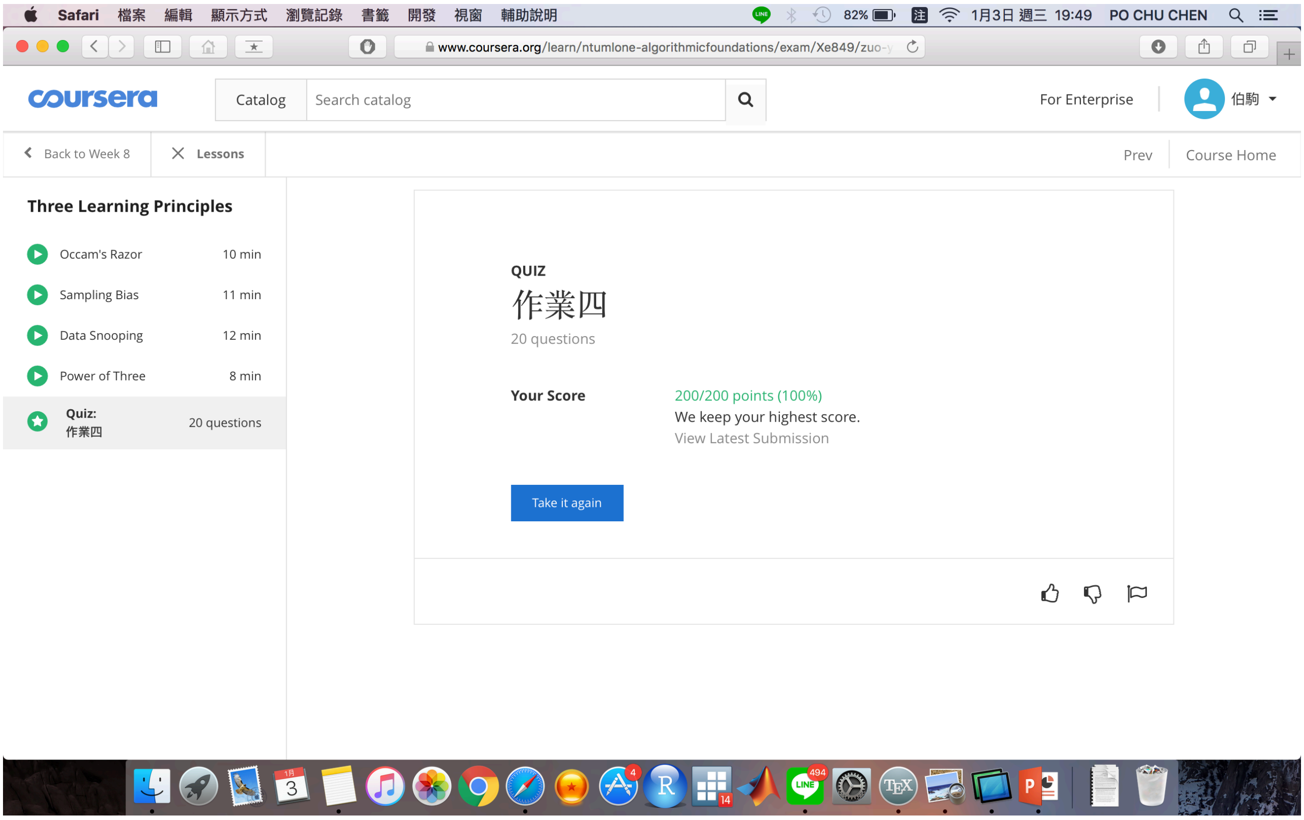
Home Work 4

Machine Learning Foundations

R04323050

經濟碩三　陳伯駒

1.



2.

$E_{aug}(\mathbf{w}) = E_{in}(\mathbf{w}) + \frac{\lambda}{N}\mathbf{w}^T\mathbf{w}$

Take the derivatives of $\mathbf{w}$: $\nabla E_{aug}(\mathbf{w}) = \nabla E_{in}(\mathbf{w}) + \frac{2\lambda}{N}\mathbf{w}$

By the Gradient Descent:

$$\begin{aligned}\mathbf{w}_{t+1} \leftarrow & \mathbf{w}_t - \eta \cdot (\nabla E_{aug}(\mathbf{w})) \\ &= \mathbf{w}_t - \eta \cdot (\nabla E_{in}(\mathbf{w}_t) + \frac{2\lambda}{N}\mathbf{w}_t) \\ &= (1 - \frac{2\eta\lambda}{N})\mathbf{w}_t - \eta\nabla E_{in}(\mathbf{w}_t)\end{aligned}$$

3.

Regularized Regression Problem:

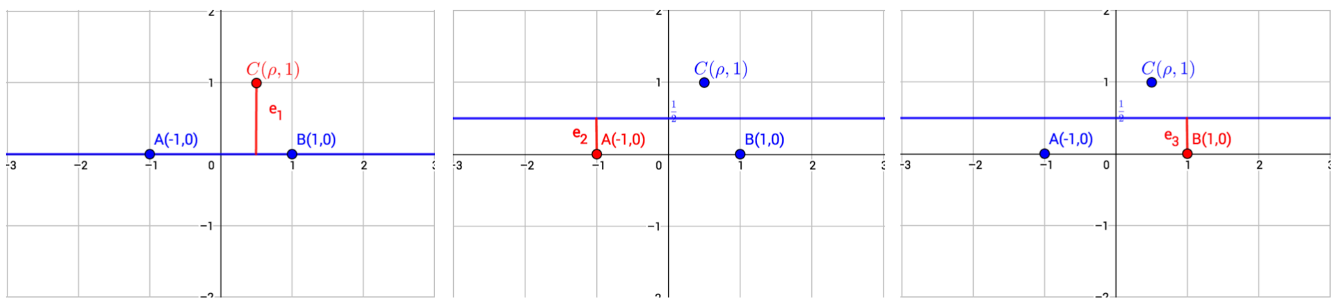
$$\min_{\mathbf{w}} \quad E_{in}(\mathbf{w}) = \frac{\mathbf{1}}{\mathbf{N}}(\mathbf{Zw} - \mathbf{y})^{\mathbf{T}}(\mathbf{Zw} - \mathbf{y}) \qquad \text{s.t.} \quad \mathbf{w}^{\mathbf{T}}\mathbf{w} \leq C$$

① If $\mathbf{w}_{lin}$ satisfies the constraints $\mathbf{w}^{\mathbf{T}}\mathbf{w} \leq C$, then $\mathbf{w}_{reg}$ is equivalent to $\mathbf{w}_{lin}$, thus $\|\mathbf{w}_{reg}\| = \|\mathbf{w}_{lin}\|$.

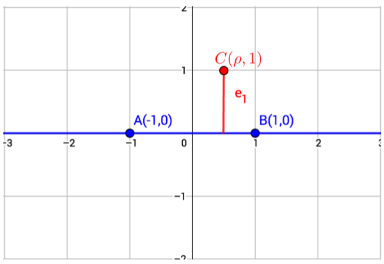
② If $\mathbf{w}_{lin}$ does not satisfy the constraints $\mathbf{w}^{\mathbf{T}}\mathbf{w} \leq C$, which means $\|\mathbf{w}_{lin}\|^2 > C$; on the other hand, we know $\mathbf{w}_{reg}$ satisfies the constraints, i.e $\|\mathbf{w}_{reg}\|^2 \leq C$. Hence, $\|\mathbf{w}_{reg}\| < \|\mathbf{w}_{lin}\|$.

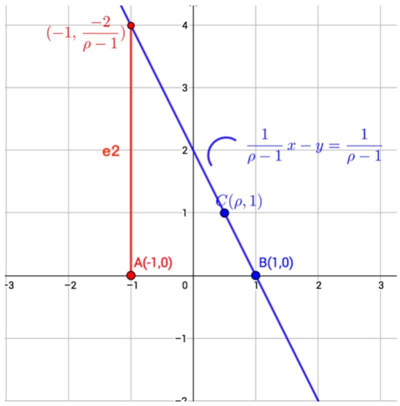
By ①、②, we have $\|\mathbf{w}_{reg}(\lambda)\| \leq \|\mathbf{w}_{lin}\|$ for all $\lambda > 0$.

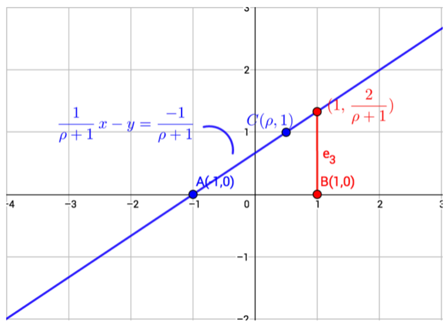
4.



$$\therefore E_{loocv}(h_0) = \frac{1}{3} \cdot (1^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2) = \frac{1}{2}$$







$$\therefore E_{loocv}(h_1) = \frac{1}{3}\cdot(1^2 + (\frac{-2}{\rho-1})^2 + (\frac{2}{\rho+1})^2)$$

Let $E_{loocv}(h_0) = E_{loocv}(h_1) \Rightarrow \quad \rho = \sqrt{9+4\sqrt{6}}$

5.

$E_{in} = \dfrac{1}{N+K}(\mathbf{w}\mathbf{x}^{\mathbf{T}}\mathbf{x}\mathbf{w} - \mathbf{2}\mathbf{w}^{\mathbf{T}}\mathbf{x}^{\mathbf{T}}\mathbf{y} + \mathbf{y}^{\mathbf{T}}\mathbf{y} + \mathbf{w}\widetilde{\mathbf{x}}^{\mathbf{T}}\widetilde{\mathbf{x}}\mathbf{w} - \mathbf{2}\widetilde{\mathbf{w}}^{\mathbf{T}}\widetilde{\mathbf{x}}\widetilde{\mathbf{y}} + \widetilde{\mathbf{y}}^{\mathbf{T}}\widetilde{\mathbf{y}})$

$\therefore \nabla E_{in}(\mathbf{w}) = \dfrac{1}{N+K}(\mathbf{x}^{\mathbf{T}}\mathbf{x}\mathbf{w} - \mathbf{x}^{\mathbf{T}}\mathbf{y} + \widetilde{\mathbf{x}}^{\mathbf{T}}\widetilde{\mathbf{x}}\mathbf{w} - \widetilde{\mathbf{x}}^{\mathbf{T}}\widetilde{\mathbf{y}})$

F.O.C. $\nabla E_{in}(\mathbf{w}) = 0 \quad \Rightarrow \mathbf{w}^* = (\mathbf{x}^{\mathbf{T}}\mathbf{x} + \widetilde{\mathbf{x}}^{\mathbf{T}}\widetilde{\mathbf{x}})^{-\mathbf{1}}(\mathbf{x}^{\mathbf{T}}\mathbf{y} + \widetilde{\mathbf{x}}^{\mathbf{T}}\widetilde{\mathbf{y}})$.

6.

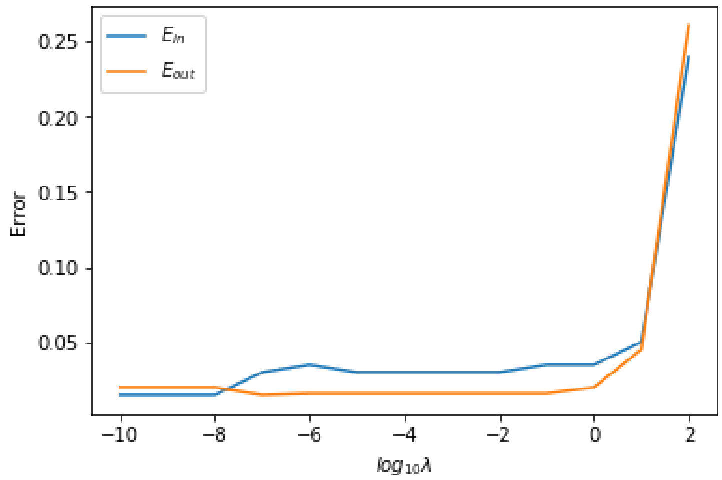
The condition of $\mathbf{w}_{reg}$ given in question is exact the Regularized Regression Problem.

By P.10 in slides 14, we know the optimal solution of regularized regression is :

$$\mathbf{w}^*_{\mathbf{reg}} = (\mathbf{x}^{\mathbf{T}}\mathbf{x} + \lambda\mathbf{I})^{-\mathbf{1}}\mathbf{x}^{\mathbf{T}}\mathbf{y}$$

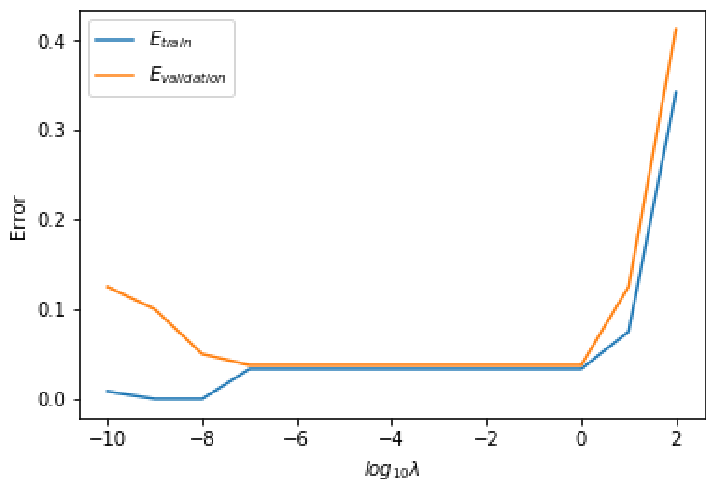
$\therefore$ Let $\widetilde{\mathbf{x}} = \sqrt{\lambda}\mathbf{I}$, $\widetilde{\mathbf{y}} = \mathbf{0}$ then $\mathbf{w}^* = \mathbf{w}^*_{\mathbf{reg}}$.

7.



In the left figure, we can observe that if we minimize E_{in} by choosing $log_{10}\lambda = 10$, then E_{in} is close enough to E_{out}. This result corresponds to the learning goal: $E_{in} \approx E_{out}$ and E_{in}, E_{out} are small enough.

8.



We can observer the $E_{validation} \leq E_{train}, \forall\, log_{10}\lambda$. It is quite intuitive since we split the original into two parts "*train* & *validation*" and tune the parameter to minimize the error in *train*. The model on *train* is tend to fit on the training set, the error on *validation* tend to be higher consequently.

9.

(a). $E_{loocv} = \frac{1}{N} \cdot (e_1 + e_2 + \cdots + e_N)$, where $N = 1126 + 1126 = 2252$

Suppose we select the validation instances "$x_1 : +, x_2 : +, \cdots, x_{1126} : +$" corresponding to "$e_1, e_2, \cdots, e_{1126}$" ; and select the validation instances "$x_{1127} : -, x_{1128} : -, \cdots, x_{2252} : -$" corresponding to "$e_{1127}, e_{1128}, \cdots, e_{2252}$".

$\mathcal{A}_{majority}$: Always predicts the majority class.

$$e_1\text{: 1 instance with + as validation; 2251 instances as train: } \begin{cases} 1125 \text{ with} + \\ 1126 \text{ with} - \Rightarrow \text{Majority} \end{cases}$$

$\therefore e_1 = 1$. Similarly, $e_2 = e_3 = \cdots = e_{1126} = 1$

$$e_{1127}\text{: 1 instance with - as validation; 2251 instances as train: } \begin{cases} 1126 \text{ with} + \Rightarrow \text{Majority} \\ 1125 \text{ with} - \end{cases}$$

$\therefore e_{1127} = 1$. Similarly, $e_{1128} = e_{1129} = \cdots = e_{2252} = 1$

Hence, $E_{loocv}(\mathcal{A}_{majority}) = \frac{1}{2252} \cdot (1 + 1 + \cdots + 1) = \frac{1}{2252} \cdot (2252) = 1$

$\mathcal{A}_{minority}$: Always predicts the minority class.

$$e_1\text{: 1 instance with + as validation; 2251 instances as train: } \begin{cases} 1125 \text{ with} + \Rightarrow \text{minority} \\ 1126 \text{ with} - \end{cases}$$

$\therefore e_1 = 0$. Similarly, $e_2 = e_3 = \cdots = e_{1126} = 0$

$$e_{1127}\text{: 1 instance with - as validation; 2251 instances as train: } \begin{cases} 1126 \text{ with} + \\ 1125 \text{ with} - \Rightarrow \text{minority} \end{cases}$$

$\therefore e_{1127} = 0$. Similarly, $e_{1128} = e_{1129} = \cdots = e_{2252} = 0$

Hence, $E_{loocv}(\mathcal{A}_{minority}) = \frac{1}{2252} \cdot (0 + 0 + \cdots + 0) = \frac{1}{2252} \cdot (0) = 0$

Therefore, we will choose $\mathcal{A}_{minority}$ based on E_{loocv}.

(b). We follow the same strategy of selecting instances as validation in (a). Suppose we have the instances $\{y_1, y_2, \cdots y_N\}$, let $\bar{y} = \sum_{i=1}^{N} \frac{y_i}{N}$.

$$e_1 = \left(y_1 - \frac{y_2 + y_3 + ... + y_N}{N-1}\right)^2 = \left[y_1 - \frac{(\bar{y} \cdot N - y_1)}{N-1}\right]^2 = \left(\frac{y_1 \cdot N - y_1 - \bar{y} \cdot N + y_1}{N-1}\right)^2 =$$

$$\left[\frac{N\left(y_1-\bar{y}\right)}{N-1}\right]^2.$$

$$e_2=\left(y_2-\frac{y_1+y_3+y_4+...+y_N}{N-1}\right)^2=\left[y_2-\frac{\left(\bar{y}\cdot N-y_2\right)}{N-1}\right]^2=\left[\frac{N\left(y_2-\bar{y}\right)}{N-1}\right]^2.$$

$$\vdots$$

$$e_N=\left[\frac{N\left(y_N-\bar{y}\right)}{N-1}\right]^2.$$

Therefore,

$$\begin{aligned}
E_{loocv} &= \frac{1}{N}\cdot\left(e_1+e_2+...+e_N\right) \\
&= \frac{1}{N}\cdot\left\{(\frac{N}{N-1})^2\left[(y_1-\bar{y})^2+(y_2-\bar{y})^2+...+(y_N-\bar{y})^2\right]\right\} \\
&= (\frac{N}{N-1})^2\cdot\frac{1}{N}\sum_{i=1}^{N}(y_i-\bar{y})^2=(\frac{N}{N-1})^2\cdot Var(y_n)
\end{aligned}$$

$\therefore$ the scale factor is $(\frac{N}{N-1})^2$