F06 #1: Here

$$E(t) = \frac{1}{2}(x')^2 + \frac{1}{4}x^4 - 2x^2.$$

Then

$$\dot{E}(t) = x'x'' + x^3x' - 4xx' = x'[x'' + x^3 - 4x] = 0.$$

Thus $E(t)$ is conserved. We can write the ODE as

$$x' = y$$
$$y' = -x^3 + 4x.$$

This system is Hamiltonian and hence all equilibrium points are centers or saddles. The equilibrium points are $(\pm 2, 0)$ and $(0,0)$. The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -3x^2+4 & 0 \end{pmatrix}$$

Then

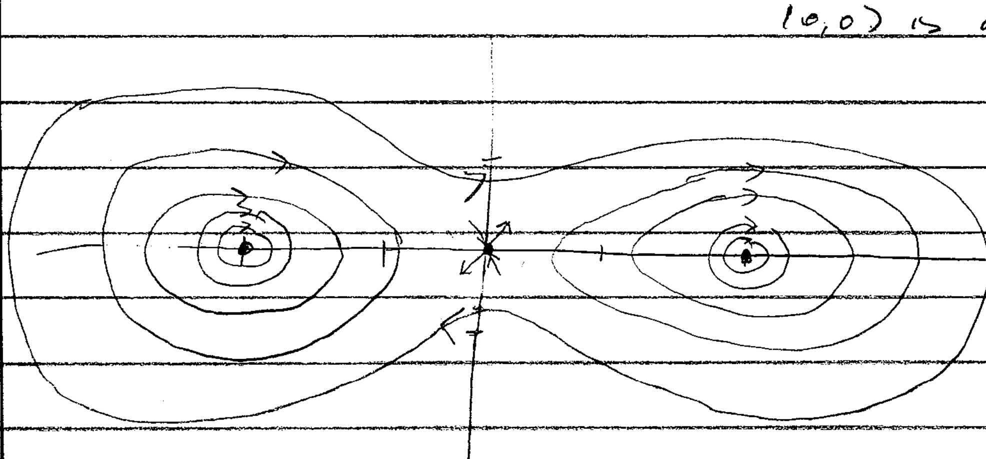
$$J(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix} \longrightarrow \text{eigenvalues } \pm 2\sqrt{2}\,i$$

$(\pm 2, 0)$ are centers.

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \longrightarrow \text{eigenvalues } \pm 2$$

eigenvectors $\begin{pmatrix} 1 \\ \pm 2 \end{pmatrix}$

$(0,0)$ is a saddle



F06 #3:

We extend u to be a function on the whole real axis in y.

u^D: Since
$$u_t = \Delta u$$
$$u(x,y,0) = u_0(x,y)$$
$$u(x,0,t) = 0,$$
we extend u to $\tilde{u}$ s.t. ~~$u(x,y,t) = -u(x,-y,t)$ for $y<0$.~~
~~Thus way $u(x,0,t) = -u(x$~~

$$\tilde{u}(x,y,t) = \begin{cases} u(x,y,t) & \text{if } y \geq 0 \\ -u(x,-y,t) & \text{if } y \leq 0. \end{cases}$$

This way $\tilde{u}(x,0,t) = -\tilde{u}(x,0,t) \longrightarrow \tilde{u}(x,0,t) = 0.$
and
$$\tilde{u}_t = \Delta \tilde{u} \quad \text{in } \mathbb{R}^2 \times \{t \geq 0\}$$
$$\tilde{u}(x,y,0) = \tilde{u}_0(x,y) = \begin{cases} u_0(x,y) & \text{if } y \geq 0 \\ -u_0(x,-y) & \text{if } y \leq 0. \end{cases}$$
$$\tilde{u}(x,0,t) = 0.$$

We have

$$\tilde{u}(x,y,t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-\frac{|(x,y)-(a,b)|^2}{4t}} \, da \, db.$$
$$= \frac{1}{4\pi t} \int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-((x-a)^2+(y-b)^2)/4t} \, da \, db.$$
$$= \frac{1}{4\pi t} \left[\int_{b \geq 0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da \, db \right.$$
$$\left. + \int_{b \leq 0} -u_0(a,-b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da \, db. \right]$$

Fob #3 con.

$$= \frac{1}{4\pi t}\left[\iint_{b\geq 0} u_0(a,b)e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\,da\,db - \int_{b>0} u_0(a,b)\,e^{-\frac{(x-a)^2+(y+b)^2}{4t}}\,da\,db\right.$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\left[1-e^{\frac{(y-b)^2-(y+b)^2}{4t}}\right]da\,db.$$

$$= \frac{1}{4\pi t}\int_0^{\infty}\int_{-\infty}^{\infty} u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\left[1-e^{-by/t}\right]da\,db$$

Thus

$$u^D(x,y,t) = \frac{1}{4\pi t}\int_0^{\infty}\int_{-\infty}^{\infty} u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\left[1-e^{-by/t}\right]da\,db.$$

u^N: We extend u to $\tilde{u}$:

$$\tilde{u}(x,y,t) = \begin{cases} u(x,y,t) & \text{if } y \geq 0 \\ u(x,-y,t) & \text{if } y \leq 0. \end{cases}$$

Thus $\tilde{u}_y(x,0,t) = -\tilde{u}_y(x,0,t) \longrightarrow \tilde{u}_y(x,0,t) = 0.$

Thus

$$\tilde{u}_t = \Delta\tilde{u} \quad \text{in } \mathbb{R}^2\times(t>0)$$

$$\tilde{u}(x,y,0) = \tilde{u}_0(x,y) = \begin{cases} u_0(x,y) & \text{if } y\geq 0 \\ u_0(x,-y) & \text{if } y\leq 0. \end{cases}$$

$$\tilde{u}_y(x,0,t) = 0.$$

F06 #3 cont.

Then

$$\tilde{u}(x,y,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db.$$

$$+ \frac{1}{4\pi t}\int_{b<0} u_0(a,-b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db.$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} + u_0(a,b) e^{-\frac{(x-a)^2+(y+b)^2}{4t}} \, da\, db.$$

$$= \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \left[1 + e^{-by/t}\right] da\, db.$$

Thus

$$u^N(x,y,t) = \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \left[1+e^{-by/t}\right] da\, db.$$

Then

$$u^D \le u^N \quad \forall\ x, y, t>0. \quad \#$$

F06 #5: Let $g(t) := a + \int_0^t f(s) y(s)^2 ds$. Then

$$g'(t) = f(t) y(t)^2 \leq f(t) g(t)^2.$$

Thus

$$\int_0^t \frac{g'(s)}{g(s)^2} ds \leq \int_0^t f(s) ds.$$

$$-\frac{1}{g(s)}\Big]_{s=0}^{t} \leq \int_0^t f(s) ds.$$

$$\longrightarrow \quad \frac{1}{a} - \frac{1}{g(t)} \leq \int_0^t f(s) ds$$

$$\frac{1}{g(t)} \geq \frac{1}{a} - \int_0^t f(s) ds.$$

$$y(t) \leq g(t) \leq \frac{1}{\frac{1}{a} - \int_0^t f(s) ds} = \frac{a}{1 - a\int_0^t f(s) ds}.$$

F06 #6: We recall $\Delta = 4\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}}$ where $\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)$.
Note w/ $K(\xi,\eta) = \frac{1}{4\pi}\log(\xi^2+\eta^2)$, $\Delta K = \delta$. Thus

$$\varphi(z) = \int_{\mathbb{C}} \varphi(\zeta)\cdot\delta(\zeta - z)\,d\zeta$$

$$= \int_{\mathbb{C}} \varphi(\zeta)\,\Delta K(\zeta - z)\,d\zeta$$

$$= \int_{\mathbb{C}} -\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)\cdot 4\frac{\partial}{\partial \zeta}K(\zeta - z)\,d\zeta$$

Note

$$\frac{\partial}{\partial \zeta}\log(\xi^2+\eta^2) = \frac{1}{2}\left(\frac{2\xi}{\xi^2+\eta^2} - i\cdot\frac{2\eta}{\xi^2+\eta^2}\right)$$

$$= \frac{1}{\xi + i\eta} = \frac{1}{\zeta}$$

Thus

$$\int_{\mathbb{C}} -\frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)\cdot 4\cdot\frac{1}{4\pi}\cdot\frac{1}{\zeta - z}\,d\zeta$$

$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta)\cdot(\zeta - z)^{-1}\,d\zeta.$$

F06 #7:

a) Let $u(x,y,t) = F(x,t)G(y)$. Then

$$u_t = \Delta u \rightarrow F_t(x,t)G(y) - F_{xx}(x,t)G(y) - F(x,t)G''(y) = 0.$$

$$\frac{F_t - F_{xx}}{F} = \frac{G''}{G}$$

Since $u_y(x,0,0) = u_y(x,\pi,0) = 0$, $G'(0) = 0$, $G'(\pi) = 0$. Thus

$$\frac{F_t - F_{xx}}{F} = \frac{G''}{G} = -\lambda^2 \text{ for } \lambda \in \mathbb{R}.$$

Then
$$\begin{aligned} G'' + \lambda^2 G &= 0 \\ G'(0) &= 0 \\ G'(\pi) &= 0 \end{aligned} \rightarrow \begin{aligned} G(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ G'(x) &= -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x) \\ 0 = G'(0) &= B\lambda \rightarrow B = 0. \\ 0 = G'(\pi) &= -A\lambda\sin(\lambda\pi) \rightarrow \lambda = n \geq 0 \end{aligned}$$

Therefore

$$u(x,y,t) = \sum_{n \geq 0} F_n(x,t)\cos(ny)$$

where

$$(F_n)_t - (F_n)_{xx} + n^2(F_n) = 0.$$

Let $H_n := e^{n^2 t}F_n$. Then

$$(H_n)_t = n^2 e^{n^2 t}F_n + e^{n^2 t}(F_n)_t.$$
$$(H_n)_{xx} = e^{n^2 t}(F_n)_{xx}.$$

Then

$$(H_n)_t - (H_n)_{xx} = 0.$$

We have

$$\frac{2}{\pi}\int_0^\pi \cos(my)\cos(ny)\,dy = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$$

$$H_n(x,0) = F_n(x,0) = \frac{2}{\pi}\int_0^\pi u_0(x,y)\cos(ny)\,dy.$$

F06 #7 cont:

$$\cancel{H_n(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \frac{2}{\pi}\int_0^{\pi} u_0(\ ,y)\cos(ny)\,dy}$$

$$H_n(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_n(y,0)\, e^{-\frac{(x-y)^2}{4t}}\, dy.$$

So

$$F_n(x,t) = e^{-n^2 t}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_n(y,0)\, e^{-\frac{(x-y)^2}{4t}}\, dy.$$

Therefore as the F_n decay exponential for $n>0$,

$$\begin{aligned}
\lim_{t\to\infty} t^{1/2} u(x,y,t) &= \lim_{t\to\infty} t^{1/2}\cdot\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_0(y,0)\, e^{-\frac{(x-y)^2}{4t}}\, dy \\
&= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{2}{\pi}\int_0^{\pi} u_0(y,s)\, ds\, dy. \\
&= \frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_0^{\pi} u_0(y,s)\, ds\, dy.
\end{aligned}$$

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F06 #8 :

a: We have

$$|u(x,t)| = \left|\int_0^x u_x(y,t)\,dy\right| \le \int_0^x |u_x(y,t)|\,dy$$

$$= \int_0^1 |u_x(y,t)|\, 1_{[0,x]}(y)\,dy \le \left(\int_0^1 |u_x(y,t)|^2 dy\right)^{1/2}\left(\int_0^1 1_{[0,x]}(y)^2 dy\right)^{1/2}$$

$$\le \left(\int_0^1 |u_x(y,t)|^2 dy\right)^{1/2}.$$

Therefore $\sup_x |u(x,t)|^2 \le \int_0^1 |u_x(y,t)|^2 dy$

We have

$$\frac{d}{dt}\int_0^1 u^2\,dx = \int_0^1 2uu_t\,dx = \int_0^1 2u(u_{xx} + cu^2)\,dx$$

$$= 2\int_0^1 uu_{xx} + cu^3\,dx$$

$$= 2\left[uu_x\Big]_{x=0}^1 - \int_0^1 u_x^2\,dx + c\int_0^1 u^3\,dx \right]$$

$$= -2\int_0^1 u_x^2\,dx + 2c\int_0^1 u^3\,dx$$

$$\le -2\int_0^1 u_x^2\,dx + 2c\left(\int_0^1 u_x^2\,dx\right)\int_0^1 u\,dx$$

$$\le -2\int_0^1 u_x^2\,dx + 2c\left(\int_0^1 u_x^2\,dx\right)\left(\int_0^1 u^2\,dx\right)^{1/2}.$$

$$= -2\int_0^1 |u_x|^2 dx\left(1 - c\left(\int_0^1 |u|^2\,dx\right)^{1/2}\right)$$

b). Let $E(t) := \int_0^1 u(x,t)^2\,dx$. ~~If~~ Suppose we did not have $E(t) < 1/c^2$ $\forall$ time. Then $\exists$ a first time T s.t.

F06 #8 cont:

$E(T) = \frac{1}{c^2}$ and $E(t) < \frac{1}{c^2}$ $\forall t < T$. Then for these $t < T$,

$$E'(t) \le -2\int_0^1 |u_x|^2 dx \left(1 - c\left(\frac{1}{c^2}\right)^{1/2}\right) \le 0.$$

Therefore

$$E(T) = \cancel{E(0) + \int_0^T E'(t)\,dt} \le E(0) < \frac{1}{c^2}$$

a contradiction.

c). here $u_0 = 1$. If u does not depend on x, then we have to solve $u_t = Cu^2$. $\rightarrow \int \frac{1}{u^2} du = \int c\,dt$

$$\rightarrow -\frac{1}{u} = Ct + \tilde{C}$$

$$u = \frac{1}{-\tilde{C} - ct}. \qquad u_0 = 1.$$

$$\text{So } u = \frac{1}{1-ct}.$$

Thus solution blows up in finite time.