SOO #1:

We use method of characteristics. We have

$$F(p,q,z,x,y) = p^2+q^2-1.$$

$\dot{x} = 2p$ $\quad x(0) = x_0$

$\dot{y} = 2q$ $\quad y(0) = x_0^2/2$

$\dot{p} = 0$ $\quad p(0) = -x_0/\sqrt{x_0^2+1}$

$\dot{q} = 0$ $\quad q(0) = 1/\sqrt{x_0^2+1}$

$\dot{z} = 2p^2+2q^2 = 2$ $\quad z(0) = 0$

from since $\phi(x,\frac{x^2}{2})=0$, $\phi_x(x,\frac{x^2}{2}) + \phi_y(x,\frac{x^2}{2})x = 0$

$\cancel{p+qx_0=0}$.

$\cancel{p^2+q^2=1}$

$p(0)+q(0)x_0 = 0$

$p(0)^2+q(0)^2 = 1$.

here we also used $\phi_y(x,\frac{x^2}{2})>0$ which implies $q(0)>0$.

We then have

$$p(s) = -\frac{x_0}{\sqrt{x_0^2+1}} \qquad x(s) = x_0 - \frac{2x_0 s}{\sqrt{x_0^2+1}}$$

$$q(s) = \frac{1}{\sqrt{x_0^2+1}} \qquad y(s) = \frac{x_0^2}{2} + \frac{2}{\sqrt{x_0^2+1}}s.$$

$$z(s) = 2s$$

Thus the solution in parametric form is:

$$x(s,t) = t - \frac{2st}{\sqrt{t^2+1}}$$

$$y(s,t) = \frac{t^2}{2} + \frac{2s}{\sqrt{t^2+1}}$$

$$z(s,t) = 2s.$$

S'00 #3: We have $u(x,t) = \sum_{k \in \mathbb{Z}^2} \hat{u}(k,t) e^{ik\cdot x}$. Since $u_t = \Delta u - u$,

$$\hat{u}_t(k,t) = -|k|^2 \hat{u}(k,t) - \hat{u}(k,t)$$
$$= (-|k|^2 - 1)\hat{u}(k,t)$$

and hence

$$\hat{u}(k,t) = e^{-(|k|^2+1)t}\hat{u}(k,0).$$

Therefore

$$u(x,t) = \sum_{k \in \mathbb{Z}^2} \hat{u}(k,t) e^{ik\cdot x} = \sum_{k \in \mathbb{Z}^2} e^{-(|k|^2+1)t} e^{ik\cdot x} \hat{u}(k,0)$$
$$= \sum_{k \in \mathbb{Z}^2} \hat{u}_0(k) e^{-(|k|^2+1)t} e^{ik\cdot x}.$$

S00 #4.

We have $f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}$, $f_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$.

We want to solve $\Delta u = 0$ in $r < 1$

$u_r = f(\theta)$ on $r = 1$.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

here $u(r,\theta) = \sum_{n \in \mathbb{Z}} a_n(r) e^{in\theta}$. Then we want

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0.$$

$$r^2 a_n'' + r a_n' - n^2 a_n = 0.$$

$$a_n = r^\alpha \longrightarrow \alpha(\alpha-1) + \alpha - n^2 = 0.$$

$$\alpha^2 - n^2 = 0 \longrightarrow \alpha = \pm n.$$

Therefore $a_n(r) = A_n r^n + B_n r^{-n}$. Thus

$$u(r,\theta) = A_0 + \sum_{n \in \mathbb{Z}\setminus\{0\}} (A_n r^n + B_n r^{-n}) e^{in\theta}$$

Since want u to be defined when $r = 0$,

$$u(r,\theta) = A_0 + \sum_{n>0} A_n r^n e^{in\theta} + \sum_{n<0} B_n r^{-n} e^{in\theta}.$$

$$= A_0 + \sum_{n>0} A_n r^n e^{in\theta} + B_{-n} r^n e^{-in\theta}.$$

$$= C_0 + \sum_{n\geq 1} C_n r^n e^{in\theta} + D_n r^n e^{-in\theta}.$$

$$= C_0 + \sum_{n\geq 1} r^n (C_n e^{in\theta} + D_n e^{-in\theta}).$$

500 #4 cont:

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = \sum_{n\geq 1} n\left(C_n e^{in\theta} + D_n e^{-in\theta}\right).$$

Thus we want

$$\sum_{n\geq 1} nC_n e^{in\theta} + nD_n e^{-in\theta} = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}.$$

For this to be true ~~we~~ it is necessary to assume

$$f_0 = 0 \longrightarrow \int_0^{2\pi} f(x)\,dx = 0.$$

Then

$$\boxed{\text{if } n>0, \quad C_n = \frac{f_n}{n}.}$$

if $n<0$, $\cancel{D_n = \frac{f_n}{-n}}$. $D_{-n} = \frac{f_n}{-n} \Rightarrow \boxed{D_n = \frac{f_{-n}}{n} \text{ for } n>0.}$

We have

$$u(r,\theta) = A_0 + \sum_{n\geq 1} r^n \frac{f_n}{n} e^{in\theta} + r^n \frac{f_{-n}}{n} e^{-in\theta}.$$

$$= A_0 + \sum_{n\geq 1} \frac{(re^{i\theta})^n}{n} \frac{1}{2\pi}\int_0^{2\pi} f(\eta)e^{-in\eta}\,d\eta + \frac{(re^{-i\theta})^n}{n}\frac{1}{2\pi}\int_0^{2\pi} f(\eta)e^{in\eta}\,d\eta.$$

$$= A_0 + \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[\sum_{n\geq 1}\frac{r^n e^{in(\theta-\eta)}}{n} + \sum_{n\geq 1}\frac{r^n e^{-in(\theta-\eta)}}{n}\right]d\eta$$

We have $\frac{1}{1-x} = \sum_{n\geq 0} x^n$ for $|x|<1$. Then

$$-\ln(1-x) = \sum_{n\geq 0}\frac{1}{n+1}x^{n+1} = \sum_{n\geq 1}\frac{1}{n}x^n.$$

$$re^{i\theta} + re^{-i\theta} = 2r\cos\theta.$$

SOO #4 cont:

Therefore

here we use r<1

$$u(r,\theta) = A_0 + \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[-\ln(1-re^{i(\theta-\eta)}) - \ln(1-re^{-i(\theta-\eta)})\right]d\eta.$$

$$= A_0 - \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[\ln(1-re^{-i(\theta-\eta)} - re^{i(\theta-\eta)} + r^2)\right]d\eta.$$

$$= A_0 - \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\ln(1-2r\cos(\theta-\eta)+r^2)\,d\eta.$$

Therefore

$$N(r,\theta) = -\ln(1-2r\cos\theta + r^2).$$

(note the solution to the Neumann problem is not unique).

S00 #5:

~~Also~~ We have $u_{tt} = c^2 v''$, $(u^2)_{xx} = (2uu_x)_x = 2(u_x^2 + uu_{xx}) = 2(v'^2 + vv'')$ and $u_{xxxx} = v''''$ where $v' = v'(y)$. Thus $y = x - ct$

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

$$\to \quad c^2 v'' + 2(vv')' = -v''''$$

$$c^2 v' + 2vv' = -v''' + c_1$$

$$c^2 v' + (v^2)' = -v''' + c_1$$

$$c^2 v + v^2 = -v'' + c_1(x - ct) + c_2.$$

Since $v \to$ ~~constant~~ M for some constant M as $|x| \to \infty$, we have $c_1 = 0$ and $c_2 = c^2 M + M^2$. Thus

$$v'' + v^2 + c^2 v = c^2 M + M^2.$$

$$v'' + (v - M)(v + (c^2 + M)) = 0.$$

Write this as a ~~nonlinear~~ system:

$$x' = y$$

$$y' = -c^2 x - x^2 + c^2 M + M^2.$$

The equilibrium pts are $(M, 0)$ and $(-c^2 - M, 0)$

The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -c^2 - 2x & 0 \end{pmatrix} \to J(M,0) = \begin{pmatrix} 0 & 1 \\ -c^2 - 2M & 0 \end{pmatrix}$$

The eigenvalues of $J(M,0)$ are the roots of $\lambda^2 + (c^2 + 2M)$.

If $M > 0$, then $(M, 0)$ is a center (as the system is Hamiltonian, so equilibrium pts are either centers or saddles). ~~Since $v \to M$ as $t \to \infty$ as time increases, the solution v is~~ Thus the solution v is ~~periodic~~ sinusoidal (at least when the initial conditions $(v(0), v'(0))$ are close to $(M, 0)$.)

S00 #6

a) We have

$$\frac{d}{d\tau}(x\cdot x) = x_1\dot{x}_1 + \dots + x_n\dot{x}_n = x\cdot\dot{x} = f(|x|^2)\,x\cdot p.$$

Since $f>0$, $\frac{d}{d\tau}(x\cdot x) > 0$ if $p\cdot x>0$ and <0 if $p\cdot x<0$.
Thus $|x| = (x\cdot x)^{1/2}$ is increasing w/ τ when $p\cdot x>0$ and decreasing w/ τ when $p\cdot x<0$.

We have

$$\frac{d}{d\tau}\left(f(|x|^2)|p|^2\right) = \frac{d}{d\tau} f\Big(\sum_{i=1}^n x_i^2\Big)\Big(\sum_{i=1}^n p_i^2\Big) = f'(|x|^2)\Big(\sum_{i=1}^n 2x_i\dot{x}_i\Big)|p|^2 + f(|x|^2)\Big(\sum_{i=1}^n 2p_i\dot{p}_i\Big)$$

$$= f'(|x|^2)|p|^2\cdot 2x\cdot f(|x|^2)p + f(|x|^2)\cdot 2p\cdot(-f'(|x|^2)|p|^2 x) = 0.$$

b) We have $\frac{d}{ds}\left(\frac{f(s)}{s}\right) = \frac{sf'(s) - f(s)}{s^2}$. Thus

$$r^2 f'(r^2) = f(r^2)$$

We have

$$\frac{d}{d\tau}(p\cdot x) = -f'(|x|^2)|p|^2|x|^2 + f(|x|^2)|p|^2.$$

~~By sol. to a) $|x|$ is constant if $x\cdot p = 0$.~~

~~Thus at $\tau=0$, for $|x|=r$,~~

$$\cancel{\frac{d}{d\tau}(p\cdot x)\Big|_{\tau=0} = -f'(r^2)|p|^2 r^2 + f(r^2)|p|^2 = 0}$$

~~Since $x(0)\cdot p(0) = 0$, by sol. to a) $|x|$ stays the~~

~~Thus $p\cdot x$ = constant for all $|x|=r$. Since $p(0)\cdot x(0)=0$,~~

~~$p\cdot x = 0$ for all $|x|=r$.~~

500 #6 cont:

We have

$$\{t : (x\cdot x)' > 0\} = \{t : x\cdot p > 0\}$$
$$\{t : (x\cdot x)' < 0\} = \{t : x\cdot p < 0\}$$
$$\frac{d}{dt}(p\cdot x) = 0 \quad \text{when} \quad x\cdot x = r^2.$$
$$(x\cdot p)(0) = 0.$$
$$(x\cdot x)(0) = r^2$$

This is not strong enough to conclude $(x\cdot x)(t) = r^2 \ \forall t$.

$$\{t : g' > 0\} = \{t : f > 0\}$$
$$\{t : g' < 0\} = \{t : f < 0\}$$
$$f' = 0 \quad \text{when} \quad g = r^2.$$
$$f(0) = 0$$
$$g(0) = r^2.$$

here $g(t) = r^2 + t^2$, $f(t) = t^3$. But $g(t) \neq r^2 \ \forall t$.

S00 #8:

a) We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} u(x,t)\,(\phi_{tt}-\Delta\phi)\,dx\,dt \qquad (*)$$

$$= \int_{\mathbb{R}} \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} u_{tt}\,\phi\,dx\,dt - \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon} u\,\Delta\phi\,dx\,dt.$$

$$\int_{|x|>\varepsilon} u\Delta\phi = \int_{|x|>\varepsilon}\phi\,\Delta u + \int_{|x|=\varepsilon}\frac{\partial\phi}{\partial\nu}u - \frac{\partial u}{\partial\nu}\phi\,d\sigma \qquad \frac{\partial}{\partial\nu} = -\frac{\partial}{\partial r}.$$

So

$$(*) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon} u_{tt}\phi - \phi\Delta u\,dx\,dt - \int_{\mathbb{R}}\int_{|x|=\varepsilon} -\frac{\partial\phi}{\partial r}u + \frac{\partial u}{\partial r}\phi\,d\sigma\,dt.$$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon}(u_{tt}-\Delta u)\phi\,dx\,dt + \int_{\mathbb{R}}\int_{|x|=\varepsilon}\frac{\partial\phi}{\partial r}u - \frac{\partial u}{\partial r}\phi\,d\sigma\,dt.$$

$u = \dfrac{f(t+r)}{r}$. We claim $u_{tt}-\Delta u = 0$ away from 0. Indeed,

$$u_r = \frac{rf'(t+r)-f(t+r)}{r^2}, \quad u_{rr} = \frac{f''(t+r)}{r} - \frac{2f'(t+r)}{r^2} + \frac{2f(t+r)}{r^3}.$$

Then

$$u_{tt}-\Delta u = u_{tt} - u_{rr} - \frac{2}{r}u_r = 0.$$

So

$$(*) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|=\varepsilon}\frac{\partial\phi}{\partial r}u - \frac{\partial u}{\partial r}\phi\,d\sigma\,dt$$

$$= \cancel{\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\left[\frac{\partial\phi(x)}{\partial r}\frac{f(t+\varepsilon)}{\varepsilon} - \left(\frac{f'(t+\varepsilon)}{\varepsilon} - \frac{f(t+\varepsilon)}{\varepsilon^2}\right)\phi(x)\right]\varepsilon^2\sin\theta\,d\theta\,d\varphi\,dt}$$

$\int^{\pi/2}\int^{2\pi}$

S00 #8 cont:

$$= \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \int_{S^2} \left[\frac{\partial \phi}{\partial r}(\varepsilon \gamma, \tau) \frac{f(t+\varepsilon)}{\varepsilon} - \left(\frac{f'(t+\varepsilon)}{\varepsilon} - \frac{f(t+\varepsilon)}{\varepsilon^2} \right) \phi(\varepsilon \gamma, \tau) \right] \varepsilon^2 \, d\sigma(\gamma) \, d\tau$$

$$= 4\pi \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{S^2} f(t+\varepsilon) \phi(\varepsilon \gamma, t) \, d\sigma(\gamma) \, d\tau.$$

$$= 4\pi \int_{\mathbb{R}} f(t) \phi(0, t) \, d\tau.$$

b) The main issue with $u(x,t) = \frac{f(t+|x|)}{|x|}$ is that it is not differentiable near 0. The solution is given by

$$u(x,t) = \frac{\partial}{\partial t} \left(t ⨍_{\partial B(x,t)} \frac{f(|y|)}{|y|} \, d\sigma_y \right) + t ⨍_{\partial B(x,t)} \frac{f'(|y|)}{|y|} \, d\sigma_y.$$

We don't expect smoothness since $f(0) = f'(0) = 0$ gives local data, yet we want data about f at any point.