P19 #1

Since $\Delta u = 0$ in the weak sense, u is weakly harmonic. Recall that a weakly harmonic function is also harmonic. By the Mean Value Property, for each $x \in \mathbb{R}^n$,

$$u(x) = \frac{1}{|B(x,1)|} \int_{B(x,1)} u(y)\, dy$$

and hence

$$|u(x)| \le \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)|\, dy < \frac{C}{|B(x,1)|}.$$

As $|B(x,1)|$ does not depend on x, it follows that u is bounded on $\mathbb{R}^n$ and hence by Liouville's Theorem, u is constant. #

F99 #2 We have

$$f(x) = (\Delta - aI)\int K_a(x-y) f(y)\, dy$$
$$f(x) = (\Delta - bI)\int K_b(x-y) f(y)\, dy$$

Note that

$$(\Delta - aI)(\Delta - bI)\int (c_1 K_a(x-y) + c_2 K_b(x-y)) f(y)\, dy$$
$$= c_1(\Delta - bI) f(x) + c_2(\Delta - aI) f(x)$$
$$= (c_1 + c_2)\Delta f(x) - c_1 b f(x) - c_2 a f(x) \qquad (*)$$

Taking $c_1 = (a-b)^{-1}$ and $c_2 = (b-a)^{-1}$, we have that $(*) = f(x)$.

Thus $(a-b)^{-1}K_a + (b-a)^{-1}K_b$ is a fundamental solution of $(\Delta - aI)(\Delta - bI)$.

We have $\Delta^2 - \Delta = \Delta(\Delta - I)$. So the fundamental solution is $-K_0 + K_1$ where K_0 is the fundamental solution for Δ and K_1 is the fundamental sol. for $\Delta - I$.

When $n = 3$, $K_0 = \frac{1}{4\pi|x|}$.

~~$K_1 = \frac{e^{-|x|}}{4\pi|x|}$~~ $\longrightarrow$ Fundamental solution of Helmholtz eqn. in $\mathbb{R}^3$.

Bessel Potential : **Formally** we solve

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$$\Delta u - u = \delta_0 . \rightarrow -\Delta u + u = -\delta_0$$

Take the Fourier transform of both sides to get

$$(1+|y|^2)\hat{u} = \widehat{-\delta_0}$$
$$\rightarrow u = \left(\frac{\widehat{-\delta_0}}{(1+|y|^2)}\right)^{\vee} = \frac{1}{(2\pi)^{3/2}}(-\delta_0 * B) = -\frac{1}{(2\pi)^{3/2}} B(x)$$

where $\hat{B} = \frac{1}{1+|y|^2}$. Then as $\frac{1}{1+|y|^2} = \int_0^\infty e^{-t(1+|y|^2)}\, dt$,

$$B = \left(\frac{1}{1+|y|^2}\right)^{\vee} = \frac{1}{(2\pi)^{3/2}}\int_0^\infty e^{-t}\int_{\mathbb{R}^3} e^{ix\cdot y - t|y|^2}\, dy\, dt.$$

$$\rightarrow u = -\frac{1}{(2\pi)^3}\int_0^\infty e^{-t}\int_{\mathbb{R}^3} e^{ix\cdot y - t|y|^2}\, dy\, dt. := K_1$$

FTT #3:

Claim: $\lambda = \min_{\substack{w \\ \partial w/\partial \nu = 0 \text{ on } \partial U}} \frac{\langle Lw, w\rangle}{\langle w, w\rangle}$ is the smallest eigenvalue of L.

Pf: We first show $\min_w \frac{\langle Lw,w\rangle}{\langle w,w\rangle}$ is indeed an eigenvalue. Let u be the function with $\partial u/\partial \nu = 0$ on ∂U s.t. $\frac{\langle Lw,w\rangle}{\langle w,w\rangle}$ attains its minimum. ~~Then for all v with $\frac{\partial v}{\partial \nu} = 0$ on ∂U~~

$$\cancel{0 = \frac{d}{d\varepsilon} \frac{\langle Lu + \varepsilon Lv, u+\varepsilon v\rangle}{\langle u+\varepsilon v, u+\varepsilon v\rangle}\Big|_{\varepsilon=0}}$$

~~Thus~~ Let $F(w) := \frac{\langle Lw,w\rangle}{\langle w,w\rangle}$. Then for all v with $\frac{\partial v}{\partial \nu} = 0$ on ∂U,

$$0 = \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\frac{\langle Lu+\varepsilon Lv, u+\varepsilon v\rangle}{\langle u+\varepsilon v, u+\varepsilon v\rangle} - \frac{\langle Lu,u\rangle}{\langle u,u\rangle}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\frac{\langle Lu,u\rangle + \varepsilon\langle Lu,v\rangle + \varepsilon\langle Lv,u\rangle + \varepsilon^2\langle Lv,v\rangle}{\langle u,u\rangle + 2\varepsilon\langle u,v\rangle + \varepsilon^2\langle v,v\rangle} - \frac{\langle Lu,u\rangle}{\langle u,u\rangle}\right]$$

$$\rightarrow 0 = \langle Lu,v\rangle\langle u,u\rangle + \langle Lv,u\rangle\langle u,u\rangle - 2\langle Lu,u\rangle\langle u,v\rangle.$$

Since L is self-adjoint, we have

$$0 = 2\langle Lu,v\rangle\langle u,u\rangle - 2\langle Lu,u\rangle\langle u,v\rangle$$

and hence

$$\int (Lu)v\,dx = \int \frac{\langle Lu,u\rangle}{\langle u,u\rangle} u\, v\, dx \quad \forall v.$$

Thus $Lu = \frac{\langle Lu,u\rangle}{\langle u,u\rangle} u$. If μ was another eigenvalue of L, w/ corr. eigenfunction w, then

$$\lambda \le \frac{\langle Lw,w\rangle}{\langle w,w\rangle} = \frac{\mu\langle w,w\rangle}{\langle w,w\rangle} = \mu. \quad \#.$$

Self-adjointness of L is proven as follows:

$$\int (Lu)v\,dx = -\int \sum_{i,j} \partial_{x_j}(a^{ij}(x)\partial_{x_i}u)v\,dx = -\sum_{i,j}\int \partial_{x_j}(a^{ij}(x)\partial_{x_i}u)v\,dx$$

since $\frac{\partial u}{\partial \nu} = 0$ $= -\sum_{i,j}\int a^{ij}(x)\partial_{x_i}u\,\partial_{x_j}v\,dx \overset{\text{since } \frac{\partial v}{\partial \nu}=0}{=} -\sum_{i,j}\int \partial_{x_i}(a^{ij}(x)\partial_{x_j}v)u\,dx = \int (Lv)u\,dx.$

F99 #3 cont:

Suppose all eigenvalues of L are ≥ 0. Then $\lambda \geq 0$. Thus by the claim,

$$\min_{\omega : \frac{\partial \omega}{\partial \nu} = 0 \text{ on } \partial U} \frac{\langle L\omega, \omega \rangle}{\langle \omega, \omega \rangle} \geq 0.$$

Fix a u with $\frac{\partial u}{\partial \nu} = 0$ and consider $u + a$ with a to be chosen later. We have

$$\langle L(u+a), u+a \rangle = \langle Lu + c(x) a, u+a \rangle$$
$$= \langle Lu, u \rangle + a \int Lu \, dx + a \int c(x) u \, dx + a^2 \int c(x) dx. \quad (*)$$

Since $\int c(x) dx \leq 0$ and $c(x) \neq 0$ for some $x \in U$, $c(x)$ smooth, choose a sufficiently large, we can make $(*) < 0$. Therefore

$$0 \leq \min_{\substack{\omega \\ \frac{\partial \omega}{\partial \nu} = 0 \text{ on } \partial U}} \frac{\langle L\omega, \omega \rangle}{\langle \omega, \omega \rangle} \leq \frac{\langle L(u+a), u+a \rangle}{\langle u+a, u+a \rangle} < 0,$$

a contradiction. Therefore $\exists$ a negative eigenvalue. #

F99 #4:

We solve

$$\begin{cases} u_t + a(x) u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

by method of characteristics. We have

$$F(p,q,z,x,t) = q + a(x)p.$$

$$\dot{t}(s) = 1 \qquad t(0) = 0$$

$$\dot{x}(s) = a(x) \qquad x(0) = x_0.$$

$$\dot{z}(s) = 0 \qquad z(0) = f(x_0).$$

Let $a(x) := x^2 + 1$. Then

$$\begin{matrix} \dot{x} = x^2 + 1 \\ x(0) = x_0 \end{matrix} \longrightarrow \tan^{-1}(x) = s + \tan^{-1}(x_0)$$

Since $t = s$, the ~~proof~~ characteristic curves are given by

$$\tan^{-1} x = t + \tan^{-1} x_0$$

Since $\tan^{-1} x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for any choice of x_0, the characteristic curves do not propogate beyond time $t = \pi$. Thus the values of $u(x,t)$ for $t > \pi$ do not depend on the values $u(x,0)$. Then the solution of the Cauchy problem is not unique.

#

F99 #5:

We have

Then $$\hat{u}(m,n,t) = \int_{[0,2\pi]^2} u(x,y,t) e^{-i(xm+ny)}\,dx\,dy.$$

$$\hat{u}_t(m,n,t) = (-\varepsilon)(-m^2-n^2)\hat{u}(m,n,t)$$
$$-(m^4+2m^2n^2+n^4)\hat{u}(m,n,t).$$

$$\hat{u}_t(m,n,t) = \varepsilon(m^2+n^2)\hat{u}(m,n,t) - (m^2+n^2)^2\hat{u}(m,n,t)$$

$$\hat{u}_t(m,n,t) = (m^2+n^2)[\cancel{\varepsilon(m^2+n^2)-\ldots}\ \varepsilon-(m^2+n^2)]\hat{u}(m,n,t)$$

$$\hat{u}(m,n,t) = e^{[(m^2+n^2)(\varepsilon-(m^2+n^2))]t}\hat{u}(m,n,0)$$

Therefore

$$u(x,y,t) = \sum_{m,n\in\mathbb{Z}} e^{(m^2+n^2)(\varepsilon-m^2-n^2)t}\hat{u}(m,n,0)e^{i(mx+ny)}.$$

i) A solution is e^{t+ix}. Indeed,

$$\partial_t(e^{t+ix}) = e^{t+ix}$$

$$\Delta u = \partial_{xx}(e^{t+ix}) = e^{t+ix}(-1)$$

$$\Delta^2 u = e^{t+ix}.$$

Therefore

$$-2\Delta u - \Delta^2 u = 2e^{t+ix} - e^{t+ix} = e^{t+ix}.$$

F99 #5 cont:

ii) We need

$$\varepsilon - m^2 - n^2 < 0 \quad \forall\, m, n.$$

$$\varepsilon < m^2 + n^2 .$$

Therefore take $\varepsilon_0 = 1$. #

F99 #6 :

a) We have

deriv. wrt y

$$\partial_t \rho = -s\rho' \qquad \partial_t u = -su'$$
$$\partial_x u = u' \qquad \partial_x(\rho u) = \rho_x u + \rho u_x = \rho' u + \rho u'.$$

Thus

$$-s\rho' + u' = 0. \longrightarrow u' = s\rho' \qquad \rho' = \tfrac{1}{s}u'$$
$$-su' + \rho' u + \rho u' = u'' \qquad u = s\rho + c_1$$

$$\rightarrow -su' + \tfrac{1}{s}u'u + (\tfrac{u}{s} - c_1)u' = u''$$
$$-su' + \tfrac{2}{s}u'u - c_1 u' = u''$$

Therefore

$$u'' = -(s + c_1)u' + \tfrac{1}{s}(u^2)'.$$

Thus

$$u' = -(s + c_1)u + \tfrac{1}{s}u^2 + c_2. \qquad (*)$$

b) Note $(*)$ is of the form $u' + Au^2 + Bu = C$ with $A = -1/s$, $B = s + c_1$, $C = c_2$. One way to solve $(*)$ is to observe $(*)$ is separable, so,

$$\int \frac{1}{\frac{1}{s}u^2 - (s + c_1)u + c_2}\, du = \int dt$$

$$\int \frac{s}{u^2 - s(s + c_1)u + sc_2}\, du = t + \tilde{c}_1$$

But the easiest way is to observe that if we guess the solution is of the form

$$u(y) = u_0 + u_1 \tanh(\alpha y + y_0),$$

F99 #6 cont:

$$u' = \alpha u_1 \operatorname{sech}^2(\alpha y + y_0).$$

Thus

$$u' + Au^2 + Bu = C$$

$$\Downarrow$$

$$\alpha u_1 \operatorname{sech}^2(\alpha y + y_0) + Au_0^2 + 2Au_0u_1 \tanh(\alpha y + y_0) + Au_1^2 \tanh(\alpha y + y_0) + Bu_0 + Bu_1 \tanh(\alpha y + y_0) = C.$$

Our ansatz is a solution if we choose u_0, u_1, α s.t.

$$\begin{aligned} \alpha u_1 &= Au_1^2 & & u_1 = \alpha/A \\ 2Au_0u_1 + Bu_1 &= 0 & \longrightarrow \quad & u_0 = -B/2A. \\ Au_1^2 + Au_0^2 + Bu_0 &= C & & \alpha = (AC + B^2/2)^{1/2}. \end{aligned}$$

Thus with ~~[illegible]~~ nice choices of c_1, c_2, (*) has solutions of the form $u(y) = u_0 + u_1 \tanh(\alpha y + y_0)$.

F99 #7.

i. We want to find ϕ s.t. $\langle Lu, v\rangle_\phi = \langle u, Lv\rangle_\phi$ for u, v satisfying $u(0)=u(1)=0$ and $v(0)=v(1)=0$.

We have

$$\langle Lu, v\rangle_\phi = \int_0^1 (u''+2u')v\phi\, dx = \int_0^1 u''v\phi + 2u'v\phi\, dx$$

$$= -\int_0^1 u'(v\phi)'\,dx - 2\int_0^1 u(v\phi)'\,dx$$

$$= \int_0^1 u(v\phi)''\,dx - 2\int_0^1 u(v\phi)'\,dx$$

$$= \int_0^1 u(v''\phi + 2v'\phi' + v\phi'') - 2u(v'\phi + v\phi')\,dx$$

$$= \int_0^1 u(v''\phi + 2v'\phi' + v\phi'' - 2v'\phi - 2v\phi')\,dx$$

$$= \int_0^1 u(v''\phi + 2v'(\phi'-\phi) + v(\phi''-2\phi'))\,dx$$

$$= \int_0^1 u\phi\left(v'' + 2v'\left(\frac{\phi'}{\phi}-1\right) + v\left(\frac{\phi''-2\phi'}{\phi}\right)\right)dx$$

Thus we would have $\langle Lu, v\rangle_\phi = \langle u, Lv\rangle_\phi$ if

$$\frac{\phi'}{\phi} - 1 = 1 \qquad\qquad \phi' = 2\phi$$

$$\frac{\phi''-2\phi'}{\phi} = 0 \quad \longrightarrow \quad \phi'' = 2\phi'.$$

Taking $\phi = e^{2x}$ gives such a ϕ.

ii. To show $L + aI$ is invertible, we will show that $\ker(L+aI)$ is trivial. Let $u \in \ker(L+aI)$. Then

~~$0 = \langle (L+aI)u, u\rangle = \int_0^1 (Lu + au)u\phi\,dx = \int_0^1 (u''+2u'+au)u\phi\,dx$~~

~~$= -\int_0^1 u'(u\phi)' + 2u'u\phi + au^2\phi\,dx$~~

~~$= -\int_0^1 u'(ue^{2x})' - 2u(ue^{2x})' + au^2 e^{2x}\,dx$~~

F99 #7 cont.

$$0 = \langle (L + aI)u, u \rangle = \int_0^1 (Lu + au) u \phi \, dx = \int_0^1 (u'' + 2u' + au) u \phi \, dx$$

$$= \int_0^1 (u'' + 2u' + au) u e^{2x} \, dx \cancel{= \int_0^1 u (ue^{2x})'' - 2u(ue^{2x})' + au^2 e^{2x} dx}$$

$$= \int_0^1 -u'(ue^{2x})' + 2u'ue^{2x} + au^2 e^{2x} \, dx$$

$$= \int_0^1 -u'(u'e^{2x} + 2e^{2x}u) + 2u'ue^{2x} + au^2 e^{2x} \, dx$$

$$= \int_0^1 -(u')^2 e^{2x} + au^2 e^{2x} \, dx$$

$$= \int_0^1 e^{2x} [-(u')^2 + au^2] \, dx$$

Since $a < 0$, if $u \not\equiv 0$, $\int$ is < 0 which would lead to a contradiction. Therefore we must have $u \equiv 0$. Thus if $a < 0$, $L + aI$ is invertible.

iii: Take $a = -1$, $u = e^{-x}$.

F99 #8:

We look for a radial symmetric u.

a) We want to solve $\begin{cases} -\Delta u = 1 & \text{on } B_{R(t)}(0) \\ u = 0 & \text{on } |x| = R(t) \end{cases}$

Since want to find radial u, we solve

$$-\left(u_{rr} + \frac{1}{r}u_r\right) = 1$$
$$r u_{rr} + u_r = -r$$
$$(r u_r)' = -r$$
$$r u_r = -\frac{r^2}{2} + c_1$$
$$u_r = -\frac{r}{2} + \frac{c_1}{r}$$
$$u = -\frac{1}{4}r^2 + c_1 \log r + c_2.$$

Since we want u to be defined on the origin, $c_1 = 0$. Since ~~also~~ $u = 0$ when $r = R(t)$, we have

$$u = -\frac{1}{4}r^2 + \frac{1}{4}R(t)^2 = -\frac{1}{4}|x|^2 + \frac{1}{4}R(t)^2.$$

b) Using $\frac{dR}{dt} = -u_r|_{r=R}$, we have

$$\frac{dR}{dt} = -u_r|_{r=R} = \frac{1}{2}R.$$

Therefore as $R(0) = R_0$,

$$R(t) = e^{\frac{1}{2}t}R_0.$$