ESO207A: Data Structures and Algorithms
Practice Set *6: MST* –

Problem 1. *A simple observation.* A nuitwit argues that by increasing the edge-weights by any value Δ does not change the minimum spanning tree. Prove or give a counterexample.

Problem 2. Give an efficient algorithm $O((|V| + |E|)\log|V|)$ to compute a maximum spanning tree of an undirected graph. Prove the correctness and time complexity of the algorithm. (*Hint*: Prove an analog of the "Cut-Property".)

Problem 3. Consider an undirected connected graph, all of whose edge weights are distinct. Prove that it has a unique minimum spanning tree.

Problem 4. State with proof or counterexample as applicable, whether each of the following statements is true or false. Assume that $G = (V, E)$ is a weighted, undirected, connected graph.

1. If G has some cycle with a unique heaviest edge e, then, e cannot be part of any MST.
2. Let e be any edge of minimum weight in G. Then e must be part of some MST.
3. If the lightest edge of the graph is unique, then it must be part of every MST.
4. Suppose e is an edge of some MST. Then it must be the minimum cut edge of some cut of G.
5. Prim's algorithm works correctly when there are negative weight edges.
6. The shortest path tree computed by Dijkstra's algorithm is always an MST.
7. If G contains a path from s to t consisting of edges all of whose weights $< r$, then every MST of G must contain a path from s to t consisting of edges, all of whose weights $< r$.

Problem 5. Consider the disjoint sets data structure. Give a sequence of (in total) m *union* and *find* operations on n elements that take $\Omega(m \log n)$ time.

Problem 6. Give a linear-time algorithm that takes as input a tree and determines whether it has a *perfect matching*, namely a set of edges that touches each node exactly once.

Problem 7. A *feedback edge set* of an undirected graph $G = (V, E)$ is a subset of edges $E' \subset E$ that intersects every cycle of the graph. Thus removing the edges in E' from the graph will render the graph acyclic. Given an undirected weighted graph $G = (V, E)$ with positive edge weights w_e, output a feedback edge set $E' \subset E$ of minimum total weight $\sum_{e \in E'} w_e$.