F02 #1

We have

$$\dot{x} = x - y$$
$$\dot{y} = x(y^2 - 1).$$

The stationary pts are

$$x = y \qquad\qquad (0,0)$$
$$\longrightarrow \quad (1,1)$$
$$x = 0 \quad y = \pm 1 \qquad (-1,-1).$$

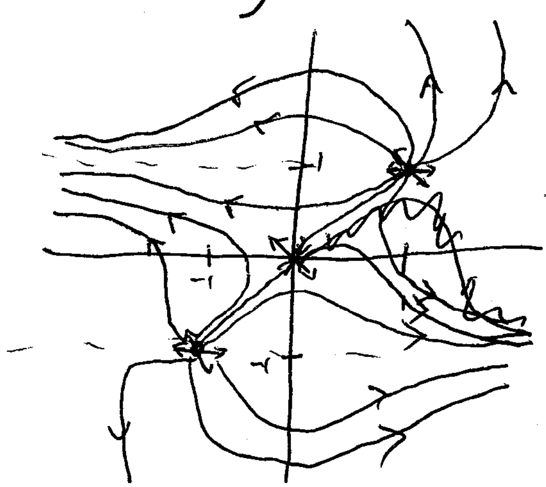
We have

$$J(x,y) = \begin{pmatrix} 1 & -1 \\ y^2-1 & 2xy \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \longrightarrow \text{eigvalues } \tfrac{1}{2}(1 \pm \sqrt{5}). \quad \text{eigvectors } \begin{pmatrix} \frac{1}{2}(-1 \mp \sqrt{5}) \\ 1 \end{pmatrix}$$

$$J(1,1) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \longrightarrow \text{eigvalues } 2, 1 \quad \text{eigvectors } \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$J(-1,-1) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$



F02 #2:

Let $L_\varepsilon u = -u'' + \varepsilon x u$.

a) We want to find the eigenvalues of L_0. So we want to find u, λ s.t. $L_0 u = \lambda u$. We want to solve $-u'' = \lambda u$ with $u(0)=0$, $u(\pi)=0$. The only time this occurs is if $\lambda = \mu^2$. Then

$$-u'' - \mu^2 u = 0.$$

$$u = A\cos\mu x + B\sin\mu x.$$

$$u(0) = 0 \longrightarrow A = 0.$$

$$u(\pi) = 0 \longrightarrow \sin(\pi\mu) = 0 \longrightarrow \mu = \pm 1, \pm 2, \ldots$$

Thus $\lambda = 1, 4, 9, \ldots$ Therefore $\lambda_0 = 1$, $\phi_0 = \sin x$.

b) Now we want to find an eigenvalue and eigenfunction λ, ϕ with

$$L_\varepsilon \phi = \lambda\phi \qquad \phi(0) = \phi(\pi) = 0.$$

where we have the expansion

$$\lambda = 1 + \varepsilon\lambda_1 + O(\varepsilon^2)$$

$$\phi = \sin x + \varepsilon\phi_1 + O(\varepsilon^2) \qquad \phi_1(0) = \phi_1(\pi) = 0.$$

We want (ignoring $O(\varepsilon^2)$ terms),

$$-\phi'' + \varepsilon x\phi = \lambda\phi$$

$$-(\phi_0 + \varepsilon\phi_1)'' + \varepsilon x(\phi_0 + \varepsilon\phi_1) = (\lambda_0 + \varepsilon\lambda_1)(\phi_0 + \varepsilon\phi_1)$$

$$\cancel{-\phi_0''} - \varepsilon\phi_1'' + \varepsilon x\phi_0 = \cancel{\lambda_0\phi_0} + \varepsilon\lambda_0\phi_1 + \varepsilon\lambda_1\phi_0.$$

$$-\phi_1'' + x\sin x = \phi_1 + \lambda_1\sin x.$$

$$L_0\phi_1 + x\sin x = \phi_1 + \lambda_1 \sin x, \quad \phi_1(0) = \phi_1(\pi) = 0.$$

Then as L_0 is self-adjoint (and using the bdy conditions for ϕ_0, ϕ_1),

$$\int_0^\pi (L_0\phi_1)\phi_0\,dx + \int_0^\pi x\sin^2 x\,dx = \int_0^\pi \phi_1\phi_0\,dx + \int_0^\pi \lambda_1 \sin^2 x\,dx.$$

$$\cancel{\int_0^\pi \phi_1 \sin x\,dx} + \int_0^\pi x\sin^2 x\,dx = \cancel{\int_0^\pi \phi_1 \sin x\,dx} + \lambda_1\int_0^\pi \sin^2 x\,dx$$

F02 #2 cont:

Thus

$$\lambda_1 = \frac{\int_0^{\pi} x \sin^2 x \, dx}{\int_0^{\pi} \sin^2 x \, dx} \quad \left(= \frac{\pi}{2} \right)$$

Now it remains to solve

$$\begin{cases} -\phi_1'' - \phi_1 = (\lambda_1 - x)\sin x \\ \phi_1(0) = 0, \ \phi_1(\pi) = 0 \end{cases}$$

$$\longrightarrow \begin{cases} \phi_1'' + \phi_1 = (x - \frac{\pi}{2})\sin x \\ \phi_1(0) = 0, \ \phi_1(\pi) = 0. \end{cases} \qquad (*)$$

A fundamental set of solutions for $y'' + y = 0$ is $\{\cos x, \sin x\}$. Then particular solution is given by

$$-\cos x \int \sin^2 x \, (x - \tfrac{\pi}{2})\, dx + \sin x \int \cos x \, (x - \tfrac{\pi}{2}) \sin x \, dx.$$

$$\int (\sin^2 x)(x - \tfrac{\pi}{2})\, dx \underset{u = x - \frac{\pi}{2}}{=} \int u \sin^2(u + \tfrac{\pi}{2})\, du = \int u \cos^2 u \, du = \frac{1}{2}\int u\cos 2u + u \, du$$

$$= \frac{1}{2}\left[\frac{u}{2}\sin 2u + \frac{1}{4}\cos 2u\right] + \frac{1}{2}\cdot\frac{u^2}{2} + C.$$

$$= \frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x - \pi) + \frac{1}{8}\cos(2x - \pi) + \frac{1}{4}(x - \tfrac{\pi}{2})^2 + C$$

$$= -\frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x) - \frac{1}{8}\cos(2x) + \frac{1}{4}(x - \tfrac{\pi}{2})^2 + C.$$

$$\int (\cos x)(x - \tfrac{\pi}{2})(\sin x)\, dx = \frac{1}{2}\int (x - \tfrac{\pi}{2})\sin 2x \, dx = -(x - \tfrac{\pi}{2})\frac{1}{4}\cos 2x + \frac{1}{8}\sin 2x + C.$$

Therefore a particular solution to the ODE $(*)$ is

$$y_p = \frac{1}{4}(x - \tfrac{\pi}{2})\sin(2x)\cos x + \frac{1}{8}\cos(2x)\cos x - \frac{1}{4}(x - \tfrac{\pi}{2})^2\cos x$$
$$- (x - \tfrac{\pi}{2})\cdot\frac{1}{2}\sin x \cos 2x + \frac{1}{4}\sin x \sin 2x.$$

Fa2 #2 cont:

We have $y_p(0) = \frac{1}{8} - \frac{\pi^2}{16}$

$y_p(\pi) = -\frac{1}{8} + \frac{\pi^2}{16}$.

The homogeneous eqn is:

$$y_h(x) = A\cos x + B\sin x.$$

$y_h(0) = A$

$y_h(\pi) = -A$.

So

$$\phi_1(x) = A\cos x + B\sin x + y_p(x)$$

$$\rightarrow \phi_1(0) = 0 \longrightarrow 0 = A + y_p(0) = A + \frac{1}{8} - \frac{\pi^2}{16}.$$

$$\phi_1(\pi) = 0 \longrightarrow 0 = -A + y_p(\pi) = -A - \frac{1}{8} + \frac{\pi^2}{16}.$$

$$\rightarrow A = -\frac{1}{8} + \frac{\pi^2}{16}.$$

Take $B = 0$. Then

$$\phi_1(x) = \left(-\frac{1}{8} + \frac{\pi^2}{16}\right)\cos x + y_p(x). \qquad \#$$

F02 #3

We first solve the system: $w := \begin{pmatrix} u \\ v \end{pmatrix}$

$$w_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} w_x = 0.$$

~~Since $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ has~~

We diagonalize $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Eigenvalues: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \left| \begin{pmatrix} \lambda-1 & -1 \\ -1 & \lambda+1 \end{pmatrix} \right| = \lambda^2 - 1 - 1$

$$\lambda = \pm\sqrt{2}.$$

Eigenvectors: $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\sqrt{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$$a + b = \pm\sqrt{2}\, a. \quad \rightsquigarrow \begin{pmatrix} 1 \\ \pm\sqrt{2}-1 \end{pmatrix}$$

$$b = (\pm\sqrt{2} - 1)\, a.$$

Thus

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix}^{-1}.$$

Let $P := \begin{pmatrix} 1 & 1 \\ \sqrt{2}-1 & -\sqrt{2}-1 \end{pmatrix}$, $D := \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix}$, $A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

let $z := P^{-1} w$.

$$w_t + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} w_x = 0 \qquad (1)$$

$$\Updownarrow$$

$$z_t + \begin{pmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{pmatrix} z_x = 0. \qquad (2)$$

Since P is invertible, well posedness of (1) is equivalent to well-posedness of (2).

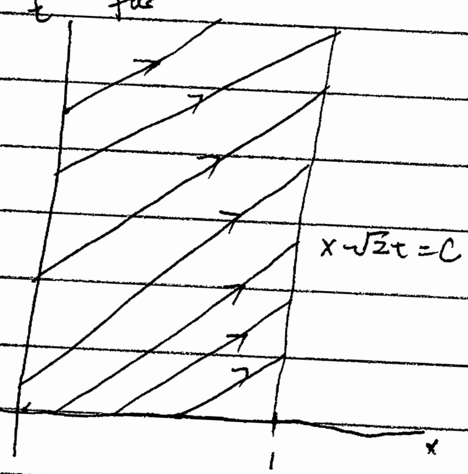
$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \begin{matrix} (z_1)_t + \sqrt{2}(z_1)_x = 0 \\ (z_2)_t - \sqrt{2}(z_2)_x = 0. \end{matrix}$$

$(z_1)_t + \sqrt{2}(z_1)_x = 0$ has characteristics: $x - \sqrt{2}t = C$ $\quad z_1, z_2$ are constant on characteristics.

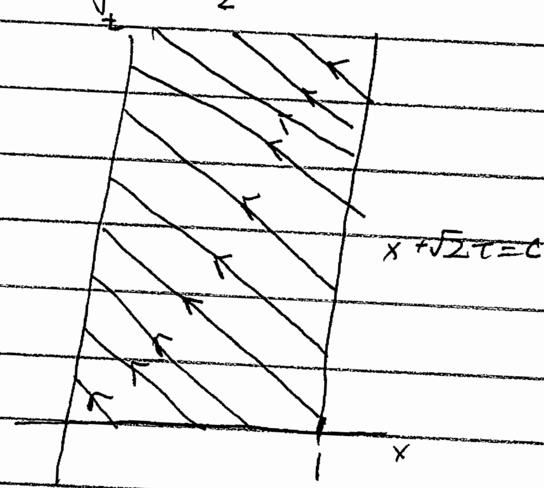
$(z_2)_t - \sqrt{2}(z_2)_x = 0$ has characteristics $x + \sqrt{2}t = C$.

F02 #3
cont:

Characteristics for z_1



Characteristics for z_2



Thus for the PDE for z_1 to be well posed we need data when $t=0$ and $x=0$. For the PDE for z_2 to be well posed we need data when $t=0$ and $x=1$.

Thus since P is invertible, for the PDE in the problem to be well posed, we need data for u on $t=0$ and $x=0$ and data for v on $t=0$ and $x=1$. The only set of initial conditions that does this is (c)

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F02 #4: We want to solve

$$u_{x_2} + u u_{x_1} = 0$$
$$u(x_1, 0) = \pm x_1$$

a) $u(x_1, 0) = x_1$

$\longrightarrow$ $F(p, z, x) = p_2 + z p_1 = 0.$

$D_p F = (z, 1)$

$D_z F = p_1$

$D_x F = 0$

$\longrightarrow$ $\dot{p} = -D_x F - D_z F p = -p_1(p_1, p_2)$

$\dot{z} = D_p F \cdot p = z p_1 + p_2 = 0.$

$\dot{x} = D_p F = (z, 1)$

w/ $x_1(0) = x_1(0)$

$x_2(0) = 0.$

$z(0) = x_1(0).$

$\longrightarrow$ $z(s) = x_1(0)$ $\quad$ $x_1(s) = x_1(0)s + x_1(0)$

$x_2(s) = s$ $\quad$ $\longrightarrow x_1(0) = \dfrac{x_1(s)}{s+1}$

Therefore $u(x_1, x_2) = \dfrac{x_1}{x_2 + 1}$. That is, $u(x, t) = \dfrac{x}{t+1}$

b) $u(x_1, 0) = -x_1$

By the same calculation,

$\dot{z} = 0$ $\quad$ w/ $x_1(0) = x_1(0)$

$\dot{x} = (z, 1)$ $\quad$ $x_2(0) = 0$

$z(0) = -x_1(0)$

$\longrightarrow$ $z(s) = -x_1(0)$ $\quad$ $x_1(s) = -x_1(0)s + x_1(0)$

$x_2(s) = s$ $\quad$ $\longrightarrow x_1(0) = \dfrac{x_1(s)}{1-s}$

Thus $u(x_1, x_2) = \dfrac{x_1}{-1 + x_2}$. That is, $u(x, t) = \dfrac{x}{-1+t}$.

$\hookrightarrow$ solution blows up at $t=1$

F02 5a: We have for each $x \in [0,1]$,

$$|u(x)| \le \int_0^x |u'(t)|\,dt = \int_0^1 |u'(t)|\, I_{[0,x]}(t)\,dt$$
$$\le \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2}\left(\int_0^1 I_{[0,x]}(t)\,dt\right)^{1/2}$$
$$\le \left(\int_0^1 |u'(t)|^2\,dt\right)^{1/2}.$$

Thus

$$|u(x)|^2 \le \int_0^1 |u'(t)|^2\,dt.$$

Since $x \in [0,1]$ was arbitrary,

$$\max_{x\in[0,1]} |u(x)|^2 \le \int_0^1 |u'(t)|^2\,dt.$$

5b Suppose L had an eigenvalue that was ≤ 0. Then for some $\lambda \ge 0$ and u satisfying $u(0)=u(1)=0$, we have $(L+\lambda I)u = 0$. Then

$$0 = \langle (L+\lambda I)u, u\rangle = \int_0^1 (-u''+pu+\lambda u)u\,dx$$
$$= \int_0^1 -u''u + pu^2 + \lambda u^2\,dx = \int_0^1 (u')^2 + p_+u^2 - p_-u^2 + \lambda u^2\,dx$$
$$\ge \int_0^1 (u')^2 - p_-u^2\,dx \overset{\text{by (a)}}{\ge} \int_0^1 (u')^2\,dx - \left(\int_0^1 (u')^2\,dt\right)\int_0^1 p_-(x)\,dx$$
$$> \int_0^1 u'^2\,dx - \int_0^1 u'^2\,dx = 0,$$

a contradiction. Therefore any eigenvalue of L must be >0.

P02 #6

Write $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\pi x)$. Then as

$$u_t = u_{xx} + e^{-2t} g(x),$$

we have

$$a_n'(t) = -a_n(t) n^2\pi^2 + e^{-2t} b_n$$

where $g(x) = \sum_{n\geq 0} b_n \cos(n\pi x)$ and $b_n = 2\int_0^1 g(x)\cos(n\pi x)\,dx$,

$b_0 = \int_0^1 g(x)\,dx$

Then

$$a_n'(t) + a_n(t) n^2\pi^2 = e^{-2t} b_n.$$

$$\left(e^{n^2\pi^2 t} a_n\right)' = e^{(n^2\pi^2-2)t} b_n.$$

$$e^{n^2\pi^2 t} a_n(t) = \frac{1}{n^2\pi^2-2} e^{(n^2\pi^2-2)t} b_n - \frac{1}{n^2\pi^2-2} b_n + a_n(0)$$

$$a_n(t) = \frac{1}{n^2\pi^2-2} e^{-2t} b_n - \frac{1}{n^2\pi^2-2} b_n e^{-n^2\pi^2 t} + a_n(0) e^{-n^2\pi^2 t}.$$

$$= \frac{b_n}{n^2\pi^2-2}\left(e^{-2t} - e^{-n^2\pi^2 t}\right) + a_n(0) e^{-n^2\pi^2 t}.$$

We have

$$a_n(0) = 2\int_0^1 f(x)\cos(n\pi x)\,dx, \quad a_0(0) = \int_0^1 f(x)\,dx.$$

Thus

$$\lim_{t\to\infty} \sum_{n\geq 0} a_n(t)\cos n\pi x. = \lim_{t\to\infty} \frac{b_0}{-2}(-1) + a_0(0) = \lim_{t\to\infty} \frac{b_0}{2} + a_0(0)$$

$$= \left(\frac{1}{2}\int_0^1 g(x)\,dx\right) + \int_0^1 f(x)\,dx.$$

F02 #7:

We have $2\frac{\partial u}{\partial \bar z} = f$. Since $\frac{c}{x+iy}$ is a fundamental solution,

$$u(z) = \int_{\mathbb{C}} \frac{c}{w} f(z-w)\,dw.$$

Then

$$\Delta u = \int_{\mathbb{C}} \frac{c}{w} 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} f(z-w)\,dw = -2c\int_{\mathbb{C}} \frac{1}{w}\, 2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar w} f(z-w)\,dw$$

$$= 2c\int_{\mathbb{C}} \frac{\partial}{\partial \bar w}\frac{1}{w}\; 2\frac{\partial}{\partial z} f(z-w)\,dw.$$

Let $K(w) := \frac{1}{2\pi}\log|w| = \frac{1}{4\pi}\log(w_1^2+w_2^2)$. We have

$$\frac{\partial}{\partial w} K(w) = \frac{1}{2}\left(\frac{\partial}{\partial w_1} - i\frac{\partial}{\partial w_2}\right)K(w) = \frac{1}{2}\cdot\frac{1}{4\pi}\left[\frac{2w_1}{w_1^2+w_2^2} - i\frac{2w_2}{w_1^2+w_2^2}\right] = \frac{1}{4\pi w}.$$

Thus

$$\Delta u = 2c\int_{\mathbb{C}} \frac{\partial}{\partial \bar w}\frac{1}{w}\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw = 2c\int_{\mathbb{C}} 4\pi\frac{\partial}{\partial \bar w}\frac{\partial}{\partial w}K(w)\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw.$$

$$= 2c\pi\int_{\mathbb{C}} \Delta_w K(w)\cdot 2\frac{\partial f}{\partial z}(z-w)\,dw.$$

Since $\Delta_w K = \delta$ and

$$2\frac{\partial u}{\partial \bar z} = f \longrightarrow \Delta u = 2\frac{\partial f}{\partial z},$$

we must have $c = \frac{1}{2\pi}$.

Now

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)e^{-2\pi i(x\xi_1+y\xi_2)} = -2\pi i\xi_1 e^{-2\pi i(x\xi_1+y\xi_2)} + i(-2\pi i\xi_2)e^{-2\pi i(x\xi_1+y\xi_2)}$$

$$= -2\pi i(\xi_1+i\xi_2)e^{-2\pi i(x\xi_1+y\xi_2)}.$$

Thus the fourier transform of $(x+iy)^{-1}$ is

$$\int_{\mathbb{R}^2} e^{-2\pi i(x\xi_1+y\xi_2)}\frac{1}{x+iy}\,dx\,dy = \int_{\mathbb{R}^2}\frac{1}{-2\pi i(\xi_1+i\xi_2)}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{-2\pi i(x\xi_1+y\xi_2)}\cdot\frac{1}{x+iy}\,dx\,dy$$

$$= \frac{2\pi}{+2\pi i(\xi_1+i\xi_2)}\int_{\mathbb{R}^2} e^{-2\pi i(x\xi_1+y\xi_2)}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\frac{1/2\pi}{x+iy}\,dx\,dy$$

$$= \frac{1}{i(\xi_1+i\xi_2)} \quad \text{since } \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\frac{1/2\pi}{x+iy}\right) = \delta. \quad \#$$

F02 #8: Let u_1, u_2 be 2 solutions of the given equation. Let $w := u_1 - u_2$. Then

$$\Delta^2 w = 0 \text{ in } D$$
$$w = \Delta w = 0 \text{ on } \partial D.$$

We have

$$0 = \int_D w\, \Delta(\Delta w)\, dx \overset{\text{since } w=0=\Delta w \text{ on } \partial D}{=} \int_D (\Delta w)^2\, dx.$$

Therefore $\Delta w = 0$ in D. Finally,

$$0 = \int_D w\, \Delta w\, dx = \int_D \nabla w \cdot \nabla w\, dx = \int_D |\nabla w|^2\, dx$$

which implies that $\nabla w = 0 \longrightarrow w = 0$ on D.
Thus the solution to the given boundary value problem is unique.