Assignment 2 : Modeling Process Quality

1. The probability distribution of the discrete random variable x is $p(x) = kr^x,\ 0 < r < 1$. Find the appropriate value of k . Find the mean and variance of x.

   (a) The value of k.

   let S be the sample space, then $P(S) = 1$

   and let $X \equiv r.v.$ of this distribution

   $$\because P(S) = P(x) = \sum_{x=0}^{\infty} kr^x = k \sum_{0}^{\infty} r^x = 1$$

   by Geometric Progression, $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$

   $$\therefore\ 1 = k \sum_{0}^{\infty} r^x = k\,\frac{1}{1-r}$$

   so we can know that $k = 1 - r$ #

   (b) The mean of x.

   by the definition of this question, we can rewrite that $p(x) = (1-r)\,r^x, x = 0, 1, 2, ...$

   and by the formula of mean, $E(x) = \sum_{x=0}^{\infty} x\,p(x)$

   $$\begin{aligned}\therefore E(x) &= \sum_{x=0}^{\infty} x\,(1-r)\,r^x \\ &= (1-r)\sum_{x=0}^{\infty} x\,r^x \\ &= r(1-r)\sum_{x=0}^{\infty} x\,r^{x-1}\end{aligned}$$

   since $f(x) = \sum_{0}^{\infty} r^x = \frac{1}{1-r}$

   by differentiate, we get $f'(x) = \sum_{0}^{\infty} x\,r^{x-1} = \frac{1}{(1-r)^2}$

   so we can rewrite and know that
   $$\begin{aligned}E(x) &= r(1-r)\frac{1}{(1-r)^2} \\ &= \frac{r}{1-r}\ _{\#}\end{aligned}$$

(c) The variance of x.

by the formula of variance, $Var(x) = E(X^2) - E(X)^2$
$= E[X(X-1)] + E(X) - E(X)^2$

and since
$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\, p(x)$$
$$= \sum_{x=0}^{\infty} x(x-1)\,(1-r)\, r^x$$
$$= (1-r) \sum_{x=0}^{\infty} x(x-1)\, r^x$$
$$= r^2(1-r) \sum_{x=0}^{\infty} x(x-1)\, r^{x-2}$$
$$= r^2(1-r)\, \frac{2}{(1-r)^3}$$
$$= \frac{2r^2}{(1-r)^2}$$

thus
$$Var(x) = E[X(X-1)] + E(X) - E(X)^2$$
$$= \frac{2r^2}{(1-r)^2} + \frac{r}{1-r} - \left(\frac{r}{1-r}\right)^2$$
$$= \frac{2r^2}{(1-r)^2} + \frac{r(1-r)}{(1-r)^2} - \frac{r^2}{(1-r)^2}$$
$$= \frac{r}{(1-r)^2}\ _{\#}$$

2. A random sample of 50 units is drawn from a production process every half hour. The fraction of nonconforming product manufactured is 0.02. What is the probability that $\hat{p} \leq 0.04$ if the fraction nonconforming really is 0.02?

$r.v.\ X \equiv$ the number of nonconforming product

$X \sim bin(n = 50, p = 0.02)$

$\because\ \hat{p} = \frac{x}{n} = \frac{x}{50} = 0.04$

$\therefore\ x = 2$

$$P(\hat{p} \leq 0.04) = P(x \leq 2)$$
$$= \sum_{x=0}^{2} C_x^{50}\, 0.02^x\, (1-0.02)^{50-x}$$
$$= 0.922\ _{\#}$$

3. The output voltage of a power supply is normally distributed with mean 5V and standard deviation 0.02V. If the lower and upper specifications for voltage are 4.95V and 5.05V, respectively.

   (a) What is the probability that a power supply selected at random will conform to the specifications on voltage?

$$
\begin{aligned}
&r.v.\ X \equiv \text{the output voltage of a power} \\
&X \sim N(\mu = 5, \sigma^2 = 0.02^2) \\
&P(conformed) = P\left(4.95 \leq x \leq 5.05\right) \\
&\qquad = P\left(\frac{4.95-5}{0.02} \leq Z \leq \frac{5.05-5}{0.02}\right) \\
&\qquad = 1 - 2\,\Phi(-2.5) \\
&\qquad = 0.98758\ _{\#}
\end{aligned}
$$

   (b) How much would the process variability need to be reduced in order to have all but one out of 1000 units conform to the specifications?

$$
\begin{aligned}
&P(nonconformed) = 1 - P\left(4.95 \leq x \leq 5.05\right) \\
&\qquad = P\left(\frac{4.95-5}{\sigma} \leq Z \leq \frac{5.05-5}{\sigma}\right) \\
&\qquad = \frac{1}{1000} = 0.001 \\
&P\left(Z \leq \frac{5.05-5}{\sigma}\right) = 0.9995 = P\left(Z \leq 3.27\right) \\
&\because\ \frac{0.05}{\sigma} = 3.27 \\
&\therefore\ \sigma = 0.0153\ _{\#}
\end{aligned}
$$

4. A stock brokerage has four computers that are used for making trades on the New York Stock Exchange. The probability that a computer fails on any single day is 0.005. Failures occur independently. Any failed computers are repaired after the exchange closes, so each day can be considered an independent trial.

   (a) What is the probability that all four computers fail on one day?

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\begin{aligned}
&r.v.\ X \equiv \text{the number that fails in 4 computer} \\
&X \sim bin(n = 4, p = 0.005) \\
&f(x) = C_x^4\, p^x\, (1-p)^{4-x} \\
&P(X = 4) = C_4^4\, p^4\, (1-p)^0 = 0.005^4 = 6.25 \times 10^{-10} \ _{\#}
\end{aligned}
$$

   (b) What is the probability that at least one computer fails on a day?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.995^4 = 0.0199 \ _{\#}$$

   (c) What is the mean number of days until a specific computer fails?

$$
\begin{aligned}
&r.v.\ Y \equiv \text{the number that fails in 4 computer} \\
&Y \sim geo(p = 0.005) \\
&f(y) = p(1-p)^{y-1} \\
&E(y) = \frac{1}{p} = \frac{1}{0.005} = 200 \text{ days} \ _{\#}
\end{aligned}
$$

5. The time to failure of a product is well described by the following probability distribution: $f(x) = 0.1e^{-0.1x}, x \geq 0$. If time is measured in hours, what is the probability that a unit fails before 100 hours?

$$
\begin{aligned}
&r.v.\ X \equiv \text{the time that a unit fails} \\
&X \sim exp(\beta = 10) \\
&P(X \leq 100) = \int_0^{100} 0.1e^{-0.1x}\, dx = 0.99995 \ _{\#}
\end{aligned}
$$