? For measures on a $\sigma$-field $\mathcal{M}$ or on a Boolean algebra $B$, one just has to determine the corresponding unigroups. Theorems 8.4 and 8.5 below show how this is done. In both cases, we understand that $\mathbb{Z}^X$ is the partially ordered abelian group under pointwise operations and relations of all functions $f : X \to \mathbb{Z}$ and that $\chi_M$ is the characteristic set function of $M$ for $M \subseteq X$.

**8.4 Theorem.** Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{F}(X, \mathcal{M})$ be the subgroup of $\mathbb{Z}^X$ consisting of the bounded $\mathcal{M}$-measurable functions. Then $\mathcal{F}(X, \mathcal{M})$ is a unigroup with unit $\chi_X$, and the $\sigma$-field $\mathcal{M}$ is isomorphic as a Boolean $\sigma$-algebra to the unit interval $\mathcal{F}(X, \mathcal{M})^+[0, \chi_X]$ under the mapping $M \mapsto \chi_M$ for all $M \in \mathcal{M}$.

If $B$ is a Boolean algebra and $X$ is the Stone space of $B$, then $B$ is isomorphic to the field of compact open subsets of $X$, hence the following theorem provides the unigroup for $B$.

**8.5 Theorem.** Let $X$ be a Stone space and let $\mathcal{F}(X)$ be the subgroup of $\mathbb{Z}^X$ consisting of the bounded functions $f$ such that $f^{-1}(n)$ is a compact open subset of $X$ for every $n \in \mathbb{Z}$. Then $\mathcal{F}(X)$ is a unigroup with unit $\chi_X$ and, as a BA, the field of compact open subsets of $X$ is isomorphic to the unit interval $\mathcal{F}(X)^+[0, \chi_X]$ under the mapping $M \mapsto \chi_M$.

Theorems 8.4 and 8.5 take care of the initial part $\mathbf{2} \to \sigma\mathrm{F} \to \mathrm{BA}$ of both the QL-spectrum and the FL-spectrum. To handle MV-algebras, we need the following standard notion from the theory of partially ordered abelian groups [29].

**8.6 Definition.** If $G$ is a partially ordered abelian group, then an element $u \in G^+$ is called an *order unit* iff, for every $g \in G$, there exists $n \in \mathbb{Z}^+$ such that $g \leq nu$.

**8.7 Lemma.** If $G$ is a unigroup with unit $u$, then $u$ is an order unit in $G$.

**8.8 Theorem (Mundici).** If $G$ is a lattice ordered abelian group and $u$ is an order unit in $G^+$, then $G$ is a unigroup with unit $u$ and $G^+[0, u]$ is an MV-algebra. Conversely, every MV-algebra is an interval effect algebra and its universal group is lattice ordered.

The next step up in the FL-spectrum takes us to the RD-effect algebras. See [58] for a proof of the following.

**8.9 Theorem (Ravindran).** If $G$ is a partially ordered abelian group with the interpolation property and $u$ is an order unit in $G^+$, then $G$ is a unigroup with unit $u$ and $G^+[0, u]$ is an RD-effect algebra. Conversely, every RD-effect algebra is an interval effect algebra and its universal group has the interpolation property.

By Theorem 10.17 in [29], Ravindran's theorem has the following corollary.

**8.10 Corollary.** If $L$ is an RD-effect algebra, then $\Omega(L)$ is a Choquet simplex.

There are lattice-ordered effect algebras, even finite OMLs, that carry no probability measures [30], hence that cannot be interval effect algebras by Theorem 7.4 (iii). Therefore, not every IEA is an interval effect algebra. It is not known how to give a perspicuous characterization of the class of unigroups that have IEAs as their unit intervals. The general question of characterizing the unigroups having OMLs, OMPs, or OAs as their unit intervals is also open.

If $L$ is a small finite effect algebra, then the universal group $(G, \gamma)$ can be calculated explicitly by standard algebraic techniques. As $L$ is finite, $G$ is finitely generated, so it has the form $G = \mathbb{Z}^r \times C$ where $r \in \mathbb{Z}^+$ and $C$ is a finite direct product of finite cyclic groups. The question of whether $L$ is an interval effect algebra can then be settled by direct calculation. Here is an example.

**8.11 Example.** The Wright triangle $W_{14}$ in Example 4.8 carries a full set $\Omega(W_{14})$ of probability measures, so it can be realized as the unit interval in a unigroup $G$ with unit $u$. For $W_{14}$, $G = \mathbb{Z}^4$, $u = (1, 1, 1, 1)$, and $G^+$ is the subcone of the standard positive cone $(\mathbb{Z}^+)^4$ consisting of all $(x, y, z, w)$ such that $w \leq x + y + z$. The atoms in $W_{14}$ are identified with vectors in $G^+[(0, 0, 0, 0), u]$ as follows: $a = (1, 0, 0, 0)$, $b = (0, 1, 0, 1)$, $c = (0, 0, 1, 0)$, $d = (1, 0, 0, 1)$, $e = (0, 1, 0, 0)$, and $f = (0, 0, 1, 1)$. Note that $a + b + c = c + d + e = e + f + a = u$. The compact convex set $\Omega(W_{14})$ is a three-dimensional polytope with five extreme points. Four of the extreme points correspond to the group homomorphisms obtained by projection of $G = \mathbb{Z}^4$ onto the four coordinates and the fifth corresponds to the group homomorphism given by $f(x, y, z, w) := \frac{1}{2}(x + y + z - w)$.

A *simplicial group* is a partially ordered abelian group of the form $\mathbb{Z}^n$ with the standard positive cone $(\mathbb{Z}^+)^n$. In Example 8.11, the positive cone $G^+$ is determined by the extreme points of $\Omega(W_{14})$, hence by Theorem 4.14 in [29], $G$ is archimedean and it is isomorphic (as a partially ordered abelian group) to a subgroup of a simplicial group. In fact, the mapping $(x, y, z, w) \mapsto (x, y, z, w, x + y + z - w)$ embeds $G$ in the simplicial group $\mathbb{Z}^5$. The characterization of those interval effect algebras $L$ having archimedean unigroups is an open question.

In the next and final section, we find the unigroup (Theorem 9.3) corresponding to the standard effect algebra $\mathbb{E}(\mathcal{H})$ of effect operators on a Hilbert space $\mathcal{H}$.

**9. Unigroups, Functional Analysis, and Statistical Physical Theories.** Here we bring our discussion full circle by indicating briefly how algebraic measure theory makes contact with functional analysis.

**9.1 Theorem.** Let $V$ be a partially ordered real vector space with positive cone $V^+$ and let $u \in V^+$ be an order unit. Then, considered as a partially ordered abelian group under addition with $V^+$ as positive cone, $V$ is a unigroup with unit $u$.

**9.2 Example.** Suppose that $G$ is a unigroup with unit $u$, $L = G^+[0, u]$, and $\Omega(L)$ is full. By Theorem 7.4 (iii), $\Omega(L)$ is a compact convex set. The real vector space $\mathrm{Aff}(\Omega(L))$ of all real-valued affine continuous functions on $\Omega(L)$ forms a partially ordered Banach space under the supremum norm, and the constant function $v(\omega) := 1$ for all $\omega \in \Omega(L)$ is an order unit in $\mathrm{Aff}(\Omega(L))$. Thus, by Theorem 9.1, $\mathrm{Aff}(\Omega(L))$ is a unigroup with unit $v$. Since $\Omega(L)$ is order determining, $L$ is embedded in $\mathrm{Aff}(\Omega(L))^+[0, v]$ by the evaluation mapping. Also, by Theorem 7.1 in [29], the evaluation mapping $\Omega(L) \to \Omega\mathrm{Aff}(\Omega(L))^+[0, v]$ is an affine homeomorphism.

In the approach to the foundations of quantum mechanics initiated by G. Ludwig [45], a *statistical physical theory* for a physical system $\mathcal{S}$ is represented by a pair $(W, V)$ consisting of a base-normed Banach space $W$ and an order-unit Banach space $\mathcal{V}$ with order unit $u$. It is assumed that $W$ and $V$ are in norm and order duality arising from a bilinear form $< \cdot, \cdot >: W \times W \to \mathbb{R}$. The order interval $V^+[0, u]$ is an effect algebra, and by Theorem 9.1 its unigroup is $V$, regarded as an additive abelian group. The *effects* $p \in V^+[0, u]$ represent elementary observables for the system $\mathcal{S}$. Under the duality determined by $< \cdot, \cdot >$, the cone base $\Delta$ of $W$ is an order-determining subset of $\Omega(V^+[0, u])$. Elements of $\Delta$ represent *states* of the physical system $\mathcal{S}$ and elements of $\partial_e \mathcal{S}$ correspond to the so-called *pure states*. If $\omega \in \Delta$ and $p \in V^+[0, u]$, then $< \omega, p >$ is interpreted as the probability that the effect $p$ will test "true" when the system is in state $\omega$. In view of Theorem 9.1, this approach fits nicely with the general program of algebraic measure theory. For a detailed up-to-date exposition of these ideas, see the papers of S. Bugajski, E. Beltrametti, *et al.* [2, 3, 4, 33]. For the connection with fuzzy probability theory, see [9].

The standard effect algebra $\mathbb{E}(\mathcal{H})$ over the Hilbert space $\mathcal{H}$ (Example 3.10) arises in the context of a statistical physical theory as follows: One takes $\mathcal{W}$ to be the Banach space under the trace norm of all bounded self-adjoint trace-class operators on $\mathcal{H}$ with the convex set of all von Neumann density operators on $\mathcal{H}$ as the base of the positive cone $\mathcal{W}^+$. The Banach space $\mathcal{V}$ under the uniform operator norm of all bounded self-adjoint operators on $\mathcal{H}$ is in duality with $\mathcal{W}$ according to $< A, B >:= \text{trace}(AB)$ for $A \in \mathcal{W}$, $B \in \mathcal{V}$. In fact, under this duality, $\mathcal{V}$ is the Banach dual of $\mathcal{W}$. The identity operator $\mathbf{1}$ on $\mathcal{H}$ is an order unit in $\mathcal{V}$, and $\mathbb{E}(\mathcal{H}) = \mathcal{V}^+[\mathbf{0}, \mathbf{1}]$. Thus, as a consequence of Theorem 9.1, we have the following:

**9.3 Theorem.** The unigroup for the standard effect algebra $\mathbb{E}(\mathcal{H})$ of all effect operators on the Hilbert space $\mathcal{H}$ is the partially ordered abelian group $\mathcal{V}$ under addition of all bounded self-adjoint operators on $\mathcal{H}$ with $\mathbf{1}$ as the unit and with the usual partial order for self-adjoint operators.

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