F'00 #1 : a. Let u_1, u_2 be 2 classical solutions of the given equation. Let $w := u_1 - u_2$. Then

$$\Delta w + a(x) w = 0, \quad x \in D$$
$$w|_S = 0 \quad x \in S.$$

Therefore

$$\int_D |\nabla w|^2 dx = -\int_D w \Delta w \, dx = \int_D a(x) w^2 dx$$

$$\leq \|a\|_{L^\infty} \int_D w^2 dx \leq \|a\|_{L^\infty} \underbrace{C(D)}_{\text{Poincaré ineq.}} \int_D |\nabla w|^2 dx$$

$$\leq \frac{1}{2} \int_D |\nabla w|^2 dx$$

if we ~~choose~~ assume $\|a\|_{L^\infty} C(D) \leq \frac{1}{2}$. Thus if $|a(x)|$ was small enough, then $\int_D |\nabla w|^2 dx = 0 \to \nabla w = 0 \to w = 0$.

b. Let $H = H_0^1(D) = \{u \in H^1(D) : u = 0 \text{ on } S\}$ and

$$B[u,v] = \int_D \nabla u \cdot \nabla v - a(x) uv \, dx, \quad u, v \in H_0^1(D).$$

Note that $H_0^1(D)$ is a Hilbert space and $B[u,v] : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$. We will use Lax-Milgram to prove the result.

① We have

$$|B[u,v]| \leq \int_D |\nabla u| |\nabla v| dx + \int_D |a(x)| |u| |v| dx$$

$$\leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|a\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq \alpha \|u\|_{H^1} \|v\|_{H^1}$$

for some constant $\alpha > 0$ which only depends on $\|a\|_{L^\infty}$.

② We want to show there is a constant $\beta > 0$ s.t. $\beta \|u\|_{H^1}^2 \leq B[u,u]$ $\forall u \in H_0^1$. We have

F'00 1 cont.

$$B[u,u] = \int_D |\nabla u|^2 - a(x)u^2\,dx = \|\nabla u\|_{L^2} - \int_D a(x)u^2\,dx.$$

$$\geq \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}\|\nabla u\|_{L^2}^2 - \|a\|_{L^\infty}\int_D u^2\,dx.$$

$$= \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}\|\nabla u\|_{L^2}^2 - \|a\|_{L^\infty}\|u\|_{L^2}^2.$$

$$\geq \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}C\|u\|_{L^2}^2 - \|a\|_{L^\infty}\|u\|_{L^2}^2$$

where from Poincaré, $\exists$ C depending only on D s.t. $C\|u\|_{L^2} \leq \|\nabla u\|_{L^2}$.

$$= \frac{1}{3}\|\nabla u\|_{L^2}^2 + \left(\frac{2}{3}C - \|a\|_{L^\infty}\right)\|u\|_{L^2}^2.$$

~~Choose~~ If $|a(x)|$ is sufficiently small, then $\exists$ constant $\beta > 0$ independent of u s.t. $B[u,u] \geq \beta\|u\|_{H^1}^2$.

Therefore since $v \longmapsto \int_D fv\,dx$ is a bounded linear functional on $H_0^1(D)$, by Lax-Milgram, if $|a(x)|$ is sufficiently small, $\exists$ a unique $u \in H_0^1(D)$ s.t.

$$\int_D \nabla u \cdot \nabla v - a(x)uv\,dx = -\int_D fv\,dx$$

$\forall\, v \in H_0^1(D)$. Since

$$\int_D \nabla u \cdot \nabla v\,dx = -\int_D v\Delta u\,dx + \int_S \frac{\partial u}{\partial \nu} v\,d\sigma$$

$$= -\int_D v\Delta u\,dx,$$

it follows that

$$\int_D (\Delta u + a(x)u)v\,dx = \int_D fv\,dx. \quad \forall\, v \in H_0^1(D).$$

This shows the existence of a solution in $H^1(D)$ assuming $f \in L^2$.

$\longrightarrow$ assume $\lim_{x\to\infty} u_1(x,t) = \lim_{x\to\infty} u_2(x,t)$ for each t.

FOO #2: Let u_1, u_2 be 2 classical bounded solutions of the given equation. Let $w := u_1 - u_2$. Then

$$w_t - \Delta w + w(u_1+u_2) = 0, \quad x \in \mathbb{R}^N,\ 0<t<T$$
$$w(x,0) = 0.$$

Let

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^N} w(x,t)^2\,dx.$$

Then

$$\dot{E}(t) = \int_{\mathbb{R}^N} w\,w_t\,dx = \int_{\mathbb{R}^N} w\big(\Delta w - w(u_1+u_2)\big)\,dx$$

$$= \int_{\mathbb{R}^N} w\Delta w\,dx - \int_{\mathbb{R}^N} w^2(u_1+u_2)\,dx$$

Since $w(\infty)=0$ ~~[illegible]~~,

$$= -\int_{\mathbb{R}^N} |\nabla w|^2\,dx - \int_{\mathbb{R}^N} w^2(u_1+u_2)\,dx$$

$$\le -\|u_1+u_2\|_{L^\infty}\int_{\mathbb{R}^N} w^2\,dx$$

$$\le -2\|u_1+u_2\|_{L^\infty} E(t).$$

Thus by Gronwall's Inequality,

$$E(t) \le E(0)\,e^{-\int_0^t 2\|u_1+u_2\|_\infty\,ds} = 0$$

Therefore $w(x,t)=0 \longrightarrow u_1 = u_2$.

FOO #4

Let v_1, v_2, v_3 be smooth compactly supported functions. Then the minimizer u_1, u_2, u_3 satisfies

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(F[u+\varepsilon v] - F[u] \right).$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Bigg(\int_D \sum_{j,k=1}^{3} (u_{j x_k} + \varepsilon v_{j x_k})^2 + \alpha \Big(\sum_{j=1}^{3} (u_j + \varepsilon v_j)^2 - 1 \Big)^2 - \int_D \sum_{j,k=1}^{3} (u_{j x_k})^2 + \alpha \Big(\sum_{j=1}^{3} u_j^2 - 1 \Big)^2 \Bigg)$$

Thus

$$0 = \int_D \sum_{j,k=1}^{3} 2 u_{j x_k} v_{j x_k} + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[\alpha \int_D \Big(\Big(\sum_{j=1}^{3} u_j^2 - 1 \Big) + 2 \sum_{j=1}^{3} u_j v_j \varepsilon + \sum_{j=1}^{3} \varepsilon^2 v_j^2 \Big)^2 - \alpha \int_D \Big(\sum_{j=1}^{3} u_j^2 - 1 \Big)^2 \Big]$$

$$= \int_D \sum_{j,k=1}^{3} 2 u_{j x_k} v_{j x_k} + 4 \Big(\sum_{j=1}^{3} u_j^2 - 1 \Big) \Big(\sum_{j=1}^{3} u_j v_j \Big) \, dx.$$

$$= \int_D - \sum_{j,k=1}^{3} 2 u_{j x_k x_k} v_j + 4 \Big(\sum_{j=1}^{3} u_j^2 - 1 \Big) \Big(\sum_{j=1}^{3} u_j v_j \Big) \, dx.$$

$$= \int_D \sum_{j=1}^{3} \Big[-2 \sum_{k=1}^{3} (u_j)_{x_k x_k} + 4 \Big(\sum_{\ell=1}^{3} u_\ell^2 - 1 \Big) u_j \Big] v_j.$$

Since the v_j are arbitrary, we have

$$-\Delta u_j + 4 (|\vec{u}|^2 - 1) u_j = 0 \quad \text{on } D, \; j = 1,2,3$$
$$u_j = \varphi_j \quad \text{on } \partial D.$$

$$-\Delta \vec{u} + 4 (|\vec{u}|^2 - 1) \vec{u} = 0 \quad \text{on } D.$$
$$\vec{u} = \vec{\varphi} \quad \text{on } \partial D.$$

~~Foo~~ Egg #6: We will use Laplace Transform. Since $u(x,0)=0$,

$$\int_0^\infty u_t(x,t)e^{-st}\,dt = -\int_0^\infty u(x,t)(-se^{-st})\,dt$$
$$= s\int_0^\infty u(x,t)e^{-st}\,dt.$$

Fix an $x \geq 0$. Then as

$$u_t - u_{xx} + au = 0$$

we have

$$s\mathcal{L}[u] - \mathcal{L}[u]_{xx} + a\mathcal{L}[u] = 0.$$
$$\longrightarrow \mathcal{L}[u]_{xx} - (s+a)\mathcal{L}[u] = 0.$$

~~Therefore as we want~~

~~$\mathcal{L}[u]$~~

We will find a bounded solution to the problem, so we will want $\mathcal{L}[u]$ to be bounded. We have

$$\mathcal{L}[u](s) = Ae^{-\sqrt{s+a}\,x}$$

Since ~~$\mathcal{L}[u]$~~ $u(0,t) = g(t)$, $\mathcal{L}[u(0,t)] = \mathcal{L}[g]$.

Therefore $A = \mathcal{L}[g]$ and hence

$$\mathcal{L}[u] = \mathcal{L}[g]e^{-\sqrt{s+a}\,x}.$$
$$\longrightarrow u(x,t) = g * \mathcal{L}^{-1}[e^{-\sqrt{s+a}\,x}].$$