W05 #1: Write $u(x,t)=F(x)G(t)$. Then

$$u_{tt}-u_{xx}-2u_x=0$$
$$\longrightarrow F(x)G''(t)-F''(x)G(t)-2F'(x)G(t)=0.$$
$$\frac{G''(t)}{G(t)}=\frac{F''(x)+2F'(x)}{F(x)}=\lambda.$$

Since $u_x(0,t)=0$, $u_x(1,t)=0 \longrightarrow F'(0)=0$, $F'(1)=0$.
We consider 3 cases:

1. $\lambda>-1$. In this case, $1+\lambda>0$. Then

$$\frac{F''(x)+2F'(x)}{F(x)}=\lambda \longrightarrow F''(x)+2F'(x)-\lambda F(x)=0.$$
$$F(x)=Ae^{(-1+\sqrt{1+\lambda})x}+Be^{(-1-\sqrt{1+\lambda})x}$$

Since

$$F'(x)=A(-1+\sqrt{1+\lambda})e^{(-1+\sqrt{1+\lambda})x}+B(-1-\sqrt{1+\lambda})e^{(-1-\sqrt{1+\lambda})x}$$
$$0=A(-1+\sqrt{1+\lambda})+B(-1-\sqrt{1+\lambda})$$
$$\longrightarrow 0=A(-1+\sqrt{1+\lambda})e^{-1+\sqrt{1+\lambda}}+B(-1-\sqrt{1+\lambda})e^{-1-\sqrt{1+\lambda}}.$$

Then

$$0=B(-1-\sqrt{1+\lambda})(e^{-1+\sqrt{1+\lambda}}-e^{-1-\sqrt{1+\lambda}})$$
$$\longrightarrow B=0 \longrightarrow A=0.$$

Thus only gives trivial solution.

2. $\lambda=-1$.

In this case $F(x)=Ae^{-x}+Bxe^{-x}$.

$$\longrightarrow F'(x)=-Ae^{-x}+B(e^{-x}-xe^{-x}).$$

Thus

$$0=F'(0)=-A+B$$
$$0=F'(1)=-Ae^{-1}+\cancel{Be^{-1}}-\cancel{Be^{-1}} \longrightarrow A=0, B=0.$$

Thus we have the trivial solution.

3. $\lambda<-1$.

In this case $0<-\lambda-1$. Then $\sqrt{1+\lambda}=i\sqrt{-\lambda-1}$.

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Then

$$F(x) = Ae^{(-1+\sqrt{1+\lambda})x} + Be^{(-1-\sqrt{1+\lambda})x}$$
$$= e^{-x}[\tilde{A}\cos(\sqrt{-1-\lambda}\,x) + \tilde{B}\sin(\sqrt{-1-\lambda}\,x)].$$
$$F'(x) = e^{-x}[-\tilde{A}\sin(\sqrt{-1-\lambda}\,x)\sqrt{-1-\lambda} + \tilde{B}\cos(\sqrt{-1-\lambda}\,x)\sqrt{-1-\lambda}]$$
$$- e^{-x}[\tilde{A}\cos(\sqrt{-1-\lambda}\,x) + \tilde{B}\sin(\sqrt{-1-\lambda}\,x)]$$
$$0 = F'(0) = \tilde{B}\sqrt{-1-\lambda} - \tilde{A}.$$
$$\rightarrow 0 = F'(1) = e^{-1}[-\tilde{A}\sin(\sqrt{-1-\lambda})\sqrt{-1-\lambda} + \tilde{B}\cos(\sqrt{-1-\lambda})\sqrt{-1-\lambda}$$
$$- \tilde{A}\cos(\sqrt{-1-\lambda}) - \tilde{B}\sin(\sqrt{-1-\lambda})]$$

Therefore

$$\tilde{A}\sin(\sqrt{-1-\lambda})\sqrt{-1-\lambda} + \tilde{B}\sin(\sqrt{-1-\lambda}) = 0$$
$$\rightarrow \tilde{B}(-1-\lambda)\sin(\sqrt{-1-\lambda}) + \tilde{B}\sin(\sqrt{-1-\lambda}) = 0$$

Since we want nontrivial solutions, $\sin(\sqrt{-1-\lambda}) = 0$.

$$\rightarrow \sqrt{-1-\lambda_n} = n\pi, \quad n = 1, 2, 3, \ldots$$
$$\lambda_n = -(n^2\pi^2+1).$$

We solve $\dfrac{G''(t)}{G(t)} = -(n^2\pi^2+1)$.

$$\rightarrow G''(t) + (n^2\pi^2+1)G(t) = 0.$$
$$G(t) = C\cos\sqrt{n^2\pi^2+1}\,t + D\sin\sqrt{n^2\pi^2+1}\,t.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} e^{-x}(a_n\cos n\pi x + b_n\sin n\pi x)(c_n\cos\sqrt{n^2\pi^2+1}\,t + d_n\sin\sqrt{n^2\pi^2+1}\,t)$$

Since $u_t(x,0) = 0$, $d_n = 0$ $\forall n$.
Since $u(x,0) = e^{-x}(\pi\cos\pi x + \sin\pi x)$, $c_1 = 1$, $b_1 = 1$, $a_1 = \pi$
and all other coeff are $= 0$.

Thus

$$u(x,t) = e^{-x}(\pi\cos\pi x + \sin\pi x)\cos(\sqrt{\pi^2+1}\,t)$$

W05 #2: We use method of characteristics (and the notation of Evans). We have

$$x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$$
$$u(x_1, x_2, 0) = \varphi(x_1, x_2)$$

$$\longrightarrow \quad F(p, z, x) = x_1 p_1 + 2x_2 p_2 + p_3 - 3z = 0$$

$$\longrightarrow \quad \begin{aligned} D_p F &= (x_1, 2x_2, 1) \\ D_x F &= (p_1, 2p_2, 0) \\ D_z F &= -3 \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} \dot{p} &= -D_x F - D_z F p = (-p_1, -2p_2, 0) + (3p_1, 3p_2, 3p_3) \\ &= (2p_1, p_2, 3p_3) \\ \dot{z} &= D_p F \cdot p = x_1 p_1 + 2x_2 p_2 + p_3 = 3z. \\ \dot{x} &= D_p F = (x_1, 2x_2, 1). \end{aligned}$$

w/ initial conditions $x_1(0) = x_1(0)$ $\quad z(0) = \varphi(x_1(0), x_2(0))$
$x_2(0) = x_2(0)$
$x_3(0) = 0$

Then

$$z(s) = \varphi(x_1(0), x_2(0))e^{3s} \qquad x_1(s) = x_1(0)e^{s}$$
$$= \varphi\left(\frac{x_1(s)}{e^{x_3(s)}}, \frac{x_2(s)}{e^{2x_3(s)}}\right)e^{3x_3(s)}. \qquad x_2(s) = x_2(0)e^{2s}$$
$$x_3(s) = s$$

Thus the solution is

$$u(x_1, x_2, x_3) = \varphi\left(\frac{x_1}{e^{x_3}}, \frac{x_2}{e^{2x_3}}\right)e^{3x_3}.$$

WOS #3

Note we only have that u is harmonic in $\mathbb{D}$ not $\overline{\mathbb{D}}$, so we have to work a bit harder.

By Poisson's formula

$$u(x) = \frac{(1-\varepsilon)^2 - |x|^2}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} \frac{u(y)}{|x-y|^2}\, d\sigma_y$$

$\forall\ |x| < 1-\varepsilon.$

Then

$$\frac{1}{(1-\varepsilon)+|x|} \leq \frac{1}{|x-y|} \leq \frac{1}{(1-\varepsilon)-|x|}$$

Thus

$$\frac{(1-\varepsilon)-|x|}{(1-\varepsilon)+|x|} \frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y \leq u(x) \leq \frac{(1-\varepsilon)+|x|}{(1-\varepsilon)-|x|} \cdot \frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y.$$

Since u is harmonic in $B_{1-\varepsilon}(0)$,

$$\frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y = u(0).$$

Thus

$$\frac{(1-\varepsilon)-|x|}{(1-\varepsilon)+|x|} u(0) \leq u(x) \leq \frac{(1-\varepsilon)+|x|}{(1-\varepsilon)-|x|} \quad \forall\ |x| < 1-\varepsilon.$$

~~Letting ε~~

$\rightarrow$ Fix $x_0 \in \mathbb{D}$, $\exists$ small ε_0 s.t. $|x_0| < 1-\varepsilon_0$. Then $\forall\ \varepsilon < \varepsilon_0$, $\frac{(1-\varepsilon)-|x_0|}{(1-\varepsilon)+|x_0|} u(0) \leq u(x_0) \leq \cdots$ let $\varepsilon \to 0$.

Fix an arbitrary $|x| < 1$, letting $\varepsilon \longrightarrow 0$ shows

$$\frac{1-|x|}{1+|x|} u(0) \leq u(x) \leq \frac{1+|x|}{1-|x|} u(0).$$

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W05 #4: ~~Let R~~ Let R be sufficiently large s.t. supp φ, supp $\psi \subset B(0, R/2)$.

From D'Alembert's formula

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\,\varphi(y) + \psi(y) + \nabla\psi(y)\cdot(y-x)\, d\sigma_y.$$

Then

$$\left| \frac{1}{4\pi t^2} \int_{\partial B(x,t)} t\,\varphi(y)\, d\sigma_y \right| = \left| \frac{1}{4\pi t} \int_{\partial B(x,t)} \varphi(y)\, 1_{supp(\varphi)}\, d\sigma_y \right|$$

$$\leq \frac{1}{4\pi t} \|\varphi\|_{L^\infty} \int_{\partial B(x,t)} 1_{supp(\varphi)}\, d\sigma_y.$$

$$\leq \frac{1}{4\pi t} \|\varphi\|_{L^\infty} \int_{\partial B(x,t)} 1_{B(0,R)}\, d\sigma_y.$$

$$\leq \frac{R^2 \|\varphi\|_{L^\infty}}{t}.$$

Similarly,

$$\left| \frac{1}{4\pi t^2} \int_{\partial B(x,t)} \psi(y)\, d\sigma_y \right| \leq \frac{1}{4\pi t^2} \|\psi\|_{L^\infty} \cdot 4\pi R^2.$$

$$= \frac{R^2 \|\psi\|_{L^\infty}}{t^2}.$$

and

$$\left| \frac{1}{4\pi t^2} \int_{\partial B(x,t)} \nabla\psi(y)\cdot(y-x)\, d\sigma_y \right| \leq \frac{1}{4\pi t^2} \|\nabla\psi\|_{L^\infty} t \cdot 4\pi R^2.$$

$$= \frac{R^2 \|\nabla\psi\|_{L^\infty}}{t}.$$

Thus

$$|u(x,t)| \leq \frac{R^2\|\varphi\|_{L^\infty}}{t} + \frac{R^2\|\nabla\psi\|_{L^\infty}}{t} + \frac{R^2\|\psi\|_{L^\infty}}{t^2} \leq \frac{C}{t}$$

for some abs. constant C.

$$r^2\cos 2\theta = x^2-y^2$$
$$\Delta(x^2-y^2) = 0.$$

<u>WOS #5</u> :

Note that there is only at most 1 solution to the given PDE. (Let u, v be 2 solutions, then $w := u-v$ satisfies $\Delta w = 0$ in D, $w=0$ on ∂D. Then $w=0$ by the maximum principle)

Writing $\Delta u = x^2-y^2$ in polar, one have

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = r^2\cos 2\theta. \qquad (*)$$
$$u = 0 \quad \text{when } r=1.$$

If $\tilde{u} = ar^2\cos 2\theta$, then

$$\tilde{u}_r = 2ar\cos 2\theta \qquad \tilde{u}_\theta = -2ar^2\sin 2\theta.$$
$$\tilde{u}_{rr} = 2a\cos 2\theta \qquad \tilde{u}_{\theta\theta} = -4ar^2\cos 2\theta.$$

$$\tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta\theta} = 2a\cos 2\theta + 2a\cos 2\theta - 4a\cos 2\theta = 0.$$

If $\tilde{u} = ar^4\cos 2\theta$.

$$\tilde{u}_r = 4ar^3\cos 2\theta \qquad \tilde{u}_\theta = -2ar^4\sin 2\theta.$$
$$\tilde{u}_{rr} = 12ar^2\cos 2\theta. \qquad \tilde{u}_{\theta\theta} = -4ar^4\cos 2\theta.$$

$$\tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta\theta} = [12ar^2 + 4ar^2 - 4ar^2]\cos 2\theta.$$

Thus a solution to $(*)$ is given by
$$= 12ar^2\cos 2\theta.$$

$$u = \frac{1}{12}r^4\cos 2\theta - \frac{1}{12}r^2\cos 2\theta = \frac{1}{12}(x^2-y^2)(x^2+y^2-1)$$

$$\cancel{= \frac{1}{12}r^2\cos 2\theta\,[r^2-1]}$$
$$\cancel{= \frac{1}{12}(x^2-y^2)}$$

WOS #6:

Let $f(x) = \sin x.$, $g(x) = \frac{\sin x}{x}$. Then $x g(x) = f(x)$.

$$\cancel{\int_{-\infty}^{\infty} x g(x) e^{-2\pi i x\xi} dx = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} -2\pi i \xi}$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \cancel{x} g(x) e^{-2\pi i x \xi} dx = \int_{-\infty}^{\infty} g(x)\left(-\frac{1}{2\pi i}\right)\frac{d}{d\xi} e^{-2\pi i x\xi} dx.$$

$$= -\frac{1}{2\pi i}\frac{d}{d\xi}\int_{-\infty}^{\infty} g(x) e^{-2\pi i x\xi} dx$$

$$= -\frac{1}{2\pi i}\frac{d}{d\xi}\hat{g}(\xi).$$

We have

$$\int_{-\infty}^{\infty} \sin x \, e^{-2\pi i x\xi} dx = \int_{-\infty}^{\infty} \frac{e^{ix} - e^{-ix}}{2i} e^{-2\pi i x\xi} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{2i}\left(e^{ix(1-2\pi\xi)} - e^{-ix(1-2\pi\xi)}\right) dx.$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} e^{-2\pi i x(\xi - \frac{1}{2\pi})} - e^{-2\pi i x(\frac{1}{2\pi}+\xi)} dx.$$

Since $\mathcal{F}[1] = \delta_0$ which comes from $1 = \mathcal{F}^{-1}[\delta_0]$.

$$= \frac{1}{2i}\left[\delta\left(\xi - \frac{1}{2\pi}\right) - \delta\left(\xi + \frac{1}{2\pi}\right)\right]$$

Thus

$$\frac{d}{d\xi}\hat{g}(\xi) = \pi\left[\delta\left(\xi + \frac{1}{2\pi}\right) - \delta\left(\xi - \frac{1}{2\pi}\right)\right]$$

$$\rightarrow \hat{g}(\xi) = \pi \, 1_{\xi + \frac{1}{2\pi} \geq 0} - \pi \, 1_{\xi - \frac{1}{2\pi} \geq 0}.$$

$$\cancel{g(x) = \pi \int_{-1/2\pi}^{1/2\pi} e^{-2\pi i x\xi} d\xi}$$

W05 #6 cont:

$$\hat f \hat g = \widehat{f * g} \qquad \widehat{fg} = \hat f * \hat g .$$

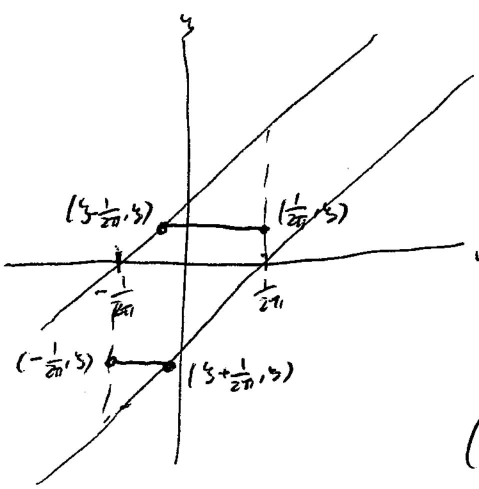
$$\left[\left(\frac{\sin x}{x}\right)^2\right]^{\wedge}(\xi) = \left[\frac{\sin x}{x}\right]^{\wedge} * \left[\frac{\sin x}{x}\right]^{\wedge}$$

$$= \int_{-\infty}^{\infty} \hat g(\xi-\eta)\, \hat g(\eta)\, d\eta .$$

$$= \pi \int_{-1/2\pi}^{1/2\pi} \hat g(\xi - \eta)\, d\eta \qquad \begin{aligned} u &= \xi - \eta \\ du &= -d\eta \end{aligned}$$

$$= \pi \int_{\xi - \frac{1}{2\pi}}^{\xi + \frac{1}{2\pi}} \hat g(u)\, du$$

$$= \pi^2 \int_{\xi-\frac{1}{2\pi}}^{\xi+\frac{1}{2\pi}} \mathbb{1}_{-\frac{1}{2\pi} < u \le \frac{1}{2\pi}}\, du .$$



$$= \begin{cases} \pi^2\left(\frac{1}{2\pi} - \xi + \frac{1}{2\pi}\right) = \pi - \pi^2 \xi & \text{if } 0 \le \xi < \frac{1}{\pi} \\ \pi^2\left(\xi + \frac{1}{\pi}\right) = \pi^2 \xi + \pi & \text{if } -\frac{1}{\pi} < \xi \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note both sol. manuals are wrong, check $\xi = 0$) #

WOS #7: We need to assume that $f(\vec{0}) = \vec{0}$.

here $V(x_1, \ldots, x_n) = \frac{1}{2}(x_1^2 + \ldots + x_n^2)$

Then $V(x_1, \ldots, x_n) > 0$ for $\vec{x} \neq \vec{0}$ and

$$\dot{V}(x_1, \ldots, x_n) = x_1 \dot{x}_1 + \ldots + x_n \dot{x}_n$$

$$= x_1 f_1(x) + \ldots + x_n f_n(x) < 0. \quad \forall \vec{x} \neq \vec{0}$$

Therefore as $V(\vec{0}) = 0$ and $\dot{V}(0) = 0$, by Lyapunov stability, we have that the zero solution is asymptotically stable and hence $\vec{x}(t) \to 0$ as $t \to \infty$. indep of the initial condition.