Math 5050 – Special Topics: Manifolds– Spring 2025

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Section 3: 1, 2, 3, 7, 8, 9

3.1. Tensor Product of covectors

Let $e_1, \ldots, e_n$ be a basis for a vector space V and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $g_{ij} \in \mathbb{R}^{n\times m}$ is an $n \times m$ matrix Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{i \le i,j,n} g_{ij} v^i w^j$$

for $v = \sum v^j e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and $\alpha^j, 1 \le i, j \le n$.

$$\begin{aligned}
\alpha^i(e_j) &= \delta_i^j && (1)\\
\alpha^i(v) &= \alpha^i\left(\sum_{j=1}^n v^j e_j\right)\\
&= \sum_{j=1}^n \alpha^i(v^j e_j) && \alpha^i \text{ is linear}\\
&= \sum_{j=1}^n v^j \alpha^i(e_j) && v^j \text{ is a scalar}\\
&= \sum_{j=1}^n v^j \delta_j^i = v^i && \text{apply (1)}\\
(\alpha^i \otimes \alpha^j)(v, w) &= \alpha^i(v)\alpha^j(w) = v^i w^j\\
\therefore \sum_{i \le i,j,n} g_{ij} v^i w^j &= \sum_{i \le i,j,n} g_{ij}\, (\alpha^i \otimes \alpha^j)(v, w)
\end{aligned}$$

3.2. Hyperplanes

(a) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n-1$. A linear subspace of V of dimesnion $n - 1$ is called a *hyperplane* in V.

$$\begin{aligned}
\dim V &= \dim \operatorname{range}(f) + \dim \ker(f)\\
\dim \ker(f) &= \dim V - \dim \operatorname{range}(f)\\
&= n - 1
\end{aligned}$$

(b) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

$$\begin{aligned}
\text{Let } v = (y + z) \in V &\text{ and } f(y) \in \operatorname{range}(f), z \in \ker(f)\\
u = (x + w) \in V &\text{ and } g(x) \in \operatorname{range}(g), z \in \ker(g)\\
g(v) = g(y) + g(z) &= g(y) \in \operatorname{range}(f)
\end{aligned}$$

3.3. A basis for k-tensors

Let V be a vector space of dimension n with basis $e_i, \ldots, e_n$. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis in $V^\vee$ Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \ldots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this show that $\dim L_k(V) = n^k$. (This problem generalizes Problem 3.1..)

We need to show three things:

(a) That $\operatorname{span}\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\} = L_k(V)$.

(b) that $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ is linearly independent.

(c) that this is independend of order.

- **Proving the Span** A proof by mathematical induction. Constructing a bilinear function from functionals is simply done with $(f \otimes g)(v, w) = f(v)g(w)$. Then

3.7. **Transformation rule for a wedge product of covectors**

Suppose two set so of covectors on a vector space V. $\beta^1, \ldots, \beta^k$ and $\gamma^i, \ldots, \gamma^k$, are related by

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^i,\ i = 1, \ldots, k$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.$$

$$\begin{aligned}
&\text{Let } \beta, \gamma \in \mathcal{M}_{n\times n}(V^\vee) \\
&\qquad \beta = \left[\,\beta^i\,\right] \text{ and } \beta(v_1, \ldots, v_k) = \left[\,\beta^i\,\right](v_1, \ldots, v_n) = \left[\,\beta^i(v_j)\,\right] \\
&\qquad \gamma = \left[\,\gamma^i\,\right] \text{ and } \gamma(v_1, \ldots, v_k) = \left[\,\gamma^i\,\right](v_1, \ldots, v_n) = \left[\,\gamma^i(v_j)\,\right] \\
&\qquad A = [a^i_j] \\
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \ldots, v_k) &= \det[\beta^i(v_j)] = \det \beta(v_1, \ldots, v_k) \\
(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \ldots, v_k) &= \det[\gamma^i(v_j)] = \det \gamma(v_1, \ldots, v_k)
\end{aligned}$$

we can see that

$$\begin{aligned}
\beta^i = \sum_{j=1}^{k} a^i_j \gamma^i &\implies \beta = A \cdot \gamma \text{ and } \beta(v_1, \ldots, v_k) = A \cdot \gamma(v_1, \ldots, v_k) \\
\det \beta &= \det(A \cdot \gamma) = \det A \cdot \det \gamma \\
\det \beta(v_1, \ldots, v_k) &= \det A \cdot \det \gamma(v_1, \ldots, v_k) \\
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \ldots, v_k) &= \det A(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \ldots, v_k) \\
\beta^1 \wedge \cdots \wedge \beta^k &= \det A(\gamma^1 \wedge \cdots \wedge \gamma^k)
\end{aligned}$$

3.8. **Transformation rule for k-covectors**

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a^i_j v_i,\ j = 1, \ldots, k,$$

for $k \times k$ matrix $A = [a^i_j]$. Show that

$$f(u_1, \ldots, u_k) = (\det A) f(v_1, \ldots, v_k).$$

$$\begin{aligned}
f(u_1, \ldots, u_k) &= f\left(\sum_{i_1=1}^{k} a^{i_1}_1 v_{i_1}, \sum_{i_2=1}^{k} a^{i_2}_2 v_{i_2}, \ldots, \sum_{i_k=1}^{k} a^{i_k}_k v_{i_k} \right) \\
&= \sum_{i_1=1}^{k} a^{i_1}_1 \sum_{i_2=1}^{k} a^{i_2}_2 \cdots \sum_{i_k=1}^{k} a^{i_k}_k f(v_{i_1}, v_{i_2}, \ldots, v_{i_k})
\end{aligned}$$

3.9. **Vanishing of a covector of top degree**

Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \ldots, e_n$ for V. then ω is the zero covector on V.