W02 #1:

The homogeneous solutions are $\{e^x, e^{-x}\}$. Then

$$G(x,\xi) = \begin{cases} ae^x + be^{-x} & 0<x<\xi \\ ce^x + de^{-x} & \xi<x<1 \end{cases}$$

We want:

$$G(0,\xi)=0 \longrightarrow a+b=0.$$
$$G(1,\xi)=0 \longrightarrow ce+de^{-1}=0. \longrightarrow ce^2+d=0.$$
$$G \text{ cont. @ } x=\xi \longrightarrow ae^{\xi}+be^{-\xi}=ce^{\xi}+de^{-\xi}.$$
$$G'(\xi^+,\xi)-G'(\xi^-,\xi)=1 \longrightarrow ce^{\xi}-de^{-\xi}-ae^{\xi}+be^{-\xi}=1.$$

$$a+be^{-2\xi}=c+de^{-2\xi}. \qquad\qquad a-ae^{-2\xi}=c-ce^{2-2\xi}$$
$$c-de^{-2\xi}-a+be^{-2\xi}=e^{-\xi} \longrightarrow c+ce^{2-2\xi}-a-ae^{-2\xi}=e^{-\xi}.$$

$$\longrightarrow a = c\cdot\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}$$

$$c(1+e^{2-2\xi})-a(1+e^{-2\xi})=e^{-\xi}.$$
$$c(1+e^{2-2\xi})-c\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})=e^{-\xi}.$$
$$c\left[1+e^{2-2\xi}-\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})\right]=e^{-\xi}.$$

So

$$c=e^{-\xi}\left[1+e^{2-2\xi}-\left(\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}\right)(1+e^{-2\xi})\right]^{-1}$$
$$a=c\cdot\frac{1-e^{2-2\xi}}{1-e^{-2\xi}}$$
$$b=-a$$
$$d=-ce^2.$$

and so a solution to $Lu=f$, $u(0)=0$, $u(1)=0$ is

$$u(x)=\int_0^1 G(x,\xi)f(\xi)\,d\xi.$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx.$$

W02 #2:

a) Extend u by odd reflection to $[-\pi,\pi]$. Then

since $\hat{f'}(n) = in\hat{f}(n)$

$$\int_{-\pi}^{\pi} |u(x)|^2\,dx = \sum_{n\neq 0} |\hat{u}(n)|^2 = \sum_{n\neq 0} \left|\frac{i}{n}\widehat{u'}(n)\right|^2$$

since $\hat{u}(0) = \left(\int_{-\pi}^{\pi} u(x)\,dx\right)\frac{1}{2\pi} = 0$

$$\leq \sum_{n\neq 0} |\widehat{u'}(n)|^2 = \int_{-\pi}^{\pi} |u'(x)|^2\,dx.$$

Therefore

Wirtinger's inequality.

$$\int_0^{\pi} |u(x)|^2\,dx \leq \int_0^{\pi} |u'(x)|^2\,dx.$$

b) Suppose L had an eigenvalue ≤ 0. Let u be a corresponding eigenfunction. Then

$$0 \geq \lambda\langle u,u\rangle = \langle Lu,u\rangle = \int_0^{\pi} -u''u + q(x)u^2\,dx$$

$$= \int_0^{\pi} (u')^2 + q(x)u^2\,dx > \int_0^{\pi} (u')^2 - u^2\,dx \geq 0$$

which is a contradiction. Therefore all eigenvalues are > 0.

W02 #3 (a) The diff eq can be written as

$$x' = y$$
$$y' = \sin x$$

This system is Hamiltonian and so all stationary pts. are either centers or saddles.

(where $\frac{\partial H}{\partial y} = y$, $-\frac{\partial H}{\partial x} = \sin x$)

here

$$H(x,y) := \frac{1}{2}y^2 + \cos x.$$

Then

$$\frac{d}{dt}H = yy' + (-\sin x)x' = y\sin x - y\sin x = 0.$$

The stationary pts are $(n\pi, 0)$ where $n \in \mathbb{Z}$.

The Jacobian is:

$$J = \begin{pmatrix} 0 & 1 \\ \cos x & 0 \end{pmatrix}$$

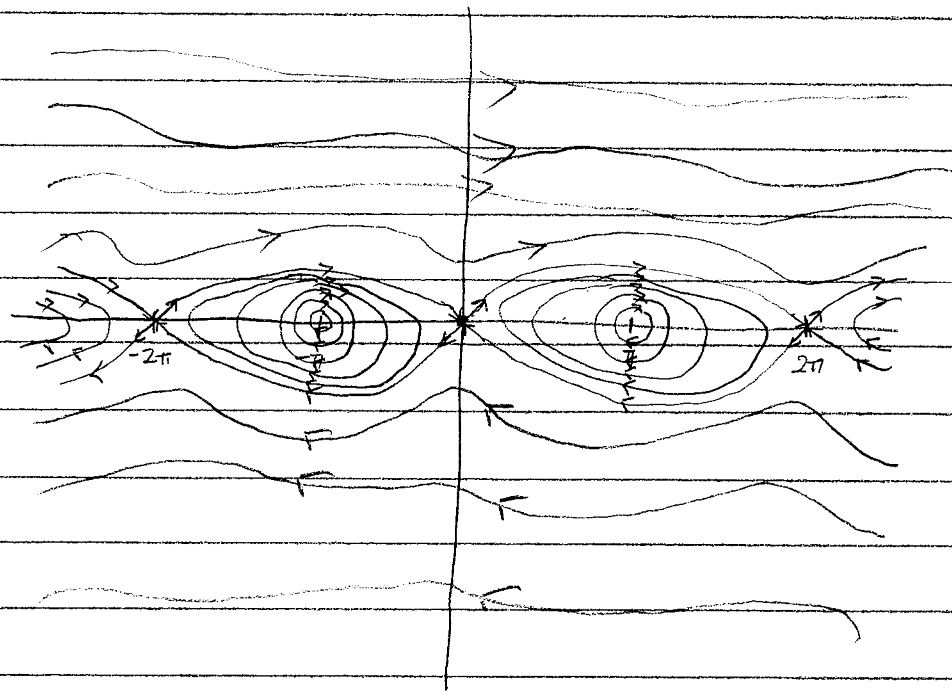
$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

If n is even, then $J(n\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has eigenvalues ± 1, so in this case $(n\pi, 0)$ is a saddle.

If n is odd, then $J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which has eigenvalues $\pm i$. Since the system is Hamiltonian, in this case $(n\pi, 0)$ is a center.

Eigenvalue	Eigenvector
1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
-1	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

W02 #3
cont:



W02 #4: One way is to use the Fourier transform, but we will use the method of characteristics.

a) We have

$$F(p_1, \ldots, p_n, q, z, x_1, \ldots, x_n, t) = q + \sum_{k=1}^{n} a_k(t) p_k + a_0(t) z.$$

With $\vec{p} = (p_1, \ldots, p_n, q)$, $\vec{x} = (x_1, \ldots, x_n, t)$, we have

$$D_{\vec{p}} F = (a_1(t), \ldots, a_n(t), 1)$$

$$D_{\vec{x}} F = \left(0, 0, \ldots, 0, \sum_{k=1}^{n} a_k'(t) p_k + a_0'(t) z\right)$$

$$D_z F = a_0(t).$$

Then

$$\dot{z} = D_{\vec{p}} F \cdot \vec{p} = -a_0(t) z.$$

$$\dot{\vec{x}} = D_{\vec{p}} F$$

and so

$$\dot{z} = -a_0(t) z. \qquad z(0) = f(x_0)$$

$$\dot{t}(s) = 1 \qquad t(0) = 0.$$

$$\dot{x}_1(s) = a_1(s) \qquad x_1(0) = (x_1)_0$$

$$\vdots \qquad \qquad \vdots$$

$$\dot{x}_n(s) = a_n(s) \qquad x_n(0) = (x_n)_0.$$

$$\longrightarrow t(s) = s.$$

$$\dot{z} = -a_0(s) z \longrightarrow z(s) = f(x_0) \exp\left(\int_0^t -a_0(s)\,ds\right).$$

$$x_k(s) = \int_0^t a_k(s)\,ds + (x_k)_0.$$

Thus

$$u(x,t) = f\left(x_1 - \int_0^t a_1(s)ds, \ldots, x_n - \int_0^t a_n(s)ds\right) \exp\left(-\int_0^t a_0(s)ds\right)$$

$$= f\left(x_k - \int_0^t a_k(s)\,ds\right) \exp\left(-\int_0^t a_0(s)\,ds\right).$$

W02 #4 b) cont:

We use ~~@~~ Duhamel's Principle. For fixed s, we first solve for $u(x,t;s)$ where $u(x,t;s)$ satisfies

$$u_t(\cdot;s) + \sum_{k=1}^{n} a_k(t) u_{x_k}(\cdot;s) + a_0(t) u(\cdot;s) = 0 \quad \text{in } \mathbb{R}^n \times (s,\infty)$$
$$u(\cdot;s) = f(\cdot;s) \text{ in } \mathbb{R}^n \times \{t=s\}$$

~~Then By the~~

By part a),

$$u(x,t;s) = f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right).$$

Thus

$$u(x,t) = \int_0^t f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

We check that u does indeed solve the PDE in b). We have $u(x,0)=0$. We have

$$u_t = f(x_k) + \int_0^t \nabla f\left(x_k - \int_s^t a_k(y)\,dy\right)\cdot(-a_k(t)) \exp\left(-\int_s^t a_0(y)\,dy\right) + f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right)[-a_0(t)]\,dy.$$

$$a_k u_{x_k} = a_k(t) \int_0^t f_{x_k}\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

$$a_0 u = a_0(t) \int_0^t f\left(x_k - \int_s^t a_k(y)\,dy\right) \exp\left(-\int_s^t a_0(y)\,dy\right) ds.$$

and so

$$u_t + \sum_k a_k u_{x_k} + a_0 u = f.$$

W02 #5: We have

$$u\Delta u + \sum_{k=1}^{n} \alpha_k u \frac{\partial u}{\partial x_k} - u^4 = 0 \quad \text{in } \Omega$$

$$\int_\Omega u\Delta u \, dx + \sum_{k=1}^{n} \alpha_k \int_\Omega u \frac{\partial u}{\partial x_k} dx - \int_\Omega u^4 dx = 0.$$

$$-\int_\Omega |Du|^2 dx + \sum_{k=1}^{n} \alpha_k \int_\Omega u \frac{\partial u}{\partial x_k} dx - \int_\Omega u^4 dx = 0.$$

Since $u = 0$ on $\partial\Omega$,

$$\int_\Omega u \frac{\partial u}{\partial x_k} dx = -\int_\Omega \frac{\partial u}{\partial x_k} u \, dx$$

$$\longrightarrow \int_\Omega u \frac{\partial u}{\partial x_k} dx = 0.$$

Thus

$$-\int_\Omega |Du|^2 dx - \int_\Omega u^4 dx = 0.$$

$$\longrightarrow u^4 = 0, \; |Du|^2 = 0 \text{ on } \Omega$$

$$\longrightarrow u = 0.$$

W02 #6:

We have $F(p,q,z,x,t) = q + z^2 p$ and

$$\dot{t}(s) = 1 \qquad t(0) = 0$$
$$\dot{x}(s) = z^2 \qquad x(0) = x_0$$
$$\dot{z}(s) = 0 \qquad z(0) = 2 + x_0.$$

$$\longrightarrow \quad t(s) = s$$
$$z(s) = 2 + x_0.$$
$$x(s) = (2+x_0)^2 s + x_0.$$

$$\longrightarrow \quad x = (4 + 4x_0 + x_0^2)t + x_0$$
$$x = t x_0^2 + (4t+1)x_0 + 4t.$$
$$x_0 = \frac{-(4t+1) \pm \sqrt{(4t+1)^2 - 4t(4t-x)}}{2t}.$$

$$\longrightarrow \quad u(x,t) = 2 + \frac{-(4t+1) \pm \sqrt{8t+1+4tx}}{2t}.$$
$$= -\frac{1}{2t} \pm \frac{\sqrt{1+4t(x+2)}}{2t}. \qquad (*)$$

As $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2/2)$, for $(*)$ to satisfy $u(x,0) = 2+x$, we need

$$u(x,t) = -\frac{1}{2t} + \frac{\sqrt{1+4t(x+2)}}{2t}.$$

W02 #7: We have

$$\Delta u = f \longrightarrow -4\pi^2 |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi)$$
$$\hat{u}(\xi) = -\frac{1}{4\pi^2} \frac{\hat{f}(\xi)}{|\xi|^2}.$$

Then

$$u = -\frac{1}{4\pi^2} \left[\frac{\hat{f}(\xi)}{|\xi|^2} \right]^\vee.$$

by Parseval

a) To show $u \in L^2(\mathbb{R}^n)$ if $n > 4$, it suffices to show $\hat{f}(\xi)/|\xi|^2$ is in $L^2(\mathbb{R}^n)$ for $n > 4$. We have

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} d\xi = \int_{|\xi| \le 1} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} d\xi + \int_{|\xi| > 1} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} d\xi.$$

$$\le \int_{|\xi| \le 1} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} d\xi + \int_{|\xi| > 1} |\hat{f}(\xi)|^2 d\xi$$

$$\le \int_{|\xi| \le 1} \frac{|\hat{f}(\xi)|^2}{|\xi|^4} d\xi + \|f\|_{L^2}^2$$

$$\le \sup_{|\xi| \le 1} |\hat{f}(\xi)|^2 \int_{S^{n-1}} \int_0^1 \frac{1}{r^4} r^{n-1} \, dr \, d\sigma + \|f\|_{L^2}^2.$$

$$\le \sup_{|\xi| \le 1} |\hat{f}(\xi)|^2 \int_{S^{n-1}} d\sigma \int_0^1 r^{n-5} dr + \|f\|_{L^2}^2 \qquad (*)$$

Since $n > 4$, $\int_0^1 r^{n-5} dr = \frac{1}{n-4}$ and since $\hat{f}$ is continuous (if f is L^1, then $\hat{f}$ is unif. cont.), it follows that $(*)$ is $< \infty$ and hence there is a solution of the PDE belonging to $L^2(\mathbb{R}^n)$ if $n > 4$.

b) We mimic the same proof as in (a). Since $\int_{\mathbb{R}^n} f(x)\,dx = 0$, $\hat{f}(0) = 0$. ~~Since~~ Expand $\hat{f}$ as a power series ~~converges~~

$$\hat{f}(\xi) = \hat{f}(0) + (\hat{f})'(0)\xi + O(\xi^2)$$

W02 #7 cont:

valid for $|\xi| \le \delta$ where δ is the radius of convergence.

Since $\hat{f}(0) = 0$,

$$\hat{f}(\xi) = (\hat{f})'(0)\,\xi + O(\xi^2)$$
$$= \xi\,((\hat{f}')(0) + O(\xi)).$$

Then

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi = \int_{|\xi|\le\delta} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi + \int_{|\xi|>\delta} \frac{|\hat{f}(\xi)|^2}{|\xi|^4}\,d\xi$$

$$\le \int_{|\xi|\le\delta} \frac{|(\hat{f}')(0) + O(\xi)|}{|\xi|^2}\,d\xi + \frac{1}{\delta^4}\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi.$$

$$\le \sup_{|\xi|\le\delta} |(\hat{f}')(0) + O(\xi)| \int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi + \delta^{-4}\|f\|_{L^2}^2.$$

Thus $\hat{f}/|\xi|^2 \in L^2(\mathbb{R}^n)$ iff $\int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi < \infty$.

For $n > 2$,

$$\int_{|\xi|\le\delta} \frac{1}{|\xi|^2}\,d\xi = \int_{S^{n-1}} \int_0^\delta \frac{1}{r^2}\, r^{n-1}\,dr\,d\sigma$$
$$= \int_{S^{n-1}} d\sigma \int_0^\delta r^{n-3}\,dr < \infty$$

since $n > 2$.

Therefore if $\int_{\mathbb{R}^n} f\,dx = 0$, there is a solution belonging to $L^2(\mathbb{R}^n)$ if $n > 2$.

WD2 #8

a) We use Duhamel's Principle. ~~We f~~ For fixed s, we first solve for $u(x,t;s)$ where

$$u_{tt}(\cdot;s) - u_{xx}(\cdot;s) = 0 \quad \text{in } \mathbb{R}\times(s,\infty)$$
$$u(\cdot;s) = 0, \quad u_t(\cdot;s) = \cancel{f(x,s)}\ f(\cdot;s) \quad \text{in } \mathbb{R}\times\{t=s\}.$$

Then

$$u(x,t) = \int_0^t u(x,t;s)\,ds.$$

Solve $\quad u_{tt} - u_{xx} = 0 \quad$ in $\mathbb{R}\times(0,\infty)$

$$u(x,0) = f(x) \quad u_t(x,0) = g(x) \quad \text{in } \mathbb{R}\times\{t=0\}.$$

Guess $u(x,t) = F(x+t) + G(x-t)$. We have

$$f(x) = F(x) + G(x) \rightarrow f'(x) = F'(x) + G'(x)$$
$$g(x) = F'(x) - G'(x)$$

$$\rightarrow \frac{1}{2}(f'(x) + g(x)) = F'(x) \qquad \rightarrow \begin{cases} F'(x) = \frac{1}{2}(f'(x) + g(x)) \\ G'(x) = \frac{1}{2}(f'(x) - g(x)). \end{cases}$$
$$f'(x) - \frac{1}{2}f'(x) - \frac{1}{2}g(x) = G'(x)$$

$$\rightarrow F(x) = \frac{1}{2}f(x) + \frac{1}{2}\int_0^x g(s)\,ds.$$
$$G(x) = \frac{1}{2}f(x) - \frac{1}{2}\int_0^x g(s)\,ds.$$

$$\begin{aligned} u(x,t) &= F(x+t) + G(x-t) \\ &= \frac{1}{2}f(x+t) + \frac{1}{2}\int_0^{x+t} g(s)\,ds + \frac{1}{2}f(x-t) - \frac{1}{2}\int_0^{x-t} g(s)\,ds \\ &= \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \end{aligned}$$

Thus

$$u(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy$$

and

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds.$$

W02#8 cont:

b) We will assume that $q(x,t)$ is uniformly bounded in $\mathbb{R} \times [0,\infty)$. From (a), we have

$$u(x,t) = -\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} q(y,s)u(y,s)\,dy\,ds. \qquad (*)$$

Let φ bdd continuous, which is complete under the sup norm.

$$F(\varphi)(x,t) := -\frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} q(y,s)\varphi(y,s)\,dy\,ds.$$

Then

$$|F(\varphi)| \leq \frac{1}{2} t^2 \|q\|_\infty \|\varphi\|_\infty. \qquad (*)$$

We now prove uniqueness in a small time interval $[0,T]$. By $(*)$, we have

$$\|F(\varphi)\|_\infty \leq \frac{1}{2} T^2 \|q\|_\infty \|\varphi\|_\infty$$

Choose T s.t. $\frac{1}{2}T^2\|q\|_\infty < \frac{1}{100}$. Then

$$\|F(\varphi)\|_\infty \leq \frac{1}{100}\|\varphi\|_\infty.$$

~~Let $u_1 = 1$, and $u_n = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}(1)$, we have~~

$$\cancel{\|u_{n+1} - u_n\|_\infty = \|F(u_n) - F(u_{n-1})\|_\infty \leq \frac{1}{100}\|u_n - u_{n-1}\|_\infty}$$

~~Therefore the $\{u_n\}$ converge to a~~

~~Let $u_1 = 1$ and $u_n = F^n(u_1) = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}(u_1)$ Then~~

$$\cancel{\|F(u_n) - F(u_{n-1})\|_\infty \leq \frac{1}{100}\|u_n - u_{n-1}\|_\infty}$$

W02 #8
cont

~~and hence F is a contraction~~

Thus for any bounded continuous φ_1, φ_2,

$$\|F(\varphi_1) - F(\varphi_2)\|_\infty \le \frac{1}{100} \|\varphi_1 - \varphi_2\|_\infty$$

which implies F is a contraction mapping. ~~Since a~~ and hence has a unique fixed point. ~~Fixed points~~ A fixed point of F satisfies (*) and hence is a solution to the PDE.

Thus is in the time interval $[0, T]$, we have shown a solution exists and is unique.

Now make a change of variable $t \mapsto t - T/2$. Since q is uniformly bounded, the above proof shows existence and uniqueness of a solution to the original PDE in the time interval $[T/2, 3T/2]$. In the original PDE making a change of variables $t \mapsto t - T$ and since q is uniformly bdd, ~~we can~~ the above proof shows existence and uniqueness of a solution to the original PDE in the time interval $[T, 2T]$. Continuing this, we can shows uniqueness of the solution for all time.

W02 #9b :

We have

$$u(\xi) = \int_D \delta(x-\xi)\,u(x)\,dx. \qquad \begin{array}{l} x=(x_1,x_2) \\ \xi=(\xi_1,\xi_2) \end{array}$$

$$= \int_D +\Delta G(x-\xi)\,u(x)\,dx$$

$$= +\left[\int_D \Delta G(x-\xi)u(x)\,dx\right]$$

$$= +\left[\int_{\partial D} \frac{\partial G}{\partial \nu}(x-\xi)u(x) - \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma + \int_D G(x-\xi)\Delta u(x)\,dx\right]$$

$$= +\left[\int_{\{x_1=0\}} \frac{\partial G}{\partial \nu}(x-\xi)u(x)\,d\sigma + \cancel{\int_{\{x_2=0\}} \frac{\partial G}{\partial \nu}(x-\xi)u(x)\,d\sigma}\right.$$

$$- \int_{\{x_1=0\}} \cancel{\frac{\partial u}{\partial \nu}G(x-\xi)}\,d\sigma - \int_{\{x_2=0\}} \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma + \left.\int_D G(x-\xi)f(x)\,dx\right]$$

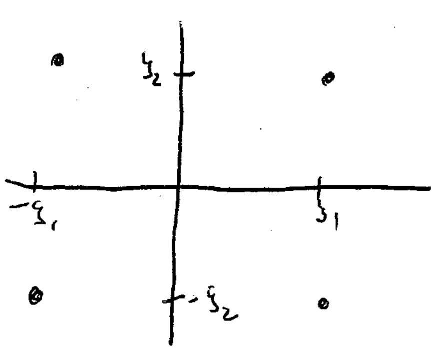
$$= +\left[\int_{\{x_1=0\}} \frac{\partial G}{\partial \nu}(x-\xi)u(x)\,d\sigma - \int_{\{x_2=0\}} \frac{\partial u}{\partial \nu}G(x-\xi)\,d\sigma\right] + \int_D G(x-\xi)f(x)\,dx.$$

Need:

$$\frac{\partial^2}{\partial x_1^2}G + \frac{\partial^2}{\partial x_2^2}G = 0.$$

$$G((x_1,0),\xi) = 0 \qquad x_1, x_2 > 0.$$

$$\frac{\partial G}{\partial x_1}((0,x_2),\xi) = 0.$$



$$G = \frac{1}{2\pi}\log|x-\xi| + \frac{A}{2\pi}\log\left((x_1+\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}$$

$$+ \frac{B}{2\pi}\log\left((x_1+\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$

$$+ \frac{C}{2\pi}\log\left((x_1-\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}.$$

W02 #9b cont:

$$G((x_1,0),\xi) = \frac{1}{2\pi}\log\left((x_1-\xi_1)^2+\xi_2^2\right)^{1/2}$$
$$+\frac{A}{2\pi}\log\left((x_1+\xi_1)^2+\xi_2^2\right)^{1/2} + \frac{B}{2\pi}\log\left((x_1+\xi_1)^2+\xi_2^2\right)^{1/2}$$
$$+\frac{C}{2\pi}\log\left((x_1-\xi_1)^2+\xi_2^2\right)^{1/2}$$

$$\longrightarrow \quad A=-B, \quad C=-1.$$

Now

$$\frac{\partial}{\partial x_1}\log\left((x_1-\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$
$$=\frac{1}{2}\frac{\partial}{\partial x_1}\log\left((x_1-\xi_1)^2+(x_2-\xi_2)^2\right)$$
$$=\frac{1}{2}\frac{2(x_1-\xi_1)}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}=\frac{x_1-\xi_1}{|x-\xi|^2}.$$

So

$$\frac{\partial G}{\partial x_1}((0,x_2),\xi) = \frac{1}{2\pi}\cdot\frac{0-\xi_1}{(0-\xi_1)^2+(x_2-\xi_2)^2}$$
$$+\frac{A}{2\pi}\cdot\frac{0+\xi_1}{(0+\xi_1)^2+(x_2+\xi_2)^2} + \frac{B}{2\pi}\cdot\frac{0+\xi_1}{(0+\xi_1)^2+(x_2-\xi_2)^2} + \frac{C}{2\pi}\cdot\frac{0-\xi_1}{(0-\xi_1)^2+(x_2+\xi_2)^2}.$$

$$\longrightarrow -1+B=0.$$

$$A-C=0. \longrightarrow B=1$$
$$A=C=-1$$

Thus

$$G(x,\xi)=\frac{1}{2\pi}\log|x-\xi| + \frac{1}{2\pi}\log\left((x_1+\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}$$
$$+\frac{1}{2\pi}\log\left((x_1+\xi_1)^2+(x_2-\xi_2)^2\right)^{1/2}$$
$$-\frac{1}{2\pi}\log\left((x_1-\xi_1)^2+(x_2+\xi_2)^2\right)^{1/2}.$$