Math 5110 – Real Analysis I– Fall 2024

w/Professor Liu

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Homework #3 – TBD: October 31, 2024

I. Let $\Omega \subset \mathbb{R}^m, a \in \Omega^o$. If $f : \Omega \to \mathbb{R}$ is continuous at $a, g : \Omega \to \mathbb{R}$ is differentiable at a and $g(a) = 0$, show that fg is differentiable at a. (Note fg is the function whose value at $x \in \Omega$ is $f(x)g(x)$).

g is differentiable at a means that there exists a transformation L such that

$$\begin{aligned} 0 &= \lim_{x\to a} \frac{g(x) - (g(a) - L(x-a))}{|x-a|} \\ &= \lim_{x\to a} \frac{g(x) + L(x-a)}{|x-a|} \\ &= L \end{aligned}$$

Let's look at the following

$$\begin{aligned} \lim_{x\to a} \frac{f(x)g(x) - (f(a)g(a) - L(x-a))}{|x-a|} &= \lim_{x\to a} \frac{f(x)g(x) + L(x-a)}{|x-a|} \\ &= \lim_{x\to a} \frac{f(x)g(x)}{|x-a|} + \lim_{x\to a} \frac{L(x-a)}{|x-a|} \\ &= 0 + L \end{aligned}$$

thus a transformation exists for $(fg)(a)$ that satisfies the definition for differentiation.

II. skip II

III. Find the total derivative (i.e., derivative matrices) of the following functions at the given points.

(a) $f(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ x_1 x_3^2 \\ x_1 + x_2 + x_3 \end{pmatrix}$ at $(x_1, x_2, x_3) = (1, 0, 1)$.

		$\partial_1 f_i$	$\partial_2 f_i$	$\partial_3 f_i$
f_1	x_2	0	1	0
f_2	$x_1 x_3$	x_3	0	x_1
f_3	$x_1 + x_2 + x_3$	1	1	1

$$J_f(x_1, x_2, x_3) = \begin{pmatrix} 0 & 1 & 0 \\ x_3 & 0 & x_1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$J_f(1, 0, 1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(b) $f(x) = \binom{x^2}{e^x}$ at $x = 3$.

$$f'(x) = \binom{2x}{e^x} \text{ and } f'(3) = \binom{6}{e^3}$$

(c) $f(x_1, x_2, x_3, x_4) = x_1^2 + 2x_2 x_4 + \sin(x_3 x_4)$ at $(x_1, x_2, x_3, x_4) = (1, 1, 0, 1)$

$$\begin{aligned} \partial_{x_1} f &= 2x_1 \\ \partial_{x_2} f &= 2x_4 \\ \partial_{x_3} f &= x_4 \cos(x_3 x_4) \\ \partial_{x_4} f &= 2x_2 + x_3 \cos(x_3 x_4) \end{aligned}$$

$$J_f(x_1, x_2, x_3, x_4) = \begin{pmatrix} 2x_1 \\ 2x_4 \\ x_4 \cos(x_3 x_4) \\ 2x_2 + x_3 \cos(x_3 x_4) \end{pmatrix}$$

$$J_f(1, 1, 0, 1) = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

IV. Section 6.2 Problem 2.

Exercise 6.2.2. Prove Lemma 6.2.4. (Hint: prove by contradiction. If $L_1 \neq L_2$, then there exists a vector v such that $L_1 v \neq L_2 v$; this vector must be non-zero (why?). Now apply the definition of derivative, and try to specialize to the case where $x = x_0 + tv$ for some scalar t, to obtain a contradiction.)

Lemma 6.2.4 (Uniqueness of derivatives). *Let E be subset of $\mathbb{R}^n, f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be an interior point of E, and let $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Suppose that f is differentiable at x_0 with derivatives L_1, and also diffentiable at x_0 with derivative L_2. Then $L_1 = L_2$*

Let $L_1, L_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations and $L_1 \neq L_2$. Also, let $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{R}^m$ be a function that is differentiable at a point $x_0 \in E^o$ with derivatives L_1 and L_2 at x_0. First, $\det f'(x_0) \neq 0$ because f is differentialable at x_0 and since $L_2 \neq L_1$ there exists a non-zero vector v such that $L_1 v \neq L_2 v$.

$$\begin{aligned}
\text{for any } x = x_0 + tv &\text{ and } x_0 \neq 0 \\
L_1 x = L_1(x_0 + tv) &\text{ and } L_2 x = L_2(x_0 + tv) \\
L_1 x_0 = L_1 x + L_1(tv) &\text{ and } L_2 x_0 = L_2 x + L_2(tv) \\
L_1 x + L_1(tv) &= L_2 x + L_2(tv) \\
L_1 x - L_2 x &= L_1(tv) + L_2(tv) \\
(L_1 - L_2)x &= (L_1 - L_2)(tv) \\
x &= tv \\
\therefore x_0 &= 0 \ \Rightarrow\Leftarrow
\end{aligned}$$

hence $L_1 = L_2$ making it unique.

V. Section 6.3, problem 3 and problem 4.

Exercise 6.3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by $f(x,y) := \frac{x^3}{x^2+y^2}$ when $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Show that f is not differentiable at $(0,0)$, despite being differentiable in every direction $v \in \mathbb{R}^2$ at $(0,0)$. Explain why this does not contradict Thoerem 6.3.8.

$$\begin{aligned} f(x,y) &= \frac{x^3}{x^2+y^2} \\ \partial_x f(x,y) &= \frac{3x^2}{x^2+y^2} - \frac{2x^3}{x^3+y^2} \\ \partial_y f(x,y) &= \frac{-2x^3}{x^2+y^2} \end{aligned}$$

if we hold x constant as $y \to 0$ we can see that $\partial_y f(x,y) \to \infty$ which means that $\partial_y f(x,y)$ is not continuous. Theorem 6.3.8 states that the first partial derivatives must be continuous at $(0,0)$ for $f(x,y)$ to be differentiable at $(0,0)$.

Exercise 6.3.4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function such that $f'(x) = 0$ for all $x \in \mathbb{R}^n$. Show that f is constant. (Hint: you may use the mean-value theorem or fundamental theorem of calculus for one-dimensional functions, but bear in mind that there is a direct analogue to these theorems for several-variable functions. I would not advise proeeding via first principles.) For a tougher challenge, replace the domain $\mathbb{R}^n$ by an open connected subet Ω of $\mathbb{R}^n$.

$$\begin{aligned} &\text{Let } \Omega \subset \mathbb{R}^n \text{ and } [a,b] \in \Omega \\ &\exists \xi \in [a,b] \to |f(b) - f(a)| \leq f'(\xi)|b-a| && \text{mean value theorem} \\ &|f(b) - f(a)| \leq 0 \\ &f(b) = f(a) \end{aligned}$$

a and b are arbitrary thus $f(x) = f(a)$ for all $x \in \mathbb{R}^n$, thus f is a constant function.

VI. Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable, $\alpha \in \mathbb{R}$. If $f(tx) = t^\alpha f(x)$ for $\forall x \in \mathbb{R}^m$ and $t > 0$, we say that f is homogeneous of order α. Show that f is homogeneous of order α iff $x \cdot \nabla f(x) = \alpha f(x)$, that is

$$x^1 \partial_1 f(x) + \cdots + x^m \partial_m f(x) = \alpha f(x).$$

This equation is classically written as

$$x^1 \frac{\partial f}{\partial x^1} + \cdots + x^m \frac{\partial f}{\partial x^m} = \alpha f(x).$$

Hint: As in the development of the theory in the text, a basic idea to study multivariable functions is to convert them into single-variable functions by restricting the variable x in a fixed direction. For example, for this problem you may consider the function $\varphi(t) = f(t)$.

($\Rightarrow$) f is homogenous of order α, that is, $f(tx) = t^\alpha f(x)$. Then,

$$\begin{aligned} \text{Let } \varphi(t) &= f(tx) = t^\alpha f(x) \\ \varphi'(t) &= f'(tx) \cdot x = \alpha t^{\alpha-1} f(x) \\ \text{Let } t = 1 \to f'(x) \cdot x &= \alpha f(x) \end{aligned}$$

($\Leftarrow$) assume that $x \cdot \nabla f(x) = \alpha f(x)$. Let $x = ty$ then

$$\begin{aligned} \text{Let } \varphi(t) &= f(xt) \\ \varphi'(t) &= x \cdot f'(tx) = \alpha f(tx) = \alpha \varphi(t) \end{aligned}$$

this is an ordinary differential equation whose solution is $\varphi(t) = Ct^\alpha$. Notice $\varphi(1) = C = f(x)$. Thus, $\varphi(t) = f(tx) = t^\alpha f(x)$.

VII. (a) Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1-map,

$$|f(x) - f(y)| \geq |x - y|, \forall x, y \in \mathbb{R}^m,$$

then $\forall a \in \mathbb{R}^m, \det f'(a) \neq 0$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable, and assume $f(0,0) = \langle 1, 2 \rangle$, and

$$Df(0,0) = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Let $g(x,y) = \left\langle xy^2, y+2, 2x-3y \right\rangle$.. Find $D(g \circ f)(0,0)$.

$$\begin{aligned}
g(x,y) &= \left\langle xy^2, y+2, 2x-3y \right\rangle \\
g'(x,y) &= \begin{pmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \\ \frac{\partial g_3(x,y)}{\partial x} & \frac{\partial g_3(x,y)}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} y^2 & x \\ 0 & 1 \\ 2-3y & 2x-3 \end{pmatrix} \\
D(g \circ f)(0,0) &= Dg(f(0,0))Df(0,0) \\
&= Dg(1,2)Df(0,0) \\
&= \begin{pmatrix} 4 & 1 \\ 0 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 12 \\ 2 & 0 \\ -6 & -12 \end{pmatrix}
\end{aligned}$$

VIII. Let $f : E \to \mathbb{R}$ be defined on some open set $E \subset \mathbb{R}^2$, and assume the partial derivatives $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 6.3.8 (continuity of partial derivatives implies f is differentiable) which we discussed in class.

Since the partial derivatives are bounded, let $M_i = \max \frac{\partial F}{\partial x_i}$. Then let $M = (M_1\ M_2)$. They are bounded and therefore continuous. Thus we can say that

$$\begin{aligned}
L &= f'(x_0) \\
\forall \epsilon > 0,\ \epsilon &> \frac{|\, f(x) - f(x_0) - L(x - x_0)\,|}{|x - x_0|}, \text{ whenever } \delta > |x - x_0| \text{ for some } \delta > 0 \\
&\leq \frac{|\, f(x) - f(x_0) - M(x - x_0)\,|}{|x - x_0|} \\
\epsilon |x - x_0| &\leq |\, f(x) - f(x_0) - M(x - x_0)\,| \\
\epsilon\delta &\geq |\, f(x) - f(x_0)\,|
\end{aligned}$$

IX. Let $F(x, y, z) = \begin{pmatrix} x+y \\ x^2 y \\ z+2x \end{pmatrix}$.

(a) At what points (x_0, y_0, z_0) does F have a local inverse, i.e., a function F^{-1} defined on an open set V containing $F(x_0, y_o, z_o)$, such that $F(F^{-1}(x, y, z)) = (x, y, z)$ for all $(x, y, z) \in V$?

The inverse exists wherever the Jacobian is valid.

$$F'(x, y, z) = \begin{pmatrix} 1 & 1 & 0 \\ 2xy & x^2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\det F'(x, y, z) = x^2 - 2xy$$

$$\det F'(x, y, z) = 0 \implies x = 2y$$

Thus, F^{-1} exists everywhere except on the line $x = 2y$.

(b) What is $D(F^{-1})(2, 1, 3)$? (Hint: $F(1, 1, 1) = (2, 1, 3)$.)

By utilizing the hint,

$$\begin{aligned} D(F^{-1})(2,1,3) &= D(F^{-1})(F(1,1,1)) \\ &= F'(1,1,1)^{-1} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \end{aligned}$$

X. When does the equation $x_1^2 + 2x_2^3x_3 - x^4 + \ln(1+x_4^2) = 1$ define a function $x_4 = g(x_1, x_2, x_3)$ implicitly? Find $\nabla g(1, 0, -1)$.

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= x_1^2 + 2x_2^3x_3 - x_4 + \ln(1+x_4^2) - 1 = 0 \\ \partial_{x_4} f &= -1 + \frac{2x_4}{1+x_4^2} \\ &= \frac{-1 - x_4^2 + 2x^4}{1+x_4^2} \\ &= \frac{(2x_4+1)(x_4-1)}{1+x_4^2} \\ \partial_{x_4} f = 0 \text{ when } x_4 &\in \left\{ \frac{-1}{2}, 1 \right\}. \end{aligned}$$

there is an implicit function for $x_4 = g(x_1, x_2, x_3)$ when $x_4 \notin \left\{ \frac{-1}{2}, 1 \right\}$. Then we have

$$\begin{aligned} \partial_{x_1} g &= \frac{-\partial_{x_1} f}{\partial_{x_4} f} = \frac{2x_1}{\frac{(2x_4+1)(x_4-1)}{1+x_4^2}} = \frac{2x_1(1+x_4^2)}{(2x_4+1)(x_4-1)} \\ \partial_{x_2} g &= \frac{-\partial_{x_2} f}{\partial_{x_4} f} = \frac{6x_2^2x_3}{\frac{(2x_4+1)(x_4-1)}{1+x_4^2}} = \frac{6x_2^2x_3(1+x_4^2)}{(2x_4+1)(x_4-1)} \\ \partial_{x_3} g &= \frac{-\partial_{x_3} f}{\partial_{x_4} f} = \frac{2x_2^3}{\frac{(2x_4+1)(x_4-1)}{1+x_4^2}} = \frac{2x_2^3(1+x_4^2)}{(2x_4+1)(x_4-1)} \end{aligned}$$

$$\nabla g(1, 0, -1) = \left\langle \frac{2(1+x_4^2)}{(2x_4+1)(x_4-1)}, 0, 0 \right\rangle$$