New lower bounds for Schur and weak Schur numbers

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Abstract

This article provides new lower bounds for both Schur and weak Schur numbers. These results were obtained by continuing on Rowley's "template"-based approach for Schur and Ramsey numbers. Finding templates allows us to get explicit partitions improving lower bounds as well as the growth rate for both Schur numbers and Ramsey numbers $R_n(3)$. We also developed a method to improve lower bounds for weak Schur numbers. Furthermore, we try to analyze former works on weak Schur numbers based on the principle that *good* partitions into $n+1$ weakly sum-free subsets start with a *good* partition into n weakly sum-free subsets. We show that exceeding the previous lower bound $WS(6) \geqslant 582$ is impossible with such an assumption upon imposing certain conditions on the *good* 5-subsets partitions. The new lower bounds include $S(9) \geqslant 17\,803$, $S(10) \geqslant 60\,948$, $WS(9) \geqslant 22\,536$ and $WS(10) \geqslant 71\,214$.

1. Introduction

We are interested in partioning the set of integers $\{1, ..., p\}$ into n subsets such that there is no subset containing three integers x, y and z verifying $x + y = z$. We say these subsets are *sum-free*. If we add the hypothesis $x \neq y$, we say the subsets are *weakly sum-free*. The greatest p for which there exists a partition into n sum-free subsets is called the n^{th} Schur number and is denoted $S(n)$ [1]. Likewise for weakly sum-free partitions we define $WS(n)$ the n^{th} weak Schur number [2]. Only the first values of these sequences are known. In this study, we improve the lower bounds on these numbers.

1.1. State of the art

Before Rowley's "template"-based approach for Schur and Ramsey numbers [3], the previous generic construction for Schur numbers was given by Abbott and Hanson [4] in 1972 with a recursive construction. It used to give the best lower bounds for all sufficiently large numbers. No equivalent was known for weak Schur numbers and as a result the best known partitions for large weak Schur numbers did not use the weakly sum-free hypothesis.

As for smaller weak Schur numbers, the best lower bounds were obtained by conducting a computer search. Eliahou [5], Bouzy [6] and Rafilipojaona [7] improved the lower bounds with Monte-Carlo methods. This was the main approach during the past decade. This search for weakly sum-free partitions relied on the recursive assumption that a good weakly sum-free partition into $n+1$ subsets starts with a good weakly sum-free partition into n subsets.

In 2020, Rowley introduced the notion of templates for Schur and Ramsey numbers [3] which generalizes Abbott and Hanson's construction and produces new lower bounds (and inequalties) for Schur numbers. He also provides two inequalities for weak Schur numbers [8] that yield significant improvements over previous lower bounds, and besides do utilize the *weakly* sum-free hypothesis.

1.2. Structure of this article

Our main contribution is a generalization of the concept of template to weak Schur numbers. Our templates provide new lower bounds (and inequalities) for weak Schur numbers. This construction also includes, as a special case, a construction similar to Abbott and Hanson's [4], but this time with *weak* Schur numbers.

n	1	2	3	4	5	6	7	8	9	10	11	12
Before Rowley	1*	4*	13*	44*	160* [9]	536 [10]	1 680 [10]	5 041 [4]	15 124 [4]	51 120 [4]	172 216 [4]	575 664 [4]
Rowley [3]								5 286	17 694	60 320	201 696	631 840
Our results									**17 803**	**60 948**	**203 828**	**638 548**

Table 1: Comparison of lower bounds for Schur numbers

n	1	2	3	4	5	6	7	8	9	10	11	12
Before Rowley	2*	8*	23*	66*	196 [11]	582 [5]	1 740 [7]	5 201 [7]	15 596 [7]	51 520 [4]	172 216 [4]	575 664 [4]
Rowley [8]						642	2 146	6 976	21 848	70 778	241 282	806 786
Our results									**22 536**	**71 214**	**243 794**	**815 314**

* denotes the exact value

Table 2: Comparison of lower bounds for weak Schur numbers

In the section 2, we explain Rowley's template-based construction in the context of Schur numbers. Then, we give new templates, thus providing new lowers bounds and inequalities as well as showing that the growth rates for both Schur and Ramsey numbers $R_n(3)$ exceed 3.28.

Then, in the section 3, we generalize the concept of templates to weak Schur numbers and provide new lower bounds for weak Schur numbers.

We now introduce notations and definitions we use throughout this article.

1.3. Definitions and notations

We start by defining sum-free and weakly sum-free subsets to introduce regular and weak Schur numbers.

Definition 1.1. *A subset A of $\mathbb{N}$ is said to be sum-free if*

$$\forall (a,b) \in A^2,\ a+b \notin A$$

Definition 1.2. *A subset B of $\mathbb{N}$ is said to be weakly sum-free if*

$$\forall (a,b) \in B^2,\ a \neq b \Longrightarrow a+b \notin B$$

Let us notice that a sum-free subset is also weakly sum-free, hence justifying the name of *weakly* sum-free subsets. Given p and n two integers, we are interested in partitioning the set of integers $\{1, 2, ..., p\}$, denoted by $[\![1, p]\!]$, into n (weakly) sum-free subsets.

Schur proved in [1] that given a number of subsets n, there exists a value of p such that there exists no partition of $[\![1, q]\!]$ into n sum-free subsets for any $q \geqslant p$. A similar property holds for weakly sum-free subsets [2]. These observations lead to the following definitions.

Definition 1.3. *Let $n \in \mathbb{N}^*$. There exists a greatest integer denoted by $S(n)$ (resp. $WS(n)$) such that $[\![1, S(n)]\!]$ (resp. $[\![1, WS(n)]\!]$) can be partitioned into n sum-free subsets (resp. weakly sum-free subsets). $S(n)$ is called the n^{th} Schur number and $WS(n)$ the n^{th} weak Schur number.*

Given a partition of $[\![1, p]\!]$ into n subsets, we generally denote these subsets $A_1, ..., A_n$. We also denote $m_i = \min(A_i)$. By ordering the subsets, we mean assuming that $m_1 < ... < m_n$. However, if not specified we do not make this hypothesis since we do not always consider partitions in which every subset plays a symmetric role.

Definition 1.4. *We sometimes refer to a partitition as a coloring. The coloring associated to a partition* $A_1, ..., A_n$ *of* $[\![1, p]\!]$ *is the function* f *such that* $\forall x \in [\![1, p]\!], x \in A_{f(x)}$*. Likewise, the partition associated to a coloring* f *of* $[\![1, p]\!]$ *with* n *colors is* $\forall c \in [\![1, n]\!], A_c = f^{-1}(c)$*.*

2. Templates for Schur numbers

In this section, we use Rowley's template-based constructions [3] in the context of Schur numbers. In order to improve lower bounds for Schur and Ramsey numbers, Rowley introduces special sum-free partitions verifying some additional properties which can be extended using a method generalizing Abbott and Hanson's construction [4]. Rowley named these partitions "templates", and we keep this name in the entire article. We then find new templates and use them to provide new lower bounds for Schur numbers.

2.1. Definition of S^+

We begin by introducing *S-templates*, standing for *"Schur templates"*. The idea is to consider the first line of Figure 1 not as a combination of two blocs anymore but as a whole, single construction. We thereby define a S-template as a new object filling the role of the first line but with less, yet sufficiant, constraints that will allow the expansion of the table.

Definition 2.1. *Let* $(p, n) \in (\mathbb{N}^*)^2$*. A S-template with width* p *and* n *colors is defined as a partition of* $[\![1, p]\!]$ *into* n *sum-free subsets* $A_1, A_2, ..., A_n$ *verifying*

$$\forall i \in [\![1, n-1]\!], \forall (x, y) \in A_i^2, x + y > p \Longrightarrow x + y - p \notin A_i$$

Here n is the "special" color: the numbers colored in the special color has less constraints than the other colors. However, please note that n is not necessarily the last color by order of appearance.

Proposition 2.2. *Let* $n \in [\![2, +\infty[\![$*. We define* $S^+(n)$ *as the greatest integer such that* $[\![1, S^+(n)]\!]$ *is a S-template with* n *colors.* $S^+(n)$ *is well defined and verifies*

$$2S(n-1) + 1 \leqslant S^+(n) \leqslant S(n)$$

Proof. The lower bound comes from Abbott and Hanson's construction. The upper bound comes from the fact that a S-template with width p and n colors is also a partition of $[\![1, p]\!]$ into n sum-free subsets.

□

Remark 2.3. S^+ *and* S *have the same asymptotic growth rate.*

2.2. Construction of Schur partitions using S-templates

We start by reminding the explicite construction of a sum-free partition with the use of a S-template and another sum free partition. We apply this contruction stated by Rowley in the context of Ramsey numbers to the Schur numbers.

Theorem 2.4. *Let* $(p, k), (q, n) \in (\mathbb{N}^*)^2$*. If there exists a S-template with width* q *and* $n + 1$ *colors and a partition of* $[\![1, p]\!]$ *into* k *sum-free subsets then there exists a partition of* $[\![1, pq + m_{n+1} - 1]\!]$ *into* $n + k$ *sum-free subsets.* m_{n+1} *denotes the minimum element colored with the special color in the S-template.*

Setting $q = S^+(n + 1)$ and $p = S(k)$ yields the following corollary.

Corollary 2.5. *Let* $n, k \in \mathbb{N}^*$*. Then*

$$S(n + k) \geqslant S^+(n + 1)S(k) + m_{n+1} - 1$$

The idea lying beneath this theorem is similar to Abbott and Hanson's contruction [4]. They vertically extend a sum-free partition by repeating it and they use an other sum-free partition to color the other half according to the line number. This way the "blocks" act as safe areas for each other. We give here an example for $p = 4$, $q = 9$, $n = 2$ and $k = 2$ showing that $S(2+2) \geqslant S(2)(2S(2)+1) + S(2)$, both with Abbott and Hanson's construction and with a S-template which is not included in Abbott and Hanson's construction. In both cases, the special color is grey.

Abbott and Hanson's construction

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

Corresponding S-template

1	2	3	4	5	6	7	8	9

Corresponding sum-free partition

1	2	3	4

In the general construction with S-templates, the special color no longer necessarily contains consecutive numbers. However, the special color is still replaced by the colors of the sum-free partition according to the line number and the other colors are still vertically extended.

S-template construction

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

Corresponding S-template

1	2	3	4	5	6	7	8	9

Corresponding sum-free partition

1	2	3	4

We now proceed to prove Theorem 2.4

Proof. Denote by f the coloring associated to the S-template with width q and g the one associated to the sum-free partition of $[\![1,p]\!]$; where $f : [\![1,q]\!] \longrightarrow [\![1,n+1]\!]$ and $g : [\![1,p]\!] \longrightarrow [\![1,k]\!]$.

NB: In the following three predicates, the conditions $x+y \leqslant q$ and $x+y \leqslant p$ are omitted for readability.

The sum-free condition is expressed as:

$$\forall (x,y) \in [\![1,q]\!]^2, f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

,

$$\forall (x,y) \in [\![1,p]\!]^2, g(x) = g(y) \Longrightarrow g(x+y) \neq g(x)$$

.

The additionnal constraint for the S-template is:

$$\forall (x,y) \in [\![1,q]\!]^2, \begin{cases} f(x) = f(y) \leqslant n \\ x+y > q \end{cases} \implies f(x+y-q) \neq f(x)$$

.

For $x \in [\![1, pq+m_{n+1}-1]\!]$, we write $x = (\alpha-1)+u$ for certain integers $\alpha \in \mathbb{Z}$ and $u \in [\![1,q]\!]$. This decomposition is of course unique. α can be interpreted as the row number of x and u as the collumn number of x. A new coloring is define as follows:

$$\begin{array}{rccl} h: & [\![1, pq+m_{n+1}-1]\!] & \longrightarrow & [\![1, n+k]\!], \\ & x & \longmapsto & \begin{cases} f(u), & \text{if } f(u) \leqslant n, \\ n+g(\alpha), & \text{if } f(u) = n+1. \end{cases} \end{array}$$

Function h is well-defined since, by definition of m_{n+1}, $\forall x \in [\![pq+1, pq+m_{n+1}-1]\!], f(u) \leqslant n$ and therefore $\forall x \in [\![1, pq+m_{n+1}-1]\!], f(u) = n+1 \implies \alpha \in [\![1,p]\!]$.

We now prove that h is a sum-free coloring. Let $x, y \in [\![1, pq+m_{n+1}-1]\!]$ such that $h(x) = h(y)$ and $x+y \leqslant pq+m_{n+1}-1$. We claim that $h(x+y) \neq h(x)$. We write $x = (\alpha-1)q+u$ and $y = (\beta-1)q+v$ where $\alpha, \beta \in \mathbb{Z}$ and $u, v \in [\![1,q]\!]$. Two cases are to be distinguished according to the value of $h(x)$.

Case 1: $h(x) \leqslant n$

Let us assume that $h(x+y) \leqslant n$, otherwise $h(x+y) \neq h(x)$ obviously holds. By definition of function h and given that $h(u) = h(v)$, we conclude $f(u) = f(v)$. Two cases are to b distinguished according to the value of $x+y$.

- If $u+v > q$, we write $w = u+v-q \in [\![1,q]\!]$. Consequentely $x+y = (\alpha+\beta-1)q+w$. By definition, $h(x+y) = f(w)$. Given that $f(u) = f(v) \leqslant n$, the additionnal constraint on f implies $f(w) \neq f(u)$, that is $h(x+y) \neq h(x)$.

- If $u+v \leqslant p$, we write $w = u+v \in [\![1,q]\!]$. Consequentely $x+y = (\alpha+\beta-2)q+w$. By definition, $h(x+y) = f(w)$. Given that $f(u) = f(v) \leqslant n$, the sum-free property of f implies $f(w) \neq f(u)$, that is $h(x+y) \neq h(x)$.

Case 2: $h(x) \geqslant n+1$

Now we have $h(x) = n+g(\alpha) = k+g(\beta) = h(y)$, hence $g(\alpha) = g(\beta)$. As in the first case, distinguish between two cases according to the value of $x+y$.

- If $u+v > q$, write $w = u+v-q \in [\![1,q]\!]$. Then $x+y = (\alpha+\beta-1)q+w$. Assume that $h(x+y) \geqslant n+1$, otherwise $h(x+y) \neq h(x)$ obviously holds. By definition, $h(x+y) = n+g(\alpha+\beta)$. Given that $g(\alpha) = g(\beta)$, the sum-free property of g implies $g(\alpha+\beta) \neq g(\alpha)$ that is $h(x+y) \neq h(x)$.

- If $u+v \leqslant q$, write $w = u+v \in [\![1,q]\!]$. Then $x+y = (\alpha+\beta-2)q+w$. The sum-free property of f implies $f(w) \neq f(u)$. Therefore $f(w) \leqslant k$ and thus $h(x+y) \leqslant n$. In particular, given that $h(x) \geqslant n+1$, $h(x+y) \neq h(x)$.

□

The following proposition may improve the additive constant of Corollary 2.5

Proposition 2.6. *Let $(q,n) \in \mathbb{N}^*)^2$ and let f be a coloring associated to a S-template with width q and $n+1$ colors. Let $b \in \mathbb{N}$ and assume there is a coloring g of $[\![1,b]\!]$ with $n+1$ colors such that:*

- $\forall (x,y) \in [\![1,q]\!]^2, \begin{cases} f(x) = f(y) \\ (x+y) \mod q \leqslant b \end{cases} \implies g((x+y) \mod q) \neq f(x),$

- $\forall (x,y) \in [\![1,q]\!] \times [\![1,b]\!], \begin{cases} f(x) = g(y) \\ x+y \leqslant b \end{cases} \implies g(x+y) \neq f(x).$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $x \longmapsto g(x - pS(n))$, we have

$$S(n+k) \geqslant S^+(n+1)S(k) + b$$

.

This proposition corresponds to the fact that the hypotheses made on the coloring of the last row can be weakened and as a result, we can change the coloring on the last row to extend the previous partitions.

There is a construction theorem for S-templates as well.

Theorem 2.7. *Let $(p,k),(q,n) \in (\mathbb{N}^*)^2$. If there is a S-template with width q and $n+1$ colors, and a S-template with width p and k colors, then there also is a S-template with width pq and $(n+k)$ colors.*

The inequality associated with this theorem is given by:

Corollary 2.8. *Let $n,k \in \mathbb{N}^*$. Then*

$$S^+(n+k) \geqslant S^+(n+1)S^+(k)$$

.

Proof. The idea is the same as in the Theorem 2.4. The only difference is the S-template property inherited from the second S-template.

□

2.3. New lower bounds for Schur numbers

In the following, we give the strongest inequalities we have so far and further insight about how they were produced.

Definition 2.9. *A sum-free partition $A_1, ..., A_n$ of $[\![1,p]\!]$ is said to be symmetric if for all $x \in [\![1,p]\!]$, x and $p+1-x$ belong to the same subset (except if $x = p+1-x$).*

A S-template with n colors is said to be symmetric if the partition into n sum-free subsets derived from this template by applying the extension procedure with a sum-free partition of length 1 is symmetric.

We produced S-templates using a SAT solver, hence providing lower bound on S^+ and inequalities of the type $S(n+k) \geqslant aS(n)+b$. We sought templates providing the largest value of (a,b) (in the lexicographic order). When the number of colors excceeded five, in order to reduce the search space we only looked for symmetric S-templates, we assumed that the special color was the last color to appear and we constrained the m_c's out of being too small. Further details about the encoding as a SAT problem can be found in [9].

Here are the best inequalities on Schur numbers so far (the templates corresponding to the third, fourth and fifth inequalities can be found in the appendix).

$$S(n+1) \geqslant 3S(n) + 1 \quad (1)$$
$$S(n+2) \geqslant 9S(n) + 4 \quad (2)$$
$$S(n+3) \geqslant 33S(n) + 6 \quad (3)$$
$$S(n+4) \geqslant 111S(n) + 43 \quad (4)$$
$$S(n+5) \geqslant 380S(n) + 148 \quad (5)$$
$$S(n+6) \geqslant 1\,140S(n) + 528 \quad (6)$$

(1) comes from Schur's original article[1]. (2) is due to Abott [4] and (3) to Rowley [3]. (4), (5) and (6) are our result.

(1), (2) and (3) are optimal. (4) is optimal among symmetric S-templates whose special color is the last in the order of apparition (and with a multiplicative factor less than or equal to 118). (5) is most likely not optimal but should not be too far from the optimum. Finally, (6) is obtained by combining a S-template with width 380 and one with width three. Although we could not find a better S-template with seven colors, (6) is definitely very far from the optimal value. One may try to seek better templates by constraining less the search space and by using Monte-Carlo methods, as in [6]. This could be the subject of a future work.

The previous inequalities give new lower bounds for $S(n)$ for $n \geqslant 9$. We compute the lower bounds for $n \in [\![8, 15]\!]$ using the four different inequalities, please notice that the best values for $n = 8$ and $n = 13$ were obtained thanks to the first one, found by Rowley. The best lower bounds are highlighted.

Table 3: New lower bounds for $n \in [\![8, 15]\!]$

n	8	9	10	11
$33\,S(n-3)+6$	5 286	17 694	55 446	174 444
$111\,S(n-4)+43$	4927	17 803	59 539	186 523
$380\,S(n-5)+148$	5 088	16 868	60 948	203 828

n	12	13	14	15
$33\,S(n-3)+6$	587 505	2 011 290	6 726 330	21 072 090
$111\,S(n-4)+43$	586 789	1 976 176	6 765 271	22 624 951
$380\,S(n-5)+148$	638 548	2 008 828	6 765 288	23 160 388

Except for 8, 9 and 13, the best lower bounds are obtained thanks to the third inequality $S(n+5) \geqslant 380S(n)+148$. The table doesn't go any further, but the same inequality allows to improve the lower bounds for every $n \geqslant 15$.

Corollary 2.10. *The growth rate for Schur numbers (and Ramsey numbers $R_n(3)$) satisfies $\gamma \geqslant \sqrt[5]{380} \approx 3.28$.*

Proof. It is a mere consequence of the inequality $S(n+5) \geqslant 380S(n)+148$. As for Ramsey numbers, the following inequality holds $S(n) \leqslant R_n(3) - 2$ (see [4]) hence the result.

□

2.4. Conclusion on S-templates

In this section, we first formalized Rowley's template-based constructions [3] in the context of Schur numbers by introducing S-templates as well as a new sequence, S^+. We provided relations between S^+ and S then stated Rowley's construction method in the context of Schur numbers. We found new S-templates allowing us to obtain new lower bounds for schur numbers. One may notice that we mostly gave only lower bounds for S^+. It should be possible to find better S-templates by making different assumptions or using a different method (Monte-Carlo methods for instance).

In the next section, we provide similar results for weak Schur numbers. We introduce WS-templates and a corresponding sequence, WS^+. We then derive similar relations and a construction method allowing us to find new lower bounds for weak Schur numbers.

3. Templates for weak Schur numbers

In this section, we generalize Rowley's constructions for weak Schur numbers [8] and give an analogous for weak Schur numbers of Abbott and Hanson's construction for Schur numbers. By analogy with the previous section, we then introduce WSF-templates as well as an associated sequence $WS^+(n)$. We find templates and use them to provide new lower bounds for weak Schur numbers.

3.1. Inequality for weak Schur numbers using Schur and weak Schur numbers

Up to now, no equivalent for weak Schur numbers of Abbott and Hanson's construction for Schur numbers [4] was known. Here we give a general lower bound for weak Schur numbers as a function of both regular and weak Schur numbers. The following theorem, inspired by Rowley's inequalities for $WS(n+1)$ and $WS(n+2)$, was found and proved by Romain Ageron.

Theorem 3.1. *Let $(p,k),(q,n) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets then there exists a partition of $[\![1, p(q+\lceil \frac{q}{2} \rceil + 1) + q]\!]$ into $n+k$ weakly sum-free subsets.*

In particular, by setting $q = WS(n)$ and $p = S(k)$ in Theorem 3.1, one obtains the following corollary.

Corollary 3.2. $\forall (n,k) \in (\mathbb{N}^*)^2,\ WS(n+k) \geqslant S(k)\left(WS(n) + \left\lceil \dfrac{WS(n)}{2} \right\rceil + 1\right) + WS(n)$

This can be seen as an equivalent for weak Schur numbers of Abott and Hanson's construction for Schur numbers. This formula includes the results of Rowley [8] as a special case. For $n > 2$, this formula does not give new lower bounds.

Remark 3.3. *The inequality from Corollary 3.2 can be improved by adding 1 to the lower bound if $WS(n)$ is odd (more generally if q is odd in the theorem). However, it makes the proof less clear and it is never useful in practice.*

Given that Theorem 3.1 will appear as a particular case of a more general theorem after the introduction of templates for weak Schur numbers, we only give here an intuitive explanation of the demonstration; a formal proof can be found in the appendix.

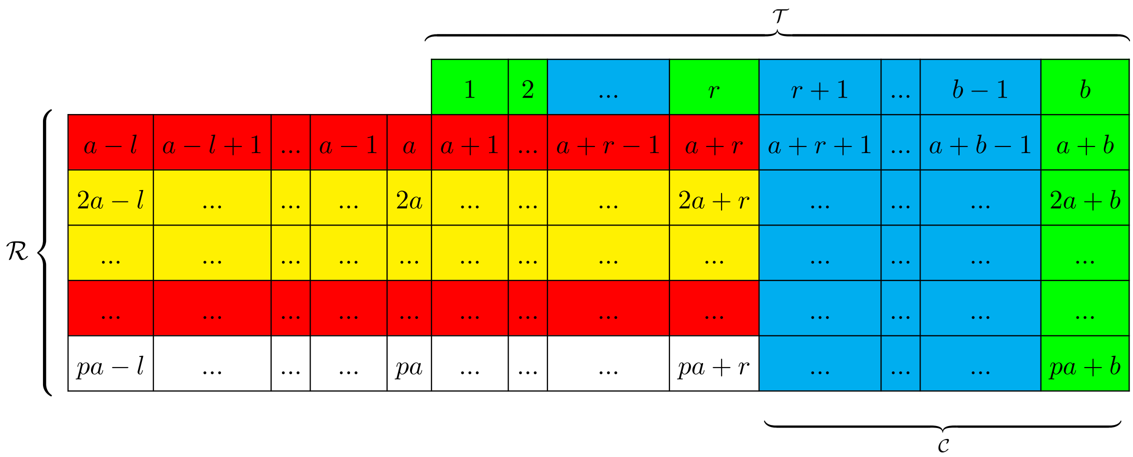
Let $(p,k),(q,n) \in (\mathbb{N}^*)^2$ such that there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets. Let $a \in \mathbb{N}$ with $a > q$ and let us try to build a coloring of $[\![1, ap+q]\!]$ into $n+k$ weakly sum-free subsets. Let $l = a-b-1$, $r \in [\![1,q]\!]$ and $w = a-l-r-1 = b-r$.

First, we put the integers of $[\![1, ap+q]\!]$ in the following table (with a columns and $p+1$ lines) and divide it into 3 blocks (the columns are numbered from $-l$ to $+q$).

- $\mathcal{T}$ (the "tail"): the integers from 1 to q. NB: this is line number 0.
- $\mathcal{R}$ (the "rows"): the integers in columns $-l$ to $+r$ (excluding $\mathcal{T}$).
- $\mathcal{C}$ (the "columns"): the integers in the last w columns (excluding $\mathcal{T}$).

Like SF-templates, $\mathcal{R}$ and $\mathcal{C}$ play the role of security zones for each other. Note that with this numbering of columns, the column of the sum of two numbers is the only integer in $[\![-l,q]\!]$ equal to two the sum of the columns modulo a.

Figure 1: Construction of the weakly sum-free coloring



$\mathcal{T}$ block

We color this block using the weakly sum-free coloring of $[\![1, q]\!]$ with colors $1, \ldots, n$.

$\mathcal{R}$ block

In this block, we use the colors $n+1, \ldots, n+k$. We color an integer x according to its line number (written $\lambda(x)$). For every $x \in \mathcal{R}$, we color x with $n+c$ where c is the color of $\lambda(x)$ in the sum-free coloring of $[\![1, p]\!]$. Let $(x, y) \in \mathcal{R}^2$. The cases are twofold.

- $\lambda(x+y) = \lambda(x) + \lambda(y)$
  In this case, we use the sum-free property of the coloring of $[\![1, p]\!]$ (in block $\mathcal{C}$, we only use colors $1, \ldots, n$).

- $\lambda(x+y) \neq \lambda(x) + \lambda(y)$
  In this case, we do not have information about the color of $\lambda(x+y)$. Thereby, we want to have $x+y \in \mathcal{C}$. A simple solution is to limit the horizontal movement, that is if the sum changes line (that is its line number is different from $\lambda(x)+\lambda(y)$), not to move too far from $(\lambda(x)+\lambda(y))a$ so that the sum stays in $\mathcal{C}$. There, the maximal displacement to the left (resp. to the right) is $2l$ (resp. $2r$). Not crossing entirely $\mathcal{C}$ by going to the left is then expressed as $-2l > -a+r$. Likewise, not going to far to the right is expressed as $2r < a-l$. It can then be written as $\max(l, r) \leqslant w$.

$\mathcal{C}$ block

In this block, we use colors $1, \ldots, n$. We color an integer x according to its column number, denoted by $\tilde{\pi}(x)$. It is linked to the projection on the first line, denoted by π, by the relation $\tilde{\pi}(x) = \pi(x) - a$. A simple solution is to color x with the same color as $\tilde{\pi}(x)$ in the weakly sum-free coloring of $[\![1, q]\!]$. As long as $2b \leqslant a+r$ (not going two far to the right) and there is no $x \in \tilde{\pi}(\mathcal{C})$ such that $2x \in \tilde{\pi}(\mathcal{C})$ (so that we do not have a sum in $\mathcal{C}$ when taking two numbers in the same column), the colors $1, \ldots, n$ are sum-free.

In particular, taking $w = l = \left\lceil \frac{q}{2} \right\rceil$ and $r = \left\lfloor \frac{q}{2} \right\rfloor$ works, thus obtaining the above theorem.

As in the previous section, we now introduce WSF-templates and the sequence WS^+ in order to generalize the above construction.

3.2. Definition of WS^+

In this subsection, we introduce WSF-templates and prove calculative results for the general construction theorem on templates for weak Schur numbers.

Definition 3.4. *Let* $(a,b) \in (\mathbb{N}^*)^2$ *such that* $a > b$. *We define :*

$$\pi_{a,b} : x \longmapsto (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x \mod a).$$

If there is no confusion on the a and b to use, $\pi_{a,b}$ is denoted by π. Notice that for all $x \in \mathbb{Z}, \pi(x) = x \mod a$ and for all $x \in [\![b+1, a+b]\!], b+1 \leqslant \pi(x) \leqslant a+b$.

π is the projection on the first line mentioned in the intuitive explanation. The following four propositions are calculative properties on π reflecting the behaviour of an element's column number in Figure 1 and that we will use later when we introduce WSF-templates.

Proposition 3.5.

$$\forall x \in [\![b+1, a+b]\!], \pi(x) = x$$

Proof. Let $x \in [\![b+1, a+b]\!]$. If $x < a$ then $x \mod a = x \notin [\![1, b]\!]$. Hence $\pi(x) = x$. Otherwise, $x \mod a = x - a \in [\![1, b]\!]$. Hence $\pi(x) = x - a + a = x$.

□

Proposition 3.6. *Let* $x \in [\![1, b]\!]$ *and* $y \in \mathbb{N}^*$. *Then*

$$\pi(x + \pi(y)) = \pi(x + y)$$

Proof. It is a direct consequence of $\pi(x) = x \mod a$.

□

Proposition 3.7. *Let* $x \in [\![1, b]\!]$ *and* $y \in \mathbb{N}^*$ *such that* $x + \pi(y) \leqslant a + b$. *Then*

$$\pi(x + y) = x + \pi(y)$$

Proof. $\pi(y) \geqslant b + 1$ and thus $b + 1 \leqslant x + \pi(y) \leqslant a + b$.

$$\begin{array}{rcll} \pi(x+y) & = & \pi(x + \pi(y)) & \text{by Proposition 3.6} \\ & = & x + \pi(y) & \text{by Proposition 3.5} \end{array}$$

□

Proposition 3.8. *Let* $(x, y) \in (\mathbb{N}^*)^2$. *Then*

$$\pi(\pi(x) + \pi(y)) = \pi(x + y)$$

Proof. It is a direct consequence of $\pi(x) = x \mod a$.

□

After defining the function related to the column number of each element in Figure 1, we introduce the function related to its line number.

Definition 3.9. *Let* $(a,b) \in (\mathbb{N}^*)^2$ *such that* $a > b$. *Define*

$$\lambda_{a,b} : x \longmapsto 1 + \left\lfloor \frac{x - b - 1}{a} \right\rfloor$$

If there is no confusion on the a and b to use, $\lambda_{a,b}$ is denoted by λ.

Function λ maps an element x to its line number as mentioned in the intuitive explanation. As we just did with π, we proove three more results on both π and λ that will appear useful in Subsection 3.3.

Proposition 3.10. *Let* $x \in \mathbb{N}^*$. *Then*

$$x = a\lambda(x) + \pi(x) - a$$

PROOF. Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$ and let $x \in \mathbb{N}^*$.
$a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x-b-1}{a} \right\rfloor + (x \mod a) + a\mathbb{1}_{[\![0,b]\!]}(x \mod a)$
if $x \mod a > b$ then $a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x}{a} \right\rfloor + x \mod a = x$
if $x \mod a \leqslant b$ then $a\lambda(x) + \pi(x) - a = a\left(\left\lfloor \dfrac{x}{a} \right\rfloor - 1\right) + x \mod a + a = x$

□

Proposition 3.11. *Let $x, y \in \mathbb{Z}$ such that $\lambda(x+y) = \lambda(y)$. Then*

$$\pi(x+y) = x + \pi(y)$$

PROOF. By applying Proposition 3.10 twice, we get $a\lambda(x+y)+\pi(x+y)-a = x+y = x+a\lambda(y)+\pi(y)-a$. The result is then obtained by simplying the equality.

□

Proposition 3.12. *Let $x, y \in \mathbb{Z}$ such that $\pi(x) + \pi(y) \in [\![a+b+1, 2a+b]\!]$. Then*

$$\lambda(x+y) = \lambda(x) + \lambda(y)$$

PROOF. By Proposition 3.10, $x + y = a(\lambda(x) + \lambda(y)) + \pi(x) + \pi(y) - 2a$. Then

$$\begin{aligned}
\lambda(x+y) &= \left\lfloor \frac{x+y-b-1}{a} \right\rfloor + 1 \\
&= \left\lfloor \frac{a(\lambda(x)+\lambda(y)) + \pi(x) + \pi(y) - 2a - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) - 1 \left\lfloor \frac{\pi(x) + \pi(y) - b - 1}{a} \right\rfloor \\
&= \lambda(x) + \lambda(y) - 1 + 1 \quad \text{since } \pi(x) + \pi(y) \in [\![a+b+1, 2a+b]\!] \\
&= \lambda(x) + \lambda(y)
\end{aligned}$$

□

Definition 3.13. *Let $(a,n,b) \in (\mathbb{N}^*)^3$ with $a > b$. Let $(A_1, ..., A_n)$ a partition of $[\![1, a+b]\!]$. This partition is said to be a b-weakly-sum-free template (b-WSF-template) with width a and n colors when :*

- $\forall i \in [\![1, n]\!], A_i$ *is weakly-sum-free,*
- $\forall i \in [\![1, n]\!], A_i \backslash [\![1, b]\!]$ *is sum-free,*
- *For A_n (the special subset) :*

$$\forall (x,y) \in A_n^2,\ x + y > b + 2a \implies x + y - 2a \notin A_n,$$

- *For the others subsets*

$$\forall i \in [\![0, n-1]\!],\ \forall (x,y) \in A_i^2,\ x + y > a + b \implies \pi(x+y) \notin A_i.$$

Please note that the special color n is not necessarily the last color by order of appearance. We now introduce the number $WS^+(n)$ that plays the same role as $\S^+(n)$ for SF-templates. However, WSF-templates being more sophisticated than SF-templates, the definition of $WS^+(n)$ is slighty more complicted.

Definition 3.14. *Let $(n,b) \in (\mathbb{N}^*)^2$. If there exists a such that there exists a b-WSF-template with width a and n colors, we define :*

$$WS_b^+(n) = \max\{a \in \mathbb{N}^* / \textit{there exists a } b\textit{-WSF-template with width } a \textit{ and } n \textit{ colors}\}.$$

If no such a exists, we set $WS_b^+(n) = 0$.

Definition 3.15. *Let $n \in \mathbb{N}^*$. We define :*

$$WS^+(n) = \max_{b \in \mathbb{N}^*} WS_b^+(n).$$

The following proposition briefly shows how $WS^+(n)$ compares to weak Schur numers.

Proposition 3.16. *Let $n \in [\![2, +\infty]\!]$. Then :*

$$\frac{3}{2} WS(n-1) + 1 \leqslant WS^+(n) \leqslant WS(n).$$

Proof. The lower bound comes from the analogous of Abott and Hanson's construction for weak Schur numbers. The upper bound comes from the fact that a WSF-template with width a and n colors contains a partition of $[\![1, a]\!]$ into n sum-free subsets.

□

Remark 3.17. *WS^+ and WS have the same asymptotic growth rate.*

We now proceed to state and prove the main result of this article.

3.3. Construction of weak Schur partitions using WSF-templates

Theorem 3.18. *Let $(a,n,b) \in (\mathbb{N}^*)^3$ with $a > b$ and $(p,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,p]\!]$ into k sum-free subsets and a b-WSF-template $(A_1, ..., A_{n+1})$ with width a and $n+1$ colors, then there exists a partition of $[\![1, pa+b]\!]$ into $k+n$ weakly sum-free subsets.*

In particular, by setting $p = S(k)$ and $a = WS^+(n+1)$ in Theorem 3.18, the next corollary follows.

Corollary 3.19. *Let $n, k \in \mathbb{N}^*$ and set $b_{max} = \max\{b \in \mathbb{N}^* / WS_b^+(n+1) = WS^+(n+1)\}$. Then :*

$$WS(n+k) \geqslant S(k)\, WS^+(n+1) + b_{max}.$$

Remark 3.20. *In the SF-template construction for Schur numbers, the additive constant comes from the fact that the special color does not necessarily appear right at the begining of the repeating pattern. Likewise, b_{max} can actually be replaced by*

$$\max_{b \in \mathbb{N}^*} \left\{ \min(A_{n+1} \backslash [\![1,b]\!]) - 1 \ | \ WS_b^+(n+1) = WS^+(n+1) \right\}.$$

Proof. Let $(a,n,b) \in (\mathbb{N}^*)^3$ and $(p,k) \in (\mathbb{N}^*)^2$. Denote by f the coloring associated to the b-WSF-template and g the one associated to the sum-free partition of $[\![1,p]\!]$; where $f : [\![1, a+b]\!] \longrightarrow [\![1, n+1]\!]$ and $g : [\![1,p]\!] \longrightarrow [\![1,k]\!]$. Moreover, assume that the sum-free coloring of $[\![1,p]\!]$ is ordered.

NB: To keep the notation short, the conditions $x + y \leqslant p$ and $x + y \leqslant a + b$ are omitted in the following five predicates.

The (weakly) sum-free conditions are expressed as:

$$\forall (x,y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) \\ x \neq y \end{cases} \Longrightarrow f(x+y) \neq f(x) \tag{7}$$

$$\forall (x,y) \in [\![b+1, a+b]\!]^2, f(x) = f(y) \Longrightarrow f(x+y) \neq f(x), \tag{8}$$

$$\forall (x, y) \in [\![1, p]\!]^2, g(x) = g(y) \Longrightarrow g(x + y) \neq g(x). \tag{9}$$

The additionnal constraints for the WSF-template are:

$$\forall (x, y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) \leqslant n \\ x + y > a + b \end{cases} \Longrightarrow f(\pi(x + y)) \neq f(x), \tag{10}$$

$$\forall (x, y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) = n + 1 \\ x + y > 2a + b \end{cases} \Longrightarrow f(x + y - 2a) \neq f(x). \tag{11}$$

Here, we consider π defined in subsection 3.2 as an application defined on $[\![b+1, pa+b]\!]$. Split $[\![1, pa+b]\!]$ into three subsets.

- $\mathcal{T} = [\![1, b]\!]$
- $\mathcal{C} = \pi^{-1}(f^{-1}([\![1, n]\!]))$
- $\mathcal{R} = \pi^{-1}(f^{-1}(\{n + 1\}))$

A new coloring h is defined as follows:

$$\begin{array}{rccl} h: & [\![1, pa+b]\!] & \longrightarrow & [\![1, n+k]\!] \\ & x & \longmapsto & \begin{cases} f(x) & \text{if } x \in \mathcal{T} \\ f(\pi(x)) & \text{if } x \in \mathcal{C} \\ n + g(\lambda(x)) & \text{if } x \in \mathcal{R} \end{cases} \end{array}$$

Function h is well defined since $(\mathcal{T}, \mathcal{C}, \mathcal{R})$ is a partition of $[\![1, pa+b]\!]$. We now prove that h is a weakly sum-free coloring. Let $x, y \in [\![1, pa+b]\!]$ be such that $x \neq y$, $h(x) = h(y)$ and $x + y \leqslant pa + b$. We claim that $h(x + y) \neq h(x)$. Nine cases are to be distinguished according to the subsets $(\mathcal{T}, \mathcal{C}, \mathcal{R})$ to which x and y belong. It is sufficient to check only six cases out of nine since x and y play symmetric roles.

Case 1: $(x, y) \in \mathcal{T}^2$

If $x + y \leqslant b$ then $h(x + y) = f(x + y)$. Otherwise, $b < x + y < a + b$ since $b < a$ and therefore $\pi(x + y) = x + y$ (proposition 3.5). Hence in both cases $h(x + y) = f(x + y)$. Given that f is a weakly sum-free coloring, $f(x + y) \neq f(x)$ since $f(x) = h(x) = h(y) = f(y)$ and $x \neq y$. That is $h(x + y) \neq h(x)$.

Case 2: $(x, y) \in \mathcal{T} \times \mathcal{C}$

Given that $h(x) = h(y)$ and by definition of h, $f(x) = f(\pi(y))$. Besides, $f(\pi(y)) \leqslant n$ since $y \in \mathcal{C}$. Two cases are to be distinguished according to the value of $x + \pi(y)$.

- If $x + \pi(y) \leqslant a + b$ then $f(x + \pi(y)) = f(\pi(x + y))$ (Proposition 3.7). Given that f is a weakly sum-free coloring, $f(x + \pi(y)) \neq f(x)$ since $f(x) = f(\pi(y))$ and $x \neq \pi(y)$ since $x \leqslant b < \pi(y)$.
- If $x + \pi(y) > a + b$ then given that f is a WSF-template and since $f(x) = f(\pi(y)) \leqslant n$, $f(\pi(x + \pi(y))) \neq f(x)$. Furthermore $f(\pi(x + \pi(y))) = f(\pi(x + y))$ (Proposition 3.6), such that $f(\pi(x + y)) \neq f(x)$.

Hence in both cases $f(\pi(x + y)) \neq f(x)$. If $f(\pi(x + y)) \leqslant n$ then $h(x + y) = f(\pi(x + y))$. Therefore $h(x + y) \neq h(x)$ since $f(x) = h(x)$. Otherwise, $f(\pi(x + y)) = n + 1$ and thus $h(x + y) > n$. In particular, $h(x + y) \neq h(x)$ since $h(x) = h(y) \leqslant n$.

Case 3: $(x, y) \in \mathcal{T} \times \mathcal{R}$

Necessarily $h(x) = h(y) = n + 1$. Two cases are to be distinguished according to the value of $\lambda(x + y)$.

- If $\lambda(y) = \lambda(x+y)$ then $\pi(x+y) = x + \pi(y)$ (Proposition 3.11). By definition of h, $f(x) = f(\pi(y))$. Given that f is a weakly sum-free coloring, $f(x + \pi(y)) \neq f(x)$ since $f(x) = f(\pi(y))$ and $x \neq \pi(y)$ since $x \leqslant b < \pi(y)$. Hence $h(x+y) \neq h(x)$.

- If $\lambda(y) \neq \lambda(x+y)$ then $\lambda(x+y) = \lambda(y)+1$ since $x \leqslant b < a$. Besides, $n+1 = h(y) = n+g(\lambda(y))$. Hence $g(\lambda(y)) = 1$. Moreover $g(1) = 1$ since g is an ordered coloring. Therefore, given that g is sum-free, $g(\lambda(y)+1) \neq 1$. If $\pi(x+y) \in A_{n+1}$ then $h(x+y) = n+g(\lambda(x+y)) \neq n+1$. Otherwise, $h(x+y) \leqslant n$. Hence in both cases $h(x+y) \neq h(x)$.

Case 4: $(x, y) \in \mathcal{C}^2$

By definition of h and since $h(x) = h(y)$, $f(\pi(x)) = f(\pi(y))$. Two cases are to be distinguished according to the value of $\pi(x) + \pi(y)$.

- If $\pi(x) + \pi(y) \leqslant a + b$ then $\pi(x) + \pi(y) = \pi(x+y)$. Hence $f(\pi(x+y)) \neq f(\pi(x))$ since f is sum-free for $x > b$

- If $\pi(x) + \pi(y) > a + b$ then given that f is a WSF-template, $f(\pi(\pi(x) + \pi(y))) \neq f(\pi(x))$ since $f(\pi(x)) = f(\pi(y))$. Besides, $f(\pi(\pi(x) + \pi(y))) = f(\pi(x+y))$ (Proposition 3.8). Hence $f(\pi(x+y)) \neq f(\pi(x))$.

Hence in both cases $f(\pi(x+y)) \neq f(x)$. If $f(\pi(x+y)) \leqslant n$ then $h(x+y) = f(\pi(x+y))$. Therefore $h(x+y) \neq h(x)$ since $f(x) = h(x)$. Otherwise, $f(\pi(x+y)) = n+1$ and thus $h(x+y) > n$. In particular, $h(x+y) \neq h(x)$ since $h(x) = h(y) \leqslant n$.

Case 5: $(x, y) \in \mathcal{C} \times \mathcal{R}$

By definition of h, $h(x) \neq h(y)$.

Case 6: $(x, y) \in \mathcal{R}^2$

In particular $f(\pi(x)) = f(\pi(y)) = n+1$. Three cases are to be distinguished according to the value of $\pi(x) + \pi(y)$.

- If $\pi(x) + \pi(y) \in [\![a+b+1, 2a+b]\!]$ then $\lambda(x+y) = \lambda(x) + \lambda(y)$ (Proposition 3.12). By definition of h and since $h(x) = h(y)$, $g(\lambda(x)) = g(\lambda(y))$. Hence, $h(\lambda(x+y)) \neq h(\lambda(x))$ since h is a sum-free coloring. If $f(x+y) \geqslant n+1$ then $h(x+y) = n + g(\lambda(x+y))$. And $h(x) = n + g(\lambda(x))$. Therefore, $h(x+y) \neq h(x)$. Otherwise $h(x+y) \leqslant n < h(x)$. In particular $h(x+y) \neq h(x)$.

- If $\pi(x) + \pi(y) > 2a + b$ then $f(\pi(\pi(x) + \pi(y))) \neq f(\pi(x)) = n+1$ since f is a b-WSF template and $f(\pi(x)) = f(\pi(y))$. Given that $\pi(\pi(x) + \pi(y)) = \pi(x+y)$ (Proposition 3.8), $f(\pi(x+y)) \neq n+1$.

- If $\pi(x) + \pi(y) \leqslant b + a$ then, given that $\pi(x) + \pi(y) \geqslant b$ and $f_{|[\![b,a+b]\!]}$ is sum-free, $f(\pi(x) + \pi(y)) \neq f(\pi(x)) = n+1$. That is $f(\pi(x+y)) \neq n+1$ (Proposition 3.5).

In both of the last two cases, $f(\pi(x+y)) \neq n+1$ that is $x + y \in \mathcal{C}$. Therefore $h(x+y) < n \leqslant h(x)$. In particular, $h(x+y) \neq h(x)$.

□

The general lower bound for weak Schur numbers in function of both regular and weak Schur numbers can be seen as a particular case of WSF-template in the same way Abott and Hanson's construction can be seen as a particular case of SF-template. Acutally, like for SF-templates, the additive constant of a WSF-template can be improved by weakening the hypotheses made on the last row. The principle behind it is the same as in the analogous proposition for SF-templates.

Proposition 3.21. *Let $(b, k, a) \in (\mathbb{N}^*)^3$ and let f be a coloring associated to a b-WSF-template with width p and k colors. Let $c \in \mathbb{N}$ ($c = \min(A_{k+1} \backslash [\![1, b]\!]) - 1$ works) and assume there there exists a coloring g of $[\![b+1, b+c]\!]$ with k colors such that for all $c \in [\![1, k]\!]$,*

- $\forall (x,y) \in [\![1, a+b]\!] \times [\![b+1, a+b]\!], \begin{cases} f(x) = f(y) \\ \pi(x+y) \leqslant b+c \end{cases} \implies g(\pi(x+y)) \neq f(x)$

- $\forall (x,y) \in [\![1, a+b]\!] \times [\![b+1, b+c]\!], \begin{cases} f(x) = g(y) \\ \pi(x+y) \leqslant b+c \end{cases} \implies g(\pi(x+y)) \neq f(x)$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $x \longmapsto g(x - pS(n))$, we have

$$WS(n+k) \geqslant WS^+(k+1)S(n) + b + c$$

The WSF-templates can actually be fine-tuned further. However, it gives only minor improvements (most likely only an additive constant) at the cost of dramatically increasing the size of the search space. Therefore, it does not seem relevant to use this sophistications given that we could not even find good WSF-templates with five colors using a computer (here good means better than those obtain by combining smaller templates).

These modifications work as follows. One may notice that the first row (excluding the "tail") has constraints that other rows do not have because of the tail, especially if the special color appears in the tail as well. Thus allowing to have a coloring on the first row different from the coloring of the other rows would weaken the constraints. Acutally, one may even go further by noticing that on the one hand the first (ordered) color of the sum-free partition used for the extension procedure has more more constraints than the other colors of the sum-free partition since the first row is of this color and is more constrained than the other rows, but that on the other hand it has more degrees of freedom than the other colors of the sum-free partition since in the sum-free partition there cannot be two consecutive numbers of this color. As a result, it removes some constraints imposed by the first row on the other rows.

To sum up, one can look for a generalised WSF-template that uses a special coloring for the tail and the first row, a coloring dedicated to the rows whose number is not 1 but is in the first color in the sum-free partition, a coloring for all the other rows and a special coloring for the last numbers (as previously explained for the improvement of the additive constant of WSF-templates).

We also have a similar theorem where only WS^+ is involved.

Theorem 3.22. *Let $(k,p) \in (\mathbb{N}^*)^2$ and $(a,n,b) \in (\mathbb{N}^*)^3$. If there exists a SF-template with width p and $k+1$ colors and a b-WSF-template with width a and n colors, then there exists pb-WSF-template with width pq and $(n+k)$ colors.*

Theorem 3.22 yields the following corollary.

Corollary 3.23. *Let $n, k \in \mathbb{N}^*$. Then*

$$WS^+(n+k) \geqslant S^+(k+1)\, WS^+(n)$$

Proof. The idea is the same as in the previous theorem. The only difference is the WSF property inherited from the WSF-template.

□

3.4. New lower bounds for Weak Schur numbers

We exhibited WSF-templates using a SAT solver, hence providing lower bound on WS^+ and inequalities of the type $WS(n+k) \geqslant aS(n) + b$. We have sought templates providing the greatest value of (a, b) (in the lexicographic order). As we will see, this way of computing lower bounds is not necessarily the best we can do. In the following, we give the strongest inequalities we have so far and further insight about how they were produced.

$$WS(n+1) \geqslant 4S(n)+2 \tag{12}$$
$$WS(n+2) \geqslant 13S(n)+8 \tag{13}$$
$$WS(n+3) \geqslant 42S(n)+24 \tag{14}$$
$$WS(n+4) \geqslant 132S(n)+26 \tag{15}$$

(12) and (13) were already found by Rowley, they are detailed in [8]. (14) is optimal and was found with a SAT solver. It uses the first sophistication explained in the Subsection 3.3 in order to add the last number in the first color. The corresponding template can be found in the appendix. As for (15), it was obtained by combining an optimal SF-template with width 33 with a WSF-template with width 4. The best template we could get with a computer search gives the inequality $WS(n+4) \geqslant 127S(n)+68$. It was also found with the SAT solver. In order to reduce the search space, we only looked for WSF-templates of five colors which start with a good $WS(4)$ partition and we assumed that the special color was the last by order of appearance. However, this approach most likely prevents us from finding the best WSF-templates as we explain in the next subsection for weakly sum-free partitions. We highly suspect that there exists more efficient WSF-templates with $n \geqslant 5$ colors. One may try to go over a different search space using a Monte-Carlo method, as in [6]. This could be the suject of a future work. Further details about the encoding as a SAT problem can be found in [9].

Like in Subsection 2.3 **LABEL NECESSAIRE**, we compute the lower bounds given by (12), (13) and (14) for $n \in [\![8, 15]\!]$. The best lower bound for each integer is highlighted.

Table 4: New lower bounds for $n \in [\![8, 15]\!]$

n	8	9	10	11
$4\,S(n-1)+2$	6 722	21 146	71 214	243 794
$13\,S(n-2)+8$	6 976	21 848	68 726	231 447
$42\,S(n-3)+24$	6 744	22 536	70 584	222 036

n	12	13	14	15
$4\,S(n-1)+2$	815 314	2 554 194	8 045 162	27 061 154
$13\,S(n-2)+8$	792 332	2 649 772	8 301 132	26 146 778
$42\,S(n-3)+24$	747 750	2 559 840	8 560 800	25 886 224

With $S(9) \geqslant 17\,803$, we found a new lower bound for $WS(10)$ using (12). Moreover, (14) gives new lower bounds for $WS(9)$ and $WS(14)$.

3.5. Conclusion on WSF-templates

In this section, we first gave a new construction which can be seen as an equivalent for weakly sum-free partitions of Abott and Hanson's construction for sum-free partitions. We then introduced WSF-templates and generalized this construction. This allows us to find new lower bounds and new inequalities for weak Schur numbers. One may notice the significant gap between the former lower bounds for weak Schur numbers obtained by conducting a computer search and the new lower bounds obtained with WSF-templates (including (12) and (13)).

4. Conclusions and future work

These new results come from an extension of Rowley's template-based approach for Ramsey graphs and Schur numbers which is relatively new. Therefore, we will not be surprised if lower bounds are later improved using better templates. Moreover, studying specifically $S^+(n)$ and $WS^+(n)$ might be of interest as they are closely related to Schur and weak Schur numbers. In order to find new templates, algorithms based on randomness such as Monte-Carlo algorithms may prove to be very useful. This could be the subject of a future work.

The fourth section gives new insight on the method that was formerly used to achieve new lower bounds for weak Schur numbers. The assumption behind it might have removed the optimal partitions from the search space and thus lowered the highest value that can be reached within it. However, the methods used to explore this search space are efficient as the optimum was found. Thereby Monte-Carlo algorithms may prove to be very useful if used in a search space with more potential. Finding a weakly sum-free partition which is better than previous lower bounds and which is not as regular as those obtained with templates would be extremely interesting (even if it does not improve current lower bounds) since it could suggest a new search space which could then be explored with the above mentioned methods.

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Appendix A. SF-templates

SF-template with width 33 and 4 colors

1	1, 6, 9, 13, 16, 20, 24, 27, 31
2	2, 5, 14, 15, 25, 26
3	3, 4, 10, 11, 12, 28, 29, 30
4	7, 8, 17, 18, 19, 21, 22, 23, 32, 33

SF-template with width 111 and 5 colors

1	1, 5, 18, 12, 14 ,21, 23, 30, 32, 36, 39, 43, 45, 52, 103 106, 110
2	2, 6, 7, 10, 15, 18, 26, 29, 34, 37, 38, 42, 46, 51, 54 101, 104, 109
3	3, 4, 9, 11, 17, 19, 25, 27, 33, 35, 40, 41, 47, 48, 55 100, 107, 108
4	13, 16, 20, 22, 24, 28, 31, 58, 61, 67, 88, 94, 97
5	44, 50, 53, 56, 57, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85 86, 87, 89, 90, 91, 92, 93, 95, 96, 98, 99, 102, 105, 111

SF-template with width 380 and 6 colors

1	1, 5, 8, 11, 15, 17, 29, 33, 36, 39, 43, 57, 61, 88, 92 106, 110, 113, 116, 120, 132, 134, 138, 141, 144, 148, 150, 154, 157, 160 164, 178, 182, 185, 188, 341, 344, 347, 351, 365, 369, 372, 375, 379
2	2, 9, 13, 16, 20, 23, 24, 27, 28, 31, 34, 35, 38, 42, 45 49, 53, 60, 67, 71, 78, 82, 89, 96, 100, 104, 107, 111, 114, 115 118, 121, 122, 125, 126, 129, 133, 136, 140, 147, 158, 162, 165, 169, 172 176, 183, 187, 194, 201, 328, 335, 342, 346, 353, 357, 360, 364, 367, 371
3	3, 4, 12, 14, 19, 25, 30, 32, 40, 41, 47, 48, 58, 91, 101 102, 108, 109, 117, 119, 124, 130, 135, 137, 145, 146, 152, 153, 161, 163 168, 179, 181, 190, 339, 348, 350, 361, 366, 368, 376, 377
4	6, 7, 10, 18, 21, 22, 26, 37, 46, 50, 51, 54, 65, 70, 79 84, 95, 98, 99, 103, 112, 123, 127, 128, 131, 139, 142, 143, 151, 155 156, 159, 167, 170, 171, 175, 186, 343, 354, 358, 359, 362, 370, 373, 374 378
5	44, 52, 55, 56, 59, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75 76, 77, 80, 81, 83, 85, 86, 87, 90, 93, 94, 97, 105, 189, 196 197, 200, 203, 206, 207, 209, 214, 219, 231, 298, 310, 315, 320, 322, 323 326, 329, 332, 333, 340
6	149, 166, 173, 174, 177, 180, 184, 191, 192, 193, 195, 198, 199, 202, 204 205, 208, 210, 211, 212, 213, 215, 216, 217, 218, 220, 221, 222, 223, 224 225, 226, 227, 228, 229, 230, 232, 233, 234, 235, 236, 237, 238, 239, 240 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 299, 300, 301 302, 303, 304, 305, 306, 307, 308, 309, 311, 312, 313, 314, 316, 317, 318 319, 321, 324, 325, 327, 330, 331, 334, 336, 337, 338, 345, 349, 352, 355 356, 363, 380

Appendix B. WSF-templates

23-WSF-template with width 42 and 4 colors

1	1, 2, 4, 8, 11, 22, 25, $(\mathbf{N+1})$
2	5, 6, 7, 19, 21, 23, 36
3	9, 10, 12, 13, 14, 15, 16, 17, 18, 20
4	24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40 41, 42

This template provides the inequality $WS(n+3) \geqslant 42S(n)+24$ by placing one last number, here represented by $(\mathbf{N+1})$, in the first subset.

Appendix C. Proof of theorem ??

Proof. Let $(p,q),(n,k) \in (\mathbb{N}^*)^2$, $N = p(q+\left\lceil \frac{q}{2} \right\rceil + 1) + q$, $\alpha = \left\lceil \frac{q}{2} \right\rceil > 0$ and $\beta = q + \alpha + 1$. We denote by f the coloring associated to the partition of $[\![1,q]\!]$ and g the one associated to the partition of $[\![1,p]\!]$.

$$f : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2, \begin{cases} x \neq y \\ f(x) = f(y) \end{cases} \Longrightarrow f(x+y) \neq f(x)$$

$$g : [\![1,p]\!] \longrightarrow [\![1,k]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

Let us start by parting the integers of $[\![1,N]\!]$ into two subsets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} = [\![1,\alpha]\!] \cup \{a\beta + u \mid (a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]\}$ and $\mathcal{B} = \{a\beta + u \mid (a,u) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]\}$.

First, $\underline{\mathcal{A} \cap \mathcal{B} = \varnothing}$:
By contradiction, suppose there exists $x \in \mathcal{A} \cap \mathcal{B} \neq \varnothing$. Then there are $(a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]$ and $(b,v) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$ such that $x = a\beta + u = b\beta + v$. By definition of α and β we have $u \in [\![\alpha+1,q]\!] \subset [\![0,\beta-1]\!]$. From there, we distinguish two cases :

- If $v \in [\![0,\alpha]\!]$ then $v \in [\![0,\beta-1]\!]$ and $v \neq u$ because $v < \alpha+1 \leqslant u$
- If $v \in [\![-\alpha,-1]\!]$, we note $\tilde{v} = \beta + v$ and thus have $x = (b-1)\beta + \tilde{v}$ with $\tilde{v} \in [\![\beta-\alpha,\beta-1]\!] \subset [\![0,\beta-1]\!]$ and $\tilde{v} \neq u$ because $u < q+1 = \beta - \alpha \leqslant \tilde{v}$.

In either cases, we run into a contradiction because of the remainder's uniqueness in the euclidean division of x byβ.

Then, we have $\underline{\mathcal{A} \cup \mathcal{B} = [\![1,N]\!]}$:

- On the one hand : $1 = \min(\mathcal{A}) \leqslant \max(\mathcal{A}) = p\beta + q = N$ and $1 \leqslant \beta - \alpha = \min(\mathcal{B}) \leqslant \max(\mathcal{B}) = p\beta + \alpha \leqslant N$, which gives $\mathcal{A} \cup \mathcal{B} \subset [\![1,N]\!]$.
- On the other hand, let $x \in [\![1,N]\!]$. If $x \leqslant \alpha$, we directly have $x \in \mathcal{A}$, let us then suppose that $x > \alpha$ and write $x = a\beta + u$ the euclidean division of x by β. We have $x \leqslant N$, thus $a \leqslant p$. We distinguish three cases :
  - If $u \in [\![0,\alpha]\!]$ then we necessarily have $a \geqslant 1$ because $x > \alpha$, and so $x \in \mathcal{B}$.
  - If $u \in [\![\alpha+1,q]\!]$, then $x \in \mathcal{A}$.
  - If $u \in [\![q+1,\beta-1]\!]$ then $x = (a+1)\beta - (\beta - u)$ with $-\alpha \leqslant \beta - u \leqslant 0$. Furthermore, $a \leqslant p-1$, else we would have $x > N$, and so $x \in \mathcal{B}$
  In any case, $x \in \mathcal{A} \cup \mathcal{B}$ and we can thus conclude that $[\![1,N]\!] \subset \mathcal{A} \cup \mathcal{B}$.

This first partition of $[\![1,N]\!]$ will help us to define our final partition by the projection of its equivalence relation. We thereby define $h : [\![1,N]\!] \longrightarrow [\![1,n+k]\!]$ as such :
- If $x \in \mathcal{A}$ then $h(x) = f(x \bmod \beta)$ (well defined because $x \bmod \beta \in [\![1,N]\!]$)
- If $x \in \mathcal{B}$ then $x = a\beta + u$ with a unique $(a,u) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$and we define $h(x) = n + g(a)$
The fact that $(\mathcal{A},\mathcal{B})$ is a partition of $[\![1,N]\!]$ ensures that this definition of h is valid. We then have to verify

that h induces weakly sum-free subsets.

The classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free :

Let $(x, y) \in \mathcal{A}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$

- If $(x, y) \in [\![1, \alpha]\!]^2$:
  We have $x + y \leqslant 2\alpha \leqslant q$ and $x + y = 0\beta + x + y$, therefore $x + y \in \mathcal{A}$. Then, by definition : $h(x) = f(x)$, $h(y) = f(y)$ and $h(x + y) = f(x + y)$, which gives us, thanks to the property verified by f, that $h(x + y) \neq h(x)$.

- If $(x, y) \in [\![1, \alpha]\!] \times (\mathcal{A} \setminus [\![1, \alpha]\!])$:
  We write $y = a\beta + u$ with $(a, u) \in [\![0, p]\!] \times [\![\alpha + 1, q]\!]$. Then $x + y = a\beta + x + u = (a + 1)\beta + x + u - \beta$, and if $x + u > q$ it follows that $a \leqslant p - 1$ since $x + y \leqslant N$, and $-\alpha \leqslant x + u - \beta \leqslant -1$. Therefore $x + y \in \mathcal{B}$ and $h(x + y) \neq h(x) = f(x)$ by definition of h. On the contrary, if $x - u \leqslant n$, then $x + y \in \mathcal{A}$ and $h(x + y) = f(x + u)$ because $x + u$ is actually the remainder of the euclidean divisionof $x + y$ by β. Moreover, $h(x) = f(x)$, $x < u$ and, with our initial hypothesis, $h(x) = h(y) = f(u)$. The property verified by f gives us $f(x + u) \neq f(x)$ which can be rewritten as $h(x + y) \neq h(x)$.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!]) \times [\![1, \alpha]\!]$:
  This case is handled exactly like the previous one by swaping the roles of x and y.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!])^2$:
  We write $x = a\beta + u$ and $y = b\beta + v$ with (a, u) and (b, v) in $[\![0, p]\!] \times [\![\alpha + 1, q]\!]$. Then $x + y = (a + b)\beta + u + v = (a + b + 1)\beta + u + v - \beta$ with $a + b \leqslant p - 1$ (else we would have $x + y > N$ because $u + v > q$) and $-\alpha \leqslant u + v - \beta \leqslant \alpha$, therefore $x + y \in \mathcal{B}$ and by definition $h(x + y) \neq h(x)$.

In any case, $h(x + y) \neq h(x)$ and the classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free.

The classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free :

Let $(x, y) \in \mathcal{B}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$.
We write $x = a\beta + u$ and $y = b\beta + v$ with (a, u) and (b, v) in $[\![1, p]\!] \times [\![-\alpha, \alpha]\!]$. We have $h(x) = q + g(a)$ and $h(y) = q + g(b)$, therefore $g(a) = g(b)$. We also have $x + y = (a + b)\beta + u + v$.
If $u + v \in [\![-\alpha, \alpha]\!]$, then $x + y \in \mathcal{B}$ and $h(x + y) = g(a + b)$, hence we can deduce that $h(x + y) \neq h(x)$ because of the property verified by g. On the contrary, if $u + v \notin [\![-\alpha, \alpha]\!]$, then necessarily $x + y \in \mathcal{A}$. Suppose $x + y \in \mathcal{B}$, then $x + y = c\beta + w$ with $(c, w) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]$. Thus, $c\beta + w = (a + b)\beta + u + v$ and $(a + b - c)\beta = w - u - v$. Furthermore $a + b - c \neq 0$, else we would have $u + v = w \in [\![-\alpha, \alpha]\!]$. This finally leads to the following inequality :

$$\beta \leqslant |a + b - c|\beta = |w - u - v| \leqslant |w| + |u| + |v| \leqslant 3\alpha \leqslant q + \alpha < \beta$$

which is absurd. We can therefore conclude that $x + y \in \mathcal{A}$ and by definition of h, $h(x + y) \neq h(x)$, proving that the classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free.

Finally, we have showed that every classe of equivalence induced by h is weakly sum-free, which ends the proof.

$\square$