New lower bounds for Schur and weak Schur numbers

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Abstract

This article provides new lower bounds for both Schur and weak Schur numbers by exploiting a "template"-based approach. The concept of "template" is also generalized to weak Schur numbers. Finding new templates leads to explicit partitions improving lower bounds as well as the growth rate for Schur numbers, weak Schur numbers and multicolor Ramsey numbers $R_n(3)$. The new lower bounds include $S(9) \geqslant 17\,803$, $S(10) \geqslant 60\,948$, $WS(6) \geqslant 646$, $WS(9) \geqslant 22\,536$ and $WS(10) \geqslant 71\,256$.

Keywords: Schur number, Weak Schur number, Ramsey theory, Sumfree partition
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1. Introduction

We are interested in partioning the set of integers $\{1, ..., p\}$ into n subsets such that there is no subset containing three integers x, y and z verifying $x + y = z$. We say these subsets are *sum-free*. If we add the hypothesis $x \neq y$, we say the subsets are *weakly sum-free*. The greatest p for which there exists a partition into n sum-free subsets is called the n^{th} Schur number and is denoted $S(n)$ [1]. Likewise for weakly sum-free partitions we define $WS(n)$ the n^{th} weak Schur number [2]. Values of $S(n)$ and $WS(n)$ are known for small n only.

1.1. State of the art

Up to recently, the most efficient generic construction for Schur numbers was given by Abbott and Hanson [3] in 1972 with a recursive construction. It gave the best lower bounds for all sufficiently large numbers. No equivalent was known for weak Schur numbers and as a result the best known partitions for large weak Schur numbers did not use the weakly sum-free hypothesis.

As for smaller weak Schur numbers, the best lower bounds were obtained by conducting a computer search. Eliahou [4], Bouzy [5] and Rafilipojaona [6] improved the lower bounds with Monte-Carlo methods. This was the main approach during the past decade.

In 2020, Rowley introduced the notion of templates for Schur and Ramsey numbers [7] which generalizes Abbott and Hanson's construction and produces new lower bounds (and inequalties) for Schur numbers. He also provided two inequalities for weak Schur numbers [8] that yield significant improvements over previous lower bounds. Besides, these two inequalities do use the *weakly* sum-free hypothesis.

The hereunder tables recap the lower bounds respectively for Schur and weak Schur numbers. Gray cells indicate the exact values.

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n	1	2	3	4	5	6	7	8	9	10	11	12
State of the art	1	4	13	44	160 [9]	536 [10]	1 696 [11]	5 286 [7]	17 694 [7]	60 320 [7]	201 696 [7]	637 856 [7]
Our results									**17 803**	**60 948**	**203 828**	**644 628**

Table 1: Comparison of lower bounds for Schur numbers

n	1	2	3	4	5	6	7	8	9	10	11	12
State of the art	2	8	23	66	196 [12]	642 [8]	2 146 [8]	6 976 [8]	22 056 [8]	70 778 [8]	241 282 [8]	806 786 [8]
Our results						**646**			**22 536**	**71 256**	**243 794**	**815 314**

Table 2: Comparison of lower bounds for weak Schur numbers

1.2. Structure of this article

Our main contribution is a generalization of the concept of template to weak Schur numbers. Our templates provide new lower bounds (and inequalities) for weak Schur numbers. This construction also includes as a special case a construction similar to Abbott and Hanson's [3], but this time for *weak* Schur numbers.

In Section 2, we explain Rowley's template-based construction in the context of Schur numbers. Then, we give new templates, thus providing new lowers bounds and inequalities as well as showing that the growth rates for both Schur and Ramsey numbers $R_n(3)$ exceed 3.28.

In Section 3, we generalize the concept of template to weak Schur numbers and provide new lower bounds for weak Schur numbers. We then use a different approach and give a new lower bound for $WS(6)$.

We now introduce notations and definitions we use throughout this article.

1.3. Definitions and notations

We start by defining sum-free and weakly sum-free subsets to introduce regular and weak Schur numbers. We denote $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

Definition 1.1. *A subset A of $\mathbb{N}$ is said to be sum-free if*

$$\forall (a, b) \in A^2,\ a + b \notin A.$$

Definition 1.2. *A subset B of $\mathbb{N}$ is said to be weakly sum-free if*

$$\forall (a, b) \in B^2,\ a \neq b \Longrightarrow a + b \notin B.$$

Let us notice that a sum-free subset is also weakly sum-free, hence justifying the name of *weakly* sum-free subsets. Given p and n two integers, we are interested in partitioning the set of integers $\{1, 2, ..., p\}$, denoted by $[\![1, p]\!]$, into n (weakly) sum-free subsets.

Schur proved in [1] that given a number of subsets n, there exists a value of p such that there exists no partition of $[\![1, q]\!]$ into n sum-free subsets for any $q \geqslant p$. A similar property holds for weakly sum-free subsets [2]. These observations lead to the following definitions.

Definition 1.3. *Let $n \in \mathbb{N}^*$. There exists a largest integer denoted by $S(n)$ such that $[\![1, S(n)]\!]$ can be partitioned into n sum-free subsets. $S(n)$ is called the n^{th} Schur number.*

Definition 1.4. *Let $n \in \mathbb{N}^*$. There exists a largest integer denoted by $WS(n)$ such that $[\![1, WS(n)]\!]$ can be partitioned into n weakly sum-free subsets. $WS(n)$ is called the n^{th} weak Schur number.*

Given a partition of $[\![1,p]\!]$ into n subsets, we generally denote these subsets $A_1, ..., A_n$. We also denote $m_i = \min(A_i)$. By ordering the subsets, we mean assuming that $m_1 < ... < m_n$. However, if not specified we do not make this hypothesis since we do not always consider partitions in which every subset plays a symmetric role.

Definition 1.5. *We sometimes refer to a partitition as a coloring. The coloring associated to a partition* $A_1, ..., A_n$ *of* $[\![1,p]\!]$ *is the function* f *such that* $\forall x \in [\![1,p]\!], x \in A_{f(x)}$. *Likewise, the partition associated to a coloring* f *of* $[\![1,p]\!]$ *with* n *colors is* $\forall c \in [\![1,n]\!], A_c = f^{-1}(c)$.

2. Templates for Schur numbers

In this section, we use Rowley's template-based constructions [7] in the context of Schur numbers. In order to improve lower bounds for Schur and Ramsey numbers, he introduces special sum-free partitions verifying some additional properties which can be extended using a method generalizing Abbott and Hanson's construction [3]. He named these partitions "templates", and we keep this name in the entire article. We then find new templates and use them to provide new lower bounds for Schur numbers.

2.1. Definition of $[\![1, S^+]\!]$

We begin by introducing *S-templates*, standing for *"Schur templates"*. The idea is to consider the first line of Figure 1 not as a combination of two blocs anymore but as a whole, single construction. A S-template is then defined as a new object filling the role of the first line but with less, yet sufficient, constraints which allow for an expansion of the partition using a sum-free partition.

Definition 2.1. *Let* $(p,n) \in (\mathbb{N}^*)^2$. *A S-template with width* p *and* n *colors is defined as a partition of* $[\![1,p]\!]$ *into* n *sum-free subsets* $A_1, A_2, ..., A_n$ *verifying for all subsets but* A_n

$$\forall i \in [\![1, n-1]\!], \forall (x,y) \in A_i^2, x+y > p \Longrightarrow x+y-p \notin A_i.$$

Here n can be seen as a "special" color in the sense that we do not impose this above additionnal constraint on this color. However, note that n is not necessarily the last color by order of appearance, any color can play this role.

Proposition 2.2. *Let* $n \in [\![2, +\infty[\![$. *We define* $S^+(n)$ *as the greatest integer* w *such that there exists a S-template with width* w *and* n *colors.* $S^+(n)$ *is well defined and verifies*

$$2S(n-1)+1 \leqslant S^+(n) \leqslant S(n).$$

PROOF. The upper bound comes from the fact that a S-template with width p and n colors is also a partition of $[\![1,p]\!]$ into n sum-free subsets. The lower bound comes from Abbott and Hanson's construction [3].
□

2.2. Construction of Schur partitions using S-templates

We start by reminding the explicit construction of a sum-free partition with the use of a S-template and another sum free partition. We rephrase in terms of Schur numbers this contruction stated by Rowley in the context of Ramsey numbers [7].

Theorem 2.3. *Let* $(p,k),(q,n) \in (\mathbb{N}^*)^2$. *If there exists a S-template with width* q *and* $n+1$ *colors and a partition of* $[\![1,p]\!]$ *into* k *sum-free subsets then there exists a partition of* $[\![1, pq+m_{n+1}-1]\!]$ *into* $n+k$ *sum-free subsets.* m_{n+1} *denotes the minimum element colored with the special color in the S-template.*

Setting $q = S^+(n+1)$ and $p = S(k)$ yields the following corollary.

Corollary 2.4. *Let* $n, k \in \mathbb{N}^*$. *Then*

$$S(n+k) \geqslant S^+(n+1)S(k) + m_{n+1} - 1.$$

The idea lying beneath this theorem is similar to Abbott and Hanson's contruction [3]. They vertically extend a sum-free partition by repeating it and they use an other sum-free partition to color the other half according to the line number. This way the "blocks" act as safe areas for each other. We give here an example for $p = 4$, $q = 9$, $n = 2$ and $k = 2$ showing that $S(2+2) \geqslant S(2)(2S(2)+1)+S(2)$, both with Abbott and Hanson's construction and with a S-template which is not included in Abbott and Hanson's construction. In both cases, the special color is grey.

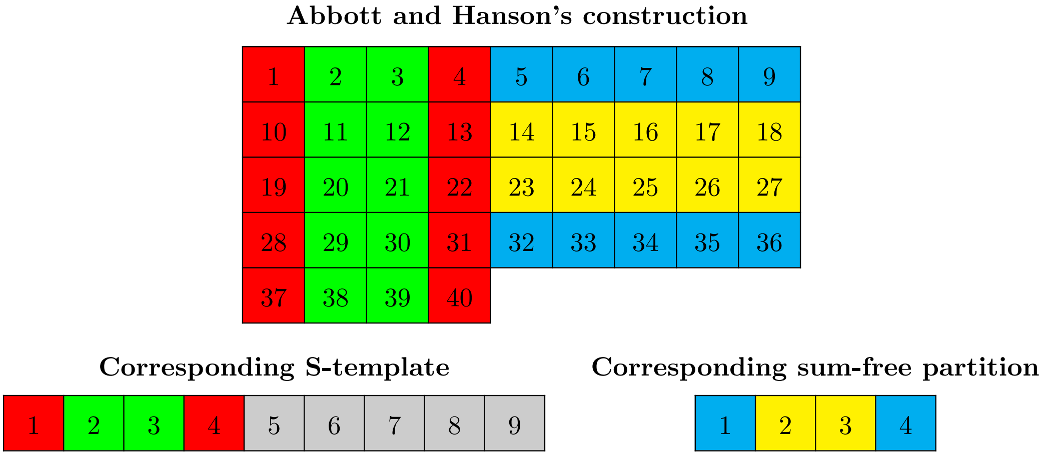

Figure 1: Visualisation of an Abbott and Hanson construction

In the general construction with S-templates, the special color no longer necessarily contains consecutive numbers. However, the special color is still replaced by the colors of the sum-free partition according to the line number and the other colors are still vertically extended.

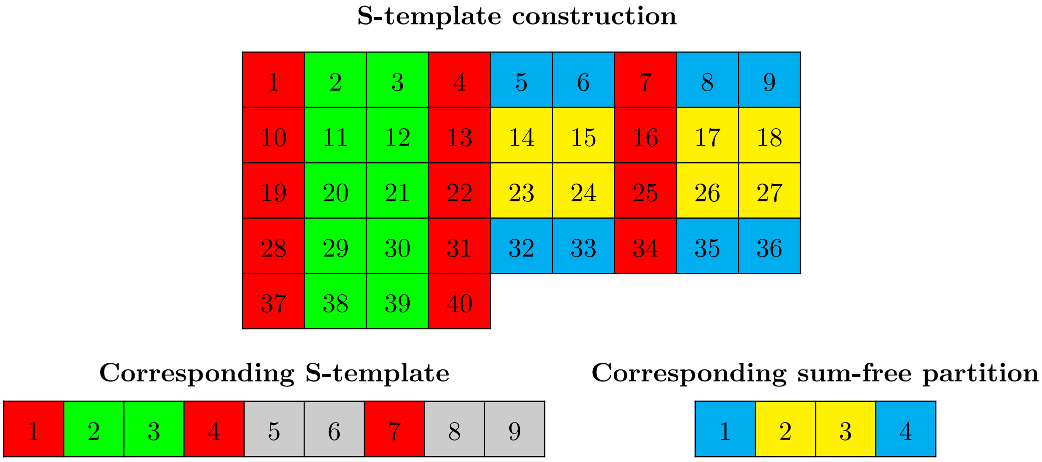

Figure 2: Visualisation of a template-based construction

We now proceed to prove Theorem 2.3.

Proof. Denote by f the coloring associated to the S-template with width q and g the one associated to the sum-free partition of $[\![1,p]\!]$; where $f : [\![1,q]\!] \longrightarrow [\![1,n+1]\!]$ and $g : [\![1,p]\!] \longrightarrow [\![1,k]\!]$.

NB: In the following three predicates, the conditions $x+y \leqslant q$ and $x+y \leqslant p$ are omitted for readability.

The sum-free condition is expressed as:

$$\forall (x,y) \in [\![1,q]\!]^2, f(x) = f(y) \Longrightarrow f(x+y) \neq f(x),$$

$$\forall (x,y) \in [\![1,p]\!]^2, g(x) = g(y) \Longrightarrow g(x+y) \neq g(x).$$

The additionnal constraint for the S-template is:

$$\forall (x,y) \in [\![1,q]\!]^2, \begin{cases} f(x) = f(y) \leqslant n \\ x+y > q \end{cases} \Longrightarrow f(x+y-q) \neq f(x).$$

For $x \in [\![1, pq+m_{n+1}-1]\!]$, we write $x = (\alpha-1)q + u$ for certain integers $\alpha \in \mathbb{Z}$ and $u \in [\![1,q]\!]$. This decomposition is of course unique. α can be interpreted as the row number of x and u as the collumn number of x. A new coloring is define as follows:

$$\begin{array}{rccl} h: & [\![1, pq+m_{n+1}-1]\!] & \longrightarrow & [\![1, n+k]\!], \\ & x & \longmapsto & \begin{cases} f(u), & \text{if } f(u) \leqslant n, \\ n+g(\alpha), & \text{if } f(u) = n+1. \end{cases} \end{array}$$

Function h is well-defined since, by definition of m_{n+1}, $\forall x \in [\![pq+1, pq+m_{n+1}-1]\!], f(u) \leqslant n$ and therefore $\forall x \in [\![1, pq+m_{n+1}-1]\!], f(u) = n+1 \implies \alpha \in [\![1,p]\!]$.

We now prove that h is a sum-free coloring. Let $x, y \in [\![1, pq+m_{n+1}-1]\!]$ such that $h(x) = h(y)$ and $x+y \leqslant pq+m_{n+1}-1$. We claim that $h(x+y) \neq h(x)$. We write $x = (\alpha-1)q+u$ and $y = (\beta-1)q+v$ where $\alpha, \beta \in \mathbb{Z}$ and $u, v \in [\![1,q]\!]$. Two cases are to be distinguished according to the value of $h(x)$.

Case 1: $h(x) \leqslant n$

Let us assume that $h(x+y) \leqslant n$, otherwise $h(x+y) \neq h(x)$ obviously holds. By definition of function h and given that $h(u) = h(v)$, we conclude $f(u) = f(v)$. Two cases are to b distinguished according to the value of $x+y$.

- If $u+v > q$, we write $w = u+v-q \in [\![1,q]\!]$. Consequentely $x+y = (\alpha+\beta-1)q+w$. By definition, $h(x+y) = f(w)$. Given that $f(u) = f(v) \leqslant n$, the additionnal constraint on f implies $f(w) \neq f(u)$, that is $h(x+y) \neq h(x)$.
- If $u+v \leqslant p$, we write $w = u+v \in [\![1,q]\!]$. Consequentely $x+y = (\alpha+\beta-2)q+w$. By definition, $h(x+y) = f(w)$. Given that $f(u) = f(v) \leqslant n$, the sum-free property of f implies $f(w) \neq f(u)$, that is $h(x+y) \neq h(x)$.

Case 2: $h(x) \geqslant n+1$

Now we have $h(x) = n+g(\alpha) = k+g(\beta) = h(y)$, hence $g(\alpha) = g(\beta)$. As in the first case, distinguish between two cases according to the value of $x+y$.

- If $u+v > q$, write $w = u+v-q \in [\![1,q]\!]$. Then $x+y = (\alpha+\beta-1)q+w$. Assume that $h(x+y) \geqslant n+1$, otherwise $h(x+y) \neq h(x)$ obviously holds. By definition, $h(x+y) = n+g(\alpha+\beta)$. Given that $g(\alpha) = g(\beta)$, the sum-free property of g implies $g(\alpha+\beta) \neq g(\alpha)$ that is $h(x+y) \neq h(x)$.
- If $u+v \leqslant q$, write $w = u+v \in [\![1,q]\!]$. Then $x+y = (\alpha+\beta-2)q+w$. The sum-free property of f implies $f(w) \neq f(u)$. Therefore $f(w) \leqslant k$ and thus $h(x+y) \leqslant n$. In particular, given that $h(x) \geqslant n+1$, $h(x+y) \neq h(x)$.

□

The following proposition may improve the additive constant of Corollary 2.4.

Proposition 2.5. *Let $(q,n) \in \mathbb{N}^*)^2$ and let f be a coloring associated to a S-template with width q and $n+1$ colors. Let $b \in \mathbb{N}$ and assume there is a coloring g of $[\![1,b]\!]$ with $n+1$ colors such that:*

- $\forall (x,y) \in [\![1,q]\!]^2, \begin{cases} f(x) = f(y) \\ (x+y) \mod q \leqslant b \end{cases} \implies g((x+y) \mod q) \neq f(x),$
- $\forall (x,y) \in [\![1,q]\!] \times [\![1,b]\!], \begin{cases} f(x) = g(y) \\ x+y \leqslant b \end{cases} \implies g(x+y) \neq f(x).$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $x \longmapsto g(x - pS(n))$, we have

$$S(n+k) \geqslant S^+(n+1)S(k) + b.$$

This proposition corresponds to the fact that the hypotheses made on the coloring of the last row can be weakened and as a result, we can change the coloring on the last row to extend the previous partitions.

There is a construction theorem for S-templates as well.

Theorem 2.6. *Let $(p,k),(q,n) \in (\mathbb{N}^*)^2$. If there is a S-template with width q and $n+1$ colors, and a S-template with width p and k colors, then there also is a S-template with width pq and $(n+k)$ colors.*

The inequality associated with this theorem is given by:

Corollary 2.7. *Let $n,k \in \mathbb{N}^*$. Then*

$$S^+(n+k) \geqslant S^+(n+1)S^+(k).$$

Proof. The idea is the same as in the Theorem 2.3. The only difference is the S-template property inherited from the second S-template.

□

2.3. New lower bounds for Schur numbers

We now give the inequalities corresponding to the current best S-templates.

Definition 2.8. *A sum-free partition $A_1, ..., A_n$ of $[\![1,p]\!]$ is said to be symmetric if for all $x \in [\![1,p]\!]$, x and $p+1-x$ belong to the same subset (except if $x = p+1-x$).*

A S-template with n colors is said to be symmetric if the partition into n sum-free subsets derived from this template by applying the extension procedure with a sum-free partition of length 1 is symmetric.

We produced S-templates using a SAT solver, hence providing lower bound on S^+ and inequalities of the type $S(n+k) \geqslant aS(n) + b$. We sought templates providing the largest value of (a,b) (in the lexicographic order). When the number of colors exceeded five, in order to reduce the search space we only looked for symmetric S-templates, we assumed that the special color was the last color to appear and we constrained the m_c's out of being too small. Details concerning the encoding as a SAT problem can be found in [9].

The following six inequalities are given by the current best S-templates with $n \leqslant 7$ colors.

$$S(n+1) \geqslant 3\,S(n) + 1 \tag{1}$$
$$S(n+2) \geqslant 9\,S(n) + 4 \tag{2}$$
$$S(n+3) \geqslant 33\,S(n) + 6 \tag{3}$$

Inequality (1) comes from Schur's original article[1]. Inequality (2) is due to Abott and Hanson [3] and inequality (3) to Rowley [7]. The three following inequalities are our result.

$$S(n+4) \geqslant 111\,S(n) + 43 \tag{4}$$
$$S(n+5) \geqslant 380\,S(n) + 148 \tag{5}$$
$$S(n+6) \geqslant 1\,160\,S(n) + 536 \tag{6}$$

The templates corresponding to inequalities (3), (4) and (5) can be found in the appendix.

Inequalities (1), (2) and (3) cannot be further improved (with this definition of S-template). Inequality (4) cannot be further improved by only searching for symmetric S-templates whose special color is the last in the order of apparition (and with a multiplicative factor less than or equal to 118). Inequality (5) can most likely be further improved but the improvement probably will not be substantial. Finally, a S-template corresponding to inequality (6) was found by extending into a S-template the Schur 6 partition used in [11] (owing to the size of this template, its interest is limited and since it can easily be derived from above mentioned partition, it is not given in the appendix). Although we could not find a better S-template with seven colors, inequality (6) can definitely be improved by a wide margin. One may try to seek better templates by constraining less the search space and by using Monte-Carlo methods, as in [5]. This could be the subject of a future work.

The previous inequalities give new lower bounds for $S(n)$ for $n \geqslant 9$. We compute the lower bounds for $n \in [\![8, 15]\!]$ using the four different inequalities. The best lower bounds are highlighted.

n	8	9	10	11
$33\,S(n-3)+6$	5 286	17 694	55 974	174 444
$111\,S(n-4)+43$	4927	17 803	59 539	188 299
$380\,S(n-5)+148$	5 088	16 868	60 948	203 828
n	12	13	14	15
$33\,S(n-3)+6$	587 505	2 011 290	6 726 330	21 272 730
$111\,S(n-4)+43$	586 789	1 976 176	6 765 271	22 624 951
$380\,S(n-5)+148$	644 628	2 008 828	6 765 288	23 160 388

Table 3: New lower bounds for $n \in [\![8, 15]\!]$

Except for $S(8)$, $S(9)$ and $S(13)$, the best lower bounds are obtained thanks to the fifth inequality $S(n+5) \geqslant 380\,S(n) + 148$. The table doesn't go any further, but the same inequality allows to improve the lower bounds for every $n \geqslant 15$.

Corollary 2.9. *The growth rate for Schur numbers (and Ramsey numbers $R_n(3)$) satisfies $\gamma \geqslant \sqrt[5]{380} \approx 3.28$.*

Proof. It is a mere consequence of the inequality $S(n+5) \geqslant 380\,S(n) + 148$. As for Ramsey numbers, the following inequality holds $S(n) \leqslant R_n(3) - 2$ (see [3]) hence the result. □

2.4. Conclusion on S-templates

In this section, we first formalized Rowley's template-based constructions [7] in the context of Schur numbers by introducing S-templates as well as a new sequence, S^+. We provided relations between S^+ and S then stated Rowley's construction method in the context of Schur numbers. We found new S-templates allowing us to obtain new lower bounds for schur numbers. One may notice that we mostly gave only lower bounds for S^+. It should be possible to find better S-templates by making different assumptions or using a different method (Monte-Carlo methods for instance).

In the next section, we provide similar results for weak Schur numbers. We introduce WS-templates and a corresponding sequence, WS^+. We then derive similar relations and a construction method allowing us to find new lower bounds for weak Schur numbers.

3. Templates for weak Schur numbers

In this section, we generalize Rowley's constructions for weak Schur numbers [8] and give an analogous for weak Schur numbers of Abbott and Hanson's construction for Schur numbers [3]. By analogy with the previous section, we then introduce *WS-templates*, standing for *"weak Schur templates"*, as well as an associated sequence $WS^+(n)$. We find templates and use them to provide new lower bounds for weak Schur numbers. Finally, we give a short explanation for the new lower bound $WS(6) \geqslant 646$ which was not directly obtained with a template contrary to the other lower bounds given in this paper.

3.1. Inequality for weak Schur numbers using Schur and weak Schur numbers

Up to now, no equivalent for weak Schur numbers of Abbott and Hanson's construction for Schur numbers [3] was known. Here we give a general lower bound for weak Schur numbers as a function of both regular and weak Schur numbers.

Theorem 3.1. *Let $(p, k), (q, n) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1, q]\!]$ into n weakly sum-free subsets and a partition of $[\![1, p]\!]$ into k sum-free subsets then there exists a partition of $[\![1, p(q + \lceil \frac{q}{2} \rceil + 1) + q]\!]$ into $n + k$ weakly sum-free subsets.*

In particular, by setting $q = WS(n)$ and $p = S(k)$ in Theorem 3.1, one obtains the following corollary.

Corollary 3.2. $\forall (n, k) \in (\mathbb{N}^*)^2,\ WS(n + k) \geqslant S(k) \left(WS(n) + \left\lceil \dfrac{WS(n)}{2} \right\rceil + 1 \right) + WS(n)$

This can be seen as an equivalent for weak Schur numbers of Abott and Hanson's construction for Schur numbers. This formula includes the results of Rowley [8] as a special case. For $n > 2$, this formula does not give new lower bounds.

Remark 3.3. *The inequality from Corollary 3.2 can be improved by adding 1 to the lower bound if $WS(n)$ is odd (more generally if q is odd in Theorem 3.1). However, it lengthens the proof and it is never useful in practice.*

Given that Theorem 3.1 will appear as a particular case of a more general theorem after the introduction of templates for weak Schur numbers, we only give here an intuitive explanation of the demonstration; a formal proof using templates for weak Schur numbers is provided in Subsection 3.3.

Let $(p, k), (q, n) \in (\mathbb{N}^*)^2$ such that there exists a partition of $[\![1, q]\!]$ into n weakly sum-free subsets and a partition of $[\![1, p]\!]$ into k sum-free subsets. Let $a \in \mathbb{N}$ with $a > q$ and let us try to build a coloring of $[\![1, ap + q]\!]$ into $n + k$ weakly sum-free subsets. Let $l = a - b - 1$, $r \in [\![1, q]\!]$ and $w = a - l - r - 1 = b - r$.

First, we put the integers of $[\![1, ap + q]\!]$ in the following table (with a columns and $p + 1$ lines) and divide it into 3 blocks (the columns are numbered from $-l$ to $+q$).

- $\mathcal{T}$ (the "tail"): the integers from 1 to q. NB: this is line number 0.
- $\mathcal{R}$ (the "rows"): the integers in columns $-l$ to $+r$ (excluding $\mathcal{T}$).
- $\mathcal{C}$ (the "columns"): the integers in the last w columns (excluding $\mathcal{T}$).

Like S-templates, $\mathcal{R}$ and $\mathcal{C}$ play the role of security zones for each other. Note that with this numbering of columns, the column of the sum of two numbers is the only integer in $[\![-l, q]\!]$ equal to two the sum of the columns modulo a.

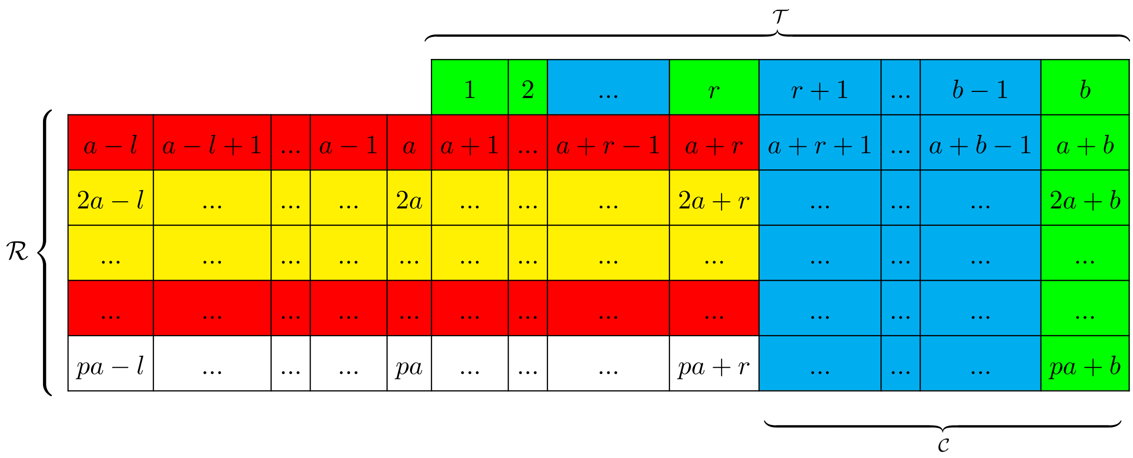

Figure 3: Construction of the weakly sum-free coloring

$\mathcal{T}$ block

We color this block using the weakly sum-free coloring of $[\![1, q]\!]$ with colors $1, ..., n$.

$\mathcal{R}$ block

In this block, we use the colors $n+1, ..., n+k$. We color an integer x according to its line number (written $\lambda(x)$). For every $x \in \mathcal{R}$, we color x with $n+c$ where c is the color of $\lambda(x)$ in the sum-free coloring of $[\![1, p]\!]$. Let $(x, y) \in \mathcal{R}^2$. The cases are twofold.

- $\lambda(x+y) = \lambda(x) + \lambda(y)$
  In this case, we use the sum-free property of the coloring of $[\![1, p]\!]$ (in block $\mathcal{C}$, we only use colors $1, ..., n$).

- $\lambda(x+y) \neq \lambda(x) + \lambda(y)$
  In this case, we do not have information about the color of $\lambda(x+y)$. Thereby, we want to have $x+y \in \mathcal{C}$. A simple solution is to limit the horizontal movement, that is if the sum changes line (that is its line number is different from $\lambda(x)+\lambda(y)$), not to move too far from $(\lambda(x)+\lambda(y))a$ so that the sum stays in $\mathcal{C}$. There, the maximal displacement to the left (resp. to the right) is $2l$ (resp. $2r$). Not crossing entirely $\mathcal{C}$ by going to the left is then expressed as $-2l > -a + r$. Likewise, not going to far to the right is expressed as $2r < a - l$. It can then be written as $\max(l, r) \leqslant w$.

$\mathcal{C}$ block

In this block, we use colors $1, ..., n$. We color an integer x according to its column number, denoted by $\tilde{\pi}(x)$. It is linked to the projection on the first line, denoted by π, by the relation $\tilde{\pi}(x) = \pi(x) - a$. A simple solution is to color x with the same color as $\tilde{\pi}(x)$ in the weakly sum-free coloring of $[\![1, q]\!]$. As long as $2b \leqslant a + r$ (not going two far to the right) and there is no $x \in \tilde{\pi}(\mathcal{C})$ such that $2x \in \tilde{\pi}(\mathcal{C})$ (so that we do not have a sum in $\mathcal{C}$ when taking two numbers in the same column), the colors $1, ..., n$ are sum-free.

In particular, taking $w = l = \left\lceil \frac{q}{2} \right\rceil$ and $r = \left\lfloor \frac{q}{2} \right\rfloor$ works, thus obtaining the above theorem.

As in the previous section, we now introduce WS-templates and the sequence WS^+ in order to generalize the above construction.

3.2. Definition of $[\![1, WS^+]\!]$

In this subsection, we introduce WS-templates and prove calculative results for the general construction theorem on templates for weak Schur numbers.

Definition 3.4. *Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$. We define :*

$$\pi_{a,b} : x \longmapsto (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x \mod a).$$

If there is no confusion on the a and b to use, $\pi_{a,b}$ is denoted by π. Notice that for all $x \in \mathbb{Z}$, $\pi(x) = x \mod a$ and for all $x \in [\![b+1, a+b]\!], b+1 \leqslant \pi(x) \leqslant a+b$.

π is the projection on the first line mentioned in the intuitive explanation. The following four propositions are calculative properties on π reflecting the behaviour of an element's column number in Figure 3 and that we will use later when we introduce WS-templates.

Proposition 3.5.

$$\forall x \in [\![b+1, a+b]\!], \pi(x) = x.$$

Proof. Let $x \in [\![b+1, a+b]\!]$. If $x < a$ then $x \mod a = x \notin [\![1,b]\!]$. Hence $\pi(x) = x$. Otherwise, $x \mod a = x - a \in [\![1,b]\!]$. Hence $\pi(x) = x - a + a = x$.

□

Proposition 3.6. *Let $x \in [\![1,b]\!]$ and $y \in \mathbb{N}^*$. Then*

$$\pi(x + \pi(y)) = \pi(x+y).$$

Proof. It is a direct consequence of $\pi(x) = x \mod a$.

□

Proposition 3.7. *Let $x \in [\![1,b]\!]$ and $y \in \mathbb{N}^*$ such that $x + \pi(y) \leqslant a+b$. Then*

$$\pi(x+y) = x + \pi(y).$$

Proof. $\pi(y) \geqslant b+1$ and thus $b+1 \leqslant x + \pi(y) \leqslant a+b$.

$$\begin{array}{rcll} \pi(x+y) & = & \pi(x+\pi(y)) & \text{by Proposition 3.6} \\ & = & x + \pi(y) & \text{by Proposition 3.5} \end{array}$$

□

Proposition 3.8. *Let $(x,y) \in (\mathbb{N}^*)^2$. Then*

$$\pi(\pi(x) + \pi(y)) = \pi(x+y).$$

Proof. It is a direct consequence of $\pi(x) = x \mod a$.

□

After defining the function related to the column number of each element in Figure 3, we introduce the function related to its line number.

Definition 3.9. *Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$. Define*

$$\lambda_{a,b} : x \longmapsto 1 + \left\lfloor \frac{x-b-1}{a} \right\rfloor.$$

If there is no confusion on the a and b to use, $\lambda_{a,b}$ is denoted by λ.

Function λ maps an element x to its line number as mentioned in the intuitive explanation. As we just did with π, we prove three more calculative results on both π and λ that are used in Subsection 3.3.

Proposition 3.10. *Let $x \in \mathbb{N}^*$. Then*

$$x = a\lambda(x) + \pi(x) - a.$$

Proof. Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$ and let $x \in \mathbb{N}^*$.
We have $a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x-b-1}{a} \right\rfloor + (x \mod a) + a\mathbb{1}_{[\![0,b]\!]}(x \mod a)$.

- If $x \mod a > b$ then $a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x}{a} \right\rfloor + x \mod a = x$.
- If $x \mod a \leqslant b$ then $a\lambda(x) + \pi(x) - a = a\left(\left\lfloor \dfrac{x}{a} \right\rfloor - 1\right) + x \mod a + a = x$.

□

Proposition 3.11. *Let $x, y \in \mathbb{Z}$ such that $\lambda(x+y) = \lambda(y)$. Then*

$$\pi(x+y) = x + \pi(y).$$

Proof. By applying Proposition 3.10 twice, we get $a\lambda(x+y)+\pi(x+y)-a = x+y = x+a\lambda(y)+\pi(y)-a$. The result is then obtained by simplifying the equality.

□

Proposition 3.12. *Let $x, y \in \mathbb{Z}$ such that $\pi(x) + \pi(y) \in [\![a+b+1, 2a+b]\!]$. Then*

$$\lambda(x+y) = \lambda(x) + \lambda(y)$$

Proof. By Proposition 3.10, $x + y = a(\lambda(x) + \lambda(y)) + \pi(x) + \pi(y) - 2a$. Then

$$\begin{aligned}
\lambda(x+y) &= \left\lfloor \frac{x+y-b-1}{a} \right\rfloor + 1 \\
&= \left\lfloor \frac{a(\lambda(x)+\lambda(y)) + \pi(x) + \pi(y) - 2a - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) - 1 \left\lfloor \frac{\pi(x)+\pi(y)-b-1}{a} \right\rfloor \\
&= \lambda(x) + \lambda(y) - 1 + 1 \quad \text{since } \pi(x)+\pi(y) \in [\![a+b+1, 2a+b]\!] \\
&= \lambda(x) + \lambda(y)
\end{aligned}$$

□

Definition 3.13. *Let $(a,n,b) \in (\mathbb{N}^*)^3$ with $a > b$. Let $(A_1, ..., A_n)$ a partition of $[\![1, a+b]\!]$. This partition is said to be a b-weakly-sum-free template (b-WS-template) with width a and n colors when :*

- $\forall i \in [\![1,n]\!], A_i$ *is weakly-sum-free,*
- $\forall i \in [\![1,n]\!], A_i \backslash [\![1,b]\!]$ *is sum-free,*
- *For A_n (the special subset) :*

$$\forall (x,y) \in A_n^2,\ x+y > b+2a \implies x+y-2a \notin A_n,$$

- *For the others subsets*

$$\forall i \in [\![1, n-1]\!],\ \forall (x,y) \in A_i^2,\ x+y > a+b \implies \pi(x+y) \notin A_i.$$

Note that the special color n is not necessarily the last color by order of appearance, any color can play this role. We now introduce the number $WS^+(n)$ that plays the same role as $S^+(n)$ for S-templates. However, WS-templates being more sophisticated than S-templates, the definition of $WS^+(n)$ is slighty more complicted.

Definition 3.14. *Let $(n,b) \in (\mathbb{N}^*)^2$. If there exists a such that there exists a b-WS-template with width a and n colors, we define :*

$$WS_b^+(n) = \max\{a \in \mathbb{N}^* / \textit{there exists a } b\textit{-WS-template with width } a \textit{ and } n \textit{ colors}\}.$$

If no such a exists, we set $WS_b^+(n) = 0$.

Definition 3.15. *Let $n \in \mathbb{N}^*$. We define :*

$$WS^+(n) = \max_{b \in \mathbb{N}^*} WS_b^+(n).$$

The following proposition briefly shows how $WS^+(n)$ compares to weak Schur numers.

Proposition 3.16. *Let $n \in [\![2, +\infty]\!]$. Then :*

$$\frac{3}{2} WS(n-1) + 1 \leqslant WS^+(n) \leqslant WS(n).$$

Proof. The lower bound comes from Corollary 3.2. The upper bound comes from the fact that a WS-template with width a and n colors is in particular a partition of $[\![1, a]\!]$ into n weakly sum-free subsets.

□

Having established these properties, we now have everything we need to state and prove our main result. We proceed to do so in the next subsection.

3.3. Construction of weak Schur partitions using WS-templates

Theorem 2.3 provides a way to extend a S-template into a larger Schur partitions using other Schur partitions. Likewise, a WS-template can be extended into larger weak Schur partitions by using Schur partitions. This is the object of the following theorem.

Theorem 3.17. *Let $(a,n,b) \in (\mathbb{N}^*)^3$ with $a > b$ and $(p,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1, p]\!]$ into k sum-free subsets and a b-WS-template $(A_1, ..., A_{n+1})$ with width a and $n+1$ colors, then there exists a partition of $[\![1, pa+b]\!]$ into $k+n$ weakly sum-free subsets.*

In particular, by setting $p = S(k)$ and $a = WS^+(n+1)$ in Theorem 3.17, the next corollary follows.

Corollary 3.18. *Let $n, k \in \mathbb{N}^*$ and set $b_{max} = \max\{b \in \mathbb{N}^* / WS_b^+(n+1) = WS^+(n+1)\}$. Then :*

$$WS(n+k) \geqslant S(k)\, WS^+(n+1) + b_{max}.$$

Remark 3.19. *In the S-template construction for Schur numbers, the additive constant comes from the fact that the special color does not necessarily appear right at the begining of the repeating pattern. Likewise, b_{max} can actually be replaced by*

$$\max_{b \in \mathbb{N}^*} \left\{ \min(A_{n+1} \backslash [\![1, b]\!]) - 1 \mid WS_b^+(n+1) = WS^+(n+1) \right\}.$$

Proof. Let $(a,n,b) \in (\mathbb{N}^*)^3$ and $(p,k) \in (\mathbb{N}^*)^2$. Denote by f the coloring associated to the b-WS-template and g the one associated to the sum-free partition of $[\![1, p]\!]$; where $f : [\![1, a+b]\!] \longrightarrow [\![1, n+1]\!]$ and $g : [\![1, p]\!] \longrightarrow [\![1, k]\!]$. Moreover, assume that the sum-free coloring of $[\![1, p]\!]$ is ordered.

NB: To keep the notation short, the conditions $x + y \leqslant p$ and $x + y \leqslant a + b$ are omitted in the following five predicates.

The (weakly) sum-free conditions are expressed as:

$$\forall (x,y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) \\ x \neq y \end{cases} \Longrightarrow f(x+y) \neq f(x), \tag{7}$$

$$\forall (x,y) \in [\![b+1, a+b]\!]^2, f(x) = f(y) \Longrightarrow f(x+y) \neq f(x), \tag{8}$$

$$\forall (x,y) \in [\![1, p]\!]^2, g(x) = g(y) \Longrightarrow g(x+y) \neq g(x). \tag{9}$$

The additionnal constraints for the WS-template are:

$$\forall (x,y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) \leqslant n \\ x+y > a+b \end{cases} \Longrightarrow f(\pi(x+y)) \neq f(x), \tag{10}$$

$$\forall (x,y) \in [\![1, a+b]\!]^2, \begin{cases} f(x) = f(y) = n+1 \\ x+y > 2a+b \end{cases} \Longrightarrow f(x+y-2a) \neq f(x). \tag{11}$$

Split $[\![1, pa+b]\!]$ into three subsets.
NB: To keep the notation short, the restriction to $[\![b+1, pa+b]\!]$ of π defined in Subsection 3.2 is denoted by π in the hereunder equations.

- $\mathcal{T} = [\![1, b]\!]$
- $\mathcal{C} = \pi^{-1}(f^{-1}([\![1, n]\!]))$
- $\mathcal{R} = \pi^{-1}(f^{-1}(\{n+1\}))$

A new coloring h is defined as follows:

$$\begin{array}{rccl} h: & [\![1, pa+b]\!] & \longrightarrow & [\![1, n+k]\!] \\ & x & \longmapsto & \begin{cases} f(x) & \text{if } x \in \mathcal{T} \\ f(\pi(x)) & \text{if } x \in \mathcal{C} \\ n + g(\lambda(x)) & \text{if } x \in \mathcal{R} \end{cases} \end{array}$$

Function h is well defined since $(\mathcal{T}, \mathcal{C}, \mathcal{R})$ is a partition of $[\![1, pa+b]\!]$. We now prove that h is a weakly sum-free coloring. Let $x, y \in [\![1, pa+b]\!]$ be such that $x \neq y$, $h(x) = h(y)$ and $x + y \leqslant pa + b$. We claim that $h(x+y) \neq h(x)$. Nine cases are to be distinguished according to the subsets $(\mathcal{T}, \mathcal{C}, \mathcal{R})$ to which x and y belong. It is sufficient to check only six cases out of nine since x and y play symmetric roles.

Case 1: $(x,y) \in \mathcal{T}^2$

If $x + y \leqslant b$ then $h(x+y) = f(x+y)$. Otherwise, $b < x+y < a+b$ since $b < a$ and therefore $\pi(x+y) = x+y$ (Proposition 3.5). Hence in both cases $h(x+y) = f(x+y)$. Given that f is a weakly sum-free coloring, $f(x+y) \neq f(x)$ since $f(x) = h(x) = h(y) = f(y)$ and $x \neq y$. That is $h(x+y) \neq h(x)$.

Case 2: $(x,y) \in \mathcal{T} \times \mathcal{C}$

Given that $h(x) = h(y)$ and by definition of h, $f(x) = f(\pi(y))$. Besides, $f(\pi(y)) \leqslant n$ since $y \in \mathcal{C}$. Two cases are to be distinguished according to the value of $x + \pi(y)$.

- If $x + \pi(y) \leqslant a + b$ then $f(x+\pi(y)) = f(\pi(x+y))$ (Proposition 3.7). Given that f is a weakly sum-free coloring, $f(x+\pi(y)) \neq f(x)$ since $f(x) = f(\pi(y))$ and $x \neq \pi(y)$ since $x \leqslant b < \pi(y)$.
- If $x + \pi(y) > a + b$ then given that f is a WS-template and since $f(x) = f(\pi(y)) \leqslant n$, $f(\pi(x+\pi(y))) \neq f(x)$. Furthermore $f(\pi(x+\pi(y))) = f(\pi(x+y))$ (Proposition 3.6), such that $f(\pi(x+y)) \neq f(x)$.

Hence in both cases $f(\pi(x+y)) \neq f(x)$. If $f(\pi(x+y)) \leqslant n$ then $h(x+y) = f(\pi(x+y))$. Therefore $h(x+y) \neq h(x)$ since $f(x) = h(x)$. Otherwise, $f(\pi(x+y)) = n+1$ and thus $h(x+y) > n$. In particular, $h(x+y) \neq h(x)$ since $h(x) = h(y) \leqslant n$.

Case 3: $(x,y) \in \mathcal{T} \times \mathcal{R}$

Necessarily $h(x) = h(y) = n+1$. Two cases are to be distinguished according to the value of $\lambda(x+y)$.

- If $\lambda(y) = \lambda(x+y)$ then $\pi(x+y) = x + \pi(y)$ (Proposition 3.11). By definition of h, $f(x) = f(\pi(y))$. Given that f is a weakly sum-free coloring, $f(x+\pi(y)) \neq f(x)$ since $f(x) = f(\pi(y))$ and $x \neq \pi(y)$ since $x \leqslant b < \pi(y)$. Hence $h(x+y) \neq h(x)$.

- If $\lambda(y) \neq \lambda(x+y)$ then $\lambda(x+y) = \lambda(y)+1$ since $x \leqslant b < a$. Besides, $n+1 = h(y) = n+g(\lambda(y))$. Hence $g(\lambda(y)) = 1$. Moreover $g(1) = 1$ since g is an ordered coloring. Therefore, given that g is sum-free, $g(\lambda(y)+1) \neq 1$. If $\pi(x+y) \in A_{n+1}$ then $h(x+y) = n+g(\lambda(x+y)) \neq n+1$. Otherwise, $h(x+y) \leqslant n$. Hence in both cases $h(x+y) \neq h(x)$.

Case 4: $(x, y) \in \mathcal{C}^2$

By definition of h and since $h(x) = h(y)$, $f(\pi(x)) = f(\pi(y))$. Two cases are to be distinguished according to the value of $\pi(x) + \pi(y)$.

- If $\pi(x) + \pi(y) \leqslant a + b$ then $\pi(x) + \pi(y) = \pi(x+y)$. Hence $f(\pi(x+y)) \neq f(\pi(x))$ since f is sum-free for $x > b$

- If $\pi(x) + \pi(y) > a + b$ then given that f is a WS-template, $f(\pi(\pi(x) + \pi(y))) \neq f(\pi(x))$ since $f(\pi(x)) = f(\pi(y))$. Besides, $f(\pi(\pi(x) + \pi(y))) = f(\pi(x+y))$ (Proposition 3.8). Hence $f(\pi(x+y)) \neq f(\pi(x))$.

Hence in both cases $f(\pi(x+y)) \neq f(x)$. If $f(\pi(x+y)) \leqslant n$ then $h(x+y) = f(\pi(x+y))$. Therefore $h(x+y) \neq h(x)$ since $f(x) = h(x)$. Otherwise, $f(\pi(x+y)) = n+1$ and thus $h(x+y) > n$. In particular, $h(x+y) \neq h(x)$ since $h(x) = h(y) \leqslant n$.

Case 5: $(x, y) \in \mathcal{C} \times \mathcal{R}$

By definition of h, $h(x) \neq h(y)$.

Case 6: $(x, y) \in \mathcal{R}^2$

In particular $f(\pi(x)) = f(\pi(y)) = n+1$. Three cases are to be distinguished according to the value of $\pi(x) + \pi(y)$.

- If $\pi(x) + \pi(y) \in [\![a+b+1, 2a+b]\!]$ then $\lambda(x+y) = \lambda(x) + \lambda(y)$ (Proposition 3.12). By definition of h and since $h(x) = h(y)$, $g(\lambda(x)) = g(\lambda(y))$. Hence, $h(\lambda(x+y)) \neq h(\lambda(x))$ since h is a sum-free coloring. If $f(x+y) \geqslant n+1$ then $h(x+y) = n + g(\lambda(x+y))$. And $h(x) = n + g(\lambda(x))$. Therefore, $h(x+y) \neq h(x)$. Otherwise $h(x+y) \leqslant n < h(x)$. In particular $h(x+y) \neq h(x)$.

- If $\pi(x) + \pi(y) > 2a + b$ then $f(\pi(\pi(x) + \pi(y))) \neq f(\pi(x)) = n+1$ since f is a b-WS template and $f(\pi(x)) = f(\pi(y))$. Given that $\pi(\pi(x) + \pi(y)) = \pi(x+y)$ (Proposition 3.8), $f(\pi(x+y)) \neq n+1$.

- If $\pi(x) + \pi(y) \leqslant b + a$ then, given that $\pi(x) + \pi(y) \geqslant b$ and $f_{|[\![b,a+b]\!]}$ is sum-free, $f(\pi(x) + \pi(y)) \neq f(\pi(x)) = n+1$. That is $f(\pi(x+y)) \neq n+1$ (Proposition 3.5).

In both of the last two cases, $f(\pi(x+y)) \neq n+1$ that is $x+y \in \mathcal{C}$. Therefore $h(x+y) < n \leqslant h(x)$. In particular, $h(x+y) \neq h(x)$.

$\square$

There is a construction theorem for WS-templates as well.

Theorem 3.20. *Let $(k,p) \in (\mathbb{N}^*)^2$ and $(a,n,b) \in (\mathbb{N}^*)^3$. If there exists a S-template with width p and $k+1$ colors and a b-WS-template with width a and n colors, then there exists a pb-WS-template with width pq and $(n+k)$ colors.*

Theorem 3.20 yields the following corollary.

Corollary 3.21. *Let $n, k \in \mathbb{N}^*$. Then*

$$WS^+(n+k) \geqslant S^+(k+1)\, WS^+(n).$$

Proof. The idea is the same as in the previous theorem. The only difference is the WS property inherited from both the S-template and the WS-template.

□

Theorem 3.1 can be seen as a particular case of WS-template in the same way Abott and Hanson's construction [3] can be seen as a particular case of S-template.

Proof of Theorem 3.1. Let $(q, n) \in (\mathbb{N}^*)^2$ such that there exists a partition of $[\![1, q]\!]$ into n weakly sum-free subsets. Let $f : [\![1, q]\!] \to [\![1, n]\!]$ a weakly sum-free colouring. Let $b = q$ and $a = q + \left\lceil \frac{q}{2} \right\rceil + 1$. A new colouring g is defined as follows:

$$\begin{array}{cccl} g: & [\![1, a+b]\!] & \longrightarrow & [\![1, n+1]\!] \\ & x & \longmapsto & \begin{cases} f(x) & \text{if } x \in [\![1, b]\!] \\ n+1 & \text{if } x \in [\![b+1, 2b+1]\!] \\ f(x-a) & \text{if } x \in [\![2b+2, a+b]\!] \end{cases} \end{array}$$

We claim that g is a b-WSF-template with width a and $n+1$ colours.

- Function $g_{|[\![b+1,a+b]\!]}$ is a sum-free colouring. Indeed, let $(x, y) \in [\![b+1, a+b]\!]^2$ such that $g(x) = g(y)$. If $g(x) = n+1$ then $z = x+y > 2b+1$ and therefore, either $z > a+b$ or $f(z) \neq n+1$. Otherwise, $x + y > a + b$.
- Function g is a weakly sum-free colouring. Indeed, let $(x, y) \in [\![1, a+b]\!]^2$ such that $z = x+y \leqslant a+b$ and $g(x) = g(y)$. Given that x and y have symmetric roles, we can assume that $x \leqslant y$. If $x > b$ then $g(z) \neq g(x)$ as seen above. If $y \leqslant b$ then $f(x) = g(x) = g(y) = f(y) \leqslant n$ and either $z \leqslant a+b$ and $f(z) \neq f(x)$ since f is a weakly sum-free colouring or $a+b+1 \leqslant z \leqslant 2b$ and $g(z) = n+1$; therefore $g(z) \neq g(x)$. If $x \leqslant b$ and $y > b$ then $g(x) = f(x)$, $g(y) = f(y-a)$ and $g(z) = f(z-a)$.We have $x \neq y-a$ (otherwise, we would have $a+b \geqslant z = 2y-a \geqslant 4b+4-a > a+b$) and thus $f(x+y-a) \neq f(x)$, that is $g(z) \neq g(x)$.
- Colour $n+1$ verifies the additionnal constraints for the special colour. Indeed, $b+2a \geqslant 4b+2$. Hence, $\forall (x, y) \in g^{-1}(\{n+1\}), x+y \leqslant b+2a$.
- Colours $1, ..., n$ verify the additionnal constraints for the regular colours. Indeed, let $(x, y) \in g^{-1}([\![1, n]\!])$ such that $x+y > a+b$. Given that x and y have symmetric roles, we can assume that $x \leqslant y$. Necessarily $y \geqslant 2b+2$. If $x \leqslant b$ then $z = x+y \in [\![a+b+1, a+2b]\!]$ and therefore $\pi(z) \in [\![b+1, 2b]\!]$. Otherwise, $z = x+y \in [\![4b+4, 2a+2b]\!]$ and therefore $\pi(z) \in [\![b+1, 2b]\!]$. Hence, in both cases, $g(\pi(z)) \neq g(x)$.

The result is then obtained by applying Theorem 3.17.

□

As in Corollary 2.4, the additive constant Theorem 3.17 can be improved by weakening the hypotheses made on the last row. The principle behind it is the same as in Proposition 2.5.

Proposition 3.22. *Let $(b, k, a) \in (\mathbb{N}^*)^3$ and let f be a coloring associated to a b-WS-template with width p and k colors. Let $c \in \mathbb{N}$ and assume there there exists a coloring g of $[\![b+1, b+c]\!]$ with k colors such that for all $c \in [\![1, k]\!]$,*

- $\forall (x, y) \in [\![1, a+b]\!] \times [\![b+1, a+b]\!], \begin{cases} f(x) = f(y) \\ \pi(x+y) \leqslant b+c \end{cases} \implies g(\pi(x+y)) \neq f(x),$
- $\forall (x, y) \in [\![1, a+b]\!] \times [\![b+1, b+c]\!], \begin{cases} f(x) = g(y) \\ \pi(x+y) \leqslant b+c \end{cases} \implies g(\pi(x+y)) \neq f(x).$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $x \longmapsto g(x - pS(n))$, we have

$$WS(n+k) \geqslant WS^+(k+1)S(n) + b + c.$$

The WS-templates can actually be fine-tuned further. However, it only gives minor improvements (most likely only an additive constant) at the cost of dramatically increasing the size of the search space. Therefore, it does not seem relevant to use this sophistications given that we could not even find good WS-templates with five colors using a computer (here good means better than those obtain by combining smaller templates).

These modifications work as follows. One may notice that the first row (excluding the "tail") has constraints that other rows do not have because of the tail, especially if the special color appears in the tail as well. Thus allowing to have a coloring on the first row different from the coloring of the other rows would weaken the constraints. Acutally, one may even go further by noticing that on the one hand the first (ordered) color of the sum-free partition used for the extension procedure has more more constraints than the other colors of the sum-free partition since the first row is of this color and is more constrained than the other rows, but that on the other hand it has more degrees of freedom than the other colors of the sum-free partition since in the sum-free partition there cannot be two consecutive numbers of this color. As a result, it removes some constraints imposed by the first row on the other rows.

To sum up, one can look for a generalised WS-template that uses a special coloring for the tail and the first row, a coloring dedicated to the rows whose number is not 1 but is in the first color in the sum-free partition, a coloring for all the other rows and a special coloring for the last numbers (as previously explained for the improvement of the additive constant of WS-templates).

3.4. New lower bounds for Weak Schur numbers

We produced WS-templates using a SAT solver, hence providing lower bound on WS^+ and inequalities of the type $WS(n+k) \geqslant aS(n) + b$. We sought templates providing the greatest value of (a, b) (in the lexicographic order). Details concerning the encoding as a SAT problem can be found in [9].

Here are the inequalities given by the current best WS-templates. The template corresponding to the third inequality can be found in the appendix.

$$WS(n+1) \geqslant 4\,S(n) + 2 \tag{12}$$
$$WS(n+2) \geqslant 13\,S(n) + 8 \tag{13}$$

Inequalities (12) and (13) were found by Rowley, they are detailed in [8].

$$WS(n+3) \geqslant 42\,S(n) + 24 \tag{14}$$
$$WS(n+4) \geqslant 132\,S(n) + 26 \tag{15}$$

Inequality (14) cannot be further improved and was found with a SAT solver. It uses the first sophistication explained in Subsection 3.3 in order to add the last number in the first color. As for inequality (15), it was obtained by combining a S-template with width 33 with a WS-template with width 4. The best template we could find with a computer search gives the inequality $WS(n+4) \geqslant 127S(n) + 68$. It was also found with the SAT solver. In order to reduce the search space, we only looked for WS-templates of five colors which start with a near-optimal $WS(4)$ partition and we assumed that the special color was the last by order of appearance. We highly suspect that better WS-templates with $n \geqslant 5$ colors can be found but one would have not to use the above assumptions. One may try to go over a different search space using a Monte-Carlo method, as in [5]. This could be the suject of a future work.

Like in Subsection 2.3, we compute the lower bounds given by inequalities (12), (13) and (14) for $n \in [\![8, 15]\!]$. The best lower bound for each value of n is highlighted.

n	8	9	10	11
$4\,S(n-1)+2$	6 786	21 146	71 214	243 794
$13\,S(n-2)+8$	6 976	22 056	68 726	231 447
$42\,S(n-3)+24$	6 744	22 536	71 256	222 036
n	12	13	14	15
$4\,S(n-1)+2$	815 314	2 578 514	8 045 162	27 061 154
$13\,S(n-2)+8$	792 332	2 649 772	8 380 172	26 146 778
$42\,S(n-3)+24$	747 750	2 559 840	8 560 800	27 074 400

Table 4: New lower bounds for $n \in [\![8, 15]\!]$

With $S(9) \geqslant 17\,803$, we found a new lower bound for $WS(10)$ using (12). Moreover, inequality (14) gives new lower bounds for $WS(9)$, $WS(10)$, $WS(14)$ and $WS(15)$.

3.5. Another approach: $WS(6) \geqslant 646$

The weak Schur partitions obtained with WS-templates have an extremely regular structure. Therefore, one may expect to find larger partitions by allowing more freedom in the structure of the partition while still preserving a structure somewhat close to a template-based partition thus reducing the search space to a manageable size.

When applying the inequality $WS(n+1) \geqslant 4\,S(n)+2$, one may realize that it is possible to build a weakly sum-free partition of length $4\,S(n)+3$ for small values of n ($n \leqslant 4$) by using the contruction of Corollary 3.18 for the integers 1, 2, $4i$ and $4i+1$ for $i \in [\![1, S(n)]\!]$ but not constraining the other integers. We did the same for the Schur partitions corresponding to Schur number five. More precisely, we imposed these constraints only for $i \leqslant 50$ in order to have more degrees of freedom. The number 50 was chosen arbitrarily so that all of the Schur number 5-partitions could be tested in a few hours.

However, trying out all of the 2 447 113 088 Schur number 5-partitions ([9]) one by one would not result in a reasonable computation time: it is necessary to test the construction on several partitions at once. We encoded the problem as a satisfiability problem and used the 1616 backdoors that were used in [9] in order to encode a group of partitions in a compact and efficient way. Among all of the 1616 backdoors, only the 911$^{\text{th}}$ backdoor gave a weakly sum-free partition of length 643. This backdoor gave a weakly sum-free partition of length 646 and cannot give a weakly sum-free partition of length 647. A weakly sum-free partition of length 646 can be found in the appendix.

3.6. Conclusion on WS-templates

We started by giving a new construction which can be seen as an equivalent for weakly sum-free partitions of Abott and Hanson's construction for sum-free partitions. We then generalized this construction by introducing WS-templates. This allows us to find new lower bounds and new inequalities for weak Schur numbers. One may notice the significant gap between the former lower bounds for weak Schur numbers obtained by conducting a computer search and the new lower bounds obtained with WS-templates. We reckon better WS-templates with $n \geqslant 5$ colors can be found by making different assumptions and using a different method (Monte-Carlo methods for instance).

4. Conclusion and future work

We exhibit new templates for Schur and Ramsey numbers. We also generalize the concept of template to weak Schur numbers. These templates provide new general inequalities of the form $S(n+k) \geqslant aS(n)+b$ and $WS(n+k) \geqslant aS(n)+b$ as well as new lower bounds for both Schur and weak Schur numbers. We also give a new lower bound $WS(6) \geqslant 646$ with a method that can yield slight improvements for fixed values of n over lower bounds obtained with the inequality $WS(n+1) \geqslant 4S(n)+2$.

Given that the introduction of templates is quite recent, we expect the bounds to be improved as this special type of partition becomes better understood and larger templates are found. One may try to find better templates by using heuristics or approximation algorithms, such as Monte-Carlo algorithms for instance.

The best lower bound $WS(6) \geqslant 583$ achieved with a computer search using Monte-Carlo methods [4] is significantly lower than those obtained with a template-like structure ($WS(6) \geqslant 642$ [8], $WS(6) \geqslant 646$ in the present article). We have evidence that 583 is the maximal value in the search space considered by [4, 5, 6]. Therefore, it questions the assumption that large partitions for $WS(n+1)$ start with large partitions for $WS(n)$. It also indicates that $WS(5)$ might need further investigation since the current lower bound $WS(5) \geqslant 196$ [12] was obtained by considering the same type of search space.

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Appendix A. S-templates

1	$1, 6, 9, 13, 16, 20, 24, 27, 31$
2	$2, 5, 14, 15, 25, 26$
3	$3, 4, 10, 11, 12, 28, 29, 30$
4	$7, 8, 17, 18, 19, 21, 22, 23, 32, 33$

Table A.5: S-template with width 33 and 4 colors

1	$1, 5, 18, 12, 14, 21, 23, 30, 32, 36, 39, 43, 45, 52, 103$ $106, 110$
2	$2, 6, 7, 10, 15, 18, 26, 29, 34, 37, 38, 42, 46, 51, 54$ $101, 104, 109$
3	$3, 4, 9, 11, 17, 19, 25, 27, 33, 35, 40, 41, 47, 48, 55$ $100, 107, 108$
4	$13, 16, 20, 22, 24, 28, 31, 58, 61, 67, 88, 94, 97$
5	$44, 50, 53, 56, 57, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70$ $71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85$ $86, 87, 89, 90, 91, 92, 93, 95, 96, 98, 99, 102, 105, 111$

Table A.6: S-template with width 111 and 5 colors

1	$1, 5, 8, 11, 15, 17, 29, 33, 36, 39, 43, 57, 61, 88, 92$ $106, 110, 113, 116, 120, 132, 134, 138, 141, 144, 148, 150, 154, 157, 160$ $164, 178, 182, 185, 188, 341, 344, 347, 351, 365, 369, 372, 375, 379$
2	$2, 9, 13, 16, 20, 23, 24, 27, 28, 31, 34, 35, 38, 42, 45$ $49, 53, 60, 67, 71, 78, 82, 89, 96, 100, 104, 107, 111, 114, 115$ $118, 121, 122, 125, 126, 129, 133, 136, 140, 147, 158, 162, 165, 169, 172$ $176, 183, 187, 194, 201, 328, 335, 342, 346, 353, 357, 360, 364, 367, 371$
3	$3, 4, 12, 14, 19, 25, 30, 32, 40, 41, 47, 48, 58, 91, 101$ $102, 108, 109, 117, 119, 124, 130, 135, 137, 145, 146, 152, 153, 161, 163$ $168, 179, 181, 190, 339, 348, 350, 361, 366, 368, 376, 377$
4	$6, 7, 10, 18, 21, 22, 26, 37, 46, 50, 51, 54, 65, 70, 79$ $84, 95, 98, 99, 103, 112, 123, 127, 128, 131, 139, 142, 143, 151, 155$ $156, 159, 167, 170, 171, 175, 186, 343, 354, 358, 359, 362, 370, 373, 374$ 378
5	$44, 52, 55, 56, 59, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75$ $76, 77, 80, 81, 83, 85, 86, 87, 90, 93, 94, 97, 105, 189, 196$ $197, 200, 203, 206, 207, 209, 214, 219, 231, 298, 310, 315, 320, 322, 323$ $326, 329, 332, 333, 340$
6	$149, 166, 173, 174, 177, 180, 184, 191, 192, 193, 195, 198, 199, 202, 204$ $205, 208, 210, 211, 212, 213, 215, 216, 217, 218, 220, 221, 222, 223, 224$ $225, 226, 227, 228, 229, 230, 232, 233, 234, 235, 236, 237, 238, 239, 240$ $241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255$ $256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270$ $271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285$ $286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 299, 300, 301$ $302, 303, 304, 305, 306, 307, 308, 309, 311, 312, 313, 314, 316, 317, 318$ $319, 321, 324, 325, 327, 330, 331, 334, 336, 337, 338, 345, 349, 352, 355$ $356, 363, 380$

Table A.7: S-template with width 380 and 6 colors

Appendix B. WS-templates

1	$1, 2, 4, 8, 11, 22, 25, (\mathbf{N+1})$
2	$3, 5, 6, 7, 19, 21, 23, 36$
3	$9, 10, 12, 13, 14, 15, 16, 17, 18, 20$
4	$24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40$ $41, 42$

Table B.8: 23-WS-template with width 42 and 4 colors

This template provides the inequality $WS(n+3) \geqslant 42\,S(n) + 24$ by placing one last number, here represented by $(\mathbf{N+1})$, in the first subset.

Appendix C. $WS(6) \geqslant 646$

1	1, 2, 6, 10, 14, 18, 26, 30, 34, 42, 46, 50, 54, 62, 70, 79, 82, 90, 95, 99, 111, 115, 119, 123, 131, 135, 139, 143, 151, 155, 159, 163, 171, 175, 179, 183, 187, 195, 199, 203, 207, 211, 215, 220, 224, 228, 232, 236, 239, 244, 252, 256, 260, 264, 267, 272, 275, 280, 284, 292, 296, 300, 304, 308, 312, 316, 320, 328, 340, 344, 348, 353, 360, 364, 368, 372, 381, 385, 388, 393, 397, 404, 408, 413, 417, 425, 428, 433, 441, 445, 449, 453, 457, 461, 465, 469, 473, 485, 489, 493, 497, 502, 505, 509, 513, 517, 521, 525, 529, 533, 537, 541, 546, 549, 553, 558, 562, 566, 569, 574, 578, 586, 590, 593, 598, 602, 606, 610, 614, 618, 622, 626, 630, 634, 638, 642, 646
2	3, 4, 5, 15, 16, 17, 27, 28, 29, 39, 40, 41, 47, 48, 49, 112, 113, 114, 120, 121, 122, 132, 133, 134, 156, 157, 158, 176, 177, 178, 200, 201, 202, 221, 222, 258, 259, 281, 282, 283, 301, 302, 303, 345, 346, 347, 365, 366, 367, 389, 426, 427, 446, 447, 448, 470, 471, 472, 490, 491, 492, 514, 515, 516, 526, 527, 528, 534, 535, 536, 599, 600, 601, 607, 608, 609, 619, 620, 621, 631, 632, 633, 643, 644, 645
3	7, 8, 9, 19, 20, 21, 22, 23, 24, 25, 35, 36, 37, 38, 87, 88, 89, 136, 137, 138, 150, 152, 153, 154, 180, 181, 182, 196, 197, 198, 208, 209, 210, 212, 213, 214, 261, 262, 263, 265, 266, 276, 277, 278, 279, 309, 321, 322, 323, 324, 325, 326, 327, 338, 339, 369, 370, 371, 382, 384, 386, 387, 434, 435, 436, 437, 438, 439, 440, 450, 451, 452, 466, 467, 468, 482, 494, 495, 496, 499, 500, 510, 511, 512, 559, 560, 561, 611, 612, 613, 623, 624, 625, 627, 628, 629, 639, 640, 641
4	11, 12, 13, 31, 32, 33, 51, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 124, 125, 126, 127, 128, 129, 130, 144, 145, 146, 147, 148, 149, 164, 165, 166, 167, 168, 169, 170, 172, 173, 174, 188, 189, 190, 191, 192, 193, 194, 454, 455, 456, 458, 459, 460, 474, 475, 476, 477, 478, 479, 480, 481, 483, 484, 498, 501, 503, 504, 518, 519, 520, 522, 523, 524, 538, 539, 540, 542, 543, 544, 545, 547, 548, 597, 615, 616, 617, 635, 636, 637
5	43, 44, 45, 52, 53, 55, 56, 57, 58, 59, 60, 61, 63, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 77, 78, 80, 81, 83, 84, 85, 86, 91, 92, 93, 94, 216, 217, 218, 219, 223, 225, 226, 227, 231, 233, 234, 235, 237, 238, 240, 241, 242, 243, 245, 246, 247, 251, 253, 254, 255, 257, 383, 390, 391, 392, 394, 395, 396, 400, 401, 402, 403, 405, 406, 407, 409, 410, 411, 412, 414, 415, 416, 421, 422, 423, 424, 429, 430, 431, 432, 554, 555, 556, 557, 563, 564, 565, 567, 568, 570, 571, 572, 573, 575, 576, 577, 579, 580, 581, 582, 583, 584, 585, 587, 588, 589, 591, 592, 594, 595, 596, 603, 604, 605
6	96, 97, 98, 116, 117, 118, 140, 141, 142, 160, 161, 162, 184, 185, 186, 204, 205, 206, 229, 230, 248, 249, 250, 268, 269, 270, 271, 273, 274, 285, 286, 287, 288, 289, 290, 291, 293, 294, 295, 297, 298, 299, 305, 306, 307, 310, 311, 313, 314, 315, 317, 318, 319, 329, 330, 331, 332, 333, 334, 335, 336, 337, 341, 342, 343, 349, 350, 351, 352, 354, 355, 356, 357, 358, 359, 361, 362, 363, 373, 374, 375, 376, 377, 378, 379, 380, 398, 399, 418, 419, 420, 442, 443, 444, 462, 463, 464, 486, 487, 488, 506, 507, 508, 530, 531, 532, 550, 551, 552

Table C.9: Weakly sum-free partition of $[\![1, 646]\!]$ into 6 subsets.