Practice Final Exam - MTH 5102 - Linear Algebra - Dr. Kanishka Perera - Fall 2024

Name: ______________________________

Each problem is worth 20 points. You may refer to your book/notes. Calculators and cell phones are not allowed. Throughout the exam $F = \mathbb{R}$ or $\mathbb{C}$ and all vector spaces are finite dimensional.

1. Let V be a vector space over F, let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle_W$, and let $T : V \to W$ be a linear transformation. Show that

$$\langle x, y \rangle_V = \langle T(x), T(y) \rangle_W \,, \quad x, y \in V$$

defines an inner product on V if and only if T is one-to-one.

2. Let V be an inner product space over F and let W be a subspace of V. Show that $(W^{\perp})^{\perp} = W$.

3. Let V be an inner product space over F and let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Show that

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \, \overline{\langle y, v_i \rangle}$$

for all $x, y \in V$.

4. Let V be an inner product space and let T be an invertible linear operator on V. Show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

5. Let $V = W \oplus W^{\perp}$ and let T be the projection on W along $W^{\perp}$. Show that $T^* = T$.