Improving the templates for multicolor Ramsey numbers and Schur numbers

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Abstract

Lower bounds for several multicolor Ramsey numbers as well as Schur numbers have recently been improved using a pattern based construction with a particular type of coloring nammed "sf-template". This article describes a generalization of the sf-templates: the b-templates.

1 Introduction

This article addresses the edge-coloring of complete graphs in an arbitrary number of colors while limiting the clique number in each color, as well as sum-free colorings of positive integers. A pattern based construction which uses a particular type of coloring nammed sf-template was described in [1]. This construction was used to improve lower bounds on several Ramsey numbers as well as Schur numbers. Lower bounds were then improved by describing larger templates in [2] for Schur numbers and in [3] for multicolor Ramsey numbers.

Ramsey numbers and Schur numbers are defined in Section 2. Section 3 defines the b-templates and describes the corrresponding construction. Section 4 states and proves the main results on b-templates and on the construction. Section 5 records the new inequalities and some new lower bounds. The improved templates can be found in Appendix A.

2 Definitions

Definition 2.1 (Coloring and partition). *Let S and T be two sets. A coloring of S with colors in T is an application $\gamma : S \to T$. The partition associated to the coloring γ is the partition $(\gamma^{-1}(t))_{t\in T \text{ and } \gamma^{-1}(\{t\})\neq\varnothing}$. Conversely, the coloring associated to a partition $(A_i)_{i\in I}$ for some $I \subset T$ is the application $\gamma : S \to T$ such that $\forall s \in S, \forall t \in T, \gamma(s) = t \iff s \in A_t$.*

The set of positive natural numbers is denoted by $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Given two integers $a, b \in \mathbb{Z}$, the set of integers $\{a, a+1, ..., b-1, b\}$ is denoted by $[\![a, b]\!]$. For $n \in \mathbb{N}$, the complete graph of order n is denoted by K_n.

Definition 2.2 (Ramsey coloring). *Let $n \in \mathbb{N}$, the order of the graph, and let $r \in \mathbb{N}$, the number of colors. Let $(k_c)_{1\leqslant c\leqslant r} \in \mathbb{N}^{*r}$. A Ramsey coloring is an edge coloring of K_n with r colors such that for any color $c \in [\![1, r]\!]$, the coloring does not contain any monochramatic complete sub-graph of order k_c with color c. The set of these colorings is denoted by $\mathcal{R}(k_1, ..., k_r; n)$.*

Theorem 2.3 (Ramsey's theorem [4]). *Let $r \in \mathbb{N}$ and let $(k_c)_{1\leqslant c\leqslant r} \in \mathbb{N}^{*r}$. Then there is $n \in \mathbb{N}$ such that $\mathcal{R}(k_1, ..., k_r; n) = \varnothing$.*

Theorem 2.3 leads to the following definition.

Definition 2.4 (Multicolor Ramsey number)**.** *Let $r \in \mathbb{N}$ and let $(k_c)_{1 \leqslant c \leqslant r} \in \mathbb{N}^{*r}$. The multicolor Ramsey number $R(k_1, ..., k_r)$ is defined as the smallest integer $n \in \mathbb{N}$ such $\mathcal{R}(k_1, ..., k_r; n) = \varnothing$. If all the k_c's are equal to some k, this Ramsey number is also denoted by $R_r(k)$.*

The construction described in [1] as well as the one described in this article use a subset of Ramsey colorings, the linear Ramsey colorings.

Definition 2.5 (Linear Ramsey coloring)**.** *Let $n \in \mathbb{N}$, and let $r \in \mathbb{N}$. Let $(k_c)_{1 \leqslant c \leqslant r} \in \mathbb{N}^{*r}$. We assume that the vertices of K_{n+1} are the integers from $[\![0, n]\!]$. A linear Ramsey coloring is any Ramsey coloring such that the color of the edge (u, v) only depends on the value of $|u - v|$. The set of these colorings is denoted by $\mathcal{L}(k_1, ..., k_r; n)$. The integer coloring associated to a linear Ramsey coloring is the coloring of $[\![1, n]\!]$ such that every $x \in [\![1, n]\!]$ is colored with the color of the edges (u, v) such that $|u - v| = x$.*

Unlike Ramsey colorings, the set $\mathcal{L}(k_1, ..., k_r; n)$ corresponds to graph colorings of order $n+1$. This definition is chosen in this article because linear Ramsey colorings are studied using their associated integer coloring. Also, in this article the linear Ramsey numbers are defined as the largest size of a coloring (contrary to Ramsey numbers) because it is more convenient in inequalities.

Definition 2.6 (Linear Ramsey number)**.** *Let $r \in \mathbb{N}$ and let $(k_c)_{1 \leqslant c \leqslant r} \in \mathbb{N}^{*r}$. The linear Ramsey number $L(k_1, ..., k_r)$ is defined as the largest integer $n \in \mathbb{N}$ such that $\mathcal{L}(k_1, ..., k_r; n) \neq \varnothing$. If all the k_c's are equal to some k, this linear Ramsey number is also denoted by $L_r(k)$.*

Linear Ramsey numbers can be used to construct lower bounds for Ramsey numbers because a linear Ramsey coloring is also a Ramsey coloring. Therefore for all $r \in \mathbb{N}$ and for all $(k_c)_{1 \leqslant c \leqslant r} \in \mathbb{N}^{*r}$, $R(k_1, ..., k_r) \geqslant L(k_1, ..., k_r) + 2$.

Let $n \in \mathbb{N}$. Any linear coloring of K_{n+1} induces a coloring of $[\![1, n]\!]$. Conversely, any coloring of $[\![1, n]\!]$ corresponds to a linear coloring of K_{n+1}. Theorem 2.7 gives a link between some sum-free properties of the coloring of $[\![1, n]\!]$ and the clique numbers of the associated coloring of K_{n+1}.

Theorem 2.7 (Link to sum-free colorings [5])**.** *Let $r \in \mathbb{N}$ and let $(k_c)_{1 \leqslant c \leqslant r} \in \mathbb{N}^{*r}$. Let $n \in \mathbb{N}^*$ and let γ be a coloring of $[\![1, n]\!]$ with r colors. Then the two following conditions are equivalent.*

(i) There is a monochromatic subgraph of order k_c with color c in the linear coloring of K_{n+1} associated to γ.

(ii) There is a monochromatic subset S of $[\![1, n]\!]$ with color c and cardinality $k_c - 1$ such that for all distinct $x, y \in S$, $\gamma(|x - y|) = c$.

Definition 2.8 (Schur number)**.** *A subset A of $\mathbb{N}$ is said to be sum-free if $\forall (a, b) \in A^2, a + b \notin A$. Let $n \in \mathbb{N}$. Schur proved in [6] that there is a largest integer denoted by $S(n)$ such that $[\![1, S(n)]\!]$ can be partioned into n sum-free subsets. $S(n)$ is called the n^{th} Schur number.*

Theorem 2.7 shows a link between Schur numbers and Ramsey numbers:

$$\forall n \in \mathbb{N}^*, S(n) = L_n(3) \leqslant R_n(3) - 2.$$

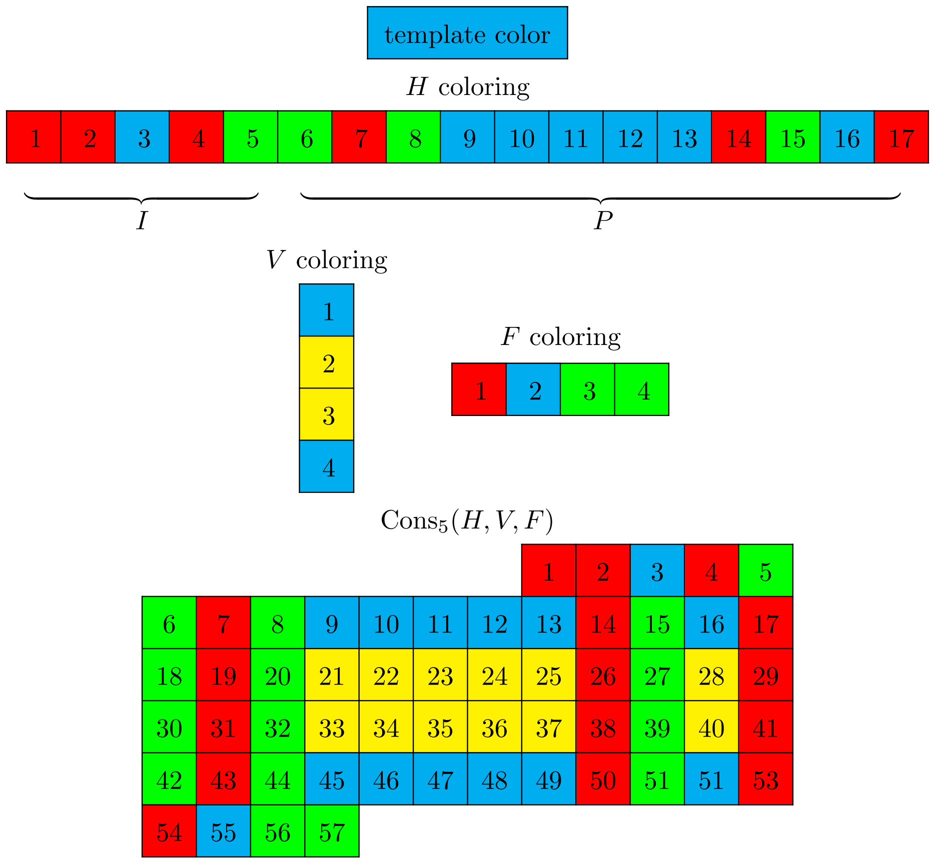
3 b-Templates

Definition 3.1 (Construction method)**.** *Let $(a, p) \in \mathbb{N}^{*2}$ and let $b \in \mathbb{N}$. Let $(r, s) \in \mathbb{N}^{*2}$. Let H (standing for "horizontal") be a coloring of $[\![1, a + b]\!]$ with $r + 1$ colors, and let V (standing for "vertical") be a coloring of $[\![1, p]\!]$ with s colors. Let $d \in \mathbb{N}$ and let F (standing for "final") be a coloring of $[\![1, d]\!]$ with $r + 1$ colors. The empty coloring (i.e. $d = 0$) is denoted by $F = \varnothing$. The construction produces a coloring $\mathrm{Cons}_b(H, V, F)$ of $[\![1, ap + b + d]\!]$ with $r + s$ colors.*

Informally, one starts by using the initial coloring $I := H_{|[\![1,b]\!]}$, then repeats p times the pattern $P := H_{|[\![b+1,a+b]\!]}$ while replacing at the i^{th} iteration the color $r + 1$ by $r + \mathcal{V}(i)$, and eventually appends the coloring $\mathcal{F}$. Formally:

$$\begin{array}{ccl} [\![1, ap + b + d]\!] & \longrightarrow & [\![1, r + s]\!] \\ x & \longmapsto & \begin{cases} H(x) & \text{if } x \leqslant b \\ H(\pi(x)) & \text{if } b + 1 \leqslant x \leqslant ap + b \text{ and } H(\pi(x)) \neq r + 1 \\ r + V\left(\left\lceil \dfrac{x-b}{a} \right\rceil\right) & \text{if } b + 1 \leqslant x \leqslant ap + b \text{ and } H(\pi(x)) = r + 1 \\ F(x - ap - b) & \text{if } x \geqslant ap + b + 1 \end{cases} \end{array}$$

Figure 1: Example of construction with $b = 5$



In some cases, it may be useful to use a specific coloring for the final coloring. However, a default coloring associated to the horizontal coloring can be used instead.

Definition 3.2 (Default coloring)**.** *Let $n \in \mathbb{N}^*$, $b \in \mathbb{N}$ and $r \in \mathbb{N}$. Let H be a coloring of $[\![1, n]\!]$ with $r+1$ colors. Set $d = \min\left(H^{-1}(\{r+1\}) \backslash [\![1, b]\!]\right) - b - 1$. The default coloring* $\text{Default}_b(H)$ *associated to H is defined as follows.*

$$\begin{array}{rccc} \text{Default}_b(H): & [\![1, d]\!] & \longrightarrow & [\![1, r+1]\!] \\ & x & \longmapsto & H(x+b) \end{array}$$

The construction described in [1] for sf-templates corresponds to the case $b = 0$ and it uses the default coloring. Contrary to sf-templates, b-templates are directly defined as particular colorings for which the clique numbers remain unchanged in the construction. The rest of this section defines the b-templates as well as the notion of compatibility for the final coloring.

In order to describe the configurations in which a given tuple can induce a monochromatic clique when using the construction, a tree is associated to every tuple.

Definition 3.3. *A tree node has two components: the first one is the label and the second one is the set of descendants of this node. A branch of a tree T is a tuple $(x_i)_{1 \leqslant i \leqslant k}$ such that there is a sequence of trees $(t_i)_{1 \leqslant i \leqslant k}$ satisfying: $t_1 = T$ and for all $i \in [\![1, k]\!]$ x_i is the label of t_i and for all $i \in [\![1, k-1]\!]$ t_{i+1} is a descendant of t_i.*

Let $a \in \mathbb{N}^*$ and $b \in \mathbb{N}$. The $\text{Tree}_{a,b}$ *application is recurssively defined as follows.*

$$\begin{cases} \forall x \in \mathbb{Z}, \text{Tree}_{a,b}(x) = (x, \varnothing), \\ \forall k \in [\![2, +\infty[\![, \forall x \in \mathbb{Z}^k, \\ \quad \begin{cases} \text{Tree}_{a,b}(x) = (x_1, \{\text{Tree}_{a,b}(x_2, ..., x_k)\}) & \text{if } x_2 - x_1 > b \\ \text{Tree}_{a,b}(x) = (x_1, \{\text{Tree}_{a,b}(x_2, ..., x_k), \text{Tree}_{a,b}(x_2 + a, ..., x_k)\}) & \text{if } 0 \leqslant x_2 - x_1 \leqslant b \\ \text{Tree}_{a,b}(x) = \text{Tree}_{a,b}\left(x_1, x_2 + a\left\lceil \dfrac{x_1 - x_2}{a} \right\rceil, x_3, ..., x_k\right) & \text{if } x_2 < x_1 \end{cases} \end{cases}$$

Let x be a non-empty tuple of integers. The set of all the branches of the tree $\text{Tree}_{a,b}(x)$ *is denoted by* $\text{TreeSet}_{a,b}(x)$*.*

Let $A \subset \mathbb{N}^*$ and let $k \in [\![2, +\infty]\!]$. When using the construction, not every tuple in A^k have a tree which describes cliques that appear in the construction.

Definition 3.4. *Let $A \subset \mathbb{N}^*$, $b \in \mathbb{N}$ and $k \in [\![2, +\infty[\![$. The set of k-tuples of A^k whose values are pariwise distinct is denoted by $\mathfrak{S}_k(A)$. The set of k-tuples of $\mathfrak{S}_k(A)$ such that the elements of a tuple that are in $[\![1, b]\!]$ appear in increasing order at begining of this tuple is denoted by $S_{b,k}(A)$. That is $S_{b,k}(A) = \{x \in \mathfrak{S}_k(A) \ : \ \forall i \in [\![2, k]\!], x_i \leqslant b \implies x_i \geqslant x_{i-1}\}$. The set of branches of trees of tuples in $S_{b,k}(A)$ is denoted by $TS_{b,k}(A)$: $TS_{b,k}(A) = \bigcup\limits_{x \in S_{b,k}(A)} \text{TreeSet}(x)$.*

Most of the integers which appear in the tuples of $TS_{b,k}(A)$ (as well as the differences of these integers) are not in $[\![1, a+b]\!]$. The role of the projection $\pi_{a,b}$ is to project these integers onto $[\![1, a+b]\!]$ in order to describe the constraints defning the b-templates.

Definition 3.5. *Let $a \in \mathbb{N}^*$ and $b \in \mathbb{N}$. The projection $\pi_{a,b}$ is defined as follows.*

$$\begin{array}{rccl} \pi_{a,b}: & \mathbb{Z} & \longrightarrow & [\![1, a+b]\!] \\ & x & \longmapsto & \begin{cases} x & \text{if } 1 \leqslant x \leqslant b \\ (x \mod a) + a\mathbb{1}_{[\![0,b]\!]}(x \mod a) & \text{if } x > b \end{cases} \end{array}$$

The b-templates are designed to be used as the horizontal coloring in the construction.

Definition 3.6 (b-Templates)**.** *Let $r \in \mathbb{N}$ and $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Let $a \in \mathbb{N}^*$ and $b \in [\![0, a-1]\!]$. A b-template with width a and $r+1$ colors is defined as a partition of $[\![1, a+b]\!]$ into $r+1$ subsets $A_1, ..., A_{r+1}$ such that:*

(i) $a \in A_{r+1}$,

(ii) $\forall (x, y) \in A_{r+1}^2, x + y \notin [\![a+b+1, 2a+b]\!] \implies \pi_{a,b}(x+y) \notin A_{r+1}$,

(iii) $\forall c \in [\![1, c]\!], \forall x \in TS_{b,k_c-1}(A_c), \exists (i,j) \in [\![1, k_c - 1]\!]^2, i < j$ *et* $\pi_{a,b}(x_j - x_i) \notin A_c$.

The set of these b-templates is denoted by $\mathcal{T}_b(k_1, ..., k_r, t; a)$. Color $r+1$ plays a special role and is nammed "template color".

Color $r+1$ is not necessarily the last color by order of appearance, the designation of $r+1$ as template color symbolized by "t" is a convention that lightens notations and avoids writting $\mathcal{T}_b(k_1, ..., k_{i-1}, t, k_{i+1}, ..., k_{r+1}; n)$ for instance.

If all the k_c's are equal to 3, these b-templates can be used to produce sum-free partitions for Schur numbers.

b-Templates are a particular case of linear colorings. In the template color, the corresponding clique number is equal to 3. That is $T_b(k_1, ..., k_r, t; a) \subset L(k_1, ..., k_r, 3; a+b)$.

The notion of compatibility with a b-template illustrates the fact that the final coloring in the construction is less constrained than the horizontal coloring, and that the constraints on this final coloring depends only on the horizontal coloring.

Definition 3.7 (Compatibility with a b-template)**.** *For $A \subset \mathbb{N}$ and $k \in \mathbb{N}$, the set of increasing sequences of length k with values in A is denoted by $\mathfrak{I}_k(A)$.*

Let $a \in \mathbb{N}^*$, $b \in [\![0, a-1]\!]$ and $r \in \mathbb{N}$. Let $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Let $T \in \mathcal{T}_b(k_1, ..., k_r, t; a)$ and denote by $A_1, ..., A_{r+1}$ the associated partition. Let $d \in [\![0, a]\!]$. Let F be a coloring of $[\![1, d]\!]$ with $r+1$ colors and denote by $B_1, ..., B_{r+1}$ the associated partition. The coloring F is said to be compatible with the b-template T if it satisfies:

(i) $a - b \notin B_{r+1}$

(ii) $\forall (x, y) \in A_{r+1}^2, (x + y > a + b$ *and* $\pi_{a,b}(x+y) \leqslant d) \implies \pi_{a,b}(x+y) \notin B_{r+1}$

(iii) $\forall (x, y) \in A_{r+1} \times B_{r+1}, x + y \leqslant d \implies x + y \notin B_{r+1}$

(iv) $\forall c \in [\![1, r]\!], \forall k \in [\![1, k_c - 1]\!], \forall x \in TS_{b,k}(A_c), \forall y \in \mathfrak{I}_{k_c-k-1}(B_c)$,

$$\left\{\begin{array}{ll} & \exists (i,j) \in [\![1, k]\!]^2, \pi_{a,b}(x_j - x_i) \notin A_c, \\ & \exists (i,j) \in [\![1, k]\!] \times [\![1, k_c - k - 1]\!], \\ \textit{or} & \left\{\begin{array}{ll} (y_j - x_i) \mod a \notin A_c & \textit{if } (y_j - x_i) \mod a > b \\ (y_j - x_i) \mod a \notin A_c \textit{ and } \pi_{a,b}(y_j - x_i) \notin A_c & \textit{if } (y_j - x_i) \mod a \leqslant b \end{array}\right. \\ & \exists (i,j) \in [\![1, k_c - k - 1]\!]^2, i < j \textit{ and } \pi_{a,b}(y_j - y_i) \notin A_c. \end{array}\right.$$

4 Properties of the b-templates

On a une cractérisation similaire à celle donnée dans [1] pour les sf-templates.

Proposition 4.1 (Condition suffisante pour les templates)**.** *Soit $a \in \mathbb{N}^*$, soit $b \in \mathbb{N}$ et soit $r \in \mathbb{N}$. Soit $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Soit T un coloriage de $[\![1, a+b]\!]$ à $r+1$ couleurs. Soit $d \in [\![1, a-1]\!]$ et soit F un coloriage de $[\![1, d]\!]$ à $r+1$ couleurs. Soit $n \in [\![3, +\infty[\![$ tel que $n \geqslant \max_{1 \leqslant c \leqslant r} k_c - 1$. Soit $s \in \mathbb{N}^*$ et soit $(l_c)_{1 \leqslant c \leqslant s} \in [\![3, +\infty[\![^s$. Soit enfin $\gamma \in \mathcal{L}(l_1, ..., l_s; n)$ tel qu'il existe une couleur $c \in [\![1, s]\!]$ et un sous-ensemble S de $[\![1, n]\!]$ monochromatique de couleur c et de cardinal $k_c - 2$ vérifiant pour tous $x, y \in S$ distincts, $\gamma(|x-y|) = c$. Supposons que $\mathrm{Cons}_b(T, \gamma, F) \in \mathcal{L}(k_1, ..., k_r, l_1, ..., l_s; an + b + d)$. Alors $T \in \mathcal{T}_b(k_1, ..., k_r, t; a)$ et F est compatible avec T.*

Proposition 4.2 (Compatibilité de la valeur par défaut)**.** *Soit $r \in \mathbb{N}$ et soit $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Soit $a \in \mathbb{N}^*$ et soit $b \in \mathbb{N}$. Soit $T \in \mathcal{T}_b(k_1, ..., k_r, t; a)$ un b-template. Alors T et $\mathrm{Default}_b(T)$ sont compatibles.*

Theorem 4.3 (Construction de coloriages linéaires)**.** *Soit $r \in \mathbb{N}$ et soit $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Soit $a \in \mathbb{N}^*$ et soit $b \in \mathbb{N}$. Soit H un coloriage de $[\![1, a+b]\!]$ à $r+1$ couleurs. Soit $d \in [\![0, a-1]\!]$ et soit F un coloriage de $[\![1, d]\!]$ à $r+1$ couleurs. Les deux conditions suivantes sont équivalentes.*

(i) $H \in \mathcal{T}_b(k_1, ..., k_r, t; a)$ *et* F *est compatible avec* H.

(ii) $\forall s \in \mathbb{N}^*, \forall l \in [\![3, +\infty[\![^s, \forall V \in \mathcal{L}(l_1, ..., l_s; p), \mathrm{Cons}_b(H, V, F) \in \mathcal{L}(k_1, ..., k_r, l_1, ..., l_s; a \times p + b + d)$

Corollary 4.4 (Inégalités pour les nombres de Ramsey linéaires)**.** *Soit $r \in \mathbb{N}$ et soit $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Soit $s \in \mathbb{N}$ et soit $(l_c)_{1 \leqslant c \leqslant s} \in [\![3, +\infty[\![^s$. Soient $a \in \mathbb{N}^*$ et $b \in \mathbb{N}$ tels que $\mathcal{T}_b(l_1, ..., l_s, t; a) \neq \varnothing$, et on fixe T un tel b-template. Soit $d \in [\![1, a]\!]$ tel qu'il existe un coloriage de $[\![1, d]\!]$ à $s+1$ couleurs compatible avec T. Alors $L(k_1, ..., k_r, l_1, ..., l_s) - 2 \geqslant a(L(k_1, ..., k_r) - 2) + b + d$.*

La construction présentée dans [7] utilise des coloriages linéaires afin de construire des 0-templates. La proposition suivante présente cette construction sous l'angle des b-templates.

Proposition 4.5. *Soit $r \in \mathbb{N}$ et soit $(k_c)_{1 \leqslant c \leqslant r} \in [\![3, +\infty[\![^r$. Soit $n \in \mathbb{N}$ tel que $\mathcal{L}(k_1, ..., k_r; n) \neq \varnothing$ et on se donne γ un tel coloriage. On définit un nouveau coloriage T de la manière suivante.*

$$\begin{array}{rccl} T: & [\![1, 2n+1]\!] & \longrightarrow & [\![1, r+1]\!] \\ & x & \longmapsto & \begin{cases} \gamma(x) & si\ x \leqslant n \\ r+1 & si\ x > n \end{cases} \end{array}$$

Alors $T \in \mathcal{T}_0(k_1, ..., k_r, t; 2n+1)$.

Lemma 4.6 (associativité de la construction)**.** *TODO*

$$\mathrm{Cons}_{b_{H_1}}(H_1, ,)$$

Theorem 4.7 (Construction de b-templates)**.** *TODO*

5 New inequalites and lower bounds

As in [2] and [3], the b-templates described in this article were found by encoding the existence of the templates into a boolean formula and then using a SAT solver (here, the parallel version of the Lingeling SAT solver [8]).

5.1 Multicolor Ramsey numbers

The templates in Appendix A provide inequalities of the form $L(k_1, ..., k_r, k_{r+1}, ..., k_{r+s}) \geqslant a\, L(k_1, ..., k_r) - 2 + d$. In table 5.1,"added k_i's" denotes the $k_{r+1}, ..., k_{r+s}$ in the previous inequality and "b-template" indicates a b-template which cannot be expressed as a sf-template.

Table 1: Summarizing table of the new inequalities

added k_i's	former (a, d)	new (a, d)	template type	final coloring
3, 3	9, 4 [5]	10, 2	b-template	default
3, 4	18, 7 [3]	19, 2	b-template	default
3, 5	30, 12 [3]	31, 3	b-template	specific
4, 5	51, 21 [3]	55, 23	sf-template	default

Table 2: Summarizing table of the new lower bounds for small Ramsey numbers

Ramsey number	former lower bound	new lower bound
$R_8(3)$	5 288 [1]	5 364
$R_{13}(3)$	2 011 292 [2]	2 038 284
$R(3, 4, 5, 5)$	729 [3]	764

The template in $\mathcal{L}_2(3, 3, t; 10)$ induces an inequality which improves the lower bounds for $R_{5n+3}(3)$ for all $n \in \mathbb{N}^*$.

5.2 Schur numbers

The existence of a 2-template with width 10 and 3 colors which cannot be expressed as a sf-template provides an improved inequality. The former corresponding inequality is $S(n + 2) \geqslant 9\, S(n) + 4$ [5].

Proposition 5.1.

$$\forall n \in \mathbb{N}, S(n + 2) \geqslant 10\, S(n) + 2$$

This inequality provides the following improved lower bounds: $S(8) \geqslant 5\,362$ and $S(13) \geqslant 2\,038\,282$ (the former lower bounds were respectively 5 286 [1] and 2 011 290 [2]). More generally, this inequality improves the lower bounds for $S(5n + 3)$ for all $n \in \mathbb{N}^*$.

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A b-Templates

Table 3: $T \in \mathcal{T}_2(3, 3, t; 10)$

1	1, 5, 8, 12
2	2, 6, 7
3	3, 4, 9, 10, 11

Final coloring: $\varnothing$

Table 4: $T \in \mathcal{T}_2(3, 4, t; 19)$

1	1, 6, 9, 13, 17, 21
2	2, 7, 8, 10, 14, 15, 16
3	3, 4, 5, 11, 12, 18, 19, 20

Final coloring: $\varnothing$

Table 5: $T \in \mathcal{T}_2(3, 5, t; 31)$

1	1, 5, 9, 12, 16, 19, 23, 26, 30
2	2, 6, 7, 8, 10, 11, 17, 18, 24, 25, 27, 28, 29, 33
3	3, 4, 13, 14, 15, 20, 21, 22, 31, 32

Final coloring: $F(1) = 1$

Table 6: $T \in \mathcal{T}_0(5, 4, t; 55)$

1	1, 5, 6, 7, 9, 10, 12, 14, 15, 17, 18, 19, 23, 25, 31, 48, 54
2	2, 3, 4, 8, 11, 13, 16, 20, 21, 22, 26, 28, 51, 52, 53
3	24, 27, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39 40, 41, 42, 43, 44, 45, 46, 47, 49, 50, 55

Final coloring: $\mathrm{Default}_0(T)$