Math 725 – Advanced Linear Algebra

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Assignment #4 – Due 9/20/23

1. Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

a) If $\mathcal{B}$ is the standard ordered basis of $\mathbb{R}^3$ and $\mathcal{B}'$ is the standard ordered basis of $\mathbb{R}^2$, what is $[T]_{\mathcal{B}'}^{\mathcal{B}}$?

$$\mathcal{M}(T) = [T]_{\mathcal{B}'}^{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

b) If $\mathcal{B} = (v_1, v_2, v_3)$ and $\mathcal{B}' = (w_1, w_2)$ where

$$v_1 = (1, 0, -1),\ v_2 = (1, 1, 1),\ v_3 = (1, 0, 0), \quad w_1 = (0, 1),\ w_2 = (1, 0)$$

what is $[T]_{\mathcal{B}'}^{\mathcal{B}}$?

Let S be the transformation from the standard basis to $\mathcal{B}$ and S' be the transformation from the standard basis to $\mathcal{B}'$. The transformation $T_{\mathcal{B}'}^{\mathcal{B}}$ these bases is $T_{\mathcal{B}'}^{\mathcal{B}} = S' \circ T \circ S$.

$$\begin{aligned}
\mathcal{M}(S') &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\mathcal{M}(S) &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
[T]_{\mathcal{B}'}^{\mathcal{B}} &= \mathcal{M}(S')\mathcal{M}(T)\mathcal{M}(S) \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -3 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}
\end{aligned}$$

2. Let V be a n-dimensional vector space over F and let $\mathcal{B} = (v_1, \ldots, v_n)$ be a basis of V.

a) We have learned that there is a unique operator T on V such that $Tv_j = v_{j+1}$ for $j = 1, \ldots, n-1$, and $Tv_n = 0$. What is the matrix A of T in the basis $\mathcal{B}$?

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

b) Prove that $T^n = 0$ but $T^{n-1} \neq 0$.

$T(1, 0, \ldots, 0) = (0, 1, \ldots, 0)$ thus $v_1 \mapsto v_2$. Intuitively speaking, each composition of T onto itself will map v_1 to the next basis vector. But we know that $T(v_n) = 0$, thus, the $n-1$th iteration, i.e., A^{n-1} will map $v_1 \to v_n \neq 0$ and then the nth composition, i.e., A^n, will map everything to zero.

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$\vdots$$

$$A^{n-1} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 0 \end{pmatrix}$$

c) Let S be an operator on V such that $S^n = 0$ but $S^{n-1} \neq 0$. Prove that there is a basis $\mathcal{B}'$ such that $[S]_{\mathcal{B}'}$ is A of part **a)**.

Let $x = (1, 0, \ldots, 0) \in V$ and let $v_{n-1} = S^{n-1}(x)$, then let $v_{n-2} = S^{n-2}(x)$ and in general $v_i = S^i(x)$ with $v_n = x$. Notice $v_i \neq 0$ for all $i = 1, \ldots, n$ because $S(v_i) = S(0) \implies S(v_{i+1}) = 0$ indicating that all $v_j = 0$ for all $j > i$ and we know that $v_{n-1} = S^{n-1}(x) \neq 0$.
Claim: This set $\{v_1, \ldots, v_n\}$ forms a basis on V.

1) Linear Independence. Given any two elements v_i, v_j where $j > i$ we can see that if there exists non-zero elements a, b then $av_1 + bv_j \neq 0$ implies $v_i = dv_j$ for some d and thus $v_i^2 = d^2 v_j^2$ and $v_i^{n-j} = d^{n-j} v_j^{n-j} = 0$, which can't be true. Thus, any $a_1 v_1 + \cdots + a_n v_n = 0$ implies that $a_1, \ldots, a_n = 0$ and hence linearly independent.
2) $\text{span}\{v_1, \ldots, v_n\} = V$. There are n linearly independent vectors in a vector space of degree n. Hence, they span V.

d) Prove that if M and N are $n \times n$ matrices over F with $M^n = N^n = 0$ but $M^{n-1} \neq 0 \neq N^{n-1}$, then M and N are similar.

Let $T, S \in \mathcal{L}(V, V)$ such that $T(v) = Mv$ and $S(v) = Nv$. By 2c) there exists a basis $\mathcal{B}$ such that $[S]_{\mathcal{B}} = A$ and a basis $\mathcal{B}'$ such that $[T]_{\mathcal{B}'} = A$. Assuming that these basis are different from the standard bases, then let P be the change of basis matrix for $\mathcal{B}$ and P' be the change of basis from the standard to $\mathcal{B}'$. Thus,

$$\begin{aligned} [S]_{\mathcal{B}} &= PMP^{-1} \text{ and } [T]_{\mathcal{B}'} = P'NP'^{-1} \\ [S]_{\mathcal{B}} &= [T]_{\mathcal{B}'} \\ PMP^{-1} &= P'NP'^{-1} \\ M &= P^{-1}P'NP'^{-1}P \end{aligned}$$

P, P' are both invertible thus $P^{-1}P'$ and $P'^{-1}P$ are invertible. Hence, forming a change in basis matrix. M and N are similar.

3. Let U and V finite dimensional vector space and let $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$.

a) Prove that $\dim \operatorname{null}(ST) \leq \dim \operatorname{null}(S) + \dim \operatorname{null}(T)$.

By definition, the range of the $\operatorname{null}(T) = 0$ and we know that $S(0) = 0$ for all transformations S hence $0 \in \operatorname{null}(S)$. Hence $\operatorname{null}(T) \subseteq \operatorname{null}(S)$. $\operatorname{null}(ST)$ will include the members of $\operatorname{null}(T)$ and those elements of U that map to the $\operatorname{null}(S)$. Let $X = \{u \in U - \operatorname{null}(T) : T(u) \in \operatorname{null}(S)\}$ then , i.e., $\operatorname{null}(ST) = \operatorname{null}(T) \cap X$. $\dim X \leq \dim \operatorname{null}(S)$. Hence, $\dim \operatorname{null}(ST) \leq \dim \operatorname{null}(S) + \dim \operatorname{null}(T)$.

b) Now also assume that W is finite dimensional. Show that $\operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}$.

$$
\begin{aligned}
\dim(V) &= \dim \operatorname{null}(S) + \operatorname{rank}(S) \implies \dim \operatorname{null}(S) = V - \operatorname{rank}(S) \\
\dim(U) &= \dim \operatorname{null}(T) + \operatorname{rank}(T) = \dim(ST) + \operatorname{rank}(ST) \\
\operatorname{rank}(ST) &= \dim \operatorname{null}(T) + \operatorname{rank}(T) - \dim \operatorname{null}(ST) \\
&\leq \dim \operatorname{null}(T) + \operatorname{rank}(T) - \dim(S) - \dim \operatorname{null}(T) \\
&\leq \operatorname{rank}(T) - \dim \operatorname{null}(S) \\
&\leq \operatorname{rank}(T) - (\dim(V) - \operatorname{rank}(S)) \\
&\leq \operatorname{rank}(T) + \operatorname{rank}(S) - \dim(V) \\
\text{Clearly, } \operatorname{rank}(T) &\leq \dim(V) \text{ and } \operatorname{rank}(S) \leq \dim(V)
\end{aligned}
$$

Thus, if $\operatorname{rank}(T) = \dim(V)$ then $\operatorname{rank}(T) > \operatorname{rank}(S)$ and $\operatorname{rank}(ST) = \operatorname{rank}(S)$ and similarly, if $\operatorname{rank}(S) = \dim(V)$ then $\operatorname{rank}(S) > \operatorname{rank}(T)$ and $\operatorname{rank}(ST) = \operatorname{rank}(T)$. Thus, $\operatorname{rank}(ST) \leq \min\{\operatorname{rank}(S), \operatorname{rank}(T)\}$

c) If $R \in \mathcal{L}(U, V)$, then show that $\operatorname{rank}(T + R) \leq \operatorname{rank}(T) + \operatorname{rank}(R)$.

We know that if U, V are subspaces of W then $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$. We also know that $\operatorname{range}(T), \operatorname{range}(R)$ are subspaces under V thus $\dim \operatorname{range}(T + R) = \dim \operatorname{range}(T) + \dim \operatorname{range}(R) - \dim \operatorname{range}(T) \cap \operatorname{range}(R)$. Hence, $\operatorname{rank}(T + R) \leq \operatorname{rank}(T) + \operatorname{rank}(R)$ indeed they are equal when $\operatorname{range}(T) \cap \operatorname{range}(R) = 0$ vector.

4. Let T be a linear operator on a finite dimensional vector space V. Show that if there is an operator U with $TU = I$ then T is invertible and $T^{-1} = U$. Show that this statement may not be true for infinite dimensional vector spaces. [Hint: differentiation]

Let V be a finite-dimensional vector space, $T \in \mathcal{L}(V, V)$ and let $U \in \mathcal{L}(V, V)$ such that $TU = I$. Then given a basis $\mathcal{B}$ we have $T(v) = [T]_{\mathcal{B}}v$ and $U(V) = [U]_{\mathcal{B}}v$. Thus,

$$\begin{aligned}(TU)(v) = [T]_{\mathcal{B}}[U]_{\mathcal{B}}v &= [I]_{\mathcal{B}}v \\ [T]_{\mathcal{B}}[U]_{\mathcal{B}} &= [I]_{\mathcal{B}} \\ [T]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}} &= [T]_{\mathcal{B}}^{-1}[I]_{\mathcal{B}} \\ [U]_{\mathcal{B}} &= [T]_{\mathcal{B}}^{-1}\end{aligned}$$

since $\mathcal{B}$ is arbitrary, this is true with all possible bases, hence $U = T^{-1}$.

If V is infinite then U must be surjective and injective. Let $V = \mathcal{P}(F)$ for some field F, and $T = D$ the differentiation transformation, that is $T(f) = f'$. Let U be the antiderivative. Clearly, $UT = I$ except that $U(0) \neq \{0\}$ and, hence, not injective.

5. Prove that for any real θ the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is similar to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ over the complex numbers. [Hint: Let T be the linear operator on $\mathbb{C}^2$ represented in the standard ordered basis $\mathcal{B}$. Then find vectors v_1 and v_2 such that $Tv_1 = e^{i\theta}v_1$ and $Tv_2 = e^{-i\theta}v_2$, and (v_1, v_2) a basis.]

Let's find an invertible matrix P that will be our change of basis matrix.

$$\text{Let } P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, P^{-1} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$$

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}$$

Since we are looking at a change of basis matrix let's take a guess and make $a = d = 1$ Then,

$$\begin{aligned}
\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} &= \begin{pmatrix} 1 & b \\ c & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & -c \\ -b & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta + c\sin\theta & -\sin\theta + b\cos\theta \\ c\cos\theta + \sin\theta & -b\sin\theta + \cos\theta \end{pmatrix} \begin{pmatrix} 1 & -c \\ -b & 1 \end{pmatrix} \\
&= \begin{pmatrix} (\cos\theta + c\sin\theta) - c(\sin\theta + b\cos\theta) & -(\cos\theta + c\sin\theta) - (\sin\theta + b\cos\theta) \\ (c\cos\theta + \sin\theta) - b(-b\sin\theta + \cos\theta) & -c(c\cos\theta + c\sin\theta) - (b\sin\theta + \cos\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta + c\sin\theta & -(\cos\theta + c\sin\theta) - (\sin\theta + b\cos\theta) \\ (c\cos\theta + \sin\theta) - b(-b\sin\theta + \cos\theta) & -c^2\cos\theta + c^2\sin\theta - bc\sin\theta + c\cos\theta) \end{pmatrix}
\end{aligned}$$

From the first row and first column it seems pretty clear that $c = -i$. At a guess, $b = -i$ and we get

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{pmatrix}$$

which we know to be true. Therefore

$$P = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$