

Assignment 2 : Modeling Process Quality

1. The probability distribution of the discrete random variable x is $p(x) = kr^x$, $0 < r < 1$. Find the appropriate value of k . Find the mean and variance of x .

(a) The value of k .

let S be the sample space, then $P(S) = 1$

and let $X \equiv r.v.$ of this distribution

$$\because P(S) = P(x) = \sum_{x=0}^{\infty} kr^x = k \sum_{x=0}^{\infty} r^x = 1$$

by Geometric Progression, $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$

$$\therefore 1 = k \sum_{x=0}^{\infty} r^x = k \frac{1}{1-r}$$

so we can know that $k = 1 - r$ #

(b) The mean of x .

by the definition of this question, we can rewrite that $p(x) = (1-r)r^x$, $x = 0, 1, 2, \dots$

and by the formula of mean, $E(x) = \sum_{x=0}^{\infty} x p(x)$

$$\begin{aligned} \therefore E(x) &= \sum_{x=0}^{\infty} x (1-r) r^x \\ &= (1-r) \sum_{x=0}^{\infty} x r^x \\ &= r(1-r) \sum_{x=0}^{\infty} x r^{x-1} \end{aligned}$$

$$\text{since } f(x) = \sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$$

$$\text{by differentiate, we get } f'(x) = \sum_{x=0}^{\infty} x r^{x-1} = \frac{1}{(1-r)^2}$$

$$\begin{aligned} \text{so we can rewrite and know that } E(x) &= r(1-r) \frac{1}{(1-r)^2} \\ &= \frac{r}{1-r} \# \end{aligned}$$

(c) The variance of x .

$$\begin{aligned}\text{by the formula of variance, } Var(x) &= E(X^2) - E(X)^2 \\ &= E[X(X-1)] + E(X) - E(X)^2\end{aligned}$$

$$\begin{aligned}\text{and since } E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) p(x) \\ &= \sum_{x=0}^{\infty} x(x-1) (1-r) r^x \\ &= (1-r) \sum_{x=0}^{\infty} x(x-1) r^x \\ &= r^2(1-r) \sum_{x=0}^{\infty} x(x-1) r^{x-2} \\ &= r^2(1-r) \frac{2}{(1-r)^3} \\ &= \frac{2r^2}{(1-r)^2}\end{aligned}$$

$$\begin{aligned}\text{thus } Var(x) &= E[X(X-1)] + E(X) - E(X)^2 \\ &= \frac{2r^2}{(1-r)^2} + \frac{r}{1-r} - \left(\frac{r}{1-r}\right)^2 \\ &= \frac{2r^2}{(1-r)^2} + \frac{r(1-r)}{(1-r)^2} - \frac{r^2}{(1-r)^2} \\ &= \frac{r}{(1-r)^2} \# \end{aligned}$$

2. A random sample of 50 units is drawn from a production process every half hour. The fraction of nonconforming product manufactured is 0.02. What is the probability that $\hat{p} \leq 0.04$ if the fraction nonconforming really is 0.02?

$r.v.$ $X \equiv$ the number of nonconforming product

$$X \sim bin(n = 50, p = 0.02)$$

$$\because \hat{p} = \frac{x}{n} = \frac{x}{50} = 0.04$$

$$\therefore x = 2$$

$$P(\hat{p} \leq 0.04) = P(x \leq 2)$$

$$\begin{aligned}&= \sum_{x=0}^2 C_x^{50} 0.02^x (1-0.02)^{50-x} \\ &= 0.922 \# \end{aligned}$$

3. The output voltage of a power supply is normally distributed with mean 5V and standard deviation 0.02V. If the lower and upper specifications for voltage are 4.95V and 5.05V, respectively.

- (a) What is the probability that a power supply selected at random will conform to the specifications on voltage?

r.v. $X \equiv$ the output voltage of a power

$$X \sim N(\mu = 5, \sigma^2 = 0.02^2)$$

$$\begin{aligned} P(\text{conformed}) &= P(4.95 \leq x \leq 5.05) \\ &= P\left(\frac{4.95 - 5}{0.02} \leq Z \leq \frac{5.05 - 5}{0.02}\right) \\ &= 2\Phi(2.5) - 1 \\ &= 0.98758 \# \end{aligned}$$

- (b) How much would the process variability need to be reduced in order to have all but one out of 1000 units conform to the specifications?

$$\begin{aligned} P(\text{nonconformed}) &= 1 - P(4.95 \leq x \leq 5.05) \\ &= P\left(\frac{4.95 - 5}{\sigma} \leq Z \leq \frac{5.05 - 5}{\sigma}\right) \\ &= \frac{1}{1000} = 0.001 \\ P\left(Z \leq \frac{5.05 - 5}{\sigma}\right) &= 0.9995 = P(Z \leq 3.27) \\ \therefore \frac{0.005}{\sigma} &= 3.27 \\ \therefore \sigma &= 0.0153 \# \end{aligned}$$

4. A stock brokerage has four computers that are used for making trades on the New York Stock Exchange. The probability that a computer fails on any single day is 0.005. Failures occur independently. Any failed computers are repaired after the exchange closes, so each day can be considered an independent trial.

- (a) What is the probability that all four computers fail on one day?

r.v. $X \equiv$ the number that fails in 4 computer

$$X \sim \text{bin}(n = 4, p = 0.005)$$

$$f(x) = C_x^4 p^x (1 - p)^{4-x}$$

$$P(X = 4) = C_4^4 p^4 (1 - p)^0 = 0.005^4 = 6.25 \times 10^{-10} \#$$

- (b) What is the probability that at least one computer fails on a day?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.995^4 = 0.0199 \#$$

- (c) What is the mean number of days until a specific computer fails?

r.v. $Y \equiv$ the number that fails in 4 computer

$$Y \sim \text{geo}(p = 0.005)$$

$$f(y) = p(1 - p)^{y-1}$$

$$E(y) = \frac{1}{p} = \frac{1}{0.005} = 200 \text{ days} \#$$

5. The time to failure of a product is well described by the following probability distribution:

$f(x) = 0.1e^{-0.1x}, x \geq 0$. If time is measured in hours, what is the probability that a unit fails before 100 hours?

r.v. $X \equiv$ the time that a unit fails

$$X \sim \text{exp}(\beta = 10)$$

$$P(X \leq 100) = \int_0^{100} 0.1e^{-0.1x} dx = 0.99995 \#$$