F03 #1

a) The stationary pts. are s.t.

$$\begin{matrix} v-u^3=0 \\ u-v=0 \end{matrix} \longrightarrow \begin{matrix} v=u^3 \\ v=u \end{matrix} \longrightarrow u=\pm 1, 0$$

and hence the stationary pts. are $(1,1)$, $(0,0)$, and $(-1,-1)$.

~~The~~ Rewrite the system as

$$\dot{x} = y - x^3$$
$$\dot{y} = x - y.$$

The Jacobian is

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}$$

At $(1,1)$ and $(-1,-1)$, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

At $(0,0)$, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

~~The eigenvalues~~ We now compute the eigenvalues and eigenvectors of $\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

1 $\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$:

Eigenvalues: $\det\begin{pmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda) - 1 = \lambda^2 + 4\lambda + 2$

$$\longrightarrow \lambda^2 + 4\lambda + 2 = 0 \longrightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} = -2 \pm \sqrt{2}.$$

Eigenvectors:

$$\begin{pmatrix} -3-(-2\pm\sqrt{2}) & 1 \\ 1 & -1-(-2\pm\sqrt{2}) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1\mp\sqrt{2} & 1 \\ 1 & 1\mp\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\longrightarrow \begin{pmatrix} 1 \\ 1\pm\sqrt{2} \end{pmatrix}$ is an eigenvector for the eigenvalue $-2\pm\sqrt{2}$.

~~Ate~~ The critical pts $(1,1)$ and $(-1,-1)$ are stable, sink nodes.

F03 #1 cont:

a) $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$:

Eigenvalues: $\det\begin{pmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = \lambda(1+\lambda) - 1 = \lambda^2 + \lambda - 1.$

$\rightarrow \lambda^2 + \lambda - 1 = 0 \rightarrow \lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$

Eigenvectors:

$$\begin{pmatrix} 0 - \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) & 1 \\ 1 & -1 - \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

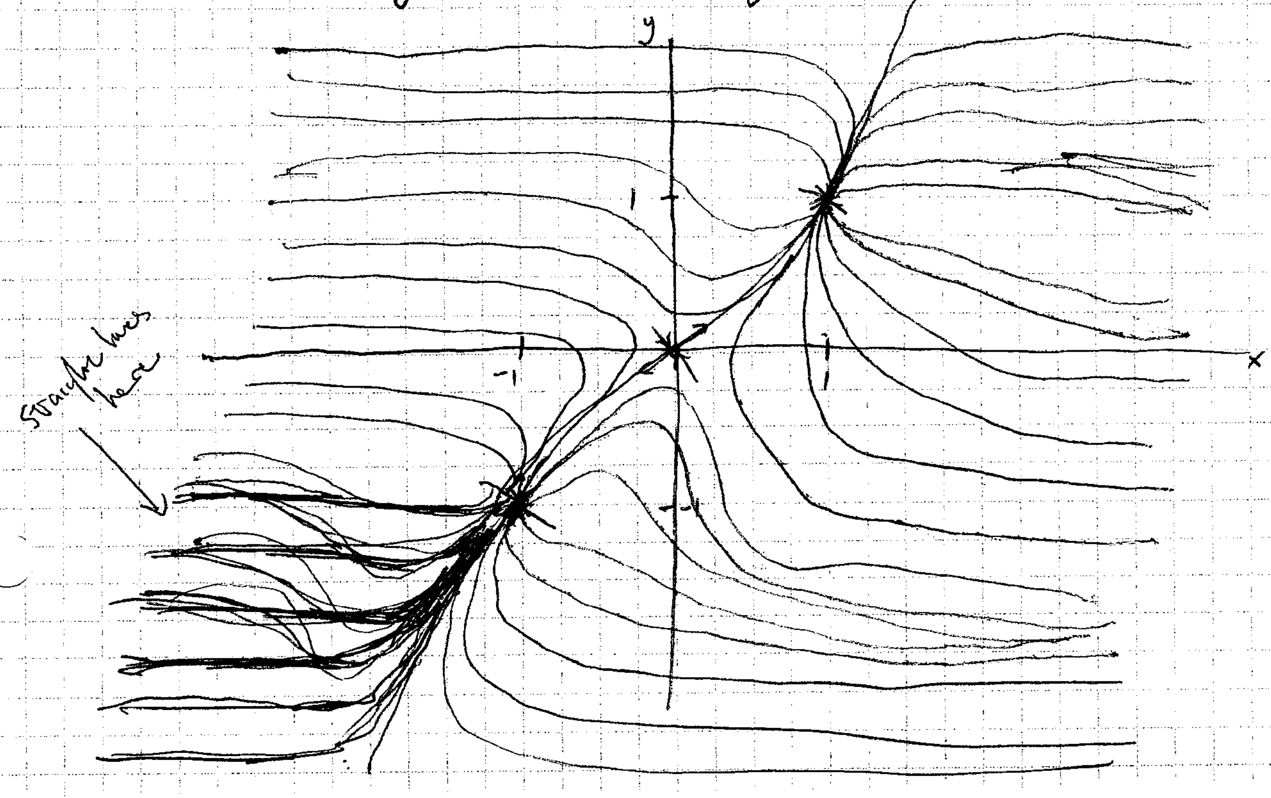
$$\rightarrow \begin{pmatrix} \frac{1}{2} \mp \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} 1 \\ \frac{-1 \pm \sqrt{5}}{2} \end{pmatrix}$ is an eigenvector for the eigenvalue $\frac{-1 \pm \sqrt{5}}{2}.$

The critical point $(0,0)$ is a saddle.

b) Note $\frac{dy}{dx} = \frac{x-y}{y-x^3}$ so for fixed y, $\frac{dy}{dx} \rightarrow 0$ as $x \rightarrow \pm\infty$.

Furthermore, ~~$y = x^3$ should~~ along $y = x^3$, $\frac{dy}{dx}$ is not defined.



F03 #3 :

① We have

$$u_t = \frac{\partial}{\partial t} \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} \bar{u}(\xi) h(x-t^2-\xi, t)\, d\xi$$

$$= \exp\left(\frac{1}{3}t^3 - xt\right)(t^2 - x) \int_{-\infty}^{\infty} \bar{u}(\xi) h(x-t^2-\xi, t)\, d\xi$$

$$+ \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} \bar{u}(\xi) \left[h_x(x-t^2-\xi, t)(-2t) + h_t(x-t^2-\xi, t)\right] d\xi$$

$$= \exp\left(\frac{1}{3}t^3 - xt\right) \Big[(t^2-x) \int_{-\infty}^{\infty} \bar{u}(\xi) h(x-t^2-\xi, t)\, d\xi + \int_{-\infty}^{\infty} \bar{u}(\xi) \left[h_x(x-t^2-\xi, t)(-2t) + h_{xx}(x-t^2-\xi, t) \right] d\xi \Big]$$

Therefore

$$u_t + xu = \exp\left(\frac{1}{3}t^3 - xt\right) \Big[t^2 \int_{-\infty}^{\infty} \bar{u}(\xi) h(x-t^2-\xi, t)\, d\xi - 2t \int_{-\infty}^{\infty} \bar{u}(\xi) h_x(x-t^2-\xi, t)\, d\xi + \int_{-\infty}^{\infty} \bar{u}(\xi) h_{xx}(x-t^2-\xi, t)\, d\xi \Big] \qquad (*)$$

Since

$$\frac{\partial}{\partial x} \exp\left(\frac{1}{3}t^3 - xt\right) = \exp\left(\frac{1}{3}t^3 - xt\right)(-t)$$

$$\frac{\partial^2}{\partial x^2} \exp\left(\frac{1}{3}t^3 - xt\right) = \exp\left(\frac{1}{3}t^3 - xt\right)(t^2),$$

and $(fg)'' = f''g + 2f'g' + fg''$, it follows that the RHS of $(*)$ is $= u_{xx}$.

Therefore $u_t + xu = u_{xx}$.

③ Note that

$$z = \frac{x-t^2-\xi}{2\sqrt{t}}, \quad dz = -\frac{1}{2\sqrt{t}} d\xi$$

$$\int_{-\infty}^{\infty} \bar{u}(\xi) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-t^2-\xi)^2}{4t}} d\xi = \int_{\infty}^{-\infty} \bar{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{4\pi t}} e^{-z^2} (-2\sqrt{t})\, dz.$$

$$= \int_{-\infty}^{\infty} \bar{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{\pi}} e^{-z^2} dz.$$

Since $\bar{u}$ is bounded and e^{-z^2} is integrable, by the DCT,

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \bar{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{\pi}} e^{-z^2} dz = \frac{\bar{u}(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = \bar{u}(x)$$

Therefore

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \exp\left(\frac{1}{3}t^3 - xt\right) \cdot \bar{u}(x) = \bar{u}(x).$$

F03 #3 cont:

(c) This is the standard argument to show h is a fundamental solution of the heat equation. Fix $\varepsilon > 0$ and an $x_0 \in \mathbb{R}$. Since u is continuous $\exists\, \delta > 0$ s.t.

$|u(x) - u(x_0)| < \varepsilon$ when $|x - x_0| < \delta$.

If $|x - x_0| < \delta/2$, then

$$\left| \int_{\mathbb{R}} u(\xi) h(x_0 - \xi, t)\, d\xi - u(x_0) \right| = \left| \int_{\mathbb{R}} (u(\xi) - u(x_0)) h(x_0 - \xi, t)\, d\xi \right|$$

$$\leq \int_{B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon \int_{B(x_0,\delta)} h(x_0 - \xi, t)\, d\xi + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon + 2\|u\|_{L^\infty} \int_{\mathbb{R} \setminus B(x_0,\delta)} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_0-\xi)^2}{4t}}\, d\xi$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \int_{\mathbb{R} \setminus B(x_0,\delta)} e^{-\frac{(x_0-\xi)^2}{4t}}\, d\xi.$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \left[\int_{-\infty}^{x_0-\delta} + \int_{x_0+\delta}^{\infty} e^{-\frac{(x_0-\xi)^2}{4t}}\, d\xi \right] \qquad z = \frac{x_0 - \xi}{2\sqrt{t}}.$$

$$dz = -\frac{1}{2\sqrt{t}}\, dt.$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \left[\int_{\infty}^{\delta/2\sqrt{t}} + \int_{-\delta/2\sqrt{t}}^{-\infty} e^{-z^2} (-2\sqrt{t})\, dz \right]$$

$$\lesssim \varepsilon + \int_{\mathbb{R} \setminus B(0, \frac{\delta}{2\sqrt{t}})} e^{-z^2}\, dz.$$

Since $\int_{\mathbb{R} \setminus B(0,\frac{\delta}{2\sqrt{t}})} e^{-z^2}\, dz \longrightarrow 0$ as $t \longrightarrow 0$, $\int_{\mathbb{R}} u(\xi) h(x_0 - \xi, t)\, d\xi \longrightarrow u(x_0)$ as $t \longrightarrow 0$.

F03 #4:

There are 2 characteristics and the equation is hyperbolic.

Let $a = x$, $b = x-y$, $c = -y$. Since $b^2 - 4ac = (x-y)^2 + 4xy > 0$ (as $x, y > 0$), the equation is hyperbolic. The characteristics are given by

$$\frac{dy}{dx} = \frac{(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2x} = \frac{(x-y) \pm (x+y)}{2x} = 1, -\frac{y}{x}.$$

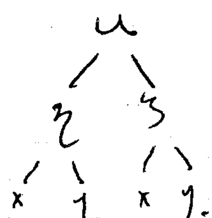
The characteristics are

$$\frac{dy}{dx} = 1 \longrightarrow y = x + c_1$$

$$\frac{dy}{dx} = -\frac{y}{x} \longrightarrow y = \frac{c_2}{x}.$$

Let $\eta = x - y$, $\xi = xy$. Then by the Chain Rule

$$u_x = u_\eta + u_\xi y, \quad u_y = -u_\eta + u_\xi x.$$



which implies

$$u_{xx} = u_{\eta\eta} + 2u_{\eta\xi} y + u_{\xi\xi} y^2$$

$$u_{xy} = -u_{\eta\eta} + x u_{\eta\xi} + u_\xi - y u_{\eta\xi} + xy u_{\xi\xi}$$

$$u_{yy} = u_{\eta\eta} - 2x u_{\eta\xi} + x^2 u_{\xi\xi}.$$

We have

$$x u_{xx} + (x-y) u_{xy} - y u_{yy} = x u_{\eta\eta} + 2xy u_{\eta\xi} + xy^2 u_{\xi\xi} + (x-y)(-u_{\eta\eta} + x u_{\eta\xi} + u_\xi - y u_{\eta\xi} + xy u_{\xi\xi}) - y(u_{\eta\eta} - 2x u_{\eta\xi} + x^2 u_{\xi\xi})$$

$$= (x+y)^2 u_{\eta\xi} + (x-y) u_\xi$$

$$= [\eta^2 + 4\xi] u_{\eta\xi} + \eta u_\xi.$$

If we interchange how ξ, η are defined, we have the desired result.

We now will solve $(\xi^2 + 4\eta) u_{\xi\eta} + \xi u_\eta = 0$. Let $v(\xi,\eta) = u_\eta(\xi,\eta)$. Then

$$(\xi^2 + 4\eta) v_\xi + \xi v = 0 \longrightarrow v_\xi = -\frac{\xi}{\xi^2 + 4\eta} v \longrightarrow \ln v = -\frac{1}{2} \ln(\xi^2 + 4\eta) + \tilde{g}(\eta)$$

$$\longrightarrow v = \frac{g(\eta)}{(\xi^2 + 4\eta)^{1/2}} \to u(\xi,\eta) = f(\xi) + \int^\eta \frac{g(\eta')}{(\xi^2 + 4\eta')^{1/2}} d\eta'$$

6 Second-Order Equations

6.1 Classification by Characteristics

Consider the second-order equation in which the derivatives of second-order all occur linearly, with coefficients only depending on the independent variables:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y). \tag{6.1}$$

The *characteristic* equation is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $b^2 - 4ac > 0 \Rightarrow$ *two* characteristics, and (6.1) is called *hyperbolic*;
- $b^2 - 4ac = 0 \Rightarrow$ *one* characteristic, and (6.1) is called *parabolic*;
- $b^2 - 4ac < 0 \Rightarrow$ *no* characteristics, and (6.1) is called *elliptic*.

These definitions are all taken at a point $x_0 \in \mathbb{R}^2$; unless a, b, and c are all constant, the *type* may change with the point x_0.

6.2 Canonical Forms and General Solutions

① $u_{xx} - u_{yy} = 0$ is hyperbolic (one-dimensional wave equation).
② $u_{xx} - u_y = 0$ is parabolic (one-dimensional heat equation).
③ $u_{xx} + u_{yy} = 0$ is elliptic (two-dimensional Laplace equation).

By the introduction of new coordinates μ and η in place of x and y, the equation (6.1) may be transformed so that its principal part takes the form ①, ②, or ③.
If (6.1) is *hyperbolic*, *parabolic*, or *elliptic*, there exists a change of variables $\mu(x,y)$ and $\eta(x,y)$ under which (6.1) becomes, respectively,

$$\begin{aligned}
u_{\mu\eta} &= \tilde{d}(\mu,\eta,u,u_\mu,u_\eta) \quad \Leftrightarrow \quad u_{\bar{x}\bar{x}} - u_{\bar{y}\bar{y}} = \bar{d}(\bar{x},\bar{y},u,u_{\bar{x}},u_{\bar{y}}),\\
u_{\mu\mu} &= \tilde{d}(\mu,\eta,u,u_\mu,u_\eta),\\
u_{\mu\mu} + u_{\eta\eta} &= \tilde{d}(\mu,\eta,u,u_\mu,u_\eta).
\end{aligned}$$

Example 1. *Reduce to canonical form and find the general solution:*

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0. \tag{6.2}$$

Proof. $a = 1,\ b = 5,\ c = 6 \quad \Rightarrow \quad b^2 - 4ac = 1 > 0 \quad \Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.
The characteristics are found by solving

$$\frac{dy}{dx} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

to find $y = 3x + c_1$ and $y = 2x + c_2$.

$$
\begin{aligned}
\text{Let} \quad & \mu(x,y) = 3x - y, \quad \eta(x,y) = 2x - y. \\
& \mu_x = 3, \qquad \eta_x = 2, \\
& \mu_y = -1, \qquad \eta_y = -1. \\
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = 3u_\mu + 2u_\eta, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = -u_\mu - u_\eta, \\
u_{xx} &= (3u_\mu + 2u_\eta)_x = 3(u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x) + 2(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = 9u_{\mu\mu} + 12u_{\mu\eta} + 4u_{\eta\eta}, \\
u_{xy} &= (3u_\mu + 2u_\eta)_y = 3(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + 2(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -3u_{\mu\mu} - 5u_{\mu\eta} - 2u_{\eta\eta}, \\
u_{yy} &= -(u_\mu + u_\eta)_y = -(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}
$$

Inserting these expressions into (6.2) and simplifying, we obtain

$$
\begin{aligned}
u_{\mu\eta} &= 0, \qquad \text{which is the } \textbf{Canonical form}, \\
u_\mu &= f(\mu), \\
u &= F(\mu) + G(\eta), \\
u(x,y) &= F(3x - y) + G(2x - y), \qquad \textbf{General solution}.
\end{aligned}
$$

□

Example 2. *Reduce to canonical form and find the general solution:*

$$
y^2 u_{xx} - 2y u_{xy} + u_{yy} = u_x + 6y. \tag{6.3}
$$

Proof. $a = y^2$, $b = -2y$, $c = 1$ $\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$ **parabolic** $\Rightarrow$ one characteristic. The characteristics are found by solving

$$
\frac{dy}{dx} = \frac{-2y}{2y^2} = -\frac{1}{y}
$$

$$
\text{to find} \quad -\frac{y^2}{2} + c = x.
$$

Let $\mu = \frac{y^2}{2} + x$. We must choose a second constant function $\eta(x,y)$ so that η is not parallel to μ. Choose $\eta(x,y) = y$.

$$
\begin{aligned}
& \mu_x = 1, \qquad \eta_x = 0, \\
& \mu_y = y, \qquad \eta_y = 1. \\
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = u_\mu, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = y u_\mu + u_\eta, \\
u_{xx} &= (u_\mu)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x = u_{\mu\mu}, \\
u_{xy} &= (u_\mu)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y = y u_{\mu\mu} + u_{\mu\eta}, \\
u_{yy} &= (y u_\mu + u_\eta)_y = u_\mu + y(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + (u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) \\
&= u_\mu + y^2 u_{\mu\mu} + 2y u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}
$$

Inserting these expressions into (6.3) and simplifying, we obtain

$$
\begin{aligned}
u_{\eta\eta} &= 6y, \\
u_{\eta\eta} &= 6\eta, \qquad \text{which is the } \textbf{Canonical form}, \\
u_{\eta} &= 3\eta^2 + f(\mu), \\
u &= \eta^3 + \eta f(\mu) + g(\mu), \\
u(x,y) &= y^3 + y \cdot f\Big(\frac{y^2}{2} + x\Big) + g\Big(\frac{y^2}{2} + x\Big), \qquad \textbf{General solution}.
\end{aligned}
$$

$\square$

Problem (F'03, #4). *Find the characteristics of the partial differential equation*

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \qquad x > 0,\ y > 0, \tag{6.4}$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^\eta \frac{g(\eta')\,d\eta'}{(\xi^2+4\eta')^{\frac{1}{2}}}$$

where f and g are arbitrary functions of ξ and η.

Proof. $a = x, \quad b = x-y, \quad c = -y \quad \Rightarrow \quad b^2 - 4ac = (x-y)^2 + 4xy > 0$ for $x > 0$, $y > 0 \;\Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.

① The **characteristics** are found by solving

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{x-y \pm \sqrt{(x-y)^2+4xy}}{2x} = \frac{x-y\pm(x+y)}{2x} = \begin{cases} \frac{2x}{2x} = 1 \\ -\frac{2y}{2x} = -\frac{y}{x} \end{cases}$$

$$\Rightarrow \qquad y = x + c_1, \qquad \frac{dy}{y} = -\frac{dx}{x},$$

$$ln\, y = ln\, x^{-1} + \tilde{c}_2,$$

$$y = \frac{c_2}{x}.$$

② Let $\mu = x - y$ and $\eta = xy$

$$\mu_x = 1, \qquad \eta_x = y,$$
$$\mu_y = -1, \qquad \eta_y = x.$$

$$\begin{aligned}
u &= u(\mu(x,y),\eta(x,y));\\
u_x &= u_\mu\mu_x + u_\eta\eta_x = u_\mu + yu_\eta,\\
u_y &= u_\mu\mu_y + u_\eta\eta_y = -u_\mu + xu_\eta,\\
u_{xx} &= (u_\mu + yu_\eta)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x + y(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta},\\
u_{xy} &= (u_\mu + yu_\eta)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_\eta + y(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta},\\
u_{yy} &= (-u_\mu + xu_\eta)_y = -u_{\mu\mu}\mu_y - u_{\mu\eta}\eta_y + x(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta},
\end{aligned}$$

Inserting these expressions into (6.4), we obtain

$$x(u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta}) + (x-y)(-u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta}) - y(u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta}) = 0,$$
$$(x^2 + 2xy + y^2)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$\big((x-y)^2 + 4xy\big)u_{\mu\eta} + (x-y)u_\eta = 0,$$
$$(\mu^2 + 4\eta)u_{\mu\eta} + \mu u_\eta = 0, \qquad \text{which is the } \textbf{Canonical form}.$$

③ We need to integrate twice to get the general solution:

$$(\mu^2 + 4\eta)(u_\eta)_\mu + \mu u_\eta = 0,$$

$$\int \frac{(u_\eta)_\mu}{u_\eta}\, d\mu = -\int \frac{\mu}{\mu^2 + 4\eta}\, d\mu,$$

$$ln\ u_\eta = -\frac{1}{2} ln\ (\mu^2 + 4\eta) + \tilde{g}(\eta),$$

$$ln\ u_\eta = ln\ (\mu^2 + 4\eta)^{-\frac{1}{2}} + \tilde{g}(\eta),$$

$$u_\eta = \frac{g(\eta)}{(\mu^2 + 4\eta)^{\frac{1}{2}}},$$

$$u(\mu, \eta) = f(\mu) + \int \frac{g(\eta)\, d\eta}{(\mu^2 + 4\eta)^{\frac{1}{2}}}, \qquad \textbf{General solution}.$$

□

F03 #5:

① Parseval's relation: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

② We have

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx = \frac{1}{\sqrt{2\pi}}\int_{|x|\le y} e^{i(\alpha-\xi)x}\frac{1}{2\sqrt{\pi y}}\,dx$$

$$= \frac{1}{2\sqrt{2}\pi\sqrt{y}}\int_{-y}^{y} e^{i(\alpha-\xi)x}\,dx = \frac{1}{2\pi\sqrt{2y}}\,\frac{1}{i(\alpha-\xi)}\left[e^{i(\alpha-\xi)y} - e^{-i(\alpha-\xi)y}\right]$$

$$= \frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}$$

③ We have

$$\int_{-\infty}^{\infty}\left|\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right|^2 d\xi = \pi^2\cdot 2y\int_{-\infty}^{\infty}\left[\frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right]^2 d\xi$$

$$= \pi^2\cdot 2y\int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$= \pi^2\cdot 2y\int_{|x|\le 2y}\left|\frac{e^{i\alpha x}}{2\sqrt{\pi y}}\right|^2 dx$$

$$= 2\pi^2 y\cdot\frac{1}{4\pi y}\cdot 2y = \pi y.$$

④ If $\xi = \alpha$, then

$$\hat{f}(\xi) = \hat{f}(\alpha) = \lim_{\xi\to\alpha}\frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi} = \frac{1}{\pi\sqrt{2y}}\,y = \frac{\sqrt{y}}{\pi\sqrt{2}}.$$

As $y\to\infty$, $\hat{f}(\xi) = \hat{f}(\alpha)\to\infty$.

If $\xi\neq\alpha$, then

$$\lim_{y\to\infty}|\hat{f}(\xi)| = \lim_{y\to\infty}\frac{1}{\pi\sqrt{2y}}\left|\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right| \le \frac{1}{\pi\sqrt{2}|\alpha-\xi|}\lim_{y\to\infty}\frac{1}{\sqrt{y}} = 0.$$

Since $\|f\|_{L^2}^2 = \int_{-y}^{y}\frac{1}{4\pi y}\,dx = \frac{1}{2\pi}$,

it follows that $\hat{f}(\xi)\to\frac{1}{\sqrt{2\pi}}\delta(\xi-\alpha)$. as $y\to\infty$.

over →

F03 #6:

a) Observe that if $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, then

$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)$$
$$x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2).$$

Therefore

$$0 = \varepsilon^3 x^3 - 2\varepsilon x^2 + 2x - 6$$
$$= \varepsilon^3 \overset{x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2)}{(\cancel{x_0 + \varepsilon x_1 + O(\varepsilon^2)})} - 2\varepsilon (x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)) + 2(x_0 + \varepsilon x_1 + O(\varepsilon^2)) - 6.$$
$$= -2\varepsilon x_0^2 + 2x_0 + 2\varepsilon x_1 - 6 + O(\varepsilon^2).$$

Thus we must have $2x_0 - 6 = 0 \longrightarrow x_0 = 3$.
$2x_1 - 2x_0^2 = 0 \longrightarrow x_1 = 9$.

Therefore $x = 3 + 9\varepsilon + O(\varepsilon^2)$.

b) Let $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$. Then

$$u_t = u_0' + \varepsilon u_1' + \varepsilon^2 u_2' + O(\varepsilon^3)$$
$$u^2 = u_0^2 + 2\varepsilon u_0 u_1 + 2\varepsilon^2 u_0 u_2 + \varepsilon^2 u_1^2 + O(\varepsilon^3)$$
$$u^3 = u_0^3 + 3\varepsilon u_0^2 u_1 + 3\varepsilon^2 u_0^2 u_2 + 3\varepsilon^2 u_0 u_1^2 + O(\varepsilon^3).$$

Therefore

$$u - \varepsilon u^3 = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$$
$$- \varepsilon u_0^3 - 3\varepsilon^2 u_0^2 u_1$$

Matching coeff. of ε in $u_t = u - \varepsilon u^3$, we have

$u_0' = u_0 \qquad u_0(0) = 1 \longrightarrow \underline{u_0 = e^t}.$
$u_1' = u_1 - u_0^3 \qquad u_1(0) = 0$
$u_2' = u_2 - 3u_0^2 u_1 \qquad u_2(0) = 0.$

$u_1' = u_1 - e^{3t} \rightarrow u_1' - u_1 = -e^{3t}.$
$e^{-t}u_1' - e^{-t}u_1 = -e^{2t}.$
$(e^{-t}u_1)' = -e^{2t}.$
$u_1 = -\frac{1}{2}e^{3t} + c_1 e^t.$
$u_1(0) = 0 \longrightarrow 0 = -\frac{1}{2} + c_1 \longrightarrow c_1 = \frac{1}{2}.$
So $\underline{u_1(t) = -\frac{1}{2}e^{3t} + \frac{1}{2}e^t.}$

$u_2' = u_2 - 3e^{2t}(-\frac{1}{2}e^{3t} + \frac{1}{2}e^t)$
$u_2' - u_2 = \frac{3}{2}e^{5t} - \frac{3}{2}e^{3t}.$
$(e^{-t}u_2)' = \frac{3}{2}e^{4t} - \frac{3}{2}e^{2t}.$
$e^{-t}u_2 = \frac{3}{2}\frac{1}{4}e^{4t} - \frac{3}{2}\frac{1}{2}e^{2t} + c_2.$
$u_2 = \frac{3}{8}e^{5t} - \frac{3}{4}e^{3t} + c_2 e^t.$
$u_2(0) = 0 \longrightarrow 0 = \frac{3}{8} - \frac{3}{4} + c_2 \longrightarrow c_2 = \frac{3}{8}.$
$\underline{u_2(t) = \frac{3}{8}e^{5t} - \frac{3}{4}e^{3t} + \frac{3}{8}e^t.}$