W03 #2:

a) We have

$$\lambda \int u^2 = \int (\lambda u) u = \int (Lu) u = \int (\Delta u - a u) u$$

$$= -\int |\nabla u|^2 + a u^2$$

Thus

$$\lambda = \frac{-\int_c |\nabla u|^2 + a u^2}{\int_c u^2} < 0 \quad \text{since } a > 0.$$

b) Note that

$$\langle Lu, v\rangle = \int (\Delta u - au) v \, dx = -\int \nabla u \cdot \nabla v + auv$$

$$= \int (\Delta v - av) u \, dx = \langle u, Lv\rangle$$

Thus

$$\lambda \langle u, v\rangle - \mu \langle u, v\rangle = 0 \longrightarrow \lambda = \mu.$$

c) ~~In this case~~ $L = \sum_{j=1}^{3} (\partial_{x_j x_j} - a_j)$.

~~Suppose we knew the eigenvalues and eigenfunctions for~~

$$\cancel{\frac{d^2}{dx_j^2} u - a_j(x_j) = \lambda u.} \quad (*)$$

~~Then the eigenvalues of L are just the sum of the eigenvalues (functions from (*)) and eigenfunctions~~

W03 #3:

Let $\{\phi_n\}$ be a set of Neumann eigenfunctions for the Laplacian in Ω. So

$$\Delta \phi_n = \lambda_n \phi_n \quad \text{in } \Omega$$
$$\frac{\partial \phi_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Order the eigenvalues, let λ_0 denote the 0 eigenvalue. Note $\phi_0 = 1$.

Let $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$ where $a_n(t) = \dfrac{\int_\Omega u(x,t)\phi_n(x)dx}{\int_\Omega \phi_n(x)^2 dx}$.

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x) \quad \text{where } f_n = \frac{\int_\Omega f(x)\phi_n(x)\,dx}{\int_\Omega \phi_n(x)^2\,dx}$$

Then

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) - a_n(t)\Delta\phi_n(x) - f_n\phi_n(x) = 0.$$

$$\sum_{n=0}^{\infty} a_n'(t)\phi_n(x) - \lambda_n a_n(t)\phi_n(x) - f_n\phi_n(x) = 0.$$

for $n \geq 1 \longrightarrow$

$$a_n'(t) - \lambda_n a_n(t) = f_n.$$

$$e^{-\lambda_n t} a_n'(t) - \lambda_n e^{-\lambda_n t} a_n(t) = f_n e^{-\lambda_n t}.$$

$$(e^{-\lambda_n t} a_n(t))' = f_n e^{-\lambda_n t}.$$

$$e^{-\lambda_n t} a_n(t) = -\frac{1}{\lambda_n} f_n e^{-\lambda_n t} + c.$$

$$a_n(t) = -\frac{1}{\lambda_n} f_n + c e^{\lambda_n t}.$$

$a_0'(t) = f_0$

$a_0(t) = f_0 t + \tilde{c}$

$a_0(0) = b_0$

$\longrightarrow a_0(t) = f_0 t + b_0$.

Let $u_0(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)$. Then $a_n(0) = b_n$. So

$$a_n(t) = -\frac{f_n}{\lambda_n} + \left(b_n + \frac{f_n}{\lambda_n}\right)e^{\lambda_n t}.$$

Note that $\lambda_n \int \phi_n^2\,dx = \int \phi_n \Delta\phi_n\,dx = -\int |\nabla\phi_n|^2\,dx \longrightarrow \lambda_n \leq 0 \;\forall n.$

WO3 #3 cont:

Thus

$$u(x,t) = (f_0 t + b_0)\phi_0(x) + \sum_{n=1}^{\infty}\left[\frac{f_n}{-\lambda_n} + \left(b_n - \frac{f_n}{-\lambda_n}\right)e^{\lambda_n t}\right]\phi_n(x)$$

and hence an approx. formula for u as $t \to \infty$ is

$$u(x,t) = (f_0 t + b_0) - \sum_{n=1}^{\infty}\frac{f_n}{\lambda_n}\phi_n(x).$$

#

W03 #4:

Claim: If f has no Fourier coefficients of negative index, then so does $1/f$.

Pf: We have $f(x) = \sum_{k\geq 0} a_k e^{ikx}$. ~~[illegible]~~ Suppose $1/f = \sum_{k \geq 0} b_k e^{ikx}$, where we will determine $\{b_k\}$. We have

$$\left(\sum_{k\geq 0} a_k e^{ikx}\right)\left(\sum_{k\geq 0} b_k e^{ikx}\right) = f \cdot 1/f = 1 + 0e^{ix} + 0 \cdot e^{2ix} + \ldots$$

$$\sum_{m \geq 0} \left(\sum_{k+j=m} a_k b_j\right) e^{imx} = 1$$

$$\longrightarrow \quad a_0 b_0 = 1$$
$$a_1 b_0 + a_0 b_1 = 0$$
$$a_2 b_0 + a_1 b_1 + a_0 b_2 = 0$$
$$\vdots$$

Thus we can successively solve for the b_k. By uniqueness of Fourier coefficients, the $\{b_k\}$ are precisely the Fourier coefficients of $1/f$. #

~~Note that if 1/f has no neg. indexed Fourier coeffs~~

a) We solve the PDE via method of characteristics. We have

$$F(p, q, z, x, t) = q - p - z^4.$$

$$\dot{t}(s) = 1 \qquad t(0) = 0$$
$$\dot{x}(s) = -1 \qquad x(0) = x_0$$
$$\dot{z}(s) = z^4 \qquad z(0) = u_0(x_0).$$

$$\longrightarrow \quad t(s) = s, \quad x(s) = -s + x_0 \; \rightarrow \; x + t = x_0$$

$$\frac{dz}{ds} = z^4 \longrightarrow \frac{1}{z^4} dz = ds \longrightarrow -\frac{1}{3z^3} = s + C.$$

W03#4 cont:

$$\longrightarrow C = -\frac{1}{3u_0(x_0)^3}.$$

Then

$$-\frac{1}{3u^3} - \cancel{\frac{1}{3t^3}} = t - \frac{1}{3u_0(x+t)^3}$$

$$+\frac{1}{u^3} = \frac{1}{u_0(x+t)^3} - 3t.$$

By the Claim if $1/f$ has no negative indexed Fourier coeff, then so does f. (apply claim to $1/f$ and $1/_{1/f} = f$)

Fix t. Suppose $u_0 \in A$. then $u_0(x+t)^3 \in A$. By the Claim, $\frac{1}{u_0(x+t)^3} \in A$. Since t is fixed, $\frac{1}{u_0(x+t)^3} - 3t \in A$ and hence $(\frac{1}{u})^3 \in A$. Therefore $u^3 \in A$. Since $u \notin A$ would imply $u^3 \notin A$, we have $u \in A$.

b) We expand $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}$. Then

$$u_t = \sum_{k\in\mathbb{Z}} \hat{u}_t(k,t)e^{ikx}$$

$$u_x = \sum_{k\in\mathbb{Z}} \hat{u}(k,t)\,ik\,e^{ikx}.$$

$$u^4 = \Big(\sum_{k\in\mathbb{Z}} \hat{u}(k,t)e^{ikx}\Big)^4 = \sum_{j\in\mathbb{Z}}\Big[\sum_{k_1+k_2+k_3+k_4=j} \hat{u}(k_1,t)\hat{u}(k_2,t)\hat{u}(k_3,t)\hat{u}(k_4,t)\Big]e^{ijx}$$

Thus

$$\hat{u}_t(k,t) = ik\hat{u}(k,t) + \sum_{a_1+a_2+a_3+a_4=k} \hat{u}(a_1,t)\hat{u}(a_2,t)\hat{u}(a_3,t)\hat{u}(a_4,t).$$

If $\hat{u}$ had no negative indexed coefficients, then we

W03 #4 cont.

have

$$\hat{u}_t(0,t) = \hat{u}(0,t)^4.$$

$$\hat{u}_t(1,t) = i\hat{u}(1,t) + 4\hat{u}(1,t)\hat{u}(0,t)^3$$

$$\cancel{\hat{u}_t(2,t) = 2i\hat{u}(2,t) + 8\hat{u}(2,t) + 6\hat{u}(1,t)^2}.$$

$$\cancel{+ 8\hat{u}(2,t)}$$

$$\hat{u}_t(2,t) = 2i\hat{u}(2,t) + 4\hat{u}(2,t)\hat{u}(0,t)^3.$$

$$+ 6\hat{u}(1,t)^2\hat{u}(0,t)^2.$$

W03 #5:

We have $F(p,q,z,x,y) = xp + (x+y)q - 1$. Then

$$\dot{x}(s) = x(s) \qquad x(0) = 1$$
$$\dot{y}(s) = x(s) + y(s) \qquad y(0) = y_0$$
$$\dot{z}(s) = 1 \qquad z(0) = y_0. \qquad 0 \le y_0 \le 1.$$

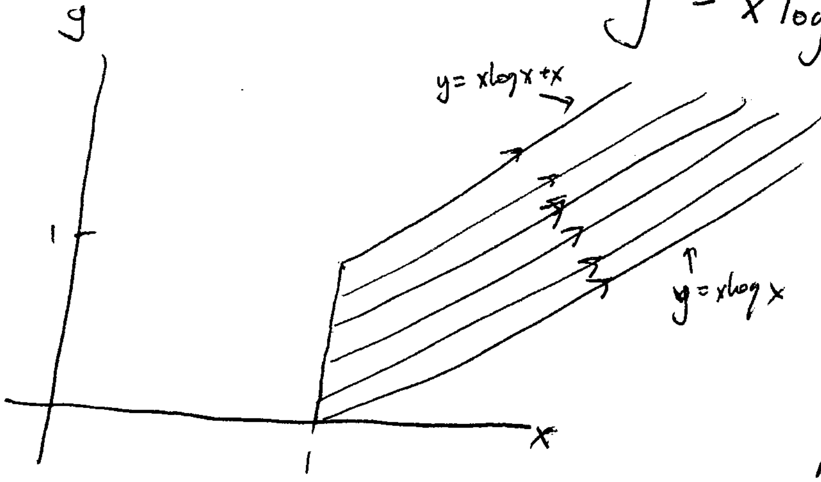
Then

$$x(s) = e^s.$$

$$\frac{dy}{ds} - y = e^s \rightarrow e^{-s}\frac{dy}{ds} - e^{-s}y = 1$$
$$(e^{-s}y)' = 1.$$
$$y = e^s(s+c). \qquad y(0) = y_0.$$
$$\rightarrow y = e^s(s+y_0).$$

$$z(s) = s + y_0 \rightarrow u(x,y) = \frac{y}{x}.$$

The characteristics are $y = x\log x + xy_0$.



Since these characteristics don't intersect, u is uniquely determined by the given conditions in the region

$\{(x,y): x \ge 1,\ x\log x \le y \le x\log x + x\}$.

W03 #6 :

~~For~~ Let

$$v(x) = v(x_1, x_2, x_3) := \begin{cases} u(x_1, x_2, x_3) & \text{if } x_3 \geq 0 \\ -u(x_1, x_2, -x_3) & \text{if } x_3 < 0. \end{cases}$$

(Since $u(x,y,0) = -u(x,y,0) \to u(x,y,0) = 0$).

For each $0 < r < 1$, let $B_r := B(0,r)$. We show v is harmonic in B_r for each r. Let

$$P_v(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r} \frac{v(y)}{|x-y|^n} \, d\sigma.$$

Then $\Delta P_v = 0$ in B_r and $P_v = v$ on ∂B_r. Note that $v - P_v$ is harmonic in $B_r \cap \{x_3 > 0\}$ and $v = P_v$ on ∂B_r. For $x \in B_r \cap \{x_3 = 0\}$, notice that

$$P_v(x) = \int_{\partial B_r} \frac{v(y)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y = \int_{\partial B_r \cap \{y_3 \geq 0\}} + \int_{\partial B_r \cap \{y_3 < 0\}}$$

$$= \int_{\substack{\partial B_r \\ \{y_3 \geq 0\}}} \frac{u(y_1, y_2, y_3)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y + \int_{\substack{\partial B_r \\ \{y_3 < 0\}}} \frac{-u(y_1, y_2, -y_3)}{((x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2)^{n/2}} \, d\sigma_y.$$

$$= 0.$$

Thus $v - P_v = 0$ on $B_r \cap \{x_3 = 0\}$. By the Strong Max Principle,

$$\max_{\overline{B_r \cap \{x_3 > 0\}}} v - P_v = \max_{\partial(B_r \cap \{x_3 > 0\})} v - P_v = 0.$$

W03 #6 con't:

Thus $v \leq \cancel{P_0}\, P_v$ in $\overline{B_r \cap \{x_3 > 0\}}$.

Interchanging P_v and v gives $v = P_v$ in $\overline{B_r \cap \{x_3 > 0\}}$.

Similarly considering $\overline{B_r \cap \{x_3 < 0\}}$ shows $v = P_v$ in $\overline{B_r}$.

Therefore v is harmonic in B_r $\forall\, 0 < r < 1$. Thus v is harmonic in $B(0,1)$.

$\#$

W03#7: We want to solve

$$u_{xx} = g$$
$$u(0) = u(L/3) = u(L) = 0.$$

By FTC,

$$u'(x) = u'(0) + \int_0^x g(t)\, dt.$$

$$\longrightarrow u(x) = u(0) + u'(0)x + \int_0^x \int_0^t g(s)\, ds\, dt.$$

Since $u(0) = 0$,

$$u(x) = u'(0)x + \int_0^x \int_0^t g(s)\, ds\, dt.$$

As $u(L) = 0$,

$$0 = u'(0)L + \int_0^L \int_0^t g(s) ds\, dt.$$

$$\longrightarrow u'(0) = -\frac{1}{L}\int_0^L \int_0^t g(s)\, ds\, dt.$$

Since $u(L/3) = 0$,

$$0 = -\frac{1}{\cancel{L}}\int_0^L \int_0^t g(s) ds\, dt \cdot \frac{\cancel{L}}{3} + \int_0^{L/3} \int_0^t g(s)\, ds\, dt.$$

$$\frac{1}{3}\int_0^L \int_0^t g(s)\, ds\, dt = \int_0^{L/3} \int_0^t g(s)\, dt. \quad (*)$$

Thus if g obeys $(*)$ ~~[illegible]~~, there is a solution of the differential equation.

W03 #8

a) We have

$u(x,t) = 0$ for $x \in \partial D$.
$\to u_t(x,t) = 0$ for $x \in \partial D$.

$$\dot{E}(t) = \int_D \varepsilon^2 2u_t u_{tt} + 2\nabla u \cdot \nabla u_t \, dx$$
$$= \int_D \varepsilon^2 2u_t u_{tt} - 2\Delta u \, u_t \, dx$$
$$= \int_D 2u_t (\varepsilon^2 u_{tt} - \Delta u) \, dx$$
$$= \int_D 2u_t (-u_t) \, dx \le 0.$$

b). By (a), $E(t) \le E(0)$. We have

$$E(0) = \int_D \varepsilon^2 u_t(x,0)^2 + |\nabla u(x,0)|^2 \, dx.$$
$$= \int_D \varepsilon^2 (\varepsilon^{-2\alpha} f(x)^2) \, dx$$
$$= \varepsilon^{2(1-\alpha)} \int_D f(x)^2 \, dx \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Since $\alpha < 1$. Thus

$$\int_D |\nabla u(x,t)|^2 \, dx \le E(t) \le \varepsilon^{2(1-\alpha)} \int_D f(x)^2 \, dx \longrightarrow 0 \text{ as } \varepsilon \to 0.$$

c) Expand $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x)$ where

$$\Delta \phi_n + \lambda_n \phi_n = 0. \quad \text{in } D$$
$$\phi_n = 0 \quad \text{on } \partial D$$
$$\|\phi_n\|_{L^2} = 1.$$

W03 #8 cont.:

$$\cancel{\varepsilon^2 u_{tt} + u_t - \Delta u = 0.}$$

$$\cancel{\sum_{n=1}^{\infty} \varepsilon^2 (\phi_n)_{tt} a_n + a_n (\phi_n)_t + \lambda_n \phi_n a_n = 0.}$$

$$\cancel{\sum_{n=1}^{\infty} a_n \left(\varepsilon^2 (\phi_n)_{tt} + (\phi_n)_t + \lambda_n \phi_n \right) = 0.}$$

$$\varepsilon^2 u_{tt} + u_t = \Delta u.$$

$$\varepsilon^2 \sum_{n \geq 1} a_n''(t) \phi_n(x) + \sum_{n \geq 1} a_n'(t) \phi_n(x) = \sum_{n \geq 1} -\lambda_n a_n(t) \phi_n(x)$$

$$\longrightarrow \quad \varepsilon^2 a_n'' + a_n' + \lambda_n a_n = 0.$$

$$r^2 \varepsilon^2 + r + \lambda = 0 \longrightarrow r = \frac{-1 \pm \sqrt{1 - 4\varepsilon^2 \lambda_n}}{2\varepsilon^2}$$

$$a_n(t) = A e^{\underbrace{\boxed{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_n}}{2\varepsilon^2}}}_{= r_{1n}} t} + B e^{\underbrace{\boxed{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_n}}{2\varepsilon^2}}}_{= r_{2n}} t}. \qquad (\text{more } \lambda_n \to \infty)$$

$u(x,0) = 0 \longrightarrow a_n(0) = 0. \longrightarrow A + B = 0.$

$u_t(x,0) = \varepsilon^{-1} f(x) \longrightarrow \displaystyle\sum_{n=1}^{\infty} a_n'(0) \phi_n(x) = \varepsilon^{-1} \sum_{n=1}^{\infty} f_n \phi_n(x).$

$$a_n'(0) = \varepsilon^{-1} f_n.$$

Since f is an eigenfunction, $f_n = 0$ except for one such n.

$$a_n(t) = A e^{r_{1n} t} + B e^{r_{2n} t} = A(e^{r_{1n} t} - e^{r_{2n} t})$$

$$a_n'(t) = A(r_{1n} e^{r_{1n} t} - r_{2n} e^{r_{2n} t})$$

$$\varepsilon^{-1} f_n = a_n'(0) = A(r_{1n} - r_{2n})$$

$$A = \frac{\varepsilon^{-1} f_n \varepsilon^2}{\sqrt{1-4\varepsilon^2\lambda_n}} = \frac{\varepsilon f_n}{\sqrt{1-4\varepsilon^2\lambda_n}}$$

W03 #8 cont.:

Since f is an eigenfunction, $f_n = \begin{cases} 0 & \forall\, n \neq n_0 \end{cases}$

for some n_0 and hence

$$u(x,t) = \frac{\varepsilon f_{n_0}}{\sqrt{1-4\varepsilon^2\lambda_{n_0}}}\,\phi_{n_0}(x)\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)$$

$$= \frac{\varepsilon f(x)}{\sqrt{1-4\varepsilon^2\lambda_{n_0}}}\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)$$

Thus

$$\int |\nabla u(x,t)|^2\,dx = \frac{\varepsilon^2}{(1-4\varepsilon^2\lambda_{n_0})}\left(e^{\frac{-1+\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t} - e^{\frac{-1-\sqrt{1-4\varepsilon^2\lambda_{n_0}}}{2\varepsilon^2}t}\right)\int |\nabla f|^2\,dx$$

$$\longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$