New lower bounds for Schur and weak Schur numbers

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Abstract

This article provides new lower bounds for both Schur and weak Schur numbers. These results were obtained by continuing on Rowley's "template"-based approach for Schur and Ramsey numbers. Finding templates allows us to apply Rowley's construction and to get explicit partitions improving lower bounds as well as the growth rate for both Schur numbers and Ramsey numbers $R_n(3)$. We also developed a new method to improve lower bounds on weak Schur numbers. Furthermore, this paper tries to analyze former works on the subject based on the principle that good partitions into $n+1$ subsets start with good partitions into n subsets. We show that exceeding the previous lower bound $WS(6) \geqslant 582$ is impossible with such an assumption upon imposing certain conditions on the good 5-subsets partition. The new lower bounds include $S(9) \geqslant 17\,803$, $S(10) \geqslant 60\,948$, $WS(9) \geqslant 22\,536$ and $WS(10) \geqslant 71\,214$.

1 Introduction

We are interested in partioning the set of integers $\{1, ..., p\}$ in n subsets such that there is no subset containing three integers x, y and z verifying $x + y = z$. We say these subsets are *sum-free*. If we add the hypothesis $x \neq y$, we say the subsets are *weakly sum-free*. The greatest p for which there exists a partition into n sum-free subsets is called the n^{th} Schur number and is denoted $S(n)$ [1]. Likewise for weakly sum-free partitions we define $WS(n)$ the n^{th} weak Schur number [2]. Only the first values of these sequences are known. This article focuses on the lower bounds for these numbers.

1.1 State of the art

Before Rowley's "template"-based approach for Schur and Ramsey numbers [3], the previous generic construction for Schur numbers was given by Abbott and Hanson [4] in 1972 with a recursive construction. It used to give the best lower bounds for all sufficiently large numbers. No equivalent was known for weak Schur numbers and as a result the best known partitions for large weak Schur numbers did not utilize the weakly sum-free hypothesis.

As for smaller numbers, the best lower bounds were obtained by conducting a computer search. Eliahou [5], Rafilipojaona [6] and Bouzy [7] improved lower bounds with Monte-Carlo methods. It has been the main approch in the previous decade. This search for weakly sum-free partitions relied on the recursive assumption that a good weakly sum-free partition into $n+1$ colors starts with a good weakly sum-free partition into n colors. This assumption was necessary in order for the Monte-Carlo approch to yield results. However, we show in the last section how this search space may not countain the optimal solution. Thus further work using similar methods might need to put that assumption aside and find a different search space.

In 2020, Rowley introduced the notion of template for Schur and Ramsey numbers which generalizes Abbott and Hanson's construction and gives new lower bounds (and inequalties) for Schur numbers. Rowley also gives two inequalities for weak Schur numbers [8] that yield significant improvements over previous lower bounds which besides do utilize the *weakly* sum-free hypothesis.

Table 1 - Comparison of lower bounds for Schur numbers

n	1	2	3	4	5	6	7	8	9	10	11	12
Before Rowley	1*	4*	13*	44*	160* [9]	536 [10]	1 680 [10]	5 041 [5]	15 124 [5]	51 120 [4]	172 216 [4]	575 664 [4]
Rowley [3]								5 286	17 694	60 320	201 696	631 840
Our results									**17 803**	**60 948**	**203 828**	**638 548**

Table 2 - Comparison of lower bounds for weak Schur numbers

n	1	2	3	4	5	6	7	8	9	10	11	12
before Rowley	2*	8*	23*	66*	196 [5]	582 [11]	1 740 [6]	5 201 [6]	15 596 [6]	51 520 [4]	172 216 [4]	575 664 [4]
Rowley [8]						642	2 146	6 976	21 848	70 778	241 282	806 786
our results									**22 536**	**71 214**	**243 794**	**815 314**

* denotes an exact value, not just a lower bound

1.2 Structure of this article

The main contribution of this article is a generalization of the concept of template to weak Schur numbers. This gives new lower bounds (and inequalities) for weak Schur numbers. This construction also includes as a special case an analogous for weak Schur numbers of Abbott and Hanson's construction for Schur numbers.

We first explain Rowley's template-based construction in the context of Schur numbers and then give new templates, thus providing new lowers bounds and inequalities as well as showing that the growth rates for both Schur and Ramsey numbers $R_n(3)$ exceed 3.28.

We then generalize the concept of templates to weak Schur numbers and provide new lower bounds for weak Schur numbers.

Finally, we analyze the significant difference between new lower bounds obtained with templates and the former lower bounds obtained by computer search and we provide evidence which indicate that the main assumption made in those articles removes the optimal partitions from the search space.

We now introduce notations and definitions we use throughout this article.

1.3 Definitions and notations

We start by defining sum-free and weakly sum-free subsets to introduce regular and weak Schur numbers.

Definition 1.1. *A subset A of $\mathbb{N}$ is said to be sum-free when:*

$$\forall (a,b) \in A^2,\ a+b \notin A$$

Definition 1.2. *A subset B of $\mathbb{N}$ is said to be weakly sum-free when:*

$$\forall (a,b) \in B^2,\ a \neq b \Longrightarrow a+b \notin B$$

Let us notice that a sum-free subset is also weakly sum-free, hence justifying the name of *weakly* sum-free subsets. Given p and n two integers, we are interested in partitioning the set of integers $\{1, 2, ..., p\}$, denoted by $[\![1, p]\!]$, into n (weakly) sum-free subsets.

Schur proved in [1] that given a number of subsets n, there exists a value of p such that there exists no partition of $[\![1, q]\!]$ into n sum-free subsets for any $q \geqslant p$. A similar property holds for weakly sum-free subsets (reference necessaire). These observations lead to the following definitions.

Definition 1.3. *Let $n \in \mathbb{N}^*$. There exists a greatest integer that we denote $S(n)$ (resp. $WS(n)$) such that $[\![1, S(n)]\!]$ (resp. $[\![1, WS(n)]\!]$) can be partitioned into n sum-free subsets (resp. weakly sum-free subsets). $S(n)$ is called the n^{th} Schur number and $WS(n)$ the n^{th} weak Schur number.*

Given a partition of $[\![1, p]\!]$ in n subsets, we generally denote these subsets $A_1, ..., A_n$. We also denote $m_i = \min(A_i)$. By ordering the subsets, we mean assuming that $m_1 < ... < m_n$. However, if not specified we do not make this hypothesis since we do not always consider partitions in which every subset plays a symmetric role.

Definition 1.4. *We sometimes refer to a partitition as a coloring. The coloring associated to a partition $A_1, ..., A_n$ of $[\![1, p]\!]$ is the function f such that $\forall x \in [\![1, p]\!], x \in A_{f(x)}$. Likewise, the partition associated to a coloring f of $[\![1, p]\!]$ with n colors is $\forall c \in [\![1, n]\!], A_c = f^{-1}(c)$.*

2 Templates for Schur numbers

In this section, we use Rowley's template-based constructions [3] in the context of Schur numbers. In order to improve lower bounds for Schur and Ramsey numbers, Rowley introduces special sum-free partitions verifying some additional properties which can be extended using a method generalizing Abbott and Hanson's construction [4]. Rowley named these partitions "templates", and we keep this name in the entire article. We then find new templates and use them to provide new lower bounds for Schur numbers.

2.1 Definition of S^+

Definition 2.1. *A SF-template with width p and n colors is defined as a partition of $[\![1,p]\!]$ into n sum-free subsets $A_1, A_2, ..., A_n$ verifying:*

$$\forall i \in [\![1, n-1]\!], \forall (x,y) \in A_i^2, x+y > p \Longrightarrow x+y-p \notin A_i$$

Here n is the "special" color: it has less constraints than the other colors. However, please note that n is not necessarily the last color by order of appearance. SF-templates include Abbott and Hanson's construction [4] as a special case.

Proposition 2.1. *Let $n \in [\![2, +\infty[\![$. We define $S^+(n)$ as the maximal width of a SF-template with n colors. $S^+(n)$ is well defined and verifies:*

$$2S(n-1)+1 \leqslant S^+(n) \leqslant S(n)$$

Proof. The lower bound comes from Abbott and Hanson's construction. The upper bound comes from the fact that a SF-template with width p and n colors is also a partition of $[\![1,p]\!]$ into n sum-free subsets. □

Remark 2.1. *S^+ and S have the same asymptotic growth rate.*

2.2 Construction of Schur partitions using SF-templates

Here we state the main result on SF-templates stated by Rowley in the context of Ramsey numbers. It consists in the explicit construction of a sum-free partition using a SF-template and a sum-free partition.

Theorem 2.1. *Let $(n,k),(p,q) \in (\mathbb{N}^*)^2$. If there exists a SF-template with width p and $k+1$ colors and a partition of $[\![1,q]\!]$ into n sum-free subsets then there exists a partition of $[\![1, pq+m_{k+1}-1]\!]$ into $n+k$ sum-free subsets. m_{k+1} is the minimum of the special subset in the SF-template.*

Setting $p = S^+(k+1)$ and $q = S(n)$ yields the following corollary.

Corollary 2.1.1. *Let $n,k \in \mathbb{N}^*$. Then*

$$S(n+k) \geqslant S^+(k+1)S(n) + m_{k+1} - 1$$

The idea lying beneath this theorem is similar to Abbott and Hanson's contruction [4]. They vertically extend a sum-free partition by repeating it and they use an other sum-free partition to color the other half according to the line number. This way the "blocks" act as safe areas for each other. We give here an example for $p=9$, $q=4$, $n=2$ and $k=2$ showing that $S(2+2) \geqslant S(2)(2S(2)+1)+S(2)$, both with Abbott and Hanson's construction and with a SF-template which is not of this type. In both cases, the special color is grey.

Abbott and Hanson's construction

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

Corresponding SF-template

1	2	3	4	5	6	7	8	9

Corresponding sum-free partition

1	2	3	4

In the general construction with SF-templates, the special color no longer necessarily contains consecutive numbers. However, the special color is still replaced by the colors of the sum-free partition according to the line number and the other colors are still vertically extended.

SF-template construction

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40					

Corresponding SF-template

1	2	3	4	5	6	7	8	9

Corresponding sum-free partition

1	2	3	4

We now proceed to prove the above mentioned theorem.

Proof. Denote by f the coloring associated to the SF-template with width p and g the one associated to the sum-free partition of $[\![1,q]\!]$; where $f : [\![1,p]\!] \longrightarrow [\![1,k+1]\!]$ and $g : [\![1,q]\!] \longrightarrow [\![1,n]\!]$.

The sum-free condition is expressed as

$$\forall (x,y) \in [\![1,p]\!]^2, f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

$$\forall (x,y) \in [\![1,q]\!]^2, \, g(x) = g(y) \Longrightarrow g(x+y) \neq g(x)$$

The additionnal constraint for the SF-template is

$$\forall (x,y) \in [\![1,p]\!]^2, \begin{cases} f(x) = f(y) \leqslant k \\ x+y > p \end{cases} \Longrightarrow f(x+y-p) \neq f(x)$$

For $x \in [\![1, pq + m_{k+1} - 1]\!]$, write $x = (\alpha - 1) + u$ for some integers $\alpha \in \mathbb{Z}$ and $u \in [\![1, p]\!]$. This decomposition is of course unique. α (resp. u) can be interpreted as the row (resp. column) number of x. Define a new coloring h as follows

$$\begin{array}{rccl} h: & [\![1, pq + m_{k+1} - 1]\!] & \longrightarrow & [\![1, n + k]\!] \\ & x & \longmapsto & \begin{cases} f(u) & \text{if } f(u) \leqslant k \\ k + g(\alpha) & \text{if } f(u) = k + 1 \end{cases} \end{array}$$

h is well-defined since, by definition of m_{k+1}, $\forall x \in [\![pq+1, pq+m_{k+1}-1]\!], f(u) \leqslant k$ and therefore $\forall x \in [\![1, pq + m_{k+1} - 1]\!], f(u) = k + 1 \implies \alpha \in [\![1, q]\!]$.

We now prove that h is a sum-free coloring. Let $x, y \in [\![1, pq + m_{k+1} - 1]\!]$ such that $h(x) = h(y)$ and $x+y \leqslant pq+m_{k+1}-1$. We claim that $h(x+y) \neq h(x)$. Write $x = (\alpha-1)p+u$ and $y = (\beta-1)p+v$ where $\alpha, \beta \in \mathbb{Z}$ and $u, v \in [\![1, p]\!]$. Distinguish between two cases according to the value of $h(x)$.

First case: $h(x) \leqslant k$

Assume that $h(x + y) \leqslant k$, otherwise $h(x + y) \neq h(x)$ obviously holds. By definition of h and given that $h(u) = h(v)$, $f(u) = f(v)$. Distinguish between two cases according to the value of $x + y$.

- If $u + v > p$, write $w = u + v - p \in [\![1, p]\!]$. Then $x + y = (\alpha + \beta - 1)p + w$. By definition, $h(x + y) = f(w)$. Given that $f(u) = f(v) \leqslant k$, the additionnal constraint on f implies $f(w) \neq f(u)$, that is $h(x + y) \neq h(x)$.
- If $u + v \leqslant p$, write $w = u + v \in [\![1, p]\!]$. Then $x + y = (\alpha + \beta - 2)p + w$. By definition, $h(x+y) = f(w)$. Given that $f(u) = f(v) \leqslant k$, the sum-free property of f implies $f(w) \neq f(u)$, that is $h(x + y) \neq h(x)$.

Second case: $h(x) \geqslant k + 1$

$h(x) = k + g(\alpha) = k + g(\beta) = h(y)$, hence $g(\alpha) = g(\beta)$. As in the first case, distinguish between two cases according to the value of $x + y$.

- If $u+v > p$, write $w = u+v-p \in [\![1, p]\!]$. Then $x+y = (\alpha+\beta-1)p+w$. Assume that $h(x+y) \geqslant k+1$, otherwise $h(x+y) \neq h(x)$ obviously holds. By definition, $h(x+y) = k+g(\alpha+\beta)$. Given that $g(\alpha) = g(\beta)$, the sum-free property of g implies $g(\alpha + \beta) \neq g(\alpha)$ that is $h(x + y) \neq h(x)$.
- If $u + v \leqslant p$, write $w = u + v \in [\![1, p]\!]$. Then $x + y = (\alpha + \beta - 2)p + w$. The sum-free property of f implies $f(w) \neq f(u)$. Therefore $f(w) \leqslant k$ and thus $h(x + y) \leqslant k$. In particular, given that $h(x) \geqslant k + 1$, $h(x + y) \neq h(x)$.

□

The following proposition can help improve the additive constant of a SF-template. Although it does not allow us to improve the SF-templates we have found, the analogous of this proposition for WSF-templates (see next section) allows us to improve one of them.

Proposition 2.2. *Let $(k, p) \in \mathbb{N}^*)^2$ and let f be a coloring associated to a SF-template with width p and $k + 1$ colors. Let $b \in \mathbb{N}$ ($b = m_{k+1} - 1$ works) and assume there there exists a coloring g of $[\![1, b]\!]$ with $k + 1$ colors such that:*

- $\forall (x, y) \in [\![1, p]\!]^2, (f(x) = f(y)$ *and* $(x + y) \mod p \leqslant b) \implies g((x + y) \mod p) \neq f(x)$
- $\forall (x, y) \in [\![1, p]\!] \times [\![1, b]\!], (f(x) = g(y)$ *and* $x + y \leqslant b) \implies g(x + y) \neq f(x)$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $i \longmapsto g(i - pS(n))$, we have

$$S(n+k) \geqslant S^+(k+1)S(n) + b$$

This proposition corresponds to the fact that sometimes a column is not the sum of two columns of a given color, but adding this column to the color would create sums in the color when applying the extension procedure. However, the last line does not intereact will all the columns when it comes to creating new sums. As a result, the hypotheses made on the coloring of the last row can be weakened.

We also have a similar construction theorem where only S^+ is involved.

Theorem 2.2. *Let $(n, k), (p, q) \in (\mathbb{N}^*)^2$. If there exists a SF-template with and p and $k+1$ colors, and a SF-template with width q and n colors, then there exists SF-template with width pq and $(n+k)$.*

And the associated inequality :

Corollary 2.2.1. *Let $n, k \in \mathbb{N}^*$, we have*

$$S^+(n+k) \geqslant S^+(k+1)S^+(n)$$

Proof. The idea is the same as in the previous theorem. The only difference is the SF property inherited from the second SF-template. □

2.3 Inequalities and new lower bounds for Schur numbers

Definition 2.2. *A SF-template with n colors is said to be symmetric if the partition in n sum-free subsets derived (with the additive constant) from this template is symmetric. A sum-free partition $A_1, ..., A_n$ of $[\![1, p]\!]$ is said to be symmetric if for all $x \in [\![1, p]\!]$, x and $p+1-x$ belong to the same subset (except if $x = p+1-x$).*

Using a SAT solver, we exhibited SF-templates, hence providing lower bound on S^+ and inequalities of the type $S(n+k) \geqslant aS(n) + b$. We have sought templates providing the greatest value of (a, b) (for the lexicographic order). When the number of colors excceeded 5, in order to reduce the search space we looked for symmetric SF-templates, we assumed that the special color was the last color to appear and we constrained the m_c's out of being too small. Further details about the encoding as a SAT problem can be found in [9].

Here are the best inequalities on Schur numbers so far (the templates corresponding to the third, fourth and fifth inequalities can be found in the appendix):

$$\begin{array}{rcrcr} S(n+1) & \geqslant & 3S(n) & + & 1 \\ S(n+2) & \geqslant & 9S(n) & + & 4 \\ S(n+3) & \geqslant & 33S(n) & + & 6 \\ S(n+4) & \geqslant & 111S(n) & + & 43 \\ S(n+5) & \geqslant & 380S(n) & + & 148 \\ S(n+6) & \geqslant & 1\,140S(n) & + & 528 \end{array}$$

The first inequality comes from Schur's original article[1]. The second one is due to Abott [4] and the third one to Rowley [3]. The other ones are new.

The first 3 inequalities are optimal. The fourth one is optimal among symmetric SF-templates whose special color is the last in the order of apparition (and with a multiplicative factor less than or equal to 118). The fifth one is most likely not optimal but should not be too far from the optimal. Finally, the sixth one is obtained by combining (see below) the SF-template with width 380 and the one with width 3. Although we could not find a better SF-template with 7 colors, the last inequality is definetely very far from the optimal value. One may try to seek better templates by constraining less the search space and by using Monte-Carlo methods, as in [7]. This could be the suject of a future work.

The previous inequalities give new lower bounds for $S(n)$ for $n \geqslant 9$. We compute the lower bounds for $n \in [\![8, 15]\!]$ using the four different inequalities, please notice that the best values for $n = 8$ and $n = 13$ were obtained thanks to the first one, found by Rowley. The best lower bounds are highlighted.

n	8	9	10	11
$33\,S(n-3)+6$	5 286	17 694	55 446	174 444
$111\,S(n-4)+43$	4927	17 803	59 539	186 523
$380\,S(n-5)+148$	5 088	16 868	60 948	203 828
$1\,140\,S(n-6)+528$	5 088	15 348	50 688	182 928

n	12	13	14	15
$33\,S(n-3)+6$	587 505	2 011 290	6 726 330	21 072 090
$111\,S(n-4)+43$	586 789	1 976 176	6 765 271	22 624 951
$380\,S(n-5)+148$	638 548	2 008 828	6 765 288	23 160 388
$1\,140\,S(n-6)+528$	611 568	1 915 728	6 026 568	20 295 948

Except for 8, 9 and 13, the best lower bounds are obtained thanks to the third inequality $S(n+5) \geqslant 380S(n)+148$. The table doesn't go any further, but the same inequality allows to improve the lower bounds for every $n \geqslant 15$.

Corollary 2.2.2. *The growth rate for Schur numbers (and Ramsey numbers $R_n(3)$) satisfies $\gamma \geqslant \sqrt[5]{380} \approx 3.28$.*

Proof. It is a mere consequence of the inequality $S(n+5) \geqslant 380S(n)+148$. As for Ramsey numbers, the following inequality holds $S(n) \leqslant R_n(3) - 2$ (see [4]). □

2.4 Conclusion on SF-templates

In this section, we first formalized Rowley's template-based constructions [3] in the context of Schur numbers by introducing SF-templates as well as a new sequence, S^+. We provided relations between S^+ and S then stated Rowley's construction method in the context of Schur numbers. We found new SF-templates allowing us to obtain new lower bounds for schur numbers. One may notice that we mostly gave only lower bounds for S^+. It should be possible to find better SF-templates by making different assumptions or using a different method (Monte-Carlo methods for instance).

In the next section, we provide similar results for weak Schur numbers. We introduce WSF-templates and a corresponding sequence, WS^+. We then derive similar relations and construction method allowing us to find new lower bounds for weak Schur numbers.

3 Templates for weak Schur numbers

In this section, we generalize Rowley's constructions for weak Schur numbers [8] and give an analogous for weak Schur numbers of Abbott and Hanson's construction for Schur numbers. By analogy with the previous section, we then introduce WSF-templates as well as the sequence $WS^+(n)$. We find suitable templates and use them to provide new lower bounds for weak Schur numbers.

3.1 Inequality for weak Schur numbers using Schur and weak Schur numbers

Up to now, there was no equivalent for weak Schur numbers of Abbott and Hanson's construction for Schur numbers [4]. Here we give a general lower bound for weak Schur numbers as a function of both regular and weak Schur numbers. The following theorem, inspired by Rowley's inequalities for $WS(n+1)$ and $WS(n+2)$, was found and proved by Romain Ageron.

Theorem 3.1. *Let $(p,q),(n,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets then there exists a partition of $[\![1,p(q+\lceil \frac{q}{2} \rceil + 1)+q]\!]$ into $n+k$ weakly sum-free subsets.*

In particular, if we choose $q = WS(n)$ and $p = S(k)$ in the last theorem, the next corollary follows.

Corollary 3.1.1. $\forall (n,k) \in (\mathbb{N}^*)^2,\ WS(n+k) \geqslant S(k)\left(WS(n) + \left\lceil \dfrac{WS(n)}{2} \right\rceil + 1\right) + WS(n)$

This can be seen as an equivalent for weak Schur numbers of Abott and Hanson's construction for Schur numbers. This formula includes the results of Rowley [8] as a special case. For $n > 2$, this formula does not give new lower bounds.

Remark 3.1. *The above inequality can be improved by adding 1 to the lower bound if $WS(n)$ is odd (more generally if q is odd in the theorem). However, it makes the proof less clear and it is never useful in practice.*

Given that this theorem will appear as a particular case of a more general theorem after the introduction of templates for weak Schur numbers, we only give here an intuitive explanation of the above theorem; a formal proof can be found in the appendix.

Let $(p,q) \in \mathbb{N}^2$ such that there exists a partition of $[\![1,q]\!]$ into n weakly sum-free subsets and a partition of $[\![1,p]\!]$ into k sum-free subsets. Let $a \in \mathbb{N}$ with $a > q$ and let's try to build a coloring of $[\![1,ap+q]\!]$ into $n+k$ weakly sum-free subsets. Let $l = a-b-1$, $r \in [\![1,q]\!]$ and $w = a-l-r-1 = b-r$.

First, put the integers of $[\![1,ap+q]\!]$ in the following table (with a columns and $p+1$ lines) and divide it into 3 blocks (the columns are numbered from $-l$ to $+q$):

- $\mathcal{T}$ (the "tail"): the integers from 1 to q. NB: this is line number 0.
- $\mathcal{R}$ (the "rows"): the integers in columns $-l$ to $+r$ (excluding $\mathcal{T}$).
- $\mathcal{C}$ (the "columns"): the integers in the last w columns (excluding $\mathcal{T}$).

Like SF-templates, $\mathcal{R}$ and $\mathcal{C}$ play the role of security zones for each other. Note that with this numbering of columns, the column of the sum of two numbers is the only integer in $[\![-l,q]\!]$ equal to two the sum of the columns modulo a.

					1	2	...	r	$r+1$	...	$b-1$	b
$a-l$	$a-l+1$	...	$a-1$	a	$a+1$	...	$a+r-1$	$a+r$	$a+r+1$	...	$a+b-1$	$a+b$
$2a-l$	...	...	...	$2a$	...	...	...	$2a+r$	...	...	...	$2a+b$
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
$pa-l$	...	...	...	pa	...	...	...	$pa+r$	...	...	...	$pa+b$

$\mathcal{T}$ block

We color this block using the weakly sum-free coloring of $[\![1,q]\!]$ with colors $1,...,n$.

$\mathcal{R}$ block

In this block, we use the colors $n+1,...,n+k$. We color an integer x according to its line number (written $\lambda(x)$). For every $x \in \mathcal{R}$, we color x with $n+c$ where c is the color of $\lambda(x)$ in the sum-free coloring of $[\![1,p]\!]$. Let $(x,y) \in \mathcal{R}^2$. The cases are twofold.

- $\lambda(x+y) = \lambda(x) + \lambda(y)$
  In this case, we use the sum-free property of the coloring of $[\![1,p]\!]$ (in block $\mathcal{C}$, we only use colors $1,...,n$).

- $\lambda(x+y) \neq \lambda(x) + \lambda(y)$
  In this case, we do not have information about the color of $\lambda(x+y)$. Thereby, we want to have $x+y \in \mathcal{C}$. A simple solution is to limit the horizontal movement, that is if the sum changes line, not to move too far so that it stays in $\mathcal{C}$. There, the maximal displacement to the left (resp. to the right) is $2l$ (resp. $2r$). Not crossing entirely $\mathcal{C}$ by going to the left is then expressed as $-2l > -a+r$. Likewise, not going to far to the right is expressed as $2r < a-l$. It can then be written as $\max(l,r) \leqslant w$.

$\mathcal{C}$ block

In this block, we use colors $1,...,n$. We color an integer x according to its column number, denoted by $\tilde{\pi}(x)$. It is linked to the projection on the first line, denoted by π, by the relation $\tilde{\pi}(x) = \pi(x) - a$. A simple solution is to color x with the same color as $\tilde{\pi}(x)$ in the weakly sum-free coloring of $[\![1,q]\!]$. As long as $2b \leqslant a+r$ (not going two far to the right) and there is no $x \in \tilde{\pi}(\mathcal{C})$ such that $2x \in \tilde{\pi}(\mathcal{C})$ (so that we do not have a sum in $\mathcal{C}$ when taking two numbers in the same column), the colors $1,...,n$ are sum-free.

In particular, taking $w = l = \left\lceil \frac{q}{2} \right\rceil$ and $r = \left\lfloor \frac{q}{2} \right\rfloor$ works, thus obtaining the above theorem.

As in the previous section, we now introduce WSF-templates and the sequence WS^+ in order to generalize the above construction.

3.2 Definition of WS^+

In this subsection, we introduce WSF-templates and define the objects and prove the results needed for the general theorem on templates for weak Schur numbers.

Definition 3.1. *Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$. Define*

$$\pi_{a,b} : x \longmapsto (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x \mod a)$$

If there is no confusion on the a and b to use, $\pi_{a,b}$ is denoted by π.

π is the projection on the first line mentioned in the intuitive explanation.

Proposition 3.1. *Let $x \in [\![1,b]\!]$ and $y \in \mathbb{N}^*$ such that $x + \pi(y) \leqslant a + b$. Then $\pi(x+y) = x + \pi(y)$.*

Proof. Let $x \in [\![1,b]\!]$, and $y \in \mathbb{N}^*$ such that $x + \pi(y) \leqslant a + b$.
if $x + \pi(y) < a$:

$$\begin{aligned}
\pi(x+y) &= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + y \mod a) \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + \pi(y) \mod a) \quad \text{since } \pi(y) = y \mod a \\
&= x + \pi(y)
\end{aligned}$$

if $x + \pi(y) \geqslant a$:

$$\begin{aligned}
\pi(x+y) &= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + y \mod a) \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + \pi(y) \mod a) && \text{since } \pi(y) = y \mod a \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + \pi(y) - a) \\
&= x + \pi(y) - a + a\mathbb{1}_{[\![0,b]\!]}(x + \pi(y) - a) \\
&= x + \pi(y) - a + a && \text{since } x + \pi(y) - a \in [\![0,b]\!] \\
&= x + \pi(y)
\end{aligned}$$

□

Proposition 3.2. *Let $(x,y) \in (\mathbb{N}^*)^2$. Then $\pi(\pi(x) + \pi(y)) = \pi(x+y)$.*

Proof. Let $(x,y) \in (\mathbb{N}^*)^2$.

$$\begin{aligned}
\pi(\pi(x) + \pi(y)) &= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(\pi(x) + \pi(y) \mod a) \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})((Id + a\mathbb{1}_{[\![0,b]\!]})(x \mod a) + (Id + a\mathbb{1}_{[\![0,b]\!]})(y \mod a) \mod a) \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(((x \mod a) + (y \mod a)) \mod a) \\
&= (\mathrm{Id} + a\mathbb{1}_{[\![0,b]\!]})(x + y \mod a) \\
&= \pi(x+y)
\end{aligned}$$

□

Definition 3.2. *Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$. Define*

$$\lambda_{a,b} : x \longmapsto 1 + \left\lfloor \frac{x - b - 1}{a} \right\rfloor$$

If there is no confusion on the a and b to use, $\lambda_{a,b}$ is denoted by λ.

λ is the function which maps an element x to its line number as mentioned in the intuitive explanation.

Proposition 3.3. *Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$ and let $x \in \mathbb{N}^*$. Then $x = a\lambda(x) + \pi(x) - a$.*

Proof. Let $(a,b) \in (\mathbb{N}^*)^2$ such that $a > b$ and let $x \in \mathbb{N}^*$.
$a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x-b-1}{a} \right\rfloor + (x \mod a) + a\mathbb{1}_{[\![0,b]\!]}(x \mod a)$
if $x \mod a > b$ then $a\lambda(x) + \pi(x) - a = a\left\lfloor \dfrac{x}{a} \right\rfloor + x \mod a = x$
if $x \mod a \leqslant b$ then $a\lambda(x) + \pi(x) - a = a\left(\left\lfloor \dfrac{x}{a} \right\rfloor - 1\right) + x \mod a + a = x$ □

Definition 3.3. *Let $(a,n,b) \in (\mathbb{N}^*)^3$, Let $(A_1, ..., A_n)$ a partition of $[\![1, a+b]\!]$. This partition is said to be a b-weakly-sum-free template (b-WSF-template) with width a and n colors if:*

- $\forall i \in [\![1,n]\!], \quad A_i$ *is weakly-sum-free*
- $\forall i \in [\![1,n]\!], \quad A_i \backslash [\![1,b]\!]$ *is sum-free*
- *For A_n (the special subset):*

$$\forall (x,y) \in A_n^2,\ x+y > b+2a \implies x+y-2a \notin A_n$$

- *For the others subsets:*

$$\forall i \in [\![0,n-1]\!], \forall (x,y) \in A_i^2,\ x+y > a+b \implies \pi(x+y) \notin A_i$$

Please note that the special color n is not necessarily the last color by order of appearance.

Definition 3.4. *Let $(n,b) \in (\mathbb{N}^*)^2$. If there exists a such that there exists a b-WSF-template with width a and n colors, define:*

$$WS_b^+(n) = -b + \max\{a \in \mathbb{N}^* / \text{there exists a b-WSF-template with width } a \text{ and } n \text{ colors}\}$$

If no such a exists, set $WS_b^+(n) = 0$.

Definition 3.5. *Let $n \in \mathbb{N}^*$. Define*

$$WS^+(n) = \max_{b \in \mathbb{N}^*} WS_b^+(n)$$

Proposition 3.4. *Let $n \in [\![2, +\infty]\!]$. Then*

$$\frac{3}{2} WS(n-1) + 1 \leqslant WS^+(n) \leqslant WS(n)$$

Proof. The lower bound comes from the analogous of Abott and Hanson's construction for weak Schur numbers. The upper bound comes from the fact that a WSF-template with width a and n colors contains a partition of $[\![1,a]\!]$ into n sum-free subsets. □

Remark 3.2. *WS^+ and WS have the same asymptotic growth rate.*

We now proceed to state and prove the main result of this article.

3.3 Construction of weak Schur partitions using WSF-templates

Theorem 3.2. *Let $(a,n,b) \in (\mathbb{N}^*)^3$, let $(p,k) \in (\mathbb{N}^*)^2$. If there exists a partition of $[\![1,p]\!]$ into k sum-free subsets and a b-WSF-template $(A_1,...,A_n)$ with width a and n colors, then there exists a partition of $[\![1,b+p\times a]\!]$ into $(k+n-1)$ weakly sum-free subsets.*

In particular, by setting $p = S(k)$ and $a = WS^+(n)$ in the last theorem, the next corollary follows.

Corollary 3.2.1. *For every $(n,k) \in (\mathbb{N}^*)^2$, define $b_{max} = \max\{b \in \mathbb{N}^* / WS^+_b(n+1) = WS^+(n+1)\}$. Then the following inequality holds.*

$$WS(n+k) \geqslant S(k)\,WS^+(n+1) + b_{max}$$

Remark 3.3. *In the SF-template construction for Schur numbers, the additive constant comes from the fact that the special color does not necessarily appear right at the begining of the repeating pattern. Likewise, b_{max} can actually be replaced by*

$$\max_{b\in\mathbb{N}^*}\left\{\min(A_{n+1}\backslash[\![1,b]\!]) - 1 \;|\; WS^+_b(n+1) = WS^+(n+1)\right\}$$

Proof. Let $(a,n,b) \in (\mathbb{N}^*)^3$ and $(p,k) \in (\mathbb{N}^*)^2$. Set $q = a+b$.
We denote by g the coloring associated to the b-WSF-template and h the one associated to the sum-free partition of $[\![1,p]\!]$.

$$\begin{aligned} g : [\![1,q]\!] &\longrightarrow [\![1,n]\!] \\ h : [\![1,p]\!] &\longrightarrow [\![1,k]\!] \end{aligned}$$

Define $f : [\![1,b+pa]\!] \longrightarrow [\![1,n]\!]$ as follows:

- if $x \leqslant b$ (we will note $x \in \mathcal{T}$) : $f(x) = g(x)$
- if $x \in [\![1,b+pa]\!]$ and $\pi(x) \notin A_n$(we will note $x \in \mathcal{C}$) : $f(x) = g(\pi(x))$
- if $x \in [\![1,b+pa]\!]$ and $\pi(x) \in A_n$(we will note $x \in \mathcal{R}$) : $f(x) = n-1+h(\lambda(x))$

f is well defined since π is defined for $x > b$ and $\forall x \in [1,b+pa]\!]$, $f(x) \leqslant n+k-1$ because $h(\lambda(x)) \leqslant k$

We have parted the integers of $[\![1,b+pa]\!]$ in three disjoints subsets $\mathcal{T}, \mathcal{C}$ and $\mathcal{R}$.

We have to verify that f induces weakly-sum-free templates:

<u>if $(x,y) \in \mathcal{T}^2$ such that $f(x) = f(y)$, $x \neq y$:</u>

$x+y < a+b$ since $b < a$ and $g(x) = f(x) = f(y) = g(y)$.
Hence $f(x+y) = g(x+y) \neq g(x) = f(x)$

<u>if $(x,y) \in \mathcal{T}\times\mathcal{C}$ such that $f(x) = f(y)$, $x \neq y$:</u>
We distinguish two cases:

- If $x + \pi(y) \leqslant a+b$
  $g(x) = f(x) = f(y) = g(\pi(y)$. Hence $g(x) \neq g(x+\pi(y)) = g(\pi(x+y))$ (qv previous proposition)
  if $g(\pi(x+y)) = n$, $f(x+y) \geqslant n > f(x)$
  else, $f(x+y) = g(\pi(x+y)) \neq g(x) = f(x)$

- If $x + \pi(y) > a + b$, $x + y > a + b$ and by definition of g, $g(\pi(x + y)) \neq g(x)$
  if $g(\pi(x + y)) = n, f(x + y) \geqslant n > f(x)$
  else, $f(x + y) = g(\pi(x + y)) \neq g(\pi(x)) = f(x)$

<u>if $(x, y) \in \mathcal{T} \times \mathcal{R}$ such that $f(x) = f(y)$, $x \neq y$:</u>

Then, $f(x) = f(y) = n$. We distinguish two cases:

- If $\lambda(y) = \lambda(x + y)$,
  $g(x) = g(\pi(y)) = n$ since $g(y) = g(\pi(y))$
  Therefore $g(\pi(x + y)) = g(x + \pi(y)) \neq g(x) = n$ (qv previous proposition)
  Hence $f(x + y) = g(\pi(x + y)) \neq n$
- If $\lambda(y) \neq \lambda(x + y)$, $\lambda(y) + 1 = \lambda(x + y)$
  $n = f(y) = n - 1 + h(\lambda(y))$. Hence $h(\lambda(y)) = 1$.
  Moreover $h(1) = 1$, therefore $h(\lambda(y) + 1) \neq 1$
  if $\pi(x + y) \in A_n$, $f(x + y) = n - 1 + h(\lambda(x + y)) > n$

<u>if $(x, y) \in \mathcal{C}^2$ such that $f(x) = f(y)$, $x \neq y$:</u>

Then $g(\pi(x)) = f(x) = f(y) = g(\pi(y))$. We distinguish two cases:

- If $\pi(x) + \pi(y) > q$, $g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$ (qv previous proposition)
  Hence $g(\pi(x + y)) = g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$
  if $g(\pi(x + y)) = n, f(x + y) \geqslant n > f(x)$
  else, $f(x + y) = g(\pi(x + y)) \neq g(\pi(x)) = f(x)$
- If $\pi(x) + \pi(y) \leqslant q$, $g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x))$ since g is sum-free for $x > b$
  if $g(\pi(x + y)) = n, f(x + y) \geqslant n > f(x)$
  else, $f(x + y) = g(\pi(x + y)) = g(\pi(\pi(x) + \pi(y)) \neq g(\pi(x)) = f(x)$

<u>if $(x, y) \in \mathcal{C} \times \mathcal{R}$, $f(x) \neq f(y)$</u>

<u>if $(x, y) \in (\mathcal{R})^2$ such that $f(x) = f(y)$, $x \neq y$:</u>

Let $r(x) = \pi(x) - a$ and $r(y) = \pi(y) - a$,
We proved that $x = a\lambda(x) + \pi(x) - a$, therefore $x = a\lambda(x) + \pi(x)$
$x + y = a(\lambda(x) + \lambda(y)) + r(x) + r(y)$. We distinguish three cases:

- If $r(x) + r(y) \in [\![b - a + 1, b]\!]$, $h(\lambda(x)) = f(x) + 1 - n = f(y) + 1 - n = h(\lambda(y))$ Hence, $h(\lambda(x) + \lambda(y)) \neq h(\lambda(x))$.

$$\begin{aligned}
\lambda(x + y) &= 1 + \left\lfloor \frac{a(\lambda(x) + \lambda(y)) + r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) + \left\lfloor \frac{r(x) + r(y) - b - 1}{a} \right\rfloor + 1 \\
&= \lambda(x) + \lambda(y) - 1 + 1 \text{ since } r(x) + r(y) \in [\![b - a + 1, b]\!] \\
&= \lambda(x) + \lambda(y)
\end{aligned}$$

$$\begin{aligned}\text{if } f(x+y) \geqslant n,\ f(x+y) &= n-1+h(\lambda(x+y)) \\ &= n-1+h(\lambda(x)+\lambda(y)) \\ &\neq n-1+h(\lambda(x)) \\ &= f(x)\end{aligned}$$

- If $r(x)+r(y) > b$, $\pi(x)+\pi(y) > 2a+b$
  Since g is a b-WSF template, $g(\pi(\pi(x)+\pi(y))) \neq n$
  Therefore, $g(\pi(x+y)) \neq n$ ie $x+y \in \mathcal{C}$
  Hence $f(x+y) < n \leqslant f(x)$
- If $r(x)+r(y) \leqslant b-a$, $\pi(x)+\pi(y) \leqslant b+a$
  Since g is sum-free for $x > b$, since $g(\pi(x)) = g(\pi(y)) = n$, $g(\pi(x+y)) \neq g(\pi(x)) = n$
  Hence, $f(x+y) < n \leqslant f(x)$

□

The general lower bound for weak Schur numbers in function of both regular and weak Schur numbers can be seen as a particular case of WSF-template in the same way Abott and Hanson's construction can be seen as a particular case of SF-template. Acutally, like for SF-templates, the additive constant of a WSF-template can be improved by weakening the hypotheses made on the last row. The principle behind it is the same as in the analogous proposition for SF-templates.

Proposition 3.5. *Let $(b,k,a) \in \mathbb{N}^*)^3$ and let f be a coloring associated to a b-WSF-template with width p and k colors. Let $c \in \mathbb{N}$ ($c = \min(A_{k+1} \backslash [\![1,b]\!]) - 1$ works) and assume there there exists a coloring g of $[\![b+1, b+c]\!]$ with k colors such that:*

- $\forall c \in [\![1,k]\!], \forall (x,y) \in [\![1,a+b]\!] \times [\![b+1,a+b]\!], (f(x) = f(y) \text{ and } \pi(x+y) \leqslant b+c) \implies g(\pi(x+y)) \neq f(x)$
- $\forall c \in [\![1,k]\!], \forall (x,y) \in [\![1,a+b]\!] \times [\![b+1,b+c]\!], (f(x) = g(y) \text{ and } \pi(x+y) \leqslant b+c) \implies g(\pi(x+y)) \neq f(x)$

Then, for every $n \in \mathbb{N}^*$, by using on the last row the coloring $i \longmapsto g(i - pS(n))$, we have

$$WS(n+k) \geqslant WS^{+}(k+1)S(n) + b + c$$

The WSF-templates can actually be fine-tuned further. However, it gives only minor improvements (most likely only an additive constant) at the cost of dramatically increasing the size of the search space. Therefore, it does not seem relevant to use this sophistications given that we could not even find good WSF-templates with 5 colors using a computer (here good means better than those obtain by combining smaller templates).

These modifications work as follows. One may notice that the first row (excluding the "tail") has constraints that other rows do not have because of the tail, especially if the special color appears in the tail as well. Thus allowing to have a coloring on the first row different from the coloring of the other rows would weaken the constraints. Acutally, one may even go further by noticing that on the one hand the first (ordered) color of the sum-free partition used for the extension procedure has more more constraints than the other colors of the sum-free partition since the first row is of this color and is more constrained than the other rows, but that on the other hand it has more degrees of freedom than the other colors of the sum-free partition since in the sum-free partition there cannot be two consecutive numbers of this color. As a result, it removes some constraints imposed by the first row on the other rows.

To sum up, one can look for a generalised WSF-template that uses a special coloring for the tail and the first row, a coloring dedicated to the rows whose number is not 1 but is in the first color in the sum-free partition, a coloring for all the other rows and a special coloring for the last numbers (as previously explained for the improvement of the additive constant of WSF-templates).

We also have a similar theorem where only WS^+ is involved.

Theorem 3.3. *Let $(k,p),(n,a) \in (\mathbb{N}^*)^2$. If there exists a SF-template with width p and $k+1$ colors and a WSF-template with width a and n colors, then there exists WSF-template with width pq and $(n+k)$ colors.*

And the associated inequality :

Corollary 3.3.1. *Let $n,k \in \mathbb{N}^*$. Then*

$$WS^+(n+k) \geqslant S^+(k+1)\, WS^+(n)$$

Proof. The idea is the same as in the previous theorem. The only difference is the WSF property inherited from the WSF-template. □

3.4 Inequalities and new lower bounds for Weak Schur numbers

Having found suitable templates, which can be found in the appendix, with a SAT solver, we claim that for all $n \in \mathbb{N}^*$:

$$\begin{array}{rcrcr} WS(n+1) & \geqslant & 4S(n) & + & 2 \\ WS(n+2) & \geqslant & 13S(n) & + & 8 \\ WS(n+3) & \geqslant & 42S(n) & + & 24 \\ WS(n+4) & \geqslant & 132S(n) & + & 26 \end{array}$$

The first two inequalities were found by Rowley, they are detailed in [8]. The third inequality is optimal and was found with a SAT solver. It uses the first sophistication explained in the previous subsection in order to add the last number in the first color. As for the fourth inequality, it was obtained by combining an optimal SF-template with width 33 with a WSF-template with width 4. The best template we could get with a computer search gives the inequality $WS(n+4) \geqslant 127S(n) + 68$. It was also found with the SAT solver. In order to reduce the search space, we only looked for WSF-templates of 5 colors which start with a good $WS(4)$ partition. However, this approach most likely prevents us from finding the best WSF-templates as we explain in the next subsection for weakly sum-free partitions. We highly suspect that there exists more efficient WSF-templates with $n \geqslant 5$ colors. One may try to go over a different search space using a Monte-Carlo method, as in [7]. This could be the suject of a future work. Further details about the encoding as a SAT problem can be found in [9].

Like in 3.3, we compute the lower bounds given by the previous inequalities for $n \in [\![8, 15]\!]$. The best lower bound for each integer is highlighted.

n	8	9	10	11
$4\,S(n-1)+2$	6 722	21 146	71 214	243 794
$13\,S(n-2)+8$	6 976	21 848	68 726	231 447
$42\,S(n-3)+24$	6 744	22 536	70 584	222 036

n	12	13	14	15
$4\,S(n-1)+2$	815 314	2 554 194	8 045 162	27 061 154
$13\,S(n-2)+8$	792 332	2 649 772	8 301 132	26 146 778
$42\,S(n-3)+24$	747 750	2 559 840	8 560 800	25 886 224

With $S(9) \geqslant 17\,803$, we found a new lower bound for $WS(10)$ using Rowley's inequality. Moreover, the third inequality gives new lower bounds for $WS(9)$ and $WS(14)$.

3.5 Conclusion on WSF-templates

In this section, we first gave a new construction which can be seen as an equivalent for weakly sum-free partitions of Abott and Hanson's construction for sum-free partitions. We then introduced WSF-templates and generalized this construction. This allows us to find new lower bounds and new inequalities for weak Schur numbers. One may notice the significant difference between the former lower bounds for weak Schur numbers obtained by conducting a computer search and the new lower bounds obtained with WSF-templates (including Rowley's two inequalities). In the next section, we try to analyze this phenomenon.

4 Analysis of the former search space

In this section, we first provide evidence which indicate that the main assumption made by papers which found the previous best known lower bounds for weak Schur numbers using a computer may not be correct. This is done primarily by studying $WS(6)$. Then, in an effort to eliminate irrelevant search spaces, we obtain stronger results than those previously known for $WS(5)$ while gaining several orders of magnitude in computation time by giving additional information to the SAT solver without losing in generality. In this section, we assume that the subsets are ordered.

4.1 Limitation of the former assumption

Rowley's new lower bound for $WS(6)$ (642) [8] was a significant improvement upon the former best known lower bound (582) [11]. This previous lower bound has been found several times using a computer (often with Monte-Carlo methods) and by recursively making the assumption that a good partition for $WS(n+1)$ starts with a good partition for $WS(n)$ which is true for small values of n. Therefore, one may wonder whether the limiting factor are the assumptions or the methods used to search for partitions. It appears that the search space induced by these assumptions does not contain the optimal partitions.

Computational Theorem 4.1. *There is no weakly sum-free partition of* $[\![1, 583]\!]$ *in 6 parts such that:*

- $m_5 \geqslant 66$
- $m_6 \geqslant 186$
- $[\![210, 349]\!] \subset A_6$

This result was obtained in 8 hours with the SAT solver plingeling [12] on a 2.60 GHz Intel i7 processor PC. However, simply encoding the existence of such a partition as explained in the previous subsection would not result in a reasonable computation time. In order to help the SAT solver, we add additional information in the propositional formula. We did not quantify the speedup, but it most likely allowed us to gain several order of magnitude in computation time as we explain in the next subsection.

For every weakly sum-free coloring f of $[\![1, 65]\!]$ with 4 colors, the sequence $f(1), f(2), f(3), ...$ always starts with the following subsequence 1121222133. Then 11 is always either in subset 1 or 3, 12 is always in subset 3 and so on. For every integer in $[\![1, 65]\!]$, we computed in which subset it can appear. By using this constraints, we could then compute for every integer in $[\![1, 185]\!]$, in which subset it can appear in a weakly sum-free partition of $[\![1, 185]\!]$ which starts with a weakly sum-free partition of $[\![1, 65]\!]$ in 4 subsets. Adding these constraints to the formula corresponding to the above theorem gives additional information to the SAT solver without losing in generality.

The above theorem shows that the previous lower bound for $WS(6)$ is optimal in the search space considered by the papers which found it. Therefore, finding a partition of $[\![1, n]\!]$ in 6 weakly sum-free subsets for some $n \geqslant 590$ which does not have a template-like structure would be extremely interesting since it could give indications on a new search space for improving lower bounds with a computer.

More generally, this theorem questions the search space previously used for finding lower bounds for $WS(n)$ with a computer. In particular, to our knowledge every paper that found the lower bound $WS(5) \geqslant 196$ used this assumption. Therefore one may wonder if this is actually a good lower bound. In the next subsection, we give properties that a hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets has to verify.

4.2 Investigating weak Schur number five

As explained in the previous subsection, the search space used for showing that $WS(5) \geqslant 196$ may not contain optimal solution. In this subsection, we give necessary conditions for a hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets using the same type of methods as in the previous subsection.

Notation 4.1. *Let P be a predicate over weakly sum-free partitions. We denote by $WS(n|P)$ the greatest number p such that there exists a partition of $[\![1, p]\!]$ in n weakly sum-free subsets which verifies P.*

[5] verified with a SAT solver that there are no partition in 5 weakly sum-free subsets of $[\![1, 197]\!]$ with $A_5 = \{67, 68\} \cup [\![70, 134]\!] \cup \{136\}$ in 17 hours and could not provide a similar result when only assuming $m_5 = 67$ even after several weeks of runtime. By using the same method as above, we were able to verify that $WS(5|m_5 = 67) = 196$ in 0.5 seconds with the SAT solver glucose [13] on a 2.60 GHz Intel i7 processor PC (we used the non-parallel version here for the sake of comparison but in the rest of this subsection, we used the parallel version of glucose). The additional information we gave to the SAT solver is that every partition of $[\![1, 66]\!]$ in 4 weakly sum-free subsets starts with a partition of $[\![1, 23]\!]$ in 3 weakly sum-free subsets (this can be checked in a few dozens of minutes with a SAT solver). Among the 3 partitions of $[\![1, 23]\!]$ in 3 weakly sum-free subsets, every number always appears in the same subset except for 16 and 17 which can appear in two different subsets. We hardcoded this external knowledge in the propositional formula which allowed us to gain several orders of magnitude in computation time. Since $WS(4) = 66$, we have $m_5 \leqslant 67$. We give the stronger following result.

Computational Theorem 4.2. *If there exists a partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets then $m_5 \leqslant 59$.*

More precisely, we verified the following results (*max* m_5 is the greatest value of m_5 for which we have not verified that $WS(5|m_5) \leq 196$).

m_4	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$WS(4\|m_4) + 1$	55	59	60	59	59	60	60	60	60	64	63	64	61	64	63	65	65	65	65	66	67
max m_5	49	51	54	53	54	54	55	55	55	55	55	56	57	57	59	59	59	59	59	58	53

For $m_4 = 24$, the value $m_5 \leqslant 53$ was obtained after 7 hours of runtime, and for $m_4 = 22$, the value $m_5 \leqslant 59$ was obtained after 11 hours of runtime. All the other values took between 1 minutes and 2 hours. To obtain these results, we once again provided additionnal information to the SAT solver. We added two different types information. The first one is the same as previously: we compute the subsets in which the first numbers can appear. The second type of information is the maximum length of a sequence using only a certain subset of the colors. For instance, if $m_4 \geqslant 22$, then there cannot be more than 17 consecutive numbers with color 1, 2 or 3.

4.3 Conclusion on the search space

As explained in the first subsection, the recursive assumption that a good partition for $WS(n+1)$ starts with a good partition for $WS(n)$ appears to be wrong. Given that no extensive search for $WS(5)$ has been conducted without making this assumption and that the size of the considered partitions is reasonable (the current lower bound is 196 and $S(5) = 160$ was verified), it seems worth investigating this special case further. In a spirit of orientating this search, we then give necessary conditions on an hypothetical partition of $[\![1, 197]\!]$ in 5 weakly sum-free subsets. Finding partitions that exceeds the lower bounds found with a computer (even if they do not exceed those obtained with templates) which are not as regular as those obtained with templates would be extremely interesing since it could designate a new search space for finding lower bounds with a computer.

5 Conclusions and future work

These new results come from an extension of Rowley's template-based approach for Ramsey graphs and Schur numbers which is relatively new. Therefore, we would not be surprised if lower bounds are later improved using better templates. Moreover, studying specifically $S^+(n)$ and $WS^+(n)$ might be of interest as they are closely related to Schur and weak Schur numbers. In order to find new templates, algorithms based on randomness such as Monte-Carlo algorithms may prove to be very useful. This could be the subject of a future work.

The fourth section gives new insight on the method that was formerly used to achieve new lower bounds for weak Schur numbers. The assumption behind it might have removed the optimal partitions from the search space and thus lowered the highest value that can be reached within it. However, the methods used to explore this search space are efficient as the optimum was found. Thereby Monte-Carlo algorithms may prove to be very useful if used in a search space with more potential. Finding a weakly sum-free partition which is better than previous lower bounds and which is not as regular as those obtained with templates would be extremely interesting (even if it does not improve current lower bounds) since it could suggest a new search space which could then be explored with the above mentioned methods.

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A SF-templates

SF-template with width 33 and 4 colors

1	1, 6, 9, 13, 16, 20, 24, 27, 31
2	2, 5, 14, 15, 25, 26
3	3, 4, 10, 11, 12, 28, 29, 30
4	7, 8, 17, 18, 19, 21, 22, 23, 32, 33

SF-template with width 111 and 5 colors

1	1, 5, 18, 12, 14 ,21, 23, 30, 32, 36, 39, 43, 45, 52, 103 106, 110
2	2, 6, 7, 10, 15, 18, 26, 29, 34, 37, 38, 42, 46, 51, 54 101, 104, 109
3	3, 4, 9, 11, 17, 19, 25, 27, 33, 35, 40, 41, 47, 48, 55 100, 107, 108
4	13, 16, 20, 22, 24, 28, 31, 58, 61, 67, 88, 94, 97
5	44, 50, 53, 56, 57, 59, 60, 62, 63, 64, 65, 66, 68, 69, 70 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85 86, 87, 89, 90, 91, 92, 93, 95, 96, 98, 99, 102, 105, 111

SF-template with width 380 and 6 colors

1	1, 5, 8, 11, 15, 17, 29, 33, 36, 39, 43, 57, 61, 88, 92 106, 110, 113, 116, 120, 132, 134, 138, 141, 144, 148, 150, 154, 157, 160 164, 178, 182, 185, 188, 341, 344, 347, 351, 365, 369, 372, 375, 379
2	2, 9, 13, 16, 20, 23, 24, 27, 28, 31, 34, 35, 38, 42, 45 49, 53, 60, 67, 71, 78, 82, 89, 96, 100, 104, 107, 111, 114, 115 118, 121, 122, 125, 126, 129, 133, 136, 140, 147, 158, 162, 165, 169, 172 176, 183, 187, 194, 201, 328, 335, 342, 346, 353, 357, 360, 364, 367, 371
3	3, 4, 12, 14, 19, 25, 30, 32, 40, 41, 47, 48, 58, 91, 101 102, 108, 109, 117, 119, 124, 130, 135, 137, 145, 146, 152, 153, 161, 163 168, 179, 181, 190, 339, 348, 350, 361, 366, 368, 376, 377
4	6, 7, 10, 18, 21, 22, 26, 37, 46, 50, 51, 54, 65, 70, 79 84, 95, 98, 99, 103, 112, 123, 127, 128, 131, 139, 142, 143, 151, 155 156, 159, 167, 170, 171, 175, 186, 343, 354, 358, 359, 362, 370, 373, 374 378
5	44, 52, 55, 56, 59, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75 76, 77, 80, 81, 83, 85, 86, 87, 90, 93, 94, 97, 105, 189, 196 197, 200, 203, 206, 207, 209, 214, 219, 231, 298, 310, 315, 320, 322, 323 326, 329, 332, 333, 340
6	149, 166, 173, 174, 177, 180, 184, 191, 192, 193, 195, 198, 199, 202, 204 205, 208, 210, 211, 212, 213, 215, 216, 217, 218, 220, 221, 222, 223, 224 225, 226, 227, 228, 229, 230, 232, 233, 234, 235, 236, 237, 238, 239, 240 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283, 284, 285 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296, 297, 299, 300, 301 302, 303, 304, 305, 306, 307, 308, 309, 311, 312, 313, 314, 316, 317, 318 319, 321, 324, 325, 327, 330, 331, 334, 336, 337, 338, 345, 349, 352, 355 356, 363, 380

B WSF-templates

23-WSF-template with width 42 and 4 colors

1	1, 2, 4, 8, 11, 22, 25, $(\mathbf{N+1})$
2	5, 6, 7, 19, 21, 23, 36
3	9, 10, 12, 13, 14, 15, 16, 17, 18, 20
4	24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40 41, 42

This template provides the inequality $WS(n+3) \geqslant 42S(n)+24$ by placing one last number, here represented by $(\mathbf{N+1})$, in the first subset.

C Proof of theorem 3.1

Proof. Let $(p,q),(n,k) \in (\mathbb{N}^*)^2$, $N = p(q + \left\lceil \frac{q}{2} \right\rceil + 1) + q$, $\alpha = \left\lceil \frac{q}{2} \right\rceil > 0$ and $\beta = q + \alpha + 1$. We denote by f the coloring associated to the partition of $[\![1,q]\!]$ and g the one associated to the partition of $[\![1,p]\!]$.

$$f : [\![1,q]\!] \longrightarrow [\![1,n]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2, \begin{cases} x \neq y \\ f(x) = f(y) \end{cases} \Longrightarrow f(x+y) \neq f(x)$$

$$g : [\![1,p]\!] \longrightarrow [\![1,k]\!] \text{ and } \forall (x,y) \in [\![1,q]\!]^2,\ f(x) = f(y) \Longrightarrow f(x+y) \neq f(x)$$

Let us start by parting the integers of $[\![1,N]\!]$ in two subsets $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{A} = [\![1,\alpha]\!] \cup \{a\beta + u \mid (a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]\}$ and $\mathcal{B} = \{a\beta + u \mid (a,u) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]\}$.

First, $\underline{\mathcal{A} \cap \mathcal{B} = \varnothing}$:
By contradiction, suppose there exists $x \in \mathcal{A} \cap \mathcal{B} \neq \varnothing$. Then there are $(a,u) \in [\![0,p]\!] \times [\![\alpha+1,q]\!]$ and $(b,v) \in [\![1,p]\!] \times [\![-\alpha,\alpha]\!]$ such that $x = a\beta + u = b\beta + v$. By definition of α and β we have $u \in [\![\alpha+1,q]\!] \subset [\![0,\beta-1]\!]$. From there, we distinguish two cases :

- If $v \in [\![0,\alpha]\!]$ then $v \in [\![0,\beta-1]\!]$ and $v \neq u$ because $v < \alpha + 1 \leqslant u$
- If $v \in [\![-\alpha,-1]\!]$, we note $\tilde{v} = \beta + v$ and thus have $x = (b-1)\beta + \tilde{v}$ with $\tilde{v} \in [\![\beta-\alpha,\beta-1]\!] \subset [\![0,\beta-1]\!]$ and $\tilde{v} \neq u$ because $u < q+1 = \beta - \alpha \leqslant \tilde{v}$.

In either cases, we run into a contradiction because of the remainder's uniqueness in the euclidean division of x byβ.

Then, we have $\underline{\mathcal{A} \cup \mathcal{B} = [\![1,N]\!]}$:

- On the one hand : $1 = \min(\mathcal{A}) \leqslant \max(\mathcal{A}) = p\beta + q = N$ and $1 \leqslant \beta - \alpha = \min(\mathcal{B}) \leqslant \max(\mathcal{B}) = p\beta + \alpha \leqslant N$, which gives $\mathcal{A} \cup \mathcal{B} \subset [\![1,N]\!]$.
- On the other hand, let $x \in [\![1,N]\!]$. If $x \leqslant \alpha$, we directly have $x \in \mathcal{A}$, let us then suppose that $x > \alpha$ and write $x = a\beta + u$ the euclidean division of x by β. We have $x \leqslant N$, thus $a \leqslant p$. We distinguish three cases :
  - If $u \in [\![0,\alpha]\!]$ then we necessarily have $a \geqslant 1$ because $x > \alpha$, and so $x \in \mathcal{B}$.
  - If $u \in [\![\alpha+1,q]\!]$, then $x \in \mathcal{A}$.
  - If $u \in [\![q+1,\beta-1]\!]$ then $x = (a+1)\beta - (\beta - u)$ with $-\alpha \leqslant \beta - u \leqslant 0$. Furthermore, $a \leqslant p-1$,

else we would have $x > N$, and so $x \in \mathcal{B}$

In any case, $x \in \mathcal{A} \cup \mathcal{B}$ and we can thus conclude that $[\![1, N]\!] \subset \mathcal{A} \cup \mathcal{B}$.

This first partition of $[\![1, N]\!]$ will help us to define our final partition by the projection of its equivalence relation. We thereby define $h : [\![1, N]\!] \longrightarrow [\![1, n + k]\!]$ as such :
- If $x \in \mathcal{A}$ then $h(x) = f(x \bmod \beta)$ (well defined because $x \bmod \beta \in [\![1, N]\!]$)
- If $x \in \mathcal{B}$ then $x = a\beta + u$ with a unique $(a, u) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]$and we define $h(x) = n + g(a)$

The fact that $(\mathcal{A}, \mathcal{B})$ is a partition of $[\![1, N]\!]$ ensures that this definition of h is valid. We then have to verify that h induces weakly sum-free subsets.

The classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free :

Let $(x, y) \in \mathcal{A}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$

- If $(x, y) \in [\![1, \alpha]\!]^2$:
  We have $x + y \leqslant 2\alpha \leqslant q$ and $x + y = 0\beta + x + y$, therefore $x + y \in \mathcal{A}$. Then, by definition : $h(x) = f(x)$, $h(y) = f(y)$ and $h(x + y) = f(x + y)$, which gives us, thanks to the property verified by f, that $h(x + y) \neq h(x)$.

- If $(x, y) \in [\![1, \alpha]\!] \times (\mathcal{A} \setminus [\![1, \alpha]\!])$:
  We write $y = a\beta + u$ with $(a, u) \in [\![0, p]\!] \times [\![\alpha+1, q]\!]$. Then $x+y = a\beta+x+u = (a+1)\beta+x+u-\beta$, and if $x+u > q$ it follows that $a \leqslant p-1$ since $x+y \leqslant N$, and $-\alpha \leqslant x+u-\beta \leqslant -1$. Therefore $x + y \in \mathcal{B}$ and $h(x + y) \neq h(x) = f(x)$ by definition of h. On the contrary, if $x - u \leqslant n$, then $x + y \in \mathcal{A}$ and $h(x + y) = f(x + u)$ because $x + u$ is actually the remainder of the euclidean divisionof $x+y$ by β. Moreover, $h(x) = f(x)$, $x < u$ and, with our initial hypothesis, $h(x) = h(y) = f(u)$. The property verified by f gives us $f(x + u) \neq f(x)$ which can be rewritten as $h(x + y) \neq h(x)$.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!]) \times [\![1, \alpha]\!]$:
  This case is handled exactly like the previous one by swaping the roles of x and y.

- If $(x, y) \in (\mathcal{A} \setminus [\![1, \alpha]\!])^2$:
  We write $x = a\beta + u$ and $y = b\beta + v$ with (a, u) and (b, v) in $[\![0, p]\!] \times [\![\alpha + 1, q]\!]$. Then $x + y = (a + b)\beta + u + v = (a + b + 1)\beta + u + v - \beta$ with $a + b \leqslant p - 1$ (else we would have $x + y > N$ because $u + v > q$) and $-\alpha \leqslant u + v - \beta \leqslant \alpha$, therefore $x + y \in \mathcal{B}$ and by definition $h(x + y) \neq h(x)$.

In any case, $h(x + y) \neq h(x)$ and the classes of equivalence $h(x)$ for $x \in \mathcal{A}$ are weakly sum-free.

The classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free :

Let $(x, y) \in \mathcal{B}^2$ such that $h(x) = h(y)$, $x \neq y$ and $x + y \leqslant N$.
We write $x = a\beta+u$ and $y = b\beta+v$ with (a, u) and (b, v) in $[\![1, p]\!] \times [\![-\alpha, \alpha]\!]$. We have $h(x) = q+g(a)$ and $h(y) = q + g(b)$, therefore $g(a) = g(b)$. We also have $x + y = (a + b)\beta + u + v$.
If $u+v \in [\![-\alpha, \alpha]\!]$, then $x+y \in \mathcal{B}$ and $h(x+y) = g(a+b)$, hence we can deduce that $h(x+y) \neq h(x)$ because of the property verified by g. On the contrary, if $u+v \notin [\![-\alpha, \alpha]\!]$, then necessarily $x+y \in \mathcal{A}$. Suppose $x+y \in \mathcal{B}$, then $x+y = c\beta+w$ with $(c, w) \in [\![1, p]\!] \times [\![-\alpha, \alpha]\!]$. Thus, $c\beta+w = (a+b)\beta+u+v$ and $(a + b - c)\beta = w - u - v$. Furthermore $a + b - c \neq 0$, else we would have $u + v = w \in [\![-\alpha, \alpha]\!]$. This finally leads to the following inequality :

$$\beta \leqslant |a + b - c|\beta = |w - u - v| \leqslant |w| + |u| + |v| \leqslant 3\alpha \leqslant q + \alpha < \beta$$

which is absurd. We can therefore conclude that $x + y \in \mathcal{A}$ and by definition of h, $h(x + y) \neq h(x)$, proving that the classes of equivalence $h(x)$ for $x \in \mathcal{B}$ are weakly sum-free.

Finally, we have showed that every classe of equivalence induced by h is weakly sum-free, which ends the proof. $\square$