Pb #1

We want to solve

$$u_{x_2} + u u_{x_1} = 3u$$
$$u(x_1, 0) = u_0(x_1)$$

Let

$$F(p, z, x) = p_2 - z p_1 - 3z = 0$$

Then

$$D_p F = (-z, 1)$$
$$\cancel{D_x} \; D_z F = -p_1$$
$$D_x F = 0.$$

$$\longrightarrow \quad \dot{p} = -D_x F - D_z F p = p_1 (p_1, p_2) = (p_1^2, p_1 p_2)$$

$$\dot{z} = D_p F \cdot p = -z p_1 + p_2 = 3z$$
$$\dot{x} = D_p F = (-z, 1).$$

$$x_1(0) = x_1(0)$$
$$x_2(0) = 0$$
$$z(0) = u_0(x_1(0)).$$

$$\longrightarrow \quad \dot{z} = 3z \longrightarrow z(s) = z(0) e^{3s}$$
$$\cancel{\dot{x}_1(s) = -z(0) e^{3s} \rightarrow x_1(s) = -z(0) \tfrac{1}{3} e^{3s}}$$
$$\cancel{\dot{x} = -z \rightarrow \ddot{x} = -\dot{z} \rightarrow}$$
$$\dot{x}_2(s) = 1 \longrightarrow x_2(s) = s.$$

Since $\dot{x}_1(s) = -z(s) = -z(0) e^{3s} = u_0(x_1(0)) e^{3s}$

$$x_1(s) = -u_0(x_1(0)) \tfrac{1}{3} e^{3s} \cancel{C} + C$$

$$\cancel{\Longrightarrow \frac{3x_1(0)}{-u_0(x_1(0))} = C} \quad \rightarrow \quad x_1(0) = -u_0(x_1(0)) \tfrac{1}{3} + C$$
$$C = x_1(0) + \tfrac{1}{3} u_0(x_1(0))$$

$$\longrightarrow \quad x_1(s) = -u_0(x_1(0)) \tfrac{1}{3} e^{3s} + x_1(0) + \tfrac{1}{3} u_0(x_1(0))$$

F01 #1
cont.

If we could write

$$x_1(0) = f(x_1(s), s)$$

for some f, then

$$z(s) = -u_0(x_1(0))e^{3s}$$

$$\cancel{= -u_0(f(x_1(s)))}$$

$$= -u_0(f(x_1(s), s))e^{3s}$$

$$= -u_0(f(x_1(s), x_2(s)))e^{3x_2(s)}.$$

Thus

$$u(x,t) = -u_0(f(x,t))e^{3t}$$

where $f(x,t)$ solves

$$x = -u_0(f)\cdot\frac{1}{3}e^{3t} + f + \frac{1}{3}u_0(f)$$

F01 #2: Notice that

$$Lu = \frac{1}{e^{2x}}[e^{2x}u']' + \alpha(x)u.$$

Thus L is a Sturm-Liouville operator ~~which~~ thus is self-adjoint wrt the inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)e^{2x}\,dx$.

a) We have

$$\int_0^1 (Lu)ve^{2x}\,dx - \int_0^1 u(Lv)e^{2x}\,dx$$

$$= \int_0^1 [e^{2x}u']'v + \alpha uve^{2x} - u[e^{2x}v']' - \alpha uve^{2x}\,dx$$

$$= \int_0^1 [e^{2x}u']'v - u[e^{2x}v']'\,dx$$

$$= e^{2x}u'v\Big]_{x=0}^{1} - \int_0^1 e^{2x}u'v' - e^{2x}v'u\Big]_{x=0}^{1} + \int_0^1 e^{2x}u'v'\,dx$$

$$= 0.$$

Therefore take $\phi(x) = e^{2x}$.

b) Suppose all eigenvalues of L are ≤ 0 (in the sense of $Lu = \lambda u$). ~~Then~~ Then the least negative eigenvalue is given by

$$\max_u \frac{\langle u, Lu\rangle}{\langle u,u\rangle} \leq 0. \quad (\text{since } \overset{p(x)=w(x)}{e^{2x}} > 0) \quad (*)$$

where u ranges over all functions w/ $\int_0^1 |u(x)|^2 e^{2x}\,dx < \infty$, $u'(0)=0$, $u'(1)=0$.

For a constant c,

$$\langle c, Lc\rangle = \int_0^1 c\cdot\alpha(x)ce^{2x}\,dx = c^2\int_0^1 \alpha(x)e^{2x}\,dx$$

$$\geq c^2\int_0^1 \alpha(x)\,dx.$$

Note that $(*)$ implies $\max\limits_{\substack{u \\ u'(0)=u'(1)=0}} \langle u, Lu\rangle \leq 0$.

F01#2 cont:

But then
$$\langle c, Lc\rangle > 0$$
for some c chosen sufficiently large. Therefore L must have a positive ~~eigenvalue~~ eigenvalue.

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Note on Sturm-Liouville:

need $p, w > 0$

~~We say~~ The eigenvalues in most sources are given by
$$[p(x)y']' + q(x)y = -\lambda w(x) y$$
Then $\lambda_1 < \lambda_2 < \lambda_3 < \dots, \lambda_n \to \infty$ and the smallest eigenvalue ~~(quickly)~~ λ_1 (not $-\lambda_1'$) is given by
$$\min_u \frac{\langle u, Lu\rangle}{\langle u,u\rangle} \qquad Lu = [pu']' + qu.$$
$$= -\max_u \frac{\langle u, Lu\rangle}{\langle u,u\rangle}.$$
However for our eigenvalues, we want
$$[e^{2x}u']' + \alpha(x)e^{2x}u = \mu e^{2x}u. \qquad L = [e^{2x}u']' + \alpha(x)e^{2x}u$$
Thus
$$\mu_1 > \mu_2 > \mu_3 > \dots \to -\infty$$
and the largest eigenvalue μ_1 is given by $\max_u \frac{\langle u, Lu\rangle}{\langle u,u\rangle}$.

~~The~~ Theorem: (Sturm =

Sturm-Liouville

Sturm-Liouville operator: $L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$.

S-L eigenvalue problem:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = -\lambda w(x) u. \qquad (*)$$

$$\boxed{\text{Assume } p(x), w(x) > 0}$$

① Then eigenvalues are increasing

$$\lambda_1 < \lambda_2 < \dots\dots, \lambda_n \longrightarrow \infty$$

② Eigenfunctions corr. to different eigenvalues are orthogonal w.r.t. to $w(x)$.

③ $f \in L^2_w[a,b]$, $\quad f = \sum_{n\geq 1} c_n \phi_n$, $\quad c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$.

④ $$\lambda_1 = \min_{\substack{u \\ \text{bdry conditions}}} -\frac{\langle u, Lu\rangle}{\langle u, u\rangle}.$$

comes from the $-$ sign in $(*)$.

FDL 2i: Let $\phi = e^{2x}$. We note that $\phi' = 2\phi$, $\phi'' = 2\phi'$. We have

$$\langle Lu, v\rangle_\phi = \langle u'' + 2u' + \alpha u, v\rangle_\phi = \int_0^1 u''v\phi + 2u'v\phi + \alpha uv\phi \, dx$$

$$= \int_0^1 -u'(v\phi)' + 2u'v\phi + \alpha uv\phi \, dx$$

$$= -u(v\phi)'\Big]_{x=0}^1 + \int_0^1 u(v\phi)'' \, dx + 2uv\phi\Big]_{x=0}^1 - \int_0^1 2u(v\phi)' dx$$

since $\phi' = 2\phi$

$$= \cancel{-uv\phi'\Big]_{x=0}^1} + \int_0^1 u(v''\phi + 2v'\phi' + v\phi'') \, dx$$

$$\cancel{+ 2uv\phi\Big]_{x=0}^1} - \int_0^1 2u(v'\phi + v\phi') \, dx + \int_0^1 \alpha uv\phi \, dx$$

$$= \int_0^1 uv''\phi + uv\phi'' + 2uv'\phi' - 2uv'\phi - 2uv\phi' + \alpha uv\phi \, dx$$

$$= \int_0^1 u\phi\left(v'' + v\cancel{\frac{\phi''}{\phi}} + 2v'\cancel{\frac{\phi'}{\phi}} - \cancel{2v'} - \cancel{2v\frac{\phi'}{\phi}} + \alpha v\right) dx$$

$\frac{\phi'}{\phi} - 1 = 1$

$$= \int_0^1 u\phi(v'' + 2v' + \alpha v) \, dx = \langle u, Lv\rangle_\phi .$$

F01 #3:

i. In $\mathbb{R}^3$,

$$\Delta u = 0 \longleftrightarrow u_{rr} + \frac{2}{r} u_r = 0.$$

$$\to r^2\left(u_{rr} + \frac{2}{r} u_r\right) = 0.$$

$$r^2 u_{rr} + 2 r u_r = 0.$$

$$(r^2 u_r)' = 0.$$

$$u_r = \frac{c_1}{r^2}.$$

Thus

$$u(r,t) = -\frac{c_1}{r} + c_2.$$

Since $u(R_1) = 1$, $u(R(t)) = 0$,

$$0 = \frac{-c_1}{R(t)} + c_2, \quad 1 = \frac{-c_1}{R_1} + c_2$$

Then

$$1 = \frac{-c_1}{R_1} + \frac{c_1}{R(t)} = c_1\left(\frac{1}{R(t)} - \frac{1}{R_1}\right)$$

$$c_1 = \frac{R(t) R_1}{R_1 - R(t)}.$$

$$c_2 = \frac{c_1}{R(t)} = \frac{R_1}{R_1 - R(t)}.$$

Thus

$$u(r,t) = -\frac{1}{r}\left(\frac{R(t) R_1}{R_1 - R(t)}\right) + \frac{R_1}{R_1 - R(t)}.$$

ii. We have

$$u_r = \frac{c_1}{r^2} \longrightarrow -u_r(r=R) = \frac{c_1}{R^2}.$$

Thus

$$\frac{dR}{dt} = \frac{c_1}{R(t)^2} = \frac{R_1}{R(t)(R_1 - R(t))}$$

w/ $R(0) = R_0$.

F01#4

The ODE can be rewritten as

$$x' = y$$
$$y' = x(1-x).$$

This system is Hamiltonian (as $\frac{\partial}{\partial x} y + \frac{\partial}{\partial y} x(1-x) = 0$)
and so all stationary pts are either centers or saddles.

The conserved energy of a Hamiltonian system is found by finding an $H(x,y)$ s.t.

$$x' = \frac{\partial H}{\partial y}$$
$$y' = -\frac{\partial H}{\partial x}$$

here

conserved energy.

$$H(x,y) := \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$

Then

$$\frac{d}{dt} H(x,y) = yy' - xx' + x^2x' = yx(1-x) - xy + x^2y = 0.$$

The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$

The stationary pts are $(0,0)$ and $(1,0)$.

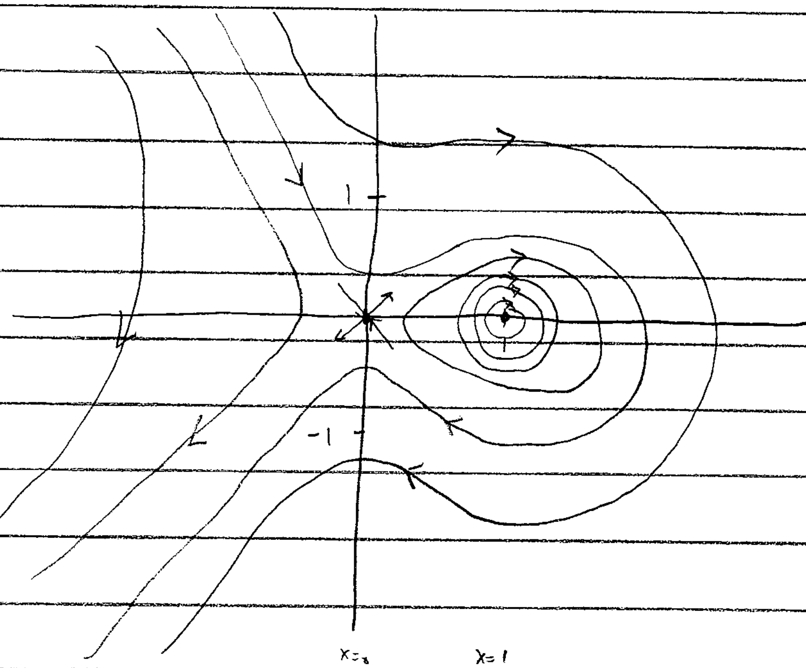
$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow$$ $(0,0)$ is a saddle
eigenvalues $1, -1$
eigenvectors $\binom{1}{1}$ $\binom{1}{-1}$

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow$$ $(1,0)$ is a center
(since eigvl. are $\pm i$ and system is Hamiltonian)

F01 #4 cont:

Nullclines: $y = 0$ $\left(\frac{dy}{dx} = \infty \quad | \right)$

$x = 0,\ x = 1$ $\left(\frac{dy}{dx} = 0 \quad — \right)$



F01 #5:

a) The quantities $f_t + h_t + f_x$ and $g_t + h_t - g_x$ are conserved since

$$f_{tt} + h_{tt} + f_{xt} = (h^2 - fg)_t - (h^2 - fg)_t = 0.$$
$$g_{tt} + h_{tt} - g_{xt} = (h^2 - fg)_t - (h^2 - fg)_t = 0.$$

b) We have

$$f_t, g_t, h_t = -sf', -sg', -sh'$$
$$f_x, g_x, h_x = f', g', h'$$

Thus we have

$$\begin{cases} -sf' + f' = h^2 - fg \\ -sg' - g' = h^2 - fg \\ -sh' = -(h^2 - fg). \end{cases}$$

c) We have

$$(1-s)f' = -sh' \longrightarrow (1-s)f = -sh + c_1$$
$$(-1-s)g' = -sh' \longrightarrow (-1-s)g = -sh + c_2.$$

Thus

$$-sh' = -h^2 + fg$$
$$= -h^2 + \frac{1}{1-s}(-sh + c_1)\frac{1}{-1-s}(-sh + c_2)$$
$$= -h^2 - \frac{1}{1-s^2}\left[s^2h^2 + \tilde{c}_1 h + \tilde{c}_2\right]$$

d) Thus ODE is of the form

$$Au' + Bu^2 + Cu = D.$$

$\frac{d}{dx}\tanh = \operatorname{sech}^2$

The solution is similar to F99 #6.

$\cosh^2 - \sinh^2 = 1.$

$\operatorname{sech}^2 = 1 - \tanh^2$

Fol #6:

We have $F(p,q,z,x,t) = q - zp - 3z$. Then

$D_{\vec{x}}F = (0,0)$ $\quad\quad \dot{\vec{p}} = -D_{\vec{x}}F - D_zF\vec{p} = (p+3)(-z,1)$

$D_{\vec{p}}F = (-z,1)$ $\longrightarrow$ $\dot{z} = D_{\vec{p}}F \cdot \vec{p} = -zp + q = 3z.$

$D_zF = -p-3$ $\quad\quad \dot{\vec{x}} = D_{\vec{p}}F = (-z,1)$

Thus

$\dot{x}(s) = -z \qquad x(0) = x_0$

$\dot{t}(s) = 1 \qquad t(0) = 0.$

$\dot{p}(s) = -z(p+3) \qquad p(0) = u_0'(x_0)$

$\dot{q}(s) = p+3. \qquad q(0) = u_0(x_0)u_0'(x_0) + 3u_0(x_0).$

$\dot{z}(s) = 3z \qquad z(0) = u_0(x_0)$

We have $t(s) = s$, $z(s) = u_0(x_0)e^{3s}$,

$$\dot{x}(s) = -u_0(x_0)e^{3s} \longrightarrow x(s) = -\frac{u_0(x_0)}{3}e^{3s} + x_0 + \frac{u_0(x_0)}{3}.$$

Thus

$$u(x,t) = u_0(x_0)e^{3t}$$

where x_0 satisfies

$$x = \frac{u_0(x_0)}{3}(1-e^{3t}) + x_0.$$

FCI #7: We want solutions of the form $u(x,t) = e^{\lambda t} v(x)$.
Then

$$u_t = \lambda e^{\lambda t} v(x)$$
$$u_{xx} = e^{\lambda t} v''(x)$$

$$\rightarrow \quad u_t = u_{xx} + c(x) u$$
$$\lambda \cancel{e^{\lambda t}} v(x) = \cancel{e^{\lambda t}} v''(x) + c(x) \cancel{e^{\lambda t}} v(x)$$

$$\rightarrow \quad v''(x) = (\lambda - c(x)) v(x).$$

If $|x| > 1$, $c(x) = 0$. Here we want solutions of the form $a e^{-k|x|}$. Then let $\tilde{v}(x) := a e^{-k|x|}$. We have

$$\tilde{v}''(x) = \lambda \tilde{v}(x)$$
$$k^2 \cancel{a} \cancel{e^{-k|x|}} = \lambda \cancel{a} \cancel{e^{-k|x|}}$$

$$\rightarrow k = \pm\sqrt{\lambda}$$

So $v(x) = A e^{-\sqrt{\lambda}|x|} + B e^{\sqrt{\lambda}|x|}$. Since we want $\|u\|_{L^2_x} < \infty$, $B = 0$, $\rightarrow v(x) = A e^{-\sqrt{\lambda}|x|}$

$$\rightarrow u(x,t) = A e^{-\sqrt{\lambda}|x|} e^{\lambda t} \text{ for } |x| > 1$$

If $|x| < 1$, $c(x) = 1$. Here we want solutions of the form $b \cos lx$. Then let $\tilde{v}(x) = b \cos lx$.

$$\rightarrow \quad \tilde{v}''(x) = (\lambda - 1)\tilde{v}(x)$$
$$-\cancel{b} l^2 \cancel{\cos lx} = (\lambda - 1) \cancel{b} \cancel{\cos lx}$$
$$l = \pm\sqrt{1-\lambda}.$$

So $v(x) = C \cos(\sqrt{1-\lambda}\, x) + D \cos(\sqrt{1-\lambda}\, x)$
$= E \cos(\sqrt{1-\lambda}\, x)$.

$$\rightarrow u(x,t) = B \cos(\sqrt{1-\lambda}\, x) e^{\lambda t} \text{ for } |x| < 1.$$

~~Note that~~ We also want the defined u to be continuous at $x = \pm 1$.

Thus we want

$$B\cos(\sqrt{1-\lambda})\cancel{e^{\lambda t}} = Ae^{-\sqrt{\lambda}}\cancel{e^{\lambda t}}$$

~~and~~

$$\cancel{B\cos(\sqrt{1-\lambda})e^{\lambda t} = Ae}$$

$$\longrightarrow \quad B = A\frac{e^{-\sqrt{\lambda}}}{\cos(\sqrt{1-\lambda})}.$$

Note that

$$u(x,t) = \begin{cases} Ae^{-\sqrt{\lambda}|x|}e^{\lambda t} & \text{for } |x|\geq 1 \\ B\cos(\sqrt{1-\lambda}\,x)e^{\lambda t} & \text{for } |x|\leq 1 \end{cases}$$

with $B = A\dfrac{e^{-\sqrt{\lambda}}}{\cos(\sqrt{1-\lambda})}$ is continuous and is a solution and $\|u\|_{L^{?}_{x}} < \infty$ as u is radial.