Sol #1: We solve this equation using method of characteristics. (and the notation of Evans).

We want to solve

$$u_{x_1} + u_{x_2} = u^2$$
$$u(x_1, 0) = h(x_1).$$

$$\rightarrow F(p, z, x) = p_1 + p_2 - z^2 = 0.$$

$$\longrightarrow \quad \begin{aligned} D_pF &= (1,1) \\ D_xF &= (0,0) \\ D_zF &= -2z. \end{aligned}$$

Then

$$\dot{p} = -D_xF - D_zFp = 2z(p_1, p_2)$$
$$\dot{z} = D_pF \cdot p = p_1 + p_2 = z^2$$
$$\dot{x} = D_pF = (1,1)$$

with

$$x_1(0) = x_1(0)$$
$$x_2(0) = 0$$
$$z(0) = h(x_1(0))$$

We have

$$x_1(s) = x_1(0) + s \qquad \dot{z} = z^2 \rightarrow \frac{1}{z^2}\,dz = ds$$
$$x_2(s) = s \qquad -\frac{1}{z(s)} = s + C$$

Since $z(0) = h(x_1(0))$, $C = -\frac{1}{h(x_1(0))}$.

Thus

$$z(s) = \frac{1}{\frac{1}{h(x_1(0))} - s} = \frac{1}{\frac{1}{h(x_1(s) - x_2(s))} - x_2(s)}.$$

That is

$$u(x_1, x_2) = \frac{1}{\frac{1}{h(x_1 - x_2)} - x_2}.$$

Sol #2:

① Mass:

We have

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t\,dx = -\int_{-\infty}^{\infty} u_{xxx}\,dx - 6\int_{-\infty}^{\infty} uu_x\,dx$$

$$= -u_{xx}\Big]_{x=-\infty}^{\infty} - 6\cdot\frac{1}{2}u^2\Big]_{x=-\infty}^{\infty} = 0.$$

② Momentum:

We have

$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx = \int_{-\infty}^{\infty} 2uu_t\,dx = 2\int_{-\infty}^{\infty} u(-u_{xxx}-6uu_x)\,dx$$

$$= -2\int_{-\infty}^{\infty} uu_{xxx}\,dx - 12\int_{-\infty}^{\infty} u^2u_x\,dx$$

$$= -2\int_{-\infty}^{\infty} uu_{xxx}\,dx - 12\cdot\frac{1}{3}u^3\Big]_{x=-\infty}^{\infty}$$

Since

$$\int_{-\infty}^{\infty} uu_{xxx}\,dx = -\int_{-\infty}^{\infty} u_xu_{xx}\,dx = +\int_{-\infty}^{\infty} u_{xx}u_x\,dx$$

$$= -\int_{-\infty}^{\infty} u_{xxx}u\,dx,$$

it follows that $\int_{-\infty}^{\infty} uu_{xxx}\,dx = 0$. Thus Momentum is conserved.

③ Energy:

We have

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2}u_x^2 - u^3\,dx = \int_{-\infty}^{\infty} u_xu_{xt} - 3u^2u_t\,dx$$

$$= \int_{-\infty}^{\infty} u_x(-u_{xxx} - 6uu_x)_x + 3u^2(u_{xxx}+6uu_x)\,dx.$$

S06 #2
cont

Note $\int_{-\infty}^{\infty} u^3 u_x \, dx = \frac{1}{4} u^4 \Big]_{u=-\infty}^{\infty} = 0$.

$$= -\int_{-\infty}^{\infty} u_x \overbrace{\cancel{[u_{xxx} + 6u_x^2 + 6uu_{xx}]}}^{(u_{xxx} + 6uu_x)_x} \, dx + 3\int_{-\infty}^{\infty} u^2 u_{xxx} \, dx$$

$$= \int_{-\infty}^{\infty} u_{xx} (u_{xxx} + 6uu_x) \, dx + 3\int_{-\infty}^{\infty} u^2 u_{xxx} \, dx$$

$$= \int_{-\infty}^{\infty} u_{xx} u_{xxx} + \cancel{6uu_x u_{xx}} \, dx - \cancel{6\int_{-\infty}^{\infty} uu_x u_{xx} \, dx}$$

$$= \int_{-\infty}^{\infty} u_{xx} u_{xxx} \, dx = 0$$

as

$$\int_{-\infty}^{\infty} u_{xx} u_{xxx} \, dx = -\int_{-\infty}^{\infty} u_{xxx} u_{xx} \, dx.$$

Sol #3:

Consider the Sturm Liouville problem

$$(p(x)u')' = -\lambda u$$
$$u'(0) = u'(L) = 0.$$

This is indeed Sturm-Liouville since $p(x) > 0$. Thus we have eigenfunctions $\{\phi_n\}$ corresponding to eigenvalues $\{\lambda_n\}_{n\geq 0}$ which form a orthogonal basis for the space of functions. ~~Since~~ Furthermore $\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty$. and

$$\lambda_0 = \min_u -\frac{\langle u, (pu')' \rangle}{\langle u, u \rangle}.$$

Since

$$\int_0^L u(pu')'\,dx = -\int_0^L p(u')^2\,dx \leq 0$$

We have $\lambda_0 \geq 0$. Since 0 is an eigenvalue!, $\lambda_0 = 0$.
Thus $\lambda_n > 0 \ \forall n > 0$.
Here

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x).$$

We have

$$u_t = \partial_x(p(x)u_x)$$
$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) = \sum_{n=0}^{\infty} \partial_x(p(x)\phi_n(x))\,a_n(t).$$
$$= \sum_{n=0}^{\infty} a_n(t)(-\lambda_n)\phi_n(x).$$

Therefore

$$a_n'(t) = -\lambda_n a_n(t)$$
$$\to a_n(t) = e^{-\lambda_n t}a_n(0) \qquad a_n(0) = \frac{\int_0^L \varphi(x)\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx}$$

S06 #3 cont:

Thus

$$\lim_{t \to \infty} u(x,t) = a_0(0) = \frac{\int_0^L \varphi(x)\phi_0(x)\,dx}{\int_0^L \phi_0(x)^2\,dx}$$

Since $\phi_0 = 1$ is the eigenfunction corresponding to $\lambda_0 = 0$, we have

$$\lim_{t \to \infty} u(x,t) = a_0(0) = \frac{1}{L}\int_0^L \varphi(x)\,dx \quad \#$$

506 #4:

Let
$$f(y) = \begin{cases} y\log(2+\frac{1}{|y|}) & \text{for } y \neq 0 \\ 0 & \text{for } y = 0. \end{cases}$$

We will show that the ODE
$$\frac{dy}{dt} = f(y)$$
$$y(0) = 0$$
has only the zero solution. This will be shown by showing if $y(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$, then $y(t) \neq 0 \ \forall t \in \mathbb{R}$

Suppose $\exists\ t^*$ w/ $y(t^*) \neq 0$. Replacing y w/ $-y$ if necessary, we may suppose WLOG that $y(t^*) > 0$. Let (a, b) be the largest open interval containing t^* s.t. $y > 0$ (here we have implicitly used continuity of y).

① $b = \infty$

Pf: Suppose $b < \infty$. Then by continuity, $y(b) = 0$, but
$$y(b) = y(t^*) + \int_{t^*}^{b} f(y(t))\,dt \geq y(t^*) > 0.$$
$\not\!\!E$

② $a = -\infty$.

Pf: Suppose $a > -\infty$. Then $y(a) = 0$.

S06 #4 cont.:

Let $g(x) = \int_1^x \frac{1}{s \log(2+\frac{1}{|s|})} ds$, $x > 0$. Then

$$\frac{d}{dt}(g(y(t))) = 1 \quad \forall t \in (a,b)$$

and so

$$g(y(t^*)) - \lim_{t \to a^+} g(y(t)) = t^* - a.$$

Since $y(a) = 0$ and $\lim_{x \to 0} g(x) = -\infty$ and $g(y(t^*)) < \infty$,

we have $g(y(t^*)) - \lim_{t \to a^+} g(y(t)) = \infty$, a contradiction.

Therefore if $\exists\, t^*$ w/ $y(t^*) \neq 0$, then $y \neq 0\ \forall t$.

Thus the zero sol. is the only solution.

It now remains to show f is not Lipschitz. We have

$$\left| \frac{f(\frac{1}{n}) - f(\frac{1}{2n})}{\frac{1}{2n}} \right| = \left| \frac{\frac{1}{n}\log(2+n) - \frac{1}{2n}\log(2+2n)}{\frac{1}{2n}} \right|$$

$$= \left| 2\log(2+n) - \log(2+2n) \right|$$

$$= \left| \log\left(\frac{(2+n)^2}{2+2n}\right) \right| \to \infty \text{ as } n \to \infty.$$

Therefore f is not Lipschitz. $\#$

S06 #5: We rewrite the ~~system~~ ODE as a system:

$$x' = y \qquad (*)$$
$$y' = -x - 2x^2.$$

This is a Hamiltonian system and hence all equilibrium points are either centers or saddles. Let

$$H(x,y) := \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3 = \frac{1}{2}(x')^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3.$$

Then

$$\frac{d}{dt}H(x,y) = x'x'' + xx' + 2x^2x'$$
$$= x'(x'' + x + 2x^2) = 0.$$

Thus $\frac{1}{2}(x')^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3$ is a conserved quantity.

The equilibrium points of $(*)$ are $(0,0)$ and $(-\frac{1}{2},0)$.
The Jacobian is

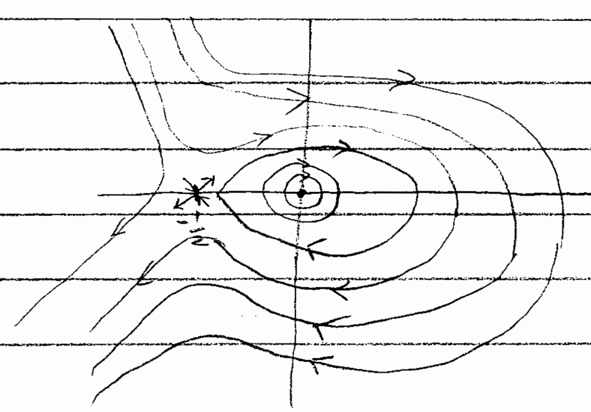
$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -1-4x & 0 \end{pmatrix}.$$

We have

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \text{eigenvalues } \pm i. \qquad \text{center}$$

$$J(-\tfrac{1}{2},0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \text{eigenvalues } \pm 1 \qquad \text{saddle.}$$

eigenvectors $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$



Sob #6: ~~Recall~~ Suppose $\max u(x) > 0$. Let x_0 be s.t. $u(x_0) = \max u(x)$

as $u=0$ on $\partial\Omega$

Then, $x_0 \in \Omega$ and, hence, $\Delta u(x_0) \leq 0$, $u_{x_k}(x_0) = 0$. Then at the point x_0,

$$\Delta u + \sum_{k=1}^{n} a_k(x) u_{x_k} + c(x)u = \Delta u + c(x_0)u(x_0)$$
$$\leq c(x_0)u(x_0) < 0$$

as $u(x_0) > 0$ and $c(x) < 0$ in Ω. This is a contradiction.

Thus $\max u(x) \leq 0$.

Suppose $\min u(x) < 0$. Let y_0 be s.t. $u(y_0) = \min u(x)$

Then as $u=0$ on $\partial\Omega$, $y_0 \in \Omega$ and, hence, $\Delta u(y_0) \geq 0$, $u_{x_k}(y_0) = 0$. At the point y_0,

$$\cancel{\Delta u + \sum_{k=1}^{n} a_k(x) u_{x_k}}$$

$$(\Delta u)(y_0) + \sum_{k=1}^{n} a_k(y_0) u_{x_k}(y_0) + c(y_0)u(y_0)$$
$$= (\Delta u)(y_0) + c(y_0)u(y_0)$$
$$\geq c(y_0)u(y_0) > 0$$

Since $u(y_0) > 0$ and $c(x) > 0$ $\forall x \in \Omega$. This is a contradiction.

Thus $\min u(x) \geq 0$.

Therefore $u=0$ on $\partial\Omega$ implies $u=0$ on Ω.

Lagrange multipliers: optimize f subj. to $g=k$.

1. Solve system $\nabla f = \lambda \nabla g$, $g=k$.

2. Plug in all solutions (x,y,z) from 1. into $f(x,y,z)$ to find min./max.

S06 #7: here $E[u] := \int_\Omega \frac{1}{2}|\nabla u|^2 - f u \, dx$, $F[u] := \int_\Omega u \, dx$. We want to minimize $E[u]$ subject to $F[u] = A$ and $u|_{\partial\Omega} = 0$.

By Lagrange multipliers, the minimum u must satisfy

$$\left(\lim_{\varepsilon\to 0} \frac{E[u+\varepsilon v]-E[u]}{\varepsilon}\right) = \lambda \left(\lim_{\varepsilon\to 0} \frac{F[u+\varepsilon v]-F[u]}{\varepsilon}\right)$$

the moral is choose v so that $u+\varepsilon v$ has the same properties as u.

for some constant λ and v s.t. $v|_{\partial\Omega} = 0$ and $\int_\Omega v \, dx = 0$ (same E eats functions "with $u|_{\partial\Omega}=0$" and $\int_\Omega u\,dx = A$ thus we need $\int_\Omega v \, dx = 0$ so that $\int_\Omega u+\varepsilon v \, dx = A$). We have

$$\lim_{\varepsilon\to 0} \frac{E[u+\varepsilon v]-E[u]}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega \frac{1}{2}|\nabla u + \varepsilon \nabla v|^2 - (f)(u+\varepsilon v) - \frac{1}{2}|\nabla u|^2 + fu \, dx$$

$$= \int_\Omega \nabla u \cdot \nabla v - f v \, dx$$

$$= \int_\Omega (-\Delta u - f) v \, dx$$

and

$$\lim_{\varepsilon\to 0} \frac{F[u+\varepsilon v]-F[u]}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_\Omega u + \varepsilon v - u \, dx = \int_\Omega v \, dx = 0.$$

Thus

$$\int_\Omega (-\Delta u - f) v \, dx = 0 \longrightarrow -\Delta u = f \text{ in } \Omega.$$

Therefore the minimum of $E[\cdot]$ subject to $u|_{\partial\Omega} = 0$ and $\int_\Omega u \, dx = A$ is the unique solution to

$$\Delta u = -f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$