New lower bounds for Schur and weak Schur numbers

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Abstract

This paper offers new lower bounds for both Schur and weak Schur numbers. These results were produced by pushing forward Rowley's "templates"-based approach for Ramsey and Schur numbers in 2020. Finding suitable templates allows us to apply Rowley's construction to get explicit partitions improving lower bounds. Furthermore, this paper tries to analyse former work on the subject based on the principle that good partitions into $n+1$ subsets start with good partitions into n subsets. We show that exceeding the previous lower bound $WS(6) \geqslant 582$ is impossible with such a method upon imposing certain conditions on the good 5-subsets partition. The new lower bounds includes $S(9) \geqslant 17803$, $S(10) \geqslant 60948$, $WS(9) \geqslant 22536$ and $WS(10) \geqslant 71214$.

1 Introduction, context and notations

We start by defining sum-free and weakly sum-free subsets to introduce regular and weak Schur numbers.

Definition 1.1 *A subset A of $\mathbb{N}$ is said to be sum-free when:*

$$\forall (a,b) \in A^2,\ a+b \notin A$$

Definition 1.2 *A subset B of $\mathbb{N}$ is said to be weakly sum-free when:*

$$\forall (a,b) \in B^2,\ a \neq b \Longrightarrow a+b \notin B$$

Let us notice that a sum-free subset is also weakly sum-free, hence justifying the name of *weakly* sum-free subsets. Given p and n two integers, we are interested in partitioning the set of integers from 1 to p into n weakly sum-free subsets.

Notation 1.3 *We denote by $[\![1,p]\!]$ the set of integers $\{1,2,...,p\}$.*

Schur proved in [10] that given a number of subsets n, there exists a value of p such that there exists no partition of $[\![1,q]\!]$ into n sum-free subsets for any $q \geqslant p$. A similar property holds for weakly sum-free subsets (reference necessaire). These observations lead to the following definitions.

Definition 1.4 *Let $n \in \mathbb{N}^*$. There exists a greatest integer that we note $S(n)$ (resp. $WS(n)$) such that $[\![1,S(n)]\!]$ (resp. $[\![1,WS(n)]\!]$) can be partitioned into n sum-free subsets (resp. weakly sum-free subsets). $S(n)$ is called the n^{th} Schur number and $WS(n)$ the n^{th} weak Schur number.*

Notation 1.5 *For a partition of $[\![1,p]\!]$ in n subsets, we generally note these subsets $A_1,...,A_n$. We also note $m_i = min(A_i)$. By ordering the subsets, we mean assuming that $m_1 < ... < m_n$. However, if not specified we do not make this hypothesis since we do not always consider partitions in which every subset plays a symmetric role.*

2 Schur numbers

In this section, we use Rowley's constructions [8] in a Schur context. To improve lower bounds for Ramsay's numbers, Rowley introduces partitions verifying some properties which can be extended using a method which generalizes Abbott-Hanson's [1] construction. Rowley named these partitions "templates", and we will keep this name in the entire article. We find suitable templates and use them to find new lower bounds for Schur numbers.

2.1 Definition of S_+

Definition 2.1 *We call SF-template of n colors and lenght p a partition of $[\![1, p]\!]$ into n sum-free subsets $A_1, A_2, ..., A_n$ which verify :*

$$\forall i \in [\![1, n-1]\!], \forall (x, y) \in A_i^2, x + y > p \Longrightarrow x + y - p \notin A_i$$

We note $S^+(n)$ the maximal lenght for a SF-template of n colors.

Remark 2.1 *SF-templates include Abott and Hanson's construction [reference necessaire] as a special case.*

Proposition 2.1 *Let $n \in [\![2, +\infty]\!]$, we have :*

$$2S(n-1) + 1 \leqslant S^+(n) \leqslant S(n)$$

Proof : The lower bound comes from Abott and Hanson's construction. The upper bound comes from the fact that a SF-template of length p with n colors is also a partition of $[\![1, p]\!]$ into n sum-free subsets.

Remark 2.2 *S^+ and S have the same asymptotic growth rate.*

2.2 Inequalities using S^+

The main result on S^+ follows. It allows us to improve lower bounds on Schur numbers by computing S^+.

Theorem 2.1 *Let $(n, k), (p, q) \in (\mathbb{N}^*)^2$. If there exists a SF-template of $k+1$ colors and lenght p, and a partition of n sum-free subsets of $[\![1, q]\!]$ then there exists a partition of $n+k$ sum-free subsets of $[\![1, pq + m_{k+1} - 1]\!]$. m_{k+1} is the first number colored with the $k+1$-th color in the SF-template.*

Setting $p = S^+(k+1)$ and $q = S(n)$ yields the following corollary.

Corollary 2.1.1 *Let $n, k \in \mathbb{N}^*$, we have*

$$S(n+k) \geqslant S^+(k+1)S(n) + m_{k+1} - 1$$

The idea lying beneath this inequality is similar to Abbott and Hanson's contruction [1]. They extend vertically a sum-free partition, and horizontally an other sum-free partition. This way each "block" acts like a security zone for the other one. Here, the horizontal partition is no longer to the side of the vertical one, but occupies the column of the special color of the SF-template, i.e the one without the extra condition. We give the following table for $p = 7, q = 4, n = 2$ and $k = 2$ to make the intuition clear.

This shows the inequality $S(2+2) \geqslant S^+(3)S(2) + 4$. The special color is blue and is first used to color 5, hence $4 = m_3 - 1$. We prooved $S^+(3) = 9$. The special color is blue.

$S^+(3)$

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

$S^+(3)$	1	2	3	4	5	6	7	8	9

$S(2)$	1	2	3	4

To compare, here is a classic Abbott and Hanson partition : $S(2+2) \geqslant S(2)(2S(2)+1)+S(2)$.

$S(2)$

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

PROOF : We denote by f the projection of the equivalence relation induced by the SF-template of lenght p and g the one induced by the sum-free partition of $[\![1,q]\!]$. Each equivalence class is represented by a single integer, therefore :

$$f : [\![1,p]\!] \longrightarrow [\![1,k+1]\!] \text{ and } \forall (x,y) \in [\![1,p]\!]^2, \left\{ \begin{array}{l} f(x) = f(y) \leqslant k \\ x+y > p \end{array} \right. \Longrightarrow f(x+y-p) \neq f(x)$$

with also the sum-free condition :

$$\forall (x,y) \in [\![1,p]\!]^2,\ f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

Moreover, by definition of m_{k+1}, if $x < m_{k+1}$, $f(x) \leqslant k$.

$$g : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ g(x) = g(y) \Longrightarrow g(x+y) \neq g(x)$$

We now define $h : [\![1, pq+m_{k+1}-1]\!] \longrightarrow [\![1, n+k]\!]$ as follows : $\forall x \in [\![1, pq+m_{k+1}-1]\!]$, we write $x = \alpha p - u$ where $\alpha \in [\![1, q+1]\!]$ and $u \in [\![0, p-1]\!]$. This decomposition if of course unique.

If $f(p-u) \leqslant k$, we set $h(x) = f(p-u)$, else (i.e $f(p-u) = k+1$) $h(x) = k + g(\alpha)$.
If $\alpha = q+1$, we have a problem plugging it into g. Hopefully, this issue never arises : if $\alpha = q+1$, we have $u > p - m_{k+1}$ since $x \in [\![1, pq+m_{k+1}-1]\!]$, hence $p-u < m_{k+1}$ and $f(p-u) \leqslant k$.

Prooving that the partition of $[\![1,pq]\!]$ induced by h is sum-free will complete the proof.

Let $x, y \in [\![1, pq + m_{k+1} - 1]\!]$, with $h(x) = h(y)$ and $x + y \leqslant pq + m_{k+1} - 1$. We write $x = \alpha p - u$ and $y = \beta p - v$. We simply need $h(x + y) \neq h(x)$.

We first examine the case where $h(x) \leqslant k$. We assume $h(x+y) \leqslant k$, otherwise we are fine. Thus by definition of h, we have $h(x) = f(p - u) = f(p - v) = h(y)$.

- If $u+v < p$, then $x+y = (\alpha+\beta)-(u+v)$ with $u+v \in [\![0, p-1]\!]$. We have $f(p-u) = f(p-v) \leqslant k$ and $p - u + p - v > p$, the extra condition on f provides $f(p - u - v) \neq f(p - u)$. We assumed $h(x+y) \leqslant k$ hence $h(x+y) = f(p-u-v)$. Since $h(x) = f(p-u)$, we have at last $h(x+y) \neq h(x)$.
- If $u + v \geqslant p$, then $x + y = (\alpha + \beta - 1) - (u + v - p)$ with $u + v - p \in [\![0, p - 1]\!]$. We assumed $h(x+y) \leqslant k$ hence $h(x+y) = f(p-(p-u-v)) = f(p-u+p-v)$. Since $f(p-u+p-v) \neq f(p-u)$, we get $h(x + y) \neq h(x)$.

We now assume $h(x) > k$. Then $h(x) = k + g(\alpha) = k + g(\beta) = h(y)$, hence $g(\alpha) = g(\beta)$. We have the two same cases as before.

- If $u + v < p$, then $x + y = (\alpha + \beta) - (u + v)$ with $u + v \in [\![0, p - 1]\!]$. We assume $h(x + y) > k$ otherwise the expected result is trivial. Since $g(\alpha) = g(\beta)$, the sum-free condition garanties $g(\alpha) \neq g(\alpha + \beta)$, thus $h(x + y) = k + g(\alpha + \beta) \neq k + g(\alpha) = h(x)$.
- If $u + v \geqslant p$, then $x + y = (\alpha + \beta - 1) - (u + v - p)$ with $u + v - p \in [\![0, p - 1]\!]$. Because of the assumption $h(x) > k$, we necessarily have $f(p-u) = k+1 = f(p-v)$. Then $f(p-(u+v-p)) = f(p - u + p - v) \leqslant k$ with the sum-free condition on f, hence $h(x + y) = f(p - (u + v - p))$ by construction of h. Thus $h(x + y) \leqslant k < h(x)$.

Using a SAT solvers, we are able to exhibit SF-template, hence providing lower bound on S^+. We seek templates providing the best lower bound possible, but also the best additive constant (m_{k+1} with the previous notation). Further details can be found in theSAT section.

Here are the best inequalities on Schur numbers so far :

$$S(n + 1) \geqslant 3S(n) + 1$$

$$S(n + 2) \geqslant 9S(n) + 4$$

$$S(n + 3) \geqslant 33S(n) + 6$$

$$S(n + 4) \geqslant 111S(n) + 43$$

$$S(n + 5) \geqslant 380S(n) + 148$$

$$S(n + 6) \geqslant 1140S(n) + 528$$

The first inequality comes from the original Schur's paper [10]. The second one is due to Abott [1] and the third one to Rowley [8]. The other ones are new.

We also have a similar theorem where only S^+ is involved.

Theorem 2.2 *Let $(n, k), (p, q) \in (\mathbb{N}^*)^2$. If there exists a SF-template of $k + 1$ colors and lenght p, and SF-template of n color and lenght q, then there exists SF-template of $(n + k)$ and lenght pq.*

And the associated inequality :

Corollary 2.2.1 *Let $n, k \in \mathbb{N}^*$, we have*

$$S^+(n+k) \geqslant S^+(k+1)S^+(n)$$

PROOF : The idea is the same as in the previous theorem. The only difference is the SF property inherited from the second SF-template.

2.3 New lower bounds for Schur numbers

The previous inequalities give new lower bounds for $S(n)$ for $n \geqslant 9$. We compute the lower bounds for $n \in [\![8, 15]\!]$ using the four different inequalities, please notice that the best values for $n = 8$ and $n = 13$ were obtained thanks to the first one, found by Rowley. The best lower bounds are highlighted.

n	8	9	10	11
$33S(n-3)+6$	5286	17694	55446	174444
$111S(n-4)+43$	4927	17803	59539	186523
$380S(n-5)+148$	5088	16868	60948	203828
$1140S(n-6)+528$	5088	15348	50688	182928

n	12	13	14	15
$33S(n-3)+6$	587505	2011290	6726330	21072090
$111S(n-4)+43$	586789	1976176	6765271	22624951
$380S(n-5)+148$	638548	2008828	6765288	23160388
$1140S(n-6)+528$	611568	1915728	6026568	20295948

Except for 8, 9 and 13, the best lower bounds are obtained thanks to the third inequality $S(n+5) \geqslant 380S(n) + 148$. The table doesn't go any further, but the same inequality allows to improve the lower bounds for every $n \geqslant 15$.

Corollary 2.2.2 *The growth rate for Schur (and Ramsey) numbers satisfies $\gamma \geqslant \sqrt[5]{380} \approx 3.28$.*

PROOF : It is a mere consequence of the inequality $S(n+5) \geqslant 380S(n) + 148$. As for Ramsey's numbers growth rate, a lower bound can be found using Schur's one, thanks to $S(n) \leqslant R_n(3) - 2$ (see [10]).

3 Weak Schur numbers

In this section, we generalize Rowley's constructions in [9]. We then introduce, by analogy with the third section, the integer $WS^+(n)$ to build suitable templates.

3.1 Lower bound for Weak Schur numbers using Schur and Weak Schur numbers

Up to now, there was no equivalent for weak Schur numbers of Abott and Hanson's construction [1]. Here we give a general lower bound for weak Schur numbers as a function of both regular and weak Schur numbers. The following theorem, inspired by Rowley's inequalities for $WS(n+1)$ and $WS(n+2)$, was found and proved by Romain Ageron.

Theorem 3.1 *Let $(p,q),(n,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets then there exists a partition of $[\![1,p(q+\lceil \frac{q}{2} \rceil+1)+q]\!]$ into $n+k$ weakly sum-free subsets.*

In particular, if we choose $q = WS(n)$ and $p = S(k)$ in the last theorem, the next corollary follows.

Corollary 3.1.1 $\forall (n,k) \in (\mathbb{N}^*)^2$, $WS(n+k) \geqslant S(k)\left(WS(n) + \left\lceil \frac{WS(n)}{2} \right\rceil + 1\right) + WS(n)$

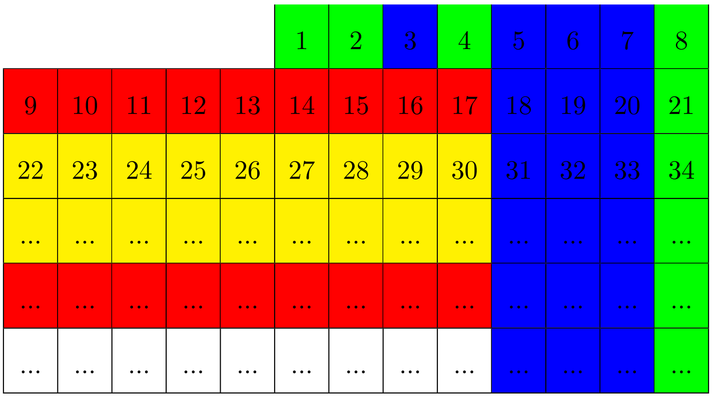
Remark 3.1 *In the above ineqality, a "+1" can be added to the lower bound if $WS(n)$ is odd (more generally if q is odd in the theorem). However, it makes the proof less clear and it is never useful in practice.*

We first give an intuitive explanation of the above theorem. Let $(p,q) \in \mathbb{N}^2$ such that there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets. Let $a \in \mathbb{N}$ with $a > q$ and let's try to build a colouring of $[\![1,ap+q]\!]$ into $n+k$ weakly sum-free subsets. Let $l = a-b-1$, $r \in [\![1,q]\!]$ and $w = a-l-r-1 = b-r$.

First, put the integers of $[\![1,ap+q]\!]$ in the following table (with a columns and $p+1$ lines) and divide it into 3 blocks (the columns are numbered from $-l$ to $+q$):

- $\mathcal{T}$ (the "tail"): the integers from 1 to q. NB: this is line number 0.
- $\mathcal{L}$ (the "lines"): the integers in columns $-l$ to $+r$ (excluding $\mathcal{T}$).
- $\mathcal{C}$ (the "columns"): the integers in the last w columns (excluding $\mathcal{T}$).

Note that with this numbering of columns, the column of the sum of two numbers is the only integer in $[\![-l,q]\!]$ equal to two the sum of the columns modulo a.



$\mathcal{T}$ block

We color this block using the weakly sum-free colouring of $[\![1, q]\!]$ with colors $1, ..., n$.

$\mathcal{L}$ block

In this block, we use the colors $n+1, ..., n+k$. We colour an integer x according to its line number (written $\lambda(x)$). For every $x \in \mathcal{L}$, we colour x with $n+c$ where c is the colour of $\lambda(x)$ in the sum-free colouring of $[\![1, p]\!]$. Let $(x, y) \in (L)^2$. The cases are twofold.

- $\underline{\lambda(x+y) = \lambda(x) + \lambda(y)}$
  In this case, we use the sum-free property of the colouring of $[\![1, p]\!]$ (in block $\mathcal{C}$, we only use colours $1, ..., n$).

- $\underline{\lambda(x+y) \neq \lambda(x) + \lambda(y)}$
  In this case, we do not have information about the colour of $\lambda(x+y)$. Thereby, we want to have $x+y \in \mathcal{C}$. A simple solution is to limit the horizontal movement, that is if the sum changes line, not to move too far so that it stays in $\mathcal{C}$. There, the maximal displacement to the left (resp. to the right) is $2l$ (resp. $2r$). Not crossing entirely $\mathcal{C}$ by going to the left is then expressed as $-2l > -a + r$. Likewise, not going to far to the right is expressed as $2r < a - l$. It can then be written as $\max(l, r) \leqslant w$.

$\mathcal{C}$ block

In this block, we use colors $1, ..., n$. We colour an integer x according to its column number (written $\tilde{\pi}(x)$, it is linked to the projection on the first line, written π, by the relation $\tilde{\pi}(x) = \pi(x) - a$). A simple solution is to colour x with the same colour as $\tilde{\pi}(x)$ in the weakly sum-free colouring of $[\![1, q]\!]$. As long as $2b \leqslant a + r$ (not going two far to the right) and there is no $x \in \tilde{\pi}(\mathcal{C})$ such that $2x \in \tilde{\pi}(\mathcal{C})$ (so that we do not have a sum in $\mathcal{C}$ when taking two numbers in the same column), the colours $1, ..., n$ are sum-free.

In particular, taking $w = l = \lceil \frac{q}{2} \rceil$ and $r = \lfloor \frac{q}{2} \rfloor$ works, thus obtaining the above theorem.

Proof : Let $(p, q), (n, k) \in (\mathbb{N}^*)^2$, $N = p(q + \lceil \frac{q}{2} \rceil + 1) + q$, $\alpha = \lceil \frac{q}{2} \rceil > 0$ and $\beta = q + \alpha + 1$. We denote by f the projection of the equivalence relation induced by the partition of $[\![1, q]\!]$ and g the one induced by the partition of $[\![1, p]\!]$. Each equivalence class is represented by a single integer, therefore :

$$f : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2, \begin{cases} x \neq y \\ \\ f(x) = f(y) \end{cases} \Longrightarrow f(x+y) \neq f(x)$$

$$g : [\![1,p]\!] \longrightarrow [\![1,k]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

Let us start by parting the integers of $[\![1,N]\!]$ in two subsets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} = [\![1,\alpha]\!] \cup \{a\beta + u \mid (a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]\}$ and $\mathcal{B} = \{a\beta + u \mid (a,u) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]\}$.

First, $\underline{\mathcal{A} \cap \mathcal{B} = \varnothing}$:
By contradiction, suppose there exists $x \in \mathcal{A} \cap \mathcal{B} \neq \varnothing$. Then there are $(a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]$ and $(b,v) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$ such that $x = a\beta + u = b\beta + v$. By definition of α and β we have $u \in [\![\alpha+1,q]\!] \subset [\![0,\beta-1]\!]$. From there, we distinguish two cases :

- If $v \in [\![0,\alpha]\!]$ then $v \in [\![0,\beta-1]\!]$ and $v \neq u$ because $v < \alpha + 1 \leqslant u$
- If $v \in [\![-\alpha,-1]\!]$, we note $\tilde{v} = \beta + v$ and thus have $x = (b-1)\beta + \tilde{v}$ with $\tilde{v} \in [\![\beta-\alpha,\beta-1]\!] \subset [\![0,\beta-1]\!]$ and $\tilde{v} \neq u$ because $u < q + 1 = \beta - \alpha \leqslant \tilde{v}$.

In either cases, we run into a contradiction because of the remainder's uniqueness in the euclidean division of x byβ.

Then, we have $\underline{\mathcal{A} \cup \mathcal{B} = [\![1,N]\!]}$:

- On the one hand : $1 = \min(\mathcal{A}) \leqslant \max(\mathcal{A}) = p\beta + q = N$ and $1 \leqslant \beta - \alpha = \min(\mathcal{B}) \leqslant \max(\mathcal{B}) = p\beta + \alpha \leqslant N$, which gives $\mathcal{A} \cup \mathcal{B} \subset [\![1,N]\!]$.
- On the other hand, let $x \in [\![1,N]\!]$. If $x \leqslant \alpha$, we directly have $x \in \mathcal{A}$, let us then suppose that $x > \alpha$ and write $x = a\beta + u$ the euclidean division of x by β. We have $x \leqslant N$, thus $a \leqslant p$. We distinguish three cases :
  - If $u \in [\![0,\alpha]\!]$ then we necessarily have $a \geqslant 1$ because $x > \alpha$, and so $x \in \mathcal{B}$.
  - If $u \in [\![\alpha+1,q]\!]$, then $x \in \mathcal{A}$.
  - If $u \in [\![q+1,\beta-1]\!]$ then $x = (a+1)\beta - (\beta - u)$ with $-\alpha \leqslant \beta - u \leqslant 0$. Furthermore, $a \leqslant p-1$, else we would have $x > N$, and so $x \in \mathcal{B}$
  In any case, $x \in \mathcal{A} \cup \mathcal{B}$ and we can thus conclude that $[\![1,N]\!] \subset \mathcal{A} \cup \mathcal{B}$.

This first partition of $[\![1,N]\!]$ will help us to define our final partition by the projection of its equivalence relation. We thereby define $h : [\![1,N]\!] \longrightarrow [\![1,n+k]\!]$ as such :
- If $x \in \mathcal{A}$ then $h(x) = f(x \bmod \beta)$ (well defined because $x \bmod \beta \in [\![1,N]\!]$)
- If $x \in \mathcal{B}$ then $x = a\beta + u$ with a unique $(a,u) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$and we define $h(x) = n + g(a)$
The fact that $(\mathcal{A},\mathcal{B})$ is a partition of $[\![1,N]\!]$ ensures that this definition of h is valid. We then have to verify that h induces weakly sum-free subsets.

The classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free :

Let $(x,y) \in \mathcal{A}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$

- If $(x,y) \in [\![1,\alpha]\!]^2$:
  We have $x + y \leqslant 2\alpha \leqslant q$ and $x + y = 0\beta + x + y$, therefore $x + y \in \mathcal{A}$. Then, by definition : $h(x) = f(x)$, $h(y) = f(y)$ and $h(x+y) = f(x+y)$, which gives us, thanks to the property verified by f, that $h(x+y) \neq h(x)$.

- If $(x,y) \in [\![1,\alpha]\!] \times (\mathcal{A} \setminus [\![1,\alpha]\!])$:
  We write $y = a\beta + u$ with $(a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]$. Then $x+y = a\beta+x+u = (a+1)\beta+x+u-\beta$, and if $x+u > q$ it follows that $a \leqslant p-1$ since $x+y \leqslant N$, and $-\alpha \leqslant x+u-\beta \leqslant -1$. Therefore $x+y \in \mathcal{B}$ and $h(x+y) \neq h(x) = f(x)$ by definition of h. On the contrary, if $x-u \leqslant n$, then $x+y \in \mathcal{A}$ and $h(x+y) = f(x+u)$ because $x+u$ is actually the remainder of the euclidean divisionof $x+y$ by β. Moreover, $h(x) = f(x)$, $x < u$ and, with our initial hypothesis, $h(x) = h(y) = f(u)$. The property verified by f gives us $f(x+u) \neq f(x)$ which can be rewritten as $h(x+y) \neq h(x)$.

- If $(x,y) \in (\mathcal{A} \setminus [\![1,\alpha]\!]) \times [\![1,\alpha]\!]$:
  This case is handled exactly like the previous one by swaping the roles of x and y.

- If $(x,y) \in (\mathcal{A} \setminus [\![1,\alpha]\!])^2$:
  We write $x = a\beta + u$ and $y = b\beta + v$ with (a,u) and (b,v) in $[\![0,p]\!] \times [\![\alpha+1,q]\!]$. Then $x+y = (a+b)\beta+u+v = (a+b+1)\beta+u+v-\beta$ with $a+b \leqslant p-1$ (else we would have $x+y > N$ because $u+v > q$) and $-\alpha \leqslant u+v-\beta \leqslant \alpha$, therefore $x+y \in \mathcal{B}$ and by definition $h(x+y) \neq h(x)$.

In any case, $h(x+y) \neq h(x)$ and the classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free.

<u>The classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free :</u>

Let $(x,y) \in \mathcal{B}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x+y \leqslant N$.
We write $x = a\beta+u$ and $y = b\beta+v$ with (a,u) and (b,v) in $[\![1,p]\!] \times [\![-\alpha,\alpha]\!]$. We have $h(x) = q+g(a)$ and $h(y) = q+g(b)$, therefore $g(a) = g(b)$. We also have $x+y = (a+b)\beta+u+v$.
If $u+v \in [\![-\alpha,\alpha]\!]$, then $x+y \in \mathcal{B}$ and $h(x+y) = g(a+b)$, hence we can deduce that $h(x+y) \neq h(x)$ because of the property verified by g. On the contrary, if $u+v \notin [\![-\alpha,\alpha]\!]$, then necessarily $x+y \in \mathcal{A}$. Suppose $x+y \in \mathcal{B}$, then $x+y = c\beta+w$ with $(c,w) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$. Thus, $c\beta+w = (a+b)\beta+u+v$ and $(a+b-c)\beta = w-u-v$. Furthermore $a+b-c \neq 0$, else we would have $u+v = w \in [\![-\alpha,\alpha]\!]$. This finally leads to the following inequality :

$$\beta \leqslant |a+b-c|\beta = |w-u-v| \leqslant |w|+|u|+|v| \leqslant 3\alpha \leqslant q+\alpha < \beta$$

which is absurd. We can therefore conclude that $x+y \in \mathcal{A}$ and by definition of h, $h(x+y) \neq h(x)$, proving that the classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free.

Finally, we have showed that every class of equivalence induced by h is weakly sum-free, which ends the proof.

Remark 3.2 *This formula includes the results of Rowley [9] as a special case. For $n > 2$, this formula does not give new lower bounds but in the same way as we introduced S^+(Definition 3.1),we will define WS^+ and find inequalities between WS^+,WS and S*

3.2 Definition of WS^+

We will now define notations and results, we will use in the following theorem.

Notation 3.1 *Let (a,b) $n \in (\mathbb{N}^*)^2$,$a > b$, we will define $\pi_{a,b}$ the projection:*

$$\pi_{a,b} : x -> (Id + a\mathbf{1}_{[\![0,b]\!]})(x \ mod \ a))$$

We will note the projection π and not $\pi_{a,b}$ when there is no doubt about the a and b we use.

Proposition 3.1 *Let $x \in [\![1,b]\!]$, let $y \in \mathbb{N}^*$ such that $x+\pi(y) \leqslant a+b$, then we have: $\pi(x+y) = x+\pi(y)$*

PROOF :Let x $\in [\![1,b]\!]$, let y $\in \mathbb{N}^*$ such that $x+\pi(y) \leqslant a+b$
if $x+\pi(y) < a$:we remark that $\pi(y) > b$ and therefore $x+\pi(y) > b$:

$$\begin{aligned}
\pi(x+y) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x+y mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x+\pi(y) \text{ mod a}) \text{ since } \pi(y)\text{=y mod a} \\
&= x+\pi(y)
\end{aligned}$$

if $x+\pi(y \geqslant a$:

$$\begin{aligned}
\pi(x+y) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x+y mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x+\pi(y) \text{ mod a}) \text{ since } \pi(y)\text{=y mod a} \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x+\pi(y)-a) \\
&= x+\pi(y)-a+a\mathbf{1}_{[\![0,b]\!]})(x+\pi(y)-a) \\
&= x+\pi(y)-a+a \quad \text{ since } x+\pi(y)-a \in [\![0,b]\!] \\
&= x+\pi(y)
\end{aligned}$$

Proposition 3.2 *Let (x,y)$\in (\mathbb{N}^*)^2$, $\pi(\pi(x)+\pi(y)) = \pi(x+y)$*

PROOF :Let (x,y)$\in (\mathbb{N}^*)^2$,

$$\begin{aligned}
\pi(\pi(x)+\pi(y)) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\pi(x)+\pi(y) \text{ mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})((Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x mod a}) + (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{y mod a}) \text{ mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{ ((x mod a) + (y mod a)) mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{ x+y mod a}) \\
&= \pi(x+y)
\end{aligned}$$

Notation 3.2 *Let (a,b) $n \in (\mathbb{N}^*)^2$,$a > b$, we will define $\lambda_{a,b}$ the projection:*

$$\lambda_{a,b} : x -> 1 + \left\lfloor \frac{x-b-1}{a} \right\rfloor$$

We will note the projection λ and not $\lambda_{a,b}$ when there is no doubt about the a and b we use.

Remark 3.3 *In the following theorem, λ is the function which return the line number of an element x.*

Proposition 3.3 *Let (a,b)$\in (\mathbb{N}^*)^2$, $a > b$, let $x \in \mathbb{N}^*$, $x = a\lambda(x) + \pi(x) - a$*

PROOF :Let (a,b)$\in (\mathbb{N}^*)^2$, $a > b$, let x $\in \mathbb{N}^*$,

$$a\lambda(x) + \pi(x) - a = a\left\lfloor \frac{x-b-1}{a} \right\rfloor + (\text{x mod a}) + \mathbf{1}_{[\![0,b]\!]}(\text{x mod a})$$

if x mod a $> b$: $a\lambda(x) + \pi(x) - a = a\left\lfloor \frac{x}{a} \right\rfloor + \text{x mod a} = x$

if x mod a $\leqslant b$: $a\lambda(x) + \pi(x) - a = a(\left\lfloor \frac{x}{a} \right\rfloor - 1) + \text{x mod a} + a = x$

Definition 3.3 *Let $(p,k,b) \in (\mathbb{N}^*)^3$, Let $(A_1, ..., A_k)$ a partition of $[\![1,p]\!]$. This partition is said to be a b-weakly-sum-free template (b-WSF-template) of k colors and lenght p when:*

<u>$\forall i \in [\![1,k]\!], \quad A_i$ *is weakly-sum-free*</u>

<u>$\forall i \in [\![1,k]\!], \quad A_i \backslash [\![1,b]\!]$ *is sum-free*</u>

<u>*For A_k (the special subset):*</u> $\forall (x,y) \in A_k^2$,

$$if \quad x + y > b + 2(p-b), \quad x + y - 2(p-b) \notin A_k$$

<u>*For the others subsets:*</u> $\forall i \in [\![0,k-1]\!], \forall (x,y) \in A_i^2$

$$x + y > p \implies \pi(x+y) \notin A_i$$

Definition 3.4 *Let $(k,b) \in (\mathbb{N}^*)^2$. If there exist p such that exists a partition of $[\![1,p]\!]$ into k subsets which is a b-WS-template of k colors and lenght p, we note:*

$WS_b^+ = -b + \max\{p \in \mathbb{N}^*/$ there exists a partition of $[\![1,p]\!]$ into k subsets which is a b-WSF-template of k colors and lenght p $\}$

If this p does not exist, we set $WS_b^+ = 0$

Definition 3.5 *Let $n \in \mathbb{N}^*$, we define $WS^+(n)$=$\max_{b\in\mathbb{N}^*}\{WS_b^+(n)\}$*

3.3 Lower bound for Weak schur numbers using Schur and Weak Schur template numbers

Theorem 3.2 *Let $(q,n,b) \in (\mathbb{N}^*)^3$, let $(p,k) \in (\mathbb{N}^*)^2$. If there exists a partition of k sum-free subsets of $[\![1,p]\!]$ and a partition of n subsets $(A_1, ..., A_n)$ of $[\![1,q]\!]$ which is a b-WSF of n colors and lenght q, then there exists a partition of $[\![1, b+p \times (q-b)]\!]$ into $(k+n-1)$ weakly sum-free subsets.*

In particular, if we choose $p = S(k)$ and $q = WS^+(n)$ in the last theorem, the next corollary follows.

Corollary 3.2.1 $\forall (n,k) \in (\mathbb{N}^*)^2$, *let $b_{max} = max\{b \in \mathbb{N}^*/WS^+(n+1) = WS_b^+(n+1)\}$,*

$$WS(n+k) \geqslant S(k)WS^+(n+1) + b_{max}$$

Proof :Let $(q,n,b) \in (\mathbb{N}^*)^3$, let $(p,k) \in (\mathbb{N}^*)^2$, let a=q-b.
We denote by g the projection of the equivalence relation induced by the partition of $[\![1,q]\!]$ and h the one induced by the partition of $[\![1,p]\!]$. Each equivalence class is represented by a single integer, therefore :

$$g : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } (A_{g^{-1}(1)}, ...A_{g^{-1}(q)}) \text{ is a b-WSF-template.}$$

$$h : [\![1,p]\!] \longrightarrow [\![1,k]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ h(x) = h(y) \Longrightarrow h(x+y) \neq h(x)$$

Let $f : [\![1, b+pa]\!] \longrightarrow [\![1,n]\!]$ such as:

-if x $\leqslant b$ (we will note x $\in \mathcal{T}$) : $f(x) = g(x)$

-if $x \in [\![1, b+pa]\!]$ and $\pi(x) \notin A_n$ (we will note x $\in \mathcal{C}$) : $f(x) = g(\pi(x))$

-if $x \in [\![1, b+pa]\!]$ and $\pi(x) \in A_n$ (we will note x $\in \mathcal{L}$) : $f(x) = n - 1 + h(\lambda(x))$

f is well defined because π is defined for x> b and $\forall x \in [1, b+pa]\!], f(x) \leqslant n+k-1$ because h($\lambda(x)) \leqslant k$

We have parted the integers of $[1, b+pa]\!]$ in three disjoints subsets $\mathcal{T}, \mathcal{C}$ and $\mathcal{L}$.

We have to verify that f induces weakly-sum-free templates:

<u>if (x,y) $\in (\mathcal{T})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

x+y<a+b since b<a and g(x)=f(x)=f(y)=g(y).
Hence f(x+y)=g(x+y)$\neq$g(x)=f(x)

<u>if (x,y) $\in \mathcal{T} \times \mathcal{C}$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>
We distinguish two cases:

- If x+$\pi(y) \leqslant a+b$
  g(x)=f(x)=f(y)=g($\pi(y)$. Hence g(x)$\neq g(x+\pi(y)) = g(\pi(x+y))$ (qv previous proposition)
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(x) = f(x)$
- If x+$\pi(y) > a+b$, x+y>a+b and by definition of g, $g(\pi(x+y)) \neq g(x)$
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(\pi(x)) = f(x)$

<u>if (x,y) $\in \mathcal{T} \times \mathcal{L}$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

Then, f(x)=f(y)=n. We distinguish two cases:

- If $\lambda(y) = \lambda(x+y)$,
  $g(x) = g(\pi(y)) = n$ since $g(y) = g(\pi(y))$
  Therefore $g(\pi(x+y)) = g(x+\pi(y)) \neq g(x) = n$(qv previous proposition)
  Hence $f(x+y) = g(\pi(x+y)) \neq n$
- If $\lambda(y) \neq \lambda(x+y)$, $\lambda(y) + 1 = \lambda(x+y)$
  n=f(y)=n-1+h($\lambda(y)$). Hence h($\lambda(y)$)=1.
  Moreover h(1)=1, therefore $h(\lambda(y)+1) \neq 1$
  if $\pi(x+y) \in A_n$, $f(x+y) = n - 1 + h(\lambda(x+y)) > n$

<u>if (x,y) $\in (\mathcal{C})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

Then $g(\pi(x)) = f(x) = f(y) = g(\pi(y))$. We distinguish two cases:

- If $\pi(x) + \pi(y) > q$, $g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$ (qv previous proposition)
  Hence $g(\pi(x+y)) = g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(\pi(x)) = f(x)$
- If $\pi(x) + \pi(y) \leqslant q$, $g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$ since g is sum-free for x$>b$
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) = g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x)) = f(x)$

<u>if (x,y) $\in \mathcal{C} \times \mathcal{L}$, $f(x) \neq f(y)$</u>

<u>if (x,y) $\in (\mathcal{L})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

Let r(x)=$\pi(x) - a$ and r(y)=$\pi(y) - a$,
We proved that $x = a\lambda(x) + \pi(x) - a$, therefore $x = a\lambda(x) + \pi(x)$
$x + y = a(\lambda(x) + \lambda(y)) + r(x) + r(y)$. We distinguish three cases:

- If $r(x) + r(y) \in [\![b - a + 1, b]\!]$, $h(\lambda(x)) = f(x) + 1 - n = f(y) + 1 - n = h(\lambda(y))$ Hence, $h(\lambda(x) + \lambda(y)) \neq h(\lambda(x))$.

$$\begin{aligned}
\lambda(x+y) &= 1 + \left\lfloor \frac{a(\lambda(x) + \lambda(y)) + r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) + \left\lfloor \frac{r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) - 1 + 1 \text{ since } r(x) + r(y) \in [\![b - a + 1, b]\!] \\
&= \lambda(x) + \lambda(y)
\end{aligned}$$

$$\begin{aligned}
\text{if f(x+y)}\geqslant\text{n, } f(x+y) &= n - 1 + h(\lambda(x+y)) \\
&= n - 1 + h(\lambda(x) + \lambda(y)) \\
&\neq n - 1 + h(\lambda(x)) \\
&= f(x)
\end{aligned}$$

- If $r(x) + r(y) > b$, $\pi(x) + \pi(y) > 2a + b$
  Since g is a b-WSF template, $g(\pi(\pi(x) + \pi(y))) \neq n$
  Therefore, $g(\pi(x+y)) \neq n$ ie $x + y \in \mathcal{C}$
  Hence $f(x+y) < n \leqslant f(x)$
- If $r(x) + r(y) \leqslant b - a,$, $\pi(x) + \pi(y) \leqslant b + a$
  Since g is sum-free for x$>b$, since $g(\pi(x)) = g(\pi(y)) = n$, $g(\pi(x+y)) \neq g(\pi(x)) = n$
  Hence, $f(x+y) < n \leqslant f(x)$

Remark 3.4 *There exists a $c(b) \geqslant min(A_{k+1} \backslash [\![1, b]\!] - b - 1$ such that $c(b)$ numbers can be added at the end of the extended partition. Therefore, we can get a better lower bound by finding a couple $(b, c(b))$ which maximizes the sum $b + c(b)$ such that $WS^+(k+1) = WS_b^+(k+1)$. Hence we have:*

$$WS(n+k) \geqslant S(n)WS^+(k+1) + b + c(b)$$

3.4 New lower bounds for Weak Schur numbers

Having found suitable templates, which can be found in the appendix, with a SAT solver, we claim that for all $n \in \mathbb{N}^*$:

$$4S(n) + 1 \leqslant WS(n+1)$$
$$13S(n) + 8 \leqslant WS(n+2)$$
$$42S(n) + 24 \leqslant WS(n+3)$$
$$127S(n) + 68 \leqslant WS(n+4)$$

The first two inequalities are due to Rowley, they are detailed in [2]. Like in 3.3, we compute the lower bounds given by the previous inequalities for $n \in [\![8, 15]\!]$. The best lower bound for each integer is highlighted.

n	8	9	10	11
$4S(n-1)+2$	6722	21146	71214	243794
$13S(n-2)+8$	6976	21848	68726	231447
$42S(n-3)+24$	6744	22536	70584	222036
$127S(n-4)+68$	5656	20388	68140	213428

n	12	13	14	15
$4S(n-1)+2$	815314	2554194	8045162	27061154
$13S(n-2)+8$	792332	2649772	8301132	26146778
$42S(n-3)+24$	747750	2559840	8560800	25886224
$127S(n-4)+68$	671390	2261049	7740464	25886224

With $S(9) \geqslant 17803$, we found a new lower bound for $WS(10)$ using Rowley's inequality. Moreover, the third inequality gives new lower bounds for $WS(9)$ and $WS(14)$. However, the last inequality doesn't give any better lower bound, even beyond $n = 15$: the best bounds are always provided by the first three. We highly suspect that these values can be improved by investigating the search space further, which would provide new, more effective templates. One may try to go over this search space using a Monte-Carlo method, as in [4], but with a different search space (as we explain in SAT section). This could be the suject of a future work.

4 About the construction of lower bounds for weak Schur numbers using a computer

In this section, we first reframe the question of the existence of (weakly) sum-free partitions as a boolean satisfiability (SAT) problem. We then provide evidence which indicates that the main assumption made by papers which found the previous best known lower bounds for weak Schur numbers may not be correct. Finally, we obtain stronger results than those previously known for $WS(5)$ while gaining several orders of magnitude in computation time by giving additional information to the SAT solver without losing in generality. In this section, we assume that the subsets are ordered.

4.1 Reformulation as a SAT problem

We encode the existence of (weakly) sum-free partitions as propositional formulae like in [7] and then use SAT solvers to determine whether these formulae are satisfiable.

Definition 4.1 *A literal is either a variable v (a positive literal) or the negation $\bar{v}$ of a variable v (a negative literal) where v takes a truth value: true or $false$. A clause is a disjunction of literals and a formula is a conjunction of clauses: it is a propositional formula in conjonctive normal form (CNF).*

Definition 4.2 *An assignment is a function from a set of variables to the truth values true (1) and $false$ (0). A literal l is satisfied (falsified) by an assignment α if l is positive and $\alpha(var(l)) = 1$ (resp. $\alpha(var(l)) = 0$) or if it is negative and $\alpha(var(l)) = 0$ (resp. $\alpha(var(l)) = 1$). A clause is satisfied by an assignment α if it contains a literal that is satisfied by α. Finally, a formula is satisfied by an assignment α if all its clauses are satisfied by α. A formula is satisfiable if there exists an assignment that satisfies it; otherwise it is unsatisfiable.*

We then encode the existence of a partition of $[\![1, p]\!]$ in k weakly sum-free subsets as follows: for every integer $i \in [\![1, p]\!]$, take k variables $x_1^{(i)}, ..., x_k^{(i)}$ and for every $\forall c \in [\![1, k]\!], x_c^{(i)} = 1 \iff i \in A_c$. The corresponding clauses are:

- **sumfree:** $\forall c \in [\![1, k]\!], \forall (i, j) \in [\![1, p]\!]^2, (i \neq j \text{ and } i + j \leq n) \implies \neg x_c^{(i)} \vee \neg x_c^{(i)} \vee \neg x_c^{(i+j)}$
- **union:** $\forall i \in [\![1, p]\!], x_1^{(i)} \vee ... \vee x_k^{(i)}$
- **disjoint:** $\forall i \in [\![1, p]\!], \forall (c_1, c_2) \in [\![1, k]\!]^2, c_1 \neq c_2 \implies \neg x_{c_1}^{(i)} \vee \neg x_{c_2}^{(i)}$

In the above formula, every color plays a symmetric role. Hence the search space can reduced by $k!$ by ordering the subsets, that is by enforcing that $m_1 < ... < m_k$. The corresponding clauses are: **symmetry breaking:** $x_1^{(1)} = 1$ and $\forall c \in [\![2, k-1]\!], \forall i \in [\![1, WS(c-1)+1]\!], x_c^{(1)} \vee ... \vee x_c^{(i)} \vee \neg x_{c+1}^{(i+1)}$

Remark 4.1 *For a given problem, it can be interesting to try out different SAT solvers because the relative performance can vary significantly according to the problem. For instance, we used two different SAT solvers in the next two next subsections.*

Remark 4.2 *Using a parallel SAT solver usually reduces the computation time, especially when trying to show that a formula is unsatisfiable. However, most of the parallel SAT solver do not have a deterministic behaviour and it can results in a strong variation of running times.*

4.2 The search space previously used in computer search for lower bounds may not contain the optimal partitions

Rowley's new lower bound for $WS(6)$ (642) [9] was a quite significant improvement upon the former best known lower bound (582) [5]. This previous lower bound was found using a computer (often with Monte-Carlo methods) and by making the assumption that a good partition for $WS(n+1)$ starts with a good partition for $WS(n)$ which is true for small values of n. Therefore, one may wonder whether the limiting factor are the assumptions or the methods used to search for partitions. It appears that the search space induced by these assumptions does not contain the optimal partitions.

Computational Theorem 4.1 *There is no weakly sum-free partition of* $[\![1, 583]\!]$ *in 6 parts such that:*

- $m_5 \geqslant 66$
- $m_6 \geqslant 186$
- $[\![210, 349]\!] \subset A_6$

This result was obtained in 8 hours with the SAT solver plingeling [3] on a 2.60 GHz Intel i7 processor PC. However, simply encoding the existence of such a partition as explained in the previous subsection would not result in a reasonable computation time. In order to help the SAT solver, we add additional information in the propositional formula. We did not quantify the speedup, but it most likely allowed us to gain several order of magnitude in computation time as we explain in the next subsection.

Every weakly sum-free partition of $[\![1, 65]\!]$ in 4 subsets starts with the following sequence 1121222133. Then 11 is always either in subset 1 or 3, 12 is always in subset 3 and so on. For every integer in $[\![1, 65]\!]$, we computed in which subset it can appear. By using this constraints, we could then compute for every integer in $[\![1, 185]\!]$, in which subset it can appear in a weakly sum-free partition of $[\![1, 185]\!]$ which starts with a weakly sum-free partition of $[\![1, 65]\!]$ in 4 subsets. Adding these constraints to the formula corresponding to the above theorem gives additional information to the SAT solver without losing in generality.

The above theorem shows that the previous lower bound for $WS(6)$ is optimal in the search space considered by the papers which found it. Therefore, finding a partition of $[\![1, n]\!]$ in 6 weakly sum-free subsets for some $n \geqslant 590$ which does not have a template-like structure would be extremely interesting since it could give indications on a new search space for improving lower bounds with a computer. More generally, it questions the search space previously used for finding lower bounds for $WS(n)$ with a computer. In particular, to our knowledge every paper that found the lower bound $WS(5) \geqslant 196$ used this assumption [reference necessaire]. Therefore one may wonder if this actually a good lower bound. In the next subsection, we give properties that a partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets has to verify.

4.3 Weak Schur number five

As explained in the previous subsection, the search space used for showing that $WS(5) \geqslant 196$ may not contain optimal solution. In this subsection, we give necessary conditions for a hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets using the same type of methods as in the previous subsection.

Notation 4.3 *Let P be a predicate over weakly sum-free partitions. We denote by $WS(n|P)$ the greatest number p such that there exists a partition of $[\![1, p]\!]$ in n weakly sum-free subsets which verifies P.*

[6] verified with a SAT solver that there are no partition in 5 weakly sum-free subsets of $[\![1, 197]\!]$ with $A_5 = \{67, 68\} \cup [\![70, 134]\!] \cup \{136\}$ in 17 hours and could not provide a similar result when only assuming $m_5 = 67$ even after several weeks of runtime. By using the same method as above, we were able to verify that $WS(5|m_5 = 67) = 196$ in 0.5 seconds with the SAT solver glucose [2] on a 2.60 GHz Intel i7 processor PC (we used the non-parallel version here for the sake of comparison but in the rest of this subsection, we used the parallel version of glucose). The additional information we gave to the SAT solver is that every partition of $[\![1, 66]\!]$ in 4 weakly sum-free subsets starts with a partition of $[\![1, 23]\!]$ in 3 weakly sum-free subsets (this can be checked in a few dozens of minutes with a SAT solver). Among the 3 partitions of $[\![1, 23]\!]$ in 3 weakly sum-free subsets, every number always appears in the same subset except for 16 and 17 which can appear in two different subsets. We hardcoded this external knowledge in the propositional formula which allowed us to gain several orders of magnitude in computation time. We also give the stronger following result.

Computational Theorem 4.2 *If there exists a partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets then $m_5 \leq$.*
NB : on doit finir d'obtenir les valeurs.

More precisely, we verified the following results (*max* m_5 is the greatest value of m_5 for which we have not verified that $WS(5|m_5) \leq 196$).

m_4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$WS(4\|m_4) + 1$	55	59	60	59	59	60	60	60	60	64	63	64	61	64	63	65	65	65	65	66	67
max m_5	49	51	54	53	54	54														57	53

To obtain these results, we once again provided additionnal information to the SAT solver. We also gave other types of information to the SAT solver. (pas encore fini)

5 Conclusions and future work

These new results come from Rowley's approach for Ramsey graphs which is relatively new. Therefore, it would not surprise us if lower bounds are later improved using better templates. Moreover, studying specifically $S_+(n)$ and $WS_+(n)$ numbers might be of interest as they are closely related to Schur and weak Schur numbers.

The fourth section gives new insight on the method that was formerly used to achieve new lower bounds for weak Schur numbers. The principle behind it might have discarded many interesting partitions of the search space and lowered the highest value that can be reached within it. However, algorithms based on randomness such as Monte-Carlo algorithms may appear very useful if used in a search space with more potential. For example, such an algorithm could be used to find better templates and improve inequalities and lower bounds for high values of n. This could be the subject fo a future work. Nevertheless, it would not help for the smallest values.

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A SF-templates

SF-template partitionning $[\![1, 111]\!]$ into 5 subsets

1	1, 5, 18, 12, 14 ,21, 23, 30, 32, 36, 39, 43, 45, 52, 103 106, 110
2	2, 6, 7, 10, 15, 18, 26, 29, 34, 37, 38, 42, 46, 51, 54 101, 104, 109
3	3, 4, 9, 11, 17, 19, 25, 27, 33, 35, 40, 41, 47, 48, 55 100, 107, 108
4	13, 16, 20, 22, 24, 28, 31, 58, 61, 67, 88, 94, 97
5	44, 50, 53, 56, 57, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85 86, 87, 89, 90, 91, 92, 93, 95, 96, 98, 99, 102, 105, 111

SF-template partitionning $[\![1, 380]\!]$ into 6 subsets

1	1, 5, 8, 11, 15, 17, 29, 33, 36, 39, 43, 57, 61, 88, 92 106, 110, 113, 116, 120, 132, 134, 138, 141, 144, 148, 150, 154, 157, 160 164, 178, 182, 185, 188, 341, 344, 347, 351, 365, 369, 372, 375, 379
2	2, 9, 13, 16, 20, 23, 24, 27, 28, 31, 34, 35, 38, 42, 45 49, 53, 60, 67, 71, 78, 82, 89, 96, 100, 104, 107, 111, 114, 115 118, 121, 122, 125, 126, 129, 133, 136, 140, 147, 158, 162, 165, 169, 172 176, 183, 187, 194, 201, 328, 335, 342, 346, 353, 357, 360, 364, 367, 371
3	3, 4, 12, 14, 19, 25, 30, 32, 40, 41, 47, 48, 58, 91, 101 102, 108, 109, 117, 119, 124, 130, 135, 137, 145, 146, 152, 153, 161, 163 168, 179, 181, 190, 339, 348, 350, 361, 366, 368, 376, 377
4	6, 7, 10, 18, 21, 22, 26, 37, 46, 50, 51, 54, 65, 70, 79 84, 95, 98, 99, 103, 112, 123, 127, 128, 131, 139, 142, 143, 151, 155 156, 159, 167, 170, 171, 175, 186, 343, 354, 358, 359, 362, 370, 373, 374 378
5	44, 52, 55, 56, 59, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75 76, 77, 80, 81, 83, 85, 86, 87, 90, 93, 94, 97, 105, 189, 196 197, 200, 203, 206, 207, 209, 214, 219, 231, 298, 310, 315, 320, 322, 323 326, 329, 332, 333, 340
6	149, 166, 173, 174, 177, 180, 184, 191, 192, 193, 195, 198, 199, 202, 204 205, 208, 210, 211, 212, 213, 215, 216, 217, 218, 220, 221, 222, 223, 224 225, 226, 227, 228, 229, 230, 232, 233, 234, 235, 236, 237, 238, 239, 240 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 299, 300, 301 302, 303, 304, 305, 306, 307, 308, 309, 311, 312, 313, 314, 316, 317, 318 319, 321, 324, 325, 327, 330, 331, 334, 336, 337, 338, 345, 349, 352, 355 356, 363, 380

B WSF-templates

WSF-template partitionning $[\![1, 42]\!]$ into 4 subsets

1	1, 2, 4, 8, 11, 22, 25, **(66)**
2	5, 6, 7, 19, 21, 23, 36
3	9, 10, 12, 13, 14, 15, 16, 17, 18, 20
4	24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40 41, 42

This template produces the inequality $WS(n+3) \geqslant 42S(n) + 24$ by placing 66 in the first subset.