New lower bounds for Schur and weak Schur numbers

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Abstract

This article provides new lower bounds for both Schur and weak Schur numbers. These results were obtained by continuing on Rowley's "templates"-based approach for Ramsey and Schur numbers. Finding suitable templates allows us to apply Rowley's construction to get explicit partitions improving lower bounds as well as the growth rate of both Schur numbers and Ramsey numbers $R_n(3)$. We also developed a new method to improve lower bounds on weak Schur numbers. Furthermore, this paper tries to analyze former works on the subject based on the principle that good partitions into $n+1$ subsets start with good partitions into n subsets. We show that exceeding the previous lower bound $WS(6) \geqslant 582$ is impossible with such an assumption upon imposing certain conditions on the good 5-subsets partition. The new lower bounds include $S(9) \geqslant 17803$, $S(10) \geqslant 60948$, $WS(9) \geqslant 22536$ and $WS(10) \geqslant 71214$.

1 Introduction

2 State of the art

3 Definitions and notations

We start by defining sum-free and weakly sum-free subsets to introduce regular and weak Schur numbers.

Definition 3.1 *A subset A of $\mathbb{N}$ is said to be sum-free when:*

$$\forall (a,b) \in A^2,\ a+b \notin A$$

Definition 3.2 *A subset B of $\mathbb{N}$ is said to be weakly sum-free when:*

$$\forall (a,b) \in B^2,\ a \neq b \Longrightarrow a+b \notin B$$

Let us notice that a sum-free subset is also weakly sum-free, hence justifying the name of *weakly* sum-free subsets. Given p and n two integers, we are interested in partitioning the set of integers from 1 to p into n (weakly) sum-free subsets.

Notation 3.3 *We denote by $[\![1,p]\!]$ the set of integers $\{1,2,...,p\}$.*

Schur proved in [10] that given a number of subsets n, there exists a value of p such that there exists no partition of $[\![1,q]\!]$ into n sum-free subsets for any $q \geqslant p$. A similar property holds for weakly sum-free subsets (reference necessaire). These observations lead to the following definitions.

Definition 3.4 *Let $n \in \mathbb{N}^*$. There exists a greatest integer that we denote $S(n)$ (resp. $WS(n)$) such that $[\![1,S(n)]\!]$ (resp. $[\![1,WS(n)]\!]$) can be partitioned into n sum-free subsets (resp. weakly sum-free subsets). $S(n)$ is called the n^{th} Schur number and $WS(n)$ the n^{th} weak Schur number.*

Notation 3.5 *For a partition of $[\![1,p]\!]$ in n subsets, we generally denote these subsets $A_1,...,A_n$. We also denote $m_i = \min(A_i)$. By ordering the subsets, we mean assuming that $m_1 < ... < m_n$. However, if not specified we do not make this hypothesis since we do not always consider partitions in which every subset plays a symmetric role.*

Definition 3.6 *We sometimes refer to a partitition as a colouring. The colouring associated to a partition $A_1,...,A_n$ of $[\![1,p]\!]$ is the function f such that $\forall x \in [\![1,p]\!], x \in A_{f(x)}$. Likewise, the partition associated to a colouring f of $[\![1,p]\!]$ with n colors is $\forall c \in [\![1,n]\!], A_c = f^{-1}(c)$.*

4 Schur numbers

In this section, we use Rowley's constructions [8] in the context of Schur numbers. To improve lower bounds for Ramsay's numbers, Rowley introduces some sum-free partitions verifying some additional properties which can be extended using a method which generalizes Abbott and Hanson's construction [1]. Rowley named these partitions "templates", and we will keep this name in the entire article. We find suitable templates and use them to find new lower bounds for Schur numbers.

4.1 Definition of S^+

Definition 4.1 *We call SF-template of n colors and lenght p a partition of $[\![1,p]\!]$ into n sum-free subsets $A_1, A_2, ..., A_n$ which verify :*

$$\forall i \in [\![1, n-1]\!], \forall (x,y) \in A_i^2, x+y > p \Longrightarrow x+y-p \notin A_i$$

We note $S^+(n)$ the maximal lenght for a SF-template of n colors.

Remark 4.1 *Here, n is the "special" color: it has less constraints than the other colors. However, please note that n is not necessarily the last color by order of appearance.*

Remark 4.2 *SF-templates include Abott and Hanson's construction [1] as a special case.*

Proposition 4.1 *Let $n \in [\![2, +\infty]\!]$. Then*

$$2S(n-1)+1 \leqslant S^+(n) \leqslant S(n)$$

Proof : The lower bound comes from Abott and Hanson's construction. The upper bound comes from the fact that a SF-template of length p with n colors is also a partition of $[\![1,p]\!]$ into n sum-free subsets.

Remark 4.3 *S^+ and S have the same asymptotic growth rate.*

4.2 Inequalities using S^+

The main result on S^+ follows. It allows us to improve lower bounds on Schur numbers by computing S^+.

Theorem 4.1 *Let $(n,k),(p,q) \in (\mathbb{N}^*)^2$. If there exists a SF-template of $k+1$ colors and lenght p, and a partition of n sum-free subsets of $[\![1,q]\!]$ then there exists a partition of $n+k$ sum-free subsets of $[\![1, pq+m_{k+1}-1]\!]$. m_{k+1} is the minimum of the special subset in the SF-template.*

Setting $p = S^+(k+1)$ and $q = S(n)$ yields the following corollary.

Corollary 4.1.1 *Let $n, k \in \mathbb{N}^*$. Then*

$$S(n+k) \geqslant S^+(k+1)S(n) + m_{k+1} - 1$$

Remark 4.4 *The following proposition can help improve the additive constant of a SF-template. Although it does not allow us to improve the SF-templates we have found, the analogous of this proposition for WSF-templates (see next section) allows us to improve one of them.*

Proposition 4.2 *Let $(k,p) \in \mathbb{N}^*)^2$ and let f be a colouring associated to a SF-template of length p with k colors. Let $b \in \mathbb{N}$ ($b = m_{k+1} - 1$ works) and assume there there exists a colouring g of $[\![1, b]\!]$ with k colors such that:*

- $\forall c \in [\![1,k]\!], \forall (x,y) \in [\![1,p]\!]^2, (f(x) = f(y)$ *and* $(x+y) \mod p \leqslant b) \implies g((x+y) \mod p) \neq f(x)$
- $\forall c \in [\![1,k]\!], \forall (x,y) \in [\![1,p]\!] \times [\![1,b]\!], (f(x) = g(y)$ *and* $x+y \leqslant b) \implies g(x+y) \neq f(x)$

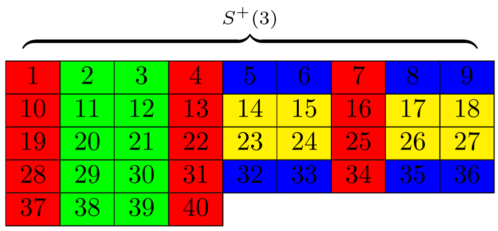
Then, for every $n \in \mathbb{N}^*$, by using on the last row the colouring $i \longmapsto g(i - pS(n))$, we have

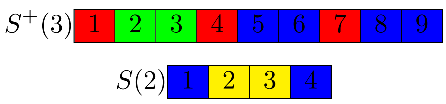
$$S(n+k) \geqslant S^+(k+1)S(n) + b$$

Remark 4.5 *This proposition corresponds to the fact that sometimes a column is not the sum of two columns of a given color, but adding this column to the color would create sums in the color when applying the extension procedure. However, the last line does not intereact will all the columns when it comes to creating new sums. As a result, the hypotheses made on the colouring of the last row can be weakened.*

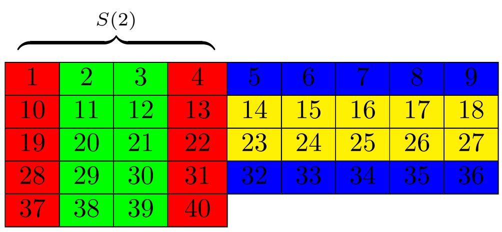
The idea lying beneath this theorem is similar to Abbott and Hanson's contruction [1]. They extend vertically a sum-free partition, and horizontally an other sum-free partition. This way each "block" acts like a security zone for the other one. Here, the horizontal partition is no longer to the side of the vertical one, but it occupies the column of the special color of the SF-template, i.e the one without the extra condition. We give the following example for $p = 9, q = 4, n = 2$ and $k = 2$.

This shows the inequality $S(2+2) \geqslant S^+(3)S(2) + 4$. The special color is blue and is first used to color 5, hence $4 = m_3 - 1$. The special color is blue.





To comparison, here is an usual Abbott and Hanson partition : $S(2+2) \geqslant S(2)(2S(2)+1)+S(2)$.



Proof of theorem: We denote by f the colouring associated to the SF-template of lenght p and g the one associated to the sum-free partition of $[\![1,q]\!]$.

$$f : [\![1,p]\!] \longrightarrow [\![1,k+1]\!] \text{ and } \forall (x,y) \in [\![1,p]\!]^2, \begin{cases} f(x) = f(y) \leqslant k \\ x+y > p \end{cases} \Longrightarrow f(x+y-p) \neq f(x)$$

with also the sum-free condition :

$$\forall (x,y) \in [\![1,p]\!]^2,\ f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

Moreover, by definition of m_{k+1}, if $x < m_{k+1}$, $f(x) \leqslant k$.

$$g : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ g(x) = g(y) \Longrightarrow g(x+y) \neq g(x)$$

We now define $h : [\![1, pq + m_{k+1} - 1]\!] \longrightarrow [\![1, n+k]\!]$ as follows : $\forall x \in [\![1, pq + m_{k+1} - 1]\!]$, we write $x = \alpha p - u$ where $\alpha \in [\![1, q+1]\!]$ and $u \in [\![0, p-1]\!]$. This decomposition if of course unique.

If $f(p-u) \leqslant k$, we set $h(x) = f(p-u)$, else (i.e $f(p-u) = k+1$) $h(x) = k + g(\alpha)$.
If $\alpha = q+1$, we have a problem plugging it into g. Hopefully, this issue never arises : if $\alpha = q+1$, we have $u > p - m_{k+1}$ since $x \in [\![1, pq + m_{k+1} - 1]\!]$, hence $p - u < m_{k+1}$ and $f(p-u) \leqslant k$.

Prooving that the partition of $[\![1, pq]\!]$ induced by h is sum-free will complete the proof.

Let $x, y \in [\![1, pq + m_{k+1} - 1]\!]$, with $h(x) = h(y)$ and $x + y \leqslant pq + m_{k+1} - 1$. We write $x = \alpha p - u$ and $y = \beta p - v$. We simply need $h(x+y) \neq h(x)$.

We first examine the case where $h(x) \leqslant k$. We assume $h(x+y) \leqslant k$, otherwise we are fine. Thus by definition of h, we have $h(x) = f(p-u) = f(p-v) = h(y)$.

- If $u+v < p$, then $x+y = (\alpha+\beta) - (u+v)$ with $u+v \in [\![0, p-1]\!]$. We have $f(p-u) = f(p-v) \leqslant k$ and $p-u+p-v > p$, the extra condition on f provides $f(p-u-v) \neq f(p-u)$. We assumed $h(x+y) \leqslant k$ hence $h(x+y) = f(p-u-v)$. Since $h(x) = f(p-u)$, we have at last $h(x+y) \neq h(x)$.
- If $u+v \geqslant p$, then $x+y = (\alpha+\beta-1) - (u+v-p)$ with $u+v-p \in [\![0, p-1]\!]$. We assumed $h(x+y) \leqslant k$ hence $h(x+y) = f(p-(p-u-v)) = f(p-u+p-v)$. Since $f(p-u+p-v) \neq f(p-u)$, we get $h(x+y) \neq h(x)$.

We now assume $h(x) > k$. Then $h(x) = k + g(\alpha) = k + g(\beta) = h(y)$, hence $g(\alpha) = g(\beta)$. We have the two same cases as before.

- If $u+v < p$, then $x+y = (\alpha+\beta) - (u+v)$ with $u+v \in [\![0, p-1]\!]$. We assume $h(x+y) > k$ otherwise the expected result is trivial. Since $g(\alpha) = g(\beta)$, the sum-free condition garanties $g(\alpha) \neq g(\alpha+\beta)$, thus $h(x+y) = k + g(\alpha+\beta) \neq k + g(\alpha) = h(x)$.
- If $u+v \geqslant p$, then $x+y = (\alpha+\beta-1) - (u+v-p)$ with $u+v-p \in [\![0, p-1]\!]$. Because of the assumption $h(x) > k$, we necessarily have $f(p-u) = k+1 = f(p-v)$. Then $f(p-(u+v-p)) = f(p-u+p-v) \leqslant k$ with the sum-free condition on f, hence $h(x+y) = f(p-(u+v-p))$ by construction of h. Thus $h(x+y) \leqslant k < h(x)$.

Definition 4.2 *A SF-template with n colors is said to be symmetric if the partition in n sum-free subsets derived (with the additive constant) from this template is symmetric. A sum-free partition $A_1, ..., A_n$ of $[\![1,p]\!]$ is said to be symmetric if for all $x \in [\![1,p]\!]$, x and $p+1-x$ belong to the same subset (except if $x = p+1-x$).*

Using a SAT solver, we exhibited SF-templates, hence providing lower bound on S^+ and inequalities of the type $S(n+k) \geqslant aS(n)+b$. We have sought templates providing the greatest value of (a,b) (for the lexicographic order). When the number of colors excceeded 5, in order to reduce the search space we looked for symmetric SF-templates, we assumed that the special color was the last color to appear and we constrained the m_c's out of being too small. Further details about the encoding as a SAT problem can be found in the SAT section.

Here are the best inequalities on Schur numbers so far (the templates corresponding to the third, fourth and fifth inequalities can be found in the appendix):

$$S(n+1) \geqslant 3S(n)+1$$

$$S(n+2) \geqslant 9S(n)+4$$

$$S(n+3) \geqslant 33S(n)+6$$

$$S(n+4) \geqslant 111S(n)+43$$

$$S(n+5) \geqslant 380S(n)+148$$

$$S(n+6) \geqslant 1140S(n)+528$$

The first inequality comes from the original Schur's paper [10]. The second one is due to Abott [1] and the third one to Rowley [8]. The other ones are new.

The first 3 inequalities are optimal. The fourth one is optimal among symmetric SF-templates whose special color is the last in the order of apparition (and with a multiplicative factor less than or equal to 118). The fifth one is most likely not optimal but should not be too far from the optimal SF-template. Finally, the sixth one is obtained by combining (see below) the SF-template of length 380 and the one of length 3. Although we could not find a better SF-template with 7 colors, the last inequality is definetely very far from the optimal value. One may try to seek better templates by constraining less the search space and by using Monte-Carlo methods, as in [4]. This could be the suject of a future work.

We also have a similar theorem where only S^+ is involved.

Theorem 4.2 *Let $(n,k),(p,q) \in (\mathbb{N}^*)^2$. If there exists a SF-template of $k+1$ colors and lenght p, and SF-template of n color and lenght q, then there exists SF-template of $(n+k)$ and lenght pq.*

And the associated inequality :

Corollary 4.2.1 *Let $n,k \in \mathbb{N}^*$, we have*

$$S^+(n+k) \geqslant S^+(k+1)S^+(n)$$

Proof : The idea is the same as in the previous theorem. The only difference is the SF property inherited from the second SF-template.

4.3 New lower bounds for Schur numbers

The previous inequalities give new lower bounds for $S(n)$ for $n \geqslant 9$. We compute the lower bounds for $n \in [\![8, 15]\!]$ using the four different inequalities, please notice that the best values for $n = 8$ and $n = 13$ were obtained thanks to the first one, found by Rowley. The best lower bounds are highlighted.

n	8	9	10	11
$33S(n-3)+6$	5286	17694	55446	174444
$111S(n-4)+43$	4927	17803	59539	186523
$380S(n-5)+148$	5088	16868	60948	203828
$1140S(n-6)+528$	5088	15348	50688	182928
n	12	13	14	15
$33S(n-3)+6$	587505	2011290	6726330	21072090
$111S(n-4)+43$	586789	1976176	6765271	22624951
$380S(n-5)+148$	638548	2008828	6765288	23160388
$1140S(n-6)+528$	611568	1915728	6026568	20295948

Except for 8, 9 and 13, the best lower bounds are obtained thanks to the third inequality $S(n+5) \geqslant 380S(n) + 148$. The table doesn't go any further, but the same inequality allows to improve the lower bounds for every $n \geqslant 15$.

Corollary 4.2.2 *The growth rate for Schur numbers (and Ramsey numbers $R_n(3)$) satisfies $\gamma \geqslant \sqrt[5]{380} \approx 3.28$.*

Proof : It is a mere consequence of the inequality $S(n+5) \geqslant 380S(n) + 148$. As for Ramsey's numbers growth rate, a lower bound can be found using Schur's one, thanks to $S(n) \leqslant R_n(3) - 2$ (see [10]).

5 Weak Schur numbers

In this section, we generalize Rowley's constructions in [9]. We then introduce, by analogy with the third section, the integer $WS^+(n)$ to build suitable templates.

5.1 Lower bound for Weak Schur numbers using Schur and Weak Schur numbers

Up to now, there was no equivalent for weak Schur numbers of Abott and Hanson's construction [1]. Here we give a general lower bound for weak Schur numbers as a function of both regular and weak Schur numbers. The following theorem, inspired by Rowley's inequalities for $WS(n+1)$ and $WS(n+2)$, was found and proved by Romain Ageron.

Theorem 5.1 *Let $(p,q),(n,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets then there exists a partition of $[\![1,p(q+\lceil \frac{q}{2} \rceil + 1)+q]\!]$ into $n+k$ weakly sum-free subsets.*

In particular, if we choose $q = WS(n)$ and $p = S(k)$ in the last theorem, the next corollary follows.

Corollary 5.1.1 $\forall (n,k) \in (\mathbb{N}^*)^2,\ WS(n+k) \geqslant S(k)\left(WS(n) + \left\lceil \frac{WS(n)}{2} \right\rceil + 1\right) + WS(n)$

Remark 5.1 *This can be seen as an equivalent for weak Schur numbers of Abott and Hanson's construction for Schur numbers.*

Remark 5.2 *In the above ineqality, a "+1" can be added to the lower bound if $WS(n)$ is odd (more generally if q is odd in the theorem). However, it makes the proof less clear and it is never useful in practice.*

We give here an intuitive explanation of the above theorem; a formal proof can be found in the appendix. Let $(p,q) \in \mathbb{N}^2$ such that there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets. Let $a \in \mathbb{N}$ with $a > q$ and let's try to build a colouring of $[\![1, ap+q]\!]$ into $n+k$ weakly sum-free subsets. Let $l = a-b-1$, $r \in [\![1,q]\!]$ and $w = a-l-r-1 = b-r$.

First, put the integers of $[\![1, ap+q]\!]$ in the following table (with a columns and $p+1$ lines) and divide it into 3 blocks (the columns are numbered from $-l$ to $+q$):

- $\mathcal{T}$ (the "tail"): the integers from 1 to q. NB: this is line number 0.
- $\mathcal{R}$ (the "rows"): the integers in columns $-l$ to $+r$ (excluding $\mathcal{T}$).
- $\mathcal{C}$ (the "columns"): the integers in the last w columns (excluding $\mathcal{T}$).

Like for SF-templates, $\mathcal{R}$ and $\mathcal{C}$ play the role of security zones for each other. Note that with this numbering of columns, the column of the sum of two numbers is the only integer in $[\![-l,q]\!]$ equal to two the sum of the columns modulo a.

					1	2	$\dots$	r	$r+1$	$\dots$	$b-1$	b
$a-l$	$a-l+1$	$\dots$	$a-1$	a	$a+1$	$\dots$	$a+r-1$	$a+r$	$a+r+1$	$\dots$	$a+b-1$	$a+b$
$2a-l$	$\dots$	$\dots$	$\dots$	$2a$	$\dots$	$\dots$	$\dots$	$2a+r$	$\dots$	$\dots$	$\dots$	$2a+b$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$pa-l$	$\dots$	$\dots$	$\dots$	pa	$\dots$	$\dots$	$\dots.$	$pa+r$	$\dots$	$\dots$	$\dots$	$pa+b$

$\mathcal{T}$ block

We color this block using the weakly sum-free colouring of $[\![1,q]\!]$ with colors $1,...,n$.

$\mathcal{R}$ block

In this block, we use the colors $n+1,...,n+k$. We colour an integer x according to its line number (written $\lambda(x)$). For every $x \in \mathcal{R}$, we colour x with $n+c$ where c is the colour of $\lambda(x)$ in the sum-free colouring of $[\![1,p]\!]$. Let $(x,y) \in \mathcal{R}^2$. The cases are twofold.

- $\lambda(x+y) = \lambda(x) + \lambda(y)$
  In this case, we use the sum-free property of the colouring of $[\![1,p]\!]$ (in block $\mathcal{C}$, we only use colours $1,...,n$).

- $\lambda(x+y) \neq \lambda(x) + \lambda(y)$
  In this case, we do not have information about the colour of $\lambda(x+y)$. Thereby, we want to have $x+y \in \mathcal{C}$. A simple solution is to limit the horizontal movement, that is if the sum changes line, not to move too far so that it stays in $\mathcal{C}$. There, the maximal displacement to the left (resp. to the right) is $2l$ (resp. $2r$). Not crossing entirely $\mathcal{C}$ by going to the left is then expressed as $-2l > -a+r$. Likewise, not going to far to the right is expressed as $2r < a-l$. It can then be written as $\max(l,r) \leqslant w$.

$\mathcal{C}$ block

In this block, we use colors $1,...,n$. We colour an integer x according to its column number (written $\tilde{\pi}(x)$, it is linked to the projection on the first line, written π, by the relation $\tilde{\pi}(x) = \pi(x) - a$). A simple solution is to colour x with the same colour as $\tilde{\pi}(x)$ in the weakly sum-free colouring of $[\![1,q]\!]$. As long as $2b \leqslant a+r$ (not going two far to the right) and there is no $x \in \tilde{\pi}(\mathcal{C})$ such that $2x \in \tilde{\pi}(\mathcal{C})$ (so that we do not have a sum in $\mathcal{C}$ when taking two numbers in the same column), the colours $1,...,n$ are sum-free.

In particular, taking $w = l = \lceil \frac{q}{2} \rceil$ and $r = \lfloor \frac{q}{2} \rfloor$ works, thus obtaining the above theorem.

Remark 5.3 *This formula includes the results of Rowley [9] as a special case. For $n > 2$, this formula does not give new lower bounds but in the same way as we introduced S^+(Definition 3.1), we define WS^+ and find inequalities between WS^+, WS and S.*

5.2 Definition of WS^+

In this subsection, we define the objects and prove the results needed for the general theorem regarding templates for weak Schur numbers.

Definition 5.1 *Let (a,b) $n \in (\mathbb{N}^*)^2$,$a > b$, we will define $\pi_{a,b}$ the projection:*

$$\pi_{a,b} : x \longmapsto (Id + a\mathbf{1}_{[\![0,b]\!]})(x \text{ } mod \text{ } a))$$

We will note the projection π and not $\pi_{a,b}$ when there is no doubt about the a and b we use.

Remark 5.4 *π is the projection on the first line mentioned in the intuitive explanation.*

Proposition 5.1 *Let $x \in [\![1, b]\!]$, let $y \in \mathbb{N}^*$ such that $x + \pi(y) \leqslant a + b$, then we have: $\pi(x + y) = x + \pi(y)$*

Proof :Let x $\in [\![1, b]\!]$, let y $\in \mathbb{N}^*$ such that $x + \pi(y) \leqslant a + b$
if $x + \pi(y) < a$:we remark that $\pi(y) > b$ and therefore $x + \pi(y) > b$:

$$\begin{aligned}
\pi(x + y) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x+y mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x + \pi(y) \text{ mod a) since } \pi(y)\text{=y mod a} \\
&= x + \pi(y)
\end{aligned}$$

if $x + \pi(y \geqslant a$:

$$\begin{aligned}
\pi(x + y) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x+y mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x + \pi(y) \text{ mod a) since } \pi(y)\text{=y mod a} \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(x + \pi(y) - a) \\
&= x + \pi(y) - a + a\mathbf{1}_{[\![0,b]\!]})(x + \pi(y) - a) \\
&= x + \pi(y) - a + a \qquad \text{since } x + \pi(y) - a \in [\![0, b]\!] \\
&= x + \pi(y)
\end{aligned}$$

Proposition 5.2 *Let (x,y)$\in (\mathbb{N}^*)^2$, $\pi(\pi(x) + \pi(y)) = \pi(x + y)$*

Proof :Let (x,y)$\in (\mathbb{N}^*)^2$,

$$\begin{aligned}
\pi(\pi(x) + \pi(y)) &= (Id + a\mathbf{1}_{[\![0,b]\!]})(\pi(x) + \pi(y) \text{ mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})((Id + a\mathbf{1}_{[\![0,b]\!]})(\text{x mod a}) + (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{y mod a) mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{ ((x mod a) + (y mod a)) mod a}) \\
&= (Id + a\mathbf{1}_{[\![0,b]\!]})(\text{ x+y mod a}) \\
&= \pi(x + y)
\end{aligned}$$

Definition 5.2 *Let (a,b) $n \in (\mathbb{N}^*)^2$,$a > b$, we will define $\lambda_{a,b}$ the projection:*

$$\lambda_{a,b} : x \longmapsto 1 + \left\lfloor \frac{x - b - 1}{a} \right\rfloor$$

We will note the projection λ and not $\lambda_{a,b}$ when there is no doubt about the a and b we use.

Remark 5.5 *In the following theorem, λ is the function which maps an element x to its line number, as mentioned in the intuitive explanation.*

Proposition 5.3 *Let (a,b)$\in (\mathbb{N}^*)^2$, $a > b$, let $x \in \mathbb{N}^*$, $x = a\lambda(x) + \pi(x) - a$*

PROOF :Let (a,b)$\in (\mathbb{N}^*)^2$, $a > b$, let x $\in \mathbb{N}^*$,

$$a\lambda(x) + \pi(x) - a = a\left\lfloor \frac{x-b-1}{a} \right\rfloor + (\text{x mod a}) + \mathbf{1}_{[\![0,b]\!]}(\text{x mod a})$$

if x mod a $> b$: $a\lambda(x) + \pi(x) - a = a\left\lfloor \frac{x}{a} \right\rfloor + \text{x mod a} = x$

if x mod a $\leqslant b$: $a\lambda(x) + \pi(x) - a = a(\left\lfloor \frac{x}{a} \right\rfloor - 1) + \text{x mod a} + a = x$

Definition 5.3 *Let $(p, n, b) \in (\mathbb{N}^*)^3$, Let $(A_1, ..., A_n)$ a partition of $[\![1, p]\!]$. This partition is said to be a b-weakly-sum-free template (b-WSF-template) of n colors and lenght p when:*

<u>$\forall i \in [\![1, n]\!]$, A_i *is weakly-sum-free*</u>

<u>$\forall i \in [\![1, n]\!]$, $A_i \backslash [\![1, b]\!]$ *is sum-free*</u>

<u>*For A_n (the special subset):*</u> $\forall (x, y) \in A_n^2$,

$$if \quad x + y > b + 2(p - b), \quad x + y - 2(p - b) \notin A_n$$

<u>*For the others subsets:*</u> $\forall i \in [\![0, n-1]\!], \forall (x, y) \in A_i^2$,

$$x + y > p \implies \pi(x + y) \notin A_i$$

Remark 5.6 *Please note that the special color n is not necessarily the last color by order of appearance.*

Definition 5.4 *Let $(k, b) \in (\mathbb{N}^*)^2$. If there exist p such that exists a partition of $[\![1, p]\!]$ into k subsets which is a b-WSF-template of k colors and lenght p, we note:*

$WS_b^+ = -b + \max\{p \in \mathbb{N}^* /$ there exists a partition of $[\![1, p]\!]$ into k subsets which is a b-WSF-template of k colors and lenght p $\}$

If this p does not exist, we set $WS_b^+ = 0$

Definition 5.5 *Let $n \in \mathbb{N}^*$, we define $WS^+(n)$=$\max_{b\in\mathbb{N}^*}\{WS_b^+(n)\}$*

Proposition 5.4 *Let $n \in [\![2, +\infty]\!]$, we have :*

$$\frac{3}{2}WS(n-1) + 1 \leqslant WS^+(n) \leqslant WS(n)$$

PROOF : The lower bound comes from the analogous of Abott and Hanson's construction for weak Schur numbers. The upper bound comes from the fact that a WSF-template of length p with n colors is also a partition of $[\![1, p]\!]$ into n sum-free subsets.

Remark 5.7 *WS^+ and WS have the same asymptotic growth rate.*

5.3 Lower bound for Weak schur numbers using Schur and Weak Schur template numbers

Theorem 5.2 *Let $(q,n,b) \in (\mathbb{N}^*)^3$, let $(p,k) \in (\mathbb{N}^*)^2$. If there exists a partition of k sum-free subsets of $[\![1,p]\!]$ and a partition of n subsets $(A_1, ..., A_n)$ of $[\![1,q]\!]$ which is a b-WSF of n colors and lenght q, then there exists a partition of $[\![1, b+p\times(q-b)]\!]$ into $(k+n-1)$ weakly sum-free subsets.*

In particular, if we choose $p = S(k)$ and $q = WS^+(n)$ in the last theorem, the next corollary follows.

Corollary 5.2.1 *$\forall(n,k) \in (\mathbb{N}^*)^2$, let $b_{max} = max\{b \in \mathbb{N}^* / WS_b^+(n+1) = WS^+(n+1)\}$,*

$$WS(n+k) \geqslant S(k)\, WS^+(n+1) + b_{max}$$

Remark 5.8 *In the SF-template construction for Schur numbers, the additive constant comes from the fact that the special color does not necessarily appear right at the begining of the repeating pattern. Likewise, b_max can actually be replaced by*

$$\max\left\{b + \min(A_{n+1}\backslash[\![1,b]\!]) - 1 \mid WS_b^+(n+1) = WS^+(n+1)\right\}$$

Remark 5.9 *Acutally, like for SF-templates, the additive constant of a WSF-template (in the form given by the above remark) can be improved by weakening the hypotheses made on the last row. The principle behind it is the same as in the analogous proposition for SF-templates.*

Proposition 5.5 *Let $(b,k,p) \in \mathbb{N}^*)^3$ and let f be a colouring associated to a b-WSF-template of length p with k colors. Let $c \in \mathbb{N}$ ($c = \min(A_{k+1}\backslash[\![1,b]\!]) - 1$ works) and assume there there exists a colouring g of $[\![b+1, b+c]\!]$ with k colors such that:*

- $\forall c \in [\![1,k]\!], \forall(x,y) \in [\![1,a+b]\!] \times [\![b+1,a+b]\!], (f(x) = f(y)$ *and* $\pi(x+y) \leqslant b+c) \implies g(\pi(x+y)) \neq f(x)$
- $\forall c \in [\![1,k]\!], \forall(x,y) \in [\![1,a+b]\!] \times [\![b+1,b+c]\!], (f(x) = g(y)$ *and* $\pi(x+y) \leqslant b+c) \implies g(\pi(x+y)) \neq f(x)$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the colouring $i \longmapsto g(i - pS(n)$, we have

$$WS(n+k) \geqslant WS^+(k+1)S(n) + b + c$$

Proof: Let $(q,n,b) \in (\mathbb{N}^*)^3$, let $(p,k) \in (\mathbb{N}^*)^2$, let a=q-b.
We denote by g the colouring associated to the partition of $[\![1,q]\!]$ and h the one associated to the partition of $[\![1,p]\!]$.

$$g : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } (A_{g^{-1}(1)}, ...A_{g^{-1}(q)}) \text{ is a b-WSF-template.}$$

$$h : [\![1,p]\!] \longrightarrow [\![1,k]\!] \text{ and } \forall(x,y) \in [\![1,q]\!]^2,\ h(x) = h(y) \implies h(x+y) \neq h(x)$$

Define $f : [\![1, b+pa]\!] \longrightarrow [\![1,n]\!]$ such that:

-if x $\leqslant b$ (we will note x $\in \mathcal{T}$) : $f(x) = g(x)$

-if $x \in [\![1, b+pa]\!]$ and $\pi(x) \notin A_n$ (we will note x $\in \mathcal{C}$) : $f(x) = g(\pi(x))$

-if $x \in [\![1, b+pa]\!]$ and $\pi(x) \in A_n$ (we will note x $\in \mathcal{R}$) : $f(x) = n-1+h(\lambda(x))$

f is well defined because π is defined for x> b and $\forall x \in [\![1, b+pa]\!], f(x) \leqslant n+k-1$ because h($\lambda(x)) \leqslant k$

We have parted the integers of $[\![1, b+pa]\!]$ in three disjoints subsets $\mathcal{T}, \mathcal{C}$ and $\mathcal{R}$.

We have to verify that f induces weakly-sum-free templates:

<u>if (x,y) $\in (\mathcal{T})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

x+y<a+b since b<a and g(x)=f(x)=f(y)=g(y).
Hence f(x+y)=g(x+y)$\neq$g(x)=f(x)

<u>if (x,y) $\in \mathcal{T} \times \mathcal{C}$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>
We distinguish two cases:

- If x+$\pi(y) \leqslant a+b$
  g(x)=f(x)=f(y)=g($\pi(y)$. Hence g(x)$\neq g(x+\pi(y)) = g(\pi(x+y))$ (qv previous proposition)
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(x) = f(x)$

- If x+$\pi(y) > a+b$, x+y>a+b and by definition of g, $g(\pi(x+y)) \neq g(x)$
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(\pi(x)) = f(x)$

<u>if (x,y) $\in \mathcal{T} \times \mathcal{R}$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

Then, f(x)=f(y)=n. We distinguish two cases:

- If $\lambda(y) = \lambda(x+y)$,
  $g(x) = g(\pi(y)) = n$ since $g(y) = g(\pi(y))$
  Therefore $g(\pi(x+y)) = g(x+\pi(y)) \neq g(x) = n$(qv previous proposition)
  Hence $f(x+y) = g(\pi(x+y)) \neq n$

- If $\lambda(y) \neq \lambda(x+y)$, $\lambda(y)+1 = \lambda(x+y)$
  n=f(y)=n-1+h($\lambda(y)$). Hence h($\lambda(y)$)=1.
  Moreover h(1)=1, therefore $h(\lambda(y)+1) \neq 1$
  if $\pi(x+y) \in A_n$, $f(x+y) = n-1+h(\lambda(x+y)) > n$

<u>if (x,y) $\in (\mathcal{C})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:</u>

Then $g(\pi(x)) = f(x) = f(y) = g(\pi(y))$. We distinguish two cases:

- If $\pi(x)+\pi(y) > q$, $g(\pi(\pi(x)+\pi(y)) \neq g(\pi(x))$ (qv previous proposition)
  Hence $g(\pi(x+y)) = g(\pi(\pi(x)+\pi(y)) \neq g(\pi(x))$
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(\pi(x)) = f(x)$

- If $\pi(x)+\pi(y) \leqslant q$, $g(\pi(\pi(x)+\pi(y)) \neq g(\pi(x))$ since g is sum-free for x> b
  if $g(\pi(x+y)) = n, f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) = g(\pi(\pi(x)+\pi(y)) \neq g(\pi(x)) = f(x)$

if (x,y) $\in \mathcal{C} \times \mathcal{R}$, $f(x) \neq f(y)$

if (x,y) $\in (\mathcal{R})^2$ such that f(x)=f(y), $x \neq y$, then $f(x+y) \neq f(x)$:

Let r(x)=$\pi(x) - a$ and r(y)=$\pi(y) - a$,
We proved that $x = a\lambda(x) + \pi(x) - a$, therefore $x = a\lambda(x) + \pi(x)$
$x + y = a(\lambda(x) + \lambda(y)) + r(x) + r(y)$. We distinguish three cases:

- If $r(x) + r(y) \in [\![b - a + 1, b]\!]$, $h(\lambda(x)) = f(x) + 1 - n = f(y) + 1 - n = h(\lambda(y))$ Hence, $h(\lambda(x) + \lambda(y)) \neq h(\lambda(x))$.

$$\begin{aligned}
\lambda(x+y) &= 1 + \left\lfloor \frac{a(\lambda(x) + \lambda(y)) + r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) + \left\lfloor \frac{r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) - 1 + 1 \text{ since } r(x) + r(y) \in [\![b - a + 1, b]\!] \\
&= \lambda(x) + \lambda(y)
\end{aligned}$$

$$\begin{aligned}
\text{if f(x+y)}\geqslant\text{n, } f(x+y) &= n - 1 + h(\lambda(x+y)) \\
&= n - 1 + h(\lambda(x) + \lambda(y)) \\
&\neq n - 1 + h(\lambda(x)) \\
&= f(x)
\end{aligned}$$

- If $r(x) + r(y) > b$, $\pi(x) + \pi(y) > 2a + b$
  Since g is a b-WSF template, $g(\pi(\pi(x) + \pi(y))) \neq n$
  Therefore, $g(\pi(x + y)) \neq n$ ie $x + y \in \mathcal{C}$
  Hence $f(x + y) < n \leqslant f(x)$

- If $r(x) + r(y) \leqslant b - a$, $\pi(x) + \pi(y) \leqslant b + a$
  Since g is sum-free for x> b, since $g(\pi(x)) = g(\pi(y)) = n$, $g(\pi(x + y)) \neq g(\pi(x)) = n$
  Hence, $f(x + y) < n \leqslant f(x)$

Corollary 5.2.2 *The general lower bound for weak Schur numbers in function of both regular and weak Schur numbers can be seen as a particular case of WSF-template in the same way Abott and Hanson's construction can be seen as a particular case of SF-template.*

Remark 5.10 *The WSF-templates can actually be fine-tuned further. However, it gives only minor improvements (most likely only additive constant) at the cost of dramatically increasing the size of the search space while. Therefore, it does not seem relevant to use this sophistications given that we could not even find good WSF-templates with 5 colors using a computer (here good means better than those obtain by combining smaller templates).*

These modifications work as follows. One may notice that the first row (excluding the "tail") has constraints that other rows do not have because of the tail, especially if the special color appears in *the tail as well. Thus allowing to have a colouring on the first row different from the colouring of the other rows would weaken the constraints. Acutally, one may even go further by noticing that on the one hand the first (ordered) color of the sum-free partition used for the extension procedure has more more constraints than the other colors of the sum-free partition since the first row is of this color and is more constrained than the other rows, but that on the other hand it has more degrees of freedom than the other colors of the sum-free partition since in the sum-free partition there cannot be two consecutive numbers of this color. As a result, it removes some constraints imposed by the first row on the other rows.*

To sum up, one can look for a generalised WSF-template that uses a special colouring for the tail and the first row, a colouring dedicated to the rows whose number is not 1 but is in the first color in the sum-free partition, a colouring for all the other rows and a special colouring for the last numbers (as previously explained for the improvement of the additive constant of WSF-templates).

We also have a similar theorem where only WS^+ is involved.

Theorem 5.3 *Let* $(n, k), (p, q) \in (\mathbb{N}^*)^2$*. If there exists a* SF*-template of* $k+1$ *colors and lenght* p*, and* WSF*-template of* n *color and lenght* q*, then there exists* WSF*-template of* $(n+k)$ *and lenght* pq*.*

And the associated inequality :

Corollary 5.3.1 *Let* $n, k \in \mathbb{N}^*$*, we have*

$$WS^+(n+k) \geqslant S^+(k+1)\, WS^+(n)$$

PROOF : The idea is the same as in the previous theorem. The only difference is the SF property inherited from the second SF-template.

5.4 New lower bounds for Weak Schur numbers

Having found suitable templates, which can be found in the appendix, with a SAT solver, we claim that for all $n \in \mathbb{N}^*$:

$$WS(n+1) \geqslant 4S(n) + 1$$
$$WS(n+2) \geqslant 13S(n) + 8$$
$$WS(n+3) \geqslant 42S(n) + 24$$
$$WS(n+4) \geqslant 132S(n) + 26$$

The first two inequalities were found by Rowley, they are detailed in [2]. The third inequality is optimal and was found with a SAT solver. It uses the first sophistication explained in the previous subsection in order to add the last number in the first color. As for the fourth inequality, it was obtained by combining an optimal SF-template of length 33 with a WSF-template of length 4. The best template we could get with a computer search gives the inequality $WS(n+4) \geqslant 127S(n) + 68$. It was also found with the SAT solver. In order to reduce the search space, we only looked for WSF-templates of 5 colors which start with a good $WS(4)$ partition. However, this approach most likely prevents us from finding the best WSF-templates as we explain in the next subsection for weakly sum-free partitions. We highly suspect that there exists more efficient WSF-templates with $n \geqslant 5$ colors. One may try to go over a different search space using a Monte-Carlo method, as in

[4]. This could be the suject of a future work. Further details about the encoding as a SAT problem can be found in the SAT section.

Like in 3.3, we compute the lower bounds given by the previous inequalities for $n \in [\![8, 15]\!]$. The best lower bound for each integer is highlighted.

n	8	9	10	11
$4S(n-1)+2$	6722	21146	71214	243794
$13S(n-2)+8$	6976	21848	68726	231447
$42S(n-3)+24$	6744	22536	70584	222036
$127S(n-4)+68$	5656	20388	68140	213428

n	12	13	14	15
$4S(n-1)+2$	815314	2554194	8045162	27061154
$13S(n-2)+8$	792332	2649772	8301132	26146778
$42S(n-3)+24$	747750	2559840	8560800	25886224
$127S(n-4)+68$	671390	2261049	7740464	25886224

With $S(9) \geqslant 17803$, we found a new lower bound for $WS(10)$ using Rowley's inequality. Moreover, the third inequality gives new lower bounds for $WS(9)$ and $WS(14)$.

5.5 Conclusion on WSF-templates

In this section, we first gave a new construction which can be seen as an equivalent for weakly sum-free partitions of Abott and Hanson's construction for sum-free partitions. We then introduced WSF-templates and generalized this construction. This allows us to find new lower bounds and new inequalities for weak Schur numbers. One may notice the significant difference between the former lower bounds for weak Schur numbers obtained by conducting a computer search and the new lower bounds obtained with WSF-templates (including Rowley's two inequalities). In the next section, we try to analyze this phenomenon.

6 Analysis of the former search space

In this section, we first provide evidence which indicates that the main assumption made by papers which found the previous best known lower bounds for weak Schur numbers using a computer may not be correct. This is done primarily by studying $WS(6)$. Then, in an effort to eliminate irrelevant search spaces, we obtain stronger results than those previously known for $WS(5)$ while gaining several orders of magnitude in computation time by giving additional information to the SAT solver without losing in generality. In this section, we assume that the subsets are ordered.

6.1 Former assumption appears to be wrong

Rowley's new lower bound for $WS(6)$ (642) [9] was a significant improvement upon the former best known lower bound (582) [5]. This previous lower bound has been found several times using a computer (often with Monte-Carlo methods) and by recursively making the assumption that a good partition for $WS(n+1)$ starts with a good partition for $WS(n)$ which is true for small values of n. Therefore, one may wonder whether the limiting factor are the assumptions or the methods used to search for partitions. It appears that the search space induced by these assumptions does not contain the optimal partitions.

Computational Theorem 6.1 *There is no weakly sum-free partition of* $[\![1, 583]\!]$ *in 6 parts such that:*

- $m_5 \geqslant 66$
- $m_6 \geqslant 186$
- $[\![210, 349]\!] \subset A_6$

This result was obtained in 8 hours with the SAT solver plingeling [3] on a 2.60 GHz Intel i7 processor PC. However, simply encoding the existence of such a partition as explained in the previous subsection would not result in a reasonable computation time. In order to help the SAT solver, we add additional information in the propositional formula. We did not quantify the speedup, but it most likely allowed us to gain several order of magnitude in computation time as we explain in the next subsection.

For every weakly sum-free colouring f of $[\![1, 65]\!]$ with 4 colors, the sequence $f(1), f(2), f(3), ...$ always starts with the following subsequence 1121222133. Then 11 is always either in subset 1 or 3, 12 is always in subset 3 and so on. For every integer in $[\![1, 65]\!]$, we computed in which subset it can appear. By using this constraints, we could then compute for every integer in $[\![1, 185]\!]$, in which subset it can appear in a weakly sum-free partition of $[\![1, 185]\!]$ which starts with a weakly sum-free partition of $[\![1, 65]\!]$ in 4 subsets. Adding these constraints to the formula corresponding to the above theorem gives additional information to the SAT solver without losing in generality.

The above theorem shows that the previous lower bound for $WS(6)$ is optimal in the search space considered by the papers which found it. Therefore, finding a partition of $[\![1, n]\!]$ in 6 weakly sum-free subsets for some $n \geqslant 590$ which does not have a template-like structure would be extremely interesting since it could give indications on a new search space for improving lower bounds with a computer. More generally, it questions the search space previously used for finding lower bounds for $WS(n)$ with a computer. In particular, to our knowledge every paper that found the lower bound $WS(5) \geqslant 196$ used this assumption. Therefore one may wonder if this actually a good lower bound. In the next subsection, we give properties that a partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets has to verify.

6.2 Weak Schur number five

As explained in the previous subsection, the search space used for showing that $WS(5) \geqslant 196$ may not contain optimal solution. In this subsection, we give necessary conditions for a hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets using the same type of methods as in the previous subsection.

Notation 6.1 *Let P be a predicate over weakly sum-free partitions. We denote by $WS(n|P)$ the greatest number p such that there exists a partition of $[\![1, p]\!]$ in n weakly sum-free subsets which verifies P.*

[6] verified with a SAT solver that there are no partition in 5 weakly sum-free subsets of $[\![1, 197]\!]$ with $A_5 = \{67, 68\} \cup [\![70, 134]\!] \cup \{136\}$ in 17 hours and could not provide a similar result when only assuming $m_5 = 67$ even after several weeks of runtime. By using the same method as above, we were able to verify that $WS(5|m_5 = 67) = 196$ in 0.5 seconds with the SAT solver glucose [2] on a 2.60 GHz Intel i7 processor PC (we used the non-parallel version here for the sake of comparison but in the rest of this subsection, we used the parallel version of glucose). The additional information we gave to the SAT solver is that every partition of $[\![1, 66]\!]$ in 4 weakly sum-free subsets starts with a partition of $[\![1, 23]\!]$ in 3 weakly sum-free subsets (this can be checked in a few dozens of minutes with a SAT solver). Among the 3 partitions of $[\![1, 23]\!]$ in 3 weakly sum-free subsets, every number always appears in the same subset except for 16 and 17 which can appear in two different subsets. We hardcoded this external knowledge in the propositional formula which allowed us to gain several orders of magnitude in computation time. Since $WS(4) = 66$, we have $m_5 \leqslant 67$. We give the stronger following result.

Computational Theorem 6.2 *If there exists a partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets then $m_5 \leqslant 59$.*

More precisely, we verified the following results (*max* m_5 is the greatest value of m_5 for which we have not verified that $WS(5|m_5) \leq 196$).

m_4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$WS(4\|m_4) + 1$	55	59	60	59	59	60	60	60	60	64	63	64	61	64	63	65	65	65	65	66	67
max m_5	49	51	54	53	54	54	55	55	55	55	55	56	57	57	59	59	59	59	59	58	53

For $m_4 = 24$, the value $m_5 \leqslant 53$ was obtained after 7 hours of runtime, and for $m_4 = 22$, the value $m_5 \leqslant 59$ was obtained after 11 hours of runtime. All the other values took between 1 minutes and 2 hours. To obtain these results, we once again provided additionnal information to the SAT solver. We added two different types information. The first one is the same as previously: we compute the subsets in which the first numbers can appear. The second type of information is the maximum length of a sequence using only a certain subset of the colors. For instance, if $m_4 \geqslant 22$, then there cannot be more than 17 consecutive numbers with color 1, 2 or 3.

6.3 Conclusion on the search space

As explained in the first subsection, the recursive assumption that a good partition for $WS(n + 1)$ starts with a good partition for $WS(n)$ appears to be wrong. Given that no extensive search for $WS(5)$ has been conducted without making this assumption and that the size of the considered partitions is reasonable (the current lower bound is 196 and $S(5) = 160$ was verified), it seems worth investigating this special case further. In a spirit of orientating this search, we then give necessary conditions on an hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets. Finding partitions that exceeds the lower bounds found with a computer (even if they do not exceed those obtained with templates) which are not as regular as those obtained with templates would be extremely interesing since it could designate a new search space for finding lower bounds with a computer.

7 Conclusions and future work

These new results come from an extension of Rowley's template-based approach for Ramsey graphs and Schur numbers which is relatively new. Therefore, we would not be surprised if lower bounds are later improved using better templates. Moreover, studying specifically $S^+(n)$ and $WS^+(n)$ might be of interest as they are closely related to Schur and weak Schur numbers.

The fourth section gives new insight on the method that was formerly used to achieve new lower bounds for weak Schur numbers. The assumption behind it might have removed the optimal partitions from the search space and thus lowered the highest value that can be reached within it. However, algorithms based on randomness such as Monte-Carlo algorithms may prove to be very useful if used in a search space with more potential. For instance, such an algorithm could be used to find better templates and improve inequalities and lower bounds for high values of n. This could be the subject of a future work.

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A SF-templates

SF-template partitionning $[\![1, 33]\!]$ into 4 subsets

1	1, 6, 9, 13, 16, 20, 24, 27, 31
2	2, 5, 14, 15, 25, 26
3	3, 4, 10, 11, 12, 28, 29, 30
4	7, 8, 17, 18, 19, 21, 22, 23, 32, 33

SF-template partitionning $[\![1, 111]\!]$ into 5 subsets

1	1, 5, 18, 12, 14 ,21, 23, 30, 32, 36, 39, 43, 45, 52, 103 106, 110
2	2, 6, 7, 10, 15, 18, 26, 29, 34, 37, 38, 42, 46, 51, 54 101, 104, 109
3	3, 4, 9, 11, 17, 19, 25, 27, 33, 35, 40, 41, 47, 48, 55 100, 107, 108
4	13, 16, 20, 22, 24, 28, 31, 58, 61, 67, 88, 94, 97
5	44, 50, 53, 56, 57, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85 86, 87, 89, 90, 91, 92, 93, 95, 96, 98, 99, 102, 105, 111

SF-template partitionning $[\![1, 380]\!]$ into 6 subsets

1	1, 5, 8, 11, 15, 17, 29, 33, 36, 39, 43, 57, 61, 88, 92 106, 110, 113, 116, 120, 132, 134, 138, 141, 144, 148, 150, 154, 157, 160 164, 178, 182, 185, 188, 341, 344, 347, 351, 365, 369, 372, 375, 379
2	2, 9, 13, 16, 20, 23, 24, 27, 28, 31, 34, 35, 38, 42, 45 49, 53, 60, 67, 71, 78, 82, 89, 96, 100, 104, 107, 111, 114, 115 118, 121, 122, 125, 126, 129, 133, 136, 140, 147, 158, 162, 165, 169, 172 176, 183, 187, 194, 201, 328, 335, 342, 346, 353, 357, 360, 364, 367, 371
3	3, 4, 12, 14, 19, 25, 30, 32, 40, 41, 47, 48, 58, 91, 101 102, 108, 109, 117, 119, 124, 130, 135, 137, 145, 146, 152, 153, 161, 163 168, 179, 181, 190, 339, 348, 350, 361, 366, 368, 376, 377
4	6, 7, 10, 18, 21, 22, 26, 37, 46, 50, 51, 54, 65, 70, 79 84, 95, 98, 99, 103, 112, 123, 127, 128, 131, 139, 142, 143, 151, 155 156, 159, 167, 170, 171, 175, 186, 343, 354, 358, 359, 362, 370, 373, 374 378
5	44, 52, 55, 56, 59, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75 76, 77, 80, 81, 83, 85, 86, 87, 90, 93, 94, 97, 105, 189, 196 197, 200, 203, 206, 207, 209, 214, 219, 231, 298, 310, 315, 320, 322, 323 326, 329, 332, 333, 340
6	149, 166, 173, 174, 177, 180, 184, 191, 192, 193, 195, 198, 199, 202, 204 205, 208, 210, 211, 212, 213, 215, 216, 217, 218, 220, 221, 222, 223, 224 225, 226, 227, 228, 229, 230, 232, 233, 234, 235, 236, 237, 238, 239, 240 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 299, 300, 301 302, 303, 304, 305, 306, 307, 308, 309, 311, 312, 313, 314, 316, 317, 318 319, 321, 324, 325, 327, 330, 331, 334, 336, 337, 338, 345, 349, 352, 355 356, 363, 380

B WSF-templates

	WSF-template partitionning $\llbracket 1, 42 \rrbracket$ into 4 subsets
1	1, 2, 4, 8, 11, 22, 25, $(\mathbf{N+1})$
2	5, 6, 7, 19, 21, 23, 36
3	9, 10, 12, 13, 14, 15, 16, 17, 18, 20
4	24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40 41, 42

This template provides the inequality $WS(n+3) \geqslant 42S(n)+24$ by placing one last number, here represented by $(\mathbf{N+1})$, in the first subset.

C Reformulation as a SAT problem

In this section, we reframe the question of the existence of (weakly) sum-free partitions as a boolean satisfiability (SAT) problem. We encode the existence of (weakly) sum-free partitions as propositional formulae like in [7] and then use SAT solvers to determine whether these formulae are satisfiable.

Definition C.1 *A literal is either a variable v (a positive literal) or the negation $\bar{v}$ of a variable v (a negative literal) where v takes a truth value: true or $false$. A clause is a disjunction of literals and a formula is a conjunction of clauses: it is a propositional formula in conjonctive normal form (CNF).*

Definition C.2 *An assignment is a function from a set of variables to the truth values true (1) and $false$ (0). A literal l is satisfied (falsified) by an assignment α if l is positive and $\alpha(var(l)) = 1$ (resp. $\alpha(var(l)) = 0$) or if it is negative and $\alpha(var(l)) = 0$ (resp. $\alpha(var(l)) = 1$). A clause is satisfied by an assignment α if it contains a literal that is satisfied by α. Finally, a formula is satisfied by an assignment α if all its clauses are satisfied by α. A formula is satisfiable if there exists an assignment that satisfies it; otherwise it is unsatisfiable.*

We then encode the existence of a partition of $\llbracket 1, p \rrbracket$ in k weakly sum-free subsets as follows: for every integer $i \in \llbracket 1, p \rrbracket$, take k variables $x_1^{(i)}, ..., x_k^{(i)}$ and for every $\forall c \in \llbracket 1, k \rrbracket, x_c^{(i)} = 1 \iff i \in A_c$. The corresponding clauses are:

- **weakly sum-free:** $\forall c \in \llbracket 1, k \rrbracket, \forall (i, j) \in \llbracket 1, p \rrbracket^2, (i \neq j \text{ and } i+j \leq n) \implies \neg x_c^{(i)} \vee \neg x_c^{(i)} \vee \neg x_c^{(i+j)}$
- **union:** $\forall i \in \llbracket 1, p \rrbracket, x_1^{(i)} \vee ... \vee x_k^{(i)}$
- **disjoint:** $\forall i \in \llbracket 1, p \rrbracket, \forall (c_1, c_2) \in \llbracket 1, k \rrbracket^2, c_1 \neq c_2 \implies \neg x_{c_1}^{(i)} \vee \neg x_{c_2}^{(i)}$

In the above formula, every color plays a symmetric role. Hence the search space can reduced by $k!$ by ordering the subsets, that is by enforcing that $m_1 < ... < m_k$. The corresponding clauses are: **symmetry breaking:** $x_1^{(1)} = 1$ and $\forall c \in \llbracket 2, k-1 \rrbracket, \forall i \in \llbracket 1, WS(c-1)+1 \rrbracket, x_c^{(1)} \vee ... \vee x_c^{(i)} \vee \neg x_{c+1}^{(i+1)}$

Remark C.1 *For a given problem, it can be interesting to try out different SAT solvers because the relative performance can vary significantly according to the problem. For instance, we used two different SAT solvers in the next two next subsections.*

Remark C.2 *Using a parallel SAT solver usually reduces the computation time, especially when trying to show that a formula is unsatisfiable. However, most of the parallel SAT solver do not have a deterministic behaviour and it can results in a strong variation of running times.*

D Proof of theorem j'arrive pas a afficher le numero du theoreme avec un lien

Proof : Let $(p, q), (n, k) \in (\mathbb{N}^*)^2$, $N = p(q + \lceil \frac{q}{2} \rceil + 1) + q$, $\alpha = \lceil \frac{q}{2} \rceil > 0$ and $\beta = q + \alpha + 1$. We denote by f the colouring associated to the partition of $[\![1, q]\!]$ and g the one associated to the partition of $[\![1, p]\!]$.

$$f : [\![1, q]\!] \longrightarrow [\![1, n]\!] \text{ and } \forall (x, y) \in [\![1, q]\!]^2, \begin{cases} x \neq y \\ f(x) = f(y) \end{cases} \Longrightarrow f(x + y) \neq f(x)$$

$$g : [\![1, p]\!] \longrightarrow [\![1, k]\!] \text{ and } \forall (x, y) \in [\![1, q]\!]^2,\ f(x) = f(y) \Longrightarrow f(x + y) \neq f(x)$$

Let us start by parting the integers of $[\![1, N]\!]$ in two subsets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} = [\![1, \alpha]\!] \cup \{a\beta + u \mid (a, u) \in [\![0, p]\!] \times [\![\alpha + 1, q]\!]\}$ and $\mathcal{B} = \{a\beta + u \mid (a, u) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]\}$.

First, $\underline{\mathcal{A} \cap \mathcal{B} = \varnothing}$:
By contradiction, suppose there exists $x \in \mathcal{A} \cap \mathcal{B} \neq \varnothing$. Then there are $(a, u) \in [\![0, p]\!] \times [\![\alpha + 1, q]\!]$ and $(b, v) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]$ such that $x = a\beta + u = b\beta + v$. By definition of α and β we have $u \in [\![\alpha + 1, q]\!] \subset [\![0, \beta - 1]\!]$. From there, we distinguish two cases :

- If $v \in [\![0, \alpha]\!]$ then $v \in [\![0, \beta - 1]\!]$ and $v \neq u$ because $v < \alpha + 1 \leqslant u$
- If $v \in [\![-\alpha, -1]\!]$, we note $\tilde{v} = \beta + v$ and thus have $x = (b - 1)\beta + \tilde{v}$ with $\tilde{v} \in [\![\beta - \alpha, \beta - 1]\!] \subset [\![0, \beta - 1]\!]$ and $\tilde{v} \neq u$ because $u < q + 1 = \beta - \alpha \leqslant \tilde{v}$.

In either cases, we run into a contradiction because of the remainder's uniqueness in the euclidean division of x byβ.

Then, we have $\underline{\mathcal{A} \cup \mathcal{B} = [\![1, N]\!]}$:

- On the one hand : $1 = \min(\mathcal{A}) \leqslant \max(\mathcal{A}) = p\beta + q = N$ and $1 \leqslant \beta - \alpha = \min(\mathcal{B}) \leqslant \max(\mathcal{B}) = p\beta + \alpha \leqslant N$, which gives $\mathcal{A} \cup \mathcal{B} \subset [\![1, N]\!]$.
- On the other hand, let $x \in [\![1, N]\!]$. If $x \leqslant \alpha$, we directly have $x \in \mathcal{A}$, let us then suppose that $x > \alpha$ and write $x = a\beta + u$ the euclidean division of x by β. We have $x \leqslant N$, thus $a \leqslant p$. We distinguish three cases :
  - If $u \in [\![0, \alpha]\!]$ then we necessarily have $a \geqslant 1$ because $x > \alpha$, and so $x \in \mathcal{B}$.
  - If $u \in [\![\alpha + 1, q]\!]$, then $x \in \mathcal{A}$.
  - If $u \in [\![q + 1, \beta - 1]\!]$ then $x = (a + 1)\beta - (\beta - u)$ with $-\alpha \leqslant \beta - u \leqslant 0$. Furthermore, $a \leqslant p - 1$, else we would have $x > N$, and so $x \in \mathcal{B}$
  In any case, $x \in \mathcal{A} \cup \mathcal{B}$ and we can thus conclude that $[\![1, N]\!] \subset \mathcal{A} \cup \mathcal{B}$.

This first partition of $[\![1, N]\!]$ will help us to define our final partition by the projection of its equivalence relation. We thereby define $h : [\![1, N]\!] \longrightarrow [\![1, n + k]\!]$ as such :
- If $x \in \mathcal{A}$ then $h(x) = f(x \bmod \beta)$ (well defined because $x \bmod \beta \in [\![1, N]\!]$)
- If $x \in \mathcal{B}$ then $x = a\beta + u$ with a unique $(a, u) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]$and we define $h(x) = n + g(a)$
The fact that $(\mathcal{A}, \mathcal{B})$ is a partition of $[\![1, N]\!]$ ensures that this definition of h is valid. We then have to verify that h induces weakly sum-free subsets.

<u>The classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free :</u>

Let $(x, y) \in \mathcal{A}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$

- If $(x, y) \in [\![1, \alpha]\!]^2$:
  We have $x + y \leqslant 2\alpha \leqslant q$ and $x + y = 0\beta + x + y$, therefore $x + y \in \mathcal{A}$. Then, by definition : $h(x) = f(x)$, $h(y) = f(y)$ and $h(x + y) = f(x + y)$, which gives us, thanks to the property verified by f, that $h(x + y) \neq h(x)$.

- If $(x, y) \in [\![1, \alpha]\!] \times (\mathcal{A} \setminus [\![1, \alpha]\!])$:
  We write $y = a\beta+u$ with $(a, u) \in [\![0, p]\!]\times[\![\alpha+1, q]\!]$. Then $x+y = a\beta+x+u = (a+1)\beta+x+u-\beta$, and if $x+u > q$ it follows that $a \leqslant p-1$ since $x+y \leqslant N$, and $-\alpha \leqslant x+u-\beta \leqslant -1$. Therefore $x + y \in \mathcal{B}$ and $h(x + y) \neq h(x) = f(x)$ by definition of h. On the contrary, if $x - u \leqslant n$, then $x + y \in \mathcal{A}$ and $h(x + y) = f(x + u)$ because $x + u$ is actually the remainder of the euclidean divisionof $x+y$ by β. Moreover, $h(x) = f(x)$, $x < u$ and, with our initial hypothesis, $h(x) = h(y) = f(u)$. The property verified by f gives us $f(x + u) \neq f(x)$ which can be rewritten as $h(x + y) \neq h(x)$.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!]) \times [\![1, \alpha]\!]$:
  This case is handled exactly like the previous one by swaping the roles of x and y.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!])^2$:
  We write $x = a\beta + u$ and $y = b\beta + v$ with (a, u) and (b, v) in $[\![0, p]\!] \times [\![\alpha + 1, q]\!]$. Then $x + y = (a + b)\beta + u + v = (a + b + 1)\beta + u + v - \beta$ with $a + b \leqslant p - 1$ (else we would have $x + y > N$ because $u + v > q$) and $-\alpha \leqslant u + v - \beta \leqslant \alpha$, therefore $x + y \in \mathcal{B}$ and by definition $h(x + y) \neq h(x)$.

In any case, $h(x + y) \neq h(x)$ and the classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free.

<u>The classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free :</u>

Let $(x, y) \in \mathcal{B}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$.
We write $x = a\beta+u$ and $y = b\beta+v$ with (a, u) and (b, v) in $[\![1, p]\!]\times[\![-\alpha, \alpha]\!]$. We have $h(x) = q+g(a)$ and $h(y) = q + g(b)$, therefore $g(a) = g(b)$. We also have $x + y = (a + b)\beta + u + v$.
If $u+v \in [\![-\alpha, \alpha]\!]$, then $x+y \in \mathcal{B}$ and $h(x+y) = g(a+b)$, hence we can deduce that $h(x+y) \neq h(x)$ because of the property verified by g. On the contrary, if $u+v \notin [\![-\alpha, \alpha]\!]$, then necessarily $x+y \in \mathcal{A}$. Suppose $x+y \in \mathcal{B}$, then $x+y = c\beta+w$ with $(c, w) \in [\![1, p]\!]\times[\![-\alpha, \alpha]\!]$. Thus, $c\beta+w = (a+b)\beta+u+v$ and $(a + b - c)\beta = w - u - v$. Furthermore $a + b - c \neq 0$, else we would have $u + v = w \in [\![-\alpha, \alpha]\!]$. This finally leads to the following inequality :

$$\beta \leqslant |a + b - c|\beta = |w - u - v| \leqslant |w| + |u| + |v| \leqslant 3\alpha \leqslant q + \alpha < \beta$$

which is absurd. We can therefore conclude that $x + y \in \mathcal{A}$ and by definition of h, $h(x + y) \neq h(x)$, proving that the classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free.

Finally, we have showed that every classe of equivalence induced by h is weakly sum-free, which ends the proof.