Sol #1

a) Note that we are working w/ inner normals.
Let u, v be s.t.

$$\nabla \cdot \gamma(x) \nabla u = 0 \text{ in } B \qquad \nabla \cdot \gamma(x) \nabla v = 0 \text{ in } B$$
$$u = f \text{ on } \partial B \qquad v = g \text{ on } \partial B.$$

Then

$$\int_{\partial B} g \gamma(x) \frac{\partial f}{\partial \nu} d\sigma = \int_{\partial B} u \gamma(x) \sum_i v_{x_i} \nu^i d\sigma$$
$$= \sum_i \int_{\partial B} u (\gamma(x) v_{x_i}) \nu^i d\sigma$$
$$= \sum_i - \int_B u_{x_i} \gamma(x) v_{x_i} dx - \int_B u (\gamma(x) v_{x_i})_{x_i} dx$$
$$= \cancel{\sum_i} - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx - \int_B u \nabla \cdot \gamma(x) \nabla v \, dx$$
$$= - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx.$$

~~We also have~~ Then

$$\int_{\partial B} f \gamma(x) \frac{\partial g}{\partial \nu} d\sigma = - \int_B \nabla v \cdot \gamma(x) \nabla u \, dx$$
$$= - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx = \int_{\partial B} g \gamma(x) \frac{\partial f}{\partial \nu} d\sigma.$$

b)

$$\int_{\partial B} f \Lambda(f) d\sigma = - \int_B |\nabla f|^2 \gamma(x) \leq 0.$$

since $\gamma(x) > 0$.

Sol #2:

a) The eigenvalues of $\begin{pmatrix}1&0\\5&3\end{pmatrix}$ are 1, 3 with corresponding eigenvectors $\begin{pmatrix}-2\\5\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. Thus

$$\begin{pmatrix}-2&0\\5&1\end{pmatrix}\begin{pmatrix}1&\\&3\end{pmatrix}\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0\\5&3\end{pmatrix}.$$

Let $w=\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}\begin{pmatrix}u_1\\u_2\end{pmatrix}$. Then

$$w_t-\begin{pmatrix}1&\\&3\end{pmatrix}w_x=0.$$

with

$$w(x,0)=\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}\begin{pmatrix}\exp(ixa)\\0\end{pmatrix}=\begin{pmatrix}-1/2&0\\5/2&1\end{pmatrix}\begin{pmatrix}\exp(ixa)\\0\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{2}\exp(iax)\\ 5/2\exp(iax)\end{pmatrix}.$$

Since $w=\begin{pmatrix}w_1\\w_2\end{pmatrix}$,

$$(w_1)_t-(w_1)_x=0 \qquad w_1(x,0)=-\frac{1}{2}\exp(iax),$$
$$(w_2)_t-3(w_2)_x=0 \qquad w_2(x,0)=\frac{5}{2}\exp(iax),$$

$$\longrightarrow\ w_1(x,t)=-\frac{1}{2}\exp(ia(x+t))$$
$$w_2(x,t)=\frac{5}{2}\exp(ia(x+3t)).$$

and hence

$$u=\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}-2&0\\5&1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}-2w_1\\5w_1+w_2\end{pmatrix}=\begin{pmatrix}\exp(ia(x+t))\\-\frac{5}{2}\exp(ia(x+t))+\frac{5}{2}\exp(ia(x+3t))\end{pmatrix}$$

b) We have

$$(u_1)_t-(u_1)_x=0. \qquad u_1(x,0)=f(x) \qquad (*)$$
$$(u_2)_t-5(u_1)_x-3(u_2)_x=0. \qquad u_2(x,0)=0.$$

Sol #2 cont:

Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx$. Then $\hat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$.

So

$$(\hat{u}_1)_t - 2\pi i \xi\, \hat{u}_1 = 0 \qquad \hat{u}_1(\xi, 0) = \hat{f}(\xi)$$

$$(\hat{u}_2)_t - 2\pi i \xi \cdot 5 \hat{u}_1 - 2\pi i \xi \cdot 3 \hat{u}_2 = 0 \qquad \hat{u}_2(\xi, 0) = 0.$$

$$\hat{u}_t - 2\pi i \xi \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} \hat{u} = 0 \qquad \hat{u}(\xi, 0) = \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix}.$$

$$\longrightarrow \hat{u}(\xi, t) = \exp\left(2\pi i \xi t \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}\right) \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix}$$

Then

$$u(x,t) = \left[\exp\left(2\pi i \xi t \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}\right) \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix} \right]^{\vee} \qquad \#$$

Sol #3:

We want to solve $2u_t - u_x^2 = x^2$.

$u(x,0) = x$.

Then $F(p,q,z,x,t) = 2q - p^2 - x^2$ and

$$\dot{t}(s) = 2 \qquad t(0) = 0$$
$$\dot{x}(s) = -2p \qquad x(0) = x_0$$
$$\dot{z}(s) = -2p^2 + 2q \qquad z(0) = x_0$$
$$\dot{p}(s) = 2x \qquad p(0) = 1$$
$$\dot{q}(s) = 0 \qquad q(0) = \frac{1+x_0^2}{2}.$$

$\rightarrow t(s) = 2s.$

$\ddot{x}(s) = -4x(s) \rightarrow x(s) = A\cos(2s) + B\sin(2s)$

$x_0 = x(0) = A$

$\dot{x}(s) = -2A\sin(2s) + 2B\cos(2s)$

$-2p(0) = \dot{x}(0) = 2B \rightarrow B = -1.$

$\rightarrow x(s) = x_0\cos(2s) - \sin(2s).$

$\ddot{p} = -4p \rightarrow p(s) = A\cos(2s) + B\sin(2s)$

$p(0) = 1 \rightarrow A = 1$

$\dot{p}(0) = 2x(0) = 2x_0$

$\dot{p}(s) = -2\sin(2s) + 2B\cos(2s)$

$2x_0 = 2B \rightarrow B = x_0.$

$\rightarrow p(s) = \cos(2s) + x_0\sin(2s).$

We have

$$\begin{aligned}\dot{z}(s) = -2p^2 + 2q &= x^2 - p^2 = x_0^2\cos^2 2s - 2x_0\cos 2s\sin 2s + \sin^2 2s \\ &\quad - \cos^2 2s - 2x_0\cos 2s\sin 2s - x_0^2\sin^2 2s \\ &= (x_0^2 - 1)\cos^2 2s - (x_0^2 - 1)\sin^2 2s - 4x_0\cos 2s\sin 2s \\ &= (x_0^2 - 1)\cos 4s - 2x_0\sin 4s.\end{aligned}$$

Sol #3 cont:

Thus

$$z(s) = \frac{1}{4}(x_0^2-1)\sin 4s + 2x_0 \frac{1}{4}\cos 4s + \frac{1}{2}x_0.$$
$$= \frac{1}{4}(x_0^2-1)\sin 4s + \frac{1}{2}x_0\cos 4s + \frac{1}{2}x_0.$$

$$\longrightarrow u(x,t) = \frac{1}{4}(x_0^2-1)\sin 2t + \frac{1}{2}x_0\cos 2t + \frac{1}{2}x_0$$

where $x_0 = \dfrac{x+\sin t}{\cos t}$.

The solution blows up in finite time. The characteristics are given by $x = x_0\cos t - \sin t$. For any x_0, the characteristics will intersect at time $t = \pi/2$, thus the solution blows up in finite time.

$$-\Delta\phi_n = +\lambda_n \phi_n \quad \text{in } D$$
$$\phi_n = 0 \quad \text{on } \partial D$$
$$\|\phi_n\|_{L^2} = 1.$$

Sol #4:

of the Laplacian (w/ $\|\phi_n\|_{L^2}=1$)

a) Expand u in terms of eigenfunctions. We have $u = \sum_{n=1}^{\infty} a_n \phi_n$.

Then

$$\int_D u^2\,dx = \int_D \Big(\sum_{n\geq 1} a_n \phi_n\Big)^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

and

$$\int_D |\nabla u|^2\,dx = -\int_D u\,\Delta u\,dx = +\int_D \Big(\sum_{n=1}^{\infty} a_n\phi_n\Big)\Big(\sum_{n=1}^{\infty}\lambda_n a_n \phi_n\Big)dx$$

$$\cancel{= \int_D \sum_{n\geq 1} \lambda_n a_n^2 \phi_n}$$

$$= +\int_D \sum_{n\geq 1} \lambda_n a_n^2 \phi_n^2\,dx$$

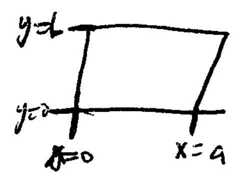
$$= +\sum_{n\geq 1} \lambda_n a_n^2.$$

Thus since $0 < \lambda_1 \leq \lambda_2 \leq \ldots$,

$$\int_D u^2\,dx = \sum_{n\geq 1} a_n^2 \leq \frac{1}{\lambda_1}\sum_{n\geq 1}\lambda_n a_n^2 = \frac{1}{\lambda_1}\int_D |\nabla u|^2\,dx.$$

b) We solve

$$-\Delta u = \lambda u \quad \text{in } D$$
$$u = 0 \quad \text{on } \partial D.$$



$$u = F(x)\,G(y)$$

$$-F''(x)G(y) - F(x)G''(y) = \lambda F(x) G(y)$$

$F(0) = F(a) = 0$

$G(0) = G(b) = 0.$

$$\frac{-F''(x) - \lambda F(x)}{F(x)} = +\frac{G''(y)}{G(y)}$$

Sol #4 con't:

$$\frac{G''(y)}{G(y)} = -\mu^2.$$

$$G(y) = A\cos(\mu y) + B\sin(\mu y)$$

$$0 = G(0) = A \longrightarrow G(y) = B\sin(\mu y)$$

$$0 = G(b) = B\sin(\mu b)$$

$$\longrightarrow \mu_n = \frac{n\pi}{b},\ n = 1, 2, \ldots$$

$$-\mu_n^2 = -\left(\frac{n\pi}{b}\right)^2.$$

$$G_n(y) = \sin\left(\frac{n\pi b}{y}\right)$$

$$\frac{-F'' - \lambda F}{F} = -\left(\frac{n\pi}{b}\right)^2$$

$$\longrightarrow F'' + \left(\lambda - \left(\frac{n\pi}{b}\right)^2\right)F = 0. \longrightarrow \text{we must have } \lambda - \left(\frac{n\pi}{b}\right)^2 > 0 \text{ otherwise } F = 0.$$

$$F = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right) + B\cos\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right)$$

$$0 = F(0) \longrightarrow B = 0 \longrightarrow F(x) = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right)$$

$$0 = F(a) = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,a\right)$$

$$\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,a = m\pi,\ m = 1, 2, \ldots$$

$$\lambda = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \quad m, n = 1, 2, \ldots$$

Therefore the smallest λ is given by

$$\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{a}\right)^2.$$

Thus the best constant is $\dfrac{1}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$. #

S01 #5:

We will assume that f is smooth (or at least unif. cont.).

Lemma: If $f \in L^2(\mathbb{R})$ and smooth $\lim_{x \to \pm\infty} |f|^2 = 0$.

Pf: ~~Suppose~~ We first show the limit exists. Suppose it did not. Then $\exists\ \varepsilon_0$ and a sequence of $x_n \to \pm\infty$ with $|x_n - x_{n+1}| > 2$ s.t. $|f(x_n)| > \varepsilon_0\ \forall n$. Since f is unif. cont, then $\exists\ \delta < 1$ s.t. if $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon_0}{2}\ \forall x, y$. Thus if $|x - x_n| < \delta$, $|f(x)| > \frac{\varepsilon_0}{2}$. Then if $|x - x_n| < \delta$, $|f(x)|^2 > \frac{\varepsilon_0^2}{4}$. Thus

$$\int_{|x-x_n|<\delta} |f(x)|^2\, dx > \frac{\varepsilon_0^2}{4} \cdot 2\delta = \frac{1}{2}\varepsilon_0^2 \delta.$$

$$\int_{\mathbb{R}} |f|^2 \geq \sum_n \int_{|x-x_n|<\delta} |f|^2 > \sum_n \frac{1}{2}\varepsilon_0^2 \delta = \infty.$$

Therefore the limit exists.

~~If~~ $\cancel{\lim_{x \to \pm\infty} |f|^2 = L \neq 0}$, ~~~~then~~ fix arb. $\varepsilon > 0$, then $\exists$ a sequence of $\{x_n\}$ s.t. $|f(x_n) - \sqrt{L}| < \frac{\varepsilon}{2}$. By uniform continuity of f, $\exists\ \delta > 0$ s.t. if $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then if $|x - x_n| < \delta$, $|f(x) - \sqrt{L}| \leq \varepsilon \to (\sqrt{L} - \varepsilon)^2 \geq |f(x)|^2$.~~

If $\lim_{x \to \pm\infty} f^2(x) = L \neq 0$, fix arb. small $\varepsilon > 0$, then $\exists$ a seq. of $\{x_n\}$ spaced w/ $|x_n - x_{n+1}| > 2$ s.t. $|f(x_n)^2 - L| < \frac{\varepsilon}{2}$. By smoothness of f^2, $\exists\ \delta > 0$ s.t. if $|x - y| < \delta$, $|f(x)^2 - f(y)^2| < \frac{\varepsilon}{2}$. Then if $|x - x_n| < \delta$, S01 #5 cont:

$$|f(x)^2 - L| \le |f(x)^2 - f(x_n)^2| + |f(x_n)^2 - L| \le \varepsilon.$$

Thus if $|x - x_n| < \delta$, $L - \varepsilon \le |f(x)|^2$. Then

$$\int |f(x)|^2 \geq \sum_n \int_{|x-x_n|<\delta} |f(x)|^2 \, dx$$

$$\geq \sum_n (L - \varepsilon)$$

Since $L > 0$, we have a contradiction. Therefore

$$\lim_{x \to \pm\infty} |f|^2 = 0.$$

---

a) Taking the Fourier transform,

$$\hat{u}_t = -4\pi \xi^2 \hat{u}. \quad \hat{u}(\xi, 0) = \hat{f}(\xi)$$

$$\hat{u}(\xi, t) = e^{-4\pi\xi^2 t} \hat{f}(\xi).$$

Thus

$$|u(x, t)| = \left| \int_{\mathbb{R}} e^{-4\pi t \xi^2} \hat{f}(\xi) e^{2\pi i x \xi} d\xi. \right|$$

$$\le \left(\int_{\mathbb{R}} e^{-8\pi t \xi^2} d\xi \right)^{1/2} \|f\|_{L^2}. \longrightarrow 0$$

as $t \to \infty$ since

$$\int_{-\infty}^{\infty} e^{-8\pi t \xi^2} d\xi = \frac{1}{\sqrt{8\pi t}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

Therefore $u \to 0$ as $t \to \infty$ uniformly in x.

Sol #5 cont:

Note that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(x-y)^2}{4t}} dy.$$

and $\lim_{\substack{x \to \pm\infty \\ t > 0}} u(x,t) = \lim_{x \to \pm\infty} f(x) = 0$ by the previous lemma.

We have

$$u_x(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\left(\frac{x-y}{2\sqrt{t}}\right)^2} \left(-\frac{x-y}{2t}\right) dy$$

$$\leq \frac{1}{2\sqrt{\pi}\, t^{1/2}} \left(\int_{\mathbb{R}} |f|^2\right)^{1/2} \left(\int_{-\infty}^{\infty} e^{-2\left(\frac{x-y}{2\sqrt{t}}\right)^2} \frac{(x-y)^2}{4t^2} dy\right)^{1/2}.$$

Let $u = \frac{x-y}{2\sqrt{t}}$. Then $du = -\frac{1}{2\sqrt{t}} dy$.

$$\int_{-\infty}^{\infty} e^{-2\left(\frac{x-y}{2\sqrt{t}}\right)^2} \frac{(x-y)^2}{4t^2} dy = \int_{-\infty}^{\infty} e^{-2u^2} \frac{(2\sqrt{t}u)^2}{4t^2} 2\sqrt{t}\, du$$

$$= 2\left(\int_{-\infty}^{\infty} e^{-2u^2} u^2 du\right) t^{-1/2}.$$

Therefore

$$|u_x| \leq C t^{-3/4} \|f\|_{L^2}.$$

b) Let $E(t) := \int_{\mathbb{R}} |\rho|^2 dx$. Then

$$E'(t) = 2\int_{\mathbb{R}} \rho \rho_t \, dx = 2 \int_{\mathbb{R}} -u\rho\rho_x \, dx.$$

Sol #5 cont:

$$= -2\left[-\int_{\mathbb{R}} (u\rho)_x \,\rho \, dx\right]$$

since $u=0$ at $x=\pm\infty$.

$$= 2\int_{\mathbb{R}} u_x \rho^2 + u\rho_x\rho \, dx.$$

$$\leq 4C_f t^{-3/4} E(t) + 2\int_{\mathbb{R}} u\rho_x\rho\, dx.$$

$$\lesssim t^{-3/4} E(t) + t^{-1/4}\int_{\mathbb{R}} \rho_x\rho\, dx.$$

$$\lesssim t^{-3/4} E(t) + t^{-1/4}\left(\tfrac{1}{2}\rho^2\right)\Big|_{x=-\infty}^{\infty}$$

Thus if $\rho(x,t) \longrightarrow 0$ as $|x| \longrightarrow \infty$, then $E'(t) \lesssim t^{-3/4} E(t)$ ~~It is not obvious that this will follow from any assumptions on ρ_0 b/c the characteristics are given by solving $\frac{dx}{dt} = u(x,t)$.~~

and hence by Gronwall,

$$E(t) \leq E(0) e^{Ct^{1/4}}$$

$$\rightarrow \int_{\mathbb{R}} |\rho(x,t)|^2 dx \leq \left(\int_{\mathbb{R}} |\rho_0(x)|^2 dx\right) e^{Ct^{1/4}}.$$

for some C depending only on f.

Assume ρ_0 is of compact support. ~~The characteristics~~ We solve $\rho_t + u\rho_x = 0$, $\rho(x,0) = \rho_0(x)$ via method of characteristics. We have

$$\dot{x} = u(x,t) \qquad x(0) = x_0$$

$$\dot{t} = 1 \qquad t(0) = 0$$

$$\dot{z} = 0 \qquad z(0) = \rho_0(x_0).$$

Sol #5 cont'd.

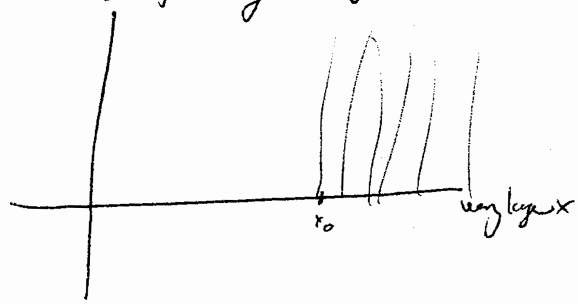
Therefore

$\rho(x,t) = \rho_0(x_0)$.

$\frac{dx}{ds} = u(x(s), s)$

$x(0) = x_0$.

When $x(s)$ is very large $u(x(s), s)$ is very close to 0. Therefore x is basically constant. Thus the characteristics are $x = x_0$ for large x_0.



Thus for very large x, $\rho(x,t) \to 0$. as $|x| \to \infty$.

Sol #6

a) Let V_1, V_2 be 2 conductor potentials, ~~then~~ and let $W := V_1 - V_2$.
Thus $\Delta W = 0$ on $\mathbb{R}^3 \setminus B$, $W = 0$ on ∂B, and $W \to 0$ as $|x| \to \infty$.
Thus

$$0 = \int_{\mathbb{R}^3 \setminus B} W \Delta W = -\int_{\mathbb{R}^3 \setminus B} |\nabla W|^2 + \int_{\partial(\mathbb{R}^3 \setminus B)} W \frac{\partial W}{\partial \nu} d\sigma = -\int_{\mathbb{R}^3 \setminus B} |\nabla W|^2 dx$$

so W is a constant on $\mathbb{R}^3 \setminus B$ and hence $W = 0$. Therefore
$V_1 = V_2$

b) Claim: $V(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Delta V(y)}{|x-y|} dy$

Pf: Let $u(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Delta V(y)}{|x-y|} dy$. Then $\Delta(V - u) = 0$.
Since $V - u \to 0$ as $|x| \to \infty$, $V - u$ is bounded.
Thus by Liouville's Theorem, $V - u$ is a constant. Since
$V - u \to 0$ as $|x| \to \infty$, $V = u$. #

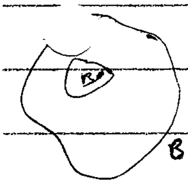
We have

$$|x| V(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Delta V(y) |x|}{|x-y|} dy$$

$$\to \lim_{|x| \to \infty} |x| V(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta V(y)\, dy = -\frac{1}{4\pi} \int_B \Delta V(y)\, dy$$
$$= -\frac{1}{4\pi} \int_{\partial B} \frac{\partial V}{\partial \nu}(y)\, d\sigma(y).$$

c) Let $\Delta V' = 0$ on $\mathbb{R}^3 \setminus B'$, $V' = 1$ on $\partial B'$; $\Delta V = 0$ on $\mathbb{R}^3 \setminus B$, $V = 1$ on ∂B. We show $\lim_{|x| \to \infty} |x| (V(x) - V'(x)) \geq 0$.

By the Maximum Principle and as $V, V' \to 0$ as $|x| \to \infty$, $0 \leq V' < 1$ on $\mathbb{R}^3 \setminus B'$ and $0 \leq V < 1$ on $\mathbb{R}^3 \setminus B$. So $V' - V$ is < 0 on ∂B and since $V' - V \to 0$ as $|x| \to \infty$, by the maximum Principle, we have $\lim_{|x| \to \infty} |x| (V - V') \geq 0$.



$\hookrightarrow$ $V' - V$ is $\leq \varepsilon$ for $|x| \geq R$.
on $|x| \leq R$ and $x \notin B$, by Max. Principle, $V' - V \leq \varepsilon$.

Sol #6 see packet of problems

SOT #7: Let $y := x - st$.

a. We have

$$f_t = -sf' \qquad g_t = -sg'$$
$$f_x = f' \qquad g_x = g'$$

$$\rightarrow \begin{cases} f_t + f_x = g^2 - f^2 \\ g_t - g_x = f^2 - g \end{cases} \longrightarrow \begin{matrix} -sf' + f' = g^2 - f^2. \\ -sg' - g' = g^2 - f. \end{matrix}$$

b) When $s = 0$, the system of ODEs is:

$$\begin{cases} f' = g^2 - f^2 \\ g' = f - g^2. \end{cases} \overset{\text{rewrite as}}{\Longrightarrow} \begin{cases} x' = y^2 - x^2 \\ y' = x - y^2. \end{cases}$$

Stationary points:

$$\begin{matrix} x^2 - y^2 = 0. \\ x - y^2 = 0 \end{matrix} \longrightarrow x^2 - x = 0 \longrightarrow \begin{matrix} x = 0 \ , \ 1 \\ y = 0 \quad y = \pm 1. \end{matrix}$$

So there are 3 equilibrium pts $(0,0), (1,1), (1,-1)$.

$$J(x,y) = \begin{pmatrix} -2x & 2y \\ 1 & -2y \end{pmatrix}$$

Eigenvalues   Eigenvectors

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ $\rightarrow$ system is degenerate, locally is ~~####~~

$$J(1,1) = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix} \qquad -2 \pm \sqrt{2}. \qquad \begin{pmatrix} \pm\sqrt{2} \\ 1 \end{pmatrix}$$

$$J(1,-1) = \begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix} \qquad \pm\sqrt{2}. \qquad \cancel{\begin{pmatrix} 1 \\ \mp\frac{\sqrt{2}}{2} - 1 \end{pmatrix}} \begin{pmatrix} -2 \pm \sqrt{2} \\ 1 \end{pmatrix}$$

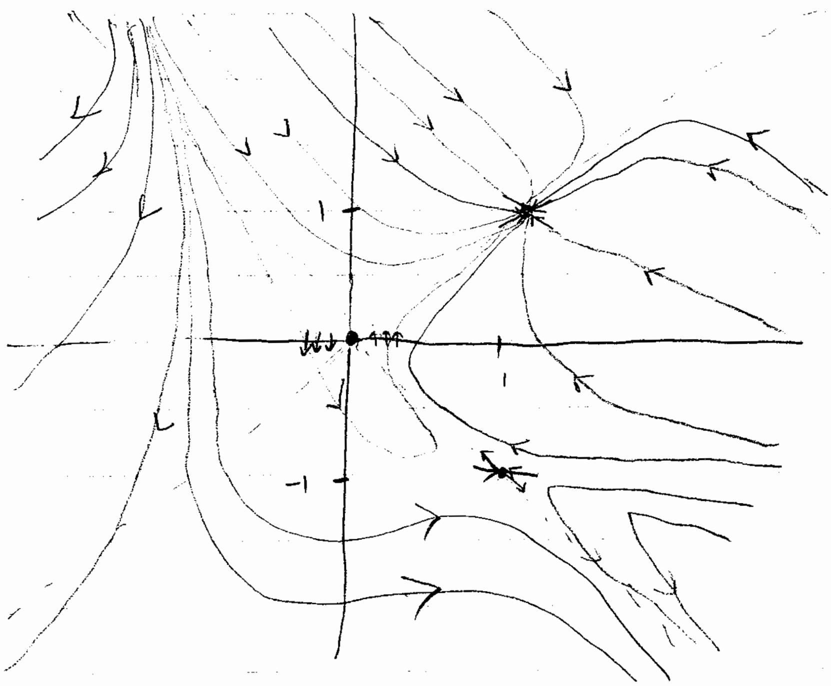
Sec #7 cont:

~~Therefore (0,0) is an improper~~

~~Therefore (1,1) and (1,−1) are saddles~~. The ~~nullcli~~ nullclines are

$$y = \pm x \qquad \frac{dy}{dx} = \frac{x-y^2}{y^2-x^2}$$

$$y = \pm\sqrt{x}.$$



(1,1) is a sink node

(1,−1) is a saddle.