Math 5050 – Special Topics: Manifolds– Spring 2025
w/Professor Berchenko-Kogan

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Section 3: 1, 2, 3, 7, 8, 9

3.1. Tensor Product of covectors

Let $e_1, \ldots, e_n$ be a basis for a vector space V and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $g_{ij} \in \mathbb{R}^{n\times m}$ is an $n \times m$ matrix define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{i \le i,j,n} g_{ij} v^i w^j$$

for $v = \sum v^j e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and $\alpha^j, 1 \le i, j \le n$.

$$\begin{aligned}
\alpha^i(e_j) &= \delta_i^j && (1)\\
\alpha^i(v) &= \alpha^i\left(\sum_{j=1}^n v^j e_j\right)\\
&= \sum_{j=1}^n \alpha^i(v^j e_j) && \alpha^i \text{ is linear}\\
&= \sum_{j=1}^n v^j \alpha^i(e_j) && v^j \text{ is a scalar}\\
&= \sum_{j=1}^n v^j \delta_j^i = v^i && \text{apply (1)}\\
(\alpha^i \otimes \alpha^j)(v, w) &= \alpha^i(v)\alpha^j(w) = v^i w^j\\
\therefore \sum_{i \le i,j,n} g_{ij} v^i w^j &= \sum_{i \le i,j,n} g_{ij}\,(\alpha^i \otimes \alpha^j)(v, w)
\end{aligned}$$

3.2. Hyperplanes

(a) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n-1$. A linear subspace of V of dimesnion $n-1$ is called a *hyperplane* in V.

$$\begin{aligned}
\dim V &= \dim \operatorname{range}(f) + \dim \ker(f)\\
\dim \ker(f) &= \dim V - \dim \operatorname{range}(f)\\
&= n - 1
\end{aligned}$$

(b) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

$$\begin{aligned}
\text{Let } v = (y+z) \in V &\text{ and } f(y) \in \operatorname{range}(f), z \in \ker(f)\\
u = (x+w) \in V &\text{ and } g(x) \in \operatorname{range}(g), z \in \ker(g)\\
\dim \ker(f) &= \dim \ker(g) = n - 1\\
\dim \operatorname{range}(f) &= \dim \operatorname{range}(g) = 1 \text{ a scalar function}\\
\therefore g &= cf \text{ for some constant } c.
\end{aligned}$$

One dimension is a single vector and either g and f contract or expand that vector and being linear they do so by a constant.

3.3. **A basis for k-tensors**

Let V be a vector space of dimension n with basis $e_i, \dots, e_n$. Let $\alpha^1, \dots, \alpha^n$ be the dual basis in $V^\vee$ Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \dots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this show that $\dim L_k(V) = n^k$. (This problem generalizes Problem 3.1..)

Let $\Phi = \{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ where $i_1, \dots, i_k = 1, \dots, n$. We want to show

- WTS Φ is a linearly independent set.

$$\begin{aligned} &\text{Let } x = \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}, \text{ for some set } \{i_k\},\, i_k \in [1, n] \\ &\text{and } y = \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_k}, \text{ for some set } \{j_k\},\, j_k \in [1, n] \\ &\text{where } \{i_k\} \neq \{j_k\} \end{aligned}$$

then for any non-zero vectors $v_1, \dots, v_n \in V$ where $v_i = (v_i^1, \dots, v_i^n)$ and any $A, B \in \mathbb{R}$ where $Ax + By = 0$

$$\begin{aligned} (Ax + By)(v_1, \dots, v_k) &= A\left((\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k})(v_1, \dots, v_k)\right) + B\left((\alpha^{j_1} \otimes \cdots \otimes \alpha^{j_k}(v_1, \dots, v_k)\right) \\ &= A\left(\prod_{m=1}^{k} \alpha^{i_m}(v_m)\right) + B\left(\prod_{p=1}^{k} \alpha^{j_p}(v_p)\right) \\ &= A\underbrace{\left(\prod_{m=1}^{k} v_m^{i_m}\right)}_{\neq 0} + B\underbrace{\left(\prod_{p=1}^{k} (v_p)^{j_p}\right)}_{\neq 0} \end{aligned}$$

thus $A = B = 0$ and the elements of Φ are linearly independent.

- WTS Φ is surjective over $L_k(V)$. Given any $f \in L_k(v)$ we can factor out k 1-covectors whose tensor product is f.

Step One: Given an element of $L_2(V)$ show that it can be factored into two elements of $L_1(V)$

Let $f \in L_2(V)$ and $v = \sum_{i-1}^{n} v^i e_i, w = \sum_{i=1}^{n} w^i e_i \in V$ then

$$\begin{aligned} f(v, w) &= f\left(\sum_{i-1}^{n} v^i e_i, w\right) \\ &= \sum_{i=1}^{n} v^i f(e_i, w) \\ &= \sum_{i=1}^{n} \alpha^i(v) f(e_i, w) \\ &= \sum_{i=1}^{n} \alpha^i(v) f\left(e_i, \sum_{n=1}^{n} w^j e_j\right) \\ &= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} w^j f(e_i, e_j) \\ &= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) f(e_i, e_j) \end{aligned}$$

Let $g, h \in L_1(V)$ such that

$$\begin{aligned} \text{Let } g(v) &= \sum_{i=1}^{n} g^i \alpha^i(v), \text{ for commpents } g^i \\ \text{and } h(v) &= \sum_{i=1}^{n} h^j \alpha^j(v),\ h^j = \frac{f(e_i, e_j)}{g^i} \end{aligned}$$

Claim: $h^j = \frac{f(e_i, e_j)}{g^i}$ has the same value regardless of e_i. (Note: we can choose g^i to counter the sign of $f(e_i, e_j)$ making h^j positive.)

Proof: let $f_{ij} = f(e_i, e_j)$ this can be represented as a matrix $\{f_{ij}\}$ which we will build using $g \otimes h$.

$$\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} = \begin{pmatrix} g^1 \\ g^2 \\ \vdots \\ g^n \end{pmatrix} \begin{pmatrix} h^1 & h^2 & \cdots & h^n \end{pmatrix}$$
$$= \begin{pmatrix} g^1h^1 & g^1h^2 & \cdots & g^1h^n \\ g^2h^1 & g^2h^2 & \cdots & g^2h^n \\ \vdots & \vdots & \ddots & \vdots \\ g^nh^1 & g^nh^2 & \cdots & g^nh^n \end{pmatrix}$$

$$\text{Notice that } h^1 = \frac{f_{11}}{g^1} = \frac{f_{21}}{g^2} = \cdots = \frac{f_{n1}}{g^n}$$

which is also true for $h^2, \ldots, h^n$. end of proof. □

$$\begin{aligned} \text{Since } f(v,w) &= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) f\left(e_i, e_j\right) \\ &= \sum_{i=1}^{n} g^i \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) \frac{f\left(e_i, e_j\right)}{g^i} \\ &= \sum_{i=1}^{n} g^i \alpha^i(v) \sum_{j=1}^{n} h^j \alpha^j(w) \\ &= g(v)h(w) \\ \therefore f(v,w) &= (g \otimes h)(v,w) \end{aligned}$$

Since g, h are both 1-covectors, that is, they are linear combinations of $\{\alpha^i\}$ thus

$$(g \otimes h)(v,w) = \left(\sum_{i=1}^{n} g^i \alpha^i \otimes \sum_{j=1}^{n} h^j \alpha^j \right)(v,w)$$

thus any $f \in L_2(V)$ can be written as a linear combination of Φ for $k = 2$.

Step Two: extend this to elements of $L_3(V)$, 3-covectors. The difficulty here is replacing the 1-covector, h, from Step One with a 2-covector and realizing the same proof.
Let $f \in L_3(V)$. As above, follow the same steps to expand the first element revealing

$$f(v,w,u) = \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) f(e_i, e_j, u)$$

Let $g \in L_1(V)$ and $h \in L_2(v)$ and g be defined as above and h defined by the h^l 1-covectors for $l = 1, \ldots, n$

$$h(w,u) = \sum_{l=1}^{n} h^l(u) \alpha^l(w), \text{ where } h^l(u) = \frac{f(e_i, e_j, u)}{g^i}$$

In precisely the same manner as the Claim from above, where the elements of f were an $n \times n$ matrix, the elements of f are now the $n \times n \times n$ structure each containing a g^i which can be factored out. These are indepenendent of the value of e_j. Thus,

$$\begin{aligned} f(v,w,u) &= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) f(e_i, e_j, u) \\ &= \sum_{i=1}^{n} g^i \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) \frac{f(e_i, e_j, u)}{g^i} \\ &= g(v)h(w,u) \\ &= (g \otimes h)(u,w,u) \end{aligned}$$

Step Three: extend this to k-covectors Let $f \in L_k(V)$ and $v = \sum_{i-1}^{n} v^i e_i, w = \sum_{i=1}^{n} w^i e_i \in V$ then, similar to above, expand the first and second vectors

$$\begin{aligned}
f(v, w, v_3, \dots, v_k) &= f\left(\sum_{i-1}^{n} v^i e_i, w, v_3, \dots, v_k\right) \\
&= \sum_{i=1}^{n} v^i f(e_i, w, v_3, \dots, v_k) \\
&= \sum_{i=1}^{n} \alpha^i(v) f(e_i, w, v_3, \dots, v_k) \\
&= \sum_{i=1}^{n} \alpha^i(v) f\left(e_i, \sum_{n=1}^{n} w^j e_j, v_3, \dots, v_k\right) \\
&= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} w^j f\left(e_i, e_j, v_3, \dots, v_k\right) \\
&= \sum_{i=1}^{n} \alpha^i(v) \sum_{j=1}^{n} \alpha^j(w) f\left(e_i, e_j, v_3, \dots, v_k\right)
\end{aligned}$$

once again, let $g \in L_1(V)$ and $g(v) = \sum_{i=1}^{n} g^i \alpha^i(v)$ and this time $h \in L_{k-1}(V)$ and

$$\begin{aligned}
h(w, v_3, \cdots, v_k) &= \sum_{j=1}^{n} h^j \alpha^j(w) f(e_i, e_j, v_3, \dots, v_k) \\
h^j &= \frac{f(e_i, e_j, v_3, \dots, v_k)}{g^i}
\end{aligned}$$

then

$$\begin{aligned}
f(v, w, v_3, \dots, v_k) &= g(v) h(w, v_3, \dots, v_k) \\
&= (g \otimes h)(v, w, v_3, \dots, v_k)
\end{aligned}$$

Step Four: Steps One, Two and Three demonstrate that we can factor out a 1-covector and k-1-covector from any k-covector f into a tensor product *from the first parameter*. By repeating this process **in sequence**, that is with identity permutation $\sigma = \{1, 2, \dots, k\}$, we can see that any k-covector can be factored into the tensor product of k 1-covectors.

This proves that Φ is surjective over $L_k(V)$.

- **WTS: Show independence of order. CLAIM:** Replace σ with a different permutation of k and it will have the same effect. That is, we can still factor out the $\sigma_{1\text{st}}$ parameter into a 1-covector, γ^1,[1] as defined by g above, on the left of the tensor product and a k-1-covector on the right, $h^1 \in L_{k-1}(V)$. Keep in mind that the components of these γ^i 1-covectors are chosen to make h^i positive. Define h^1 as

$$h^1(v_1, v_2, \dots, \underbrace{w}_{\sigma_{2\text{nd}}}, \dots, v_k) = \sum_{j=1}^{n} h^1_j(v_1, v_2, \dots, w, \dots, v_k) \qquad \sigma_{2\text{nd}} \text{ parameter is used}$$

$$h^1_j(v_1, v_2, \dots, w, \dots, v_k) = \sum_{i=1}^{n} \alpha^i w f(v_1, v_2, \dots, e_{\sigma_1}, \dots, e_{\sigma_2} \dots, v_k)$$

and

$$f(v_1, \dots, v_{\sigma_1}, \dots, v_k) = (\gamma^1 \otimes h^1)(v_{\sigma_1}, v_1, \dots, v_{\sigma_1 - 1}, v_{\sigma_1 + 1}, \dots, v_k)$$

repeating the process reducing each $h^1, \dots, h^{k-1}$ (each one lower level covector than previous one) with a progression of left hand operands, $\gamma^1, \gamma^2, \dots, \gamma^{k-1}$, to the product tensor following the sequence in σ. Thus,

$$f(v_1, \dots, v_k) = (\gamma^1 \otimes \gamma^2 \otimes \cdots \otimes \gamma^{k-1} \otimes h^{k-1})(v_{\sigma_1}, v_{\sigma_2}, \cdots, v_{\sigma_k})$$

Therefore, any k-covector $f \in L_k(V)$ is the tensor product of k 1-covectors in any order, each of which is a linear combination of elements from Φ.

[1] We use γ here because using g^i for covectors would confuse the g^i used in Step One

3.7. **Transformation rule for a wedge product of covectors**

Suppose two set so of covectors on a vector space V. $\beta^1, \dots, \beta^k$ and $\gamma^i, \dots, \gamma^k$, are related by

$$\beta^i = \sum_{j=1}^{k} a_j^i \gamma^i,\ i = 1, \dots, k$$

for a $k \times k$ matrix $A = [a_j^i]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.$$

$$\begin{aligned}
&\text{Let } \beta, \gamma \in \mathcal{M}_{n\times n}(V^\vee) \\
&\qquad \beta = \left[\, \beta^i \,\right] \text{ and } \beta(v_1, \dots, v_k) = \left[\, \beta^i \,\right](v_1, \dots, v_n) = \left[\, \beta^i(v_j) \,\right] \\
&\qquad \gamma = \left[\, \gamma^i \,\right] \text{ and } \gamma(v_1, \dots, v_k) = \left[\, \gamma^i \,\right](v_1, \dots, v_n) = \left[\, \gamma^i(v_j) \,\right] \\
&\qquad A = [a_j^i] \\
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \dots, v_k) &= \det[\beta^i(v_j)] = \det \beta(v_1, \dots, v_k) \\
(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \dots, v_k) &= \det[\gamma^i(v_j)] = \det \gamma(v_1, \dots, v_k)
\end{aligned}$$

we can see that

$$\begin{aligned}
\beta^i = \sum_{j=1}^{k} a_j^i \gamma^i &\implies \beta = A \cdot \gamma \text{ and } \beta(v_1, \dots, v_k) = A \cdot \gamma(v_1, \dots, v_k) \\
\det \beta &= \det(A \cdot \gamma) = \det A \cdot \det \gamma \\
\det \beta(v_1, \dots, v_k) &= \det A \cdot \det \gamma(v_1, \dots, v_k) \\
(\beta^1 \wedge \cdots \wedge \beta^k)(v_1, \dots, v_k) &= \det A(\gamma^1 \wedge \cdots \wedge \gamma^k)(v_1, \dots, v_k) \\
\beta^1 \wedge \cdots \wedge \beta^k &= \det A(\gamma^1 \wedge \cdots \wedge \gamma^k)
\end{aligned}$$

3.8. **Transformation rule for k-covectors**

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \dots, u_k$ and $v_1, \dots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a_j^i v_i,\ j = 1, \dots, k,$$

for $k \times k$ matrix $A = [a_j^i]$. Show that

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k).$$

$$\begin{aligned}
f(u_1, \dots, u_k) &= f\left(\sum_{i_1=1}^{k} a_1^{i_1} v_{i_1}, \sum_{i_2=1}^{k} a_2^{i_2} v_{i_2}, \dots, \sum_{i_k=1}^{k} a_k^{i_k} v_{i_k} \right) \\
&= \sum_{i_1=1}^{k} a_1^{i_1} \sum_{i_2=1}^{k} a_2^{i_2} \cdots \sum_{i_k=1}^{k} a_k^{i_k} f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \\
&= \sum_{\sigma \in S_k} a_1^{\sigma_1} \cdots a_k^{\sigma_k} f(v_{i_1}, v_{i_2}, \dots, v_{i_k}) \\
&= (\det A) f(v_{i_1}, v_{i_2}, \dots, v_{i_k})
\end{aligned}$$

3.9. **Vanishing of a covector of top degree**

Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \ldots, e_n$ for V. then ω is the zero covector on V.

$$\begin{aligned} 0 &= \omega(v_1, \ldots, v_n) \\ \exists \omega_i &\in L_1(V),\ i = 1, \ldots, n \\ \text{such that } \omega(v_1, \ldots, v_n) &= \left(\bigotimes_{i=1}^{n} \omega_i \right) (v_1, \ldots, v_n) \\ &= \prod_{i=1}^{n} \omega_i(v_i) \end{aligned}$$

which means that there exists an element j such that $\omega_j(v_j) = 0$ and

$$\omega_j(v_j) = \sum_{i=1}^{n} c^i \alpha^i (v_j) = 0$$

the α^i are linearly independent thus either the components of v_j must be zero or the components c^i of ω_j must be zero. Since v_j is arbitrary, all of the $c^i = 0$ and $\omega_j = 0$. Since, this is true for all v_j then $\omega = 0$ always.