W04 #1:

a) $\lambda = 0$: $\Delta u = 0$ $\quad u = F(x)G(y)$.

$u(x,0) = u(x,\pi) = 0$ $\quad F''(x)G(y) + F(x)G''(y) = 0$.

$\downarrow$

$G(0) = G(\pi) = 0$. $\quad -\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)} = -\mu$.

Need $\mu > 0$.

$G''(y) + \mu G(y) = 0$.

$G(y) = A\cos(\sqrt{\mu}y) + B\sin(\sqrt{\mu}y)$.

$0 = G(0) = A \longrightarrow G(y) = B\sin(\sqrt{\mu}y)$.

$0 = G(\pi) \Longleftrightarrow 0 = \sin(\sqrt{\mu}\pi) \longrightarrow \sqrt{\mu}\pi = n\pi, \ n = 1, 2, \ldots$

$\mu_n = n^2, \ n = 1, 2, \ldots$

$G_n(y) = \sin(ny)$

$\longrightarrow \frac{F''}{F} = \mu \longrightarrow F'' - n^2 F = 0$.

$\longrightarrow F(x) = Ae^{-nx} + Be^{nx}$.

$B = 0$ since we want bdd solns.

Therefore

$$u(x,y) = \sum_{n \geq 1} A_n e^{-nx} \sin(ny).$$

b) $\lambda > 0$. We have

$$F''(x)G(y) + F(x)G''(y) + \lambda F(x)G(y) = 0.$$

$$-\frac{F'' + \lambda F}{F} = +\frac{G''}{G} = -\mu \text{ need } \mu > 0.$$

$\rightarrow \mu_n = n^2$

$G_n(y) = \sin(ny)$. $\longrightarrow F'' + \lambda F - n^2 F = 0$.

$F'' + (\lambda - n^2)F = 0$.

W04 #1 cont:

We have:

if $\lambda > n^2$: $F'' + (\lambda - n^2)F = 0$.

$F = A\cos(\sqrt{\lambda - n^2}\,x) + B\sin(\sqrt{\lambda - n^2}\,x)$

if $\lambda = n^2$ $\quad F = A + Bx$

if $\lambda < n^2$ $\quad F = Ae^{-\sqrt{n^2-\lambda}\,x} + \cancel{Be^{+\sqrt{n^2-\lambda}\,x}}$

since we want bdd solutions

Thus

$$u(x,y) = \sum_{n<\sqrt{\lambda}} [A_n\cos(\sqrt{\lambda-n^2}\,x) + B_n\sin(\sqrt{\lambda-n^2}\,x)]\sin(ny)$$
$$+ (C + Dx)\sin(\sqrt{\lambda}\,y)\,1_{\sqrt{\lambda}\in\mathbb{Z}}$$
$$+ \sum_{n>\sqrt{\lambda}} E_n e^{-\sqrt{n^2-\lambda}\,x}\sin(ny).$$

c) $\lambda < 0$. here $\gamma := -\lambda > 0$. Then the only thing that changes is the equation for F is now

$$F'' + (\lambda - n^2)F = 0 \longrightarrow F'' - (\gamma + n^2)F = 0$$

Thus

$$\longrightarrow F_n(x) = A_n e^{-\sqrt{\gamma+n^2}\,x} + \cancel{B_n e^{\sqrt{\gamma+n^2}\,x}}$$

$$u(x,y) = \sum_{n\geq 1} A_n e^{-\sqrt{-\lambda+n^2}\,x}\sin(ny).$$

(since we only want bdd solutions).

W04 #2: Fix arbitrary $v \in C_0^\infty(\Omega)$. Since u_0 is the minimizer of the functional,

$$0 = \lim_{\varepsilon \to 0} \frac{D(u_0+\varepsilon v) - D(u_0)}{\varepsilon}$$

$$= \int_\Omega 2(u_0)_x v_x + 2(u_0)_y v_y + f v \, dx + \int_{\partial\Omega} 2 a u_0 v \, d\sigma$$

$$= \int_\Omega 2(\nabla u_0 \cdot \nabla v) + f v \, dx + 2\int_{\partial\Omega} a u_0 v \, d\sigma$$

$$= 2\int_\Omega -v \Delta u_0 \, dx + 2\int_{\partial\Omega} (\nabla u_0 \cdot \nu) v \, d\sigma + \int_\Omega f v \, dx + 2\int_{\partial\Omega} a u_0 v \, d\sigma.$$

$$= \int_\Omega (-2\Delta u_0 + f) v \, dx + 2\int_{\partial\Omega} (\nabla u_0 \cdot \nu + a u_0) v \, d\sigma.$$

Thus

$$-2\Delta u_0 + f = 0 \quad \text{in } \Omega$$
$$\nabla u_0 \cdot \nu + a u_0 = 0 \quad \text{on } \partial\Omega.$$

W04 #3:

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right) \quad w = y_1 + iy_2.$$

We have

$$u(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{f(w)}{w - z}\, dw$$

Let $K(w) = \frac{1}{2\pi}\log|w| = \frac{1}{4\pi}\log(y_1^2 + y_2^2)$. We have

$$\frac{\partial}{\partial w} K(w) = \frac{1}{2}\cdot\frac{1}{4\pi}\left[\frac{2y_1}{y_1^2+y_2^2} - i\frac{2y_2}{y_1^2+y_2^2}\right] = \frac{1}{4\pi}\cdot\frac{1}{w}.$$

Thus

$$u(z) = \frac{4\pi}{2\pi}\int_{\mathbb{C}} f(w) K_w(z-w)\, dw$$

$$= 2\int_{\mathbb{C}} f(z-w) K_w(w)\, dw.$$

Therefore

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)u = 2\frac{\partial u}{\partial \bar{z}} = 4\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} f(z-w) K_w(w)\, dw$$

$$= -4\int_{\mathbb{C}} \frac{\partial}{\partial \bar{w}} f(z-w) K_w(w)\, dw$$

$$= 4\int_{\mathbb{C}} f(z-w) \frac{\partial}{\partial \bar{w}}\frac{\partial}{\partial w} K(w)\, dw.$$

$$= \int_{\mathbb{C}} f(z-w)\Delta K(w)\, dw.$$

$$= \int_{\mathbb{C}} f(z-w)\,\delta(w)\, dw.$$

$$= f(z).$$

W04#4:

Let y_1, y_2 be 2 ~~unique~~ distinct solutions. Then $w = y_1 - y_2$ satisfies

$$-w'' + pw = 0 \qquad 0 < x < \pi$$
$$w(0) = 0, \; w'(\pi) = 0.$$

We have

$$0 = \int_0^\pi -w''w + pw^2\,dx = \int_0^\pi (w')^2 + pw^2\,dx$$

~~Let ε~~

By Poincaré's

We now find the eigenfunctions of the $-\Delta$:

$$-y'' = \lambda y$$
$$y(0) = y'(\pi) = 0. \rightarrow \lambda > 0 \quad y(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x.$$
$$0 = y(0) = A \rightarrow y(x) = B\sin\sqrt{\lambda}\,x.$$
$$\downarrow$$
$$y'(x) = B\sqrt{\lambda}\cos\sqrt{\lambda}\,x.$$
$$\rightarrow \sqrt{\lambda}\,\pi = \frac{2n+1}{2}\pi, \; n = 0,1,2\ldots$$
$$\lambda_n = (n+\tfrac{1}{2})^2; \; n = 0,1,2,\ldots$$

Let $\phi_n := \sin((n+\frac{1}{2})x)$, $\lambda_n := (n+\frac{1}{2})^2$. Then

$$-\phi_n'' = \lambda_n \phi_n$$
$$\phi_n(0) = \phi_n'(\pi) = 0.$$

We have $w(x) = \sum_{n \geq 0} a_n \phi_n(x)$. Then

W04 #4 cont:

$$\int_0^\pi -w''w\,dx = \int_0^\pi \Big(\sum_{n\geq 0} a_n \lambda_n \phi_n(x)\Big)\Big(\sum_{m\geq 0} a_m \phi_m(x)\Big)\,dx$$

$$= \int_0^\pi \sum_{n\geq 0} a_n^2 \lambda_n \phi_n^2(x)\,dx$$

$$= \sum_{n\geq 0} a_n^2 \lambda_n \int_0^\pi \phi_n(x)^2\,dx$$

$$\geq \frac{1}{4} \sum_{n\geq 0} a_n^2 \int_0^\pi \phi_n(x)^2\,dx$$

$$= \frac{1}{4}\int_0^\pi w^2\,dx$$

and the $\frac{1}{4}$ is sharp. Thus $(*)$

$$0 = \int_0^\pi -w''w + pw^2\,dx \geq \int_0^\pi \Big(\frac{1}{4}+p\Big) w^2\,dx.$$

when $p > -\frac{1}{4}$, the assume

$$0 = \int_0^\pi \Big(\frac{1}{4}+p\Big) w^2\,dx \geq 0$$

and hence $w = 0$. Since $\frac{1}{4}$ is sharp in $(*)$, $\lambda_0 = -\frac{1}{4}$.

W04 #5: Taking the Fourier Transform in the x-variable, we have

$$u_{xx} + u_{yy} = 0 \longrightarrow -4\pi^2 \xi^2 \hat{u} + \hat{u}_{yy} = 0$$

Note A, B could depend on ξ

$$\longrightarrow \hat{u}(\xi, y) = Ae^{-2\pi|\xi|y} + Be^{2\pi|\xi|y}$$

Since we want a bounded solution, $B = 0$. Therefore

$$\hat{u}(\xi, y) = Ae^{-2\pi|\xi|y}.$$

Since

$$u_y(x,0) - u(x,0) = f(x) \longrightarrow \hat{u}_y(\xi,0) - \hat{u}(\xi,0) = \hat{f}(\xi)$$

Thus as $\hat{u}_y(\xi, y) = -2\pi|\xi| Ae^{-2\pi|\xi|y}$, we have

$$-2\pi|\xi| A - A = \hat{f}(\xi)$$

$$A = \frac{-\hat{f}(\xi)}{2\pi|\xi| + 1}.$$

Thus

$$\hat{u}(\xi, y) = \frac{-\hat{f}(\xi)}{2\pi|\xi|+1} e^{-2\pi|\xi|y}.$$

~~Since f is smooth of compact support, the RHS is Schwartz, so~~ $\cancel{u(x,y) = \left(\frac{-\hat{f}(\xi)}{2\pi|\xi|+1} e^{-2\pi|\xi|y}\right)^{\vee} \in \mathcal{S}(\mathbb{R}^2)}$

Since $y \geq 0$ and $\hat{f} \in \mathcal{S}(\mathbb{R})$ (as $f \in \mathcal{S}(\mathbb{R})$), we have $|\hat{u}(\xi, y)| \leq |\hat{f}(\xi)|$. Thus $\hat{u}(\xi, y)$ is integrable on ξ. Therefore

$$u(x,y) = \int_{\mathbb{R}} \hat{u}(\xi, y) e^{2\pi i \xi x} \, d\xi$$

$$= \int_{\mathbb{R}} \frac{-\hat{f}(\xi)}{2\pi|\xi|+1} e^{-2\pi|\xi|y} e^{2\pi i \xi x} \, d\xi$$

We also have

$$|u(x,y)| \leq \int_{\mathbb{R}} |\hat{f}(\xi)| e^{-2\pi|\xi|y} \, d\xi$$

$$\leq \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \left(\int_{\mathbb{R}} e^{-4\pi^2 \xi^2 y^2} d\xi \right)^{1/2}$$

$u = 2\pi \xi y$
$du = 2\pi y \, d\xi$

Since
$$\int_{-\infty}^{\infty} e^{-(2\pi \xi y)^2} d\xi = \frac{1}{2\pi y} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{2\sqrt{\pi}\, y}.$$

Thus
$$|u(x,y)| \leq \|f\|_{L^2} (2\sqrt{\pi})^{-1/2} y^{-1/2} \longrightarrow 0$$
as $y \longrightarrow \infty$ uniformly in x

Characteristics point ↑ since they pt in increasing time.

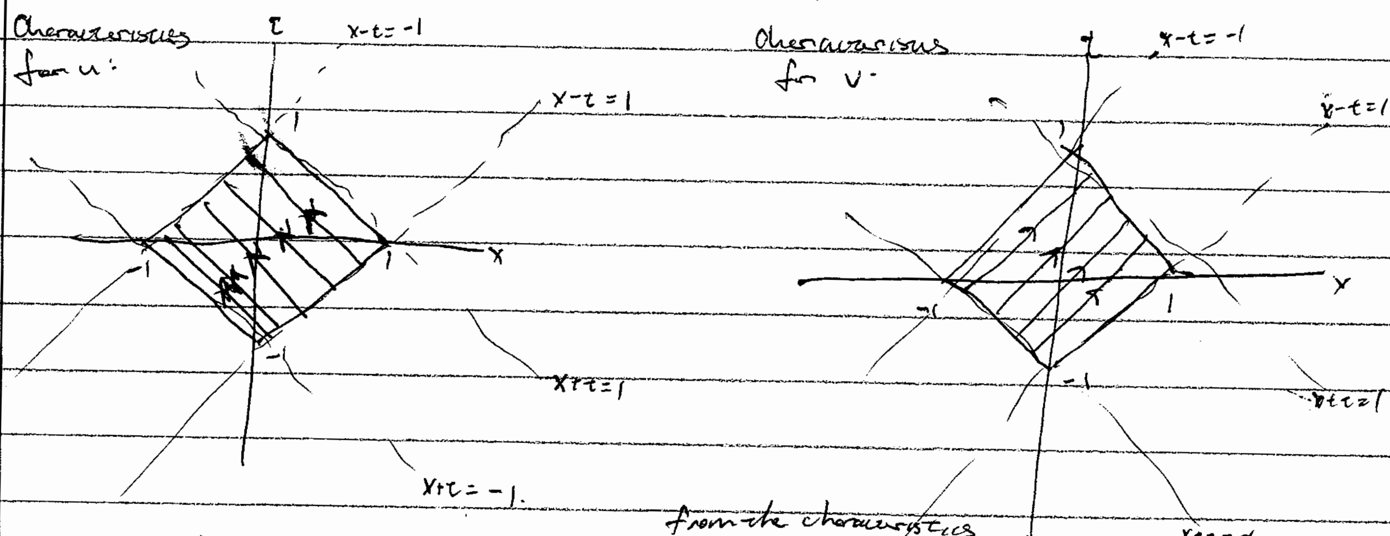
W04 #6. We have

Characteristics:

$$u_t - u_x = 0 \longrightarrow x + t = C$$

$$v_t + v_x = 0 \qquad x - t = C.$$

u and v are constant on their respective characteristics



a) This problem is well posed since from the characteristics to ~~find u~~ solve for u it suffices to know u on the line $x - t = -1$ and to solve for v it suffices to know v on the line $x + t = -1$. The ~~solution~~ solution is given by

$$u(x,t) = u_0(x+t)$$

$$v(x,t) = v_0(x-t)$$

b) From the characteristics, this problem is not well-posed.

WO4 #7:

a) We have

$$f(x') = \int_{-\infty}^{\infty} \delta(x-x') f(x)\, dx$$
$$= \int_{-\infty}^{\infty} L G(x,x') f(x)\, dx = \int_{-\infty}^{\infty} G(x,x') L f(x)\, dx$$

We also have

$$\int_{-\infty}^{\infty} Lf(x) G(x-x', 0)\, dx = \int_{-\infty}^{\infty} (Lf)(x+x') G(x,0)\, dx$$
$$= \int_{-\infty}^{\infty} (Lg)(x) G(x,0)\, dx \qquad g(x) = f(x+x')$$

So

$$= g(0) = f(x').$$

$$\int_{-\infty}^{\infty} Lf(x) G(x-x', 0)\, dx = \int_{-\infty}^{\infty} Lf(x) G(x,x')\, dx. \qquad \forall f.$$

Therefore $G(x-x', 0) = G(x,x')$.

b) Since we want

$$\frac{d^2}{dx^2} G - G = \delta(x-x') = \begin{cases} 0 & \text{if } x \neq x' \end{cases}$$

and $G \to 0$ as $x \to -\infty$ and $G \to 0$ as $x \to \infty$, G must be a homogeneous ~~fldo~~ solution to $y'' - y = 0$ and satisfy the limit conditions. Thus

$$G(x,x') = \begin{cases} a_- e^{x} & \text{if } x < x' \\ a_+ e^{-x} & \text{if } x > x' \end{cases}$$

W04 #7 cont'd

We need $a_- e^{x'} = a_+ e^{-x'}$.

We also need

$$G'(x'_+,x')-G'(x'_-,x') = \int_{x'_-}^{x'_+} LG(x,x')\,dx = \int_{x'_-}^{x'_+} \delta(x-x') = 1$$

We have

$$G'(x,x') = \begin{cases} a_- e^{x} & \text{if } x<x' \\ -a_+ e^{-x} & \text{if } x>x' \end{cases}$$

Then

$$\left.\begin{array}{l} -a_+e^{-x'} - a_- e^{x'} = 1. \\ a_- e^{x'} = a_+ e^{-x'}. \end{array}\right] \longrightarrow -a_- e^{x'} = \frac{1}{2}$$

$$a_- = -\frac{1}{2e^{x'}}.$$

$$a_+ = a_- e^{2x'} = -\frac{1}{2e^{x'}} e^{2x'} = -\frac{e^{x'}}{2}.$$

So

$$G(x,\xi) = \begin{cases} -\frac{1}{2e^{\xi}} e^{x} & \text{if } x \leq \xi \\ -\frac{e^{\xi}}{2} e^{-x} & \text{if } x>\xi. \end{cases}$$

W04 #8:

The system is

$$x' = x - y^2$$
$$y' = y - x^2.$$

The equilibrium pts are:

$$x = y^2.$$
$$y = x^2 \longrightarrow y = y^4 \longrightarrow y(y^3 - 1) = 0.$$

$$y = 0, \quad y = 1.$$
$$x = 0 \qquad x = 1.$$

$$(0,0) \text{ and } (1,1).$$

The Jacobian is:

$$J(x,y) = \begin{pmatrix} 1 & -2y \\ -2x & 1 \end{pmatrix}$$

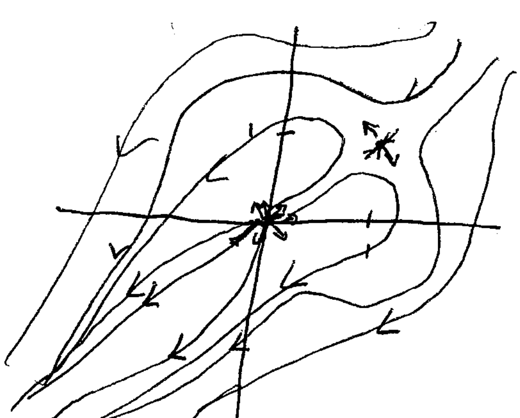
$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J(1,1) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

unstable source node

eigenvalues $3, -1$

eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

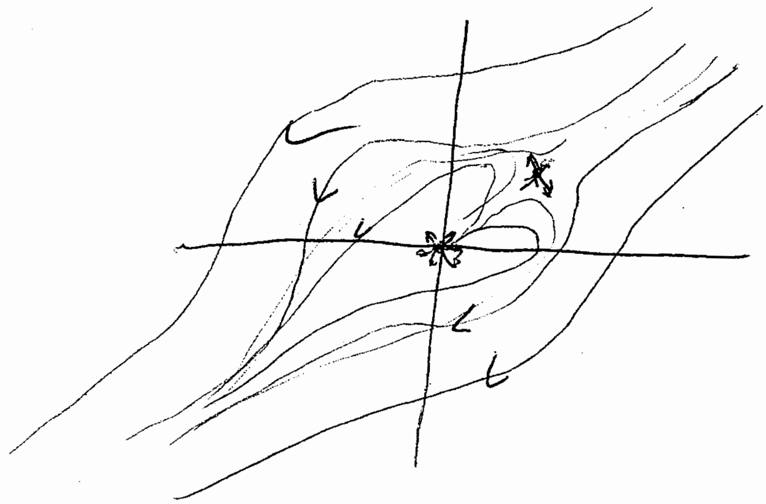
saddle.



Can also sketch direction field to guess phase plane.

W04 #8 cont'd:

Since when x, y large, the paths look like $x^3 - y^3 = C$



Let $w(t) = u(t) - v(t)$. Then

$$\dot{w} = u_t - v_t = u - v^2 - v + u^2$$
$$= (u-v) + (u^2 - v^2)$$
$$= (u-v) + (u-v)(u+v)$$
$$= (u-v)(u+v+1)$$

$$\int_0^t w(s)\,ds = \int_0^t (u-v)(u+v+1)\,ds.$$

$$w_t = w(u+v+1)$$
$$w(0) = 0.$$

$$w(t) = \int_0^t w(s)(u+v+1)\,ds.$$

Therefore if $w \geq 0\ \forall t$, then

$$w_t \leq w\,|u+v+1|$$

and so by Gronwall, $w(t) \leq e^{\int_0^t |u+v+1|\,ds}\, w(0) = 0.$

and hence $w(t) = 0\ \forall t.$

W04 #8 cont:

Thus

$$\int_0^t u_t(s) - v_t(s)\, ds = \int_0^t (u-v)(u+v+1)\, ds.$$

As $u(0) - v(0) = 0$,

Then $$u(t) - v(t) = \int_0^t (u-v)(u+v+1)\, ds$$

$$|u(t) - v(t)| \leq \int_0^t |u(s) - v(s)|\,|u(s) + v(s) + 1|\, ds.$$

Then by the integral form of Gronwall,

$$|u(t) - v(t)| \leq 0.$$

$$\rightarrow u = v \quad \forall t. \quad \#$$