Practice Final

1. $p \in [1, \infty)$

$f_n \to f$ in $L^p(\mu)$: $\int_X |f_n - f|^p \, d\mu \to 0$

$g_n(x) \to g(x) \quad \forall x \in X$

(g_n) is bdd in $L^\infty(\mu)$: $\operatorname{ess\,sup}_{x \in X} |g_n(x)| \leq C \quad \forall n, \; C > 0$

Show: $f_n g_n \longrightarrow fg$ in $L^p(\mu)$: $\|f_n g_n - fg\|_p \longrightarrow 0$

$$\|f_n g_n - fg\|_p = \|(f_n - f)g_n + f(g_n - g)\|_p$$

$$\leq \|(f_n - f)g_n\|_p + \|f(g_n - g)\|_p$$

Claim: $\|(f_n - f)g_n\|_p \longrightarrow 0$

Pf: $\|(f_n - f)g_n\|_p^p = \int_X |(f_n - f)g_n|^p \, d\mu$

$\leq C^p \int_X |f_n - f|^p \, d\mu$

$\longrightarrow 0$

Claim: $\|f(g_n - g)\|_p \longrightarrow 0$

Pf: $\|f(g_n - g)\|_p^p = \int_X |f(g_n - g)|^p \, d\mu$

$f(g_n - g) \longrightarrow 0 \ \forall x \in X$ since $g_n(x) \longrightarrow g(x) \ \forall x \in X$

$|g(x)| \leq C$ a.e. since $|g_n(x)| \leq C$ a.e.

$|f(g_n - g)|^p \leq |f|(|g_n| + |g|)^p \leq (2C)^p |f|^p \in L^1(\mu)$ since $f \in L^p(\mu)$

So $\displaystyle\int_X |f(g_n - g)|^p \, d\mu \to 0$ by DCT.

2. $\|f\|_p^p = \int_X |f|^p \, d\mu = \int_X |f|^{(1-\lambda)r+\lambda s} \, d\mu$, where $\lambda = \frac{p-r}{s-r}$

$r < p < s$

$1-\lambda = 1 - \frac{p-r}{s-r}$

$= \frac{s-p}{s-r}$

$p = (1-\lambda)r + \lambda s$ for some $\lambda \in (0,1)$

$\lambda = \frac{p-r}{s-r}$

$\int_X gh \le \left(\int_X |g|^\tau\right)^{1/\tau} \left(\int_X |h|^\sigma\right)^{1/\sigma}, \quad \frac{1}{\tau}+\frac{1}{\sigma}=1$

$$= \int_X \underbrace{|f|^{(1-\lambda)r}}_{g} \underbrace{|f|^{\lambda s}}_{h} \, d\mu$$

$$\tau = \frac{1}{1-\lambda}, \quad \sigma = \frac{1}{\lambda} \qquad \left(\frac{1}{\tau} + \frac{1}{\sigma} = (1-\lambda) + \lambda = 1 \right)$$

$$\leq \left(\int_X |f|^{\cancel{(1-\lambda)} r \frac{1}{\cancel{1-\lambda}}} \, d\mu \right)^{1-\lambda} \left(\int_X |f|^{\cancel{\lambda} s \frac{1}{\cancel{\lambda}}} \, d\mu \right)^{\lambda}$$

$$= \left(\int_X |f|^r \, d\mu \right)^{(s-p)/(s-r)} \left(\int_X |f|^s \, d\mu \right)^{(p-r)/(s-r)}$$

$$\|f\|_p^p \le \|f\|_r^{r(s-p)/(s-r)} \, \|f\|_s^{s(p-r)/(s-r)} \le M^{r(s-p)/(s-r)} \, M^{s(p-r)/(s-r)} = M^p$$

$$r(s-p)/(s-r) + s(p-r)/(s-r) = p$$

This shows that if $f \in \mathcal{L}^r(\mu) \cap \mathcal{L}^s(\mu)$, then $f \in \mathcal{L}^p(\mu)$.

Let $M = \max\{\|f\|_r, \|f\|_s\}$. Then $\|f\|_r \le M$ and $\|f\|_s \le M$.

$$\|f\|_p^p \le M^p \Rightarrow \|f\|_p \le \max\{\|f\|_r, \|f\|_s\}$$

3. $\mu(X) = 1$, $f, g : X \to (0, \infty)$ measurable, $fg \geq 1$

<u>Show:</u> $\left(\int_X f \, d\mu\right)\left(\int_X g \, d\mu\right) \geq 1$

$$fg \geq 1 \Rightarrow f^{1/2} g^{1/2} \geq 1 \Rightarrow \int_X 1 \, d\mu \leq \int_X f^{1/2} g^{1/2} \, d\mu$$

$$1 = \mu(X) \leq \left(\int_X f \, d\mu\right)^{1/2} \left(\int_X g \, d\mu\right)^{1/2}$$

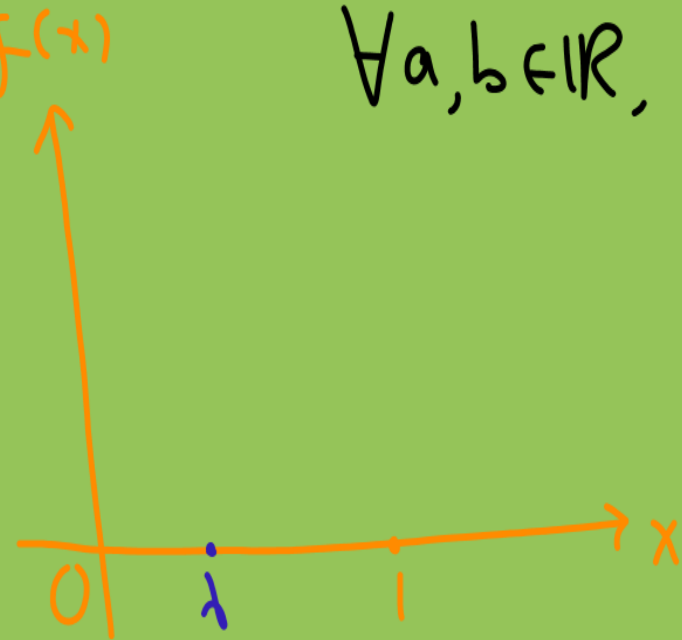
4. $$\varphi\left(\int_0^1 f(x)\,dx\right) \leq \int_0^1 \varphi(f(x))\,dx$$

__Show:__ φ is convex: $\varphi((1-\lambda)a+\lambda b) \leq (1-\lambda)\varphi(a)+\lambda\varphi(b)$

$\forall a,b\in\mathbb{R},\ \lambda\in[0,1]$

$$\int_0^1 f(x)\,dx \overset{?}{=} (1-\lambda)a+\lambda b$$

$$\int_0^\lambda f(x)\,dx + \boxed{\int_\lambda^1 f(x)\,dx} \overset{?}{=} \boxed{(1-\lambda)a} + \lambda b$$



$$\int_0^\lambda f(x)\,dx \overset{?}{=} \lambda b$$

$$\int_\lambda^1 f(x)\,dx = (1-\lambda)a$$

Take $f(x) = \begin{cases} b & \text{if } x \in [0,\lambda) \\ a & \text{if } x \in [\lambda, 1]. \end{cases}$

Then $\varphi\left(\int_0^1 f(x)\,dx\right) = \varphi\big((1-\lambda)a + \lambda b\big)$

and $\int_0^1 \varphi(f(x))\,dx = \int_0^\lambda \varphi(b)\,dx + \int_\lambda^1 \varphi(a)\,dx$

$= \lambda\varphi(b) + (1-\lambda)\varphi(a).$

p. 72–16. Prove Egoroff's thm: If $\mu(X) < \infty$, $f_n : X \to \mathbb{C}$ measurable, $f_n \to f$ a.e., then for any $\varepsilon > 0$, $\exists$ a measurable set $E \subset X$ st $\mu(X \setminus E) < \varepsilon$ and $f_n \to f$ uniformly on E.

<u>Hint:</u> $S(n,k) = \bigcap_{i,j>n} \{x \in X : |f_i(x) - f_j(x)| < \frac{1}{k}\}$

Show that $\exists$ an increasing seq (n_k) st we can take

$E = \bigcap_k S(n_k, k)$.

consider the case where $f_n(x) \to f(x)$ for all $x \in X$

First note that $S(n,k) \subset S(n+1,k)$. Next we and show that

$\bigcup_n S(n,k) = X$.