S02 #1

a. We will ~~solve~~ find an u s.t. $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{2\pi}\log r$. We have

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{2\pi} r\log r.$$

$$\rightarrow r\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right) + c_1$$

$$\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r\log r - \frac{1}{4}r\right) + \frac{c_1}{r}.$$

Choose $c_1 = 0$. Then

$$\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r\log r - \frac{1}{4}r\right) = \frac{1}{4\pi}r\log r - \frac{1}{8\pi}r.$$

$$\rightarrow u = \frac{1}{4\pi}\left(\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right) - \frac{1}{8\pi}\cdot\frac{1}{2}r^2.$$

$$= \frac{r^2}{8\pi}(\log r - 1).$$

So $\cancel{\frac{r^2}{8\pi}}\, u = \frac{|x|^2}{8\pi}(\log|x| - 1)$ is a radially symmetric solution to $\Delta u = \frac{1}{2\pi}\log|x|$ in $\mathbb{R}^2$.

We claim u is a fundamental solution for Δ^2. For arbitrary $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} u\,\Delta^2\phi\,dx = \int_{B_\varepsilon(0)} u\,\Delta^2\phi\,dx + \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta^2\phi\,dx := I_\varepsilon + J_\varepsilon.$$

Note

$$|I_\varepsilon| \le \int_{B_\varepsilon(0)} |u||\Delta^2\phi|\,dx \lesssim \varepsilon^2\cdot\varepsilon^2\log\varepsilon\cdot\|\Delta^2\phi\|_{L^\infty} \rightarrow 0 \text{ as } \varepsilon\rightarrow 0.$$

We also have

$$J_\varepsilon = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta^2\phi\,dx = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta(\Delta\phi)\,dx = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \Delta u\cdot\Delta\phi\,dx$$

$$+ \int_{\partial(\mathbb{R}^2\setminus B_\varepsilon(0))} u\frac{\partial(\Delta\phi)}{\partial\nu} - \Delta\phi\frac{\partial u}{\partial\nu}\,d\sigma$$

where ν is the inward normal for $\mathbb{R}^2\setminus B_\varepsilon(0)$.

Since $\Delta u = \frac{1}{2\pi}\log|x|$ which is the fundamental solution for Δ, we have

$$\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \Delta u\cdot\Delta\phi\,dx \Rightarrow \phi(0) \text{ as } \varepsilon\rightarrow 0.$$ We also have

$$\left|\int_{\partial(\mathbb{R}^2\setminus B_\varepsilon(0))} u\frac{\partial(\Delta\phi)}{\partial\nu}\,d\sigma\right| \lesssim \varepsilon\cdot\varepsilon^2\log\varepsilon\;\|\nabla(\Delta\phi)\|_{L^\infty} \rightarrow 0 \quad \text{as } \varepsilon\rightarrow 0.$$

S02 #1b:

We first prove an identity:

Lemma: $\displaystyle\int_U u\Delta^2 v - v\Delta^2 u \, dx = \int_{\partial U} u\frac{\partial(\Delta v)}{\partial\nu} - \Delta v\frac{\partial u}{\partial\nu} - v\frac{\partial(\Delta u)}{\partial\nu} + \Delta u\frac{\partial v}{\partial\nu}\, d\sigma.$

Pf: We have

$$\int_U u\Delta^2 v - v\Delta^2 u\,dx = \int_U u\Delta^2 v - \Delta u\Delta v\,dx - \int_U v\Delta^2 u - \Delta v\Delta u\,dx$$

$$= \int_{\partial U} u\frac{\partial(\Delta v)}{\partial\nu} - \Delta v\frac{\partial u}{\partial\nu}\,d\sigma - \int_{\partial U} v\frac{\partial(\Delta u)}{\partial\nu} - \Delta u\frac{\partial v}{\partial\nu}\,d\sigma.$$

#

Let Φ denote the fundamental solution in part a). Let w solve

$$\begin{cases} \Delta^2 w = f & \text{in } \cancel{\ } U \\ w=0,\ \frac{\partial w}{\partial\nu}=0 & \text{on } \cancel{\ }\ \partial U. \end{cases} \qquad (\Delta)$$

Fix $x\in U$. Choose $\varepsilon>0$ small enough s.t. $B(x,\varepsilon)\subset U$. Let $V_\varepsilon := U\setminus B(x,\varepsilon)$.

Then

$$\int_{V_\varepsilon} w(y)(\Delta^2\Phi)(y-x)\,dy - \int_{V_\varepsilon}(\Delta^2 w)(y)\,\Phi(y-x)\,dy$$

$$= \int_{\partial U} w(y)\frac{\partial\Delta\Phi(y-x)}{\partial\nu}\,d\sigma - \int_{\partial B(x,\varepsilon)} w(y)\frac{\partial\Delta\Phi}{\partial\nu}(y-x)\,d\sigma$$

$$- \int_{\partial U}\Delta\Phi(y-x)\frac{\partial w}{\partial\nu}\,d\sigma + \int_{\partial B(x,\varepsilon)}\Delta\Phi(y-x)\frac{\partial w}{\partial\nu}\,d\sigma \qquad (*)$$

$$- \int_{\partial U}\Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma + \int_{\partial B(x,\varepsilon)}\Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma$$

$$+ \int_{\partial U}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma - \int_{\partial B(x,\varepsilon)}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma.$$

Since $\Delta^2\Phi=0$ for $x\neq y$, $\int_{V_\varepsilon} w(y)(\Delta^2\Phi)(y-x)\,dy = 0$. Since $w=0$, $\frac{\partial w}{\partial\nu}=0$ on ∂U,

$\int_{\partial U} w(y)\frac{\partial\Delta\Phi}{\partial\nu}(y-x)\,d\sigma = 0$ and $\int_{\partial U}\Delta\Phi(y-x)\frac{\partial w}{\partial\nu}\,d\sigma = 0.$

1b cont.:

Since $\Delta\Phi = \frac{1}{2\pi}\log|x|$, $\nabla(\Delta\Phi) = \frac{x}{2\pi}\cdot\frac{1}{|x|^2}$ and hence on $\partial B(x,\varepsilon)$

as $\nu = \frac{x}{\varepsilon}$, $\frac{\partial \Delta\Phi}{\partial\nu} = \nabla(\Delta\Phi)\cdot\nu = \frac{|x|^2}{2\pi|x|^2\varepsilon} = \frac{1}{2\pi\varepsilon}$. Therefore

$$\int_{\partial B(x,\varepsilon)} w(y)\frac{\partial\Delta\Phi}{\partial\nu}(y-x)\,d\sigma = \frac{1}{2\pi\varepsilon}\int_{\partial B(x,\varepsilon)} w(y)\,d\sigma \longrightarrow w(x) \text{ as } \varepsilon\to 0.$$

We also have

$$\left|\int_{\partial B(x,\varepsilon)} \Delta\Phi(y-x)\frac{\partial w}{\partial\nu}(y)\,d\sigma\right| \lesssim_w \varepsilon\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

$$\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma\right| \lesssim_w \varepsilon^3\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

$$\left|\int_{\partial B(x,\varepsilon)} \Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma\right| \lesssim_w \varepsilon^2\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

Therefore letting $\varepsilon\to 0$ in $(*)$ yields

$$-\int_U (\Delta^2 w)(y)\Phi(y-x)\,dy = -w(x) - \int_{\partial U}\Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma + \int_{\partial U}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma.$$

Since w satisfies $(\diamond)$, we have

$$w(x) = \int_U f(y)\Phi(y-x)\,dy + \int_{\partial U}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma. \quad (**)$$

For each x, define a ϕ^x s.t.

$$\begin{cases} \Delta^2\phi^x = 0 & \text{in } U. \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U. \end{cases}$$

Then

$$\int_U w(y)\Delta^2\phi^x(y)\,dy - \int_U \phi^x(y)\Delta^2 w(y)\,dy = \int_{\partial U} w(y)\frac{\partial\Delta\phi^x}{\partial\nu} - \Delta\phi^x\cdot\frac{\partial w}{\partial\nu}\,d\sigma$$
$$- \int_{\partial U}\phi^x(y)\frac{\partial\Delta w}{\partial\nu} - \Delta w\,\frac{\partial\phi^x}{\partial\nu}(y)\,d\sigma$$

1b cont'd

Since $\Delta^2 \phi^x = 0$ on U and $w = 0, \frac{\partial w}{\partial \nu} = 0$ on ∂U, we have

$$\int_U \phi^x(y) \Delta^2 w(y)\, dy = \int_{\partial U} \phi^x(y) \frac{\partial \Delta w}{\partial \nu} - \Delta w \frac{\partial \phi^x}{\partial \nu}\, d\sigma. \quad (***)$$

Since $\Delta^2 w = f$ on U, we have after adding $(**)$ to $(***)$

$$w(x) + \int_U \phi^x(y) f(y)\, dy = \int_U f(y) \Phi(y-x)\, dy + \int_{\partial U} \Delta w \cancelto{0}{\left[\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y)\right]} d\sigma(y) \quad = 0 \text{ since } \Phi(y-x) = \phi^x \text{ on } \partial U$$

Let $G(x,y) := \Phi(y-x) - \phi^x(y)$. Then

$$w(x) = \int_U f(y) G(x,y)\, dy,$$

$G(x,y)$ is our Green's function.

S02 #2a:

By Duhamel's Principle, if U is a solution to

$$U_{tt}(x,t,s) - U_{xx}(x,t,s) = 0 \quad \text{in } \mathbb{R}\times[0,\infty) \qquad (*)$$

$$U(x,0,s) = 0, \quad U_t(x,0,s) = f(x)\cos s \quad \text{on } \mathbb{R}\times\{t=0\},$$

then

$$u(x,t) := \int_0^t U(x,t-s,s)\,ds$$

is a solution to

$$u_{tt} - u_{xx} = f(x)\cos t \quad \text{in } \mathbb{R}\times[0,\infty)$$

$$u(x,0) = u_t(x,0) = 0 \quad \text{on } \mathbb{R}\times\{t=0\}.$$

By D'Alembert's equation, the solution U to $(*)$ is

$$U(x,t,s) = \frac{1}{2}\int_{x-t}^{x+t} f(\xi)\cos s\,d\xi = \frac{\cos s}{2}\int_{x-t}^{x+t} f(\xi)\,d\xi.$$

we have

$$u(x,t) = \frac{1}{2}\int_0^t \cos s\int_{x-t+s}^{x+t-s} f(\xi)\,d\xi\,ds$$

is a solution to the PDE.

S02 #2a alt:

Guess $u(x,t) = F(x)\cos t$. Then

$$f(x)\cos t = -F(x)\cos t - F''(x)\cos t.$$

$$\rightarrow \quad F''(x) + F(X) = -f(x).$$

Then

$$F(X) = A\cos x + B\sin x + G(x)$$

where $G(x)$ is s.t. $G'' + G = -f$. Therefore a particular solution to the PDE is

$$u_p(x,t) = A\cos x\cos t + B\sin x\cos t + G(x)\cos t.$$

$\longrightarrow$ u_p not unique b/c A, B can be anything.

The homogenous solution u_h satisfies

$$(u_h)_{tt} - (u_h)_{xx} = 0$$

$$u_h(x,0) = -(A\cos x + B\sin x + G(x))$$

$$(u_h)_t(x,0) = 0.$$

By D'Alembert's formula,

$$u_h(x,t) = \frac{1}{2}\Big[-(A\cos(x+t) + B\sin(x+t) + G(x+t)) - (A\cos(x-t) + B\sin(x-t) + G(x-t))\Big]$$

Then the solution $u(x,t) = u_h(x,t) + u_p(x,t)$.

[Finding a G s.t. $G'' + G = -f$. The easiest way is via variation of parameters. We have

$$G(x) = v_1(x)\cos x + v_2(x)\sin x$$

where

$$v_1(x) = +\int_0^x \sin t\, f(t)\,dt, \quad v_2(x) = -\int_0^x \cos t\, f(t)\,dt$$

Note that f is compactly supported, so one can indeed find v_1, v_2.]

So

$$G(x) = \int_0^x f(t)(\sin t\cos x - \cos t\sin x)\,dt = \int_0^x f(t)\sin(t-x)\,dt.$$

$$= -\int_0^x f(t)\sin(x-t)\,dt.$$

502 #2b:

Let u^1, u^2 be 2 distinct solutions. Let $w := u^1 - u^2$. Then

$$w_{tt} - w_{xx} = \cancel{f(x)\cos} \; 0 \qquad -\infty < x < \infty \qquad (*)$$
$$w(x,0) = w_t(x,0) = 0 \qquad 0 \le t < \infty.$$

We claim that $w \equiv 0$ (this follows from D'Alembert's formula).

We rederive D'Alembert's formula: We want to find a solution u of

$$u_{tt} - u_{xx} = 0.$$
$$u(x,0) = g, \; u_t(x,0) = h.$$

Let $u(x,t) := F(x+t) + G(x-t)$. Then

$$u_x(x,t) = F'(x+t) + G'(x-t) \qquad u_t = F'(x+t) - G'(x-t)$$
$$u_{xx} = F''(x+t) + G''(x-t) \qquad u_{tt} = F''(x+t) + G''(x-t).$$

$$g = u(x,0) = F(x) + G(x) \longrightarrow g'(x) = F'(x) + G'(x).$$
$$h = u_t(x,0) = F'(x) - G'(x)$$

So

$$F'(x) = \frac{g'(x)+h(x)}{2} \qquad \cancel{G'(x) = F'(x) - h(x) = \frac{g'(x)-h(x)}{2}}.$$

$$\longrightarrow F(x) = \int_0^x \frac{g'(t)+h(t)}{2}\,dt, \qquad \cancel{G(x) = \int_0^x \frac{g'(t)-h(t)}{2}\,dt}$$
$$= \frac{1}{2}[g(x) - g(0)] + \frac{1}{2}\int_0^x h(t)\,dt \qquad \cancel{= \frac{1}{2}[g(x)-g(0)] - \frac{1}{2}\int_0^x h(t)\,dt}$$

$$\cancel{F(x+t) + G(x-t) = \frac{1}{2}[g(x+t) - g(0)] + \frac{1}{2}\int_0^{x+t} h(t)\,dt}$$
$$\cancel{+ \frac{1}{2}[g(x-t) - g(0)] - \frac{1}{2}\int_0^{x-t} h(t)\,dt.}$$

Since

$$G(x) = g(x) - F(x) = \frac{1}{2}[g(x) + g(0)] - \frac{1}{2}\int_0^x h(t)\,dt,$$

we have

$$u(x,t) = F(x+t) + G(x-t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds. \rightarrow$$

Now to show $w \equiv 0$ in $(*)$, take $g, h = 0$

S02 #3:

a) We have
$$u_1 \frac{\partial u_1}{\partial x_1} + \frac{1}{\rho}\frac{\partial p}{\partial x_1} - \frac{\eta}{\rho}\Delta u_1 = 0.$$
$$\frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial p}{\partial x_3} = 0.$$
Then
$$\frac{\partial p}{\partial x_1} = \eta \Delta u - u_1 \rho \frac{\partial u}{\partial x_1} = \eta \Delta u.$$
and hence

since $u = u(x_2, x_3)$.
$$\frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}(\eta \Delta u) = \eta \frac{\partial}{\partial x_1}(\Delta u) = 0$$
$$\frac{\partial}{\partial x_2}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_2}\right) = 0$$
$$\frac{\partial}{\partial x_3}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_3}\right) = 0.$$
Therefore $\frac{\partial p}{\partial x_1}$ is a constant c which implies $\Delta u = c/\eta$.

b) Since $\Delta u = c/\eta$ and u is radially symmetric,
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{c}{\eta} \longrightarrow \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{c}{\eta} r$$
$$r\frac{\partial u}{\partial r} = \frac{1}{2}\frac{c}{\eta} r^2 + c_1$$
$$\frac{\partial u}{\partial r} = \frac{1}{2}\frac{c}{\eta} r + \frac{c_1}{r}.$$
$$u = \frac{1}{4}\frac{c}{\eta} r^2 + c_1 \log r + c_2.$$
Since $\vec{u}$ is a velocity vector, to prevent a singularity at $r=0$, we must have $c_1 = 0$. Since $u(R) = 0$,
$$c_2 = -\frac{1}{4}\frac{c}{\eta}R^2.$$
Therefore
$$u(r) = \frac{c}{4\eta}(r^2 - R^2)$$
and hence
$$Q = \rho \int_{\{x_2^2 + x_3^2 \le R^2\}} u \, dx_2 dx_3 = 2\pi\rho \int_0^R \frac{c}{4\eta}(r^3 - R^2 r)\, dr = 2\pi\rho \cdot \frac{c}{4\eta}\left[\frac{1}{4}R^4 - R^2 \cdot \frac{1}{2}R^2\right]$$
$$= -\frac{c\pi\rho}{8\eta}R^4.$$

S02 #4: We look for a solution $u \in H^1(\mathbb{R})$. Taking the Fourier Transform ~~do~~ yields

$\hat{u}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\,dx$

$$(i\xi + c + e^{-i\xi})\hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}}. \qquad (*)$$

Note c is real and $|c| > 1$. Thus

$$|i\xi + c + e^{-i\xi}| \geq |\mathrm{Re}(i\xi + c + e^{-i\xi})|$$
$$= |c + \cos\xi| \geq |c| - 1.$$

Thus as

$$\left\| \frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}} \right\|_{L^2}^2 = \int \frac{|\hat{f}(\xi)|^2}{|i\xi + c + e^{-i\xi}|^2}\,d\xi \leq \frac{1}{(|c|-1)^2} \int |\hat{f}(\xi)|^2\,d\xi$$

$< \infty$ as $f \in L^2$

Therefore ~~•~~ we can invert the Fourier Transform in $(*)$ and the unique solution $u \in L^2$ is given by

$$\cancel{u(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \frac{\hat{f}(\xi)}{i\xi}}$$

$$u(x) = \left[\frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}} \right]^{\vee}$$

S02 #5:

Here $u(x,t) = \phi(x+ct)$. Then $u_t = u(1-u) + u_{xx}$ becomes

$$\phi'' - c\phi' + \phi(1-\phi) = 0.$$

We can rewrite this as the system

$$x' = y$$
$$y' = cy - x(1-x).$$

The critical points are $(0,0)$ and $(1,0)$. The Jacobian is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x-1 & c \end{pmatrix}$$

The linearized system at $(1,0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}$ are

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & c-\lambda \end{pmatrix} = -\lambda(c-\lambda) - 1 = \lambda^2 - c\lambda - 1$$

$$\rightarrow \lambda = \frac{c \pm \sqrt{c^2+4}}{2}$$

Therefore $(1,0)$ is an unstable saddle for all $c \geq 0$.

The linearized system at $(0,0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}$ are

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & c-\lambda \end{pmatrix} = -\lambda(c-\lambda) + 1 = \lambda^2 - c\lambda + 1.$$

$$\rightarrow \lambda = \frac{c \pm \sqrt{c^2-4}}{2}. \qquad (*)$$

If $0 < c < 2$, then $(0,0)$ is an unstable spiral (clockwise)

If $c > 2$, then $(0,0)$ is an unstable (proper) node. (clockwise)

502 #5 cont:

Technically in the nonlinear case, should be either a node or a spiral, but the sys. is
$x' = y$
$y' = 2y - x + x^2$.
Contrib. from x^2 term is very small when x is close to 0, so it should behave like
$x' = y$
$y' = 2y - x$.

If $c = 2$, then $(0,0)$ is an unstable (improper) node.

If $c = 0$, then ~~the~~ the system is given by

$$x' = y$$
$$y' = -x(1-x).$$

Since $\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x(1-x)) = 0$, this is a Hamiltonian system and hence all critical points are centers or saddles.

Since if $c = 0$, the associated eigenvalue in $(*)$ is ~~0~~ purely imaginary, the critical pt. $(0,0)$ must either be a center or spiral. Therefore $(0,0)$ is a center in this case.

S02 #6:

We reformulate into the notation of Evans. We want to solve
$$u_{x_1} + u_{x_1} u_{x_2} = 1 \quad , \quad u(x_1, 0) = f(x_1).$$

Let $F(p,z,x) = p_1 + p_1 p_2 - 1 = 0$. Then
$$\begin{aligned} \dot{p} &= -D_x F - D_z F p & & D_p F = (1+p_2, p_1) \\ \dot{z} &= D_p F \cdot p & \longrightarrow \quad & D_x F = (0,0) \\ \dot{x} &= D_p F & & D_z F = 0. \end{aligned}$$

The problem is noncharacteristic if $f'(x_1(0))$ is $\neq 0$. The condition that $f'(x) \neq 0 \ \forall x$ will ensure that the problem is noncharacteristic.

The initial conditions are
$$\begin{aligned} p_1(0) &= f'(x_1(0)) & x_1(0) &= x_1(0) \\ p_2(0) &= \frac{1}{f'(x_1(0))} - 1 & x_2(0) &= 0 \\ & & z(0) &= f(x_1(0)). \end{aligned}$$

Then
$$\begin{aligned} p_1(s) &= f'(x_1(0)) \\ p_2(s) &= \frac{1}{f'(x_1(0))} - 1 \\ \dot{z}(s) &= 2 - f'(x_1(0)) \longrightarrow z(s) = (2 - f'(x_1(0)))s + f(x_1(0)) \\ x_1(s) &= \frac{1}{f'(x_1(0))} s + x_1(0) \\ x_2(s) &= f'(x_1(0)) s. \end{aligned}$$

Therefore
$$z(s) = (2 - f'(x_1(0))) \frac{x_2(s)}{f'(x_1(0))} + f(x_1(0))$$
and
$$x_1(s) = \frac{x_2(s)}{f'(x_1(0))^2} + x_1(0).$$

That is,
$$u(x,y) = (2 - f'(r)) \frac{y}{f'(r)} + f(r) = \frac{2y}{f'(r)} - y + f(r)$$
where r satisfies $f'(r)^2 (x - r) = y$.

S02 #6 cont.:

Let $G(x,y,s) := f'(s)^2(x-s) - y$. Since

$$G_s(x_0, 0, x_0) = -f'(s)^2 + (x-s)\cdot 2f'(s)f''(s)\Big|_{(x,y,s)=(x_0,0,x_0)}$$

$$= -f'(x_0)^2 \neq 0,$$

by the Implicit Function Theorem, one can solve $y = (f'(r))^2(x-r)$ for r in terms of (x,y) in a sufficiently small neighborhood of $(x_0, 0)$ with $r(x_0,0) = x_0$.

502 #7:

Let $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ and let $A = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$. We can diagonalise A in the following manner. Let $P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$. Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} := D$$

Then the system becomes

$$\mathcal{U}_t + A\mathcal{U}_x = 0.$$

Set $\tilde{\mathcal{U}} := P^{-1}\mathcal{U}$. We have

$$P\tilde{\mathcal{U}}_t + PDP^{-1}\tilde{\mathcal{U}}_x = 0.$$

$$\rightarrow P\tilde{\mathcal{U}}_t + PD\tilde{\mathcal{U}}_x = 0.$$

$$\rightarrow \tilde{\mathcal{U}}_t + D\tilde{\mathcal{U}}_x = 0$$

$$\tilde{\mathcal{U}} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

That is

$$\begin{pmatrix} \tilde{u}_t \\ \tilde{v}_t \end{pmatrix} = +\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{u}_x \\ \tilde{v}_x \end{pmatrix}.$$

$$\tilde{u}_t = -2\tilde{u}_x \longrightarrow \tilde{u}(x,t) = F(x-2t)$$

$$\tilde{v}_t = 3\tilde{v}_x \longrightarrow \tilde{v}(x,t) = G(x+3t).$$

Therefore

$$\mathcal{U} = P\tilde{\mathcal{U}} \longrightarrow \mathcal{U} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$$\rightarrow u = -2F(x-2t) + G(x+3t)$$

$$v = F(x-2t) + 2G(x+3t)$$

now we have 2 cases $x-2t>0$ and $x-2t<0$ (of course both w/ $x>0, t>0$)

We know if $x>0$,

$$\left.\begin{aligned} u(x,0) &= f(x) \\ v(x,0) &= 0 \end{aligned}\right| \rightarrow \begin{aligned} f(x) &= -2F(x) + G(x) \\ 0 &= F(x) + 2G(x). \end{aligned} \rightarrow \begin{aligned} F(x) &= -\tfrac{2}{5}f(x) \\ G(x) &= \tfrac{1}{5}f(x). \end{aligned}$$

if $x>0$. Key!

So if $x-2t>0, x>0, t>0$,

$$u(x,t) = \tfrac{4}{5}f(x-2t) + \tfrac{1}{5}f(x+3t)$$

$$v(x,t) = -\tfrac{2}{5}f(x-2t) + \tfrac{2}{5}f(x+3t).$$

if $x-2t>0, x>0, t>0$

~~Since u(0,t)=0 for t>0, also need 4f(-2t) = f(3t) for all t>0~~

§02 #7 cont.

If $x-2t<0$, $x>0$, $t>0$, then we use the condition that $u(0,t)=0$ for $t>0$. We have

$$0=-2F(-2t)+G(3t) \quad \text{for } t>0.$$

Since $F(x)=-\frac{2}{5}f(x)$, $G(x)=\frac{1}{5}f(x)$ for $x>0$,

$$0=-2F(-2t)+\frac{1}{5}f(3t) \quad \text{for all } t>0.$$

$$F(-2t)=\frac{1}{10}f(3t) \quad \text{for all } t>0.$$

Since $x-2t<0$, $\frac{x-2t}{-2}>0$ and here

$$F(x-2t)=\frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right).$$

Therefore

$$\left.\begin{aligned} u(x,t)&=-\frac{1}{5}f\left(-\frac{3}{2}(x-2t)\right)+\frac{1}{5}f(x+3t) \\ v(x,t)&=\frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right)+\frac{2}{5}f(x+3t)\end{aligned}\right\} \text{ if } x-2t<0,\ x>0,\ t>0.$$

Thus the solution is

$$u(x,t)=\begin{cases}\frac{4}{5}f(x-2t)+\frac{1}{5}f(x+3t) & \text{if } x-2t>0,\ x>0,\ t>0 \\ -\frac{1}{5}f\left(-\frac{3}{2}(x-2t)\right)+\frac{1}{5}f(x+3t) & \text{if } x-2t<0,\ x>0,\ t>0.\end{cases}$$

$$v(x,t)=\begin{cases}-\frac{2}{5}f(x-2t)+\frac{2}{5}f(x+3t) & \text{if } x-2t>0,\ x>0,\ t>0 \\ \frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right)+\frac{2}{5}f(x+3t) & \text{if } x-2t<0,\ x>0,\ t>0.\end{cases}$$

Note that u, v are both diff. when $x-2t=0$ since $f(x)$ is smooth and vanishes in a neighborhood of $x=0$.

S02 #8:

a) Let $\mathcal{Y} = \{u \in C^2(\Omega), u \neq 0, \frac{\partial u}{\partial \nu} + au = 0 \text{ on } \partial\Omega\}$. We claim that the smallest eigenvalue is given by

$$m := \min_{u \in \mathcal{Y}} \frac{\int_\Omega |\nabla u|^2 dx + a\int_{\partial\Omega} u^2 d\sigma}{\int_\Omega u^2 dx}.$$

Let u be the function in $\mathcal{Y}$ associated to m. Let v be an arbitrary elt. of $\mathcal{Y}$. Let

$$f(u+\varepsilon v) = \frac{\int_\Omega |\nabla(u+\varepsilon v)|^2 dx + a\int_{\partial\Omega} (u+\varepsilon v)^2 d\sigma}{\int_\Omega (u+\varepsilon v)^2 dx}.$$

Since $\frac{d}{d\varepsilon} f(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$, by a similar calculation as in S04 #7, we must have

$$(*) \quad \left(\int_\Omega u^2 dx\right)\left(\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} auv\, d\sigma\right) = \left(\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u^2 d\sigma\right)\left(\int_\Omega uv\, dx\right).$$

Since

$$\int_\Omega \nabla u \cdot \nabla v\, dx = -\int_\Omega v\Delta u\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\, d\sigma$$

and

$$\int_\Omega \nabla u \cdot \nabla u\, dx = -\int_\Omega u\Delta u\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u\, d\sigma,$$

combining this with (*), and the fact that $\frac{\partial u}{\partial \nu} + au = 0$, on $\partial\Omega$, we have

$$\left(\int_\Omega u^2 dx\right)\left(\int_\Omega v\Delta u\, dx\right) = \left(\int_\Omega u\Delta u\, dx\right)\left(\int_\Omega uv\, dx\right)$$

Therefore with $\alpha := \int_\Omega u^2 dx$, $\beta := \int_\Omega u\Delta u\, dx$, we have

$$\int_\Omega (\alpha\Delta u - \beta u)v\, dx = 0 \quad \forall v \in \mathcal{Y}.$$

Thus
$$\Delta u = \frac{\beta}{\alpha} u = \frac{\int_\Omega u\Delta u\, dx}{\int_\Omega u^2 dx} u = -\frac{\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} au^2 d\sigma}{\int_\Omega u^2 dx} u.$$

$$\longrightarrow \; -\Delta u = mu.$$

S02 #8 cont:

Thus m is an eigenvalue of $-\Delta u$.

Now we claim that m is the smallest eigenvalue of $-\Delta u$.

Let $\tilde{\lambda}$ be an eigenvalue of $-\Delta$ with eigenfunction v. Then

$$m \le \frac{\int_\Omega |\nabla v|^2 dx + a\int_{\partial\Omega} v^2 d\sigma}{\int_\Omega v^2 dx} = \frac{-\int_\Omega v\Delta v\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v\, d\sigma + a\int_{\partial\Omega} v^2 d\sigma}{\int_\Omega v^2 dx}$$

$$= \frac{\tilde{\lambda}\int_\Omega v^2 dx}{\int_\Omega v^2 dx} = \tilde{\lambda}.$$

Therefore m is the smallest eigenvalue.

b). Let $a>0$

By Hopf's Lemma, $\exists\, x_0 \in \partial\Omega$ s.t. $u(x_0) = \max_{\bar\Omega} u(x)$ and $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Then as

$$\frac{\partial u}{\partial \nu}(x_0) + a u(x_0) = g(x_0),$$

$$\longrightarrow\quad u(x_0) < \frac{1}{a} g(x_0) \le \cancel{\max_{\partial\Omega}\left\{\frac{1}{a}g(x),0\right\}}\ \frac{1}{a}\max_{\partial\Omega}|g|$$

and hence

$$\max_{\bar\Omega} u(x) \le \cancel{\max_{\partial\Omega}\left\{\frac{1}{a}g(x),0\right\}}.\ \frac{1}{a}\max_{\partial\Omega}|g|.$$

Let $v := -u$. Then

$$-\Delta v + k^2 v = 0 \text{ in } \Omega$$

$$\frac{\partial v}{\partial \nu} + av = -g \text{ on } \partial\Omega.$$

So

$$\max_{\bar\Omega} v(x) \le \cancel{\max_{\partial\Omega}\left\{\frac{1}{a}(-g(x)),0\right\}}.\ \frac{1}{a}\max_{\partial\Omega}|g|.$$

$$\longrightarrow\quad \max_{\bar\Omega} -u(x) \le \cancel{\max_{\partial\Omega}\left\{-\frac{1}{a}g(x),0\right\}}.\ \frac{1}{a}\max_{\partial\Omega}|g|.$$

Therefore

$$\cancel{\max_{\Omega} |u| \le \frac{1}{a}\max_{\partial\Omega}|g|}$$

$$\cancel{\min_{\Omega} u(x) \ge -\max_{\partial\Omega}\left\{\frac{1}{a}g(x),0\right\}}$$

$$\min_{\bar\Omega} u(x) \ge -\frac{1}{a}\max_{\partial\Omega}|g|.$$

This implies

$$\max_{\Omega} |u(x)| \le \frac{1}{a}\max_{\partial\Omega}|g|.$$