Advanced Linear Algebra

The Unforgetable Someone

Summer 2023

Contents

Chapter 1

Vector Spaces

1.1 Exercises

1. Suppose a and b are real numebers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c+di.$$

$$\begin{aligned}
\text{Let } a+bi = re^{i\theta} &= r(cos\theta + i\sin\theta) \text{ and } c+di = se^{i\phi} = s(\cos\phi + i\sin\phi) \\
1 &= (a+bi)(c+di) = (r(cos\theta + i\sin\theta))(s(\cos\phi + i\sin\phi)) \\
&= rs(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\
&= re^{i\theta}se^{i\phi} = rse^{i(\theta+\phi)} \implies \theta+\phi = 0 \text{ and } rs = 1 \\
\frac{d}{c^2+d^2} &= \frac{-b}{a^2+b^2} \text{ and } (a^2+b^2)(c^2+d^2) = 1 \implies c^2+d^2 = 1/(a^2+b^2) \\
d(a^2+b^2) &= \frac{b}{a^2+b^2} \\
d &= \frac{-b}{(a^2+b^2)^2} \text{ and } c = \frac{a}{(a^2+b^2)^2}
\end{aligned}$$

2. Show that

$$\frac{-1+1\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

$$\begin{aligned}
\text{Let } \omega = r(\cos\theta + i\sin\theta) &= \frac{-1+1\sqrt{3}i}{2} \\
\theta &= \frac{2\pi}{3} \text{ and } r = 1 \\
\omega^3 = (e^{\frac{2\pi}{3}i})^3 = e^{2\pi i} = 1
\end{aligned}$$

3. Find two distinct square roots of i.

$$\begin{aligned}
\text{Let } \omega &= a+bi \text{ and } \omega^2 = i \\
(a+bi)^2 = i &\implies (a^2-b^2) + 2abi = i \\
a^2 = b^2 &\implies a = \pm b \text{ and } 2ab = 1 \implies a = \pm\sqrt{2}/2 \\
\therefore \omega &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \text{ or } \omega = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}
\end{aligned}$$

4. Show that $\alpha+\beta = \beta+\alpha$ for all $\alpha, \beta, \in \mathbb{C}$.

Let $\alpha = a+bi, \beta = c+di$ for some $a, b, c, d \in \mathbb{R}$. Then,

$$\begin{aligned}
\alpha+\beta &= (a+bi) + (c+di) \\
&= a+c+ib+id = (c+di) + (a+bi) \\
&= \beta+\alpha
\end{aligned}$$

5. Show that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

6. Show that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

7. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ sucht aht $\alpha + \beta = 0$.

8. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

9. Show that $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ for all $\gamma, \alpha, \beta \in \mathbb{C}$.

10. Find $x \in \mathbb{R}^4$ such that
$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$$
Let $x = (a, b, c, d)$ then
$$\begin{aligned}
(4, -3, 1, 7) + 2(a, b, c, d) &= (5, 9, -6, 8) \\
(4 + 2a, -3 + 2b, 1 + 2c, 7 + 2d) &= (5, 9, -6, 8) \\
4 + 2a = 5 &\implies a = 1/2 \\
-3 + 2b = 9 &\implies b = 6 \\
1 + 2c = -6 &\implies c = -7/2 \\
7 + 2d = 8 &\implies d = 1/2
\end{aligned}$$

11. Explain why there does not exist $\gamma \in \mathbb{C}$ such that
$$\gamma(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$
these two points are linearly independent (i.e., a line going through one and the origin will not go through the other).

12. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

13. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b, \in \mathbf{F}^n$.

14. Show that $1x = x$ for all $x \in \mathbf{F}^n$

15. Show that $\gamma(x + y) = \gamma x + \gamma y$ for all $\gamma \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$

16. Show that $(a + b)x = ax + bx$ for all $a, b, \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

1.2 Exercises B

Definition 1.2.1. A ***Vector Space*** is commutative, associative, has an identity, has an additive inverse for each element, supports scalar multiplication (with a multiplicative identity), supports the distributive property.

Remark 1.2.2. Notation: $\mathbf{F}^S \equiv \{f : S \to \mathbf{F}\}$. *this is crappy notation for a textbook.* This implies that $\mathbb{R}^n \equiv \{\{1, \ldots, n\} \to \mathbb{R}\}$. This is counter intuitive when $\mathbb{R}^n$ is usually defined as an n-tuple. That is, $\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$. I have ot ask, who let him publish a book with an obviously confusing notation? Perhaps, I'll see some wisdom in this notation as we continue.

Be that as it may, since every $f \in \mathbf{F}^S$ generate values in F it is easy to see that F^S is a vector space. Thus, our intuitive notion of it representing an n-tuple is now expanded to a more abstract understanding.

1. Prove that $-(-v) = v$ for every $v \in V$.

   Additive Inverse Exists. Thus there exists $w \to v + w = 0$. Since it is closed under addition we may add the addtive inverse of w, $-w$ to both sides. This gives us $(v + w) - w = -2$. Since it is associative we can say $v + (w - w) = -w$ and $v = -w = -(-v)$ as $-v$ is another way of writing additive inverse of v.

2. Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

   if $a = 0$ we are done. if $a \neq 0$, let's assume that $v \neq 0$. then for any $w \in V$ we have $av + aw = aw$ or $v + w = w$ or $v = 0$.

3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

   let x, x' be distinct solutions $v + 3x = v + 3x' \to 3x = 3x' \to x = x'$

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements lisetd in 1.19. Which one?

   A vector space MUST have an additive identity, namely 0, which means that it cannot be empty.

5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replace with the condition that

$$0v = 0 \text{ for all } v \in V$$

   Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition meas that the collection of objcets satisfying the definiton is unchanged if the original condition is replace with the new condition.)

   This is equivalent to saying that the additive inverse is actually -1 times the element rather than notationally assigned $-v$.

6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multipication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specfically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} \infty & \text{if } t > 0, \\ 0 & \text{if } t = 0 \\ -\infty & \text{if } t < 0 \end{cases} \quad t(-\infty) = \begin{cases} -\infty & \text{if } t > 0, \\ 0 & \text{if } t = 0 \\ \infty & \text{if } t < 0 \end{cases}$$

$$t + \infty = \infty + t = \infty, t + (-\infty) = (-\infty) + t = -\infty$$

$$\infty + \infty = \infty, (-\infty) + (\infty) = -\infty, \infty + (-\infty) = 0$$

   Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

   Yes. It passes all rules and it even has an additive inverse for ∞.

1.3 Subspaces

1. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

   (a) $\Xi = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

   - Zero Must Exist: $(0, 0, 0) \to 0 + 2(0) + 3(0) = 0 + 0 + 0 = 0$. Yes.
   - Closed under Addition: Let $\alpha = (a, b, c), \beta = (d, e, f) \in \Xi$. then $\alpha + \beta = (a + d, b + e, c + f)$ and $(a + d) + 2(b + e) + 3(c + f) = a + d + 2b + 2e + 3c + 3f$ since this is also associative we have $(a + 2b + 3c) + (d + 2d + 3f) = 0 + 0 = 0$. Yes.
   - Closed under Scalar Multiplication: Let $\alpha = (a, b, c) \in \Xi$. then let $x \in \mathbf{F}$ be a scalar then $x(a+2b+3c) = x(0) = 0$. Yes.

   (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
   No. Not closed under addition.

   (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$;
   Not closed under addition.

   (d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$
   Yes

2. Verify all the assertionsi n Example 1.35.

   - if $b \in \mathbf{F}$, then

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

     is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.
   - The set of continuous real-valued function on the interval $[0.1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
     Let f, g be continuous real-valued functions on $[0, 1]$. So is $f + g$ and xf for any $x \in \mathbb{R}$.
   - The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
     Let f, g be differentialbe real-valued functions on $\mathbb{R}$ then so is $f + g$ and xf for all $x \in \mathbb{R}$.
   - The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
     let f, g be members of the set. then $(f + g)'(x) = f'(x) + g'(x)$ and in order for $f + g$ to be a member of the set then $f'(2) + g'(2) = 2b = b$ which is only true when $b = 0$.

- The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^\infty$
  Let $\{x_i\}, \{y_i\}$ be sequences that converge to zero. That is $\lim_{i\to\infty} x_i = \lim_{i\to\infty} y_i = 0$ then $\lim_{i\to\infty}(x_i + y_i) = \lim_{i\to\infty} x_i + \lim_{i\to\infty} y_i = 0$, Similarly for scalar multiplication.

3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

   Let check addition: $3(f+g)(2) = 3f(2) + 3g(2) = f'(-1) + g'(-1) = (f+g)'(-1)$ yes.
   Let's check scalar mulitplication $x3f(2) = xf'(-1)$, yes.

4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subpace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

5. if $\mathbb{R}^2$ is subspace of the complex vector space $\mathbb{C}^2$.

6. (a) Is $\{(a, b, c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

   No. Let $\alpha = (a, b, c), \beta = (d, e, f)$ be members of the set. Then, $\alpha + \beta = (a + d, b + e, c + f)$ which is not in the set because

$$
\begin{aligned}
0 &= (a+d)^3 - (b+e)^3 \\
&= a^3 + a^2d + 3ad^2 + d^3 - (b^3 + 3b^2e + 3be^2 + e^3) \\
&= 3a^2d + 3ad^2 - 3b^2e - 3be^2 \\
&= a^2d + ad^2 - b^2e - be^2 \\
&= ad(a+D) - be(b+e)
\end{aligned}
$$

   which must be true for all of $\mathbb{R}^3$ which it is not.

   (b) Is $\{(a, b, c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

7. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meanint $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

   Let $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ this does not contain zero.

8. Give an example of a nonempty subset of U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace fo $\mathbb{R}^2$.

   Let $U = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$ not closed under addition $(1, 1), (-2, 2) \in U$ but $(1 - 2, 1 + 2) = (-1, 3)$ and $(-1)^2 \neq 3^2$.

9. A function $f : \mathbb{R} \to \mathbb{R}$ is ***periodic*** if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functiosn from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^\mathbb{R}$? Explain.

   Let U be the set of periodic functions. Answer three questions:

   Does the set contain the additive identity?
   Let $f(x) \equiv 0$ for all $x \in \mathbb{R}$ then for any $p \in \mathbb{R}, f(x + p) = 0$. Yes.

   Is the set closed under addition?
   Let $f, g \in U$. Then, $f(x + p) = f(x), g(x + p) = x$ for all $x \in \mathbb{R}$. So, $(f + g)(x + p) = f(x + p) + g(x + p) = f(x) + g(x) = (f + g)(x)$. Yes.

   Is the set closed under scalar multiplication? Yes

10. Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

    Let $x, y \in U_1 \cap U_2$. then $x + y \in U_1$ and $x + y \in U_2$ therefore $x + y \in U_1 \cap U_2$. Also, $ax \in U_1 \cap U_2$ for all $a \in \mathbb{R}$ for the same reason.

11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

    Let $\{U_i\}$ be a collection of subspaces of a vector space V. Let $x, y \in \cap U_i$. $0 \in \cap U_i$, $x+y \in U_i$ for all $i \to x+y \in \cap U_i$ and $ax \in \cap U_i$.

12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

    Given A, B are subspaces of V. Choose $x, y \in A \cup B$ and assume that $x + y \in A \cup B$. This means that $x + y \in A$ or $x + y \in B$ or $x + y \in A \cap B$. The last case is handled in a different exercise. The first case implies that both $x, y \in A$ as A is a subspace under V and closed under addition. Since, $x, y \in A \cup B$, it must be the case that $x, y \in B$, hence $A \subset B$.

13. Prove that the union of three suspaces of V is a subspace of V if and only if one of the subpsaces contains the other two. *[This exercise is surprisingly harder than the previous exercsie, possibly because this exercise i not true if we replace* $\boldsymbol{F}$ *with a field containing only two elements.]*

    Given A, B, C subspaces of V. We know from previous exercises that if $A \cup B$ is a subspace the one contained in the other. So, let's assume that this is not the case. Then $A \cup B = \{0\}$ the additive identity and $A \cup B$ does NOT make a subspace. Assuming that $A \cup B \cup C$ is a subspace of V and letting $x, y \in A \cup B \cup C$ and neither x nor y is zero. if $x, y \notin A$ then $x + y \notin A$ and $x + y \in B \cup C$. Therefore $B \subset C$ or $C \subset B$. A similar argument can be made when $x, y, \notin B$ concluding that $A \subset C$ or $C \subset A$. Either $C = \{0\}$ so that it can be contained by both or C contains both subspaces.

14. Verify the assertion in Example 1.38.

    1.38 Suppose that $U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ and $W = \{(x, x, x, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$ then

    $$U + W = \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\}$$

    the question really is the range of each coordinate. the range of the first two coordinate in each subspace are the same and can be assigned the variable x. The range of the third coordinate is $x + y$ which is independent of x but has the same range and thus can be designated y. The range of the fourth coordiante is independent also thus cannot be the same as the other two and must be designated z.

15. Suppose U is a subspace of V. What is $U + U$?

    Any element of $U + U$ is made up of elements $x, y \in U$ having the form $x + y$. But, $x + y \in U$ therefore $U + U = U$.

16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

    Yes

17. Is the operation of addition on the subspaces of V associative? In other words, in U_1, U_2, U_3 are subspaces of V, is

    $$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

    Yes

18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have the additive inverses?

    0 is a member of all subspaces and $0 + 0$ is a member of all added subspaces.
    Let U, V be subspaces of W. $U + V = 0$ can only happen when all of V cancels out all of U in all possible combinations. That is for every $u \in U$ every $v \in V$ but be $u + v = 0$. We know that there is a unique additive inverse for each element of a subspace. Therefore, 0 is the only subspace with and additive inverse.

19. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

    $$U_1 + W = U_2 + W,$$

    then $U_1 = U_2$.

    Let $x \in U_1 + W$ and $x = u_1 + w$ for some $u_1 \in U_1$ and $w_1 \in W$. then $u_1 + w_1 \in U_2 + W$. Therefore, there exists $u_2 \in U_2$ and $w_2 \in W$ such that $u_2 + w_2 = u_1 + w_1$.
    Counter example: $W = \{(x, 0)\}, U_1 = \{(x, x)\}, U_2 = \{(0, y)\}$

20. Suppose

    $$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\},$$

    Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

    Let $w = (a, b, c, d) \in U \oplus W$ and $w \notin U$. Then, $a \neq b$ and/or $c \neq d$. A simple representation is $W = \{(0, b, 0, d) : c, d \in \mathbf{F}\}$

21. Suppose

    $$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$$

    Find a subspace W of $\mathbf{F}^5 = U \oplus W$.

22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 \,:\, x, y \in \mathbf{F}\}$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$ none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Let $x = (a, b, c, d, e) \in \mathbf{F}^5$ and $x \notin U$, that is $c \neq a + b, d \neq x - y, e \neq 2x$. simply let $W_1 = \{(0, 0, c, 0, 0)\}, W_2 = \{(0, 0, 0, d, 0)\}$ and $W_3 = \{(0, 0, 0, 0, e)\}$.

23. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W.$$

then $U_1 = U_2$.

Let $x = V$ then $x = u_1 + w_1$ and $x = u_2 + w_2$ where $w \in W, u_1 \in U_1$ and $u_2 \in U_2$. Since, these are direct sums, there exists only one way for x to be determined in both cases, thus $w_1 = w_2$. Therefore, $u_1 + w_1 = u_2 + w_2$ implies that $u_1 = u_2$.

24. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. a function $f : \mathbb{R} \to \mathbb{R}$ is called ***odd*** if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even fucntions on $\mathbb{R}$ and U_o denote the set of real-valued odd fucntions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Given a $F \in \mathbb{R}^{\mathbb{R}}$. Then define $f, g \in \mathbb{R}^{\mathbb{R}}$ as

$$f(x) = \begin{cases} F(x) & \text{if } F(x) = F(-x) \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} F(x) & \text{if } F(-x) = -F(x) \\ 0 & \text{otherwise} \end{cases}$$

clearly f is even and g is odd and $F = f + g$.

Chapter 2

Finite Dimensional Vector Spaces

2.1 Span and Linear Independence Exercises

1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Let $u_1 = v_1 - v_2, u_2 = v_2 - v_3, u_3 = v_3 - v_4, u_4 = v_4$.
Which can also be written as $v_1 = u_1 + u_2 + u_3 + u_4, v_2 = u_2 + u_3 + u_4, v_3 = u_3 + u_4, v_4 = u_4$.
let $x \in V$ then $x = av_1 + bv_2 + cv_3 + dv_4$
Given any $x \in V$ we have

$$\begin{aligned} x &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\ &= a_1(u_1 + u_2 + u_3 + u_4) + a_2(u_2 + u_3 + u_4) + a_3(u_3 + u_4) + a_4(u_4) \\ &= (a_1)u_1 + (a_1 + a_2)u_2 + (a_1 + a_2 + a_3)u_3 + (a_1 + a_2 + a_3 + a_4)u_4 \end{aligned}$$

which is a linear combination of our list of vectors and is true for all $x \in V$. All of these equations are reversible indicating that the span$(u_1, u_2, u_3, u_4) \subseteq V$ hence it is equal.

2. Verify the assertions in Example 2.18.

   - a list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.
     For v to be linearly independent $av = 0$ if and only if $a = 0$. if $v = 0$ then a can be anything.
   - A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.
     Vectors u, v are linearly independent implies that $au + bv = 0$ is only true when $a = b = 0$. Let $a, b \neq 0$ then $au = -bv$ or $u = \frac{-b}{a}v$ which means that they are scalar multiples of each other.
   - $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ is linearly independent in $\mathbf{F}^4$.
     $a(1, 0, 0, 0) + b(0, 1, 0, 0) + c(0, 0, 1, 0) = (a, b, c, 0)$ this can only be (0,0,0,0) when $a = b = c = 0$. Hence, linearly independent.
   - The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathbf{F})$ for each nonegative integer m.
     Given any two integers i, j there is no scalar a such that $z^i = az^j$. Thus, any linear combination of $1, z, \ldots, z^m$, i.e., polynomial, will equal zero only when all of the coefficients are zero.

3. Find a number t such that

$$(3, 1, 4), (2. -3, 5), (5, 9, t)$$

is not linearly independent $\mathbb{R}^3$.

$= 4a + 5b + t = 0$ and $a - 3b + 9 = 0$ or $t = -4a - 5b$ and $a = 3b - 9$ $t = -4(3b - 9) - 5b = -17b + 36$. When $t = \frac{-36}{17}b$.

4. Verify the assertion in the second bullet point in Example 2.20.

   - The list $(2, 3, 1), (1, -1, 2)(7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

5. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1 + i, 1 - i)$ is linearly independent.
   $a(1 + i) + b(1 - i) = a + ai + b - bi$ This will equal zero when $a = -b$ is zero in $\mathbb{R}$ ignoring the complex case.

   (b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $(1 + i, 1 - i)$ is lineraly dependent.
   similarly from above, but $a = b = 0$ is the only solution

6. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2, v_3, v_3 - v_4, v_4$$

is also linearly independent.

7. Prove or give a conterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

8. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda m$ is linearly independent.

   $\lambda v_1 + \lambda v_2 + \cdots + \lambda v_m = \lambda(v_1 + v_2 + \cdots v_m) = 0$

9. Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V , then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

   $v_1 + w_1 + v_2 + w_2 + \cdots v_m + w_m = (v_1 + v_2 + \cdots v_m + w_1 + w_2 + \cdots + w_m = 0 + 0$

10. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w_m$ is linearly independent, then $w \in \text{span}(v_1, \ldots, v_m)$.

    $0 = v_1 + w + v_2 + w + \cdots + v_m + w = mw + v_1 + v_2 + \cdots + v_m$ which means that w is a linear combination of $v_1, v_2, \ldots, v_m$ and hence $w \in \text{span}(v_1, v_2, \ldots, v_m)$.

11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, \ldots, v_m).$$

12. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

    the degree of $\mathcal{P}_4(\mathbf{F})$ is 5. It needs 5 linearly independent vectors to span it.

13. Explay why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

    the degree of $\mathcal{P}_4(\mathbf{F})$ is 5. It needs 5 linearly independent vectors to span it.

14. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

    ($\Rightarrow$)Let there be the infinite list of linearly independent vectors $v1, v2, \ldots$ all in the vector space V. Given any $m \in \mathbb{Z}^+$ it is clear that $v_1, v_2, \ldots, v_m$ are linearly independent.
    ($\Leftarrow$)Assume that given any $m \in \mathbb{Z}^+$ there is a sequence of linearly independent vectors $U_m = (v_1, v_2, \ldots, v_m)$ in the finitely dimensional vector space V and $\text{span}(v_1, v_2, \ldots, v_m) = V$. Let $n > m$. Then, there exists $U_n = (u_1, u_2, \ldots, u_n)$ linearly independent vectors such that $\text{span}(U_n) = V$. Thus, $\text{span}(U_n) = \text{span}(U_m)$. Remove the m vectors from U_n that span U_m. Let w be one of the remaining vectors. w is linearly independent of all U_m and the removed vectors. Thus $w \notin \text{span}(U_m)$ which is a contradiction. therefore V is infinitely dimensional.

15. Prove the $\mathbf{F}^\infty$ is infinte-dimensional.

    Proof by contradiction. IF it were finitely dimensional, then we would have finite number of linearly independet vectors taht span the entire set. That isn't the case.

16. Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

    Suppose this is true. Then there exists a finite set of functions $f_1, \ldots, f_n$ such that $\text{span}(f_1, \ldots, f_n) = \mathbb{R}^{\{}[0, 1]]$. Then, given any $u \in [0, 1]$ there are n different values in the set $X = \{f_1(u), \ldots, f_n(u)\}$. Let y be such that $y \notin X$ and $y \in [0, 1]$. Define $g \in \mathbb{R}^{[0,1]}$ as $g(x) = y$. Clearly, $g(u) = y$ and $g \notin \text{span}(f_1, \ldots, f_n)$. A contradiction.

17. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

    There cannot be a linear equation which passes through the point (2,0) and be basis vector. All basis vectors have the property $p(0) = 0$.

2.2 Bases – Exercise

1. Find all vector spaces that have exactly one basis.

   Let x be a basis for U. Then $ax \in U$ for all $a \in \mathbf{F}$. This is true when $x = 0$ and $U = \{0\}$. This is also true when $\mathbf{F} = \mathbb{R}$ and $x = 1$ (Or any other member of $\mathbb{R}$). This is also true when $\mathbf{F} = \mathbb{C}$ and $x = i$ (or any complex constant) and we allow scalars to be members of $\mathbb{C}$.

2. (a) Let U be the subspace of $\mathbb{R}^5$ defined by

   $$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \,:\, x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

   Find a basis of U.

   Every $u = (x_1, x_2, x_3, x_4, x_5) \in U$ then we know that $u = (3x_2, x_2, 7x_4, x_4, x_5) = (3, 1, 0, 0, 0)x_2 + (0, 0, 7, 1, 0)x_4 + (0, 0, 0, 0, 1)x_5$ therefore $(2, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ forms a basis for U

   (b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$

   Given $u \in \mathbb{R}^5$ and $u \notin U$ then $x_1 \neq 3x_2$ or $x_3 \neq 7x_4$. Thus, $u = a(2, 1, 0, 0) + b(0, 0, 7, 1, 0 + c(0, 0, 0, 0, 1) + d(1, 0, 0, 0, 0) + e(0, 0, 1, 0, 0)$ where d and e are scalars of any value.

   (c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

   $W = \{(x_1, x_2, x_3, x_4, 0) \in \mathbb{R}^5 \,:\, x_1 \neq 3x_2 \text{ and } x_3 \neq 7x_4\} \cup \{0\}$

3. (a) Let U be a subspace of $\mathbb{C}^5$ defined by

   $$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 \,:\, 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

   Find a basis for U.

   (b) Extend the basis in part (a) to a basis of $\mathbb{C}^5$.

   (c) Find a subspace W of $\mathbb{C}^5$ such that $\mathbb{C}^5 = U \oplus W$.

4. Prove or disprove: there exists a bassis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

   Any basis for $\mathcal{P}_3(\mathbf{F})$ must have at least four vectors. However, each vector with an element for x^2 must also have a non-zero element for x^3. That is $(0, 0, 1, b)$ where $b \neq 0$. Thus, this set is equal to $\text{span}(\{(1, 0, 0, 0)(0, 1, 0, 0), (0, 0, 1, 1)\}$ which only has three elements. Disproven.

5. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

   $$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

   is also a basis of V.

   Four linearly independent vectors forms a basis for V.

   $$\begin{aligned} a(v_1 + v_2) + c(v_2 + v_3) + d(v_3 + v_4) + ev_4 &= av_1 + av_2 + c(v_2 + cv_3) + dv_3 + dv_4 + ev_4 \\ &= av_1 + (a + c)v_2 + (c + d)v_3 + (d + e)v_4 \end{aligned}$$

   v_1, v_2, v_3, v_4 are linearly independent. Ths sum can only equal zero when $a = 0$ which implies that $c = 0$ and $d = 0$ and $e = 0$. Which means that the vectors are linearly independent.

6. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

7. Suppose U and W are subspacs of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

   $$u_1, \ldots, u_m, w_a, \ldots, w_n$$

   is a basis of V.

2.3 Dimension – Exercises

1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

   Let $n = \dim U$. Then a basis for U will have n linearly independent vectors. U is a subspace of V hence the basis vectors are part of V. Since, $\dim V = n$ then any n linearly independent vectors will forma basis for V. Hence, the span(n-vectors)$= U = V$.

2. Show that the subspacs of $\mathbb{R}^2$ are precisely $\{0\}, \mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

   The subspaces of $\mathbb{R}^2$ must have dimension of 0, 1, or 2. $\dim\{0\} = 0, \dim \mathbb{R}^2 = 2$. Each line can written in the form $y = mx + b$ which is one-dimensional and is a vector space if and only if $b = 0$.

3. Show that the subspaces in $\mathbb{R}^3$ are precisely $\{0\}, \mathbb{R}^3$, and all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

4. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis for U.

   Notice that if $p \in U$ then p has the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ and $p(6) = 0$ implies that $0 = a_0 + a_1 6 + a_2 6^2 + a_3 6^3 + a_4 6^4$. If we look at polynomials of each degree we see that
   degree 0 implies $a_0 = 0$ or (0,0,0,0,0)
   degree 1 implies $0 = a_0 + 6a_1$ or (-6,1,0,0,0)
   degree 2 implies $0 = a_0 + 6a_1 + 36a_2$ or (-6,-5,1,0,0)
   degree 3 implies $0 = a_0 + 6a_1 + 36a_2 + 216a_3$ or (-36, -6, -4, 1, 0)
   degree 4 implies $(1, 6, 6^2, 6^3, 6^4)$
   We can find a vector that describes the degree 4 polynomials. This means that we have 4 vectors in a vector space of dimension 5.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$

   Add the vector (1,0,0,0,0) to the basis.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})(\mathbf{F})$.

   W is the set of constant polynomials.

   (d) Find a subspace W of $\mathcal{P}_4(\mathbf{F})(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F})(\mathbf{F}) = U \oplus W$

5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p''(6) = 0$. Find a basis of U.

   Clearly, polynomials of degree 0 and 1 are members in U which give sus (1,0,0,0,0) and (0,1,0,0,0). That leaves
   degree 2 implies $0 = a_2$ which means that (0,0,0,0,0)
   degree 3 implies $0 = a_2 + a_3x$ or (0,0,-6, 1,0) degree 4 implies $0 = a_2 + a_3x + a_4x^2$ or (0,0,-6,-5,1).

   (b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

   add the vector (0,0,1,0,0)

   (c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

6. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

7. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

8. (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : \int_{-1}^{1} p = 0\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

9. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$$

   if $w \notin \operatorname{span}(v_1, \ldots, v_m)$ then $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m$ and if $-w \in \{v_1, \ldots, v_m\}$ then $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) = m - 1$

10. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis for $\mathcal{P}_m(\mathbf{F})$.

11. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3, \dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

12. Supoose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

13. Suppose U and W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exists two vectors in $U \cap W$ wuch that neither of these vectors is a scalar multiple of the other.

14. Supose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m$$

15. Suppose V is finite-dimensiona, with $\dim V = n \geq 1$. Prove that there exists 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that
$$V = U_1 \oplus \cdots \oplus U_n 0$$

16. Supoose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and
$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

17. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finte-dimensional vector space, then
$$\begin{aligned} \dim(U_1 + U_2 + U_3) = &\dim U_1 + \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3) \end{aligned}$$
Prove this or give acounter example.

Chapter 3

Linear Maps

3.1 The Vector Space of Linear Maps – Exercises

1. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

$$\begin{aligned}
\text{Let } T(x, y, z) &= (2x - 4y + 3z + b, 6x + cxyz) \\
T(u, v, w) &= (2u - 4v + 3w + b, 6u + cuvw) \\
T((x, y, z) + (u, v, w)) &= T((x + u, y + v, z + w)) \\
&= (2(x + u) - 4(y + v) + 3(z + w) + b, 6(x + u) + c(x + u)(y + v)(z + w)) \\
&= (2x - 4y - 3z + b, 6x + c(xyz)) + (2u - 4v + 3w, 6u + cuvw) + (0, c(\dots)) \\
&= T(x, y, z) + T(u, v, w) \iff b = 0 \text{ and } c = 0
\end{aligned}$$

2. Suppose $b, c \in \mathbb{R}$. Define $T : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^2$ by

$$Tp = (3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)dx + c \sin p(0)).$$

Show that T is linear if and only if $b = c = 0$.

b depends on the term $bp(1)p(2)$ which becomes $b(p + q)(1)(p + q)(2) = b(p(1) + q(1))(p(2) + q(2))$ which is not linear unless $b = 0$ and cancels it out.
c depends on the term $c \sin(p(0))$ which becomes $c \sin((p + q)(0)) = c \sin((p(0) + q(0))$ which is not lineary unless $c = 0-$ and cancels it out.

3. Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{ij} \in \mathbf{F}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$ such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \dots, x_n) \in \mathbf{F}^n$.
[The exercise above shows that T has the form promised in the last item of Example 3.4.]

Given $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ and any vectors $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbf{F}^n$. Let $n = 1$ then $T(u) = (f_1(u_1), \dots, f_m(u_1))$ for some m functions $f_i : \mathbb{R} \to \mathbb{R}$. Each of these is of the form $f_i(x) = A_i x$. $T(u + v) = (\dots, f_i(u_i + v_i), \dots) = (\dots, f_i(u_i), \dots) + (\dots, f_i(v_i), \dots) = Tu + Tv$. Now assume that this is true for n (the A_i now translate to A_{ij}) it is easy to show that it is true for $n + 1$. Hence an inductive proof.

4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_,$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

T is a linear map and Tv_i are linearly independent. Then $aTv_1 + bTv_2 = 0$ implies $a = b = 0$ thus, $T(av_1 + bv_2) = 0$ implies $T(0) = 0$ or that $av_1 + bv_2 = 0$ only if $a = b = 0$.

5. Prove the assertion in 3.7. $\mathcal{L}(V, W)$ is a vector space.

Let $S, T \in \mathcal{L}(V, W)$ and $v \in V$ and $w \in W$. Then $(T + S)(v) = T(v) + S(v) \in W$ therefore $T + S \in \mathcal{L}(V, W)$. And $aT(v) \in W$ when $a \in \mathbf{F}$ hence $aT \in \mathcal{L}(V, W)$. therefore, $\mathcal{L}(V, W)$ is a vector space.

6. Prove the assertions in 3.9.

   Let $T_1 \in \mathcal{L}(V,U), T_2 \in \mathcal{L}(W,V), T_3 \in \mathcal{L}(X,W)$.
   Associative: Show that $(T_1T_2)T_3 = T_1(T_2T_3)$

   $(T_1T_2)T_3(x) = (T_1(T_2))T_3(x) = T_1(T_2(T_3(x))) = T_1(T_2T_3(x)) = T_1(T_2T_3)(x)$

   Identity: $IT = TI = T$, where $I(v) = v$

   $I(T(v)) = T(v)$ and $T(I(v)) = T(v)$

   Distributive

7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More Precisely, prove that if $\dim V = 1$ and T$\in \mathcal{L}(V,V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

   Let $T(0) = b$ for some value $b \in \mathbf{F}$. then $T(v-v) = T(v) - T(v) = 0$ therefore T must be a single term of the form $T(v) = \alpha v$ for some $\alpha \in \mathbf{F}$.

8. Give an example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

   $$\varphi(av) = a\varphi(v)$$

   for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

   $\varphi(x,y) = \sqrt[3]{x^3 + y^3}$

9. Given an example of a funtion $\varphi : \mathbb{C} \to \mathbb{C}$ such that

   $$\varphi(w+z) = \varphi(w) + \varphi(z)$$

   for all $w, z \in \mathbb{C}$ but φ is not linear.

10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U,W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

    $$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

    Prove that T is not a linear map of V.

    Choose $v, w \in V$ such that $v, w \notin U$ and $v + w = u \in U$. then $T(v+w) = T(v) + T(w) = 0$ but $T(u) = Su$ a contradiction.

11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that U is a subspace of V and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.

    Let X be a subspace of V such that $X \oplus U = V$. Let $R \in \mathcal{L}(X,W)$ and

    $$Tv = \begin{cases} Sv & \text{if } v \in U \\ Rv & \text{if } v \in X \end{cases}$$

12. Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V,W)$ is infinite-dimensional.

    Let $m = \dim \mathcal{L}(V,W)$. There, there exists a basis of linear maps $T_1, \ldots, T_m$ such that $\operatorname{span}(T_1, \ldots, T_m) = \mathcal{L}(V,W)$. Thus, $\dim T_1(v) + \cdots + \dim T_m(v) \leq m$. Thus, given any $w \in W$ w can be written with a finite sum of linearly independent vectors from T_i. But $\dim W = \infty$ a contradiction.

13. Suppose $v_1, \ldots, v_m$ is linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V,W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

14. Suppose V is finite-dimensional with $\dim V \geq 2$. Prove that there exist $S, T \in \mathcal{L}(V,V)$ such that $ST \neq TS$.

    Let $S = (y,x)$ and $T = (x,-y)$ then $S(T(x,y)) = (-y,x)$ and $T(S(x,y)) = (y,-x)$.

3.2 Null Space – Exercises

1. Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$;
   $T \in \mathcal{L}(\mathbb{R}^2, \mathcal{P}_4(\mathbf{F}))$ and $T(a,b) = ax^3 + bx^4$
2. Suppose V is a vector space and $S, T \in \mathcal{L}(V,V)$ are such that
   $$\operatorname{range} S \subset \operatorname{null} T$$
   Prove that $(ST)^2 = 0$
   Show that $STST = 0$. Basically, no matter what happens $T(S(v)) = 0$ so $S(\underline{T(S}(T(v))))$ which will give us zero.
3. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by
   $$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$
   (a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?
   (b) What property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?
4. Show that
   $$U = \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$$
   is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$
   $\dim \operatorname{null} T$ can take on the values 3, 4 an 5. which means that the $\dim \operatorname{range} T$ takes on the values 2, 1, and 0, respectively. For $\dim \operatorname{null} T = 3$ we could have two entirely disjoint linear maps, e.g., $S(a,b,c,d,e) = (a,b,0,0)$ and $T(a,b,c,d,e) = (0,0,c,d)$ These are both in U but added $S + T \notin U$. hence not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. Similarly, for $\dim \operatorname{null} T = 4$ and 5
5. Give an example of linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that
   $$\operatorname{range} T = \operatorname{null} T$$
   $T(a,b,c,d) = T(a,0,c,0)$
6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that
   $$\operatorname{range} T = \operatorname{null} T$$
   $1 \leq \dim \operatorname{range} T \leq 5$ and $1 \leq \dim \operatorname{null} T \leq 4$ and $\dim \operatorname{range} T + \dim \operatorname{null} T = 4$ thus if $\dim \operatorname{range} T = \dim \operatorname{null} T = 2.5$ which isn't possible.
7. Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not injective}\}$ is not a suspace of $\mathcal{L}(V,W)$
   Counter example, pick two linear maps from the set, one that cancels out the other when you add them leaving behind an injective linear map. Hence, not a subspace.

   $V, W = \mathbb{R}^2, T(x,y) = (x \mod 4, y), S(x,y) = (-x \mod 4, y)$ then $(T+S)(x,y) = (0, 2y) \notin \mathcal{L}(V,W)$
8. Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V,W)$.
   $T(x,y) = (x,0), S(x,y) = (0,y), (S+T)(x,y) = (x,y)$
9. Suppose $T \in \mathcal{L}(V,W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.
   Given any $u \in \operatorname{span} v_1, \ldots, v_n$ we have constants $c_1, \ldots, c_n$ such that $u = c_1 v_1 + \cdots + c_n v_n$. Then, $T(u) = T(c_1 v_1 + \cdots + c_n v_n)$
10. Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V,W)$. Prove that the list $Tv_1, \ldots, T_n$ spans range T.
    Let $w \in \operatorname{range}(T)$. Then, there exists an $x \in V$ such that $Tx = w$. Since $\operatorname{span}(v_1, \ldots, v_n) = V$ then $x \in \operatorname{span}(v_1, \ldots, v_n)$. Hence, $w \in \operatorname{span}(Tv_1, \ldots, Tv_n)$.
11. Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \cdots S_n$ makes sense. Prove that $S_1 S_2 \cdots S_{n-1}$ is injective.
12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V,W)$. Prove that there exist a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.
    Let $u, v \in \operatorname{range} T$. Then there must be $x, y \in U$ such that $Tx = u, Ty = v$. We know that $\operatorname{range} T$ is a subspace, therefore $u + v \in \operatorname{range} T$. Thus $Tx + Ty \in \operatorname{range} T$ and $Tx + Ty = T(x+y)$. Hence, $x + y \in U$. A similar argument holds for scalar multpilcation.

13. Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\text{null}\, T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 \; : \; x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Give any $(x, y) \in \mathbf{F}^2$ we can clearly find at least one point (actually a whole set of points) $(a, b, c, d) \in \mathbb{R}^4$ such that $(x, y) = (a - 5b, c - 7d)$. Setting $b = d = 0$ we can see that $(x, y) = (a, d)$ thus spanning $\mathbf{F}^2$

14. Suppose U is a 3-dimensional subspace of $\mathbb{R}^8$ and that T is a linear map from $\mathbb{R}^8$ to $\mathbb{R}^5$ such that $\text{null}\, T = U$. Prove that T is surjective.

Since $\dim V = \dim \text{null}\, T + \dim \text{range}\, T$ and $\dim \text{null}\, T = \dim U = 3$ then $8 - 3 = 5 = \dim \text{range}\, T$. Since, the $\dim W = \dim \mathbb{R}^5 = 5$ then T must be surjective.

15. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 \; : \; x_1 = x_2 \text{ and } x_3 = x_4 = x_5\}.$$

the $\text{null}\, T = \{(a, a, b, b, b) \; : \text{ for some } a, b\}$. Hence $\dim \text{null}\, T = 2$. Since $\dim V = 5$ then the $\dim \text{range}\, T$ would have to be 3 but $\dim W = 2$ hence no such T exists.

16. Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Let $T = \mathcal{L}(V, W)$ for some vector space W not necessarily finite. Since $\dim V = \dim \text{range}\, T + \dim \text{null}\, T$ then $\dim V$ must have some finite value hence be finite dimensional.

17. Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

18. Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

19. Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\text{null}\, T = U$ if and only if $\dim U \geq \dim V - \dim W$.

20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

T injective implies that for every $w \in \text{range}\, T$ there exists one and only one $v \in V$ such that $Tv = w$. So, let $S \in \mathcal{L}(W, V)$ defined as

$$S(w) = \begin{cases} v & \text{if } w \in \text{range}\, T \text{ and } Tv = w \\ 0 & \text{otherwise} \end{cases}$$

21. Suppose V is finite-dimensional and $G \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

T surjective on W means that for every $w \in W$ there exists at least one $v \in V$ such that $Tv = w$. Define $S \in \mathcal{L}(W, V)$ such that $Sw = v$ where $Tv = w$.

22. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null}\, ST \leq \dim \text{null}\, S + \dim \text{null}\, T.$$

23. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{range}\, ST \leq \min\{\dim \text{range}\, S, \dim \text{range}\, T\}.$$

24. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\text{null}\, T_1 \subset \text{null}\, T_2$ if and only if there exists $S \in \mathcal{L}(W, W)$ such that $T_2 = ST_1$.

25. Suppose V is finite-dimsnsional and $T_1, T_@ \in \mathcal{L}(V, W)$. Prove that $\text{range}\, T_1 \subset \text{range}\, T_2$ if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.

26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(R))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(R)$. Prove that D is surjective.

27. Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

28. Suppose $T \in \mathcal{L}(V, W)$, and $w_1, \ldots, w_m$ is a basis of range T. Prove that there exists $\varphi_1, \ldots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

29. Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in null φ. Prove that

$$V = \operatorname{null}\varphi \oplus \{au : a \in \mathbf{F}\}.$$

30. Suppose φ_1 and φ_2 are linear maps from V to $\mathbf{F}$ that have the same null space. Show that there exists a constant $c \in \mathbf{F}$ such that $\varphi_1 = c\varphi_2$.

31. Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

3.3 Matrices – Exercises

1. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least dim range T nonzero entries.

   Since $Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$ for each $k = 1, \ldots, n$ If any row, j, is filled with zeros then $Tv_j = 0$ and not in the range T.

2. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respet to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

   [*Compare the exercise above to Example 3.34. The next exerecise generalizes the exercise above.*

$$\begin{aligned} Tv_k &= \sum_{j=1}^{m} A_{k,j}w_j \\ Tv_1 &= A_{1,1}w_1 + A_{1,2}w_2 + A_{1,3}w_3 = w_1 \\ Tv_2 &= w_2 \\ Tv_3 &= w_3 \end{aligned}$$

   Thus, T is the identity mapping any basis to itself (except for w_4).

3. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

   Since,

$$Tv_k = \sum_{j=1}^{m} A_{k,j}w_k$$

   there can be only dim range T linearly indepdent vectors to make up the basis of range T.

4. Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppse $T \in \mathcal{L}(V, W)$, Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the basis $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$) are 0 except or possibly a 1 in the first row, first colum. [In this exercise, unlike Exercise 3, you are given the basis of V instead of being able to choose a basis of V.]

   Note that $Tv_1 = w_1\mathcal{M}(T)$ for some $w_1 \in W$. Let $\dim W = 2$. Since

$$Tv_1 = \sum_{j=1}^{m} A_{1,j}w_j$$

   We can pick w_1 such that $w_1 = Tv_1$ which is precisely the first column with a 1 followed by zeros. Now produce w_2 from Tv_2 and verify that this forms a basis for W. Now proceed with an inductive proof to show that is true for all dimensions of W.

5. Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all the entries in the first row of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 esxcept for possible a 1 in the first row, first column. [In this exercise, unlike Exercise 3, you are given the basis for W instead of being able to choose a basis of W.]

6. Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

7. Verify 3.36. "Suppose $S, T \in \mathcal{L}(V, W)$, Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$."

   From 3.7 $\mathcal{L}(V, W)$ is a vector space. Therefore, given any $x \in V$

$$\begin{aligned}
(T+S)(x) &= T(x) + S(x) \\
\mathcal{M}(T+S)(x) &= \mathcal{M}(T)x + \mathcal{M}(S)x \\
&= (\mathcal{M}(T) + \mathcal{M}(S))x \\
\mathcal{M}(T+S) &= \mathcal{M}(T) + \mathcal{M}(S)
\end{aligned}$$

8. Verity 3.38. "Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

9. Prove 3.52.

10. Suppose A is a m-by-n matrix and C is an n-by-p matrix. Prove that

$$(AC)_{j,.} = A_{j,.}C$$

    for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

11. Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and C is a n-by$p$ matrix. Prove that

$$aC = a_1 C_1 + \cdots + a_n C_n.$$

    In other words, show that aC is a linaear combination of rows of C, with the scalars that multiply the rows coming from a.

12. Give an example with 2-by-2 matrices to show that matrix multiplication is not commutative. In other words, find 2-by-2 matrices A adn C such that $AC \neq CA$.

13. Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Prove that $AB + AC$ and $DF + EF$ both make sense and that $A(B+C) = AB + AC$ and $(D+E)F = DF + EF$.

14. Prove that matrix multiplication is asociative. IN other words, suppose A, B, And C are matriecs whose Sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

15. Suppose A is an n-by-n matrix and $a \leq j, k \leq n$. Show that the entries in row j, column k, of A^3 (which is defiend by AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

3.4 Invertibility and Isomorphic Vector Spaces – Exercises

1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in (V, W)$ are both invertible linear maps.Prove taht $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = t * -1S * -1$.

2. Suppose V is finite-dimensional dn $\dim V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

3. Suppose V is finite-dimensional, U is asubspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $TGu = Su$ for eveyr $u \in U$ if and only if S is injective.

4. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \text{Ł}(w)$ such tha t$T_1 = ST_2$.

5. Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$ prove that $\operatorname{range} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in \text{Ł}(V)$ such that $T_1 = T_2 S$.

6. Suppose V and W are finite-dimensional dn $T_1, T_2 \in \text{Ł}(V, W)$. Prove that there exist invertible operators $F \in \text{Ł}(V)$ and $S \in \text{Ł}(W)$ such that $T_1 = sT_2 R$ if and only if $\dim \operatorname{null} T_1 = \dim \operatorname{null} T_2$.

7. Suppose v and W are finite-dimensiona. Let $v \in V$. Let

$$E = \{PT \in Ł(V, w) \, : \, Tv = 0\}.$$

   (a) Show that E is a subspace of $\mathcal{L}(V, W)$.

   (b) Suppose $v \neq 0$. What is $\dim E$.

8. Suppose V is finite-dimensional and $T : V \to W$ is a surjecitve linear map of V onto W. Porve that there is a subspace U of V such that $T|_U$ is an isomoprnhism of U onto W. (Here $T|_U$ means the function T restricted to U. in orehte rwords, $T|U$ is the function whos domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ fo reveyr $u \in U$.)

9. Suppose V is finite0dimensional and $S, T \in Ł(V)$. Prove that ST is invertible if and only if both S and T are invertible.

10. Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

11. Suppose V is finite-dimesnional and $S, T, U \in \mathcal{L}(V)$ and STU=I.$Showthat$T is invertible and that $T_{-1} = US$.

12. Show that the resutl in the previous exercis cna fail without the hyothesis that V is finite-dimensional.

13. Suppose V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that TST is surjective. Prove that S is injective.

14. Suppose $v_1, \ldots, v_n$ is abasis of V. Prove that the map $T : V \to \mathbf{F}^{n,1}$ define dby

$$Tv = \mathcal{M}(v)$$

   is an isomorphsim of V onto $\mathbf{F}^{n,1}$; here $\mathcal{M}(V)$ is the matrix of $v \in V$ with respect to the basis $v_a, \ldots, v_n$.

15. Prove tha tevery linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. IN other words, prove ath $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}m, 1$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbf{F}^{n,1}$.

16. Supoose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a sdcalar mutiple of th eidentity if and only if $ST = TS$ for eveyr $S \in \mathcal{L}(V)$.

17. Suppose V is finite dimensional dn $\mathcal{E}$ is a subspace of $\mathcal{L}(V)$ such that $ST \in \mathcal{E}$ and $TS \in \mathcal{E}$ for all $s \in \mathcal{L}(V)$ and all $T \in]mathcal E$. Prove that $\mathcal{E} = \{0\}$ or $\mathcal{E} = Ł(V)$.

18. Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

19. Suppose $T \in Ł(\mathcal{P}(R))$ is such that T is injective and $\deg Tp \leq \deg p$ for eveyr nonzero polynomial $p \in \mathcal{P}(\mathbb{R})$.

   (a) Profve that T is surjective.

   (b) Proe that $\deg Tp = \deg p$ for every nonzero $p \in \mathcal{P}(\mathbb{R})$.

20. Suppose n is a positive integer and $A_{a,j} \in \mathbf{F}$ for $i, j = 1, \ldots, n$. Prove that the following are equivalent (note that in both pars belo2w, the number of equations equals the number of variables):

   (a) The trivial soluion $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} A_n, k x_k = 0$$

   (b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to th esystem of equations

$$\sum_{k=1}^{n} A_{a,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = c_k.$$