F05 #2 :

We have $Lu = \lambda u.$

$$u'' + u' - a(1+x^2)u = \lambda u.$$

$$\cancel{-u'' - u' + a(1+x^2)u = -\lambda u.}$$

$$\cancel{(-e^x u')' + a(1+x^2)u = -\lambda u.}$$

$$\cancel{Tu = -u'' - u' + a(1+x^2)u.}$$

$$(e^x u')' - a(1+x^2)u = \lambda u.$$

a) We have

$$\int_0^1 (Lu) v e^x \, dx = \int_0^1 (u'' + u' - a(1+x^2)u) v e^x \, dx$$

$$\boxed{v(0) = v(1) = 0.}$$

$$= \int_0^1 u'' v e^x \, dx + \int_0^1 u' v e^x \, dx - a\int_0^1 (1+x^2) uv e^x \, dx$$

$$= \int_0^1 -u'(v'e^x + ve^x) \, dx + \int_0^1 u' v e^x \, dx - a\int_0^1 (1+x^2) uv e^x \, dx$$

$$= -\int_0^1 u' v' e^x - a\int_0^1 (1+x^2) uv e^x \, dx.$$

$$= \int_0^1 u(v'e^x)' \, dx - \int_0^1 a(1+x^2) uv e^x \, dx$$

$$= \int_0^1 u(v''e^x + v'e^x) \, dx - \int_0^1 a(1+x^2) uv e^x \, dx$$

$$= \int_0^1 u\left[v'' + v' - a(1+x^2)v\right] e^x \, dx$$

$$= \int_0^1 u Lv \, e^x \, dx.$$

b) We have

$$\lambda_{a_0} \langle u_{a_0}, u_{a_0} \rangle_{e^x} = \int_0^1 u_{a_0} L u_{a_0} e^x \, dx = \int_0^1 u_{a_0} \left(u_{a_0}'' + u_{a_0}' - a(1+x^2) u_{a_0} \right) e^x dx$$

FOS #2 cont..

$$= \int_0^1 \left[u_{a_0} u_{a_0}'' + \left(\tfrac{1}{2} u_{a_0}^2\right)' - a(1+x^2) u_{a_0}^2 \right] e^x \, dx. \quad (*)$$

$$\int_0^1 u_{a_0} u_{a_0}'' e^x \, dx = - \int_0^1 \left[(u_{a_0}') e^x + u_{a_0} e^x \right] u_{a_0}'$$

$$= - \int_0^1 (u_{a_0}')^2 e^x \, dx - \int_0^1 u_{a_0} u_{a_0}' e^x \, dx$$

Thus

$$(*) = - \int_0^1 (u_{a_0}')^2 e^x \, dx - \int_0^1 a(1+x^2) u_{a_0}^2 e^x dx < 0.$$

c) ~~We have~~

$$\cancel{\text{Smallest } \lambda \text{ for } T = \min_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Tu\rangle}{\langle u,u\rangle} = \min_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu\rangle}{\langle u,u\rangle}}$$

Smallest μ s.t.

$$Lu = -\mu u$$

is given by

$$\mu_0 = \min_{\substack{u \\ u(0)=u(1)=0}} - \frac{\langle u, Lu\rangle}{\langle u,u\rangle} = - \max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu\rangle}{\langle u,u\rangle}.$$

$$\rightarrow -\mu_0 = \max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu\rangle}{\langle u,u\rangle}.$$

Therefore

$$\lambda_{a_0} = \max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu\rangle}{\langle u,u\rangle}.$$

F05 #2 cont..

We have

$$\frac{\langle u, Lu\rangle}{\langle u,u\rangle} = \frac{1}{\langle u,u\rangle}\int_0^1 u e^x \left(u''+u' - a(1+x^2)u\right) dx.$$

$$\frac{d}{da}\frac{\langle u,Lu\rangle}{\langle u,u\rangle} = \frac{1}{\langle u,u\rangle}\int_0^1 -(1+x^2)u^2 e^x\, dx < 0.$$

Therefore λ_{a_0} is a decreasing function of a. Thus $\cancel{a \geq \lambda_a}$ as $\lambda_{a_0} < 0$, if $0 < a_1 < a_2$, $|\lambda_{a_1 0}| < |\lambda_{a_2 0}|$.

#

F05 #3 cont:

The periodic orbits:

$$y^2 = \frac{1}{2}x^4 - x^2 + 2C.$$

$$y = \pm\sqrt{\frac{1}{2}x^4 - x^2 + 2C}$$

Need $\frac{1}{2}x^4 - x^2 + 2C$ to be ≥ 0 for $x \in [-1, 1]$.

~~Thus we need $C \geq 0$~~ Thus we need $C > 0$.

~~Since $\frac{1}{2}x^4 - x^2$ is minimized on $x = \pm 1$, We need~~

$$\cancel{2C + \left(\frac{1}{2} - 1\right) \geq 0 \rightarrow}$$

~~We have~~ For the orbit to be periodic we also need

2 distinct sol. of $\frac{1}{2}x^4 - x^2 + 2C = 0$. $\rightarrow$ at these pts. $y = 0$. So

$$x^2 = \frac{1 \pm \sqrt{1 - 4 \cdot \frac{1}{2} \cdot 2C}}{2} = \frac{1 \pm \sqrt{1 - 4C}}{2}$$

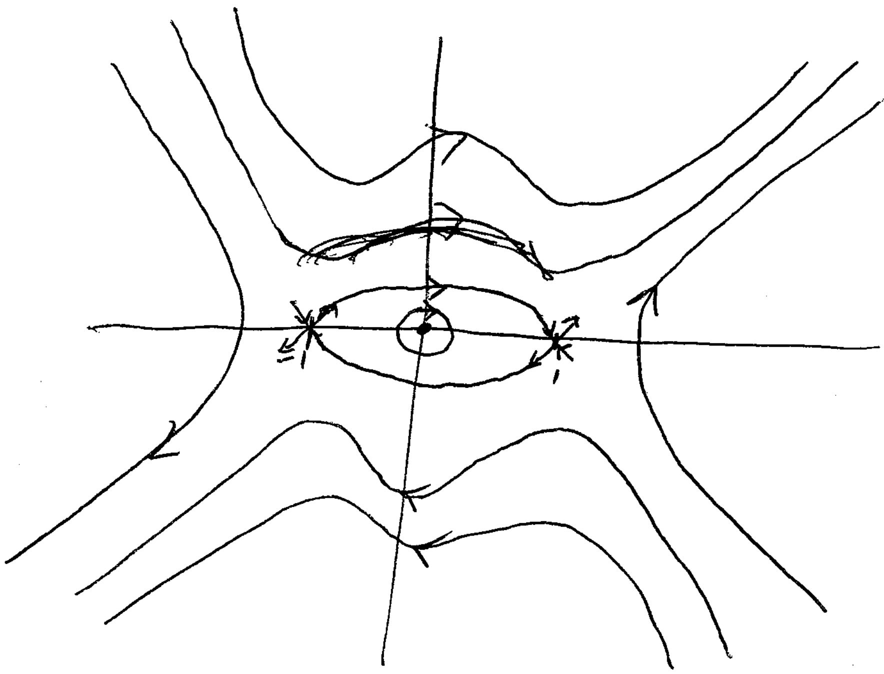
$\rightarrow$ need $1 - 4C > 0 \rightarrow C < \frac{1}{4}$.

Thus the periodic orbits are given by

$$y^2 = \frac{1}{2}x^4 - x^2 + D$$

$$0 < D < \frac{1}{2}$$

F05 #3 cont:



F05 #4: u solves the heat equation.

a) We have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\left(\frac{x-y}{\sqrt{4t}}\right)^2} dy \qquad u = \frac{x-y}{\sqrt{4t}}.$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\infty}^{x/\sqrt{4t}} e^{-u^2}(-\sqrt{4t})\, du \qquad du = -\frac{1}{\sqrt{4t}}\, dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-u^2}\, du$$

Then for each fixed x,

$$\lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2}\, du = \frac{1}{2}.$$

b) We claim the limit is not uniform ~~in~~ x. To do so we rephrase the question of asking whether the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4n}}^{\infty} e^{-u^2}\, du = \frac{1}{2} \qquad f_n(x) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4n}}^{\infty} e^{-u^2}\, du$$

is uniform in x. Suppose it was, then $\forall \varepsilon > 0$ $\exists N$ depending only on ε s.t. for $n \geq N$, $|f_n(x) - \frac{1}{2}| < \varepsilon$ $\forall x$.

Then $|f_N(x) - \frac{1}{2}| < \varepsilon$ $\forall x$. But

$$\lim_{x\to\infty} f_N(x) = \lim_{x\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4N}}^{\infty} e^{-u^2}\, du = 0.$$

Therefore the $\{f_n\}$ do not conv. unif to $\frac{1}{2}$. So the convergence is not uniform.

F05 #5.

Since u is a smooth soln on the torus, integration by parts gives no bdry conditions.

We have

$$\frac{d}{dt}E(t) = \varepsilon\int \nabla u \cdot \nabla u_t + \frac{1}{\varepsilon}\int W'(u)u_t.$$

$$= \varepsilon\int -\Delta u \, u_t + \frac{1}{\varepsilon}\int W'(u)u_t$$

$$= \int \left(\frac{1}{\varepsilon}W'(u) - \varepsilon\Delta u\right)u_t \, dx$$

$$= \int \left(-\frac{1}{\varepsilon}W'(u) + \varepsilon\Delta u\right)\Delta\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\right) dx$$

$$= -\int \left|\nabla\left(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\right)\right|^2 dx \le 0.$$

Therefore E is monotonically $\downarrow$. #

05 #6:

here $F(r) = \frac{1}{4\pi r^2}\int_{|x|=r} f(x)\, d\sigma = \frac{1}{4\pi}\int_{|x|=1} f(rx)\, d\sigma.$

We have

$$F'(r) = \frac{1}{4\pi}\int_{\partial B(0,1)} \nabla f(rx)\cdot x\, d\sigma. = \frac{1}{4\pi}\int_{\partial B(0,1)} \frac{\partial f}{\partial \nu}(rx)\, d\sigma$$

$$= \frac{1}{4\pi}\int_{B(0,1)} \nabla\cdot\left((\nabla f)(rx)\right) dx.$$

$$= \frac{r}{4\pi}\int_{B(0,1)} (\Delta f)(rx)\, dx.$$

$$= \frac{r}{4\pi r^3}\int_{B(0,r)} (\Delta f)(y)\, dy$$

$$= \frac{r}{3}⨍_{B(0,r)} (\Delta f)(y)\, dy$$

Mean Value Property

$$= \frac{r}{3}(\Delta f)(0)$$

Therefore

$$F(a) - F(0) = \int_0^a F'(r)\, dr = \int_0^a \frac{r}{3}(\Delta f)(0)\, dr$$

$$= \frac{a^2}{6}(\Delta f)(0).$$

$$\longrightarrow \frac{1}{4\pi a^2}\int_{|x|=a} f(x)\, d\sigma = f(0) + \frac{a^2}{6}(\Delta f)(0).$$

#.

F05 #7:

Method of characteristics:

$$u_t + u u_x = 0$$

$$F(p, q, z, x, t) = q + zp.$$

$$\longrightarrow \quad \dot{x} = z \qquad x(0) = x_0$$
$$\dot{t} = 1 \qquad t(0) = 0$$
$$\dot{z} = 0 \qquad z(0) = f(x_0)$$

where $f(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 < x < 0 \\ 1 - \frac{1}{2}x & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$

Therefore

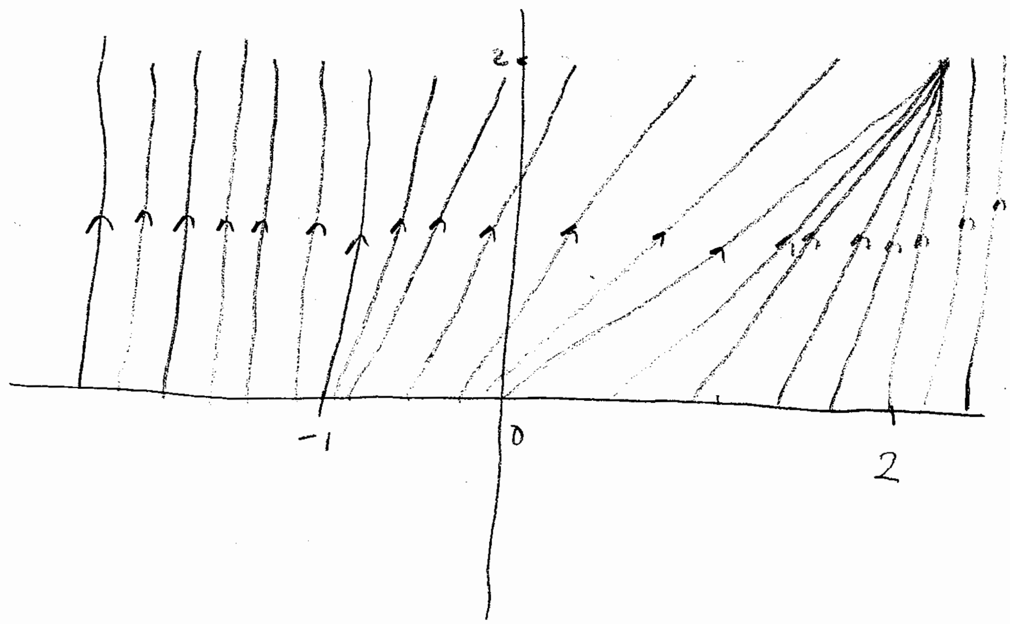
$$z(s) = f(x_0), \quad t(s) = s$$
$$x(s) = f(x_0) s + x_0.$$

The characteristics are

$$x = f(x_0) t + x_0.$$

		Characteristics		
if $x_0 < -1$,	$\longrightarrow$	$x = x_0$		
$-1 < x_0 < 0$	$\longrightarrow$	$\frac{x - x_0}{x_0 + 1} = t.$	$\longrightarrow$	$x_0 = \frac{x - t}{t + 1}$
$0 < x_0 < 2$	$\longrightarrow$	$\frac{x - x_0}{1 - \frac{1}{2}x_0} = t.$	$\longrightarrow$	$x_0 = \frac{x - t}{1 - \frac{1}{2}t}.$
$x_0 > 2$	$\longrightarrow$	$x = x_0.$		

FOS #7 cont;



The characteristics cross at time $t=2$. We have for $t<2$,

$$u(x,t) = \begin{cases} 0 & \text{if } x<-1. \\ \dfrac{x+1}{t+1} & \text{if } -1<x<t. \quad x_0<0 \longleftrightarrow x<t \\ \dfrac{2-x}{2-t} & \text{if } t<x<2 \quad x_0<2 \longleftrightarrow x<2. \\ 0 & \text{if } x>2. \end{cases}$$

Now we compute the shock ~~for~~ which occurs for $t>2$. Let $x=s(t)$ be the equation for the shock. Then the Rankine–Hugoniot,

$$\frac{\frac{1}{2}\left(\frac{s+1}{t+1}\right)^2 - \frac{1}{2}\cdot 0^2}{\frac{s+1}{t+1} - 0} = \dot{s} \qquad s(2)=2.$$

F05 #7 cont:

$$\frac{ds}{dt} = \frac{1}{2}\,\frac{s+1}{t+1}$$

$$\frac{1}{s+1}\,ds = \frac{1}{2}\,\frac{1}{(t+1)}\,dt$$

$$\ln s+1 = \ln\sqrt{t+1} + C.$$

$$\ln 3 = \frac{1}{2}\ln 3 + C \longrightarrow C = \frac{1}{2}\ln 3.$$

$$s+1 = \sqrt{3(t+1)}.$$

$$s(t) = \sqrt{3(t+1)} - 1.$$

Thus for $t > 2$, the entropy solution is given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < -1. \\ \frac{x+1}{t+1} & \text{if } -1 < x < \sqrt{3(t+1)} - 1 \\ 0 & \text{if } x > \sqrt{3(t+1)} - 1. \end{cases}$$

#

FOS #8

We have

$$F(p,q,z,x,y) = p^2 + q^2 - 1.$$

Then

$$\dot{x} = 2p \qquad x(0) = x_0$$

$$\dot{y} = 2q \qquad y(0) = 0$$

$$\dot{z} = 2 \qquad z(0) = \cos x_0$$

$$\dot{p} = 0 \qquad p(0) = -\sin x_0 \quad (\text{since } \phi(x,0) = \cos x,\ \phi_x(x,0) = -\sin x)$$

$$\dot{q} = 0 \qquad q(0) = \pm \cos x_0.$$

$$\longrightarrow \quad x(s) = (-2\sin x_0)s + x_0.$$

$$y(s) = (\pm 2\cos x_0)s \longrightarrow \text{since } 0 \le y < \infty,$$

$$y(s) = |2\cos x_0| s.$$

$$z(s) = 2s + \cos x_0.$$

Therefore

$$\phi(x,y) = 2s + \cos r$$

where

$$x = -2(\sin r)s + r$$

$$y = |2\cos r| s$$