F06 #1: Here

$$E(t) = \frac{1}{2}(x')^2 + \frac{1}{4}x^4 - 2x^2.$$

Then

$$\dot{E}(t) = x'x'' + x^3x' - 4xx' = x'[x'' + x^3 - 4x] = 0.$$

Thus $E(t)$ is conserved. We can write the ODE as

$$x' = y$$
$$y' = -x^3 + 4x.$$

This system is Hamiltonian and hence all equilibrium points are centers or saddles. The equilibrium points are $(\pm 2, 0)$ and $(0,0)$. The Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -3x^2+4 & 0 \end{pmatrix}$$

Then

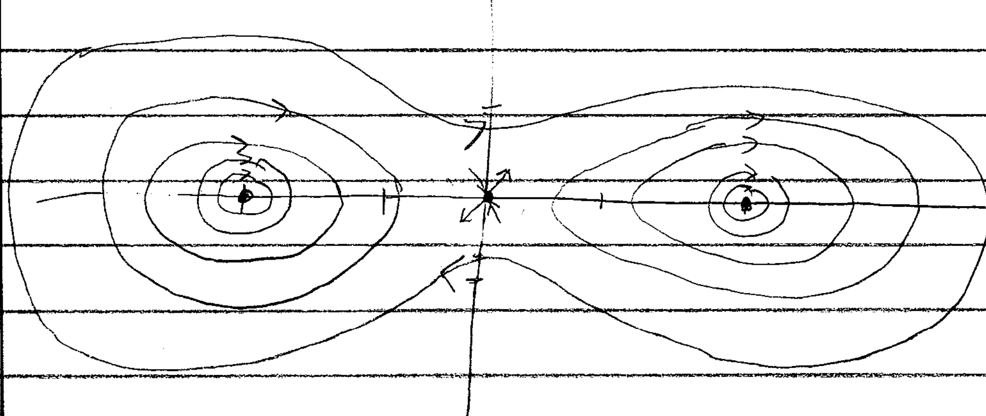
$$J(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix} \to \text{eigenvalues } \pm 2\sqrt{2}i$$

$(\pm 2, 0)$ are centers.

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \to \text{eigenvalues } \pm 2$$

eigenvectors $\begin{pmatrix} 1 \\ \pm 2 \end{pmatrix}$

$(0,0)$ is a saddle



F06 #3:

We extend u to be a function on the whole real axis in y.

u^D: Since
$$u_t = \Delta u$$
$$u(x,y,0) = u_0(x,y)$$
$$u(x,0,t) = 0,$$
we extend u to $\tilde{u}$ s.t. ~~$u(x,y,t) = -u(x,-y,t)$ for $y<0$.~~
~~Thus by $u(x,0,t) = -u(x$~~

$$\tilde{u}(x,y,t) = \begin{cases} u(x,y,t) & \text{if } y \geq 0 \\ -u(x,-y,t) & \text{if } y \leq 0. \end{cases}$$

Thus by $\tilde{u}(x,0,t) = -\tilde{u}(x,0,t) \longrightarrow \tilde{u}(x,0,t) = 0.$
and
$$\tilde{u}_t = \Delta \tilde{u} \quad \text{in } \mathbb{R}^2 \times \{t \geq 0\}$$
$$\tilde{u}(x,y,0) = \tilde{u}_0(x,y) = \begin{cases} u_0(x,y) & \text{if } y \geq 0 \\ -u_0(x,-y) & \text{if } y \leq 0. \end{cases}$$
$$\tilde{u}(x,0,t) = 0.$$

We have

$$\tilde{u}(x,y,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-\frac{|(x,y)-(a,b)|^2}{4t}}\, da\, db.$$
$$= \frac{1}{4\pi t}\int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-((x-a)^2+(y-b)^2)/4t}\, da\, db.$$
$$= \frac{1}{4\pi t}\left[\int_{b\geq 0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\, da\, db \right.$$
$$\left. + \int_{b\leq 0} -u_0(a,-b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\, da\, db.\right]$$

F06 #3 cont.

$$= \frac{1}{4\pi t}\Big[\int\!\!\int_{b>0} u_0(a,b)e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\,da\,db - \int_{b>0} u_0(a,b)\,e^{-\frac{(x-a)^2+(y+b)^2}{4t}}\,da\,db.$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\Big[1-e^{\frac{(y-b)^2-(y+b)^2}{4t}}\Big]\,da\,db.$$

$$= \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\Big[1-e^{-by/t}\Big]\,da\,db$$

Thus

$$u^D(x,y,t) = \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b)\,e^{-\frac{(x-a)^2+(y-b)^2}{4t}}\Big[1-e^{-by/t}\Big]\,da\,db.$$

u^N: We extend u to $\tilde u$:

$$\tilde u(x,y,t) = \begin{cases} u(x,y,t) & \text{if } y \geq 0 \\ u(x,-y,t) & \text{if } y \leq 0. \end{cases}$$

Thus u even.

Thus $\tilde u_y(x,0,t) = -\tilde u_y(x,0,t) \longrightarrow \tilde u_y(x,0,t) = 0.$

$$\tilde u_t = \Delta \tilde u \quad \text{in } \mathbb{R}^2\times(t>0)$$
$$\tilde u(x,y,0) = \tilde u_0(x,y) = \begin{cases} u_0(x,y) & \text{if } y\geq 0 \\ u_0(x,-y) & \text{if } y\leq 0. \end{cases}$$
$$\tilde u_y(x,0,t) = 0.$$

F06 #3 cont.

Then

$$\tilde{u}(x,y,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} \tilde{u}_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db.$$

$$+ \frac{1}{4\pi t}\int_{b<0} u_0(a,-b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \, da\, db.$$

$$= \frac{1}{4\pi t}\int_{b>0} u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} + u_0(a,b) e^{-\frac{(x-a)^2+(y+b)^2}{4t}} \, da\, db.$$

$$= \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \left[1 + e^{-by/t}\right] da\, db.$$

Thus

$$u^N(x,y,t) = \frac{1}{4\pi t}\int_0^\infty\int_{-\infty}^\infty u_0(a,b) e^{-\frac{(x-a)^2+(y-b)^2}{4t}} \left[1+e^{-by/t}\right] da\, db.$$

Then

$$u^D \le u^N \quad \forall\ x, y,\ t>0.$$ #

F06 #5: Let $g(t) := a + \int_0^t f(s) y(s)^2 \, ds$. Then

$$g'(t) = f(t) y(t)^2 \le f(t) g(t)^2.$$

Thus

$$\int_0^t \frac{g'(s)}{g(s)^2} \, ds \le \int_0^t f(s) \, ds.$$

$$-\frac{1}{g(s)} \Big]_{s=0}^{t} \le \int_0^t f(s) \, ds.$$

$$\longrightarrow \quad \frac{1}{a} - \frac{1}{g(t)} \le \int_0^t f(s) \, ds$$

$$\frac{1}{g(t)} \ge \frac{1}{a} - \int_0^t f(s) \, ds.$$

$$y(t) \le g(t) \le \frac{1}{\frac{1}{a} - \int_0^t f(s) \, ds} = \frac{a}{1 - a \int_0^t f(s) \, ds}.$$

F06 #6: We recall $\Delta = 4\frac{\partial}{\partial \xi}\frac{\partial}{\partial \bar{\xi}}$ where $\frac{\partial}{\partial \xi} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)$.
Note w/ $K(\xi,\eta) = \frac{1}{4\pi}\log(\xi^2+\eta^2)$, $\Delta K = \delta$. Thus

$$\varphi(z) = \int_{\mathbb{C}} \varphi(\xi) \cdot \delta(\xi - z)\, d\xi$$
$$= \int_{\mathbb{C}} \varphi(\xi)\,\Delta K(\xi - z)\, d\xi$$
$$= \int_{\mathbb{C}} -\frac{\partial \varphi}{\partial \bar{\xi}}(\xi) \cdot 4\frac{\partial}{\partial \xi} K(\xi - z)\, d\xi$$

Note

$$\frac{\partial}{\partial \xi}\log(\xi^2+\eta^2) = \frac{1}{2}\left(\frac{2\xi}{\xi^2+\eta^2} - i\cdot\frac{2\eta}{\xi^2+\eta^2}\right)$$
$$= \frac{1}{\xi + i\eta} = \frac{1}{\xi}$$

Thus

$$\int_{\mathbb{C}} -\frac{\partial \varphi}{\partial \bar{\xi}}(\xi) \cdot 4 \cdot \frac{1}{4\pi} \cdot \frac{1}{\xi - z}\, d\xi$$
$$= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{\xi}}(\xi) \cdot (\xi - z)^{-1}\, d\xi.$$

F06 #7:

a) here $u(x,y,t) = F(x,t)G(y)$. Then

$$u_t = \Delta u \longrightarrow F_t(x,t)G(y) - F_{xx}(x,t)G(y) - F(x,t)G''(y) = 0.$$

$$\frac{F_t - F_{xx}}{F} = \frac{G''}{G}$$

Since $u_y(x,0,0) = u_y(x,\pi,0) = 0$, $G'(0) = 0$, $G'(\pi) = 0$. Thus

$$\frac{F_t - F_{xx}}{F} = \frac{G''}{G} = -\lambda^2 \text{ for } \lambda \in \mathbb{R}.$$

Then $G'' + \lambda^2 G = 0$

$G'(0) = 0$

$G'(\pi) = 0$

$$\longrightarrow \quad G(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$G'(x) = -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x)$$

$$0 = G'(0) = B\lambda \longrightarrow B = 0.$$

$$0 = G'(\pi) = -A\lambda\sin(\lambda\pi) \longrightarrow \lambda = n \geq 0$$

Therefore

$$u(x,y,t) = \sum_{n \geq 0} F_n(x,t)\cos(ny)$$

where

$$(F_n)_t - (F_n)_{xx} + n^2(F_n) = 0.$$

here $H_n := e^{n^2 t} F_n$. Then

$$(H_n)_t = n^2 e^{n^2 t} F_n + e^{n^2 t}(F_n)_t.$$

$$(H_n)_{xx} = e^{n^2 t}(F_n)_{xx}.$$

Then

$$(H_n)_t - (H_n)_{xx} = 0.$$

We have

$$\frac{2}{\pi}\int_0^\pi \cos(my)\cos(ny)\,dy = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$$

$$H_n(x,0) = F_n(x,0) = \frac{2}{\pi}\int_0^\pi u_0(x,y)\cos(ny)\,dy.$$

F06 #7 cont.;

$$\cancel{H_n(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\frac{2}{\pi}\int_0^{\pi} u_0(x,y)\cos(ny)\,dy}$$

$$H_n(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_n(y,0)\, e^{-\frac{(x-y)^2}{4t}}\,dy.$$

So

$$F_n(x,t) = e^{-n^2 t}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_n(y,0)\, e^{-\frac{(x-y)^2}{4t}}\,dy.$$

Therefore as the F_n decay exponential for $n>0$,

$$\begin{aligned}\lim_{t\to\infty} t^{1/2} u(x,y,t) &= \lim_{t\to\infty} t^{1/2}\cdot\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} H_0(y,0)\, e^{-\frac{(x-y)^2}{4t}}\,dy \\ &= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{2}{\pi}\int_0^{\pi} u_0(y,s)\,ds\,dy \\ &= \frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_0^{\pi} u_0(y,s)\,ds\,dy.\end{aligned}$$

#

F06 #8:

a: We have

$$|u(x,t)| = \left|\int_0^x u_x(y,t)\,dy\right| \le \int_0^x |u_x(y,t)|\,dy$$
$$= \int_0^1 |u_x(y,t)|\,1_{[0,x]}(y)\,dy \le \left(\int_0^1 |u_x(y,t)|^2 dy\right)^{1/2}\left(\int_0^1 1_{[0,x]}(y)^2 dy\right)^{1/2}$$
$$\le \left(\int_0^1 |u_x(y,t)|^2 dy\right)^{1/2}.$$

Therefore $\sup_x |u(x,t)|^2 \le \int_0^1 |u_x(y,t)|^2 dy$.

We have

$$\frac{d}{dt}\int_0^1 u^2\,dx = \int_0^1 2uu_t\,dx = \int_0^1 2u(u_{xx} + cu^2)\,dx$$
$$= 2\int_0^1 uu_{xx} + cu^3\,dx$$
$$= 2\left[\, uu_x\Big]_{x=0}^1 - \int_0^1 u_x^2\,dx + c\int_0^1 u^3\,dx\right]$$
$$= -2\int_0^1 u_x^2\,dx + 2c\int_0^1 u^3\,dx$$
$$\le -2\int_0^1 u_x^2\,dx + 2c\left(\int_0^1 u_x^2\,dx\right)\int_0^1 u\,dx$$
$$\le -2\int_0^1 u_x^2\,dx + 2c\left(\int_0^1 u_x^2\,dx\right)\left(\int_0^1 u^2\,dx\right)^{1/2}$$
$$= -2\int_0^1 |u_x|^2 dx\left(1 - c\left(\int_0^1 |u|^2\,dx\right)^{1/2}\right).$$

b). Let $E(t) := \int_0^1 u(x,t)^2\,dx$. ~~If~~ Suppose we did not have $E(t) < 1/c^2$ $\forall$ time. Then $\exists$ a first time T s.t.

F06 #8 cont:

$E(T) = \frac{1}{c^2}$ and $E(t) < \frac{1}{c^2}$ $\forall t < T$. Then for these $t < T$,

$$E'(t) \le -2\int_0^1 |u_x|^2 dx \left(1 - c\left(\frac{1}{c^2}\right)^{1/2}\right) \le 0.$$

Therefore

$$E(T) = \cancel{E(0) + \int_0^T E'(t)\,dt} \le E(0) < \frac{1}{c^2}$$

a contradiction.

c). here $u_0 = 1$. If u does not depend on x, then we have to solve $u_t = cu^2$. $\rightarrow \int \frac{1}{u^2} du = \int c\,dt$

$$\rightarrow -\frac{1}{u} = ct + \tilde{C}$$

$$u = \frac{1}{-\tilde{C} - ct}. \qquad u_0 = 1.$$

So $u = \frac{1}{1 - ct}$.

Thus solution blows up in finite time.

Sol #1: We solve this equation using method of characteristics. (and the notation of Evans).

We want to solve

$$u_{x_1} + u_{x_2} = u^2$$
$$u(x_1, 0) = h(x_1).$$

$$\rightarrow F(p, z, x) = p_1 + p_2 - z^2 = 0.$$

$$\longrightarrow \quad D_p F = (1, 1)$$
$$D_x F = (0, 0)$$
$$D_z F = -2z.$$

Then

$$\dot{p} = -D_x F - D_z F p = 2z(p_1, p_2)$$
$$\dot{z} = D_p F \cdot p = p_1 + p_2 = z^2$$
$$\dot{x} = D_p F = (1, 1)$$

with

$$x_1(0) = x_1(0)$$
$$x_2(0) = 0$$
$$z(0) = h(x_1(0))$$

We have

$$x_1(s) = x_1(0) + s \qquad \dot{z} = z^2 \rightarrow \frac{1}{z^2}\, dz = ds$$
$$x_2(s) = s \qquad -\frac{1}{z(s)} = s + C$$

Since $z(0) = h(x_1(0))$, $C = -\frac{1}{h(x_1(0))}$.

Thus

$$z(s) = \frac{1}{\frac{1}{h(x_1(0))} - s} = \frac{1}{\frac{1}{h(x_1(s) - x_2(s))} - x_2(s)}.$$

That is

$$u(x_1, x_2) = \frac{1}{\frac{1}{h(x_1 - x_2)} - x_2}.$$

Sol #2:

① Mass:

We have
$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t\,dx = -\int_{-\infty}^{\infty} u_{xxx}\,dx - 6\int_{-\infty}^{\infty} uu_x dx$$
$$= -u_{xx}\Big]_{x=-\infty}^{\infty} - 6\cdot\frac{1}{2}u^2\Big]_{x=-\infty}^{\infty} = 0.$$

② Momentum:

We have
$$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\,dx = \int_{-\infty}^{\infty} 2uu_t\,dx = 2\int_{-\infty}^{\infty} u(-u_{xxx}-6uu_x)\,dx$$
$$= -2\int_{-\infty}^{\infty} uu_{xxx}\,dx - 12\int_{-\infty}^{\infty} u^2u_x\,dx$$
$$= -2\int_{-\infty}^{\infty} uu_{xxx}\,dx - 12\cdot\frac{1}{3}u^3\Big]_{x=-\infty}^{\infty}$$

Since
$$\int_{-\infty}^{\infty} uu_{xxx}\,dx = -\int_{-\infty}^{\infty} u_x u_{xx}\,dx = +\int_{-\infty}^{\infty} u_{xx}u_x\,dx$$
$$= -\int_{-\infty}^{\infty} u_{xxx}u\,dx,$$

it follows that $\int_{-\infty}^{\infty} uu_{xxx}\,dx = 0$. Thus Momentum is conserved.

③ Energy:

We have
$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}u_x^2 - u^3\,dx = \int_{-\infty}^{\infty} u_xu_{xt} - 3u^2u_t\,dx$$
$$= \int_{-\infty}^{\infty} u_x(-u_{xxx} - 6uu_x)_x + 3u^2(u_{xxx}+6uu_x)\,dx.$$

S06#2
cont

Note $\int_{-\infty}^{\infty} u^3 u_x \, dx = \frac{1}{4} u^4 \Big]_{u=-\infty}^{\infty} = 0$.

$$= -\int_{-\infty}^{\infty} u_x \overset{(u_{xxx} + 6uu_x)_x}{\cancel{[u_{xxxx} + 6u_x^2 + 6uu_{xx}]}} \, dx + 3\int_{-\infty}^{\infty} u^2 u_{xxx} \, dx$$

$$= \int_{-\infty}^{\infty} u_{xx} (u_{xxx} + 6uu_x) \, dx + 3\int_{-\infty}^{\infty} u^2 u_{xxx} \, dx$$

$$= \int_{-\infty}^{\infty} u_{xx} u_{xxx} + \cancel{6uu_x u_{xx}} - 6\int_{-\infty}^{\infty} \cancel{uu_x u_{xx}} \, dx$$

$$= \int_{-\infty}^{\infty} u_{xx} u_{xxx} \, dx = 0$$

as

$$\int_{-\infty}^{\infty} u_{xx} u_{xxx} \, dx = -\int_{-\infty}^{\infty} u_{xxx} u_{xx} \, dx.$$

Sob #3:

Consider the Sturm Liouville problem

$$(p(x)u')' = -\lambda u$$
$$u'(0) = u'(L) = 0.$$

This is indeed Sturm-Liouville since $p(x) > 0$. Thus we have eigenfunctions $\{\phi_n\}$ corresponding to eigenvalues $\{\lambda_n\}_{n\geq 0}$ which form a orthogonal basis for the space of functions. ~~Since~~ Furthermore
$\lambda_0 < \lambda_1 < \lambda_2 < \ldots \to \infty$. and

$$\lambda_0 = \min_u -\frac{\langle u, (pu')' \rangle}{\langle u, u \rangle}.$$

Since

$$\int_0^L u(pu')' \, dx = -\int_0^L p(u')^2 \, dx \leq 0$$

We have $\lambda_0 \geq 0$. Since 0 is an eigenvalue!, $\lambda_0 = 0$.
Thus $\lambda_n > 0 \ \forall n > 0$.
Here

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t) \phi_n(x).$$

We have

$$u_t = \partial_x(p(x) u_x)$$
$$\sum_{n=0}^{\infty} a_n'(t) \phi_n(x) = \sum_{n=0}^{\infty} \partial_x(p(x)\phi_n'(x))\, a_n(t).$$
$$= \sum_{n=0}^{\infty} a_n(t)(-\lambda_n)\phi_n(x).$$

Therefore

$$a_n'(t) = -\lambda_n a_n(t)$$
$$\longrightarrow a_n(t) = e^{-\lambda_n t} a_n(0) \qquad a_n(0) = \frac{\int_0^L \varphi(x)\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx}$$

S06 #3 cont:

Thus

$$\lim_{t \to \infty} u(x,t) = a_0(0) = \frac{\int_0^L \varphi(x)\phi_0(x)\,dx}{\int_0^L \phi_0(x)^2\,dx}$$

Since $\phi_0 = 1$ is the eigenfunction corresponding to $\lambda_0 = 0$, we have

$$\lim_{t \to \infty} u(x,t) = a_0(0) = \frac{1}{L}\int_0^L \varphi(x)\,dx \quad \#$$

S06 #4:

Let
$$f(y) = \begin{cases} y \log(2 + \frac{1}{|y|}) & \text{for } y \neq 0 \\ 0 & \text{for } y = 0. \end{cases}$$

We will show that the ODE
$$\frac{dy}{dt} = f(y)$$
$$y(0) = 0$$
has only the zero solution. This will be shown by showing if $y(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$, then $y(t) \neq 0 \ \forall t \in \mathbb{R}$.

Suppose $\exists \ t^*$ w/ $y(t^*) \neq 0$. Replacing y w/ $-y$ if necessary, we may suppose WLOG that $y(t^*) > 0$. Let (a, b) be the largest open interval containing t^* s.t. $y > 0$ (here we have implicitly used continuity of y).

① $b = \infty$.

Pf: Suppose $b < \infty$. Then by continuity, $y(b) = 0$, but
$$y(b) = y(t^*) + \int_{t^*}^{b} f(y(t))\,dt \geq y(t^*) > 0.$$
⊥

② $a = -\infty$.

Pf: Suppose $a > -\infty$. Then $y(a) = 0$.

S06 #4 cont:

Let $g(x) = \int_1^x \frac{1}{s\log(2+\frac{1}{|s|})}\,ds$, $x>0$. Then

$$\frac{d}{dt}\left(g(y(t))\right) = 1 \quad \forall t \in (a,b)$$

and so

$$g(y(t^*)) - \lim_{t\to a^+} g(y(t)) = t^* - a.$$

Since $y(a)=0$ and $\lim_{x\to 0} g(x) = -\infty$ and $g(y(t^*))<\infty$,

we have $g(y(t^*)) - \lim_{t\to a^+} g(y(t)) = \infty$, a contradiction.

Therefore if $\exists\, t^*$ w/ $y(t^*) \neq 0$, then $y \neq 0\ \forall t$.

Thus the zero sol. is the only solution.

It now remains to show f is not Lipschitz. We have

$$\left|\frac{f(\frac{1}{n}) - f(\frac{1}{2n})}{\frac{1}{2n}}\right| = \left|\frac{\frac{1}{n}\log(2+n) - \frac{1}{2n}\log(2+2n)}{\frac{1}{2n}}\right|$$

$$= \left|2\log(2+n) - \log(2+2n)\right|$$

$$= \left|\log\left(\frac{(2+n)^2}{2+2n}\right)\right| \to \infty \text{ as } n\to\infty.$$

Therefore f is not Lipschitz. #

S06 #5: We rewrite the ~~system~~ ODE as a system:

$$x' = y \qquad (*)$$
$$y' = -x - 2x^2.$$

This is a Hamiltonian system and hence all equilibrium points are either centers or saddles. Let

$$H(x,y) := \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3 = \frac{1}{2}(x')^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3.$$

Then

$$\frac{d}{dt}H(x,y) = x'x'' + xx' + 2x^2x'$$
$$= x'(x'' + x + 2x^2) = 0.$$

Thus $\frac{1}{2}(x')^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3$ is a conserved quantity.

The equilibrium points of $(*)$ are $(0,0)$ and $(-\frac{1}{2},0)$.
The Jacobian is

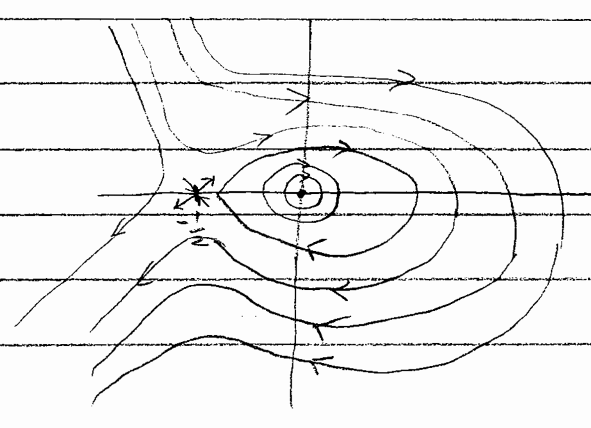
$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -1-4x & 0 \end{pmatrix}.$$

We have

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \text{eigenvalues } \pm i. \qquad \text{center}$$

$$J(-\tfrac{1}{2},0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{eigenvalues } \pm 1 \qquad \text{saddle.}$$

eigenvectors $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$



S06 #6: ~~Lemma~~ Suppose $\max u(x) > 0$. Let x_0 be s.t. $u(x_0) = \max u(x)$

Then $x_0 \in \Omega$ and hence $\Delta u(x_0) \leq 0$, $u_{x_k}(x_0) = 0$. Then at the point x_0,

as $u=0$ on $\partial\Omega$

$$\Delta u + \sum_{k=1}^{n} a_k(x) u_{x_k} + c(x) u = \Delta u + c(x_0) u(x_0)$$
$$\leq c(x_0) u(x_0) < 0$$

as $u(x_0) > 0$ and $c(x) < 0$ in Ω. This is a contradiction.

Thus $\max u(x) \leq 0$.

Suppose $\min u(x) < 0$. Let y_0 be s.t. $u(y_0) = \min u(x)$

Then as $u=0$ on $\partial\Omega$, $y_0 \in \Omega$ and hence $\Delta u(y_0) \geq 0$, $u_{x_k}(y_0) = 0$. At the point y_0,

$$\cancel{\Delta u + \sum_{k=1}^{n} a_k(x) u_{x_k}}$$

$$(\Delta u)(y_0) + \sum_{k=1}^{n} a_k(y_0) u_{x_k}(y_0) + c(y_0) u(y_0)$$
$$= (\Delta u)(y_0) + c(y_0) u(y_0)$$
$$\geq c(y_0) u(y_0) > 0$$

Since $u(y_0) < 0$ and $c(x) < 0$ $\forall x \in \Omega$. This is a contradiction.

Thus $\min u(x) \geq 0$.

Therefore $u=0$ on $\partial\Omega$ implies $u=0$ on Ω.

Lagrange multipliers: optimize f subj. to $g=k$.

1. Solve system $\nabla f = \lambda \nabla g$, $g = k$.

2. Plug in all solutions (x,y,z) from 1. into $f(x,y,z)$ to find min./max.

S06 #7: here $E[u] := \int_\Omega \frac{1}{2}|\nabla u|^2 - f u \, dx$, $F[u] := \int_\Omega u \, dx$. We want to minimize $E[u]$ subject to $F[u] = A$ and $u|_{\partial\Omega} = 0$.

By Lagrange multipliers, the minimum u must satisfy

$$\left(\lim_{\varepsilon \to 0} \frac{E[u+\varepsilon v] - E[u]}{\varepsilon}\right) = \lambda \left(\lim_{\varepsilon \to 0} \frac{F[u+\varepsilon v] - F[u]}{\varepsilon}\right)$$

the moral is choose v so that $u+\varepsilon v$ has the same properties as u.

for some constant λ and v s.t. $v|_{\partial\Omega} = 0$ and $\int_\Omega v \, dx = 0$ (same E eats functions with $u|_{\partial\Omega}=0$ and $\int_\Omega u \, dx = A$ thus we need $\int_\Omega v \, dx = 0$ so that $\int_\Omega u + \varepsilon v \, dx = A$). We have

$$\lim_{\varepsilon \to 0} \frac{E[u+\varepsilon v] - E[u]}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_\Omega \frac{1}{2}|\nabla u + \varepsilon \nabla v|^2 - (f)(u+\varepsilon v) - \frac{1}{2}|\nabla u|^2 + f u \, dx$$

$$= \int_\Omega \nabla u \cdot \nabla v - f v \, dx$$

$$= \int_\Omega (-\Delta u - f) v \, dx$$

and

$$\lim_{\varepsilon \to 0} \frac{F[u+\varepsilon v] - F[u]}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_\Omega u + \varepsilon v - u \, dx = \int_\Omega v \, dx = 0.$$

Thus

$$\int_\Omega (-\Delta u - f) v \, dx = 0 \longrightarrow -\Delta u = f \text{ in } \Omega.$$

Therefore the minimum of $E[\cdot]$ subject to $u|_{\partial\Omega} = 0$ and $\int_\Omega u \, dx = A$ is the unique solution to

$$\Delta u = -f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

F05 #2 :

We have $Lu = \lambda u.$

$$u'' + u' - a(1+x^2)u = \lambda u.$$

$$\cancel{-u'' - u' + a(1+x^2)u = -\lambda u.}$$

$$\cancel{(-e^x u')' + a(1+x^2)u = -\lambda u.}$$

$\cancel{Tu = -u'' - u' + a(1+x^2)u}$

$(e^x u')' - a(1+x^2)u = \lambda u.$

a) We have

$$\int_0^1 (Lu) v e^x \, dx = \int_0^1 (u'' + u' - a(1+x^2)u) v e^x \, dx$$

$\boxed{v(0) = v(1) = 0.}$

$$= \int_0^1 u'' v e^x \, dx + \int_0^1 u' v e^x \, dx - a\int_0^1 (1+x^2) uv e^x \, dx$$

$$= \int_0^1 - u'(v'e^x + ve^x) \, dx + \int_0^1 u' v e^x \, dx - a\int_0^1 (1+x^2) uv e^x \, dx$$

$$= -\int_0^1 u'v'e^x - a\int_0^1 (1+x^2) uv e^x \, dx.$$

$$= \int_0^1 u(v'e^x)' \, dx - \int_0^1 a(1+x^2) uv e^x \, dx$$

$$= \int_0^1 u(v''e^x + v'e^x) \, dx - \int_0^1 a(1+x^2) uv e^x \, dx$$

$$= \int_0^1 u\left[v'' + v' - a(1+x^2)v\right] e^x \, dx$$

$$= \int_0^1 u Lv \, e^x \, dx.$$

b) We have

$$\lambda_{a_0} \langle u_{a_0}, u_{a_0} \rangle_{e^x} = \int_0^1 u_{a_0} L u_{a_0} e^x \, dx = \int_0^1 u_{a_0} \left(u_{a_0}'' + u_{a_0}' - a(1+x^2) u_{a_0} \right) e^x dx$$

FOS #2 cont..

$$= \int_0^1 \left[u_{a_0} u_{a_0}'' + \left(\tfrac{1}{2} u_{a_0}^2\right)' - a(1+x^2) u_{a_0}^2 \right] e^x \, dx. \quad (*)$$

$$\int_0^1 u_{a_0} u_{a_0}'' e^x \, dx = -\int_0^1 \left[(u_{a_0}') e^x + u_{a_0} e^x \right] u_{a_0}'$$

$$= -\int_0^1 (u_{a_0}')^2 e^x \, dx - \int_0^1 u_{a_0} u_{a_0}' e^x \, dx$$

Thus

$$(*) = -\int_0^1 (u_{a_0}')^2 e^x \, dx - \int_0^1 a(1+x^2) u_{a_0}^2 e^x dx < 0.$$

c) ~~We have~~

~~Smallest $\lambda = \min\limits_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Tu \rangle}{\langle u, u \rangle} = \min\limits_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu \rangle}{\langle u, u \rangle}$ for T~~

Smallest μ s.t.

$$Lu = -\mu u$$

is given by

$$\mu_0 = \min_{\substack{u \\ u(0)=u(1)=0}} -\frac{\langle u, Lu \rangle}{\langle u, u \rangle} = -\max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu \rangle}{\langle u, u \rangle}.$$

$$\rightarrow -\mu_0 = \max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu \rangle}{\langle u, u \rangle}.$$

Therefore

$$\lambda_{a_0} = \max_{\substack{u \\ u(0)=u(1)=0}} \frac{\langle u, Lu \rangle}{\langle u, u \rangle}.$$

F05 #2 cont..

We have

$$\frac{\langle u, Lu\rangle}{\langle u,u\rangle} = \frac{1}{\langle u,u\rangle}\int_0^1 ue^x\left(u''+u'-a(1+x^2)u\right)dx.$$

$$\frac{d}{da}\frac{\langle u,Lu\rangle}{\langle u,u\rangle} = \frac{1}{\langle u,u\rangle}\int_0^1 -(1+x^2)u^2e^x\,dx < 0.$$

Therefore λ_{a_0} is a decreasing function of a. Thus ~~$a \geq \lambda_a$~~ as $\lambda_{a_0} < 0$, if $0 < a_1 < a_2$, $|\lambda_{a_1 0}| < |\lambda_{a_2 0}|$.

F05 #3 cont':

The periodic orbits:

$$y^2 = \frac{1}{2}x^4 - x^2 + 2C.$$

$$y = \pm\sqrt{\frac{1}{2}x^4 - x^2 + 2C}$$

Need $\frac{1}{2}x^4 - x^2 + 2C$ to be ≥ 0 for $x \in [-1, 1]$.

~~Thus we need $C \geq 0$~~ Thus we need $C > 0$.

~~Since $\frac{1}{2}x^4 - x^2$ is minimized on $x = \pm 1$, We need~~

$$\cancel{2C + \left(\frac{1}{2} - 1\right) \geq 0 \rightarrow}$$

~~We have~~ For the orbit to be periodic we also need 2 distinct sol. of $\frac{1}{2}x^4 - x^2 + 2C = 0$ at these pts. $y = 0$. So

$$x^2 = \frac{1 \pm \sqrt{1 - 4 \cdot \frac{1}{2} \cdot 2C}}{2} = \frac{1 \pm \sqrt{1 - 4C}}{2}$$

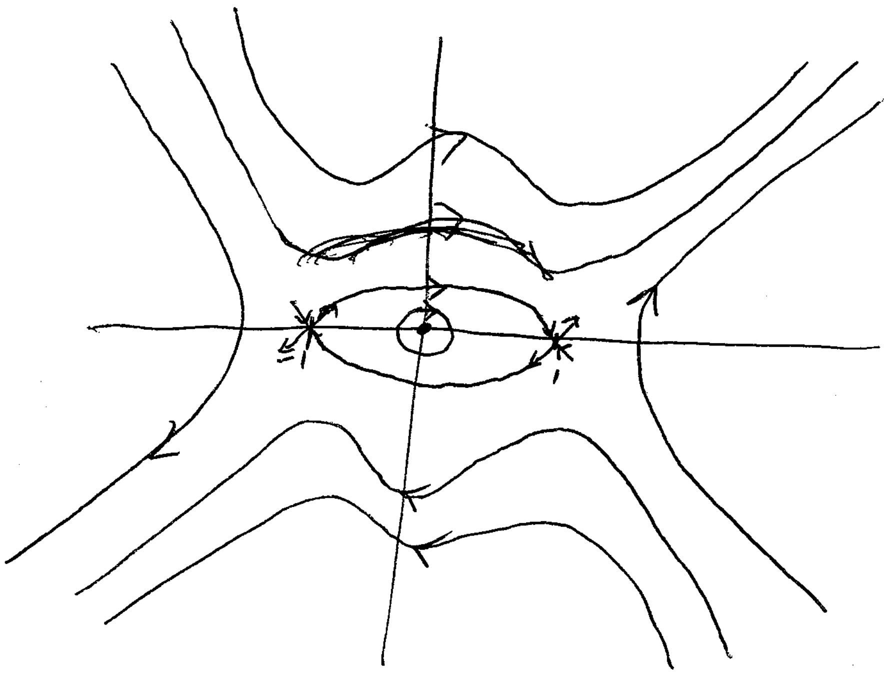
$\rightarrow$ need $1 - 4C > 0 \longrightarrow C < \frac{1}{4}$.

Thus the periodic orbits are given by

$$y^2 = \frac{1}{2}x^4 - x^2 + D$$

$$0 < D < \frac{1}{2}$$

F05 #3 cont:



F05 #4: u solves the heat equation.

a) We have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-\left(\frac{x-y}{\sqrt{4t}}\right)^2} dy \qquad u = \frac{x-y}{\sqrt{4t}}, \quad du = -\frac{1}{\sqrt{4t}}\, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{\infty}^{x/\sqrt{4t}} e^{-u^2} (-\sqrt{4t})\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-u^2}\, du$$

Then for each fixed x,

$$\lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4t}}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2}\, du = \frac{1}{2}.$$

b) We claim the limit is not uniform ~~in~~ x. To do so we rephrase the question of asking whether the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4n}}^{\infty} e^{-u^2}\, du = \frac{1}{2} \qquad f_n(x) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4n}}^{\infty} e^{-u^2}\, du$$

is uniform in x. Suppose it was, then $\forall \varepsilon > 0\ \exists N$ depending only on ε s.t. for $n \geq N$, $|f_n(x) - \frac{1}{2}| < \varepsilon\ \forall x$.

Then $|f_N(x) - \frac{1}{2}| < \varepsilon\ \forall x$. But

$$\lim_{x\to\infty} f_N(x) = \lim_{x\to\infty} \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4N}}^{\infty} e^{-u^2}\, du = 0.$$

Therefore the $\{f_n\}$ do not conv. unif to $\frac{1}{2}$. So the convergence is not uniform.

F05 #5.

Since u is a smooth soln on the torus, integration by parts gives no bdry conditions.

We have

$$\frac{d}{dt}E(t) = \varepsilon\int \nabla u\cdot\nabla u_t + \frac{1}{\varepsilon}\int W'(u)u_t.$$

$$= \varepsilon\int -\Delta u\, u_t + \frac{1}{\varepsilon}\int W'(u)u_t$$

$$= \int\left(\frac{1}{\varepsilon}W'(u) - \varepsilon\Delta u\right)u_t\,dx$$

$$= \int\left(-\frac{1}{\varepsilon}W'(u) + \varepsilon\Delta u\right)\Delta\left(\varepsilon\Delta u - \tfrac{1}{\varepsilon}W'(u)\right)dx$$

$$= -\int\left|\nabla\left(\varepsilon\Delta u - \tfrac{1}{\varepsilon}W'(u)\right)\right|^2 dx \le 0.$$

Therefore E is monotonically $\downarrow$. #

05 #6:

here $F(r) = \frac{1}{4\pi r^2}\int_{|x|=r} f(x)\, d\sigma = \frac{1}{4\pi}\int_{|x|=1} f(rx)\, d\sigma.$

We have

$$F'(r) = \frac{1}{4\pi}\int_{\partial B(0,1)} \nabla f(rx)\cdot x \, d\sigma. = \frac{1}{4\pi}\int_{\partial B(0,1)} \frac{\partial f}{\partial \nu}(rx)\, d\sigma$$

$$= \frac{1}{4\pi}\int_{B(0,1)} \nabla\cdot((\nabla f)(rx))\, dx.$$

$$= \frac{r}{4\pi}\int_{B(0,1)} (\Delta f)(rx)\, dx.$$

$$= \frac{r}{4\pi r^3}\int_{B(0,r)} (\Delta f)(y)\, dy$$

$$= \frac{r}{3} ⨍_{B(0,r)} (\Delta f)(y)\, dy$$

Mean Value Property

$$= \frac{r}{3}(\Delta f)(0)$$

Therefore

$$F(a) - F(0) = \int_0^a F'(r)\, dr = \int_0^a \frac{r}{3}(\Delta f)(0)\, dr.$$

$$= \frac{a^2}{6}(\Delta f)(0).$$

$$\longrightarrow \frac{1}{4\pi a^2}\int_{|x|=a} f(x)\, d\sigma = f(0) + \frac{a^2}{6}(\Delta f)(0).$$

#.

F05 #7:

Method of characteristics:

$$u_t + u u_x = 0$$

$$F(p, q, z, x, t) = q + zp.$$

$\longrightarrow$ $\dot{x} = z$ $\quad x(0) = x_0$

$\dot{t} = 1$ $\quad t(0) = 0$ $\quad$ where $f(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 < x < 0 \\ 1 - \frac{1}{2}x & \text{if } 0 < x < 2 \\ 0 & \text{if } x > 2 \end{cases}$

$\dot{z} = 0$ $\quad z(0) = f(x_0)$

Therefore

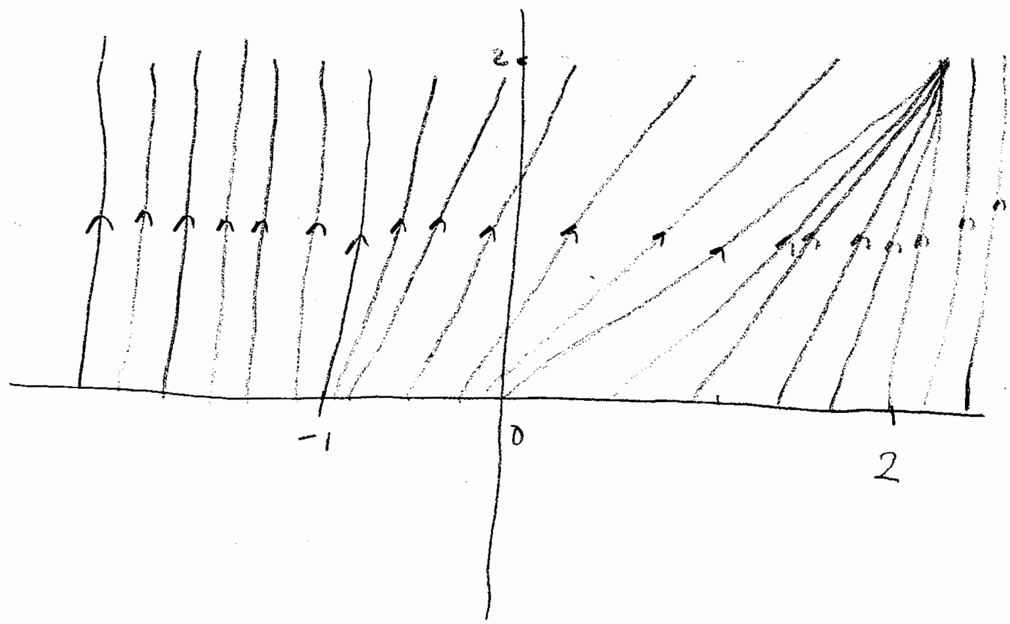
$$z(s) = f(x_0), \quad t(s) = s$$
$$x(s) = f(x_0) s + x_0.$$

The characteristics are

$$x = f(x_0) t + x_0.$$

		Characteristics		
if $x_0 < -1$,	$\longrightarrow$	$x = x_0$		
$-1 < x_0 < 0$	$\longrightarrow$	$\dfrac{x - x_0}{x_0 + 1} = t.$	$\longrightarrow$	$x_0 = \dfrac{x - t}{t + 1}$
$0 < x_0 < 2$	$\longrightarrow$	$\dfrac{x - x_0}{1 - \frac{1}{2}x_0} = t.$	$\longrightarrow$	$x_0 = \dfrac{x - t}{1 - \frac{1}{2}t}.$
$x_0 > 2$	$\longrightarrow$	$x = x_0.$		

FOS #7 cont;



The characteristics cross at time $t=2$. We have for $t<2$,

$$u(x,t) = \begin{cases} 0 & \text{if } x<-1. \\ \dfrac{x+1}{t+1} & \text{if } -1<x<t. \quad \nearrow x_0<0 \longleftrightarrow x<t \\ \dfrac{2-x}{2-t} & \text{if } t<x<2 \quad \nearrow x_0<2 \longleftrightarrow x<2. \\ 0 & \text{if } x>2. \end{cases}$$

Now we compute the shock ~~for~~ which occurs for $t>2$. Let $x=s(t)$ be the equation for the shock. Then the Rankine-Hugoniot,

$$\frac{\frac{1}{2}\left(\frac{s+1}{t+1}\right)^2 - \frac{1}{2}\cdot 0^2}{\frac{s+1}{t+1} - 0} = \dot{s} \qquad s(2)=2.$$

F05 #7 cont:

$$\frac{ds}{dt} = \frac{1}{2}\,\frac{s+1}{t+1}$$

$$\frac{1}{s+1}\,ds = \frac{1}{2}\frac{1}{(t+1)}\,dt$$

$$\ln s+1 = \ln\sqrt{t+1} + C.$$

$$\ln 3 = \frac{1}{2}\ln 3 + C \longrightarrow C = \frac{1}{2}\ln 3.$$

$$s+1 = \sqrt{3(t+1)}.$$

$$s(t) = \sqrt{3(t+1)} - 1.$$

Thus for $t > 2$, the entropy solution is given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < -1. \\ \dfrac{x+1}{t+1} & \text{if } -1 < x < \sqrt{3(t+1)} - 1 \\ 0 & \text{if } x > \sqrt{3(t+1)} - 1. \end{cases}$$

#

FOS #8

We have

$$F(p, q, z, x, y) = p^2 + q^2 - 1.$$

Then

$$\dot{x} = 2p \qquad x(0) = x_0$$
$$\dot{y} = 2q \qquad y(0) = 0$$
$$\dot{z} = 2 \qquad z(0) = \cos x_0$$
$$\dot{p} = 0 \qquad p(0) = -\sin x_0 \ (\text{since } \phi(x,0) = \cos x,\ \phi_x(x,0) = -\sin x)$$
$$\dot{q} = 0 \qquad q(0) = \pm \cos x_0.$$

$\longrightarrow$ $x(s) = (-2\sin x_0)s + x_0.$

$y(s) = (\pm 2\cos x_0)s \longrightarrow$ since $0 \le y < \infty$,

$y(s) = |2\cos x_0| s.$

$z(s) = 2s + \cos x_0.$

Therefore

$$\phi(x, y) = 2s + \cos r$$

where

$$x = -2(\sin r)s + r$$
$$y = |2\cos r| s$$

W05 #1: Write $u(x,t) = F(x)G(t)$. Then

$$u_{tt} - u_{xx} - 2u_x = 0$$
$$\longrightarrow F(x)G''(t) - F''(x)G(t) - 2F'(x)G(t) = 0.$$
$$\frac{G''(t)}{G(t)} = \frac{F''(x)+2F'(x)}{F(x)} = \lambda.$$

Since $u_x(0,t) = 0$, $u_x(1,t) = 0 \longrightarrow F'(0) = 0$, $F'(1) = 0$.

We consider 3 cases:

1. $\lambda > -1$. In this case, $1+\lambda > 0$. Then

$$\frac{F''(x)+2F'(x)}{F(x)} = \lambda \longrightarrow F''(x) + 2F'(x) - \lambda F(x) = 0.$$
$$F(x) = Ae^{(-1+\sqrt{1+\lambda})x} + Be^{(-1-\sqrt{1+\lambda})x}$$

Since

$$F'(x) = A(-1+\sqrt{1+\lambda})e^{(-1+\sqrt{1+\lambda})x} + B(-1-\sqrt{1+\lambda})e^{(-1-\sqrt{1+\lambda})x}$$
$$0 = A(-1+\sqrt{1+\lambda}) + B(-1-\sqrt{1+\lambda})$$
$$\longrightarrow 0 = A(-1+\sqrt{1+\lambda})e^{-1+\sqrt{1+\lambda}} + B(-1-\sqrt{1+\lambda})e^{-1-\sqrt{1+\lambda}}.$$

Then

$$0 = B(-1-\sqrt{1+\lambda})(e^{-1+\sqrt{1+\lambda}} - e^{-1-\sqrt{1+\lambda}})$$
$$\longrightarrow B = 0 \longrightarrow A = 0.$$

Thus only gives trivial solution.

2. $\lambda = -1$.

In this case $F(x) = Ae^{-x} + Bxe^{-x}$.

$$\longrightarrow F'(x) = -Ae^{-x} + B(e^{-x} - xe^{-x}).$$

Thus

$$0 = F'(0) = -A + B$$
$$0 = F'(1) = -Ae^{-1} + \cancel{Be^{-1}} - \cancel{Be^{-1}} \longrightarrow A = 0, B = 0.$$

Thus we have the trivial solution.

3. $\lambda < -1$.

In this case $0 < -\lambda - 1$. Then $\sqrt{1+\lambda} = i\sqrt{-\lambda-1}$.

W05 #1 Laura

Then

$$F(x) = Ae^{(-1+\sqrt{1+\lambda})x} + Be^{(-1-\sqrt{1+\lambda})x}$$
$$= e^{-x}[\tilde{A}\cos(\sqrt{-1-\lambda}\,x) + \tilde{B}\sin(\sqrt{-1-\lambda}\,x)].$$
$$F'(x) = e^{-x}[-\tilde{A}\sin(\sqrt{-1-\lambda}\,x)\sqrt{-1-\lambda} + \tilde{B}\cos(\sqrt{-1-\lambda}\,x)\sqrt{-1-\lambda}]$$
$$- e^{-x}[\tilde{A}\cos(\sqrt{-1-\lambda}\,x) + \tilde{B}\sin(\sqrt{-1-\lambda}\,x)]$$
$$0 = F'(0) = \tilde{B}\sqrt{-1-\lambda} - \tilde{A}.$$
$$\rightarrow 0 = F'(1) = e^{-1}[-\tilde{A}\sin(\sqrt{-1-\lambda})\sqrt{-1-\lambda} + \tilde{B}\cos(\sqrt{-1-\lambda})\sqrt{-1-\lambda}$$
$$- \tilde{A}\cos(\sqrt{-1-\lambda}) - \tilde{B}\sin(\sqrt{-1-\lambda})]$$

Therefore

$$\tilde{A}\sin(\sqrt{-1-\lambda})\sqrt{-1-\lambda} + \tilde{B}\sin(\sqrt{-1-\lambda}) = 0$$
$$\rightarrow \tilde{B}(-1-\lambda)\sin(\sqrt{-1-\lambda}) + \tilde{B}\sin(\sqrt{-1-\lambda}) = 0$$

Since we want nontrivial solutions, $\sin(\sqrt{-1-\lambda}) = 0$.

$$\rightarrow \sqrt{-1-\lambda_n} = n\pi, \quad n = 1, 2, 3, \ldots$$
$$\lambda_n = -(n^2\pi^2+1).$$

We solve $\dfrac{G''(t)}{G(t)} = -(n^2\pi^2+1).$

$$\rightarrow G''(t) + (n^2\pi^2+1)G(t) = 0.$$
$$G(t) = C\cos\sqrt{n^2\pi^2+1}\,t + D\sin\sqrt{n^2\pi^2+1}\,t.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} e^{-x}(a_n\cos n\pi x + b_n\sin n\pi x)(c_n\cos\sqrt{n^2\pi^2+1}\,t + d_n\sin\sqrt{n^2\pi^2+1}\,t)$$

Since $u_t(x,0) = 0$, $d_n = 0$ $\forall n$.
Since $u(x,0) = e^{-x}(\pi\cos\pi x + \sin\pi x)$, $c_1 = 1$, $b_1 = 1$, $a_1 = \pi$
and all other coeff are $= 0$.

Thus

$$u(x,t) = e^{-x}(\pi\cos\pi x + \sin\pi x)\cos(\sqrt{\pi^2+1}\,t)$$

W05 #2: We use method of characteristics (and the notation of Evans). We have

$$x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u$$
$$u(x_1, x_2, 0) = \varphi(x_1, x_2)$$

$$\longrightarrow \quad F(p, z, x) = x_1 p_1 + 2x_2 p_2 + p_3 - 3z = 0$$

$$D_p F = (x_1, 2x_2, 1)$$
$$\longrightarrow \quad D_x F = (p_1, 2p_2, 0)$$
$$D_z F = -3$$

$$\longrightarrow \quad \dot{p} = -D_x F - D_z F p = (-p_1, -2p_2, 0) + (3p_1, 3p_2, 3p_3)$$
$$= (2p_1, p_2, 3p_3)$$
$$\dot{z} = D_p F \cdot p = x_1 p_1 + 2x_2 p_2 + p_3 = 3z.$$
$$\dot{x} = D_p F = (x_1, 2x_2, 1).$$

w/ initial conditions $x_1(0) = x_1(0)$, $x_2(0) = x_2(0)$, $x_3(0) = 0$, $z(0) = \varphi(x_1(0), x_2(0))$

Then

$$z(s) = \varphi(x_1(0), x_2(0))e^{3s} \qquad x_1(s) = x_1(0)e^{s}$$
$$= \varphi\left(\frac{x_1(s)}{e^{x_3(s)}}, \frac{x_2(s)}{e^{2x_3(s)}}\right)e^{3x_3(s)}. \qquad x_2(s) = x_2(0)e^{2s}$$
$$x_3(s) = s$$

Thus the solution is

$$u(x_1, x_2, x_3) = \varphi\left(\frac{x_1}{e^{x_3}}, \frac{x_2}{e^{2x_3}}\right)e^{3x_3}.$$

WOS #3

Note we only have that u is harmonic in $\mathbb{D}$ not $\overline{\mathbb{D}}$, so we have to work a bit harder.

By Poisson's formula

$$u(x) = \frac{(1-\varepsilon)^2 - |x|^2}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} \frac{u(y)}{|x-y|^2} \, d\sigma_y$$

$\forall \; |x| < 1-\varepsilon.$

Then

$$\frac{1}{(1-\varepsilon)+|x|} \le \frac{1}{|x-y|} \le \frac{1}{(1-\varepsilon)-|x|}$$

Thus

$$\frac{(1-\varepsilon)-|x|}{(1-\varepsilon)+|x|} \frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y \le u(x) \le \frac{(1-\varepsilon)+|x|}{(1-\varepsilon)-|x|} \cdot \frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y.$$

Since u is harmonic in $B_{1-\varepsilon}(0)$,

$$\frac{1}{2\pi(1-\varepsilon)} \int_{\partial B(0,1-\varepsilon)} u(y)\, d\sigma_y = u(0).$$

Thus

$$\frac{(1-\varepsilon)-|x|}{(1-\varepsilon)+|x|} u(0) \le u(x) \le \frac{(1-\varepsilon)+|x|}{(1-\varepsilon)-|x|} \quad \forall \; |x| < 1-\varepsilon.$$

~~Letting ε~~

$\to$ Fix $x_0 \in \mathbb{D}$, $\exists$ small ε_0 s.t. $|x_0| < 1-\varepsilon_0$. Then $\forall\, \varepsilon < \varepsilon_0$, $\frac{(1-\varepsilon)-|x_0|}{(1-\varepsilon)+|x_0|} u(0) \le u(x_0) \le \cdots$ let $\varepsilon \to 0$.

Fix an arbitrary $|x| < 1$, letting $\varepsilon \to 0$ shows

$$\frac{1-|x|}{1+|x|} u(0) \le u(x) \le \frac{1+|x|}{1-|x|} u(0).$$

#

W05 #4: ~~Let R~~ Let R be sufficiently large s.t. supp φ, supp $\psi \subset B(0, R/2)$.

From D'Alembert's formula

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} t\,\varphi(y) + \psi(y) + \nabla\psi(y)\cdot(y-x)\,d\sigma_y.$$

Then

$$\left|\frac{1}{4\pi t^2}\int_{\partial B(x,t)} t\,\varphi(y)\,d\sigma_y\right| = \left|\frac{1}{4\pi t}\int_{\partial B(x,t)} \varphi(y)\,\mathbb{1}_{\mathrm{supp}(\varphi)}\,d\sigma_y\right|$$

$$\leq \frac{1}{4\pi t}\|\varphi\|_{L^\infty}\int_{\partial B(x,t)} \mathbb{1}_{\mathrm{supp}(\varphi)}\,d\sigma_y.$$

$$\leq \frac{1}{4\pi t}\|\varphi\|_{L^\infty}\int_{\partial B(x,t)} \mathbb{1}_{B(0,R)}\,d\sigma_y.$$

$$\leq \frac{R^2\|\varphi\|_{L^\infty}}{t}.$$

Similarly

$$\left|\frac{1}{4\pi t^2}\int_{\partial B(x,t)} \psi(y)\,d\sigma_y\right| \leq \frac{1}{4\pi t^2}\|\psi\|_{L^\infty}\cdot 4\pi R^2.$$

$$= \frac{R^2\|\psi\|_{L^\infty}}{t^2}.$$

and

$$\left|\frac{1}{4\pi t^2}\int_{\partial B(x,t)} \nabla\psi(y)\cdot(y-x)\,d\sigma_y\right| \leq \frac{1}{4\pi t^2}\|\nabla\psi\|_{L^\infty}\,t\cdot 4\pi R^2.$$

$$= \frac{R^2\|\nabla\psi\|_{L^\infty}}{t}.$$

Thus

$$|u(x,t)| \leq \frac{R^2\|\varphi\|_{L^\infty}}{t} + \frac{R^2\|\nabla\psi\|_{L^\infty}}{t} + \frac{R^2\|\psi\|_{L^\infty}}{t^2} \leq \frac{C}{t}$$

for some abs. constant C.

$$r^2\cos 2\theta = x^2-y^2$$

WOS #5:

$$\Delta(x^2-y^2)=0.$$

Note that there is only at most 1 solution to the given PDE. (Let u, v be 2 solutions, then $w := u-v$ satisfies $\Delta w = 0$ in D, $w=0$ on ∂D. Then $w \equiv 0$ by the maximum principle)

Writing $\Delta u = x^2-y^2$ in polar, we have

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = r^2\cos 2\theta. \qquad (*)$$

$$u = 0 \quad \text{when } r = 1.$$

If $\tilde{u} = ar^2\cos 2\theta$, then

$$\tilde{u}_r = 2ar\cos 2\theta \qquad \tilde{u}_\theta = -2ar^2\sin 2\theta.$$
$$\tilde{u}_{rr} = 2a\cos 2\theta \qquad \tilde{u}_{\theta\theta} = -4ar^2\cos 2\theta.$$

$$\tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta\theta} = 2a\cos 2\theta + 2a\cos 2\theta - 4a\cos 2\theta = 0.$$

If $\tilde{u} = ar^4\cos 2\theta$.

$$\tilde{u}_r = 4ar^3\cos 2\theta \qquad \tilde{u}_\theta = -2ar^4\sin 2\theta.$$
$$\tilde{u}_{rr} = 12ar^2\cos 2\theta. \qquad \tilde{u}_{\theta\theta} = -4ar^4\cos 2\theta.$$

$$\tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r + \frac{1}{r^2}\tilde{u}_{\theta\theta} = [12ar^2 + 4ar^2 - 4ar^2]\cos 2\theta.$$

Thus a solution to $(*)$ is given by

$$= 12ar^2\cos 2\theta.$$

$$u = \frac{1}{12}r^4\cos 2\theta - \frac{1}{12}r^2\cos 2\theta = \frac{1}{12}(x^2-y^2)(x^2+y^2-1)$$

$$\cancel{= \frac{1}{12}r^2\cos 2\theta\,[r^2-1]}$$

$$\cancel{= \frac{1}{12}(x^2-y^2)}$$

W05 #6:

Let $f(x) = \sin x.$, $g(x) = \frac{\sin x}{x}$. Then $x g(x) = f(x)$.

$$\cancel{\int_{-\infty}^{\infty} x g(x) e^{-2\pi i x\xi} dx = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}}$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} x g(x) e^{-2\pi i x\xi} dx = \int_{-\infty}^{\infty} g(x)\left(-\frac{1}{2\pi i}\right)\frac{d}{d\xi} e^{-2\pi i x\xi} dx.$$

$$= -\frac{1}{2\pi i}\frac{d}{d\xi}\int_{-\infty}^{\infty} g(x) e^{-2\pi i x\xi} dx$$

$$= -\frac{1}{2\pi i}\frac{d}{d\xi}\hat{g}(\xi).$$

We have

$$\int_{-\infty}^{\infty} \sin x \, e^{-2\pi i x\xi} dx = \int_{-\infty}^{\infty} \frac{e^{ix}-e^{-ix}}{2i} e^{-2\pi i x\xi} dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{2i}\left(e^{ix(1-2\pi\xi)} - e^{-ix(1-2\pi\xi)}\right) dx.$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} e^{-2\pi i x(\xi-\frac{1}{2\pi})} - e^{-2\pi i x(\frac{1}{2\pi}+\xi)} dx.$$

Since $\mathcal{F}[1]=\delta_0$ which comes from $1 = \mathcal{F}^{-1}[\delta_0]$.

$$= \frac{1}{2i}\left[\delta\left(\xi-\frac{1}{2\pi}\right) - \delta\left(\xi+\frac{1}{2\pi}\right)\right]$$

Thus

$$\frac{d}{d\xi}\hat{g}(\xi) = \pi\left[\delta\left(\xi+\frac{1}{2\pi}\right) - \delta\left(\xi-\frac{1}{2\pi}\right)\right]$$

$$\rightarrow \hat{g}(\xi) = \pi \mathbb{1}_{\xi+\frac{1}{2\pi}\geq 0} - \pi \mathbb{1}_{\xi-\frac{1}{2\pi}\geq 0}.$$

$$\cancel{g(x) = \pi\int_{-1/2\pi}^{1/2\pi} e^{-2\pi i x\xi} d\xi}$$

W05 #6 cont:

$$\hat f \hat g = \widehat{f * g} \qquad \widehat{fg} = \hat f * \hat g .$$

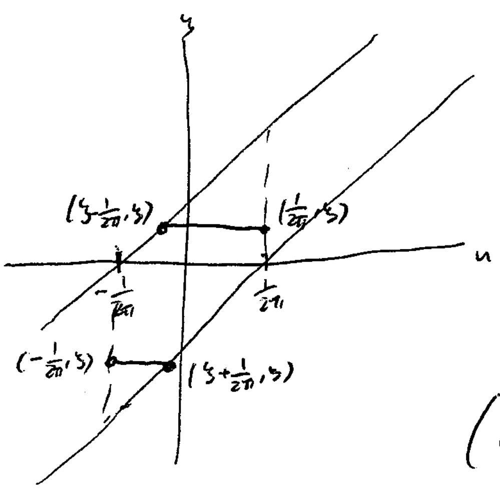
$$\left[\left(\frac{\sin x}{x}\right)^2\right]^\wedge(\xi) = \left[\frac{\sin x}{x}\right]^\wedge * \left[\frac{\sin x}{x}\right]^\wedge$$

$$= \int_{-\infty}^{\infty} \hat g(\xi-\eta)\,\hat g(\eta)\,d\eta .$$

$$= \pi \int_{-1/2\pi}^{1/2\pi} \hat g(\xi-\eta)\,d\eta \qquad \begin{aligned} u &= \xi-\eta \\ du &= -d\eta . \end{aligned}$$

$$= \pi \int_{\xi-\frac{1}{2\pi}}^{\xi+\frac{1}{2\pi}} \hat g(u)\,du$$

$$= \pi^2 \int_{\xi-\frac{1}{2\pi}}^{\xi+\frac{1}{2\pi}} \mathbb{1}_{-\frac{1}{2\pi} < u \le \frac{1}{2\pi}}\,du .$$



$$= \begin{cases} \pi^2\left(\frac{1}{2\pi} - \xi + \frac{1}{2\pi}\right) = \pi - \pi^2\xi & \text{if } 0 \le \xi < \frac{1}{\pi} \\ \pi^2\left(\xi + \frac{1}{\pi}\right) = \pi^2\xi + \pi & \text{if } -\frac{1}{\pi} < \xi \le 0 \\ 0 & \text{otherwise} \end{cases}$$

(Note both sol manuals are wrong, check $\xi = 0$) #

WOS #7: We need to assume that $f(\vec{0}) = \vec{0}$.

here $V(x_1, \dots, x_n) = \frac{1}{2}(x_1^2 + \dots + x_n^2)$.

Then $V(x_1, \dots, x_n) > 0$ for $\vec{x} \neq \vec{0}$ and

$$\dot{V}(x_1, \dots, x_n) = x_1 \dot{x}_1 + \dots + x_n \dot{x}_n$$

$$= x_1 f_1(x) + \dots + x_n f_n(x) < 0. \quad \forall \vec{x} \neq \vec{0}$$

Therefore as $V(\vec{0}) = 0$ and $\dot{V}(0) = 0$, by Lyapunov stability, we have that the zero solution is asymptotically stable and hence $\vec{x}(t) \to 0$ as $t \to \infty$. indep. of the initial condition.

F04#1

We use separation of variables.

If $u(x,t) = F(x)G(t)$, then

$$u_{tt} - u_{xx} + u = 0 \longrightarrow F(x)G''(t) - F''(x)G(t) + F(x)G(t) = 0.$$

$$F(x)[G''(t) + G(t)] = F''(x)G(t)$$

$$\frac{G''(t) + G(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda.$$

Then if $\lambda > 0$

$$F''(x) + \lambda F(x) = 0 \longrightarrow F(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x.$$

Since $u(0,t) = u(\pi,t) = 0$, $F(0) = 0$, $F(\pi) = 0$.

Then

$$F(0) = 0 \longrightarrow A = 0.$$

since we're nontrivial solns.

$$F(\pi) = 0 \longrightarrow 0 = B\sin\sqrt{\lambda}\,\pi \longrightarrow 0 = \sin\sqrt{\lambda}\,\pi$$

$$\longrightarrow \sqrt{\lambda} = n \text{ for } n = 1, 2, 3, \ldots$$

We get no nontrivial solutions when $\lambda \le 0$.

~~Let $F_n(x) = \sin nx$ and let $\lambda_n := n^2$. Then~~

$$\cancel{\frac{G_n''(t) + G_n(t)}{G_n(t)} = -\lambda_n.}$$

[illegible]

$$\cancel{\longrightarrow G_n''(t) + (n^2+1)G_n(t) = 0.}$$

$$\cancel{\longrightarrow G_n(t) = A\cos(\sqrt{n^2+1}\,t) + B\sin(\sqrt{n^2+1}\,t).}$$

~~Since~~

Now let's solve $\dfrac{G'' + G}{G} = -\lambda_n \longrightarrow G'' + (n^2+1)G = 0.$

Since $u_t(x,0) = 0$, $G'(0) = 0$. Therefore

$$G(t) = A\cos(\sqrt{n^2+1}\,t),\ n = 1, 2, 3, \ldots$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n(\sin nx)\cos(\sqrt{n^2+1}\,t)$$

F04 1 cont.

Since $u(x,0) = f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Note ~~formum~~,

$$\int_0^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi/2 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Thus

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x) \sin mx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin mx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin mx \, dx\right]$$

We have

$+$	x	$\sin nx$
$-$	1	$-\frac{1}{n}\cos nx$
$+$	0	$-\frac{1}{n^2}\sin nx$

$$\int x \sin nx \, dx = -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx + C.$$

Then

$$\int_0^{\pi/2} x \sin nx \, dx = \cancel{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}}$$

$$-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}.$$

$$\int_{\pi/2}^{\pi} x \sin nx \, dx = -\frac{\pi}{n}\cos n\pi - \left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\rightarrow \int_0^{\pi/2} x \sin nx \, dx - \int_{\pi/2}^{\pi} x \sin nx \, dx$$

$$= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n}\cos n\pi + \left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n}(-1)^n.$$

F04 #1
cont.

$$\pi \int_{\pi/2}^{\pi} \sin nx \, dx = \pi \cdot \frac{-1}{n} \cos nx \Big]_{x=\pi/2}^{\pi}$$

$$= \pi \left[\frac{1}{n} \cos \frac{n\pi}{2} - \frac{1}{n} \cos n\pi \right]$$

Therefore

$$a_n = \frac{2}{\pi}\left(\cancel{-\frac{\pi}{n} \cos \frac{n\pi}{2}} + \frac{2}{n^2} \sin \frac{n\pi}{2} + \cancel{\frac{\pi}{n}(-1)^n} + \cancel{\frac{\pi}{n} \cos \frac{n\pi}{2}} - \cancel{\frac{\pi}{n} \cos n\pi} \right)$$

$$= \frac{4}{\pi n^2} \sin \frac{n\pi}{2}.$$

Thus

$$u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} (\sin n x)(\cos \sqrt{n^2+1}\, t)$$

F04 #2: Let $\mathcal{U}(x,\tau) = u(ax,\tau)$. Then

$$\mathcal{U}_{xx} = a^2 u_{xx}(ax,\tau)$$
$$\mathcal{U}_t = u_\tau(ax,\tau).$$

Thus

$$\mathcal{U}_t = \mathcal{U}_{xx} \quad , \; t>0,\; x\in\mathbb{R}$$
$$\mathcal{U}(x,0) = \varphi(ax)$$

~~then~~ It suffices to compute $\lim_{\tau\to\infty} \mathcal{U}(x,\tau)$. ~~Thy~~ We have

$$\mathcal{U}(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}}\varphi(ay)\,dy$$

$$\stackrel{u=\frac{x-y}{2\sqrt{t}}}{=} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\varphi(ax-2\sqrt{t}ua)\,du. \qquad (*)$$

Since u is a bounded solution, φ is bounded. Thus the integral in $(*)$ converges. Then

$$\lim_{t\to\infty}\mathcal{U}(x,t) = \lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-u^2}\varphi(ax-2\sqrt{t}ua)\,du$$
$$+\lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u^2}\varphi(ax-2\sqrt{t}ua)\,du$$

by DCT as φ is bdd.

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-u^2}\lim_{t\to\infty}\varphi(ax-2\sqrt{t}ua)\,du$$
$$+\frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u^2}\lim_{t\to\infty}\varphi(ax-2\sqrt{t}ua)\,du$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-u^2} b\,du + \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-u^2} c$$
$$= \frac{1}{2}(b+c).$$

F04 #3: We have

$$E'(t) = \frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 + |\partial_t u|^2\, dx\right]$$

$$= \frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}^3} Du\cdot Du + u_t^2\, dx\right]$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} 2Du_t\cdot Du + 2u_t u_{tt}\, dx$$

$$= \int_{\mathbb{R}^3} Du_t\cdot Du + u_t u_{tt}\, dx$$

$$= -\int_{\mathbb{R}^3} u_t \Delta u - u_t u_{tt}\, dx$$

since $u_{tt} - \Delta u + a(x)u_t = 0$

$$= -\int_{\mathbb{R}^3} u_t\,(a(x)u_t)\, dx$$

$$= -\int_{\mathbb{R}^3} u_t^2\, a(x)\, dx \le 0.$$

Therefore $E(t)$ is a decreasing function of $t \geq 0$.

F04 #4: Let $r^2 = x_1^2 + x_2^2$. Then $r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$ and hence

$$r\dot{r} = x_1(x_2 + x_1(x_1^2+x_2^2)) + x_2(-x_1 + x_2(x_1^2+x_2^2))$$
$$= x_1^2 r^2 + x_2^2 r^2 = r^4.$$

Then $\dot{r} = r^3 \longrightarrow \frac{dr}{dt} = r^3$. Thus for some constant C,

$$-\frac{1}{2} r^{-2} = t - C.$$
$$r(t)^2 = \frac{1}{2(C-t)}.$$

Therefore each solution $r(t)$ blows up in finite time and hence each solution of the given autonomous system blows up in finite time.

If $x_1(0) = 1$, $x_2(0) = 0$, $r(0)^2 = 1$ and hence $C = \frac{1}{2}$. Therefore $r(t)^2 = \frac{1}{1-2t}$, which implies that the blow up time is $t = \frac{1}{2}$.

F04 #5: This is Dulac's Criterion.

We observe

$$\nabla\cdot(\varphi f) = \frac{\partial}{\partial x_1}\left(\frac{1}{x_1 x_2} f_1\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{x_1 x_2} f_2\right)$$
$$= \frac{\partial}{\partial x_1}\left(\frac{a - b x_2 - e x_1}{x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{-c + d x_1 - f x_2}{x_1}\right)$$
$$= -\frac{e}{x_2} - \frac{f}{x_1} < 0$$

since $x_1, x_2 > 0$.

Suppose there was a closed orbit in the 1st quadrant, let Ω be the region enclosed by this closed orbit. Then

$$0 > \int_\Omega \nabla\cdot(\varphi f)\, dx = \int_{\partial\Omega} \varphi f \cdot \nu \, d\sigma$$

Since $f = (\dot{x}_1, \dot{x}_2)$ and $\partial\Omega$ is the closed orbit represented by (x_1, x_2), $f\cdot\nu = 0$ and hence

$$\int_{\partial\Omega} \varphi(f\cdot\nu)\, d\sigma = 0.$$

This is a contradiction. Therefore there are no closed orbits in the 1st quadrant. #

F04 #6 :

Let u, v be 2 vector flds satisfying the 3 given properties. Let $w = u - v$. Then w is conservative, $\nabla \cdot w = 0$ and $|w(x)| = O(|x|^{-2})$. Since w is conservative, $\exists$ F s.t. $w = \nabla F$. Since $\nabla \cdot w = 0$, $\Delta F = 0$. and hence F is harmonic.

Since $|w(x)| = O(|x|^{-2})$, each derivative of F is bdd. Since each derivative of F is harmonic, by Liouville's Theorem, $\partial_{x_i} F = 0 \ \forall i$ since $w \to 0$ as $x \to \infty$. Therefore F is constant. Since $w = \nabla F = 0$, we have $u = v$.

3) Let R be s.t. $\text{supp } q \subset B(0, R/2)$. For $|x| \geq 10R$,

$$|u(x)| \leq \frac{1}{4\pi} \int_{B(0,R/2)} \frac{|q(y)|}{|x-y|^2} \, dy.$$

For $|x| \geq 10R$, $|x-y| \geq |x| - |y| \geq |x| - R/2 \geq \frac{|x|}{2}$ since $|x| \geq 10R$.

Thus for these x,

$$|u(x)| \leq \frac{1}{4\pi} \frac{4}{|x|^2} \int_{B(0,R/2)} |q(y)| \, dy.$$

$$\leq \frac{1}{4\pi} \frac{4}{|x|^2} \int_{\mathbb{R}^3} |q(y)| \, dy.$$

Therefore for large x,

$$|u(x)| = O(|x|^{-2}).$$

F04 #6 cont'd

2) We have

$$\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\,dy = \frac{+1}{4\pi}\int_{\mathbb{R}^3} \frac{q(x-y)\,y}{|y|^3}\,dy.$$

Then

$$\nabla_x \cdot \frac{+1}{4\pi}\int_{\mathbb{R}^3} q(x-y)\frac{y}{|y|^3}\,dy = +\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|y|^3}\sum_{j=1}^{3}\partial_{x_j}\left(q(x-y)y_j\right)dy.$$

$$= +\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|y|^3}\, y\cdot\nabla_x q(x-y)\,dy.$$

$$= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|y|^3}\, y\cdot\nabla_y\left(q(x-y)\right)dy.$$

$$= \lim_{\varepsilon\to 0}\frac{-1}{4\pi}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{1}{|y|^3}\, y\cdot\nabla_y\left(q(x-y)\right)dy.$$

$$= \lim_{\varepsilon\to 0}\frac{-1}{4\pi}\left[-\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\nabla_y\cdot\frac{y}{|y|^3}\,q(x-y)\,dy + \int_{\partial(\mathbb{R}^3\setminus B(0,\varepsilon))} q(x-y)\frac{y}{|y|^3}\cdot\nu(y)\,dy\right] \quad (*)$$

As $\frac{y}{4\pi|y|^3}$ is the gradient of the fundamental sol in $\mathbb{R}^3$,

$$-\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\nabla_y\frac{y}{|y|^3}\,q(x-y)\,dy = 0.$$

So

$$(*) = \frac{-1}{4\pi}\lim_{\varepsilon\to 0}\int_{\partial(\mathbb{R}^3\setminus B(0,\varepsilon))} q(x-y)\frac{y}{|y|^3}\cdot\nu(y)\,d\sigma(y)$$

$$= \frac{-1}{4\pi}\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} q(x-y)\frac{y}{|y|^3}\cdot\left(-\frac{y}{|y|}\right)d\sigma(y)$$

F04 #6 cont'.

$$= + \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} q(x-y) \frac{1}{\varepsilon^2} \, d\sigma(y)$$

$$= + \frac{1}{4\pi \varepsilon^2} \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} q(x-y) \, d\sigma(y) = + q(x).$$

Therefore

$$\nabla \cdot u = q.$$

1) here $F(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(y)}{|x-y|} dy. = -\frac{1}{4\pi} \int_{\mathbb{R}^3} q(x-y) \frac{1}{|y|} dy.$

We claim $\nabla F = u$. But this comes from

$$\nabla \frac{1}{|x|} = -\frac{x}{|x|^3}.$$

~~and here~~

$$\cancel{\nabla_x F = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla_x q(x-y) \frac{1}{|y|} dy}$$

We have

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} q(x-y) \frac{y}{|y|^3} dy = -\frac{1}{4\pi} \int_{\mathbb{R}^3} q(x-y) \nabla_y \left(\frac{1}{|y|}\right) dy$$

$$= \lim_{\varepsilon \to 0^+} -\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} q(x-y) \nabla_y \left(\frac{1}{|y|}\right) dy$$

$$= \lim_{\varepsilon \to 0^+} -\frac{1}{4\pi} \left[-\int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \nabla_y q(x-y) \cdot \frac{1}{|y|} dy + \int_{\partial(\mathbb{R}^3 \setminus B(0,\varepsilon))} \frac{q(x-y)}{|y|} \nu \, d\sigma(y) \right)$$

F04 #6 cont:

Since
$$\left| \int_{\partial B(0,\varepsilon)} \frac{q(x-y)}{|y|} \frac{-y}{|y|} \, d\sigma(y) \right|$$
$$\leq \|q\|_{L^\infty} \frac{1}{\varepsilon} 4\pi\varepsilon^2 \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Thus
$$u(x) = \lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \nabla_x q(x-y) \frac{1}{|y|} \, dy$$
$$= \nabla F(x).$$

#

F04 #7: Write $u u_{x_1} + u_{x_2} + u = 0$

$$u(0, x_2) = e^{-2x_2}.$$

Let $F(p, z, x) := z p_1 + p_2 + z = 0$. Then

$D_p F = (z, 1) \qquad D_z F = p_1 + 1 \qquad D_x F = 0$

~~$x_1(0) = 0$~~

Therefore:

$$\dot{p} = -D_x F - D_z F p = (p_1 + 1) p \qquad x_1(0) = 0$$

$$\dot{z} = D_p F \cdot p = z p_1 + p_2 = -z. \qquad x_2(0) = x_2(0)$$

$$\dot{x} = D_p F = (z, 1) \qquad z(0) = e^{-2x_2(0)}.$$

① $\dot{z} = -z \longrightarrow z(s) = z(0) e^{-s} = e^{-2x_2(0)} e^{-s}$.

② $\dot{x}_1 = z = z(0) e^{-s}$, $x_1(0) = 0$.

$\hookrightarrow x_1(s) = -z(0) e^{-s} + x_1(0) + z(0)$

$= z(0)(1 - e^{-s}) = e^{-2x_2(0)}(1 - e^{-s})$.

③ $\dot{x}_2 = 1$, $x_2(0) = x_2(0)$

$\hookrightarrow x_2(s) = s + x_2(0)$.

Therefore

$$z(s) = e^{-2x_2(0)} e^{-s} = e^{-2x_2(0)} e^{x_2(0) - x_2(s)} = e^{-x_2(0)} e^{-x_2(s)}.$$

We have

$$x_1(s) = e^{-2x_2(0)}\left(1 - e^{x_2(0) - x_2(s)}\right) = (e^{-x_2(0)})^2 - e^{-x_2(s)}(e^{-x_2(0)})$$

Thus

$$e^{-x_2(0)} = \frac{e^{-x_2(s)} + \sqrt{e^{-2x_2(s)} + 4x_1(s)}}{2}$$

(~~since~~ we chose + since if we chose the −, we would get a contradiction as $e^{-x_2(0)} > 0$)

Therefore

$$z(s) = e^{-x_2(s)}\left[\frac{e^{-x_2(s)} + \sqrt{e^{-2x_2(s)} + 4x_1(s)}}{2}\right]$$

$$\longrightarrow u(x, t) = e^{-t}\left[\frac{e^{-t} + \sqrt{e^{-2t} + 4x}}{2}\right]$$

Note

$$u\left(\tfrac{1}{9}, \ln 2\right) = \frac{1}{2}\left[\frac{1/2 + \sqrt{1/4 + 4/9}}{2}\right] = \frac{1}{4}\left(\frac{1}{2} + \frac{5}{6}\right) = \frac{1}{3}.$$

F04 #8

We have $\int_0^\infty e^{-s\tau}\frac{dy}{d\tau}(x,\tau)\,d\tau = e^{-s\tau}y(x,\tau)\Big]_{\tau=0}^{\infty} - \int_0^\infty -se^{-s\tau}y(x,\tau)\,d\tau$

$$= -1 + s\int_0^\infty e^{-s\tau}y(x,\tau)\,d\tau.$$

Thus taking the Laplace transform, we have

$$x\frac{d}{dx}\bar{y} + \frac{d}{dx}s\bar{y} + 2\bar{y} = 0.$$

$$(x+s)\frac{d\bar{y}}{dx} + 2\bar{y} = 0. \qquad \bar{y}(0,t) = \int_0^\infty e^{-s\tau}e^{-a\tau}\,d\tau = \frac{1}{s+a}.$$

$$(\ln\bar{y})' = -\frac{2}{x+s}.$$

$$\ln\bar{y} = \ln(x+s)^{-2}$$

$$\bar{y} = (x+s)^{-2}c \longrightarrow c = \frac{s^2}{s+a}.$$

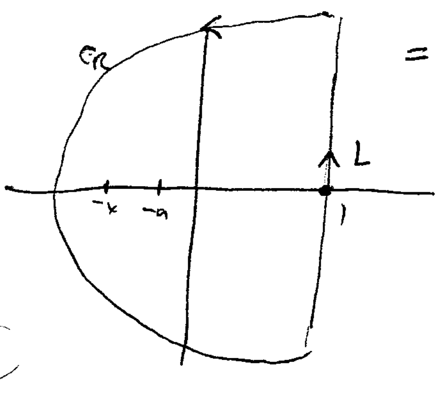
Thus

$$\bar{y}(x,s) = \frac{s^2}{(x+s)^2(s+a)}.$$

For $x,\tau \geq 0$,

chosen b/c poles are $s=-x,-a$ which are ≤ 0.

$$y(x,t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{1-iR}^{1+iR}\frac{e^{st}s^2}{(x+s)^2(s+a)}\,ds$$



$$= -\frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R}\frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds + \left(\operatorname*{Res}_{s=-a}\frac{e^{st}s^2}{(s+x)^2(s+a)} + \operatorname*{Res}_{s=-x}\frac{e^{st}s^2}{(s+x)^2(s+a)}\right)$$

We have

$$\left|\int_{C_R}\frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds\right| = \left|\int_{\pi/2}^{3\pi/2}\frac{e^{(1+Re^{i\theta})t}(1+Re^{i\theta})^2}{(1+Re^{i\theta}+x)^2(1+Re^{i\theta}+a)}\,iRe^{i\theta}\,d\theta\right|$$

$$\leq \frac{(R-1)^2Re^t}{(R-|x|-1)(R-|a|-1)}\int_{\pi/2}^{3\pi/2}e^{Rt\cos\theta}\,d\theta.$$

34 #8 cont:

$$\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\,d\theta = 2\int_0^{\pi/2} e^{-Rt\sin\theta}\,d\theta$$

for $\theta \in [0, \frac{\pi}{2}]$, $\sin\theta \geq \frac{2}{\pi}\theta$. Thus $e^{-Rt\sin\theta} \leq e^{-Rt\frac{2}{\pi}\theta}$.

Then

$$2\int_0^{\pi/2} e^{-Rt\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}\theta}\,d\theta.$$

$$= 2\left(-\frac{\pi}{2Rt}\right)e^{-\frac{2Rt}{\pi}\theta}\Big]_{\theta=0}^{\pi/2}$$

$$= +\frac{\pi}{Rt}\left[1 - e^{-Rt}\right].$$ → need $t>0$ here

Therefore

$$\lim_{R\to\infty}\int_{C_R} \frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds \to 0 \quad \text{as } t > 0.$$

We also have

$$\operatorname*{Res}_{s=-a} \frac{e^{st}s^2}{(s+x)^2(s+a)} = \lim_{s\to -a} \frac{e^{st}s^2}{(s+x)^2} = \frac{e^{-at}a^2}{(x-a)^2}.$$

$$\operatorname*{Res}_{s=-x} \frac{e^{st}s^2}{(s+x)^2(s+a)} = \lim_{s\to -x} \frac{d}{ds}\frac{e^{st}s^2}{s+a} = \lim_{s\to -x} \frac{(s+a)[te^{st}s^2 + e^{st}2s] - e^{st}s^2}{(s+a)^2}$$

$$= \frac{(-x+a)(te^{-xt}x^2 - e^{-xt}2x) - e^{-xt}x^2}{(a-x)^2}.$$

Therefore

$$y(x,t) = \frac{a^2e^{-at}}{(x-a)^2} + \frac{(x-a)(2x - tx^2)e^{-xt} - x^2e^{-xt}}{(x-a)^2}.$$ #

W04 #1:

a) $\lambda=0$:

$$\Delta u = 0 \qquad u = F(x)G(y).$$

$$u(x,0)=u(x,\pi)=0 \qquad F''(x)G(y) + F(x)G''(y) = 0.$$

$$\downarrow \qquad -\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = -\mu.$$

$$G(0) = G(\pi) = 0.$$

Need $\mu > 0$.

$$G''(y) + \mu G(y) = 0.$$

$$G(y) = A\cos(\sqrt{\mu}y) + B\sin(\sqrt{\mu}y).$$

$$0 = G(0) = A \longrightarrow G(y) = B\sin(\sqrt{\mu}y).$$

$$0 = G(\pi) \Longrightarrow 0 = \sin(\sqrt{\mu}\pi). \longrightarrow \sqrt{\mu}\pi = n\pi,\ n=1,2,\ldots$$

$$\mu_n = n^2,\ n=1,2,\ldots$$

$$G_n(y) = \sin(ny)$$

$$\longrightarrow \frac{F''}{F} = \mu \longrightarrow F'' - n^2F = 0.$$

$$\longrightarrow F(x) = Ae^{-nx} + Be^{nx}.$$

$B=0$ since we want bdd solns.

Therefore

$$u(x,y) = \sum_{n\geq 1} A_n e^{-nx}\sin(ny).$$

b) $\lambda > 0$. We have

$$F''(x)G(y) + F(x)G''(y) + \lambda F(x)G(y) = 0.$$

$$-\frac{F''+\lambda F}{F} = +\frac{G''}{G} = -\mu \text{ need } \mu > 0.$$

$$\rightarrow \mu_n = n^2$$

$$G_n(y) = \sin(ny). \longrightarrow F'' + \lambda F - n^2F = 0.$$

$$F'' + (\lambda - n^2)F = 0.$$

W04 #1 cont:

We have:

if $\lambda > n^2$: $F'' + (\lambda - n^2)F = 0$.

$F = A\cos(\sqrt{\lambda - n^2}\,x) + B\sin(\sqrt{\lambda - n^2}\,x)$

if $\lambda = n^2$ $\quad F = A + Bx$

if $\lambda < n^2$ $\quad F = Ae^{-\sqrt{n^2-\lambda}\,x} + \cancel{Be^{+\sqrt{n^2-\lambda}\,x}}$

since we want bdd solutions

Thus

$$u(x,y) = \sum_{n<\sqrt{\lambda}} \left[A_n\cos(\sqrt{\lambda-n^2}\,x) + B_n\sin(\sqrt{\lambda-n^2}\,x)\right]\sin(ny)$$
$$+ (C + Dx)\sin(\sqrt{\lambda}\,y)\,1_{\sqrt{\lambda}\in\mathbb{Z}}$$
$$+ \sum_{n>\sqrt{\lambda}} E_n e^{-\sqrt{n^2-\lambda}\,x}\sin(ny).$$

c) $\lambda < 0$. here $\gamma := -\lambda > 0$. Then the only thing that changes is the equation for F is now

$$F'' + (\lambda - n^2)F = 0 \longrightarrow F'' - (\gamma + n^2)F = 0$$

Thus

$$\longrightarrow F_n(x) = A_n e^{-\sqrt{\gamma+n^2}\,x} + \cancel{B_n e^{\sqrt{\gamma+n^2}\,x}}$$

$$u(x,y) = \sum_{n\geq 1} A_n e^{-\sqrt{-\lambda+n^2}\,x}\sin(ny).$$

(Since we only want bdd solutions).

W04 #2: Fix arbitrary $v \in C_0^\infty(\Omega)$. Since u_0 is the minimizer of the functional,

$$0 = \lim_{\varepsilon \to 0} \frac{D(u_0 + \varepsilon v) - D(u_0)}{\varepsilon}$$

$$= \int_\Omega 2(u_0)_x v_x + 2(u_0)_y v_y + f v \, dx + \int_{\partial\Omega} 2 a u_0 v \, d\sigma$$

$$= \int_\Omega 2(\nabla u_0 \cdot \nabla v) + f v \, dx + 2\int_{\partial\Omega} a u_0 v \, d\sigma$$

$$= 2\int_\Omega -v \Delta u_0 \, dx + 2\int_{\partial\Omega} (\nabla u_0 \cdot \nu) v \, d\sigma + \int_\Omega f v \, dx + 2\int_{\partial\Omega} a u_0 v \, d\sigma.$$

$$= \int_\Omega (-2\Delta u_0 + f) v \, dx + 2\int_{\partial\Omega} (\nabla u_0 \cdot \nu + a u_0) v \, d\sigma.$$

Thus

$$-2\Delta u_0 + f = 0 \quad \text{in } \Omega$$
$$\nabla u_0 \cdot \gamma + a u_0 = 0 \quad \text{on } \partial\Omega.$$

W04 #3 :

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right) \quad w = y_1 + iy_2.$$

We have

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{f(w)}{w-z}\,dw$$

Let $K(w) = \frac{1}{2\pi}\log|w| = \frac{1}{4\pi}\log(y_1^2+y_2^2)$. We have

$$\frac{\partial}{\partial w}K(w) = \frac{1}{2}\cdot\frac{1}{4\pi}\left[\frac{2y_1}{y_1^2+y_2^2} - i\frac{2y_2}{y_1^2+y_2^2}\right] = \frac{1}{4\pi}\cdot\frac{1}{w}.$$

Thus

$$u(z) = \frac{4\pi}{2\pi}\int_{\mathbb{C}} f(w)K_w(z-w)\,dw$$

$$= 2\int_{\mathbb{C}} f(z-w)K_w(w)\,dw.$$

Therefore

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)u = 2\frac{\partial u}{\partial \bar{z}} = 4\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z-w)K_w(w)\,dw$$

$$= -4\int_{\mathbb{C}} \frac{\partial}{\partial \bar{w}} f(z-w)K_w(w)\,dw$$

$$= 4\int_{\mathbb{C}} f(z-w)\frac{\partial}{\partial \bar{w}}\frac{\partial}{\partial w}K(w)\,dw.$$

$$= \int_{\mathbb{C}} f(z-w)\Delta K(w)\,dw.$$

$$= \int_{\mathbb{C}} f(z-w)\,\delta(w)\,dw.$$

$$= f(z).$$

W04#4:

Let y_1, y_2 be 2 ~~distinct~~ unique solns. Then $w = y_1 - y_2$ satisfies

$$-w'' + pw = 0 \qquad 0 < x < \pi$$
$$w(0) = 0, \; w'(\pi) = 0.$$

We have

$$0 = \int_0^\pi -w''w + pw^2 \, dx = \int_0^\pi (w')^2 + pw^2 \, dx$$

~~Let ε~~

By Poincaré's

We now find the eigenfuns of the $-\Delta$:

$$-y'' = \lambda y$$
$$y(0) = y'(\pi) = 0. \quad \rightarrow \quad \lambda > 0 \quad y(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x.$$

$$0 = y(0) = A \rightarrow y(x) = B\sin\sqrt{\lambda}\, x.$$

$$\downarrow$$

$$y'(x) = B\sqrt{\lambda}\cos\sqrt{\lambda}\, x.$$

$$\rightarrow \sqrt{\lambda}\,\pi = \frac{2n+1}{2}\pi, \; n = 0, 1, 2 \ldots$$

$$\lambda_n = (n + \tfrac{1}{2})^2; \; n = 0, 1, 2, \ldots$$

Let $\phi_n := \sin((n+\frac{1}{2})x)$, $\lambda_n := (n+\frac{1}{2})^2$. Then

$$-\phi_n'' = \lambda_n \phi_n$$
$$\phi_n(0) = \phi_n'(\pi) = 0.$$

We have $w(x) = \sum_{n \geq 0} a_n \phi_n(x)$. Then

W04 #4 cont:

$$\int_0^{\pi} -w''w\,dx = \int_0^{\pi}\Big(\sum_{n\geq 0} a_n \lambda_n \phi_n(x)\Big)\Big(\sum_{m\geq 0} a_m \phi_m(x)\Big)\,dx$$

$$= \int_0^{\pi} \sum_{n\geq 0} a_n^2 \lambda_n \phi_n^2(x)\,dx$$

$$= \sum_{n\geq 0} a_n^2 \lambda_n \int_0^{\pi} \phi_n(x)^2\,dx$$

$$\geq \frac{1}{4}\sum_{n\geq 0} a_n^2 \int_0^{\pi} \phi_n(x)^2\,dx$$

$$= \frac{1}{4}\int_0^{\pi} w^2\,dx$$

and the $\frac{1}{4}$ is sharp. Thus $\qquad (*)$

$$0 = \int_0^{\pi} -w''w + pw^2\,dx \geq \int_0^{\pi}\Big(\frac{1}{4}+p\Big)w^2\,dx.$$

When $p > -\frac{1}{4}$, then

$$0 = \int_0^{\pi}\Big(\frac{1}{4}+p\Big)w^2\,dx \geq 0$$

and hence $w = 0$. Since $\frac{1}{4}$ is sharp in $(*)$, $\lambda_0 = -\frac{1}{4}$.

W04 #5: Taking the Fourier Transform in the x-variable, we have

$$u_{xx}+u_{yy}=0 \longrightarrow -4\pi^2\xi^2\hat{u}+\hat{u}_{yy}=0.$$

Note A, B could depend on ξ

$$\longrightarrow \hat{u}(\xi,y)=Ae^{-2\pi|\xi|y}+Be^{2\pi|\xi|y}$$

Since we want a bounded solution, $B=0$. Therefore

$$\hat{u}(\xi,y)=Ae^{-2\pi|\xi|y}.$$

Since

$$u_y(x,0)-u(x,0)=f(x) \longrightarrow \hat{u}_y(\xi,0)-\hat{u}(\xi,0)=\hat{f}(\xi)$$

Thus as $\hat{u}_y(\xi,y)=-2\pi|\xi|Ae^{-2\pi|\xi|y}$, we have

$$-2\pi|\xi|A-A=\hat{f}(\xi)$$

$$A=\frac{-\hat{f}(\xi)}{2\pi|\xi|+1}.$$

Thus

$$\hat{u}(\xi,y)=\frac{-\hat{f}(\xi)}{2\pi|\xi|+1}e^{-2\pi|\xi|y}.$$

~~Since f is smooth of compact support, the RHS is Schwartz, so~~

$$\cancel{u(x,y)=\left(\frac{\hat{f}(\xi)}{2\pi|\xi|+1}e^{-2\pi|\xi|y}\right)^{\vee}\in\mathcal{S}(\mathbb{R}^2)}$$

Since $y\geq 0$ and $\hat{f}\in\mathcal{S}(\mathbb{R})$ (as $f\in\mathcal{S}(\mathbb{R})$), we have $|\hat{u}(\xi,y)|\leq|\hat{f}(\xi)|$. Thus $\hat{u}(\xi,y)$ is integrable in ξ. Therefore

$$u(x,y)=\int_{\mathbb{R}}\hat{u}(\xi,y)e^{2\pi i\xi x}\,d\xi$$

$$=\int_{\mathbb{R}}\frac{-\hat{f}(\xi)}{2\pi|\xi|+1}e^{-2\pi|\xi|y}e^{2\pi i\xi x}\,d\xi$$

We also have

$$|u(x,y)|\leq\int_{\mathbb{R}}|\hat{f}(\xi)|e^{-2\pi|\xi|y}\,d\xi$$

$$\le \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \left(\int_{\mathbb{R}} e^{-4\pi^2\xi^2y^2} d\xi\right)^{1/2}$$

Since

$u = 2\pi\xi y$
$du = 2\pi y \, d\xi$

$$\int_{-\infty}^{\infty} e^{-(2\pi\xi y)^2} d\xi = \frac{1}{2\pi y}\int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{2\sqrt{\pi}\, y}.$$

Thus

$$|u(x,y)| \le \|f\|_{L^2} (2\sqrt{\pi})^{-1/2} y^{-1/2} \longrightarrow 0$$

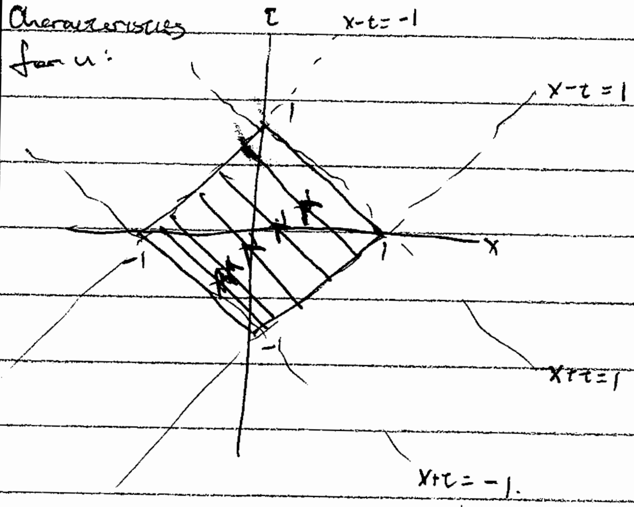
as $y \longrightarrow \infty$ uniformly in x

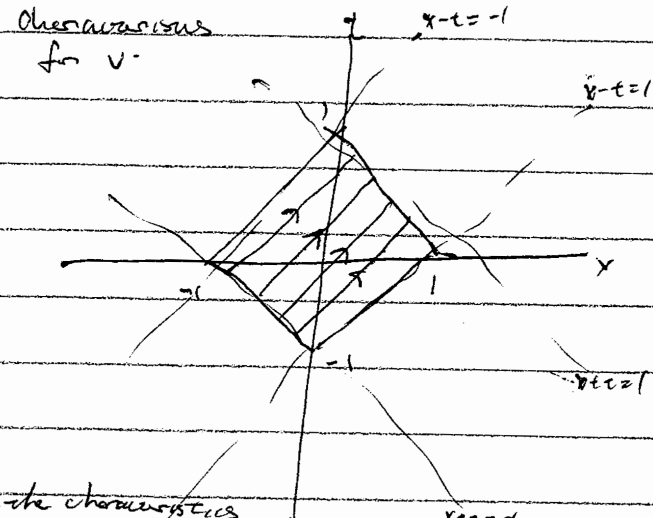
Characteristics point ↑ since they pt in increasing time.

W04 #6. We have

$$u_t - u_x = 0 \longrightarrow \text{Characteristics: } x + t = C$$
$$v_t + v_x = 0 \qquad\qquad x - t = C.$$

u and v are constant on their respective characteristics





a) This problem is well posed since from the characteristics to ~~find u~~ solve for u it suffices to know u on the line $x - t = -1$ and to solve for v it suffices to know v on the line $x + t = -1$. The ~~solution~~ solution is given by

$$u(x,t) = u_0(x+t)$$
$$v(x,t) = v_0(x-t)$$

b) From the characteristics, this problem is not well-posed.

W04 #7:

a) We have

$$f(x') = \int_{-\infty}^{\infty} \delta(x-x') f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \mathcal{L} G(x,x') f(x)\, dx = \int_{-\infty}^{\infty} G(x,x') \mathcal{L} f(x)\, dx$$

We also have

$$\int_{-\infty}^{\infty} \mathcal{L} f(x) G(x-x',0)\, dx = \int_{-\infty}^{\infty} (\mathcal{L} f)(x+x') G(x,0)\, dx$$

$$= \int_{-\infty}^{\infty} (\mathcal{L} g)(x) G(x,0)\, dx \qquad g(x) = f(x+x')$$

So

$$= g(0) = f(x').$$

$$\int_{-\infty}^{\infty} \mathcal{L} f(x) G(x-x',0)\, dx = \int_{-\infty}^{\infty} \mathcal{L} f(x) G(x,x')\, dx. \qquad \forall f.$$

Therefore $G(x-x',0) = G(x,x')$.

b) Since we want

$$\frac{d^2}{dx^2} G - G = \delta(x-x') = \begin{cases} 0 & \text{if } x \neq x' \end{cases}$$

and $G \to 0$ as $x \to -\infty$ and $G \to 0$ as $x \to \infty$, G must be a homogeneous ~~fkkk~~ solution to $y'' - y = 0$ and satisfy the limit conditions. Thus

$$G(x,x') = \begin{cases} a_- e^{x} & \text{if } x < x' \\ a_+ e^{-x} & \text{if } x > x' \end{cases}$$

W04 #7 con't

We need $a_- e^{x'} = a_+ e^{-x'}$.

We also need

$$G'(x'_+,x') - G'(x'_-,x') = \int_{x'_-}^{x'_+} L\,G(x,x')\,dx = \int_{x'_-}^{x'_+} \delta(x-x') = 1$$

We have

$$G'(x,x') = \begin{cases} a_- e^{x} & \text{if } x<x' \\ -a_+ e^{-x} & \text{if } x>x'. \end{cases}$$

Then

$$\left.\begin{array}{l} -a_+ e^{-x'} - a_- e^{x'} = 1. \\ a_- e^{x'} = a_+ e^{-x'}. \end{array}\right] \to -a_- e^{x'} = \frac{1}{2}$$

$$a_- = -\frac{1}{2e^{x'}}.$$

$$a_+ = a_- e^{2x'} = -\frac{1}{2e^{x'}} e^{2x'} = -\frac{e^{x'}}{2}.$$

So

$$G(x,\xi) = \begin{cases} -\frac{1}{2e^{\xi}} e^{x} & \text{if } x \leq \xi \\ -\frac{e^{\xi}}{2} e^{-x} & \text{if } x > \xi. \end{cases}$$

W04 #8:

The system is

$$x' = x - y^2$$
$$y' = y - x^2.$$

The equilibrium pts are:

$$x = y^2.$$
$$y = x^2 \longrightarrow y = y^4 \longrightarrow y(y^3-1) = 0.$$

$$y = 0, \quad y = 1.$$
$$\downarrow \qquad \downarrow$$
$$x = 0 \qquad x = 1.$$

$$(0,0) \text{ and } (1,1).$$

The Jacobian is:

$$J(x,y) = \begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J(1,1) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

$\downarrow$ unstable source node

$\downarrow$ eigenvalues $3, -1$

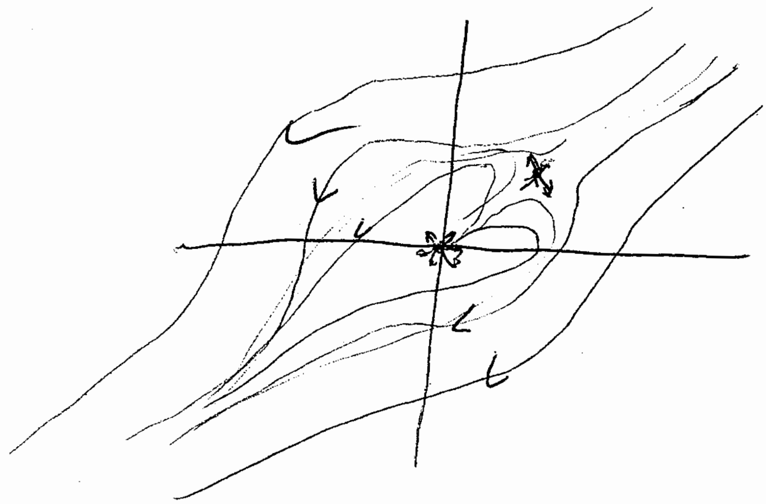
eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

saddle.

Can also sketch direction field to guess phase plane.

W04 #8 cont'd:

Since when x, y large, the paths look like $x^3 - y^3 = C$



Let $w(t) = u(t) - v(t)$. Then

$$\dot{w} = u_t - v_t = u - v^2 - v + u^2$$
$$= (u-v) + (u^2 - v^2)$$
$$= (u-v) + (u-v)(u+v)$$
$$= (u-v)(u+v+1)$$

~~$\int_0^t w(s)\,ds = \int_0^t (u-v)(u+v+1)\,ds$~~

$$\boxed{w_t = w(u+v+1), \quad w(0) = 0.}$$

~~$w(t) = \int_0^t w(s)(u+v+1)\,ds$~~

Therefore if $w \geq 0 \ \forall t$, then

$$w_t \leq w\,|u+v+1|$$

and so by Gronwall, $w(t) \leq e^{\int_0^t |u+v+1|\,ds}\, w(0) = 0.$

and hence $w(t) = 0 \ \forall t.$

W04 #8 cont:

Thus

$$\int_0^t u_t(s) - v_t(s)\,ds = \int_0^t (u-v)(u+v+1)\,ds.$$

As $u(0) - v(0) = 0$,

Then $$u(t) - v(t) = \int_0^t (u-v)(u+v+1)\,ds$$

$$|u(t) - v(t)| \le \int_0^t |u(s) - v(s)|\,|u(s)+v(s)+1|\,ds.$$

Then by the integral form of Gronwall,

$$|u(t) - v(t)| \le 0.$$

$$\rightarrow u = v \quad \forall t. \quad \#$$

F03 #1

a) The stationary pts. are s.t.

$$\begin{matrix} v-u^3=0 \\ u-v=0 \end{matrix} \longrightarrow \begin{matrix} v=u^3 \\ v=u \end{matrix} \longrightarrow u=\pm 1, 0$$

and here the stationary pts. are $(1,1)$, $(0,0)$, and $(-1,-1)$.

~~The~~ Rewrite the system as

$$\dot{x} = y - x^3$$
$$\dot{y} = x - y.$$

the Jacobian is

$$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}$$

At $(1,1)$ and $(-1,-1)$, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

At $(0,0)$, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

~~The eigenvalues~~ We now compute the eigenvalues and eigenvectors of $\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

1 $\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$:

Eigenvalues: $\det\begin{pmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda) - 1 = \lambda^2 + 4\lambda + 2.$

$$\longrightarrow \lambda^2 + 4\lambda + 2 = 0 \longrightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 2}}{2} = -2 \pm \sqrt{2}.$$

Eigenvectors:

$$\begin{pmatrix} -3-(-2\pm\sqrt{2}) & 1 \\ 1 & -1-(-2\pm\sqrt{2}) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1\mp\sqrt{2} & 1 \\ 1 & 1\mp\sqrt{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\longrightarrow \begin{pmatrix} 1 \\ 1\pm\sqrt{2} \end{pmatrix}$ is an eigenvector for the eigenvalue $-2\pm\sqrt{2}$.

~~Are~~ The critical pts $(1,1)$ and $(-1,-1)$ are stable, ~~nodes~~ sink.

F03 #1 cont:

② $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$:

Eigenvalues: $\det\begin{pmatrix} -\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} = \lambda(1+\lambda) - 1 = \lambda^2 + \lambda - 1.$

$\rightarrow \lambda^2 + \lambda - 1 = 0 \rightarrow \lambda = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$

Eigenvectors:

$$\begin{pmatrix} 0 - \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) & 1 \\ 1 & -1 - \left(-\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

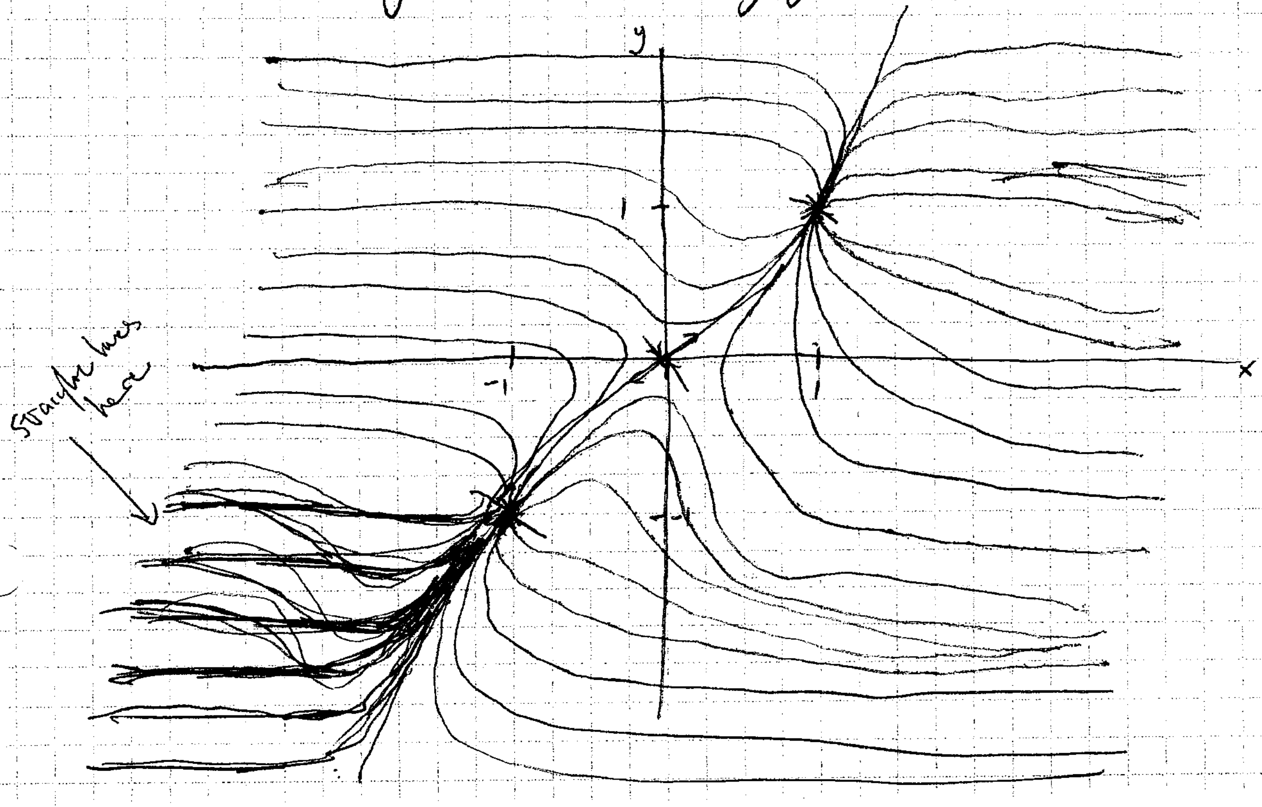
$$\rightarrow \begin{pmatrix} \frac{1}{2} \mp \frac{\sqrt{5}}{2} & 1 \\ 1 & -\frac{1}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightarrow \begin{pmatrix} 1 \\ \frac{-1 \pm \sqrt{5}}{2} \end{pmatrix}$ is an eigenvector for the eigenvalue $\frac{-1 \pm \sqrt{5}}{2}$.

The critical point $(0,0)$ is a saddle.

b) Note $\frac{dy}{dx} = \frac{x-y}{y-x^3}$ so for fixed y, $\frac{dy}{dx} \rightarrow 0$ as $x \rightarrow \pm\infty$.

Furthermore, ~~$y = x^3$ should~~ along $y = x^3$, $\frac{dy}{dx}$ is not defined.



F03 #3:

① We have

$$u_t = \frac{\partial}{\partial t} \exp\left(\frac{1}{3}t^3 - xt\right)\int_{-\infty}^{\infty} \overline{u}(\xi) h(x-t^2-\xi, t)\, d\xi$$

$$= \exp\left(\frac{1}{3}t^3 - xt\right)(t^2 - x)\int_{-\infty}^{\infty} \overline{u}(\xi) h(x-t^2-\xi, t)\, d\xi$$

$$+ \exp\left(\frac{1}{3}t^3 - xt\right)\int_{-\infty}^{\infty} \overline{u}(\xi)\left[h_x(x-t^2-\xi, t)(-2t) + h_t(x-t^2-\xi, t)\right] d\xi$$

$$= \exp\left(\frac{1}{3}t^3 - xt\right)\left[(t^2-x)\int_{-\infty}^{\infty} \overline{u}(\xi) h(x-t^2-\xi, t)\, d\xi + \int_{-\infty}^{\infty} \overline{u}(\xi)\left[h_x(x-t^2-\xi, t)(-2t) + h_{xx}(x-t^2-\xi, t)\right] d\xi\right]$$

Therefore

$$u_t + xu = \exp\left(\frac{1}{3}t^3 - xt\right)\left[t^2\int_{-\infty}^{\infty} \overline{u}(\xi) h(x-t^2-\xi, t)\, d\xi - 2t\int_{-\infty}^{\infty} \overline{u}(\xi) h_x(x-t^2-\xi, t)\, d\xi + \int_{-\infty}^{\infty} \overline{u}(\xi) h_{xx}(x-t^2-\xi, t)\, d\xi\right] \qquad (*)$$

Since

$$\frac{\partial}{\partial x} \exp\left(\frac{1}{3}t^3 - xt\right) = \exp\left(\frac{1}{3}t^3 - xt\right)(-t)$$

$$\frac{\partial^2}{\partial x^2} \exp\left(\frac{1}{3}t^3 - xt\right) = \exp\left(\frac{1}{3}t^3 - xt\right)(t^2),$$

and $(fg)'' = f''g + 2f'g' + fg''$, it follows that the RHS of $(*)$ is $= u_{xx}$.

Therefore $u_t + xu = u_{xx}$.

③ Note that

$$z = \frac{x-t^2-\xi}{2\sqrt{t}}, \quad dz = -\frac{1}{2\sqrt{t}}\, d\xi$$

$$\int_{-\infty}^{\infty} \overline{u}(\xi) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-t^2-\xi)^2}{4t}}\, d\xi = \int_{\infty}^{-\infty} \overline{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{4\pi t}} e^{-z^2}(-2\sqrt{t})\, dz.$$

$$= \int_{-\infty}^{\infty} \overline{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{\pi}} e^{-z^2}\, dz.$$

Since $\overline{u}$ is bounded and e^{-z^2} is integrable, by the DCT,

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \overline{u}(x-t^2-2\sqrt{t}z) \frac{1}{\sqrt{\pi}} e^{-z^2}\, dz = \frac{\overline{u}(x)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\, dz = \overline{u}(x)$$

Therefore

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \exp\left(\frac{1}{3}t^3 - xt\right) \cdot \overline{u}(x) = \overline{u}(x).$$

F03 #3 cont:

(c) This is the standard argument to show h is a fundamental solution of the heat equation. Fix $\varepsilon > 0$ and $x_0 \in \mathbb{R}$. Since u is continuous $\exists\, \delta > 0$ s.t. $|u(x) - u(x_0)| < \varepsilon$ when $|x - x_0| < \delta$.

If $|x - x_0| < \delta/2$, then

$$\left| \int_{\mathbb{R}} u(\xi) h(x_0 - \xi, t)\, d\xi - u(x_0) \right| = \left| \int_{\mathbb{R}} (u(\xi) - u(x_0)) h(x_0 - \xi, t)\, d\xi \right|$$

$$\leq \int_{B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon \int_{B(x_0,\delta)} h(x_0 - \xi, t)\, d\xi + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon + \int_{\mathbb{R} \setminus B(x_0,\delta)} h(x_0 - \xi, t) |u(\xi) - u(x_0)|\, d\xi$$

$$\leq \varepsilon + 2\|u\|_{L^\infty} \int_{\mathbb{R} \setminus B(x_0,\delta)} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x_0 - \xi)^2}{4t}}\, d\xi$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \int_{\mathbb{R} \setminus B(x_0,\delta)} e^{-\frac{(x_0 - \xi)^2}{4t}}\, d\xi.$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \left[\int_{-\infty}^{x_0 - \delta} + \int_{x_0 + \delta}^{\infty} e^{-\frac{(x_0 - \xi)^2}{4t}}\, d\xi \right] \qquad z = \frac{x_0 - \xi}{2\sqrt{t}}.$$

$$\lesssim \varepsilon + \frac{1}{\sqrt{t}} \left[\int_{\infty}^{\delta/2\sqrt{t}} + \int_{-\delta/2\sqrt{t}}^{-\infty} e^{-z^2} (-2\sqrt{t})\, dz \right] \qquad dz = -\frac{1}{2\sqrt{t}} dt.$$

$$\lesssim \varepsilon + \int_{\mathbb{R} \setminus B(0, \frac{\delta}{2\sqrt{t}})} e^{-z^2}\, dz.$$

Since $\int_{\mathbb{R} \setminus B(0, \frac{\delta}{2\sqrt{t}})} e^{-z^2}\, dz \longrightarrow 0$ as $t \longrightarrow 0$, $\int_{\mathbb{R}} u(\xi) h(x_0 - \xi, t)\, d\xi \longrightarrow u(x_0)$ as $t \longrightarrow 0$.

F03 #4:

There are 2 characteristics and the equation is hyperbolic.
Let $a = x$, $b = x-y$, $c = -y$. Since $b^2-4ac = (x-y)^2+4xy > 0$ (as $x,y>0$), the equation is hyperbolic. The characteristics are given by

$$\frac{dy}{dx} = \frac{(x-y) \pm \sqrt{(x-y)^2+4xy}}{2x} = \frac{(x-y)\pm(x+y)}{2x} = 1, -\frac{y}{x}.$$

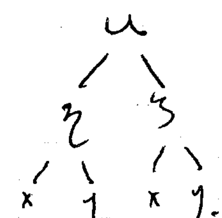
The characteristics are

$$\frac{dy}{dx} = 1 \longrightarrow y = x + c_1$$

$$\frac{dy}{dx} = -\frac{y}{x} \longrightarrow y = \frac{c_2}{x}.$$

Let $\eta = x-y$, $\xi = xy$. Then by the Chain Rule

$$u_x = u_\eta + u_\xi y, \quad u_y = -u_\eta + u_\xi x.$$



which implies

$$u_{xx} = u_{\eta\eta} + 2u_{\eta\xi}y + u_{\xi\xi}y^2$$

$$u_{xy} = -u_{\eta\eta} + xu_{\eta\xi} + u_\xi - yu_{\eta\xi} + xyu_{\xi\xi}$$

$$u_{yy} = u_{\eta\eta} - 2xu_{\eta\xi} + x^2u_{\xi\xi}.$$

We have

$$\begin{aligned} xu_{xx} + (x-y)u_{xy} - yu_{yy} &= xu_{\eta\eta} + 2xyu_{\eta\xi} + xy^2u_{\xi\xi} + (x-y)(-u_{\eta\eta} + xu_{\eta\xi} + u_\xi - yu_{\eta\xi} + xyu_{\xi\xi}) \\ &\qquad - y(u_{\eta\eta} - 2xu_{\eta\xi} + x^2u_{\xi\xi}) \\ &= (x+y)^2u_{\eta\xi} + (x-y)u_\xi \\ &= [\eta^2 + 4\xi]u_{\eta\xi} + \eta u_\xi. \end{aligned}$$

If we interchange how ξ, η are defined, we have the desired result.
e now will solve $(\xi^2+4\eta)u_{\xi\eta} + \xi u_\eta = 0$. Let $v(\xi,\eta) = u_\eta(\xi,\eta)$. Then

$$(\xi^2+4\eta)v_\xi + \xi v = 0 \longrightarrow v_\xi = -\frac{\xi}{\xi^2+4\eta}v \longrightarrow \ln v = -\frac{1}{2}\ln(\xi^2+4\eta) + \tilde{g}(\eta)$$

$$\longrightarrow v = \frac{g(\eta)}{(\xi^2+4\eta)^{1/2}} \rightarrow u(\xi,\eta) = f(\xi) + \int^{\eta} \frac{g(\eta')}{(\xi^2+4\eta')^{1/2}}\,d\eta'$$

6 Second-Order Equations

6.1 Classification by Characteristics

Consider the second-order equation in which the derivatives of second-order all occur linearly, with coefficients only depending on the independent variables:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} = d(x,y,u,u_x,u_y). \tag{6.1}$$

The *characteristic* equation is

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- $b^2 - 4ac > 0 \;\Rightarrow\;$ *two* characteristics, and (6.1) is called *hyperbolic*;
- $b^2 - 4ac = 0 \;\Rightarrow\;$ *one* characteristic, and (6.1) is called *parabolic*;
- $b^2 - 4ac < 0 \;\Rightarrow\;$ *no* characteristics, and (6.1) is called *elliptic*.

These definitions are all taken at a point $x_0 \in \mathbb{R}^2$; unless a, b, and c are all constant, the *type* may change with the point x_0.

6.2 Canonical Forms and General Solutions

① $u_{xx} - u_{yy} = 0$ is hyperbolic (one-dimensional wave equation).
② $u_{xx} - u_y = 0$ is parabolic (one-dimensional heat equation).
③ $u_{xx} + u_{yy} = 0$ is elliptic (two-dimensional Laplace equation).

By the introduction of new coordinates μ and η in place of x and y, the equation (6.1) may be transformed so that its principal part takes the form ①, ②, or ③.
If (6.1) is *hyperbolic*, *parabolic*, or *elliptic*, there exists a change of variables $\mu(x,y)$ and $\eta(x,y)$ under which (6.1) becomes, respectively,

$$\begin{aligned}
&u_{\mu\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta) \qquad \Leftrightarrow \qquad u_{\bar{x}\bar{x}} - u_{\bar{y}\bar{y}} = \bar{d}(\bar{x},\bar{y},u,u_{\bar{x}},u_{\bar{y}}),\\
&u_{\mu\mu} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta),\\
&u_{\mu\mu} + u_{\eta\eta} = \tilde{d}(\mu,\eta,u,u_\mu,u_\eta).
\end{aligned}$$

Example 1. *Reduce to canonical form and find the general solution:*

$$u_{xx} + 5u_{xy} + 6u_{yy} = 0. \tag{6.2}$$

Proof. $a = 1,\ b = 5,\ c = 6 \;\Rightarrow\; b^2 - 4ac = 1 > 0 \;\Rightarrow\;$ **hyperbolic** $\Rightarrow$ two characteristics.
The characteristics are found by solving

$$\frac{dy}{dx} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

to find $y = 3x + c_1$ and $y = 2x + c_2$.

Let $\quad \mu(x,y) = 3x - y, \quad \eta(x,y) = 2x - y.$

$$\begin{aligned}
& \mu_x = 3, \qquad \eta_x = 2, \\
& \mu_y = -1, \qquad \eta_y = -1. \\
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = 3u_\mu + 2u_\eta, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = -u_\mu - u_\eta, \\
u_{xx} &= (3u_\mu + 2u_\eta)_x = 3(u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x) + 2(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = 9u_{\mu\mu} + 12u_{\mu\eta} + 4u_{\eta\eta}, \\
u_{xy} &= (3u_\mu + 2u_\eta)_y = 3(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + 2(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -3u_{\mu\mu} - 5u_{\mu\eta} - 2u_{\eta\eta}, \\
u_{yy} &= -(u_\mu + u_\eta)_y = -(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.2) and simplifying, we obtain

$$\begin{aligned}
u_{\mu\eta} &= 0, && \text{which is the } \textbf{Canonical form}, \\
u_\mu &= f(\mu), \\
u &= F(\mu) + G(\eta), \\
u(x,y) &= F(3x - y) + G(2x - y), && \textbf{General solution}.
\end{aligned}$$

□

Example 2. *Reduce to canonical form and find the general solution:*

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} = u_x + 6y. \tag{6.3}$$

Proof. $a = y^2$, $b = -2y$, $c = 1 \;\Rightarrow\; b^2 - 4ac = 0 \;\Rightarrow$ **parabolic** $\Rightarrow$ one characteristic. The characteristics are found by solving

$$\frac{dy}{dx} = \frac{-2y}{2y^2} = -\frac{1}{y}$$

to find $\; -\dfrac{y^2}{2} + c = x.$

Let $\mu = \frac{y^2}{2} + x$. We must choose a second constant function $\eta(x,y)$ so that η is not parallel to μ. Choose $\eta(x,y) = y$.

$$\begin{aligned}
& \mu_x = 1, \qquad \eta_x = 0, \\
& \mu_y = y, \qquad \eta_y = 1. \\
u &= u(\mu(x,y), \eta(x,y)); \\
u_x &= u_\mu \mu_x + u_\eta \eta_x = u_\mu, \\
u_y &= u_\mu \mu_y + u_\eta \eta_y = y u_\mu + u_\eta, \\
u_{xx} &= (u_\mu)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x = u_{\mu\mu}, \\
u_{xy} &= (u_\mu)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y = y u_{\mu\mu} + u_{\mu\eta}, \\
u_{yy} &= (y u_\mu + u_\eta)_y = u_\mu + y(u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y) + (u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) \\
&= u_\mu + y^2 u_{\mu\mu} + 2y u_{\mu\eta} + u_{\eta\eta}.
\end{aligned}$$

Inserting these expressions into (6.3) and simplifying, we obtain

$$
\begin{aligned}
u_{\eta\eta} &= 6y, \\
u_{\eta\eta} &= 6\eta, \qquad \text{which is the } \textbf{Canonical form}, \\
u_{\eta} &= 3\eta^2 + f(\mu), \\
u &= \eta^3 + \eta f(\mu) + g(\mu), \\
u(x,y) &= y^3 + y \cdot f\Big(\frac{y^2}{2} + x\Big) + g\Big(\frac{y^2}{2} + x\Big), \qquad \textbf{General solution}.
\end{aligned}
$$

$\square$

Problem (F'03, #4). *Find the characteristics of the partial differential equation*

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \qquad x > 0,\ y > 0, \tag{6.4}$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^\eta \frac{g(\eta')\,d\eta'}{(\xi^2+4\eta')^{\frac{1}{2}}}$$

where f and g are arbitrary functions of ξ and η.

Proof. $a = x$, $b = x-y$, $c = -y$ $\Rightarrow$ $b^2 - 4ac = (x-y)^2 + 4xy > 0$ for $x > 0$, $y > 0$ $\Rightarrow$ **hyperbolic** $\Rightarrow$ two characteristics.

① The **characteristics** are found by solving

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{x-y \pm \sqrt{(x-y)^2+4xy}}{2x} = \frac{x-y\pm(x+y)}{2x} = \begin{cases} \frac{2x}{2x} = 1 \\ -\frac{2y}{2x} = -\frac{y}{x} \end{cases}$$

$$\Rightarrow \qquad y = x + c_1, \qquad \begin{aligned} \frac{dy}{y} &= -\frac{dx}{x}, \\ \ln y &= \ln x^{-1} + \tilde{c}_2, \\ y &= \frac{c_2}{x}. \end{aligned}$$

② Let $\mu = x - y$ and $\eta = xy$

$$\begin{aligned} \mu_x &= 1, & \eta_x &= y, \\ \mu_y &= -1, & \eta_y &= x. \end{aligned}$$

$$\begin{aligned}
u &= u(\mu(x,y),\eta(x,y)); \\
u_x &= u_\mu\mu_x + u_\eta\eta_x = u_\mu + yu_\eta, \\
u_y &= u_\mu\mu_y + u_\eta\eta_y = -u_\mu + xu_\eta, \\
u_{xx} &= (u_\mu + yu_\eta)_x = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x + y(u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x) = u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta}, \\
u_{xy} &= (u_\mu + yu_\eta)_y = u_{\mu\mu}\mu_y + u_{\mu\eta}\eta_y + u_\eta + y(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = -u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta}, \\
u_{yy} &= (-u_\mu + xu_\eta)_y = -u_{\mu\mu}\mu_y - u_{\mu\eta}\eta_y + x(u_{\eta\mu}\mu_y + u_{\eta\eta}\eta_y) = u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta},
\end{aligned}$$

Inserting these expressions into (6.4), we obtain

$$\begin{aligned}
& x(u_{\mu\mu} + 2yu_{\mu\eta} + y^2u_{\eta\eta}) + (x-y)(-u_{\mu\mu} + xu_{\mu\eta} + u_\eta - yu_{\eta\mu} + xyu_{\eta\eta}) - y(u_{\mu\mu} - 2xu_{\mu\eta} + x^2u_{\eta\eta}) = 0, \\
& (x^2 + 2xy + y^2)u_{\mu\eta} + (x-y)u_\eta = 0, \\
& \big((x-y)^2 + 4xy\big)u_{\mu\eta} + (x-y)u_\eta = 0, \\
& (\mu^2 + 4\eta)u_{\mu\eta} + \mu u_\eta = 0, \qquad \text{which is the } \textbf{Canonical form}.
\end{aligned}$$

③ We need to integrate twice to get the general solution:

$$(\mu^2 + 4\eta)(u_\eta)_\mu + \mu u_\eta = 0,$$

$$\int \frac{(u_\eta)_\mu}{u_\eta}\, d\mu = -\int \frac{\mu}{\mu^2 + 4\eta}\, d\mu,$$

$$ln\ u_\eta = -\frac{1}{2} ln\ (\mu^2 + 4\eta) + \tilde{g}(\eta),$$

$$ln\ u_\eta = ln\ (\mu^2 + 4\eta)^{-\frac{1}{2}} + \tilde{g}(\eta),$$

$$u_\eta = \frac{g(\eta)}{(\mu^2 + 4\eta)^{\frac{1}{2}}},$$

$$u(\mu, \eta) = f(\mu) + \int \frac{g(\eta)\, d\eta}{(\mu^2 + 4\eta)^{\frac{1}{2}}}, \qquad \textbf{General solution}.$$

□

F03 #5:

① Parseval's relation: $\|f\|_{L^2} = \|\hat f\|_{L^2}$.

② We have

$$\hat f(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x) e^{-ix\xi}\,dx = \frac{1}{\sqrt{2\pi}}\int_{|x|\le y} e^{i(\alpha-\xi)x}\frac{1}{2\sqrt{\pi y}}\,dx$$
$$= \frac{1}{2\sqrt{2}\pi\sqrt{y}}\int_{-y}^{y} e^{i(\alpha-\xi)x}\,dx = \frac{1}{2\pi\sqrt{2y}}\,\frac{1}{i(\alpha-\xi)}\left[e^{i(\alpha-\xi)y} - e^{-i(\alpha-\xi)y}\right]$$
$$= \frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}$$

③ We have

$$\int_{-\infty}^{\infty}\left|\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right|^2 d\xi = \pi^2\cdot 2y\int_{-\infty}^{\infty}\left[\frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right]^2 d\xi$$
$$= \pi^2\cdot 2y\int_{-\infty}^{\infty}|f(x)|^2\,dx$$
$$= \pi^2\cdot 2y\int_{|x|\le 2y}\left|\frac{e^{i\alpha x}}{2\pi y}\right|^2 dx$$
$$= 2\pi^2 y\cdot\frac{1}{4\pi y}\cdot 2y = \pi y.$$

④ If $\xi = \alpha$, then

$$\hat f(\xi) = \hat f(\alpha) = \lim_{\xi\to\alpha}\frac{1}{\pi\sqrt{2y}}\,\frac{\sin((\alpha-\xi)y)}{\alpha-\xi} = \frac{1}{\pi\sqrt{2y}}\,y = \frac{\sqrt{y}}{\pi\sqrt{2}}.$$

As $y\to\infty$, $\hat f(\xi) = \hat f(\alpha)\to\infty$.

If $\xi\ne\alpha$, then

$$\lim_{y\to\infty}|\hat f(\xi)| = \lim_{y\to\infty}\frac{1}{\pi\sqrt{2y}}\left|\frac{\sin((\alpha-\xi)y)}{\alpha-\xi}\right| \le \frac{1}{\pi\sqrt{2}|\alpha-\xi|}\lim_{y\to\infty}\frac{1}{\sqrt{y}} = 0.$$

Since $\|f\|_{L^2}^2 = \int_{-y}^{y}\frac{1}{4\pi y}\,dx = \frac{1}{2\pi}$,

it follows that $\hat f(\xi) \longrightarrow \frac{1}{\sqrt{2\pi}}\delta(\xi-\alpha)$. as $y\to\infty$.

over $\longrightarrow$

F03 #6 :

a) Observe that if $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, then

$$x^2 = x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)$$
$$x^3 = x_0^3 + 3\varepsilon x_0^2 x_1 + O(\varepsilon^2).$$

Therefore

$$0 = \varepsilon^3 x^3 - 2\varepsilon x^2 + 2x - 6$$
$$= \varepsilon^3(\overset{x_0^3+3\varepsilon x_0^2 x_1 + O(\varepsilon^2)}{\cancel{x_0 + \varepsilon x_1 + O(\varepsilon^2)}}) - 2\varepsilon(x_0^2 + 2\varepsilon x_0 x_1 + O(\varepsilon^2)) + 2(x_0 + \varepsilon x_1 + O(\varepsilon^2)) - 6.$$
$$= -2\varepsilon x_0^2 + 2x_0 + 2\varepsilon x_1 - 6 + O(\varepsilon^2).$$

Thus we must have $2x_0 - 6 = 0 \longrightarrow x_0 = 3$.
$2x_1 - 2x_0^2 = 0 \longrightarrow x_1 = 9$.

Therefore $x = 3 + 9\varepsilon + O(\varepsilon^2)$.

b) Let $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$. Then

$$u_t = u_0' + \varepsilon u_1' + \varepsilon^2 u_2' + O(\varepsilon^3)$$
$$u^2 = u_0^2 + 2\varepsilon u_0 u_1 + 2\varepsilon^2 u_0 u_2 + \varepsilon^2 u_1^2 + O(\varepsilon^3)$$
$$u^3 = u_0^3 + 3\varepsilon u_0^2 u_1 + 3\varepsilon^2 u_0^2 u_2 + 3\varepsilon^2 u_0 u_1^2 + O(\varepsilon^3).$$

Therefore

$$u - \varepsilon u^3 = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3)$$
$$- \varepsilon u_0^3 - 3\varepsilon^2 u_0^2 u_1$$

Matching coeff. of ε in $u_t = u - \varepsilon u^3$, we have

$u_0' = u_0 \qquad u_0(0) = 1 \qquad \longrightarrow \underline{u_0 = e^t}.$
$u_1' = u_1 - u_0^3 \qquad u_1(0) = 0$
$u_2' = u_2 - 3u_0^2 u_1 \qquad u_2(0) = 0.$

$u_1' = u_1 - e^{3t} \rightarrow u_1' - u_1 = -e^{3t}.$
$e^{-t}u_1' - e^{-t}u_1 = -e^{2t}.$
$(e^{-t}u_1)' = -e^{2t}.$
$u_1 = -\frac{1}{2}e^{3t} + c_1 e^t.$
$u_1(0) = 0 \longrightarrow 0 = -\frac{1}{2} + c_1 \longrightarrow c_1 = \frac{1}{2}.$
So $\underline{u_1(t) = -\frac{1}{2}e^{3t} + \frac{1}{2}e^t}.$

$u_2' = u_2 - 3e^{2t}(-\frac{1}{2}e^{3t} + \frac{1}{2}e^t)$
$u_2' - u_2 = \frac{3}{2}e^{5t} - \frac{3}{2}e^{3t}.$
$(e^{-t}u_2)' = \frac{3}{2}e^{4t} - \frac{3}{2}e^{2t}.$
$e^{-t}u_2 = \frac{3}{2}\frac{1}{4}e^{4t} - \frac{3}{2}\frac{1}{2}e^{2t} + c_2.$
$u_2 = \frac{3}{8}e^{5t} - \frac{3}{4}e^{3t} + c_2 e^t.$
$u_2(0) = 0 \longrightarrow 0 = \frac{3}{8} - \frac{3}{4} + c_2 \longrightarrow c_2 = \frac{3}{8}.$
$\underline{u_2(t) = \frac{3}{8}e^{5t} - \frac{3}{4}e^{3t} + \frac{3}{8}e^t}.$

W03 #2:

a) We have

$$\lambda \int u^2 = \int (\lambda u) u = \int (Lu) u = \int (\Delta u - au) u$$

$$= -\int |\nabla u|^2 + au^2$$

Thus

$$\lambda = \frac{-\int_c |\nabla u|^2 + au^2}{\int_c u^2} < 0 \quad \text{since } a > 0.$$

b) Note that

$$\langle Lu, v\rangle = \int (\Delta u - au) v \, dx = -\int \nabla u \cdot \nabla v + auv$$

$$= \int (\Delta v - av) u \, dx = \langle u, Lv \rangle$$

Thus

$$\lambda \langle u, v\rangle - \mu \langle u, v\rangle = 0 \longrightarrow \lambda = \mu.$$

c) ~~In this case~~ $L = \sum_{j=1}^{3} (\partial^2_{x_j x_j} - a_j)$.

~~Suppose we knew the eigenvalues and eigenfunctions for~~

$$\cancel{\frac{d^2}{dx_j^2} u - a_j(x_j) = \lambda u.} \quad (*)$$

~~Then the eigenvalues and eigenfunctions of L are just the sum of the 3 eigenvalues (functions from (*)~~

W03 #3:

Let $\{\phi_n\}$ be a set of Neumann eigenfunctions for the Laplacian in Ω. So

$$\Delta\phi_n = \lambda_n \phi_n \quad \text{in } \Omega$$
$$\frac{\partial\phi_n}{\partial\nu} = 0 \quad \text{on } \partial\Omega.$$

Order the eigenvalues, let λ_0 denote the 0 eigenvalue. Note $\phi_0 = 1$.

Let $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$ where $a_n(t) = \dfrac{\int_\Omega u(x,t)\phi_n(x)dx}{\int_\Omega \phi_n(x)^2 dx}$.

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x) \quad \text{where } f_n = \frac{\int_\Omega f(x)\phi_n(x)\,dx}{\int_\Omega \phi_n(x)^2\,dx}$$

Then

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) - a_n(t)\Delta\phi_n(x) - f_n\phi_n(x) = 0.$$

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) - \lambda_n a_n(t)\phi_n(x) - f_n\phi_n(x) = 0.$$

for $n \geq 1 \longrightarrow$

$$a_n'(t) - \lambda_n a_n(t) = f_n.$$

$$e^{-\lambda_n t}a_n'(t) - \lambda_n e^{-\lambda_n t}a_n(t) = f_n e^{-\lambda_n t}.$$

$$(e^{-\lambda_n t}a_n(t))' = f_n e^{-\lambda_n t}.$$

$$e^{-\lambda_n t}a_n(t) = -\frac{1}{\lambda_n}f_n e^{-\lambda_n t} + c.$$

$$a_n(t) = -\frac{1}{\lambda_n}f_n + ce^{\lambda_n t}.$$

$a_0'(t) = f_0$

$a_0(t) = f_0 t + \tilde{c}$

$a_0(0) = b_0$

$\longrightarrow a_0(t) = f_0 t + b_0$.

Let $u_0(x) = \sum_{n=0}^{\infty} b_n\phi_n(x)$. Then $a_n(0) = b_n$. So

$$a_n(t) = -\frac{f_n}{\lambda_n} + \left(b_n + \frac{f_n}{\lambda_n}\right)e^{\lambda_n t}.$$

Note that $\lambda_n\int\phi_n^2\,dx = \int\phi_n\Delta\phi_n\,dx = -\int|\nabla\phi_n|^2dx \longrightarrow \lambda_n \leq 0 \ \forall n.$

WD3 #3 cont:

Thus

$$u(x,t) = (f_0 t + b_0)\phi_0(x) + \sum_{n=1}^{\infty}\left[\frac{f_n}{-\lambda_n} + \left(b_n - \frac{f_n}{-\lambda_n}\right)e^{\lambda_n t}\right]\phi_n(x)$$

and hence an approx. formula for u as $t \to \infty$ is

$$u(x,t) = (f_0 t + b_0) - \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n}\phi_n(x).$$

#

W03 #4:

Claim: If f has no Fourier coefficients of negative index, then so does $1/f$.

Pf: We have $f(x) = \sum_{k\geq 0} a_k e^{ikx}$ Suppose $1/f = \sum_{k\geq 0} b_k e^{ikx}$, where we will determine $\{b_k\}$. We have

$$\Big(\sum_{k\geq 0} a_k e^{ikx}\Big)\Big(\sum_{k\geq 0} b_k e^{ikx}\Big) = f\cdot \tfrac{1}{f} = 1 + 0e^{ix} + 0\cdot e^{2ix} + \dots$$

$$\sum_{m\geq 0}\Big(\sum_{k+j=m} a_k b_j\Big) e^{imx} = 1$$

$$\longrightarrow \quad a_0 b_0 = 1$$
$$a_1 b_0 + a_0 b_1 = 0$$
$$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0$$
$$\vdots$$

Thus we can successively solve for the b_k. By uniqueness of Fourier coefficients, the $\{b_k\}$ are precisely the Fourier coefficients of $1/f$. #

~~Note that if 1/f has no neg. indexed Fourier coeffic~~

a) We solve the PDE via method of characteristics. We have

$$F(p,q,z,x,t) = q - p - z^4.$$

$$\dot{t}(s) = 1 \qquad t(0) = 0$$
$$\dot{x}(s) = -1 \qquad x(0) = x_0$$
$$\dot{z}(s) = z^4 \qquad z(0) = u_0(x_0).$$

$$\longrightarrow \quad t(s) = s, \quad x(s) = -s + x_0 \;\rightarrow\; x + t = x_0$$

$$\frac{dz}{ds} = z^4 \;\longrightarrow\; \frac{1}{z^4}\,dz = ds \;\longrightarrow\; -\frac{1}{3z^3} = s + C.$$

W03#4 cont:

$$\longrightarrow C = -\frac{1}{3u_0(x_0)^3}.$$

Then

$$-\frac{1}{3u^3} - \cancel{\frac{1}{3\,\cdot^3}} = t - \frac{1}{3u_0(x+t)^3}$$

$$+\frac{1}{u^3} = \frac{1}{u_0(x+t)^3} - 3t.$$

By the Claim if $1/f$ has no negative indexed Fourier coeff, then so does f (apply claim to $1/f$ and $\frac{1}{1/f} = f$)

Fix t. Suppose $u_0 \in A$. then $u_0(x+t)^3 \in A$. By the Claim, $\frac{1}{u_0(x+t)^3} \in A$. Since t is fixed, $\frac{1}{u_0(x+t)^3} - 3t \in A$ and hence $(\frac{1}{u})^3 \in A$. Therefore $u^3 \in A$. Since $u \notin A$ would imply $u^3 \notin A$, we have $u \in A$.

b) We expand $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}$. Then

$$u_t = \sum_{k\in\mathbb{Z}} \hat{u}_t(k,t)e^{ikx}$$

$$u_x = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)ike^{ikx}.$$

$$u^4 = \left(\sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}\right)^4 = \sum_{j\in\mathbb{Z}}\left[\sum_{k_1+k_2+k_3+k_4=j} \hat{u}(k_1,t)\hat{u}(k_2,t)\hat{u}(k_3,t)\hat{u}(k_4,t)\right]e^{ijx}$$

Thus

$$\hat{u}_t(k,t) = ik\hat{u}(k,t) + \sum_{a_1+a_2+a_3+a_4=k} \hat{u}(a_1,t)\hat{u}(a_2,t)\hat{u}(a_3,t)\hat{u}(a_4,t).$$

If $\hat{u}$ had no negative indexed coefficients, then we

W03 #4 cont.

have

$$\hat{u}_t(0,t) = \hat{u}(0,t)^4.$$

$$\hat{u}_t(1,t) = i\hat{u}(1,t) + 4\hat{u}(1,t)\hat{u}(0,t)^3$$

$$\cancel{\hat{u}_t(2,t) = 2i\hat{u}(2,t) + 8\hat{u}(2,t) + 6\hat{u}(1,t)^2}.$$

$$\cancel{+ 8\hat{u}(2,t)}$$

$$\hat{u}_t(2,t) = 2i\hat{u}(2,t) + 4\hat{u}(2,t)\hat{u}(0,t)^3 + 6\hat{u}(1,t)^2\hat{u}(0,t)^2.$$

W03 #5:

We have $F(p, q, z, x, y) = xp + (x+y)q - 1$. Then

$$\dot{x}(s) = x(s) \qquad x(0) = 1$$
$$\dot{y}(s) = x(s) + y(s) \qquad y(0) = y_0$$
$$\dot{z}(s) = 1 \qquad z(0) = y_0. \qquad 0 \le y_0 \le 1.$$

Then

$$x(s) = e^s.$$

$$\frac{dy}{ds} - y = e^s \to e^{-s}\frac{dy}{ds} - e^{-s}y = 1$$

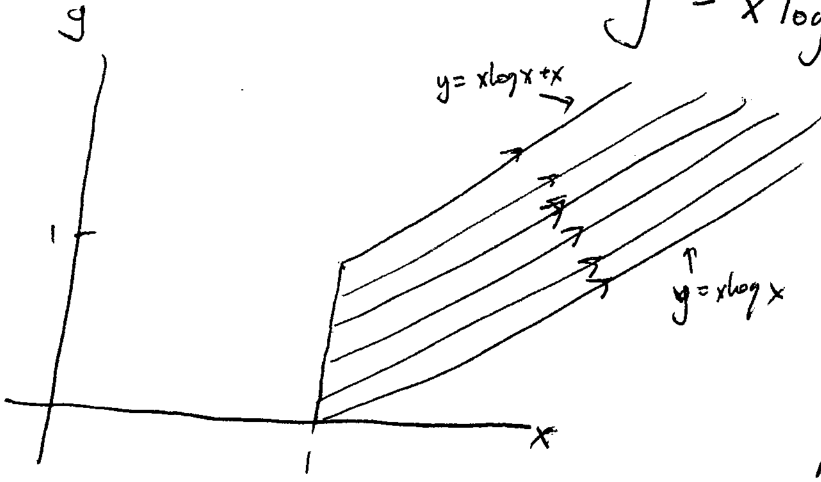
$$(e^{-s}y)' = 1.$$

$$y = e^s(s+c). \qquad y(0) = y_0.$$

$$\to y = e^s(s+y_0).$$

$$z(s) = s + y_0 \to u(x,y) = \frac{y}{x}.$$

The characteristics are $y = x\log x + xy_0$.



Since these characteristics don't intersect, u is uniquely determined by the given conditions in the region

$\{(x,y) : x \ge 1,\ x\log x \le y \le x\log x + x\}$.

2003 #6 :

~~For~~ Let

$$v(x) = v(x_1, x_2, x_3) := \begin{cases} u(x_1, x_2, x_3) & \text{if } x_3 \geq 0. \\ -u(x_1, x_2, -x_3) & \text{if } x_3 < 0. \end{cases}$$

(Since $u(x,y,0) = -u(x,y,0) \rightarrow u(x,y,0) = 0$).

For each $0 < r < 1$, let $B_r := B(0,r)$. ~~Thus~~ We show v is harmonic in B_r for each r. Let

$$P_v(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r} \frac{v(y)}{|x-y|^n} \, d\sigma.$$

Then $\Delta P_v = 0$ in B_r and $P_v = v$ on ∂B_r. Note that $v - P_v$ is harmonic in $B_r \cap \{x_3 > 0\}$ and $v = P_v$ on ∂B_r.

For $x \in B_r \cap \{x_3 = 0\}$, notice that

$$P_v(x) = \int_{\partial B_r} \frac{v(y)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y = \int_{\partial B_r \cap \{y_3 \geq 0\}} + \int_{\partial B_r \cap \{y_3 < 0\}}$$

$$= \int_{\substack{\partial B_r \\ \{y_3 \geq 0\}}} \frac{u(y_1, y_2, y_3)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y + \int_{\substack{\partial B_r \\ \{y_3 < 0\}}} \frac{-u(y_1, y_2, -y_3)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y.$$

$$= 0.$$

Thus $v - P_v = 0$ on $B_r \cap \{x_3 = 0\}$. By the Strong Max Principle,

$$\max_{\overline{B_r \cap \{x_3 > 0\}}} v - P_v = \max_{\partial(B_r \cap \{x_3 > 0\})} v - P_v = 0.$$

W03 #6 cont:

Thus $v \leq P_v$ in $\overline{B_r \cap \{x_3 > 0\}}$.

Interchanging P_v and v gives $v = P_v$ in $\overline{B_r \cap \{x_3 > 0\}}$.

Similarly considering $\overline{B_r \cap \{x_3 < 0\}}$ shows $v = P_v$ in $\overline{B_r}$.

Therefore v is harmonic in B_r $\forall\, 0 < r < 1$. Thus v is harmonic in $B(0,1)$.

$\#$

W03 #7: We want to solve

$$u_{xx} = g$$
$$u(0) = u(L/3) = u(L) = 0.$$

By FTC,

$$u'(x) = u'(0) + \int_0^x g(t)\,dt.$$

$$\longrightarrow u(x) = u(0) + u'(0)x + \int_0^x \int_0^t g(s)\,ds\,dt.$$

Since $u(0) = 0$,

$$u(x) = u'(0)x + \int_0^x \int_0^t g(s)\,ds\,dt.$$

As $u(L) = 0$,

$$0 = u'(0)L + \int_0^L \int_0^t g(s)\,ds\,dt.$$

$$\longrightarrow u'(0) = -\frac{1}{L}\int_0^L \int_0^t g(s)\,ds\,dt.$$

Since $u(L/3) = 0$,

$$0 = -\frac{1}{\cancel{L}}\int_0^L \int_0^t g(s)\,ds\,d\tau \cdot \frac{\cancel{L}}{3} + \int_0^{L/3} \int_0^t g(s)\,ds\,d\tau.$$

$$\frac{1}{3}\int_0^L \int_0^t g(s)\,ds\,d\tau = \int_0^{L/3} \int_0^t g(s)\,d\tau. \quad (*)$$

Thus if g obeys $(*)$, there is a solution of the differential equation.

W03 #8

a) We have

$u(x,t) = 0$ for $x \in \partial D$.
$\rightarrow u_t(x,t) = 0$ for $x \in \partial D$.

$$\dot{E}(t) = \int_D \varepsilon^2 2u_t u_{tt} + 2\nabla u \cdot \nabla u_t \, dx$$
$$= \int_D \varepsilon^2 2u_t u_{tt} - 2\Delta u \, u_t \, dx$$
$$= \int_D 2u_t (\varepsilon^2 u_{tt} - \Delta u) \, dx$$
$$= \int_D 2u_t (-u_t) \, dx \leq 0.$$

b). By (a), $E(t) \leq E(0)$. We have

$$E(0) = \int_D \varepsilon^2 u_t(x,0)^2 + |\nabla u(x,0)|^2 \, dx.$$
$$= \int_D \varepsilon^2 (\varepsilon^{-2\alpha} f(x)^2) \, dx$$
$$= \varepsilon^{2(1-\alpha)} \int_D f(x)^2 \, dx \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Since $\alpha < 1$. Thus

$$\int_D |\nabla u(x,t)|^2 \, dx \leq E(t) \leq \varepsilon^{2(1-\alpha)} \int_D f(x)^2 \, dx \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

c) Expand $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x)$ where

$$\Delta \phi_n + \lambda_n \phi_n = 0. \quad \text{in } D$$
$$\phi_n = 0 \quad \text{on } \partial D$$
$$\|\phi_n\|_{L^2} = 1.$$

W03 #8 cont:

$$\cancel{\varepsilon^2 u_{tt} + u_t - \Delta u = 0}$$

$$\cancel{\sum_{n=1}^{\infty} \varepsilon^2 (\phi_n)_{tt} a_n + a_n (\phi_n)_t + \lambda_n \phi_n a_n = 0}$$

$$\cancel{\sum_{n=1}^{\infty} a_n \left(\varepsilon^2 (\phi_n)_{tt} + (\phi_n)_t + \lambda_n \phi_n \right) = 0}$$

$$\varepsilon^2 u_{tt} + u_t = \Delta u.$$

$$\varepsilon^2 \sum_{n \geq 1} a_n''(t) \phi_n(x) + \sum_{n \geq 1} a_n'(t) \phi_n(x) = \sum_{n \geq 1} -\lambda_n a_n(t) \phi_n(x)$$

$\longrightarrow$ $$\varepsilon^2 a_n'' + a_n' + \lambda_n a_n = 0.$$

$$r^2 \varepsilon^2 + r + \lambda = 0 \longrightarrow r = \frac{-1 \pm \sqrt{1 - 4\varepsilon^2 \lambda_n}}{2\varepsilon^2}$$

$$a_n(t) = A e^{\underbrace{\boxed{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_n}}{2\varepsilon^2}}}_{= r_{1n}} t} + B e^{\underbrace{\boxed{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_n}}{2\varepsilon^2}}}_{= r_{2n}} t}.$$ (more $\lambda_n \to \infty$)

$u(x,0) = 0 \longrightarrow a_n(0) = 0. \longrightarrow A + B = 0.$

$u_t(x,0) = \varepsilon^{-1} f(x) \longrightarrow \sum_{n=1}^{\infty} a_n'(0) \phi_n(x) = \varepsilon^{-1} \sum_{n=1}^{\infty} f_n \phi_n(x).$

$a_n'(0) = \varepsilon^{-1} f_n.$ [Since f is an eigenfunction, $f_n = 0$ except for one such n.]

$$a_n(t) = A e^{r_{1n} t} + B e^{r_{2n} t} = A(e^{r_{1n}t} - e^{r_{2n}t})$$

$$a_n'(t) = A(r_{1n} e^{r_{1n}t} - r_{2n} e^{r_{2n}t})$$

$$\varepsilon^{-1} f_n = a_n'(0) = A(r_{1n} - r_{2n})$$

$$A = \frac{\varepsilon^{-1} f_n \varepsilon^2}{\sqrt{1-4\varepsilon^2\lambda_n}} = \frac{\varepsilon f_n}{\sqrt{1-4\varepsilon^2}}$$

W03 #8 cont:

Since f is an eigenfunction, $f_n = \begin{cases} 0 & \forall\, n \neq n_0 \end{cases}$

for some n_0 and hence

$$u(x,t) = \frac{\varepsilon f_{n_0}}{\sqrt{1-4\varepsilon^2\lambda_{n_0}}}\,\phi_{n_0}(x)\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)$$

$$= \frac{\varepsilon f(x)}{\sqrt{1-4\varepsilon^2\lambda_{n_0}}}\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)$$

Thus

$$\int |\nabla u(x,t)|^2 dx = \frac{\varepsilon^2}{(1-4\varepsilon^2\lambda_{n_0})}\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)\int |\nabla f|^2 dx$$

$$\longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

F02 #1

We have

$$\dot{x} = x - y$$
$$\dot{y} = x(y^2 - 1).$$

The stationary pts are

$$x = y \qquad \longrightarrow \qquad (0,0)$$
$$x = 0 \quad y = \pm 1 \qquad\qquad (1,1)$$
$$(-1,-1).$$

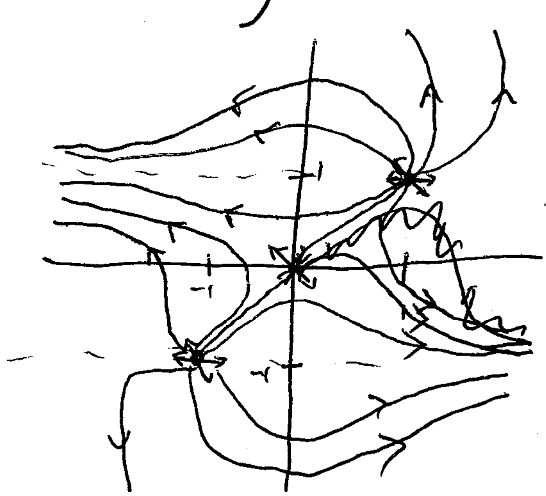
We have

$$J(x,y) = \begin{pmatrix} 1 & -1 \\ y^2-1 & 2xy \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \longrightarrow \text{eigvalues } \tfrac{1}{2}(1 \pm \sqrt{5}). \quad \text{eigvectors } \begin{pmatrix} \frac{1}{2}(-1 \mp \sqrt{5}) \\ 1 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \longrightarrow \text{eigvalues } 2, 1 \quad \text{eigvectors } \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$J(-1,-1) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$



F02 #2:

Let $L_\varepsilon u = -u'' + \varepsilon x u$.

a) We want to find the eigenvalues of L_0. So we want to find u, λ s.t. $L_0 u = \lambda u$. We want to solve $-u'' = \lambda u$ with $u(0)=0$, $u(\pi)=0$. The only time this occurs is if $\lambda = \mu^2$. Then

$$-u'' - \mu^2 u = 0.$$

$$u = A\cos\mu x + B\sin\mu x.$$

$$u(0) = 0 \longrightarrow A = 0.$$

$$u(\pi) = 0 \longrightarrow \sin(\pi\mu) = 0 \longrightarrow \mu = \pm 1, \pm 2, \ldots$$

Thus $\lambda = 1, 4, 9, \ldots$ Therefore $\lambda_0 = 1$, $\phi_0 = \sin x$.

b) Now we want to find an eigenvalue and eigenfunction λ, ϕ with

$$L_\varepsilon \phi = \lambda\phi \qquad \phi(0) = \phi(\pi) = 0.$$

where we have the expansion

$$\lambda = 1 + \varepsilon\lambda_1 + O(\varepsilon^2)$$

$$\phi = \sin x + \varepsilon\phi_1 + O(\varepsilon^2) \qquad \phi_1(0) = \phi_1(\pi) = 0.$$

We want (ignoring $O(\varepsilon^2)$ terms),

$$-\phi'' + \varepsilon x\phi = \lambda\phi$$

$$-(\phi_0 + \varepsilon\phi_1)'' + \varepsilon x(\phi_0 + \varepsilon\phi_1) = (\lambda_0 + \varepsilon\lambda_1)(\phi_0 + \varepsilon\phi_1)$$

$$\cancel{-\phi_0''} - \varepsilon\phi_1'' + \varepsilon x\phi_0 = \cancel{\lambda_0\phi_0} + \varepsilon\lambda_0\phi_1 + \varepsilon\lambda_1\phi_0.$$

$$-\phi_1'' + x\sin x = \phi_1 + \lambda_1\sin x.$$

$$L_0\phi_1 + x\sin x = \phi_1 + \lambda_1\sin x \quad , \quad \phi_1(0) = \phi_1(\pi) = 0.$$

Then as L_0 is self-adjoint (and using the bdry conditions for ϕ_0, ϕ_1),

$$\int_0^\pi (L_0\phi_1)\phi_0\,dx + \int_0^\pi x\sin^2 x\,dx = \int_0^\pi \phi_1\phi_0\,dx + \int_0^\pi \lambda_1\sin^2 x\,dx$$

$$\cancel{\int_0^\pi \phi_1 \sin x\,dx} + \int_0^\pi x\sin^2 x\,dx = \cancel{\int_0^\pi \phi_1 \sin x\,dx} + \lambda_1\int_0^\pi \sin^2 x\,dx$$

F02 #2 cont:

Thus

$$\lambda_1 = \frac{\int_0^{\pi} x \sin^2 x \, dx}{\int_0^{\pi} \sin^2 x \, dx} . \quad \left(= \frac{\pi}{2} \right)$$

Now it remains to solve

$$\begin{cases} -\phi_1'' - \phi_1 = (\lambda_1 - x)\sin x \\ \phi_1(0) = 0, \ \phi_1(\pi) = 0 \end{cases}$$

$$\longrightarrow \begin{cases} \phi_1'' + \phi_1 = (x - \tfrac{\pi}{2})\sin x \\ \phi_1(0) = 0, \ \phi_1(\pi) = 0 . \end{cases} \qquad (*)$$

A fundamental set of solutions for $y'' + y = 0$ is $\{\cos x, \sin x\}$. Then particular solution is given by

$$-\cos x \int \sin^2 x \, (x - \tfrac{\pi}{2}) \, dx + \sin x \int \cos x \, (x - \tfrac{\pi}{2}) \sin x \, dx .$$

$$\int (\sin^2 x)(x - \tfrac{\pi}{2}) \, dx \underset{u = x - \frac{\pi}{2}}{=} \int u \sin^2(u + \tfrac{\pi}{2}) \, du = \int u \cos^2 u \, du = \frac{1}{2}\int u\cos 2u + u \, du$$

$$= \frac{1}{2}\left[\frac{u}{2}\sin 2u + \frac{1}{4}\cos 2u \right] + \frac{1}{2}\cdot\frac{u^2}{2} + C.$$

$$= \frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x - \pi) + \frac{1}{8}\cos(2x - \pi) + \frac{1}{4}(x - \tfrac{\pi}{2})^2 + C$$

$$= -\frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x) - \frac{1}{8}\cos(2x) + \frac{1}{4}(x - \tfrac{\pi}{2})^2 + C.$$

$$\int (\cos x)(x - \tfrac{\pi}{2})(\sin x) dx = \frac{1}{2}\int (x - \tfrac{\pi}{2}) \sin 2x \, dx = -(x - \tfrac{\pi}{2})\frac{1}{4}\cos 2x + \frac{1}{8}\sin 2x + C.$$

Therefore a particular solution to the ODE $(*)$ is

$$y_p = \frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x)\cos x + \frac{1}{8}\cos(2x)\cos x - \frac{1}{4}(x - \tfrac{\pi}{2})^2\cos x$$
$$- (x - \tfrac{\pi}{2})\cdot\frac{1}{4}\sin x \cos 2x + \frac{1}{8}\sin x \sin 2x .$$

Fa2 #2 cont:

We have $y_p(0) = \frac{1}{8} - \frac{\pi^2}{16}$

$y_p(\pi) = -\frac{1}{8} + \frac{\pi^2}{16}$.

The homogeneous eqn is:

$$y_h(x) = A\cos x + B\sin x.$$

$y_h(0) = A$

$y_h(\pi) = -A$.

So

$$\phi_1(x) = A\cos x + B\sin x + y_p(x)$$

$\rightarrow \phi_1(0) = 0 \longrightarrow 0 = A + y_p(0) = A + \frac{1}{8} - \frac{\pi^2}{16}$.

$\phi_1(\pi) = 0 \longrightarrow 0 = -A + y_p(\pi) = -A - \frac{1}{8} + \frac{\pi^2}{16}$.

$$\rightarrow A = -\frac{1}{8} + \frac{\pi^2}{16}.$$

Take $B = 0$. Then

$$\phi_1(x) = \left(-\frac{1}{8} + \frac{\pi^2}{16}\right)\cos x + y_p(x). \qquad \#$$

F02 #3

We first solve the system: $w := \begin{pmatrix} u \\ v \end{pmatrix}$

$$w_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} w_x = 0.$$

~~Since~~ $\cancel{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$ ~~has.~~

We diagonalize $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Eigenvalues: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \left| \begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda+1 \end{pmatrix} \right| = \lambda^2 - 1 - 1$

$$\lambda = \pm\sqrt{2}.$$

Eigenvectors: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\sqrt{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$$a + b = \pm\sqrt{2}\, a. \quad \longrightarrow \begin{pmatrix} 1 \\ \pm\sqrt{2}-1 \end{pmatrix}$$

$$b = (\pm\sqrt{2} - 1)\, a.$$

Thus

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix}^{-1}.$$

Let $P := \begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix}$, $D := \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix}$, $A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

let $z := P^{-1} w$.

$$w_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} w_x = 0 \qquad (1)$$

$$\Updownarrow$$

$$z_t + \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix} z_x = 0. \qquad (2)$$

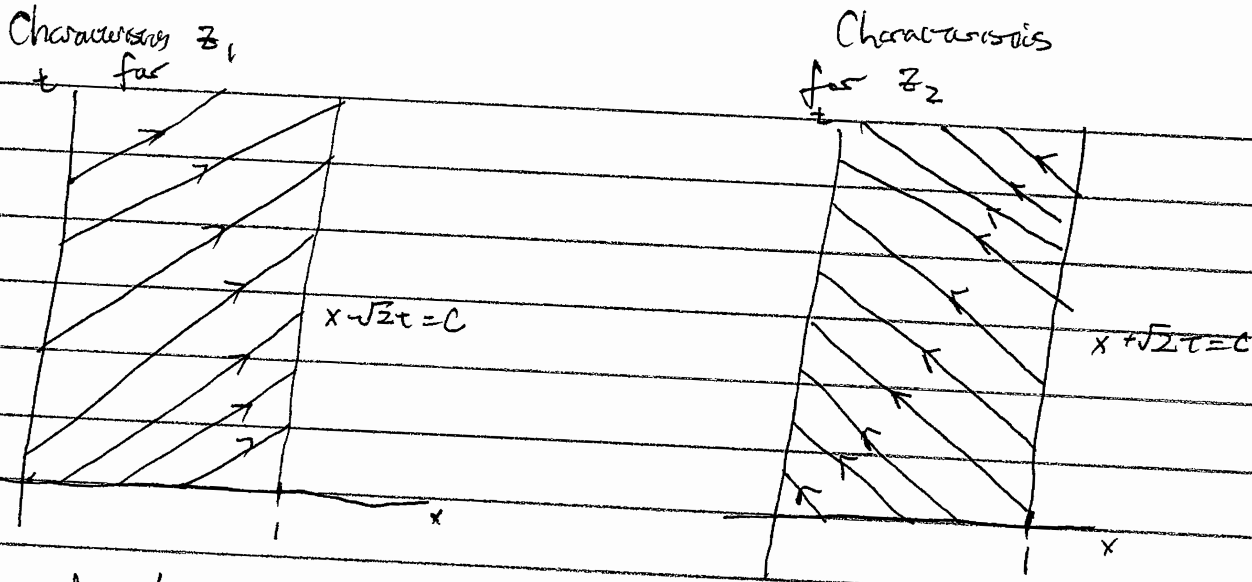
Since P is invertible, well posedness of (1) is equivalent to well-posedness of (2).

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \begin{aligned} &(z_1)_t + \sqrt{2}(z_1)_x = 0 \\ &\cancel{(z_2)_t - \sqrt{2}(z_2)_x = 0.} \end{aligned}$$

$(z_1)_t + \sqrt{2}(z_1)_x = 0$ has characteristics: $x - \sqrt{2}t = C$ $\quad z_1, z_2$ are constant on characteristics.

$(z_2)_t - \sqrt{2}(z_2)_x = 0$ has characteristics $x + \sqrt{2}t = C$.

F02 #3
cont:



Thus for the PDE for z_1 to be well posed we need data when $t=0$ and $x=0$. For the PDE for z_2 to be well posed we need data when $t=0$ and $x=1$.

Thus since P is invertible, for the PDE in the problem to be well posed, we need data for u on $t=0$ and $x=0$ and data for V on $t=0$ and $x=1$. The only set of initial conditions that does this is (c)

//

F02 #4: We want to solve

$$u_{x_2} + u u_{x_1} = 0$$
$$u(x_1, 0) = \pm x_1.$$

a) $u(x_1, 0) = x_1$

$\longrightarrow$ $F(p, z, x) = p_2 + z p_1 = 0.$
$D_p F = (z, 1)$
$D_z F = p_1$
$D_x F = 0$

$\longrightarrow$ $\dot{p} = -D_x F - D_z F p = -p_1(p_1, p_2)$
$\dot{z} = D_p F \cdot p = z p_1 + p_2 = 0.$
$\dot{x} = D_p F = (z, 1)$
w/ $x_1(0) = x_1(0)$
$x_2(0) = 0.$
$z(0) = x_1(0).$

$\longrightarrow$ $z(s) = x_1(0)$ $\quad$ $x_1(s) = x_1(0)s + x_1(0)$
$x_2(s) = s$ $\quad$ $\longrightarrow x_1(0) = \dfrac{x_1(s)}{s+1}$

Therefore $u(x_1, x_2) = \dfrac{x_1}{x_2 + 1}$. That is, $u(x,t) = \dfrac{x}{t+1}$

b) $u(x_1, 0) = -x_1$
By the same calculation,
$\dot{z} = 0$ $\quad$ w/ $\quad$ $x_1(0) = x_1(0)$
$\dot{x} = (z, 1)$ $\quad$ $x_2(0) = 0$
$z(0) = -x_1(0)$

$\longrightarrow$ $z(s) = -x_1(0)$ $\quad$ $x_1(s) = -x_1(0)s + x_1(0)$
$x_2(s) = s$ $\quad$ $\longrightarrow x_1(0) = \dfrac{x_1(s)}{1-s}$

Thus $u(x_1, x_2) = \dfrac{x_1}{-1 + x_2}$. That is, $u(x,t) = \dfrac{x}{-1+t}$.
$\hookrightarrow$ solution blows up at $t=1$

F02 5a: We have for each $x \in [0,1]$,

$$|u(x)| \le \int_0^x |u'(t)|\,dt = \int_0^1 |u'(t)|\, I_{[0,x]}(t)\,dt$$
$$\le \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2} \left(\int_0^1 I_{[0,x]}(t)\,dt\right)^{1/2}$$
$$\le \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2}.$$

Thus

$$|u(x)|^2 \le \int_0^1 |u'(t)|^2\,dt.$$

Since $x \in [0,1]$ was arbitrary,

$$\max_{x\in[0,1]} |u(x)|^2 \le \int_0^1 |u'(t)|^2\,dt.$$

5b Suppose L had an eigenvalue that was ≤ 0. Then for some $\lambda \ge 0$ and u satisfying $u(0)=u(1)=0$, we have $(L+\lambda I)u = 0$. Then

$$0 = \langle (L+\lambda I)u, u\rangle = \int_0^1 (-u'' + pu + \lambda u)u\,dx$$
$$= \int_0^1 -u''u + pu^2 + \lambda u^2\,dx = \int_0^1 (u')^2 + p_+u^2 - p_-u^2 + \lambda u^2\,dx$$
$$\ge \int_0^1 (u')^2 - p_-u^2\,dx \overset{\text{by (a)}}{\ge} \int_0^1 (u')^2 dx - \left(\int_0^1 (u')^2 dt\right)\int_0^1 p_-(x)\,dx$$
$$> \int_0^1 u'^2\,dx - \int_0^1 u'^2\,dx = 0,$$

a contradiction. Therefore any eigenvalue of L must be > 0.

F02 #6

Write $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\pi x)$. Then as

$$u_t = u_{xx} + e^{-2t} g(x),$$

we have

$$a_n'(t) = -a_n(t) n^2\pi^2 + e^{-2t} b_n$$

where $g(x) = \sum_{n\geq 0} b_n \cos(n\pi x)$ and $b_n = 2\int_0^1 g(x)\cos(n\pi x)dx$, $b_0 = \int_0^1 g(x)dx$

Then

$$a_n'(t) + a_n(t)n^2\pi^2 = e^{-2t}b_n.$$

$$(e^{n^2\pi^2 t} a_n)' = e^{(n^2\pi^2-2)t} b_n.$$

$$e^{n^2\pi^2 t} a_n(t) = \frac{1}{n^2\pi^2-2} e^{(n^2\pi^2-2)t} b_n - \frac{1}{n^2\pi^2-2} b_n + a_n(0)$$

$$a_n(t) = \frac{1}{n^2\pi^2-2} e^{-2t} b_n - \frac{1}{n^2\pi^2-2} b_n e^{-n^2\pi^2 t} + a_n(0)e^{-n^2\pi^2 t}.$$

$$= \frac{b_n}{n^2\pi^2-2}\left(e^{-2t} - e^{-n^2\pi^2 t}\right) + a_n(0)e^{-n^2\pi^2 t}.$$

We have

$$a_n(0) = 2\int_0^1 f(x)\cos(n\pi x)dx, \quad a_0(0) = \int_0^1 f(x)dx.$$

Thus

$$\lim_{t\to\infty} \sum_{n\geq 0} a_n(t)\cos n\pi x = \lim_{t\to\infty} \frac{b_0}{-2}(-1) + a_0(0) = \lim_{t\to\infty} \frac{b_0}{2} + a_0(0)$$

$$= \left(\frac{1}{2}\int_0^1 g(x)dx\right) + \int_0^1 f(x)dx.$$

F02 #7:

We have $2\frac{\partial u}{\partial \bar z} = f$. Since $\frac{c}{x+iy}$ is a fundamental solution,

$$u(z) = \int_{\mathbb{C}} \frac{c}{w} f(z-w)\,dw.$$

Then

$$\Delta u = \int_{\mathbb{C}} \frac{c}{w}\, 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} f(z-w)\,dw = -2c\int_{\mathbb{C}} \frac{1}{w}\, 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar w} f(z-w)\,dw$$

$$= 2c\int_{\mathbb{C}} \frac{\partial}{\partial \bar w}\frac{1}{w}\; 2\frac{\partial}{\partial z} f(z-w)\,dw.$$

Let $K(w) := \frac{1}{2\pi}\log|w| = \frac{1}{4\pi}\log(w_1^2+w_2^2)$. We have

$$\frac{\partial}{\partial w}\overset{K(w)}{\cancel{\tfrac{z}{w}}} = \frac{1}{2}\left(\frac{\partial}{\partial w_1} - i\frac{\partial}{\partial w_2}\right)K(w) = \frac{1}{2}\cdot\frac{1}{4\pi}\left[\frac{2w_1}{w_1^2+w_2^2} - i\frac{2w_2}{w_1^2+w_2^2}\right] = \frac{1}{4\pi w}.$$

Thus

$$\Delta u = 2c\int_{\mathbb{C}} \frac{\partial}{\partial \bar w}\frac{1}{w}\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw = 2c\int_{\mathbb{C}} 4\pi\frac{\partial}{\partial \bar w}\frac{\partial}{\partial w}K(w)\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw.$$

$$= 2c\pi\int_{\mathbb{C}} \Delta_w K(w)\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw.$$

Since $\Delta_w K = \delta$ and

$$2\frac{\partial u}{\partial \bar z} = f \longrightarrow \Delta u = 2\frac{\partial f}{\partial z},$$

we must have $c = \frac{1}{2\pi}$.

Note

$$\cancel{\tfrac{\partial}{\partial z}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)e^{-2\pi i(x\xi_1+y\xi_2)} = -2\pi i\xi_1 e^{-2\pi i(x\xi_1+y\xi_2)} + i(-2\pi i\xi_2)e^{-2\pi i(x\xi_1+y\xi_2)}$$

$$= -2\pi i(\xi_1+i\xi_2)e^{-2\pi i(x\xi_1+y\xi_2)}.$$

Thus the fourier transform of $(x+iy)^{-1}$ is

$$\int_{\mathbb{R}^2} e^{-2\pi i(x\xi_1+y\xi_2)}\frac{1}{x+iy}\,dx\,dy = \int_{\mathbb{R}^2}\frac{1}{-2\pi i(\xi_1+i\xi_2)}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)e^{-2\pi i(x\xi_1+y\xi_2)}\cdot\frac{1}{x+iy}\,dx\,dy$$

$$= \frac{2\pi}{+2\pi i(\xi_1+i\xi_2)}\int_{\mathbb{R}^2} e^{-2\pi i(x\xi_1+y\xi_2)}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{1/2\pi}{x+iy}\,dx\,dy$$

$$= \frac{1}{i(\xi_1+i\xi_2)} \quad \text{since } \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{1/2\pi}{x+iy}\right) = \delta. \quad \#$$

F02 #8: Let u_1, u_2 be 2 solutions of the given equation.
Let $w := u_1 - u_2$. Then

$$\Delta^2 w = 0 \text{ in } D$$
$$w = \Delta w = 0 \text{ on } \partial D.$$

We have

$$0 = \int_D w\,\Delta(\Delta w)\,dx \overset{\text{since } w=0=\Delta w \text{ on } \partial D}{=} \int_D (\Delta w)^2\,dx.$$

Therefore $\Delta w = 0$ in D. Finally,

$$0 = \int_D w\,\Delta w\,dx = \int_D \nabla w \cdot \nabla w\,dx = \int_D |\nabla w|^2\,dx$$

which implies that $\nabla w = 0 \longrightarrow w = 0$ on D.
Thus the solution to the given boundary value problem is unique.

S02 #1

a. We will ~~solve~~ find an u s.t. $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{2\pi}\log r$. We have

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{2\pi} r\log r.$$

$$\rightarrow r\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right) + c_1$$

$$\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r\log r - \frac{1}{4}r\right) + \frac{c_1}{r}.$$

Choose $c_1 = 0$. Then

$$\frac{\partial u}{\partial r} = \frac{1}{2\pi}\left(\frac{1}{2}r\log r - \frac{1}{4}r\right) = \frac{1}{4\pi}r\log r - \frac{1}{8\pi}r.$$

$$\rightarrow u = \frac{1}{4\pi}\left(\frac{1}{2}r^2\log r - \frac{1}{4}r^2\right) - \frac{1}{8\pi}\cdot\frac{1}{2}r^2.$$

$$= \frac{r^2}{8\pi}(\log r - 1).$$

So $\cancel{\frac{r^2}{8\pi}}\, u = \frac{|x|^2}{8\pi}(\log|x| - 1)$ is a radially symmetric solution to $\Delta u = \frac{1}{2\pi}\log|x|$ in $\mathbb{R}^2$.

We claim u is a fundamental solution for Δ^2. For arbitrary $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} u\,\Delta^2\phi\,dx = \int_{B_\varepsilon(0)} u\,\Delta^2\phi\,dx + \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta^2\phi\,dx := I_\varepsilon + J_\varepsilon.$$

Note

$$|I_\varepsilon| \le \int_{B_\varepsilon(0)} |u||\Delta^2\phi|\,dx \lesssim \varepsilon^2\cdot\varepsilon^2\log\varepsilon\cdot\|\Delta^2\phi\|_{L^\infty} \rightarrow 0 \text{ as } \varepsilon\rightarrow 0.$$

We also have

$$J_\varepsilon = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta^2\phi\,dx = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} u\,\Delta(\Delta\phi)\,dx = \int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \Delta u\cdot\Delta\phi\,dx$$

$$+ \int_{\partial(\mathbb{R}^2\setminus B_\varepsilon(0))} u\frac{\partial(\Delta\phi)}{\partial\nu} - \Delta\phi\frac{\partial u}{\partial\nu}\,d\sigma$$

where ν is the inward normal for $\mathbb{R}^2\setminus B_\varepsilon(0)$.

Since $\Delta u = \frac{1}{2\pi}\log|x|$ which is the fundamental solution for Δ, we have

$$\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \Delta u\cdot\Delta\phi\,dx \Rightarrow \phi(0) \text{ as } \varepsilon\rightarrow 0.$$ We also have

$$\left|\int_{\partial(\mathbb{R}^2\setminus B_\varepsilon(0))} u\frac{\partial(\Delta\phi)}{\partial\nu}\,d\sigma\right| \lesssim \varepsilon\cdot\varepsilon^2\log\varepsilon\,\|\nabla(\Delta\phi)\|_{L^\infty} \rightarrow 0 \quad \text{as } \varepsilon\rightarrow 0.$$

S02 #1b:

We first prove an identity:

<u>Lemma:</u> $\displaystyle\int_U u\Delta^2 v - v\Delta^2 u \, dx = \int_{\partial U} u\frac{\partial(\Delta v)}{\partial \nu} - \Delta v \frac{\partial u}{\partial \nu} - v\frac{\partial(\Delta u)}{\partial \nu} + \Delta u \frac{\partial v}{\partial \nu}\, d\sigma.$

<u>Pf:</u> We have

$$\int_U u\Delta^2 v - v\Delta^2 u \, dx = \int_U u\Delta^2 v - \Delta u \Delta v \, dx - \int_U v\Delta^2 u - \Delta v \Delta u \, dx$$

$$= \int_{\partial U} u\frac{\partial(\Delta v)}{\partial \nu} - \Delta v\frac{\partial u}{\partial \nu}\, d\sigma - \int_{\partial U} v\frac{\partial(\Delta u)}{\partial \nu} - \Delta u \frac{\partial v}{\partial \nu}\, d\sigma.$$

#

Let Φ denote the fundamental solution in part a). Let w solve

$$\begin{cases} \Delta^2 w = f & \text{in } U \\ w = 0,\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial U. \end{cases} \qquad (\Delta)$$

Fix $x \in U$. Choose $\varepsilon > 0$ small enough s.t. $B(x,\varepsilon) \subset U$. Let $V_\varepsilon := U \setminus B(x,\varepsilon)$.
Then

$$\int_{V_\varepsilon} w(y)(\Delta^2\Phi)(y-x)\,dy - \int_{V_\varepsilon} (\Delta^2 w)(y)\,\Phi(y-x)\,dy$$

$$= \int_{\partial U} w(y)\frac{\partial \Delta\Phi(y-x)}{\partial \nu}\, d\sigma - \int_{\partial B(x,\varepsilon)} w(y)\frac{\partial \Delta \Phi}{\partial \nu}(y-x)\, d\sigma$$

$$- \int_{\partial U} \Delta\Phi(y-x)\frac{\partial w}{\partial \nu}\, d\sigma + \int_{\partial B(x,\varepsilon)} \Delta\Phi(y-x)\frac{\partial w}{\partial \nu}\, d\sigma \qquad (*)$$

$$- \int_{\partial U} \Phi(y-x)\frac{\partial \Delta w}{\partial \nu}(y)\, d\sigma + \int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial \Delta w}{\partial \nu}(y)\, d\sigma$$

$$+ \int_{\partial U} \Delta w(y)\frac{\partial \Phi}{\partial \nu}(y-x)\, d\sigma - \int_{\partial B(x,\varepsilon)} \Delta w(y)\frac{\partial\Phi}{\partial \nu}(y-x)\, d\sigma.$$

Since $\Delta^2\Phi = 0$ for $x \neq y$, $\int_{V_\varepsilon} w(y)(\Delta^2\Phi)(y-x)\,dy = 0$. Since $w = 0$, $\frac{\partial w}{\partial \nu} = 0$ on ∂U,
$\int_{\partial U} w(y)\frac{\partial\Delta\Phi}{\partial \nu}(y-x)\, d\sigma = 0$ and $\int_{\partial U} \Delta\Phi(y-x)\frac{\partial w}{\partial \nu}\, d\sigma = 0.$

1b cont.:

Since $\Delta\Phi = \frac{1}{2\pi}\log|x|$, $\nabla(\Delta\Phi) = \frac{x}{2\pi}\cdot\frac{1}{|x|^2}$ and hence on $\partial B(x,\varepsilon)$ as $\nu = \frac{x}{\varepsilon}$, $\frac{\partial\Delta\Phi}{\partial\nu} = \nabla(\Delta\Phi)\cdot\nu = \frac{|x|^2}{2\pi|x|^2\varepsilon} = \frac{1}{2\pi\varepsilon}$. Therefore

$$\int_{\partial B(x,\varepsilon)} w(y)\frac{\partial\Delta\Phi}{\partial\nu}(y-x)\,d\sigma = \frac{1}{2\pi\varepsilon}\int_{\partial B(x,\varepsilon)} w(y)\,d\sigma \longrightarrow w(x) \text{ as } \varepsilon\to 0.$$

We also have

$$\left|\int_{\partial B(x,\varepsilon)} \Delta\Phi(y-x)\frac{\partial w}{\partial\nu}(y)\,d\sigma\right| \lesssim_w \varepsilon\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

$$\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma\right| \lesssim_w \varepsilon^3\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

$$\left|\int_{\partial B(x,\varepsilon)} \Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma\right| \lesssim_w \varepsilon^2\log\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0.$$

Therefore letting $\varepsilon\to 0$ in $(*)$ yields

$$-\int_U (\Delta^2 w)(y)\,\Phi(y-x)\,dy = -w(x) - \int_{\partial U}\Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma + \int_{\partial U}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x)\,d\sigma.$$

Since w satisfies $(\diamond)$, we have

$$w(x) = \int_U f(y)\Phi(y-x)\,dy + \int_{\partial U}\Delta w(y)\frac{\partial\Phi}{\partial\nu}(y-x) - \Phi(y-x)\frac{\partial\Delta w}{\partial\nu}(y)\,d\sigma. \quad (**)$$

For each x, define a ϕ^x s.t.

$$\begin{cases}\Delta^2\phi^x = 0 & \text{in } U.\\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U.\end{cases}$$

Then

$$\int_U w(y)\Delta^2\phi^x(y)\,dy - \int_U \phi^x(y)\Delta^2 w(y)\,dy = \int_{\partial U} w(y)\frac{\partial\Delta\phi^x}{\partial\nu} - \Delta\phi^x\cdot\frac{\partial w}{\partial\nu}\,d\sigma$$
$$- \int_{\partial U}\phi^x(y)\frac{\partial\Delta w}{\partial\nu} - \Delta w\,\frac{\partial\phi^x}{\partial\nu}(y)\,d\sigma$$

3

1b cont'd

Since $\Delta^2 \phi^x = 0$ on Ω and $w = 0, \frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$, we have

$$\int_\Omega \phi^x(y)\, \Delta^2 w(y)\, dy = \int_{\partial\Omega} \phi^x(y) \frac{\partial \Delta w}{\partial \nu} - \Delta w \frac{\partial \phi^x}{\partial \nu}\, d\sigma. \quad (***)$$

Since $\Delta^2 w = f$ on Ω, we have after adding $(**)$ to $(***)$

=0 since $\Phi(y-x) = \phi^x$ on $\partial\Omega$

$$w(x) + \int_\Omega \phi^x(y) f(y)\, dy = \int_\Omega f(y)\, \Phi(y-x)\, dy + \int_{\partial\Omega} \Delta w \left[\frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y)\right] d\sigma(y)$$

Let $G(x,y) := \Phi(y-x) - \phi^x(y)$. Then

$$w(x) = \int_\Omega f(y)\, G(x,y)\, dy,$$

$G(x,y)$ is our Green's function.

S02 #2a:

By Duhamel's Principle, if $\mathcal{U}$ is a solution to

$$\mathcal{U}_{tt}(x,t,s) - \mathcal{U}_{xx}(x,t,s) = 0 \quad \text{in } \mathbb{R}\times[0,\infty) \qquad (*)$$

$$\mathcal{U}(x,0,s) = 0,\ \mathcal{U}_t(x,0,s) = f(x)\cos s \quad \text{on } \mathbb{R}\times\{t=0\},$$

then

$$u(x,t) := \int_0^t \mathcal{U}(x,t-s,s)\,ds$$

is a solution to

$$u_{tt} - u_{xx} = f(x)\cos t \quad \text{in } \mathbb{R}\times[0,\infty)$$

$$u(x,0) = u_t(x,0) = 0 \quad \text{on } \mathbb{R}\times\{t=0\}.$$

By D'Alembert's equation, the solution $\mathcal{U}$ to $(*)$ is

$$\mathcal{U}(x,t,s) = \frac{1}{2}\int_{x-t}^{x+t} f(\xi)\cos s\,d\xi = \frac{\cos s}{2}\int_{x-t}^{x+t} f(\xi)\,d\xi.$$

we have

$$u(x,t) = \frac{1}{2}\int_0^t \cos s\int_{x-t+s}^{x+t-s} f(\xi)\,d\xi\,ds$$

is a solution to the PDE.

S02 #2a alt:

Guess $u(x,t) = F(x)\cos t$. Then
$$f(x)\cos t = -F(x)\cos t - F''(x)\cos t.$$
$$\rightarrow \quad F''(x) + F(x) = -f(x).$$
Then
$$F(x) = A\cos x + B\sin x + G(x)$$
where $G(x)$ is s.t. $G'' + G = -f$. Therefore a particular solution to the PDE is
$$u_p(x,t) = A\cos x\cos t + B\sin x\cos t + G(x)\cos t. \longrightarrow \text{u_p not unique b/c A, B can be anything.}$$
The homogeneous solution u_h satisfies
$$(u_h)_{tt} - (u_h)_{xx} = 0$$
$$u_h(x,0) = -(A\cos x + B\sin x + G(x))$$
$$(u_h)_t(x,0) = 0.$$
By D'Alembert's formula,
$$u_h(x,t) = \frac{1}{2}\Big[-(A\cos(x+t) + B\sin(x+t) + G(x+t)) - (A\cos(x-t) + B\sin(x-t) + G(x-t))\Big]$$
Then the solution $u(x,t) = u_h(x,t) + u_p(x,t)$.

[Finding a G s.t. $G'' + G = -f$. The easiest way is via variation of parameters.
We have
$$G(x) = v_1(x)\cos x + v_2(x)\sin x$$
where
$$v_1(x) = +\int_0^x \sin t\, f(t)\,dt, \quad v_2(x) = -\int_0^x \cos t\, f(t)\,dt$$
Note that f is compactly supported, so one can indeed find v_1, v_2.]

So
$$G(x) = \int_0^x f(t)(\sin t\cos x - \cos t\sin x)\,dt = \int_0^x f(t)\sin(t-x)\,dt.$$
$$= -\int_0^x f(t)\sin(x-t)\,dt.$$

502 #2b:

Let u^1, u^2 be 2 distinct solutions. Let $w := u^1 - u^2$. Then

$$w_{tt} - w_{xx} = \cancel{f(x)\cos}\ 0 \qquad -\infty < x < \infty \qquad (*)$$
$$w(x,0) = w_t(x,0) = 0 \qquad 0 \le t < \infty.$$

We claim that $w \equiv 0$ (this follows from D'Alembert's formula).

We rederive D'Alembert's formula: We want to find a solution u of

$$u_{tt} - u_{xx} = 0.$$
$$u(x,0) = g, \quad u_t(x,0) = h.$$

Let $u(x,t) := F(x+t) + G(x-t)$. Then

$$u_x(x,t) = F'(x+t) + G'(x-t) \qquad u_t = F'(x+t) - G'(x-t)$$
$$u_{xx} = F''(x+t) + G''(x-t) \qquad u_{tt} = F''(x+t) + G''(x-t).$$

$$g = u(x,0) = F(x) + G(x) \longrightarrow g'(x) = F'(x) + G'(x).$$
$$h = u_t(x,0) = F'(x) - G'(x)$$

So

$$F'(x) = \frac{g'(x) + h(x)}{2} \qquad \cancel{G'(x) = F'(x) - h(x) = \frac{g'(x) - h(x)}{2}}$$

$$\longrightarrow F(x) = \int_0^x \frac{g'(t) + h(t)}{2}\,dt, \qquad \cancel{G(x) = \int_0^x \frac{g'(t) - h(t)}{2}\,dt}$$
$$= \frac{1}{2}[g(x) - g(0)] + \frac{1}{2}\int_0^x h(t)\,dt \qquad \cancel{= \frac{1}{2}[g(x) - g(0)] - \frac{1}{2}\int_0^x h(t)\,dt}$$

$$\cancel{F(x+t) + G(x-t) = \frac{1}{2}[g(x+t) - g(0)] + \frac{1}{2}\int_0^{x+t} h(t)\,dt}$$
$$\cancel{+ \frac{1}{2}[g(x-t) - g(0)] - \frac{1}{2}\int_0^{x-t} h(t)\,dt}$$

Since

$$G(x) = g(x) - F(x) = \frac{1}{2}[g(x) + g(0)] - \frac{1}{2}\int_0^x h(t)\,dt,$$

we have

$$u(x,t) = F(x+t) + G(x-t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds. \longrightarrow$$

Now to show $w \equiv 0$ in $(*)$, take $g, h = 0$

S02 #3:

a) We have
$$u_1 \frac{\partial u_1}{\partial x_1} + \frac{1}{\rho}\frac{\partial p}{\partial x_1} - \frac{\eta}{\rho}\Delta u_1 = 0.$$
$$\frac{\partial p}{\partial x_2} = 0, \quad \frac{\partial p}{\partial x_3} = 0.$$
Then
$$\frac{\partial p}{\partial x_1} = \eta \Delta u - u_1 \rho \frac{\partial u}{\partial x_1} = \eta \Delta u.$$
and hence

since $u = u(x_2, x_3)$.
$$\frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}(\eta \Delta u) = \eta \frac{\partial}{\partial x_1}(\Delta u) = 0$$
$$\frac{\partial}{\partial x_2}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_2}\right) = 0$$
$$\frac{\partial}{\partial x_3}\left(\frac{\partial p}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_3}\right) = 0.$$
Therefore $\frac{\partial p}{\partial x_1}$ is a constant c which implies $\Delta u = c/\eta$.

b) Since $\Delta u = c/\eta$ and u is radially symmetric,
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{c}{\eta} \longrightarrow \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{c}{\eta} r$$
$$r\frac{\partial u}{\partial r} = \frac{1}{2}\frac{c}{\eta} r^2 + c_1$$
$$\frac{\partial u}{\partial r} = \frac{1}{2}\frac{c}{\eta} r + \frac{c_1}{r}.$$
$$u = \frac{1}{4}\frac{c}{\eta} r^2 + c_1 \log r + c_2.$$
Since $\vec{u}$ is a velocity vector, to prevent a singularity at $r=0$, we must have $c_1 = 0$. Since $u(R) = 0$,
$$c_2 = -\frac{1}{4}\frac{c}{\eta} R^2.$$
Therefore
$$u(r) = \frac{c}{4\eta}(r^2 - R^2)$$
and hence
$$Q = \rho \int_{\{x_2^2 + x_3^2 \le R^2\}} u \, dx_2 dx_3 = 2\pi\rho \int_0^R \frac{c}{4\eta}(r^3 - R^2 r)\, dr = 2\pi\rho \cdot \frac{c}{4\eta}\left[\frac{1}{4}R^4 - R^2 \cdot \frac{1}{2} R^2\right]$$
$$= -\frac{c\pi\rho}{8\eta} R^4.$$

S02 #4: We look for a solution $u \in H^1(\mathbb{R})$. Taking the Fourier Transform yields

$\hat{u}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ix\xi} u(x)\,dx$

$$(i\xi + c + e^{-i\xi})\hat{u}(\xi) = \hat{f}(\xi)$$

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}}. \qquad (*)$$

Note c is real and $|c|>1$. Thus

$$|i\xi + c + e^{-i\xi}| \geq |\mathrm{Re}(i\xi + c + e^{-i\xi})|$$
$$= |c + \cos\xi| \geq |c| - 1.$$

Thus as

$$\left\| \frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}} \right\|_{L^2}^2 = \int \frac{|\hat{f}(\xi)|^2}{|i\xi + c + e^{-i\xi}|^2}\,d\xi \leq \frac{1}{(|c|-1)^2} \int |\hat{f}(\xi)|^2\,d\xi$$
$$< \infty \quad \text{as } f \in L^2$$

Therefore we can invert the Fourier Transform in $(*)$ and the unique solution $u \in L^2$ is given by

$$\cancel{u(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \frac{\hat{f}(\xi)}{i\xi}}$$

$$u(x) = \left[\frac{\hat{f}(\xi)}{i\xi + c + e^{-i\xi}} \right]^{\vee}$$

S02 #5:

Here $u(x,t) = \phi(x+ct)$. Then $u_t = u(1-u) + u_{xx}$ becomes

$$\phi'' - c\phi' + \phi(1-\phi) = 0.$$

We can rewrite this as the system

$$x' = y$$
$$y' = cy - x(1-x).$$

The critical points are $(0,0)$ and $(1,0)$. The Jacobian is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2x-1 & c \end{pmatrix}$$

The linearized system at $(1,0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}$ are

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & c-\lambda \end{pmatrix} = -\lambda(c-\lambda) - 1 = \lambda^2 - c\lambda - 1$$

$$\rightarrow \lambda = \frac{c \pm \sqrt{c^2+4}}{2}$$

Therefore $(1,0)$ is an unstable saddle for all $c \geq 0$.

The linearized system at $(0,0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ -1 & c \end{pmatrix}$ are

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & c-\lambda \end{pmatrix} = -\lambda(c-\lambda) + 1 = \lambda^2 - c\lambda + 1.$$

$$\rightarrow \lambda = \frac{c \pm \sqrt{c^2-4}}{2} \qquad (*)$$

If $0 < c < 2$, then $(0,0)$ is an unstable spiral (clockwise)

If $c > 2$, then $(0,0)$ is an unstable (proper) node. (clockwise)

502 #5 cont:

Technically in the nonlinear case, should be either a node or a spiral, but the sys. is
$x' = y$
$y' = 2y - x + x^2$. Contrib. from x^2 term is very small when x is close to 0, so it should behave like
$x' = y$
$y' = 2y - x$.

If $c = 2$, then $(0,0)$ is an unstable (improper) node.

If $c = 0$, then ~~the~~ the system is given by

$$x' = y$$
$$y' = -x(1-x).$$

Since $\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x(1-x)) = 0$, this is a Hamiltonian system.
and hence all critical points are centers or saddles.

Since if $c = 0$, the associated eigenvalue in $(*)$ is ~~0~~ purely imaginary, the critical pt. $(0,0)$ must either be a center or spiral. Therefore $(0,0)$ is a center in this case.

S02 #6:

We reformulate into the notation of Evans. We want to solve

$$u_{x_1} + u_{x_1} u_{x_2} = 1 \quad , \quad u(x_1, 0) = f(x_1).$$

Let $F(p, z, x) = p_1 + p_1 p_2 - 1 = 0$. Then

$$\dot{p} = -D_x F - D_z F p \qquad D_p F = (1+p_2, p_1)$$
$$\dot{z} = D_p F \cdot p \quad \longrightarrow \quad D_x F = (0, 0)$$
$$\dot{x} = D_p F \qquad D_z F = 0.$$

The problem is noncharacteristic if $f'(x_1(0))$ is $\neq 0$. The condition that $f'(x) \neq 0 \ \forall x$ will ensure that the problem is noncharacteristic.

The initial conditions are

$$p_1(0) = f'(x_1(0)) \qquad x_1(0) = x_1(0)$$
$$p_2(0) = \frac{1}{f'(x_1(0))} - 1 \qquad x_2(0) = 0$$
$$z(0) = f(x_1(0)).$$

Then

$$p_1(s) = f'(x_1(0))$$
$$p_2(s) = \frac{1}{f'(x_1(0))} - 1$$
$$\dot{z}(s) = 2 - f'(x_1(0)) \longrightarrow z(s) = (2 - f'(x_1(0)))s + f(x_1(0))$$
$$x_1(s) = \frac{1}{f'(x_1(0))} s + x_1(0)$$
$$x_2(s) = f'(x_1(0)) s.$$

Therefore

$$z(s) = (2 - f'(x_1(0))) \frac{x_2(s)}{f'(x_1(0))} + f(x_1(0))$$

and

$$x_1(s) = \frac{x_2(s)}{f'(x_1(0))^2} + x_1(0).$$

That is,

$$u(x,y) = (2 - f'(r)) \frac{y}{f'(r)} + f(r) = \frac{2y}{f'(r)} - y + f(r)$$

where r satisfies $f'(r)^2 (x - r) = y$.

S02 #6 cont.:

Let $G(x,y,s) := f'(s)^2(x-s) - y$. Since

$$G_s(x_0,0,x_0) = -f'(s)^2 + (x-s)\cdot 2f'(s)f''(s)\Big|_{(x,y,s)=(x_0,0,x_0)}$$
$$= -f'(x_0)^2 \neq 0,$$

by the Implicit Function Theorem, one can solve $y = (f'(r))^2(x-r)$ for r in terms of (x,y) in a sufficiently small neighborhood of $(x_0,0)$ with $r(x_0,0) = x_0$.

502 #7:

Let $\vec{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ and let $A = \begin{pmatrix} 1 & -2 \\ -2 & -2 \end{pmatrix}$. We can diagonalise A in the following manner. Let $P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$. Then

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} := D$$

Then the system becomes

$$\vec{U}_t + A\vec{U}_x = 0.$$

Set $\tilde{U} := P^{-1}\vec{U}$. We have

$$P\tilde{U}_t + PDP^{-1}\vec{U}_x = 0.$$

$$\rightarrow P\tilde{U}_t + PD\tilde{U}_x = 0.$$

$$\rightarrow \tilde{U}_t + D\tilde{U}_x = 0$$

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

That is

$$\begin{pmatrix} \tilde{u}_t \\ \tilde{v}_t \end{pmatrix} = +\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \tilde{u}_x \\ \tilde{v}_x \end{pmatrix}.$$

$$\tilde{u}_t = -2\tilde{u}_x \longrightarrow \tilde{u}(x,t) = F(x-2t)$$

$$\tilde{v}_t = 3\tilde{v}_x \longrightarrow \tilde{v}(x,t) = G(x+3t).$$

Therefore

$$\vec{U} = P\tilde{U} \longrightarrow \vec{U} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$$\rightarrow u = -2F(x-2t) + G(x+3t)$$

$$v = F(x-2t) + 2G(x+3t)$$

now

We have 2 cases $x-2t>0$ and $x-2t<0$ (of course both w/ $x>0, t>0$)

We know if $x>0$,

$$\left.\begin{matrix} u(x,0) = f(x) \\ v(x,0) = 0 \end{matrix}\right| \rightarrow \begin{matrix} f(x) = -2F(x) + G(x) \\ 0 = F(x) + 2G(x). \end{matrix} \rightarrow \begin{matrix} F(x) = -\frac{2}{5}f(x) \\ G(x) = \frac{1}{5}f(x). \end{matrix}$$

if $x>0$. Key!

So if $x-2t>0, x>0, t>0$,

$$u(x,t) = \frac{4}{5}f(x-2t) + \frac{1}{5}f(x+3t)$$

$$v(x,t) = -\frac{2}{5}f(x-2t) + \frac{2}{5}f(x+3t).$$

if $x-2t>0$, $x>0$, $t>0$

~~Since $u(0,t)=0$ for $t>0$, also need $4f(-2t) = f(3t)$ for all $t>0$~~

§02 #7 cont.

If $x-2t<0$, $x>0$, $t>0$, then we use the condition that $u(0,t)=0$ for $t>0$. We have

$$0 = -2F(-2t) + G(3t) \quad \text{for } t>0.$$

Since $F(x) = -\frac{2}{5}f(x)$, $G(x) = \frac{1}{5}f(x)$ for $x>0$,

$$0 = -2F(-2t) + \frac{1}{5}f(3t) \quad \text{for all } t>0.$$

$$F(-2t) = \frac{1}{10}f(3t) \quad \text{for all } t>0.$$

Since $x-2t<0$, $\frac{x-2t}{-2}>0$ and hence

$$F(x-2t) = \frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right).$$

Therefore

$$\left.\begin{aligned} u(x,t) &= -\frac{1}{5}f\left(-\frac{3}{2}(x-2t)\right) + \frac{1}{5}f(x+3t) \\ v(x,t) &= \frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right) + \frac{2}{5}f(x+3t) \end{aligned}\right\} \text{ if } x-2t<0,\ x>0,\ t>0.$$

Thus the solution is

$$u(x,t) = \begin{cases} \frac{4}{5}f(x-2t) + \frac{1}{5}f(x+3t) & \text{if } x-2t>0,\ x>0,\ t>0 \\ -\frac{1}{5}f\left(-\frac{3}{2}(x-2t)\right) + \frac{1}{5}f(x+3t) & \text{if } x-2t<0,\ x>0,\ t>0. \end{cases}$$

$$v(x,t) = \begin{cases} -\frac{2}{5}f(x-2t) + \frac{2}{5}f(x+3t) & \text{if } x-2t>0,\ x>0,\ t>0 \\ \frac{1}{10}f\left(-\frac{3}{2}(x-2t)\right) + \frac{2}{5}f(x+3t) & \text{if } x-2t<0,\ x>0,\ t>0. \end{cases}$$

Note that u, v are both diff. when $x-2t=0$ since $f(x)$ is smooth and vanishes in a neighborhood of $x=0$.

S02 #8:

a) Let $\mathcal{Y} = \{u \in C^2(\Omega), u \neq 0, \frac{\partial u}{\partial \nu} + au = 0 \text{ on } \partial\Omega\}$. We claim that the smallest eigenvalue is given by

$$m := \min_{u \in \mathcal{Y}} \frac{\int_\Omega |\nabla u|^2 dx + a\int_{\partial\Omega} u^2 d\sigma}{\int_\Omega u^2 dx}.$$

Let u be the function in $\mathcal{Y}$ associated to m. Let v be an arbitrary elt. of $\mathcal{Y}$. Let

$$f(u+\varepsilon v) = \frac{\int_\Omega |\nabla(u+\varepsilon v)|^2 dx + a\int_{\partial\Omega} (u+\varepsilon v)^2 d\sigma}{\int_\Omega (u+\varepsilon v)^2 dx}.$$

Since $\frac{d}{d\varepsilon} f(u+\varepsilon v)\Big|_{\varepsilon=0} = 0$, by a similar calculation as in S04 #7, we must have

$$(*) \quad \left(\int_\Omega u^2 dx\right)\left(\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} auv\, d\sigma\right)$$
$$= \left(\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u^2 d\sigma\right)\left(\int_\Omega uv\, dx\right).$$

Since

$$\int_\Omega \nabla u \cdot \nabla v\, dx = -\int_\Omega v \Delta u\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v\, d\sigma$$

and

$$\int_\Omega \nabla u \cdot \nabla u\, dx = -\int_\Omega u \Delta u\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u\, d\sigma,$$

combining this with $(*)$, and the fact that $\frac{\partial u}{\partial \nu} + au = 0$, on $\partial\Omega$, we have

$$\left(\int_\Omega u^2 dx\right)\left(\int_\Omega v\Delta u\, dx\right) = \left(\int_\Omega u\Delta u\, dx\right)\left(\int_\Omega uv\, dx\right)$$

Therefore with $\alpha := \int_\Omega u^2 dx$, $\beta := \int_\Omega u\Delta u\, dx$, we have

$$\int_\Omega (\alpha\Delta u - \beta u)v\, dx = 0 \quad \forall v \in \mathcal{Y}.$$

Thus

$$\Delta u = \frac{\beta}{\alpha} u = \frac{\int_\Omega u\Delta u\, dx}{\int_\Omega u^2 dx} u = -\frac{\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} au^2 d\sigma}{\int_\Omega u^2 dx} u.$$

$$\longrightarrow \quad -\Delta u = mu.$$

S02 #8 cont:

Thus m is an eigenvalue of $-\Delta u$.

Now we claim that m is the smallest eigenvalue of $-\Delta u$.

Let $\tilde{\lambda}$ be an eigenvalue of $\cancel{\Delta u \text{ w}}$ $-\Delta$ with eigenfunction v. Then

$$m \leq \frac{\int_\Omega |\nabla v|^2 dx + a\int_{\partial\Omega} v^2 d\sigma}{\int_\Omega v^2 dx} = \frac{-\int_\Omega v\Delta v\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v\, d\sigma + a\int_{\partial\Omega} v^2 d\sigma}{\int_\Omega v^2 dx}$$

$$= \frac{\tilde{\lambda}\int_\Omega v^2 dx}{\int_\Omega v^2 dx} = \tilde{\lambda}.$$

Therefore m is the smallest eigenvalue.

b). Let $a>0$

By Hopf's Lemma, $\exists\, x_0 \in \partial\Omega$ s.t. $u(x_0) = \max_{\bar\Omega} u(x)$ and $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Then as

$$\frac{\partial u}{\partial \nu}(x_0) + a\, u(x_0) = g(x_0),$$

$$\longrightarrow \quad u(x_0) < \frac{1}{a} g(x_0) \leq \cancel{\max_{\partial\Omega}\left\{\frac{1}{a} g(x), 0\right\}} \; \frac{1}{a}\max_{\partial\Omega} |g|$$

and hence

$$\max_{\bar\Omega} u(x) \leq \cancel{\max_{\partial\Omega}\left\{\frac{1}{a} g(x), 0\right\}} \,.\, \frac{1}{a}\max_{\partial\Omega} |g|.$$

Let $v := -u$. Then

$$-\Delta v + k^2 v = 0 \text{ in } \Omega$$

$$\frac{\partial v}{\partial \nu} + a v = -g \text{ on } \partial\Omega.$$

So

$$\max_{\bar\Omega} v(x) \leq \cancel{\max_{\partial\Omega}\left\{\frac{1}{a}(-g(x)), 0\right\}} \,.\, \frac{1}{a}\max_{\partial\Omega} |g|.$$

$$\rightarrow \quad \max_{\bar\Omega} -u(x) \leq \cancel{\max_{\partial\Omega}\left\{-\frac{1}{a} g(x), 0\right\}} \,.\, \frac{1}{a}\max_{\partial\Omega} |g|.$$

Therefore

$$\cancel{\max_{\Omega} |u| \leq \frac{1}{a}\max_{\partial\Omega} |g|}$$

$$\cancel{\min_{\Omega} u(x) \geq -\max_{\partial\Omega}\left\{\frac{1}{a} g(x), 0\right\}}$$

$$\min_{\bar\Omega} u(x) \geq -\frac{1}{a}\max_{\partial\Omega} |g|.$$

This implies

$$\max_{\bar\Omega} |u(x)| \leq \frac{1}{a}\max_{\partial\Omega} |g|.$$

W02 #1:

The homogeneous solutions are $\{e^x, e^{-x}\}$. Then

$$G(x,\xi) = \begin{cases} ae^x + be^{-x} & 0<x<\xi \\ ce^x + de^{-x} & \xi < x < 1 \end{cases}$$

We want:

$$G(0,\xi)=0 \longrightarrow a+b=0.$$
$$G(1,\xi)=0 \longrightarrow ce+de^{-1}=0. \longrightarrow ce^2+d=0.$$
$$G \text{ cont. } @ x=\xi \longrightarrow ae^{\xi}+be^{-\xi} = ce^{\xi}+de^{-\xi}.$$
$$G'(\xi^+,\xi)-G'(\xi^-,\xi)=1 \longrightarrow ce^{\xi}-de^{-\xi}-ae^{\xi}+be^{-\xi}=1.$$

$$a+be^{-2\xi} = c+de^{-2\xi}. \qquad a-ae^{-2\xi}=c-ce^{2-2\xi}$$
$$c-de^{-2\xi}-a+be^{-2\xi}=e^{-\xi} \quad \rightarrow \quad c+ce^{2-2\xi}-a-ae^{-2\xi}=e^{-\xi}.$$

$$\rightarrow a = c\cdot\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}$$

$$c(1+e^{2-2\xi})-a(1+e^{-2\xi})=e^{-\xi}.$$
$$c(1+e^{2-2\xi})-c\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})=e^{-\xi}.$$
$$c\left[1+e^{2-2\xi}-\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})\right]=e^{-\xi}.$$

So

$$c=e^{-\xi}\left[1+e^{2-2\xi}-\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})\right]^{-1}$$
$$a=c\cdot\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}$$
$$b=-a$$
$$d=-ce^2.$$

and so a solution to $Lu=f$, $u(0)=0$, $u(1)=0$ is

$$u(x)=\int_0^1 G(x,\xi)f(\xi)\,d\xi.$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

W02 #2:

a) Extend u by odd reflection to $[-\pi,\pi]$. Then

since $\hat{f'}(n) = in\hat{f}(n)$

$$\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{n\neq 0} |\hat{u}(n)|^2 = \sum_{n\neq 0} \left|\frac{i}{n}\widehat{u'}(n)\right|^2$$

since $\hat{u}(0) = \left(\int_{-\pi}^{\pi} u(x)\,dx\right)\frac{1}{2\pi} = 0$

$$\leq \sum_{n\neq 0} |\widehat{u'}(n)|^2 = \int_{-\pi}^{\pi} |u'(x)|^2\,dx.$$

Therefore

Wirtinger's inequality.

$$\int_0^{\pi} |u(x)|^2\,dx \leq \int_0^{\pi} |u'(x)|^2\,dx.$$

b) Suppose L had an eigenvalue ≤ 0. Let u be a corresponding eigenfunction. Then

$$0 \geq \lambda\langle u,u\rangle = \langle Lu,u\rangle = \int_0^{\pi} -u''u + q(x)u^2\,dx$$

$$= \int_0^{\pi} (u')^2 + q(x)u^2\,dx > \int_0^{\pi} (u')^2 - u^2\,dx \geq 0$$

which is a contradiction. Therefore all eigenvalues are > 0.

W02 #3 (a) The diff eq can be written as

$$x' = y$$
$$y' = \sin x$$

This system is Hamiltonian and so all stationary pts. are either centers or saddles.

(Where $\frac{\partial H}{\partial y} = y$, $-\frac{\partial H}{\partial x} = \sin x$)

here

$$H(x,y) := \frac{1}{2}y^2 + \cos x.$$

Then

$$\frac{d}{dt}H = yy' + (-\sin x)x' = y\sin x - y\sin x = 0.$$

The stationary pts are $(n\pi, 0)$ where $n \in \mathbb{Z}$.

The Jacobian is:

$$J = \begin{pmatrix} 0 & 1 \\ \cos x & 0 \end{pmatrix}$$

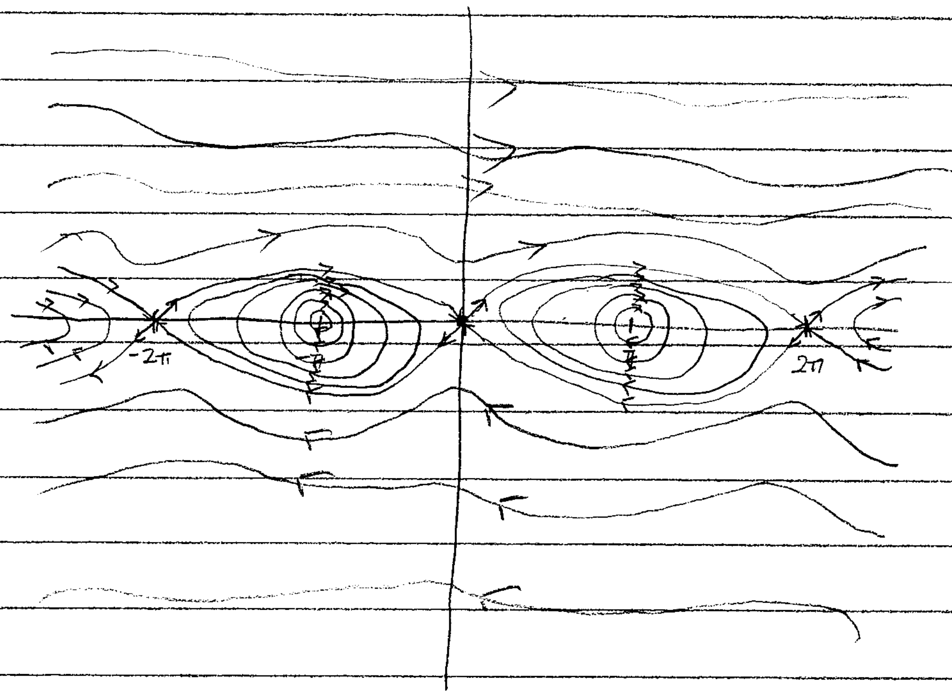
$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

If n is even, then $J(n\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has eigenvalues ± 1, so in this case $(n\pi, 0)$ is a saddle.

If n is odd, then $J(n\pi,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which has eigenvalues $\pm i$. Since the system is Hamiltonian, in this case $(n\pi, 0)$ is a center.

Eigenvalue	Eigenvector
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
-1	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

W02 #3
cont:



WQ2 #4. One way is to use the Fourier transform, but we will use the method of characteristics.

a) We have

$$F(p_1,\ldots,p_n,q,z,x_1,\ldots,x_n,t) = q + \sum_{k=1}^{n} a_k(t)p_k + a_0(t)z.$$

With $\vec{p} = (p_1,\ldots,p_n,q)$, $\vec{x} = (x_1,\ldots,x_n,t)$, we have

$$D_{\vec{p}}F = (a_1(t),\ \ldots,\ a_n(t),\ 1)$$
$$D_{\vec{x}}F = \left(0,\ 0,\ \ldots,\ 0,\ \sum_{k=1}^{n} a_k'(t)p_k + a_0'(t)z\right)$$
$$D_zF = a_0(t).$$

Then

$$\dot{z} = D_{\vec{p}}F\cdot\vec{p} = -a_0(t)z.$$
$$\dot{\vec{x}} = D_{\vec{p}}F$$

and so

$$\dot{z} = -a_0(t)z. \qquad z(0) = f(x_0)$$
$$\dot{t}(s) = 1 \qquad t(0) = 0.$$
$$\dot{x}_1(s) = a_1(s) \qquad x_1(0) = (x_1)_0$$
$$\vdots \qquad\qquad \vdots$$
$$\dot{x}_n(s) = a_n(s) \qquad x_n(0) = (x_n)_0.$$

$$\longrightarrow t(s) = s.$$
$$\dot{z} = -a_0(s)z \longrightarrow z(s) = f(x_0)\exp\left(\int_0^t -a_0(s)\,ds\right).$$
$$x_k(s) = \int_0^t a_k(s)\,ds + (x_k)_0.$$

Thus

$$u(x,t) = f\left(x_1 - \int_0^t a_1(s)ds,\ \ldots,\ x_n - \int_0^t a_n(s)ds\right)\exp\left(-\int_0^t a_0(s)ds\right)$$
$$= f\left(x_k - \int_0^t a_k(s)\,ds\right)\exp\left(-\int_0^t a_0(s)\,ds\right).$$

W02 #4 b) cont:

We use ~~@~~ Duhamel's Principle. For fixed s, we first solve for $u(x,t;s)$ where $u(x,t;s)$ satisfies

$$u_t(\cdot;s) + \sum_{k=1}^{n} a_k(t) u_{x_k}(\cdot;s) + a_0(t) u(\cdot;s) = 0 \quad \text{in } \mathbb{R}^n \times (s,\infty)$$
$$u(\cdot;s) = f(\cdot;s) \quad \text{in } \mathbb{R}^n \times \{t=s\}$$

~~Then By the~~

By part a),

$$u(x,t;s) = f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right).$$

Thus

$$u(x,t) = \int_0^t f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

We check that u does indeed solve the PDE in b). We have $u(x,0)=0$. We have

$$u_t = f(x_k) + \int_0^t \nabla f\left(x_k - \int_s^t a_k(y)\,dy\right) \cdot (-a_k(t)) \exp\left(-\int_s^t a_0(y)\,dy\right) + f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right)[-a_0(t)]\,dy.$$

$$a_k u_{x_k} = a_k(t) \int_0^t f_{x_k}\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

$$a_0 u = a_0(t) \int_0^t f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

and so

$$u_t + \sum_k a_k u_{x_k} + a_0 u = f.$$

W02#5: We have

$$u\Delta u+\sum_{k=1}^{n}\alpha_k u\frac{\partial u}{\partial x_k}-u^4=0 \quad \text{in } \Omega$$

$$\int_\Omega u\Delta u\,dx+\sum_{k=1}^{n}\alpha_k\int_\Omega u\frac{\partial u}{\partial x_k}\,dx-\int_\Omega u^4\,dx=0.$$

$$-\int_\Omega |Du|^2\,dx+\sum_{k=1}^{n}\alpha_k\int_\Omega u\frac{\partial u}{\partial x_k}\,dx-\int_\Omega u^4\,dx=0.$$

Since $u=0$ on $\partial\Omega$,

$$\int_\Omega u\frac{\partial u}{\partial x_k}\,dx=-\int_\Omega\frac{\partial u}{\partial x_k}u\,dx$$

$$\rightarrow \int_\Omega u\frac{\partial u}{\partial x_k}\,dx=0.$$

Thus

$$-\int_\Omega |Du|^2\,dx-\int_\Omega u^4\,dx=0.$$

$$\rightarrow u^4=0,\ |Du|^2=0 \text{ on } \Omega$$

$$\rightarrow u=0.$$

W02 #6:

We have $F(p,q,z,x,z) = q + z^2 p$ and

$$\dot{t}(s) = 1 \qquad t(0) = 0$$
$$\dot{x}(s) = z^2 \qquad x(0) = x_0$$
$$\dot{z}(s) = 0 \qquad z(0) = 2 + x_0.$$

$$\longrightarrow \quad t(s) = s$$
$$z(s) = 2 + x_0.$$
$$x(s) = (2+x_0)^2 s + x_0.$$

$$\longrightarrow \quad x = (4 + 4x_0 + x_0^2)t + x_0$$
$$x = t x_0^2 + (4t+1)x_0 + 4t.$$
$$x_0 = \frac{-(4t+1) \pm \sqrt{(4t+1)^2 - 4t(4t-x)}}{2t}.$$

$$\longrightarrow \quad u(x,t) = 2 + \frac{-(4t+1) \pm \sqrt{8t + 1 + 4tx}}{2t}$$
$$= -\frac{1}{2t} \pm \frac{\sqrt{1 + 4t(x+2)}}{2t} \qquad (*)$$

As $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2/2)$, for $(*)$ to satisfy $u(x,0) = 2+x$, we need

$$u(x,t) = -\frac{1}{2t} + \frac{\sqrt{1+4t(x+2)}}{2t}.$$

W02 #7: We have

$$\Delta u = f \longrightarrow -4\pi^2|\xi|^2\hat{u}(\xi) = \hat{f}(\xi)$$
$$\hat{u}(\xi) = -\frac{1}{4\pi^2}\frac{\hat{f}(\xi)}{|\xi|^2}.$$

Then

$$u = -\frac{1}{4\pi^2}\left[\frac{\hat{f}(\xi)}{|\xi|^2}\right]^{\vee}.$$

a) To show $u \in L^2(\mathbb{R}^n)$ if $n>4$, it suffices to show (by Parseval) $\hat{f}(\xi)/|\xi|^2$ is in $L^2(\mathbb{R}^n)$ for $n>4$. We have

$$\int_{\mathbb{R}^n}\frac{|\hat{f}(\xi)|^2}{|\xi|^4}d\xi = \int_{|\xi|\le 1}\frac{|\hat{f}(\xi)|^2}{|\xi|^4}d\xi + \int_{|\xi|>1}\frac{|\hat{f}(\xi)|^2}{|\xi|^4}d\xi.$$

$$\le \int_{|\xi|\le 1}\frac{|\hat{f}(\xi)|^2}{|\xi|^4}d\xi + \int_{|\xi|>1}|\hat{f}(\xi)|^2 d\xi$$

$$\le \int_{|\xi|\le 1}\frac{|\hat{f}(\xi)|^2}{|\xi|^4}d\xi + \|f\|_{L^2}^2$$

$$\le \sup_{|\xi|\le 1}|\hat{f}(\xi)|^2\int_{S^{n-1}}\int_0^1\frac{1}{r^4}r^{n-1}\,dr\,d\sigma + \|f\|_{L^2}^2.$$

$$\le \sup_{|\xi|\le 1}|\hat{f}(\xi)|^2\int_{S^{n-1}}d\sigma\int_0^1 r^{n-5}\,dr + \|f\|_{L^2}^2 \qquad (*)$$

Since $n>4$, $\int_0^1 r^{n-5}dr = \frac{1}{n-4}$ and since $\hat{f}$ is continuous (if f is L^1, then $\hat{f}$ is unif. cont.), it follows that $(*)$ is $<\infty$ and hence there is a solution of the PDE belonging to $L^2(\mathbb{R}^n)$ if $n>4$.

b) We mimic the same proof as in (a). Since $\int_{\mathbb{R}^n} f(x)\,dx = 0$, $\hat{f}(0)=0$. ~~Since~~ Expand $\hat{f}$ as a power series ~~converging~~

$$\hat{f}(\xi) = \hat{f}(0) + (\hat{f})'(0)\xi + O(\xi^2)$$

W02 #7 cont:

valid for $|\xi| \le \delta$ where δ is the radius of convergence.

Since $\hat{f}(0) = 0$,

$$\hat{f}(\xi) = (\hat{f})'(0)\,\xi + O(\xi^2)$$
$$= \xi\,((\hat{f}')(0) + O(\xi)).$$

Then

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi = \int_{|\xi|\le\delta} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi + \int_{|\xi|>\delta} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi$$
$$\le \int_{|\xi|\le\delta} \frac{|(\hat{f}')(0) + O(\xi)|}{|\xi|^2}\,d\xi + \frac{1}{\delta^4}\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi.$$
$$\le \sup_{|\xi|\le\delta} |(\hat{f}')(0) + O(\xi)| \int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi + \delta^{-4}\|f\|_{L^2}^2.$$

Thus $\hat{f}/|\xi|^2 \in L^2(\mathbb{R}^n)$ iff $\int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi < \infty$.

For $n > 2$,

$$\int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi = \int_{S^{n-1}}\int_0^\delta \frac{1}{r^2} r^{n-1}\,dr\,d\sigma$$
$$= \int_{S^{n-1}} d\sigma \int_0^\delta r^{n-3}\,dr < \infty$$

since $n > 2$.

Therefore if $\int_{\mathbb{R}^n} f\,dx = 0$, there is a solution belonging to $L^2(\mathbb{R}^n)$ if $n > 2$.

WD2 #8

a) We use Duhamel's Principle. ~~We f~~ For fixed s, we first solve for $u(x,t;s)$ where

$$u_{tt}(\cdot;s) - u_{xx}(\cdot;s) = 0 \quad \text{in } \mathbb{R}\times(s,\infty)$$
$$u(\cdot;s) = 0, \quad u_t(\cdot;s) = \overset{f(\cdot;s)}{\cancel{f(x,s)}} \quad \text{in } \mathbb{R}\times\{t=s\}.$$

Then

$$u(x,t) = \int_0^t u(x,t;s)\,ds.$$

[Solve $\quad u_{tt} - u_{xx} = 0 \quad$ in $\mathbb{R}\times(0,\infty)$

$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \quad \text{in } \mathbb{R}\times\{t=0\}.$$

Guess $u(x,t) = F(x+t) + G(x-t)$. We have

$$f(x) = F(x) + G(x) \to f'(x) = F'(x) + G'(x)$$
$$g(x) = F'(x) - G'(x)$$

$$\to \frac{1}{2}(f'(x) + g(x)) = F'(x) \qquad \to \begin{cases} F'(x) = \frac{1}{2}(f'(x) + g(x)) \\ G'(x) = \frac{1}{2}(f'(x) - g(x)). \end{cases}$$
$$f'(x) - \frac{1}{2}f'(x) - \frac{1}{2}g(x) = G'(x)$$

$$\to \quad F(x) = \frac{1}{2}f(x) + \frac{1}{2}\int_0^{x} g(s)\,ds.$$
$$G(x) = \frac{1}{2}f(x) - \frac{1}{2}\int_0^{x} g(s)\,ds.$$

$$\begin{aligned} u(x,t) &= F(x+t) + G(x-t) \\ &= \frac{1}{2}f(x+t) + \frac{1}{2}\int_0^{x+t} g(s)\,ds + \frac{1}{2}f(x-t) - \frac{1}{2}\int_0^{x-t} g(s)\,ds \\ &= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \end{aligned}$$]

Thus

$$u(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy$$

and

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds.$$

W02 #8
cont:

b) We will assume that $q(x,t)$ is uniformly bounded in $\mathbb{R}\times[0,\infty)$. From (a), we have

$$u(x,t) = -\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} q(y,s)\,u(y,s)\,dy\,ds. \qquad (*)$$

Let

φ bdd continuous, which is complete under the sup norm.

$$F(\varphi)(x,t) := -\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} q(y,s)\,\varphi(y,s)\,dy\,ds.$$

Then

$$|F(\varphi)| \le \frac{1}{2}t^2\|q\|_\infty\|\varphi\|_\infty. \qquad (*)$$

We now prove uniqueness in a small time interval $[0,T]$. By $(*)$, we have

$$\|F(\varphi)\|_\infty \le \frac{1}{2}T^2\|q\|_\infty\|\varphi\|_\infty$$

Choose T s.t. $\frac{1}{2}T^2\|q\|_\infty < \frac{1}{100}$. Then

$$\|F(\varphi)\|_\infty \le \frac{1}{100}\|\varphi\|_\infty.$$

~~Let $u_1 = 1$, and $u_n = \underbrace{F\circ F\circ\cdots\circ F}_{n\text{ times}}(1)$, we have~~

$$\cancel{\|u_{n+1}-u_n\|_\infty = \|F(u_n)-F(u_{n-1})\|_\infty \le \frac{1}{100}\|u_n-u_{n-1}\|_\infty}$$

~~Therefore the $\{u_n\}$ converge to a~~

Let $u_1 = 1$ and $u_n = F^n(u_1) = \underbrace{F\circ F\circ\cdots\circ F}_{n\text{ times}}(u_1)$. Then

$$\|F(u_n)-F(u_{n-1})\|_\infty \le \frac{1}{100}\|u_n-u_{n-1}\|_\infty$$

W02 #8 cont

~~and hence F is a contraction~~

Thus for any bounded continuous φ_1, φ_2,

$$\|F(\varphi_1) - F(\varphi_2)\|_\infty \leq \frac{1}{100}\|\varphi_1 - \varphi_2\|_\infty$$

which implies F is a contraction mapping. ~~Since a~~ and hence has a unique fixed point. ~~Fixed points~~ A fixed point of F satisfies (+) and hence is a solution to the PDE.

Thus is in the time interval $[0, T]$, we have shown a solution exists and is unique.

Now make a change of variable $t \mapsto t - T/2$. Since q is uniformly bounded, the above proof shows existence and uniqueness of a solution to the original PDE in the time interval $[T/2, 3T/2]$. In the original PDE making a change of variables $t \mapsto t - T$ and since q is uniformly bdd, ~~we can~~ the above proof shows existence and uniqueness of a solution to the original PDE in the time interval $[T, 2T]$. Continuing this, we can shows uniqueness of the solution for all time.

W02 #9b:

We have

$$u(\xi) = \int_D \delta(x-\xi)\,u(x)\,dx. \qquad \begin{aligned} x&=(x_1,x_2)\\ \xi&=(\xi_1,\xi_2)\end{aligned}$$

$$= \int_D +\Delta G(x-\xi)\,u(x)\,dx$$

$$= +\left[\int_D \Delta G(x-\xi)\,u(x)\,dx\right]$$

$$= +\left[\int_{\partial D} \frac{\partial G}{\partial \nu}(x-\xi)\,u(x) - \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma + \int_D G(x-\xi)\,\Delta u(x)\,dx\right]$$

$$= +\left[\int_{\{x_1=0\}} \frac{\partial G}{\partial \nu}(x-\xi)\,u(x)\,d\sigma + \cancel{\int_{\{x_2=0\}} \frac{\partial G}{\partial \nu}(x-\xi)\,u(x)\,d\sigma} - \cancel{\int_{\{x_1=0\}} \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma} - \int_{\{x_2=0\}} \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma + \int_D G(x-\xi)\,f(x)\,dx\right]$$

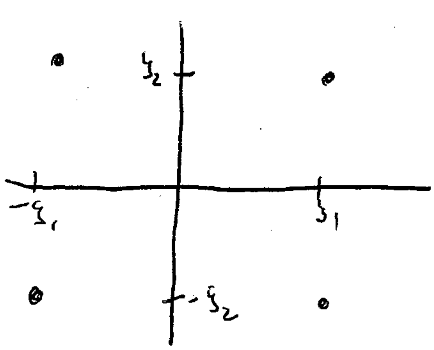
$$= +\left[\int_{\{x_1=0\}} \frac{\partial G}{\partial \nu}(x-\xi)\,u(x)\,d\sigma - \int_{\{x_2=0\}} \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma\right] + \int_D G(x-\xi)\,f(x)\,dx.$$

Need:

$$\frac{\partial^2}{\partial x_1^2}G + \frac{\partial^2}{\partial x_2^2}G = 0.$$

$$G((x_1,0),\xi) = 0 \qquad x_1, x_2 > 0.$$

$$\frac{\partial G}{\partial x_1}((0,x_2),\xi) = 0.$$



$$G = \frac{1}{2\pi}\log|x-\xi| + \frac{A}{2\pi}\log\left((x_1+\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}$$
$$+ \frac{B}{2\pi}\log\left((x_1+\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$
$$+ \frac{C}{2\pi}\log\left((x_1-\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}.$$

W02 #9b cont:

$$G((x_1,0),\xi) = \frac{1}{2\pi}\log\left((x_1-\xi_1)^2+\xi_2^2\right)^{1/2}$$
$$+\frac{A}{2\pi}\log\left((x_1+\xi_1)^2+\xi_2^2\right)^{1/2} + \frac{B}{2\pi}\log\left((x_1+\xi_1)^2+\xi_2^2\right)^{1/2}$$
$$+\frac{C}{2\pi}\log\left((x_1-\xi_1)^2+\xi_2^2\right)^{1/2}$$

$$\longrightarrow \quad A=-B, \quad C=-1.$$

Note

$$\frac{\partial}{\partial x_1}\log\left((x_1-\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$
$$=\frac{1}{2}\frac{\partial}{\partial x_1}\log\left((x_1-\xi_1)^2+(x_2-\xi_2)^2\right).$$
$$=\frac{1}{2}\frac{2(x_1-\xi_1)}{(x_1-\xi_1)^2+(x_2-\xi_2)^2} = \frac{x_1-\xi_1}{|x-\xi|^2}.$$

So

$$\frac{\partial G}{\partial x_1}((0,x_2),\xi) = \frac{1}{2\pi}\cdot\frac{0-\xi_1}{(0-\xi_1)^2+(x_2-\xi_2)^2}$$
$$+\frac{A}{2\pi}\cdot\frac{0+\xi_1}{(0+\xi_1)^2+(x_2+\xi_2)^2} + \frac{B}{2\pi}\cdot\frac{0+\xi_1}{(0+\xi_1)^2+(x_2-\xi_2)^2} + \frac{C}{2\pi}\cdot\frac{0-\xi_1}{(0-\xi_1)^2+(x_2+\xi_2)^2}.$$

$$\longrightarrow -1+B=0.$$

$$A-C=0. \longrightarrow \begin{array}{l} B=1 \\ A=C=-1 \end{array}$$

Thus

$$G(x,\xi) = \frac{1}{2\pi}\log|x-\xi| + \frac{1}{2\pi}\log\left((x_1+\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}$$
$$+\frac{1}{2\pi}\log\left((x_1+\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$
$$-\frac{1}{2\pi}\log\left((x_1-\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}.$$

Pbl #1

We want to solve

$$u_{x_2} + u u_{x_1} = 3u$$
$$u(x_1, 0) = u_0(x_1)$$

Let

$$F(p, z, x) = p_2 - z p_1 - 3z = 0$$

Then

$$D_p F = (-z, 1)$$
$$\cancel{D_z} \; D_z F = -p_1$$
$$D_x F = 0.$$

$$\longrightarrow \quad \dot{p} = -D_x F - D_z F\, p = p_1 (p_1, p_2) = (p_1^2, p_1 p_2)$$

$$\dot{z} = D_p F \cdot p = -z p_1 + p_2 = 3z$$
$$\dot{x} = D_p F = (-z, 1).$$

$$x_1(0) = x_1(0)$$
$$x_2(0) = 0$$
$$z(0) = u_0(x_1(0)).$$

$$\longrightarrow \quad \dot{z} = 3z \longrightarrow z(s) = z(0) e^{3s}$$
$$\cancel{\dot{x}_1(s) = -z(0) e^{3s} \to x_1(s) = -z(0) \tfrac{1}{3} e^{3s}}$$
$$\cancel{\dot{x} = -z \to \ddot{x} = \dot{z} \to}$$
$$\dot{x}_2(s) = 1 \longrightarrow x_2(s) = s.$$

Since $\dot{x}_1(s) = -z(s) = -z(0) e^{3s} = -u_0(x_1(0)) e^{3s}$

$$x_1(s) = -u_0(x_1(0)) \tfrac{1}{3} e^{3s} + C$$

$$\cancel{\Longrightarrow \frac{3x_1(0)}{-u_0(x_1(0))} = C} \quad \longrightarrow \quad x_1(0) = -u_0(x_1(0)) \tfrac{1}{3} + C$$
$$C = x_1(0) + \tfrac{1}{3} u_0(x_1(0))$$

$$\longrightarrow \quad x_1(s) = -u_0(x_1(0)) \tfrac{1}{3} e^{3s} + x_1(0) + \tfrac{1}{3} u_0(x_1(0))$$

F01 #1 cont.

If we could write
$$x_1(0) = f(x_1(s), s)$$
for some f, then

$$z(s) = -u_0(x_1(0))e^{3s}$$
$$\cancel{= -u_0(f(x_1(s)))}$$
$$= -u_0(f(x_1(s), s))e^{3s}$$
$$= -u_0(f(x_1(s), x_2(s)))e^{3x_2(s)}.$$

Thus
$$u(x,t) = -u_0(f(x,t))e^{3t}$$
where $f(x,t)$ solves
$$x = -u_0(f)\cdot\frac{1}{3}e^{3t} + f + \frac{1}{3}u_0(f)$$

F01 #2: Notice that
$$Lu = \frac{1}{e^{2x}}[e^{2x}u']' + \alpha(x)u.$$
Thus L is a Sturm-Liouville operator ~~with~~ thus is self-adjoint wrt the inner product $\langle f,g\rangle = \int_0^1 f(x)g(x)e^{2x}\,dx$.

a) We have
$$\int_0^1 (Lu)v e^{2x}\,dx - \int_0^1 u(Lv)e^{2x}\,dx$$
$$= \int_0^1 [e^{2x}u']'v + \alpha uve^{2x} - u[e^{2x}v']' - \alpha uve^{2x}\,dx$$
$$= \int_0^1 [e^{2x}u']'v - u[e^{2x}v']'\,dx$$
$$= e^{2x}u'v\Big]_{x=0}^{1} - \int_0^1 e^{2x}u'v' - e^{2x}v'u\Big]_{x=0}^{1} + \int_0^1 e^{2x}u'v'\,dx$$
$$= 0.$$
Therefore take $\phi(x) = e^{2x}$.

in the sense of $Lu = \lambda u$.

b) Suppose all eigenvalues of L are ≤ 0. ~~Then~~ Then the least negative eigenvalue is given by
$$\max_{u} \frac{\langle u, Lu\rangle}{\langle u,u\rangle} \leq 0. \quad (\text{since } \overset{p(x)=w(x)}{e^{2x}} > 0) \quad (*)$$
where u ranges over all functions w/ $\int_0^1 |u(x)|^2 e^{2x}\,dx < \infty$, $u'(0)=0$, $u'(1)=0$.
For a constant c,
$$\langle c, Lc\rangle = \int_0^1 c\cdot\alpha(x)c\,e^{2x}\,dx = c^2\int_0^1 \alpha(x)e^{2x}\,dx$$
$$\geq c^2\int_0^1 \alpha(x)\,dx.$$

Note that $(*)$ implies $\displaystyle\max_{\substack{u \\ u'(0)=u'(1)=0}} \langle u, Lu\rangle \leq 0.$

F01#2 cont:

But then
$$\langle c, Lc \rangle > 0$$
for some c chosen sufficiently large. Therefore L must have a positive ~~[illegible]~~ eigenvalue.

#

Note on Sturm-Liouville:

~~We say~~ The eigenvalues in most sources are given by

need $p, w > 0$

$$[p(x) y']' + q(x) y = -\lambda w(x) y$$
Then $\lambda_1 < \lambda_2 < \lambda_3 < \dots, \lambda_n \to \infty$ and the smallest eigenvalue ~~[illegible]~~ λ_1 (not $-\lambda_1$) is given by
$$\min_u \frac{\langle u, Lu \rangle}{\langle u, u \rangle} \qquad Lu = [pu']' + qu.$$
$$= -\max_u \frac{\langle u, Lu \rangle}{\langle u, u \rangle}.$$
However for our eigenvalues, we want
$$[e^{2x} u']' + \alpha(x) e^{2x} u = \mu e^{2x} u. \qquad L = [e^{2x} u']' + \alpha(x) e^{2x} u$$
Thus
$$\mu_1 > \mu_2 > \mu_3 > \dots \to -\infty$$
and the largest eigenvalue μ_1 is given by $\max_u \frac{\langle u, Lu \rangle}{\langle u, u \rangle}$.

~~Theorem: 2 Sturm =~~

Sturm-Liouville

Sturm-Liouville operator: $L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$.
S-L eigenvalue problem:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = -\lambda w(x) u. \qquad (*)$$

$\boxed{\text{Assume } p(x), w(x) > 0}$

1. Then eigenvalues are increasing $\lambda_1 < \lambda_2 < \dots, \lambda_n \to \infty$
2. Eigenfunctions corr. to different eigenvalues are orthogonal wrt to $w(x)$.
3. $f \in L^2_w[a,b]$, $f = \sum_{n\geq 1} c_n \phi_n$, $c_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}$.
4. $\lambda_1 = \min_{\substack{u \\ \text{bdy conditions}}} -\frac{\langle u, Lu\rangle}{\langle u, u\rangle}$.

comes from the $-$ sign in $(*)$.

F01 2i: Let $\phi = e^{2x}$. We note that $\phi' = 2\phi$, $\phi'' = 2\phi'$. We have

$$\langle Lu, v\rangle_\phi = \langle u'' + 2u' + \alpha u, v\rangle_\phi = \int_0^1 u''v\phi + 2u'v\phi + \alpha uv\phi\,dx$$

$$= \int_0^1 -u'(v\phi)' + 2u'v\phi + \alpha uv\phi\,dx$$

$$= -u(v\phi)'\Big]_{x=0}^1 + \int_0^1 u(v\phi)''\,dx + 2uv\phi\Big]_{x=0}^1 - \int_0^1 2u(v\phi)'\,dx$$

Since $\phi' = 2\phi$

$$= \cancel{-uv\phi'\Big]_{x=0}^1} + \int_0^1 u(v''\phi + 2v'\phi' + v\phi'')\,dx$$

$$\cancel{+ 2uv\phi\Big]_{x=0}^1} - \int_0^1 2u(v'\phi + v\phi')\,dx + \int_0^1 \alpha uv\phi\,dx$$

$$= \int_0^1 uv''\phi + uv\phi'' + 2uv'\phi' - 2uv'\phi - 2uv\phi' + \alpha uv\phi\,dx$$

$$= \int_0^1 u\phi\left(v'' + \cancel{v\cdot\frac{\phi''}{\phi}} + 2v'\cancel{\frac{\phi'}{\phi}} - \cancel{2v'} - \cancel{2v\frac{\phi'}{\phi}} + \alpha v\right)dx$$

$\frac{\phi'}{\phi} - 1 = 1$

$$= \int_0^1 u\phi(v'' + 2v' + \alpha v)\,dx = \langle u, Lv\rangle_\phi.$$

FOI #3:

i. In $\mathbb{R}^3$,

$$\Delta u = 0 \longleftrightarrow u_{rr} + \frac{2}{r} u_r = 0.$$

$$\rightarrow r^2 \left(u_{rr} + \frac{2}{r} u_r\right) = 0.$$

$$r^2 u_{rr} + 2 r u_r = 0.$$

$$(r^2 u_r)' = 0.$$

$$u_r = \frac{c_1}{r^2}.$$

Thus

$$u(r,t) = -\frac{c_1}{r} + c_2.$$

Since $u(R_1) = 1$, $u(R(t)) = 0$,

$$0 = \frac{-c_1}{R(t)} + c_2, \quad 1 = \frac{-c_1}{R_1} + c_2$$

Then

$$1 = \frac{-c_1}{R_1} + \frac{c_1}{R(t)} = c_1\left(\frac{1}{R(t)} - \frac{1}{R_1}\right)$$

$$c_1 = \frac{R(t)R_1}{R_1 - R(t)}.$$

$$c_2 = \frac{c_1}{R(t)} = \frac{R_1}{R_1 - R(t)}.$$

Thus

$$u(r,t) = -\frac{1}{r}\left(\frac{R(t)R_1}{R_1 - R(t)}\right) + \frac{R_1}{R_1 - R(t)}.$$

ii. We have

$$u_r = \frac{c_1}{r^2} \longrightarrow -u_r(r=R) = \frac{c_1}{R^2}.$$

Thus

$$\frac{dR}{dt} = \frac{c_1}{R(t)^2} = \frac{R_1}{R(t)(R_1 - R(t))}$$

w/ $R(0) = R_0$.

F01#4

The ODE can be rewritten as

$$x' = y$$
$$y' = x(1-x).$$

This system is Hamiltonian (as $\frac{\partial}{\partial x} y + \frac{\partial}{\partial y} x(1-x) = 0$) and so all stationary pts are either centers or saddles.

The conserved energy of a Hamiltonian system is found by finding an $H(x,y)$ s.t.

$$x' = \frac{\partial H}{\partial y}$$
$$y' = -\frac{\partial H}{\partial x}$$

here

$$H(x,y) := \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3.$$ $\longrightarrow$ conserved energy.

Then

$$\frac{d}{dt} H(x,y) = yy' - xx' + x^2x' = yx(1-x) - xy + x^2y = 0.$$

The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$

The stationary pts are $(0,0)$ and $(1,0)$.

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow (0,0) \text{ is a saddle}$$

eigenvalues $1, -1$
eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

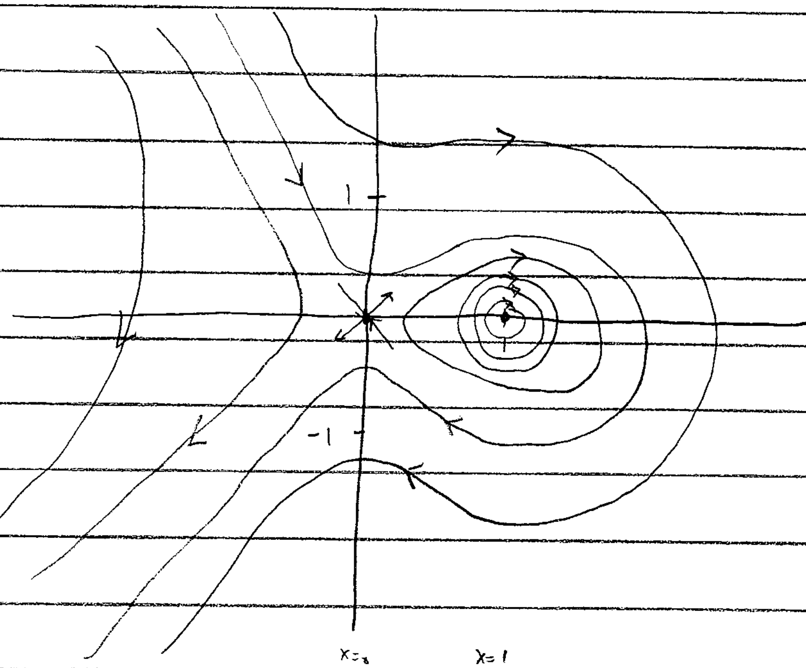
$$J(1,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \longrightarrow (1,0) \text{ is a center}$$

(since eigvl. are $\pm i$ and system is Hamiltonian)

FOI #4 cont:

Nullclines: $y = 0$ $\left(\frac{dy}{dx} = \infty \quad | \right)$

$x = 0,\ x = 1$ $\left(\frac{dy}{dx} = 0 \quad — \right)$



F01 #5:

a) The quantities $f_t + h_t + f_x$ and $g_t + h_t - g_x$ are conserved since

$$f_{tt} + h_{tt} + f_{xt} = (h^2 - fg)_t - (h^2 - fg)_t = 0.$$
$$g_{tt} + h_{tt} - g_{xt} = (h^2 - fg)_t - (h^2 - fg)_t = 0.$$

b) We have

$$f_t, g_t, h_t = -sf', -sg', -sh' \qquad \cancel{\phi(y)}$$
$$f_x, g_x, h_x = f', g', h'$$

Thus we have

$$\begin{cases} -sf' + f' = h^2 - fg \\ -sg' - g' = h^2 - fg \\ -sh' = -(h^2 - fg). \end{cases}$$

c) We have

$$(1-s)f' = -sh' \longrightarrow (1-s)f = -sh + c_1$$
$$(-1-s)g' = -sh' \longrightarrow (-1-s)g = -sh + c_2.$$

~~Thus~~ Thus

$$-sh' = -h^2 + fg$$
$$= -h^2 + \frac{1}{1-s}(-sh + c_1)\frac{1}{-1-s}(-sh + c_2)$$
$$\cancel{= -h^2 - \frac{s}{1-s}h}$$
$$= -h^2 - \frac{1}{1-s^2}\left[s^2h^2 + \tilde{c}_1 h + \tilde{c}_2\right]$$

d) Thus ODE is of the form

$$Au' + Bu^2 + Cu = D. \qquad \frac{d}{dx}\tanh = \operatorname{sech}^2$$

The solution is similar to F99 #6. $\qquad \cosh^2 - \sinh^2 = 1.$

$$\operatorname{sech}^2 = 1 - \tanh^2$$

Fol #6:

We have $F(p,q,z,x,t) = q - zp - 3z$. Then

$D_{\vec{x}} F = (0,0)$ $\quad\quad \dot{\vec{p}} = -D_{\vec{x}}F - D_zF\vec{p} = (p+3)(-z,1)$

$D_{\vec{p}} F = (-z,1) \longrightarrow \dot{z} = D_{\vec{p}}F \cdot \vec{p} = -zp+q = 3z.$

$D_z F = -p-3$ $\quad\quad \dot{\vec{x}} = D_{\vec{p}}F = (-z,1).$

Thus

$\dot{x}(s) = -z \qquad x(0) = x_0$

$\dot{t}(s) = 1 \qquad t(0) = 0.$

$\dot{p}(s) = -z(p+3) \qquad p(0) = u_0'(x_0)$

$\dot{q}(s) = p+3. \qquad q(0) = u_0(x_0)u_0'(x_0) + 3u_0(x_0).$

$\dot{z}(s) = 3z \qquad z(0) = u_0(x_0)$

We have $t(s) = s$, $z(s) = u_0(x_0)e^{3s}$,

$$\dot{x}(s) = -u_0(x_0)e^{3s} \longrightarrow x(s) = -\frac{u_0(x_0)}{3}e^{3s} + x_0 + \frac{u_0(x_0)}{3}.$$

Thus

$$u(x,t) = u_0(x_0)e^{3t}$$

where x_0 satisfies

$$x = \frac{u_0(x_0)}{3}(1-e^{3t}) + x_0.$$

FCI #7: We want solutions of the form $u(x,t) = e^{\lambda t} v(x)$.
Then

$$u_t = \lambda e^{\lambda t} v(x)$$
$$u_{xx} = e^{\lambda t} v''(x)$$

$$\rightarrow \quad u_t = u_{xx} + c(x) u$$
$$\lambda \cancel{e^{\lambda t}} v(x) = \cancel{e^{\lambda t}} v''(x) + c(x) \cancel{e^{\lambda t}} v(x)$$

$$\rightarrow \quad v''(x) = (\lambda - c(x)) v(x).$$

If $|x|>1$, $c(x)=0$. Here we want solutions of the form $ae^{-k|x|}$. Then let $\tilde{v}(x) := ae^{-k|x|}$. We have

$$\tilde{v}''(x) = \lambda \tilde{v}(x)$$
$$k^2 \cancel{a} \cancel{e^{-k|x|}} = \lambda \cancel{a} \cancel{e^{-k|x|}}$$

$$\rightarrow k = \pm\sqrt{\lambda}$$

So $v(x) = Ae^{-\sqrt{\lambda}|x|} + Be^{\sqrt{\lambda}|x|}$. Since we want $\|u\|_{L^2_x} < \infty$, $B=0$, $\rightarrow v(x) = Ae^{-\sqrt{\lambda}|x|}$
$\rightarrow u(x,t) = Ae^{-\sqrt{\lambda}|x|}e^{\lambda t}$ for $|x|>1$

If $|x|<1$, $c(x)=1$. Here we want solutions of the form $b\cos lx$. Then let $\tilde{v}(x) = b\cos lx$.

$$\rightarrow \quad \tilde{v}''(x) = (\lambda - 1)\tilde{v}(x)$$
$$-\cancel{b} l^2 \cancel{\cos lx} = (\lambda-1)\cancel{b} \cancel{\cos lx}$$
$$l = \pm\sqrt{1-\lambda}.$$

So $v(x) = C\cos(\sqrt{1-\lambda}\,x) + D\cos(\sqrt{1-\lambda}\,x)$
$= E\cos(\sqrt{1-\lambda}\,x).$

$$\rightarrow u(x,t) = B\cos(\sqrt{1-\lambda}\,x)e^{\lambda t} \text{ for } |x|<1.$$

~~Note that~~ We also want the defined u to be continuous at $x = \pm 1$.

Thus we want

$$B\cos(\sqrt{1-\lambda})\cancel{e^{\lambda t}} = Ae^{-\sqrt{\lambda}}\cancel{e^{\lambda t}}$$

~~and~~

$$\cancel{B\cos(\sqrt{1-\lambda})e^{\lambda t} = Ae}$$

$$\longrightarrow \quad B = A\frac{e^{-\sqrt{\lambda}}}{\cos(\sqrt{1-\lambda})}.$$

Note that

$$u(x,t) = \begin{cases} Ae^{-\sqrt{\lambda}|x|}e^{\lambda t} & \text{for } |x|\geq 1 \\ B\cos(\sqrt{1-\lambda}\,x)e^{\lambda t} & \text{for } |x|\leq 1 \end{cases}$$

with $B = A\dfrac{e^{-\sqrt{\lambda}}}{\cos(\sqrt{1-\lambda})}$ is continuous and is a solution and $\|u\|_{L^2_x} < \infty$ as it is radial.

Sol #1

a) Note that we are working w/ inner normals.

Let u, v be s.t.

$$\nabla \cdot \gamma(x) \nabla u = 0 \text{ in } B \qquad \nabla \cdot \gamma(x) \nabla v = 0 \text{ in } B$$
$$u = f \text{ on } \partial B \qquad v = g \text{ on } \partial B.$$

Then

$$\int_{\partial B} g\, \gamma(x) \frac{\partial f}{\partial \nu}\, d\sigma = \int_{\partial B} u\, \gamma(x) \sum_i v_{x_i} \nu^i \, d\sigma$$
$$= \sum_i \int_{\partial B} u \left(\gamma(x) v_{x_i} \right) \nu^i \, d\sigma$$
$$= \sum_i - \int_B u_{x_i} \gamma(x) v_{x_i}\, dx - \int_B u \left(\gamma(x) v_{x_i} \right)_{x_i} dx$$
$$= \cancel{\sum_i} - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx - \int_B u\, \nabla \cdot \gamma(x) \nabla v \, dx$$
$$= - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx.$$

~~We also have~~ Then

$$\int_{\partial B} f\, \gamma(x) \frac{\partial g}{\partial \nu}\, d\sigma = - \int_B \nabla v \cdot \gamma(x) \nabla u \, dx$$
$$= - \int_B \nabla u \cdot \gamma(x) \nabla v \, dx = \int_{\partial B} g\, \gamma(x) \frac{\partial f}{\partial \nu}\, d\sigma.$$

b)

$$\int_{\partial B} f\, \Lambda(f)\, d\sigma = - \int_B |\nabla f|^2 \gamma(x) \le 0.$$

since $\gamma(x) > 0$.

Sol #2:

a) The eigenvalues of $\begin{pmatrix}1&0\\5&3\end{pmatrix}$ are 1, 3 with corresponding eigenvectors $\begin{pmatrix}-2\\5\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. Thus

$$\begin{pmatrix}-2&0\\5&1\end{pmatrix}\begin{pmatrix}1&\\&3\end{pmatrix}\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}=\begin{pmatrix}1&0\\5&3\end{pmatrix}.$$

Let $w=\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}\begin{pmatrix}u_1\\u_2\end{pmatrix}$. Then

$$w_t-\begin{pmatrix}1&\\&3\end{pmatrix}w_x=0.$$

with

$$w(x,0)=\begin{pmatrix}-2&0\\5&1\end{pmatrix}^{-1}\begin{pmatrix}\exp(ixa)\\0\end{pmatrix}=\begin{pmatrix}-1/2&0\\5/2&1\end{pmatrix}\begin{pmatrix}\exp(ixa)\\0\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{2}\exp(iax)\\ 5/2\exp(iax)\end{pmatrix}.$$

Since $w=\begin{pmatrix}w_1\\w_2\end{pmatrix}$,

$$(w_1)_t-(w_1)_x=0 \qquad w_1(x,0)=-\frac{1}{2}\exp(iax),$$
$$(w_2)_t-3(w_2)_x=0 \qquad w_2(x,0)=\frac{5}{2}\exp(iax),$$

$$\longrightarrow\ w_1(x,t)=-\frac{1}{2}\exp(ia(x+t))$$
$$w_2(x,t)=\frac{5}{2}\exp(ia(x+3t)).$$

and hence

$$u=\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}-2&0\\5&1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}-2w_1\\5w_1+w_2\end{pmatrix}=\begin{pmatrix}\exp(ia(x+t))\\-\frac{5}{2}\exp(ia(x+t))+\frac{5}{2}\exp(ia(x+3t))\end{pmatrix}$$

b) We have

$$(u_1)_t-(u_1)_x=0. \qquad u_1(x,0)=f(x) \qquad (*)$$
$$(u_2)_t-5(u_1)_x-3(u_2)_x=0. \qquad u_2(x,0)=0.$$

Sol #2 cont:

Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx$. Then $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$.

So

$$(\hat{u}_1)_t - 2\pi i \xi\, \hat{u}_1 = 0 \qquad \hat{u}_1(\xi, 0) = \hat{f}(\xi)$$

$$(\hat{u}_2)_t - 2\pi i \xi \cdot 5\hat{u}_1 - 2\pi i \xi \cdot 3\hat{u}_2 = 0 \qquad \hat{u}_2(\xi, 0) = 0.$$

$$\hat{u}_t - 2\pi i \xi \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix} \hat{u} = 0 \qquad \hat{u}(\xi, 0) = \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix}.$$

$$\longrightarrow \hat{u}(\xi, t) = \exp\left(2\pi i \xi t \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}\right) \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix}$$

Then

$$u(x, t) = \left[\exp\left(2\pi i \xi t \begin{pmatrix} 1 & 0 \\ 5 & 3 \end{pmatrix}\right) \begin{pmatrix} \hat{f}(\xi) \\ 0 \end{pmatrix} \right]^{\vee} \qquad \#$$

Sol #3:

We want to solve $2u_t - u_x^2 = x^2$.

$u(x,0) = x$.

Then $F(p,q,z,x,t) = 2q - p^2 - x^2$ and

$$\dot t(s) = 2 \qquad t(0) = 0$$
$$\dot x(s) = -2p \qquad x(0) = x_0$$
$$\dot z(s) = -2p^2 + 2q \qquad z(0) = x_0$$
$$\dot p(s) = 2x \qquad p(0) = 1$$
$$\dot q(s) = 0 \qquad q(0) = \frac{1+x_0^2}{2}.$$

$\rightarrow t(s) = 2s.$

$$\ddot x(s) = -4x(s) \longrightarrow x(s) = A\cos(2s) + B\sin(2s)$$
$$x_0 = x(0) = A$$
$$\dot x(s) = -2A\sin(2s) + 2B\cos(2s)$$
$$-2p(0) = \dot x(0) = 2B \longrightarrow B = -1.$$

$\rightarrow x(s) = x_0\cos(2s) - \sin(2s).$

$$\ddot p = -4p \longrightarrow p(s) = A\cos(2s) + B\sin(2s)$$
$$p(0) = 1 \longrightarrow A = 1$$
$$\dot p(0) = 2x(0) = 2x_0$$
$$\dot p(s) = -2\sin(2s) + 2B\cos(2s)$$
$$2x_0 = 2B \longrightarrow B = x_0.$$

$\rightarrow p(s) = \cos(2s) + x_0\sin(2s).$

We have

$$\begin{aligned}\dot z(s) = -2p^2 + 2q = x^2 - p^2 &= x_0^2\cos^2 2s - 2x_0\cos 2s\sin 2s + \sin^2 2s \\ &\quad - \cos^2 2s - 2x_0\cos 2s\sin 2s - x_0^2\sin^2 2s \\ &= (x_0^2-1)\cos^2 2s - (x_0^2-1)\sin^2 2s - 4x_0\cos 2s\sin 2s \\ &= (x_0^2-1)\cos 4s - 2x_0\sin 4s.\end{aligned}$$

Sol #3 cont:

Thus

$$z(s) = \frac{1}{4}(x_0^2-1)\sin 4s + 2x_0 \frac{1}{4}\cos 4s + \frac{1}{2}x_0.$$
$$= \frac{1}{4}(x_0^2-1)\sin 4s + \frac{1}{2}x_0 \cos 4s + \frac{1}{2}x_0.$$

$$\rightarrow u(x,t) = \frac{1}{4}(x_0^2-1)\sin 2t + \frac{1}{2}x_0\cos 2t + \frac{1}{2}x_0$$

where $x_0 = \dfrac{x+\sin t}{\cos t}$.

The solution blows up in finite time. The characteristics are given by $x = x_0 \cos t - \sin t$. For any x_0, the characteristics will intersect at time $t = \pi/2$, thus the solution blows up in finite time.

$$-\Delta \phi_n = +\lambda_n \phi_n \quad \text{in } D$$
$$\phi_n = 0 \quad \text{on } \partial D$$
$$\|\phi_n\|_{L^2} = 1.$$

Sol #4:

of the Laplacian (w/ $\|\phi_n\|_{L^2}=1$)

a) Expand u in terms of eigenfunctions. We have $u = \sum_{n=1}^{\infty} a_n \phi_n$.

Then

$$\int_D u^2\,dx = \int_D \Big(\sum_{n\geq 1} a_n \phi_n\Big)^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

and

$$\int_D |\nabla u|^2\,dx = -\int_D u\Delta u\,dx = +\int_D \Big(\sum_{n=1}^{\infty} a_n\phi_n\Big)\Big(\sum_{n=1}^{\infty}\lambda_n a_n \phi_n\Big)dx$$

$$\cancel{= \int_D \sum_{n\geq 1} \lambda_n a_n^2 \phi_n}$$

$$= +\int_D \sum_{n\geq 1} \lambda_n a_n^2 \phi_n^2\,dx$$

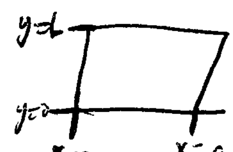
$$= +\sum_{n\geq 1} \lambda_n a_n^2.$$

Thus since $0 < \lambda_1 \leq \lambda_2 \leq \dots$,

$$\int_D u^2\,dx = \sum_{n\geq 1} a_n^2 \leq \frac{1}{\lambda_1}\sum_{n\geq 1}\lambda_n a_n^2 = \frac{1}{\lambda_1}\int_D |\nabla u|^2\,dx.$$

b) We solve

$$-\Delta u = \lambda u \quad \text{in } D$$
$$u = 0 \quad \text{on } \partial D.$$



$$u = F(x)G(y)$$

$$-F''(x)G(y) - F(x)G''(y) = \lambda F(x)G(y)$$

$F(0) = F(a) = 0$

$G(0) = G(b) = 0.$

$$\frac{-F''(x) - \lambda F(x)}{F(x)} = +\frac{G''(y)}{G(y)}$$

Sol #4 cont:

$$\frac{G''(y)}{G(y)} = -\mu^2.$$

$$G(y) = A\cos(\mu y) + B\sin(\mu y)$$

$$0 = G(0) = A \rightarrow G(y) = B\sin(\mu y)$$

$$0 = G(b) = B\sin(\mu b)$$

$$\rightarrow \mu_n = \frac{n\pi}{b}, \ n = 1, 2, \dots$$

$$-\mu_n^2 = -\left(\frac{n\pi}{b}\right)^2.$$

$$G_n(y) = \sin\left(\frac{n\pi b}{y}\right)$$

$$\frac{-F'' - \lambda F}{F} = -\left(\frac{n\pi}{b}\right)^2$$

$$\rightarrow F'' + \left(\lambda - \left(\frac{n\pi}{b}\right)^2\right)F = 0. \longrightarrow \text{we must have } \lambda - \left(\frac{n\pi}{b}\right)^2 > 0 \text{ otherwise } F = 0.$$

$$F = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right) + B\cos\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right)$$

$$0 = F(0) \longrightarrow B = 0 \longrightarrow F(x) = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,x\right)$$

$$0 = F(a) = A\sin\left(\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,a\right)$$

$$\sqrt{\lambda - \left(\tfrac{n\pi}{b}\right)^2}\,a = m\pi, \ m = 1, 2, \dots$$

$$\lambda = \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \quad m, n = 1, 2, \dots$$

Therefore the smallest λ is given by

$$\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{a}\right)^2.$$

Thus the best constant is $\dfrac{1}{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$. #

S01 #5:

We will assume that f is smooth (or at least unif. cont.).

Lemma: If $f \in L^2(\mathbb{R})$ and smooth, $\lim_{x \to \pm\infty} |f|^2 = 0$.

Pf: ~~Suppose~~ We first show the limit exists. Suppose it did not. Then $\exists\, \varepsilon_0$ and a sequence of $x_n \to \pm\infty$ with $|x_n - x_{n+1}| > 2$ s.t. $|f(x_n)| > \varepsilon_0$ $\forall n$. Since f is unif. cont, then $\exists\, \delta < 1$ s.t. if $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon_0}{2}$ $\forall x, y$. Thus if $|x - x_n| < \delta$, $|f(x)| > \frac{\varepsilon_0}{2}$. Then if $|x - x_n| < \delta$, $|f(x)|^2 > \frac{\varepsilon_0^2}{4}$. Thus

$$\int_{|x-x_n|<\delta} |f(x)|^2\,dx > \frac{\varepsilon_0^2}{4} \cdot 2\delta = \frac{1}{2}\varepsilon_0^2 \delta.$$

$$\int_{\mathbb{R}} |f|^2 \geq \sum_n \int_{|x-x_n|<\delta} |f|^2 > \sum_n \frac{1}{2}\varepsilon_0^2 \delta = \infty.$$

Therefore the limit exists.

~~If $\lim_{x \to \pm\infty} |f|^2 = L \neq 0$, fix arb. $\varepsilon > 0$, then $\forall \varepsilon > 0$ $\exists$ a sequence of $\{x_n\}$ s.t. $|f(x_n) - \sqrt{L}| < \frac{\varepsilon}{2}$. By uniform continuity of f, $\exists\, \delta > 0$ s.t. if $|x-y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{2}$. Then if $|x - x_n| < \delta$, $|f(x)\ \sqrt{L}| \leq \varepsilon$. $\to (\sqrt{L} - \varepsilon)^2 \leq |f(x)|^2$.~~

If $\lim_{x \to \pm\infty} f^2(x) = L \neq 0$, fix arb. small $\varepsilon > 0$, then $\exists$ a seq. of $\{x_n\}$ spaced w/ $|x_n - x_{n+1}| > 2$ s.t. $|f(x_n)^2 - L| < \frac{\varepsilon}{2}$. By smoothness of f^2, $\exists\, \delta > 0$ s.t. if $|x-y| < \delta$, $|f(x)^2 - f(y)^2| < \frac{\varepsilon}{2}$. Then if $|x - x_n| < \delta$,

S01 #5 cont:

$$|f(x)^2 - L| \le |f(x)^2 - f(x_n)^2| + |f(x_n)^2 - L| \le \varepsilon.$$

Thus if $|x - x_n| < \delta$, $L - \varepsilon \le |f(x)|^2$. Then

$$\int |f(x)|^2 \geq \sum_n \int_{|x-x_n|<\delta} |f(x)|^2 \, dx$$

$$\geq \sum_n (L - \varepsilon)$$

Since $L > 0$, we have a contradiction. Therefore

$$\lim_{x \to \pm\infty} |f|^2 = 0.$$

---

a) Taking the Fourier transform,

$$\hat{u}_t = -4\pi \xi^2 \hat{u}. \quad \hat{u}(\xi, 0) = \hat{f}(\xi)$$

$$\hat{u}(\xi, t) = e^{-4\pi \xi^2 t} \hat{f}(\xi).$$

Thus

$$|u(x, t)| = \left| \int_{\mathbb{R}} e^{-4\pi t \xi^2} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi. \right|$$

$$\le \left(\int_{\mathbb{R}} e^{-8\pi t \xi^2} \, d\xi \right)^{1/2} \|f\|_{L^2}. \longrightarrow 0$$

as $t \to \infty$ since

$$\int_{-\infty}^{\infty} e^{-8\pi t \xi^2} d\xi = \frac{1}{\sqrt{8\pi t}} \int_{-\infty}^{\infty} e^{-u^2} du.$$

Therefore $u \to 0$ as $t \to \infty$ uniformly in x.

Sol #5 con't:

Note that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{(x-y)^2}{4t}} \, dy .$$

and $\lim\limits_{\substack{x \to \pm\infty \\ t > 0}} u(x,t) = \lim\limits_{x \to \pm\infty} f(x) = 0$ by the previous lemma.

We have

$$u_x(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\left(\frac{x-y}{2\sqrt{t}}\right)^2} \left(-\frac{x-y}{2t}\right) dy$$

$$\cancel{= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}}$$

$$\leq \frac{1}{2\sqrt{\pi}\, t^{1/2}} \left(\int_{\mathbb{R}} |f|^2 \right)^{1/2} \left(\int_{-\infty}^{\infty} e^{-2\left(\frac{x-y}{2\sqrt{t}}\right)^2} \frac{(x-y)^2}{4t^2} \, dy \right)^{1/2} .$$

Let $u = \frac{x-y}{2\sqrt{t}}$. Then $du = -\frac{1}{2\sqrt{t}} dy$.

$$\int_{-\infty}^{\infty} e^{-2\left(\frac{x-y}{2\sqrt{t}}\right)^2} \frac{(x-y)^2}{4t^2} \, dy = \int_{-\infty}^{\infty} e^{-2u^2} \frac{(2\sqrt{t}u)^2}{4t^2} \, 2\sqrt{t} \, du$$

$$= 2 \left(\int_{-\infty}^{\infty} e^{-2u^2} u^2 \, du \right) t^{-1/2} .$$

Therefore

$$|u_x| \leq C\, t^{-3/4} \|f\|_{L^2} .$$

b) Let $E(t) := \int_{\mathbb{R}} |\rho|^2 dx$. Then

$$E'(t) = 2\int_{\mathbb{R}} \rho \rho_t \, dx = 2 \int_{\mathbb{R}} -u \rho \rho_x \, dx .$$

Sol #5 cont:

$$\underset{\substack{\text{since } u=0\\ \text{at } x=\pm\infty.}}{=} -2\left[-\int_{\mathbb{R}} (u\rho)_x\,\rho\,dx\right]$$

$$= 2\int_{\mathbb{R}} u_x\rho^2 + u\rho_x\rho\,dx.$$

$$\le 4C_f\,t^{-3/4}E(t) + 2\int_{\mathbb{R}} u\rho_x\rho\,dx.$$

$$\lesssim t^{-3/4}E(t) + \cdot\, t^{-1/4}\int_{\mathbb{R}}\rho_x\rho\,dx.$$

$$\lesssim t^{-3/4}E(t) + t^{-1/4}\left(\frac{1}{2}\rho^2\right)\Big|_{x=-\infty}^{\infty}$$

Thus if $\rho(x,t)\longrightarrow 0$ as $|x|\longrightarrow\infty$, then $E'(t)\lesssim t^{-3/4}E(t)$

~~It is not obvious that this will follow from~~ any ~~assumptions on ρ_0 b/c the characteristics~~ a~~re given by solving $\frac{dx}{dt}=u(x,t)$~~.

and hence by Gronwall,

$$E(t)\le E(0)e^{Ct^{1/4}}$$

$$\rightarrow \int_{\mathbb{R}}|\rho(x,t)|^2dx \le \left(\int_{\mathbb{R}}|\rho_0(x)|^2dx\right)e^{Ct^{1/4}}.$$

for some C depending only on f.

Assume ρ_0 is of compact support. ~~The characteristics~~ We solve $\rho_t + u\rho_x = 0$, $\rho(x,0)=\rho_0(x)$ via method of characteristics. We have

$$\dot{x} = u(x,t) \qquad x(0)=x_0$$
$$\dot{t} = 1 \qquad t(0)=0$$
$$\dot{z} = 0 \qquad z(0)=\rho_0(x_0).$$

Sol #5 cont'.

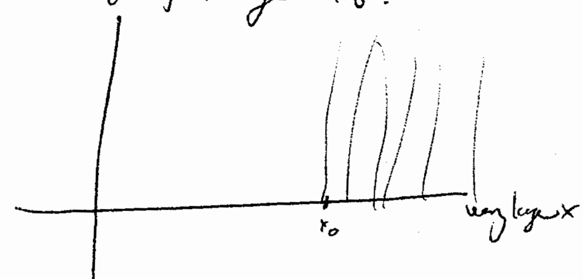
Therefore

$\rho(x,t) = \rho_0(x_0)$.

$\frac{dx}{ds} = u(x(s), s)$

$x(0) = x_0$.

When $x(s)$ is very large $u(x(s), s)$ is very close to 0. Therefore x is basically constant. Thus the characteristics are $x = x_0$ for large x_0.



Thus for very large x, $\rho(x,t) \to 0$. as $|x| \to \infty$.

SOL #6

a) Let V_1, V_2 be 2 conductor potentials, ~~then~~ and let $W := V_1 - V_2$.
Then $\Delta W = 0$ on $\mathbb{R}^3 \setminus B$, $W = 0$ on ∂B, and $W \to 0$ as $|x| \to \infty$.
Thus

$$0 = \int_{\mathbb{R}^3 \setminus B} W \Delta W = -\int_{\mathbb{R}^3 \setminus B} |\nabla W|^2 + \int_{\partial(\mathbb{R}^3 \setminus B)} W \frac{\partial W}{\partial \nu}\, d\sigma = -\int_{\mathbb{R}^3 \setminus B} |\nabla W|^2\, dx$$

so W is a constant on $\mathbb{R}^3 \setminus B$ and hence $W = 0$. Therefore
$V_1 = V_2$

b) Claim: $V(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta V(y)}{|x-y|}\, dy$

Pf: Let $u(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta V(y)}{|x-y|}\, dy$. Then $\Delta(V-u) = 0$.
Since $V - u \to 0$ as $|x| \to \infty$, $V - u$ is bounded.
Thus by Liouville's Theorem, $V - u$ is a constant. Since
$V - u \to 0$ as $|x| \to \infty$, $V = u$. #

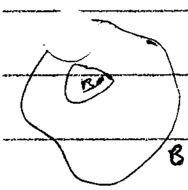
We have

$$|x|\, V(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta V(y)|x|}{|x-y|}\, dy$$

$$\to \lim_{|x| \to \infty} |x|\, V(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \Delta V(y)\, dy = -\frac{1}{4\pi}\int_B \Delta V(y)\, dy$$

$$= -\frac{1}{4\pi}\int_{\partial B} \frac{\partial V}{\partial \nu}(y)\, d\sigma(y).$$

c) Let $\Delta V' = 0$ on $\mathbb{R}^3 \setminus B'$, $\Delta V = 0$ on $\mathbb{R}^3 \setminus B$. We show $\lim_{|x| \to \infty} |x|(V(x) - V'(x)) \geq 0$.
$V' = 1$ on $\partial B'$ $\quad V = 1$ on ∂B.



By the Maximum Principle and as $V, V' \to 0$ as $|x| \to \infty$, $0 \leq V' < 1$ on $\mathbb{R}^3 \setminus B'$ and $0 \leq V < 1$ on $\mathbb{R}^3 \setminus B$. So $V' - V$ is < 0 on ∂B and since $V' - V \to 0$ as $|x| \to \infty$, by the maximum Principle, we have $\lim_{|x| \to \infty} |x|(V - V') \geq 0$.

$\hookrightarrow$ $V' - V$ is $\leq \varepsilon$ for $|x| \geq R$.
on $|x| \leq R$ and $x \notin B$, by Max. Principle, $V' - V \leq \varepsilon$.

Sol #6 see packet of problems

SOT # 7: Let $y := x - st$.

a. We have

$$f_t = -sf' \qquad g_t = -sg'$$
$$f_x = f' \qquad g_x = g'$$

$$\rightarrow \begin{cases} f_t + f_x = g^2 - f^2 \\ g_t - g_x = f^2 - g \end{cases} \longrightarrow \begin{array}{l} -sf' + f' = g^2 - f^2. \\ -sg' - g' = g^2 - f \end{array}.$$

b) When $s = 0$, the system of ODEs is:

$$\begin{cases} f' = g^2 - f^2 \\ g' = f - g^2. \end{cases} \xRightarrow{\text{rewrite as}} \begin{cases} x' = y^2 - x^2 \\ y' = x - y^2. \end{cases}$$

Stationary points:

$$\begin{array}{l} x^2 - y^2 = 0. \\ x - y^2 = 0 \end{array} \longrightarrow x^2 - x = 0 \longrightarrow \begin{array}{ll} x = 0 & , 1 \\ \downarrow & \downarrow \\ y = 0 & y = \pm 1. \end{array}$$

So there are 3 equilibrium pts $(0,0), (1,1), (1,-1)$.

$$J(x,y) = \begin{pmatrix} -2x & 2y \\ 1 & -2y \end{pmatrix}$$

Eigenvalues   Eigenvectors

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ $\longrightarrow$ system is degenerate, locally is

$$J(1,1) = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix} \qquad -2 \pm \sqrt{2}. \qquad \begin{pmatrix} \pm\sqrt{2} \\ 1 \end{pmatrix}$$

$$J(1,-1) = \begin{pmatrix} -2 & -2 \\ 1 & 2 \end{pmatrix} \qquad \pm\sqrt{2}. \qquad \cancel{\begin{pmatrix} 1 \\ \mp\frac{\sqrt{2}}{2} - 1 \end{pmatrix}} \begin{pmatrix} -2 \pm \sqrt{2} \\ 1 \end{pmatrix}$$

Soc #7 cont:

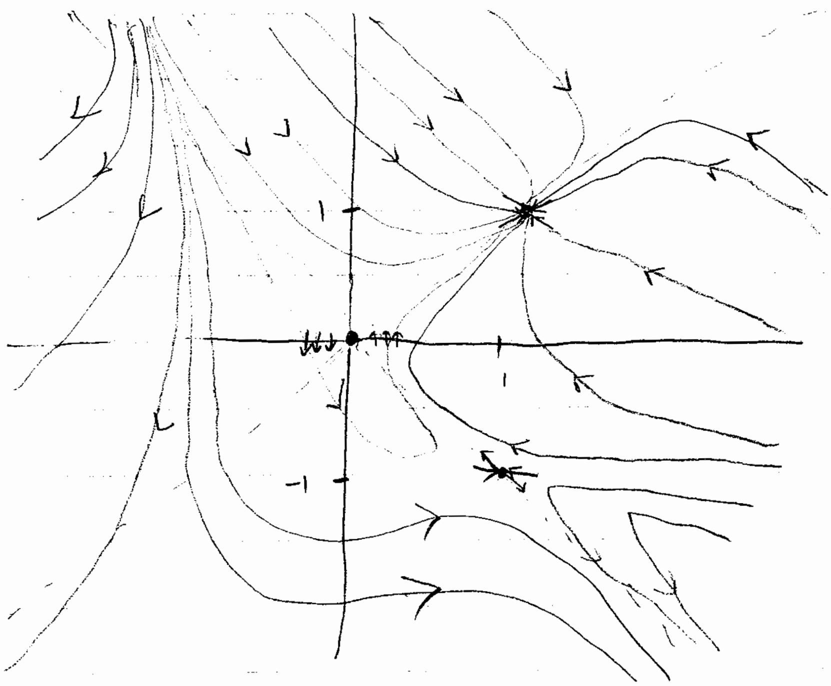
~~Therefore (0,0) is an improper~~

~~Therefore (1,1) and (1,-1) are saddles.~~ The ~~nullclin~~ nullclines are

$y = \pm x$

$y = \pm\sqrt{x}$.

$\frac{dy}{dx} = \frac{x-y^2}{y^2-x^2}$



(1, 1) is a sink node

(1, -1) is a saddle.

F'00 #1 : a) Let u_1, u_2 be 2 classical solutions of the given equation. Let $w := u_1 - u_2$. Then

$$\Delta w + a(x) w = 0, \quad x \in D$$
$$w|_S = 0 \quad x \in S.$$

~~Let~~ Therefore

$$\int_D |\nabla w|^2 dx = -\int_D w \Delta w \, dx = \int_D a(x) w^2 dx$$

$$\leq \|a\|_{L^\infty} \int_D w^2 dx \leq \|a\|_{L^\infty} \underbrace{C(D)}_{\text{Poincaré ineq.}} \int_D |\nabla w|^2 dx$$

$$\leq \frac{1}{2} \int_D |\nabla w|^2 dx$$

if we ~~choose~~ assume $\|a\|_{L^\infty} C(D) \leq \frac{1}{2}$. Thus if $|a(x)|$ was small enough, then $\int_D |\nabla w|^2 dx = 0 \to \nabla w = 0 \to w = 0$.

b. Let $H = H_0^1(D) = \{ u \in H^1(D) : u = 0 \text{ on } S \}$ and

$$B[u,v] = \int_D \nabla u \cdot \nabla v - a(x) u v \, dx, \quad u, v \in H_0^1(D).$$

Note that $H_0^1(D)$ is a Hilbert space and $B[u,v] : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$. We will use Lax-Milgram to prove the result.

① We have

$$|B[u,v]| \leq \int_D |\nabla u| |\nabla v| dx + \int_D |a(x)| |u| |v| dx$$

$$\leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|a\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2}.$$

$$\leq \alpha \|u\|_{H^1} \|v\|_{H^1}$$

for some constant $\alpha > 0$ which only depends on $\|a\|_{L^\infty}$.

② We want to show there is a constant $\beta > 0$ s.t. $\beta \|u\|_{H^1}^2 \leq B[u,u]$ $\forall u \in H_0^1$. We have

F'00 1 cont.

$$B[u,u] = \int_D |\nabla u|^2 - a(x)u^2\,dx = \|\nabla u\|_{L^2} - \int_D a(x)\,u^2\,dx.$$

$$\geq \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}\|\nabla u\|_{L^2}^2 - \|a\|_{L^\infty}\int_D u^2\,dx.$$

$$= \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}\|\nabla u\|_{L^2}^2 - \|a\|_{L^\infty}\|u\|_{L^2}^2.$$

where from Poincaré, $\exists$ C depending only on D s.t. $C\|u\|_{L^2} \le \|\nabla u\|_{L^2}$.

$$\geq \frac{1}{3}\|\nabla u\|_{L^2}^2 + \frac{2}{3}C\|u\|_{L^2}^2 - \|a\|_{L^\infty}\|u\|_{L^2}^2$$

$$= \frac{1}{3}\|\nabla u\|_{L^2}^2 + \left(\frac{2}{3}C - \|a\|_{L^\infty}\right)\|u\|_{L^2}^2.$$

~~Choose~~ If $|a(x)|$ is sufficiently small, then $\exists$ constant $\beta > 0$ independent of u s.t. $B[u,u] \geq \beta\|u\|_{H^1}^2$.

Therefore since $v \longmapsto \int_D fv\,dx$ is a bounded linear functional on $H_0^1(D)$, by Lax-Milgram, if $|a(x)|$ is sufficiently small, $\exists$ a unique $u \in H_0^1(D)$ s.t.

$$\int_D \nabla u\cdot\nabla v - a(x)uv\,dx = -\int_D fv\,dx$$

$\forall\, v \in H_0^1(D)$. Since

$$\int_D \nabla u\cdot\nabla v\,dx = -\int_D v\Delta u\,dx + \int_S \frac{\partial u}{\partial\nu}v\,d\sigma$$

$$= -\int_D v\Delta u\,dx,$$

it follows that

$$\int_D (\Delta u + a(x)u)v\,dx = \int_D fv\,dx.\ \forall\, v \in H_0^1(D).$$

This shows the existence of a solution in $H^1(D)$ assuming $f \in L^2$.

assume $\lim_{x\to\infty} u_1(x,t) = \lim_{x\to\infty} u_2(x,t)$ for each t.

FOO #2: Let u_1, u_2 be 2 classical bounded solutions of the given equation. Let $w := u_1 - u_2$. Then

$$w_t - \Delta w + w(u_1+u_2) = 0 \quad , \; x \in \mathbb{R}^N, \; 0 < t < T$$
$$w(x,0) = 0.$$

Let

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^N} w(x,t)^2\, dx.$$

Then

$$\dot{E}(t) = \int_{\mathbb{R}^N} w w_t \, dx = \int_{\mathbb{R}^N} w\left(\Delta w - w(u_1+u_2)\right) dx$$

$$= \int_{\mathbb{R}^N} w \Delta w\, dx - \int_{\mathbb{R}^N} w^2 (u_1+u_2)\, dx$$

Since $w(\infty)=0$ ~~[illegible]~~,

$$= -\int_{\mathbb{R}^N} |\nabla w|^2 dx - \int_{\mathbb{R}^N} w^2 (u_1+u_2) dx$$

$$\le -\|u_1+u_2\|_{L^\infty} \int_{\mathbb{R}^N} w^2\, dx$$

$$\le -2\|u_1+u_2\|_{L^\infty} E(t).$$

Thus by Gronwall's Inequality,

$$E(t) \le E(0) e^{-\int_0^t 2\|u_1+u_2\|_\infty ds} = 0$$

Therefore $w(x,t) = 0 \longrightarrow u_1 = u_2$.

FOO #4

Let v_1, v_2, v_3 be smooth compactly supported functions. Then the minimizer u_1, u_2, u_3 satisfies

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(F[u+\varepsilon v] - F[u] \right).$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Bigg(\int_D \sum_{j,k=1}^{3} (u_{j x_k} + \varepsilon v_{j x_k})^2 + \alpha\Big(\sum_{j=1}^{3}(u_j+\varepsilon v_j)^2 - 1\Big)^2 - \int_D \sum_{j,k=1}^{3}(u_{j x_k})^2 + \alpha\Big(\sum_{j=1}^{3} u_j^2 - 1\Big)^2 \Bigg)$$

Thus

$$0 = \int_D \sum_{j,k=1}^{3} 2 u_{j x_k} v_{j x_k} + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\alpha \int_D \Big(\big(\sum_{j=1}^{3} u_j^2 - 1\big) + 2\sum_{j=1}^{3} u_j v_j \varepsilon + \sum_{j=1}^{3} \varepsilon^2 v_j^2 \Big)^2 - \alpha \int_D \Big(\sum_{j=1}^{3} u_j^2 - 1\Big)^2 \right]$$

$$= \int_D \sum_{j,k=1}^{3} 2 u_{j x_k} v_{j x_k} + 4\Big(\sum_{j=1}^{3} u_j^2 - 1\Big)\Big(\sum_{j=1}^{3} u_j v_j\Big)\, dx.$$

$$= \int_D -\sum_{j,k=1}^{3} 2 u_{j x_k x_k} v_j + 4\Big(\sum_{j=1}^{3} u_j^2 - 1\Big)\Big(\sum_{j=1}^{3} u_j v_j\Big)\, dx.$$

$$= \int_D \sum_{j=1}^{3}\left[-2\sum_{k=1}^{3} (u_j)_{x_k x_k} + 4\Big(\sum_{\ell=1}^{3} u_\ell^2 - 1\Big) u_j \right] v_j.$$

Since the v_j are arbitrary, we have

$$-\Delta u_j + 4(|\vec{u}|^2 - 1) u_j = 0 \quad \text{on } D,\ j = 1, 2, 3$$
$$u_j = \varphi_j \quad \text{on } \partial D.$$

$$-\Delta \vec{u} + 4(|\vec{u}|^2 - 1)\vec{u} = 0 \quad \text{on } D.$$
$$\vec{u} = \vec{\varphi} \quad \text{on } \partial D.$$

~~F99~~ F00 #6: We will use Laplace Transform. Since $u(x,0)=0$,

$$\int_0^\infty u_t(x,t)e^{-st}\,dt = -\int_0^\infty u(x,t)(-se^{-st})\,dt$$
$$= s\int_0^\infty u(x,t)e^{-st}\,dt.$$

Fix an $x \geq 0$. Then as

$$u_t - u_{xx} + au = 0$$

we have

$$s\mathcal{L}[u] - \mathcal{L}[u]_{xx} + a\mathcal{L}[u] = 0.$$
$$\longrightarrow \mathcal{L}[u]_{xx} - (s+a)\mathcal{L}[u] = 0.$$

~~Therefore as we want~~

~~$\mathcal{L}[u]$~~

We will find a bounded solution to the problem, so we will want $\mathcal{L}[u]$ to be bounded. We have

$$\mathcal{L}[u](s) = Ae^{-\sqrt{s+a}\,x}$$

Since ~~$\mathcal{L}[u]$~~ $u(0,t) = g(t)$, $\mathcal{L}[u(0,t)] = \mathcal{L}[g]$.
Therefore $A = \mathcal{L}[g]$ and hence

$$\mathcal{L}[u] = \mathcal{L}[g]e^{-\sqrt{s+a}\,x}.$$
$$\longrightarrow u(x,t) = g * \mathcal{L}^{-1}[e^{-\sqrt{s+a}\,x}].$$

S00 #1:

We use method of characteristics. We have

$$F(p,q,z,x,y) = p^2+q^2-1.$$

$\dot{x} = 2p$ $\qquad x(0) = x_0$

$\dot{y} = 2q$ $\qquad y(0) = x_0^2/2$

$\dot{p} = 0$ $\qquad p(0) = -x_0/\sqrt{x_0^2+1}$

$\dot{q} = 0$ $\qquad q(0) = 1/\sqrt{x_0^2+1}$

$\dot{z} = 2p^2+2q^2 = 2$ $\qquad z(0) = 0$

from since $\phi(x,\frac{x^2}{2})=0$, $\phi_x(x,\frac{x^2}{2})+\phi_y(x,\frac{x^2}{2})x=0$

~~$p+qx_0=0$.~~

~~$p^2+q^2=1$~~

$p(0)+q(0)x_0=0$

$p(0)^2+q(0)^2=1$.

here we also used $\phi_y(x,\frac{x^2}{2})>0$ which implies $q(0)>0$.

We then have

$$p(s) = -\frac{x_0}{\sqrt{x_0^2+1}} \qquad x(s) = x_0 - \frac{2x_0 s}{\sqrt{x_0^2+1}}$$

$$q(s) = \frac{1}{\sqrt{x_0^2+1}} \qquad y(s) = \frac{x_0^2}{2} + \frac{2}{\sqrt{x_0^2+1}}s.$$

$$z(s) = 2s$$

Thus the solution in parametric form is:

$$x(s,t) = t - \frac{2st}{\sqrt{t^2+1}}$$

$$y(s,t) = \frac{t^2}{2} + \frac{2s}{\sqrt{t^2+1}}$$

$$z(s,t) = 2s.$$

S'00 #3: We have $u(x,t) = \sum_{k\in\mathbb{Z}^2} \hat{u}(k,t)e^{ik\cdot x}$. Since $u_t = \Delta u - u$,

$$\hat{u}_t(k,t) = -|k|^2\hat{u}(k,t) - \hat{u}(k,t)$$
$$= (-|k|^2 - 1)\hat{u}(k,t)$$

and hence

$$\hat{u}(k,t) = e^{-(|k|^2+1)t}\hat{u}(k,0).$$

Therefore

$$u(x,t) = \sum_{k\in\mathbb{Z}^2} \hat{u}(k,t)e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^2} e^{-(|k|^2+1)t}e^{ik\cdot x}\hat{u}(k,0)$$
$$= \sum_{k\in\mathbb{Z}^2} \hat{u}_0(k)e^{-(|k|^2+1)t}e^{ik\cdot x}.$$

S00 #4.

We have $f(\theta) = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$, $f_k = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$.

We want to solve $\Delta u = 0$ in $r<1$

$u_r = f(\theta)$ on $r=1$.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

here $u(r,\theta) = \sum_{n\in\mathbb{Z}} a_n(r) e^{in\theta}$. Then we want

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0.$$

$$r^2 a_n'' + r a_n' - n^2 a_n = 0.$$

$$a_n = r^\alpha \rightarrow \alpha(\alpha-1) + \alpha - n^2 = 0.$$

$$\alpha^2 - n^2 = 0 \longrightarrow \alpha = \pm n.$$

Therefore $a_n(r) = A_n r^n + B_n r^{-n}$. Thus

$$u(r,\theta) = A_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} (A_n r^n + B_n r^{-n}) e^{in\theta}$$

Since want u to be defined when $r=0$,

$$u(r,\theta) = A_0 + \sum_{n>0} A_n r^n e^{in\theta} + \sum_{n<0} B_n r^{-n} e^{in\theta}.$$

$$= A_0 + \sum_{n>0} A_n r^n e^{in\theta} + B_{-n} r^n e^{-in\theta}.$$

$$= C_0 + \sum_{n\geq 1} C_n r^n e^{in\theta} + D_n r^n e^{-in\theta}.$$

$$= C_0 + \sum_{n\geq 1} r^n \left(C_n e^{in\theta} + D_n e^{-in\theta}\right).$$

S00 #4 cont:

$$\frac{\partial u}{\partial r}\Big|_{r=1} = \sum_{n\geq 1} n\left(C_n e^{in\theta} + D_n e^{-in\theta}\right).$$

Thus we want

$$\sum_{n\geq 1} nC_n e^{in\theta} + nD_n e^{-in\theta} = \sum_{n\in\mathbb{Z}} f_n e^{in\theta}.$$

For this to be true ~~we~~ it is necessary to assume

$$f_0 = 0 \longrightarrow \int_0^{2\pi} f(x)\,dx = 0.$$

Then

$$\boxed{\text{if } n>0, \quad C_n = \frac{f_n}{n}.}$$

$$\text{if } n<0, \quad \cancel{D_n = \frac{f_n}{-n}}.\ D_{-n} = \frac{f_n}{-n} \Rightarrow \boxed{D_n = \frac{f_{-n}}{n}. \text{ for } n>0.}$$

We have

$$u(r,\theta) = A_0 + \sum_{n\geq 1} r^n \frac{f_n}{n} e^{in\theta} + r^n \frac{f_{-n}}{n} e^{-in\theta}.$$

$$= A_0 + \sum_{n\geq 1} \frac{(re^{i\theta})^n}{n} \frac{1}{2\pi}\int_0^{2\pi} f(\eta)e^{-in\eta}\,d\eta + \frac{(re^{-i\theta})^n}{n} \frac{1}{2\pi}\int_0^{2\pi} f(\eta)e^{in\eta}\,d\eta.$$

$$= A_0 + \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[\sum_{n\geq 1}\frac{r^n e^{in(\theta-\eta)}}{n} + \sum_{n\geq 1}\frac{r^n e^{-in(\theta-\eta)}}{n}\right]d\eta$$

We have $\frac{1}{1-x} = \sum_{n\geq 0} x^n$ for $|x|<1$. Then

$$-\ln(1-x) = \sum_{n\geq 0}\frac{1}{n+1}x^{n+1} = \sum_{n\geq 1}\frac{1}{n}x^n.$$

$re^{i\theta}+re^{-i\theta} = 2r\cos\theta.$

S00 #4 cont:

Therefore

here we use $r<1$

$$u(r,\theta) = A_0 + \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[-\ln(1-re^{i(\theta-\eta)}) - \ln(1-re^{-i(\theta-\eta)})\right]d\eta.$$

$$= A_0 - \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\left[\ln(1-re^{-i(\theta-\eta)}-re^{i(\theta-\eta)}+r^2)\right]d\eta.$$

$$= A_0 - \frac{1}{2\pi}\int_0^{2\pi} f(\eta)\ln(1-2r\cos(\theta-\eta)+r^2)\,d\eta.$$

Therefore

$$N(r,\theta) = -\ln(1-2r\cos\theta+r^2).$$

(note the solution to the Neuman problem is not unique).

S00 #5:

~~Let~~ We have $u_{tt} = c^2 v''$, $(u^2)_{xx} = (2uu_x)_x = 2(u_x^2 + uu_{xx}) = 2(v'^2 + vv'')$ and $u_{xxxx} = v''''$ where $v' = v'(y)$. Thus $y = x - ct$

$$u_{tt} + (u^2)_{xx} = -u_{xxxx}$$

$$\rightarrow \quad c^2 v'' + 2(vv')' = -v''''$$

$$c^2 v' + 2vv' = -v''' + c_1$$

$$c^2 v' + (v^2)' = -v''' + c_1$$

$$c^2 v + v^2 = -v'' + c_1(x-ct) + c_2.$$

Since $v \to$ ~~constant~~ M for some constant M as $|x| \to \infty$, we have $c_1 = 0$ and $c_2 = c^2 M + M^2$. Thus

$$v'' + v^2 + c^2 v = c^2 M + M^2.$$

$$v'' + (v - M)(v + (c^2 + M)) = 0.$$

Write this as a ~~[illegible]~~ system:

$$x' = y$$

$$y' = -c^2 x - x^2 + c^2 M + M^2.$$

The equilibrium pts are $(M,0)$ and $(-c^2 - M, 0)$

The Jacobian is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -c^2 - 2x & 0 \end{pmatrix} \rightarrow J(M,0) = \begin{pmatrix} 0 & 1 \\ -c^2 - 2M & 0 \end{pmatrix}$$

The eigenvalues of $J(M,0)$ are the roots of $\lambda^2 + (c^2 + 2M)$.

If $M > 0$, then $(M,0)$ is a center (as the system is Hamiltonian, so equilibrium pts are either centers or saddles). ~~Since $v \to M$ as $t \to \infty$ as time increases, the solution v is~~ Thus the solution v is ~~periodic~~ sinusoidal (at least when the initial conditions $(v(0), v'(0))$ are close to $(M,0)$.)

S00 #6

a) We have

$$\frac{d}{d\tau}(x\cdot x) = x_1\dot{x}_1 + \dots + x_n\dot{x}_n = x\cdot\dot{x} = f(|x|^2)\,x\cdot p.$$

Since $f>0$, $\frac{d}{d\tau}(x\cdot x)>0$ if $p\cdot x>0$ and <0 if $p\cdot x<0$.
Thus $|x| = (x\cdot x)^{1/2}$ is increasing w/ τ when $p\cdot x>0$ and decreasing w/ τ when $p\cdot x<0$.

We have

$$\frac{d}{d\tau}\left(f(|x|^2)|p|^2\right) = \frac{d}{d\tau} f\Big(\sum_{i=1}^n x_i^2\Big)\Big(\sum_{i=1}^n p_i^2\Big) = f'(|x|^2)\Big(\sum_{i=1}^n 2x_i\dot{x}_i\Big)|p|^2 + f(|x|^2)\Big(\sum_{i=1}^n 2p_i\dot{p}_i\Big)$$

$$= f'(|x|^2)|p|^2\cdot 2x\cdot f(|x|^2)p + f(|x|^2)\cdot 2p\cdot\left(-f'(|x|^2)|p|^2 x\right) = 0.$$

b) We have $\frac{d}{ds}\left(\frac{f(s)}{s}\right) = \frac{sf'(s) - f(s)}{s^2}$. Thus

$$r^2 f'(r^2) = f(r^2)$$

We have

$$\frac{d}{d\tau}(p\cdot x) = -f'(|x|^2)|p|^2|x|^2 + f(|x|^2)|p|^2.$$

~~By sol. to a) $|x|$ is constant if $x\cdot p = 0$.~~

Thus ~~at $\tau=0$, for $|x|=r$,~~

~~$$\frac{d}{d\tau}(p\cdot x)\Big|_{\tau=0} = -f'(r^2)|p|^2r^2 + f(r^2)|p|^2 = 0$$~~

~~Since $x(0)\cdot p(0) = 0$, by sol. to a) $|x|$ stays the~~

~~Thus $p\cdot x$ = constant for all $|x|=r$. Since $p(0)\cdot x(0)=0$,~~

~~$p\cdot x = 0$ for all $|x|=r$.~~

500#6 cont:

We have

$$\{t : (x\cdot x)' > 0\} = \{t : x\cdot p > 0\}$$

$$\{t : (x\cdot x)' < 0\} = \{t : x\cdot p < 0\}$$

$$\frac{d}{dt}(p\cdot x) = 0 \quad \text{when} \quad x\cdot x = r^2.$$

$$(x\cdot p)(0) = 0.$$

$$(x\cdot x)(0) = r^2$$

This is not strong enough to conclude $(x\cdot x)(t) = r^2\ \forall t$.

$$\{t : g' > 0\} = \{t : f > 0\}$$

$$\{t : g' < 0\} = \{t : f < 0\}$$

$$f' = 0 \quad \text{when} \quad g = r^2.$$

$$f(0) = 0$$

$$g(0) = r^2.$$

here $g(t) = r^2 + t^2$, $f(t) = t^3$. But $g(t) \neq r^2\ \forall t$.

S00 #8:

a) We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} u(x,t)\,(\phi_{tt}-\Delta\phi)\,dx\,dt \qquad (*)$$

$$=\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} u_{tt}\,\phi\,dx\,dt-\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon} u\,\Delta\phi\,dx\,dt.$$

$$\int_{|x|>\varepsilon} u\Delta\phi=\int_{|x|>\varepsilon}\phi\,\Delta u+\int_{|x|=\varepsilon}\frac{\partial\phi}{\partial\nu}u-\frac{\partial u}{\partial\nu}\phi\,d\sigma \qquad \frac{\partial}{\partial\nu}=-\frac{\partial}{\partial r}.$$

So

$$(*)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon} u_{tt}\phi-\phi\Delta u\,dx\,dt-\int_{\mathbb{R}}\int_{|x|=\varepsilon}-\frac{\partial\phi}{\partial r}u+\frac{\partial u}{\partial r}\phi\,d\sigma\,dt.$$

$$=\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|>\varepsilon}(u_{tt}-\Delta u)\phi\,dx\,dt+\int_{\mathbb{R}}\int_{|x|=\varepsilon}\frac{\partial\phi}{\partial r}u-\frac{\partial u}{\partial r}\phi\,d\sigma\,dt.$$

$u=\dfrac{f(t+r)}{r}$. We claim $u_{tt}-\Delta u=0$ away from 0. Indeed,

$$u_r=\frac{rf'(t+r)-f(t+r)}{r^2},\quad u_{rr}=\frac{f''(t+r)}{r}-\frac{2f'(t+r)}{r^2}+\frac{2f(t+r)}{r^3}.$$

Then

$$u_{tt}-\Delta u=u_{tt}-u_{rr}-\frac{2}{r}u_r=0.$$

So

$$(*)=\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{|x|=\varepsilon}\frac{\partial\phi}{\partial r}u-\frac{\partial u}{\partial r}\phi\,d\sigma\,dt$$

$$=\cancel{\int_{\mathbb{R}}\lim_{\varepsilon\to 0}\int_{|x|=\varepsilon}\left[\frac{\partial\phi(x)}{\partial r}\frac{f(t+\varepsilon)}{\varepsilon}-\left(\frac{f'(t+\varepsilon)}{\varepsilon}-\frac{f(t+\varepsilon)}{\varepsilon^2}\right)\phi(x)\right]\varepsilon^2\sin\theta\,d\theta\,d\varphi\,dt}$$

$$\int^{\pi/2}\int^{2\pi}$$

S00 #8 cont:

$$= \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \int_{S^2} \left[\frac{\partial \phi}{\partial r}(\varepsilon \gamma, \tau) \frac{f(t+\varepsilon)}{\varepsilon} - \left(\frac{f'(t+\varepsilon)}{\varepsilon} - \frac{f(t+\varepsilon)}{\varepsilon^2} \right) \phi(\varepsilon \gamma, \tau) \right] \varepsilon^2 \, d\sigma(\gamma) \, d\tau$$

$$= 4\pi \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{S^2} f(t+\varepsilon) \phi(\varepsilon \gamma, t) \, d\sigma(\gamma) \, d\tau.$$

$$= 4\pi \int_{\mathbb{R}} f(t) \phi(0,t) \, d\tau.$$

b) The main issue with $u(x,t) = \frac{f(t+|x|)}{|x|}$ is that it is not differentiable near 0. The solution is given by

$$u(x,t) = \frac{\partial}{\partial t} \left(t \fint_{\partial B(x,t)} \frac{f(|y|)}{|y|} \, d\sigma_y \right) + t \fint_{\partial B(x,t)} \frac{f'(|y|)}{|y|} \, d\sigma_y .$$

We don't expect singularities since $f(0) = f'(0) = 0$ gives local data, yet we want data about f at any point.

F-09 #1

Since $\Delta u = 0$ in the weak sense, u is weakly harmonic. Recall that a weakly harmonic function is also harmonic. By the Mean Value Property, for each $x \in \mathbb{R}^n$,

$$u(x) = \frac{1}{|B(x,1)|} \int_{B(x,1)} u(y)\, dy$$

and hence

$$|u(x)| \le \frac{1}{|B(x,1)|} \int_{B(x,1)} |u(y)|\, dy < \frac{C}{|B(x,1)|}.$$

As $|B(x,1)|$ does not depend on x, it follows that u is bounded on $\mathbb{R}^n$ and hence by Liouville's Theorem, u is constant. #

F99 #2 We have

$$f(x) = (\Delta - aI)\int K_a(x-y) f(y)\,dy$$
$$f(x) = (\Delta - bI)\int K_b(x-y) f(y)\,dy$$

Note that

$$(\Delta - aI)(\Delta - bI)\int (c_1 K_a(x-y) + c_2 K_b(x-y)) f(y)\,dy$$
$$= c_1(\Delta - bI) f(x) + c_2(\Delta - aI) f(x)$$
$$= (c_1 + c_2)\Delta f(x) - c_1 b f(x) - c_2 a f(x) \qquad (*)$$

Taking $c_1 = (a-b)^{-1}$ and $c_2 = (b-a)^{-1}$, we have that $(*) = f(x)$.

Thus $(a-b)^{-1}K_a + (b-a)^{-1}K_b$ is a fundamental solution of $(\Delta - aI)(\Delta - bI)$.

We have $\Delta^2 - \Delta = \Delta(\Delta - I)$. So the fundamental solution is $-K_0 + K_1$ where K_0 is the fundamental solution for Δ and K_1 is the fundamental sol. for $\Delta - I$.

When $n = 3$, $K_0 = \frac{1}{4\pi|x|}$.

~~$K_1 \neq \frac{e^{-|x|}}{4\pi|x|}$~~ $\longrightarrow$ Fundamental solution of Helmholtz eqn. in $\mathbb{R}^3$.

Bessel Potential: **Formally** we solve

P.191 Evans

$$\Delta u - u = \delta_0 \rightarrow -\Delta u + u = -\delta_0$$

Take the Fourier transform of both sides to get

$$(1+|y|^2)\hat{u} = \widehat{-\delta_0}$$

$$\rightarrow u = \left(\frac{\widehat{-\delta_0}}{(1+|y|^2)}\right)^\vee = \frac{1}{(2\pi)^{3/2}}(-\delta_0 * B) = -\frac{1}{(2\pi)^{3/2}} B(x)$$

where $\hat{B} = \frac{1}{1+|y|^2}$. Then as $\frac{1}{1+|y|^2} = \int_0^\infty e^{-t(1+|y|^2)}\,dt$,

$$B = \left(\frac{1}{1+|y|^2}\right)^\vee = \frac{1}{(2\pi)^{3/2}}\int_0^\infty e^{-t}\int_{\mathbb{R}^3} e^{ix\cdot y - t|y|^2}\,dy\,dt.$$

$$\rightarrow u = -\frac{1}{(2\pi)^3}\int_0^\infty e^{-t}\int_{\mathbb{R}^3} e^{ix\cdot y - t|y|^2}\,dy\,dt. := K_1$$

FTT #3:

Claim: $\lambda = \min\limits_{\substack{w \\ \partial w/\partial\nu = 0 \text{ on } \partial\Omega}} \dfrac{\langle Lw, w\rangle}{\langle w, w\rangle}$ is the smallest eigenvalue of L.

Pf: We first show $\min\limits_{w} \dfrac{\langle Lw,w\rangle}{\langle w,w\rangle}$ is indeed an eigenvalue. Let u be the function with $\partial u/\partial\nu = 0$ on $\partial\Omega$ s.t. $\dfrac{\langle Lw,w\rangle}{\langle w,w\rangle}$ attains its minimum. ~~Then for all v with $\frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega$~~

$$\cancel{0 = \frac{d}{d\varepsilon} \frac{\langle Lu+\varepsilon Lv, u+\varepsilon v\rangle}{\langle u+\varepsilon v, u+\varepsilon v\rangle}\Big|_{\varepsilon=0}}$$

~~Thus~~ Let $F(w) := \dfrac{\langle Lw,w\rangle}{\langle w,w\rangle}$. Then for all v with $\frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega$,

$$0 = \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon v) - F(u)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\frac{\langle Lu+\varepsilon Lv, u+\varepsilon v\rangle}{\langle u+\varepsilon v, u+\varepsilon v\rangle} - \frac{\langle Lu,u\rangle}{\langle u,u\rangle}\right]$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\frac{\langle Lu,u\rangle + \varepsilon\langle Lu,v\rangle + \varepsilon\langle Lv,u\rangle + \varepsilon^2\langle Lv,v\rangle}{\langle u,u\rangle + 2\varepsilon\langle u,v\rangle + \varepsilon^2\langle v,v\rangle} - \frac{\langle Lu,u\rangle}{\langle u,u\rangle}\right]$$

$$\rightarrow 0 = \langle Lu,v\rangle\langle u,u\rangle + \langle Lv,u\rangle\langle u,u\rangle - 2\langle Lu,u\rangle\langle u,v\rangle.$$

Since L is self-adjoint, we have

$$0 = 2\langle Lu,v\rangle\langle u,u\rangle - 2\langle Lu,u\rangle\langle u,v\rangle$$

and hence

$$\int (Lu)v\,dx = \int \frac{\langle Lu,u\rangle}{\langle u,u\rangle} u\,v\,dx \quad \forall v.$$

Thus $Lu = \dfrac{\langle Lu,u\rangle}{\langle u,u\rangle} u$. If μ was another eigenvalue of L, w/ corr. eigenfunction w, then

$$\lambda \le \frac{\langle Lw,w\rangle}{\langle w,w\rangle} = \frac{\mu\langle w,w\rangle}{\langle w,w\rangle} = \mu. \quad \#.$$

Self-adjointness of L is proven as follows:

$$\int (Lu)v\,dx = -\int \sum_{i,j} \partial_{x_j}(a^{ij}(x)\partial_{x_i}u)v\,dx = -\sum_{i,j}\int \partial_{x_j}(a^{ij}(x)\partial_{x_i}u)v\,dx$$

since $\frac{\partial u}{\partial\nu} = 0$ $\;= -\sum_{i,j}\int a^{ij}(x)\partial_{x_i}u\,\partial_{x_j}v\,dx \overset{\text{since } \frac{\partial v}{\partial\nu}=0}{=} -\sum_{i,j}\int \partial_{x_i}(a^{ij}(x)\partial_{x_j}v)u\,dx = \int (Lv)u\,dx.$

F99#3 cont:

Suppose all eigenvalues of L are ≥ 0. Then $\lambda \geq 0$. Thus by the claim,

$$\min_{w : \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial U} \frac{\langle Lw, w\rangle}{\langle w, w\rangle} \geq 0.$$

Fix a u with $\frac{\partial u}{\partial \nu} = 0$ and consider $u + a$ with a to be chosen later. We have

$$\langle L(u+a), u+a\rangle = \langle Lu + c(x)a, u+a\rangle$$
$$= \langle Lu, u\rangle + a\int Lu\,dx + a\int c(x)u\,dx + a^2\int c(x)\,dx. \quad (*)$$

Since $\int c(x)\,dx \leq 0$ and $c(x) \neq 0$ for some $x \in U$, $c(x)$ smooth, choose a sufficiently large, we can make $(*) < 0$. Therefore

$$0 \leq \min_{\substack{w \\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial U}} \frac{\langle Lw, w\rangle}{\langle w, w\rangle} \leq \frac{\langle L(u+a), u+a\rangle}{\langle u+a, u+a\rangle} < 0,$$

a contradiction. Therefore $\exists$ a negative eigenvalue. #

F99 #4:

We solve

$$\begin{cases} u_t + a(x)u_x = 0 \\ u(x,0) = f(x) \end{cases}$$

by method of characteristics. We have

$$F(p,q,z,x,t) = q + a(x)p.$$

$$\dot{t}(s) = 1 \qquad t(0) = 0$$

$$\dot{x}(s) = a(x) \qquad x(0) = x_0.$$

$$\dot{z}(s) = 0 \qquad z(0) = f(x_0).$$

Let $a(x) := x^2 + 1$. Then

$$\begin{matrix} \dot{x} = x^2 + 1 \\ x(0) = x_0 \end{matrix} \longrightarrow \tan^{-1}(x) = s + \tan^{-1}(x_0)$$

Since $t = s$, the ~~proofs~~ characteristic curves are given by

$$\tan^{-1} x = t + \tan^{-1} x_0$$

Since $\tan^{-1} x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for any choice of x_0, the characteristic curves do not propogate beyond time $t = \pi$. Thus the values of $u(x,t)$ for $t > \pi$ do not depend on the values $u(x,0)$. Then the solution of the Cauchy problem is not unique.

#

F99 #5 :

We have

Then $$\hat{u}(m,n,t) = \int_{[0,2\pi]^2} u(x,y,t) e^{-i(xm+ny)} \, dx dy.$$

$$\hat{u}_t(m,n,t) = (-\varepsilon)(-m^2-n^2)\hat{u}(m,n,t)$$
$$-(m^4+2m^2n^2+n^4)\hat{u}(m,n,t).$$

$$\hat{u}_t(m,n,t) = \varepsilon(m^2+n^2)\hat{u}(m,n,t) - (m^2+n^2)^2\hat{u}(m,n,t)$$

$$\hat{u}_t(m,n,t) = (m^2+n^2)[\varepsilon - (m^2+n^2)]\hat{u}(m,n,t)$$

$$\hat{u}(m,n,t) = e^{[(m^2+n^2)(\varepsilon-(m^2+n^2))]t}\hat{u}(m,n,0)$$

Therefore

$$u(x,y,t) = \sum_{m,n\in\mathbb{Z}} e^{(m^2+n^2)(\varepsilon-m^2-n^2)t}\hat{u}(m,n,0)e^{i(mx+ny)}.$$

i) A solution is e^{t+ix}. Indeed,

$$\partial_t(e^{t+ix}) = e^{t+ix}$$

$$\Delta u = \partial_{xx}(e^{t+ix}) = e^{t+ix}(-1)$$

$$\Delta^2 u = e^{t+ix}.$$

Therefore

$$-2\Delta u - \Delta^2 u = 2e^{t+ix} - e^{t+ix} = e^{t+ix}.$$

F99 #5 cont:

ii) We need

$$\cancel{\ldots}\ \varepsilon - m^2 - n^2 < 0 \quad \forall\, m, n.$$

$$\varepsilon < m^2 + n^2 .$$

Therefore take $\varepsilon_0 = 1$. #

F99 #6 :

a) We have

deriv. w.r.t. y

$$\partial_t \rho = -s\rho' \qquad \partial_t u = -su'$$
$$\partial_x u = u' \qquad \partial_x(\rho u) = \rho_x u + \rho u_x = \rho' u + \rho u'.$$

Thus

$$-s\rho' + u' = 0. \longrightarrow u' = s\rho' \qquad \rho' = \frac{1}{s}u'$$
$$-su' + \rho' u + \rho u' = u'' \qquad\qquad u = s\rho + c_1$$

$$\rightarrow -su' + \frac{1}{s}u'u + \left(\frac{u}{s} - c_1\right)u' = u''$$
$$-su' + \frac{2}{s}u'u - c_1 u' = u''$$

Therefore

$$u'' = -(s+c_1)u' + \frac{1}{s}(u^2)'.$$

Thus

$$u' = -(s+c_1)u + \frac{1}{s}u^2 + c_2. \qquad (*)$$

b) Note $(*)$ is of the form $u' + Au^2 + Bu = C$ with $A = -1/s$, $B = s + c_1$, $C = c_2$. One way to solve $(*)$ is to observe $(*)$ is separable, so,

$$\int \frac{1}{\frac{1}{s}u^2 - (s+c_1)u + c_2}\, du = \int dt$$

$$\int \frac{s}{u^2 - s(s+c_1)u + sc_2}\, du = t + \tilde{c}_1$$

but the easiest way is to observe that if we guess the solution is of the form

$$u(y) = u_0 + u_1 \tanh(\alpha y + y_0),$$

F99 #6 cont:

$$u' = \alpha u_1 \operatorname{sech}^2(\alpha y + y_0).$$

Thus

$$u' + Au^2 + Bu = C$$

$$\downarrow$$

$$\alpha u_1 \operatorname{sech}^2(\alpha y + y_0) + Au_0^2 + 2Au_0u_1 \tanh(\alpha y + y_0) + Au_1^2 \tanh(\alpha y + y_0)$$
$$+ Bu_0 + Bu_1 \tanh(\alpha y + y_0) = C.$$

Our ansatz is a solution if we choose u_0, u_1, α s.t.

$$\alpha u_1 = Au_1^2 \qquad\qquad u_1 = \alpha/A$$
$$2Au_0u_1 + Bu_1 = 0 \longrightarrow \quad u_0 = -B/2A.$$
$$Au_1^2 + Au_0^2 + Bu_0 = C \qquad \alpha = (AC + B^2/2)^{1/2}.$$

Thus with ~~[illegible]~~ nice choices of c_1, c_2, $(*)$ has solutions of the form $u(y) = u_0 + u_1 \tanh(\alpha y + y_0)$.

F99 #7.

i. We want to find ϕ s.t. $\langle Lu, v\rangle_\phi = \langle u, Lv\rangle_\phi$ for u, v satisfying $u(0)=u(1)=0$ and $v(0)=v(1)=0$.

We have

$$\langle Lu, v\rangle_\phi = \int_0^1 (u''+2u')v\phi\,dx = \int_0^1 u''v\phi + 2u'v\phi\,dx$$

$$= -\int_0^1 u'(v\phi)'\,dx - 2\int_0^1 u(v\phi)'\,dx$$

$$= \int_0^1 u(v\phi)''\,dx - 2\int_0^1 u(v\phi)'\,dx$$

$$= \int_0^1 u(v''\phi + 2v'\phi' + v\phi'') - 2u(v'\phi + v\phi')\,dx$$

$$= \int_0^1 u(v''\phi + 2v'\phi' + v\phi'' - 2v'\phi - 2v\phi')\,dx$$

$$= \int_0^1 u(v''\phi + 2v'(\phi'-\phi) + v(\phi''-2\phi'))\,dx$$

$$= \int_0^1 u\phi\left(v'' + 2v'\left(\frac{\phi'}{\phi}-1\right) + v\left(\frac{\phi''-2\phi'}{\phi}\right)\right)dx$$

Thus we would have $\langle Lu, v\rangle_\phi = \langle u, Lv\rangle_\phi$ if

$$\frac{\phi'}{\phi} - 1 = 1 \qquad\qquad \phi' = 2\phi$$
$$\frac{\phi''-2\phi'}{\phi} = 0 \quad \longrightarrow \quad \phi'' = 2\phi'.$$

Taking $\phi = e^{2x}$ gives such a ϕ.

ii. To show $L+aI$ is invertible, we will show that $\ker(L+aI)$ is trivial. Let $u \in \ker(L+aI)$. Then

$$\cancel{0 = \langle (L+aI)u, u\rangle = \int_0^1 (Lu+au)u\phi\,dx = \int_0^1 (u''+2u'+au)u\phi\,dx}$$

$$\cancel{= -\int_0^1 u'(u\phi)' + 2u'u\phi + au^2\phi\,dx.}$$

$$\cancel{= -\int_0^1 u'(ue^{2x})' - 2u(ue^{2x})' + au^2e^{2x}\,dx.}$$

F99 #7
cont.

$$0 = \langle (L+aI)u, u\rangle = \int_0^1 (Lu+au)u\phi\,dx = \int_0^1 (u''+2u'+au)u\phi\,dx$$

$$= \int_0^1 (u''+2u'+au)ue^{2x}\,dx \cancel{= \int_0^1 u(ue^{2x})'' - 2u(ue^{2x})' + au^2e^{2x}\,dx}$$

$$= \int_0^1 -u'(ue^{2x})' + 2u'ue^{2x} + au^2e^{2x}\,dx$$

$$= \int_0^1 -u'(u'e^{2x} + 2e^{2x}u) + 2u'ue^{2x} + au^2e^{2x}\,dx$$

$$= \int_0^1 -(u')^2e^{2x} + au^2e^{2x}\,dx$$

$$= \int_0^1 e^{2x}\left[-(u')^2 + au^2\right]dx$$

Since $a<0$, if $u \not\equiv 0$, is <0 which would lead to a contradiction. Therefore we must have $u \equiv 0$. Thus if $a<0$, $L+aI$ is invertible.

iii: Take $a=-1$, $u=e^{-x}$.

F99 #8:

We look for a radial symmetric u.

a) We want to solve $\begin{cases} -\Delta u = 1 & \text{on } B_{R(t)}(0) \\ u = 0 & \text{on } |x| = R(t) \end{cases}$

Since want to find radial u, we solve

$$-\left(u_{rr} + \frac{1}{r} u_r\right) = 1$$
$$r u_{rr} + u_r = -r$$
$$(r u_r)' = -r$$
$$r u_r = -\frac{r^2}{2} + c_1$$
$$u_r = -\frac{r}{2} + \frac{c_1}{r}$$
$$u = -\frac{1}{4} r^2 + c_1 \log r + c_2.$$

Since we want u to be defined on the origin, $c_1 = 0$. Since ~~also~~ $u = 0$ when $r = R(t)$, we have

$$u = -\frac{1}{4} r^2 + \frac{1}{4} R(t)^2 = -\frac{1}{4} |x|^2 + \frac{1}{4} R(t)^2.$$

b) Using $\frac{dR}{dt} = -u_r|_{r=R}$, we have

$$\frac{dR}{dt} = -u_r|_{r=R} = \frac{1}{2} R.$$

Therefore as $R(0) = R_0$,

$$R(t) = e^{\frac{1}{2}t} R_0.$$