# Math 5111 – Real Analysis II– Sprint 2025
w/Professor Perera

Paul Carmody
Extra Credit – February 24, 2025

Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that if $f : X \to [0, \infty]$ is measurable then

$$\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f}{n}\right) d\mu = \int_X f d\mu$$

*(Hint: use the Dominated Convergence Theorem)*

$$\begin{aligned}
\text{recall } \log(1+y) &= y \text{ as } y \to 0 \\
\text{Let } h_n(x) &= n\log(1 + f(x)/n) \\
n\log(1 + f(x)/n) &\to n\frac{f(x)}{n} = f(x) \text{ as } n \to \infty \\
\lim_{n\to\infty} h_n(x) &= f(x)
\end{aligned}$$

further $h_n(x) \leq f(x)$ for all $n = 1, \ldots, \infty$. Thus, by the Dominate Convergence Theorem

$$\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f}{n}\right) d\mu = \int_X f d\mu$$