F04#1 We use separation of variables.

If $u(x,t) = F(x)G(t)$, then

$$u_{tt} - u_{xx} + u = 0 \longrightarrow F(x)G''(t) - F''(x)G(t) + F(x)G(t) = 0.$$
$$F(x)[G''(t) + G(t)] = F''(x)G(t)$$
$$\frac{G''(t) + G(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda.$$

Then if $\lambda > 0$

$$F''(x) + \lambda F(x) = 0 \longrightarrow F(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x.$$

Since $u(0,t) = u(\pi,t) = 0$, $F(0) = 0$, $F(\pi) = 0$.

Then

$$F(0) = 0 \longrightarrow A = 0.$$
$$F(\pi) = 0 \longrightarrow 0 = B\sin\sqrt{\lambda}\,\pi \xrightarrow{\text{since we want nontrivial solns.}} 0 = \sin\sqrt{\lambda}\,\pi$$
$$\longrightarrow \sqrt{\lambda} = n \text{ for } n = 1, 2, 3, \ldots$$

We get no nontrivial solutions when $\lambda \le 0$.

~~Let $F_n(x) = \sin nx$ and let $\lambda_n := n^2$. Then~~

$$\cancel{\frac{G_n''(t) + G_n(t)}{G_n(t)} = -\lambda_n.}$$

$$\cancel{\longrightarrow G_n''(t) + (n^2+1)G_n(t) = 0.}$$
$$\cancel{\longrightarrow G_n(t) = A\cos(\sqrt{n^2+1}\,t) + B\sin(\sqrt{n^2+1}\,t).}$$

~~Since~~

Now let's solve $\dfrac{G'' + G}{G} = -\lambda_n \longrightarrow G'' + (n^2+1)G = 0.$

Since $u_t(x,0) = 0$, $G'(0) = 0$. Therefore

$$G(t) = A\cos(\sqrt{n^2+1}\,t),\ n = 1, 2, 3, \ldots$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} a_n(\sin nx)\cos(\sqrt{n^2+1}\,t)$$

F04 1
cont.

Since $u(x,0) = f(x)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Note ~~for~~,

$$\int_0^{\pi} \sin nx \sin mx \, dx = \begin{cases} \pi/2 & \text{if } n=m \\ 0 & \text{if } n \neq m. \end{cases}$$

Thus

$$a_m = \frac{2}{\pi}\int_0^{\pi} f(x) \sin mx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin mx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin mx \, dx\right]$$

We have

$$\begin{array}{ccc} + & x & \sin nx \\ - & 1 & -\frac{1}{n}\cos nx \\ + & 0 & -\frac{1}{n^2}\sin nx \end{array}$$

$$\int x \sin nx \, dx = -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx + C.$$

Then

$$\int_0^{\pi/2} x \sin nx \, dx = \cancel{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}}$$

$$-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}.$$

$$\int_{\pi/2}^{\pi} x \sin nx \, dx = -\frac{\pi}{n}\cos n\pi - \left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$\rightarrow \int_0^{\pi/2} x \sin nx \, dx - \int_{\pi/2}^{\pi} x \sin nx \, dx$$

$$= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n}\cos n\pi + \left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{\pi}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2} + \frac{\pi}{n}(-1)^n.$$

F04 #1
cont.

$$\pi \int_{\pi/2}^{\pi} \sin nx \, dx = \pi \cdot \frac{-1}{n} \cos nx \Big]_{x=\pi/2}^{\pi}$$

$$= \pi \left[\frac{1}{n} \cos \frac{n\pi}{2} - \frac{1}{n} \cos n\pi \right]$$

Therefore

$$a_n = \frac{2}{\pi}\left(\cancel{-\frac{\pi}{n} \cos \frac{n\pi}{2}} + \frac{2}{n^2} \sin \frac{n\pi}{2} + \cancel{\frac{\pi}{n}(-1)^n} + \cancel{\frac{\pi}{n} \cos \frac{n\pi}{2}} - \cancel{\frac{\pi}{n} \cos n\pi} \right)$$

$$= \frac{4}{\pi n^2} \sin \frac{n\pi}{2}.$$

Thus

$$u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} (\sin n x)(\cos \sqrt{n^2+1}\, t)$$

F04 #2: Let $\mathcal{U}(x,\tau) = u(ax,\tau)$. Then

$$\mathcal{U}_{xx} = a^2 u_{xx}(ax,\tau)$$
$$\mathcal{U}_t = u_\tau(ax,\tau).$$

Thus

$$\mathcal{U}_t = \mathcal{U}_{xx}, \quad t>0,\ x\in\mathbb{R}$$
$$\mathcal{U}(x,0) = \varphi(ax).$$

~~Then~~ It suffices to compute $\lim_{\tau\to\infty} \mathcal{U}(x,\tau)$. ~~Thy~~ We have

$$\mathcal{U}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} \varphi(ay)\,dy$$

$u = \frac{x-y}{2\sqrt{t}}$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \varphi(ax - 2\sqrt{t}ua)\,du. \quad (*)$$

Since u is a bounded solution, φ is bounded. Thus the integral in $(*)$ converges. Then

$$\lim_{t\to\infty} \mathcal{U}(x,t) = \lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-u^2}\varphi(ax-2\sqrt{t}ua)\,du$$
$$+ \lim_{t\to\infty} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2}\varphi(ax-2\sqrt{t}ua)\,du$$

by DCT as φ is bdd.

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-u^2} \lim_{t\to\infty} \varphi(ax-2\sqrt{t}ua)\,du$$
$$+ \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2} \lim_{t\to\infty} \varphi(ax-2\sqrt{t}ua)\,du$$
$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-u^2} b\,du + \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u^2} c$$
$$= \frac{1}{2}(b+c).$$

F04 #3: We have

$$E'(t) = \frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}^3} |Du|^2 + |\partial_t u|^2 \, dx\right]$$

$$= \frac{d}{dt}\left[\frac{1}{2}\int_{\mathbb{R}^3} Du \cdot Du + u_t^2 \, dx\right]$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} 2Du_t \cdot Du + 2u_t u_{tt} \, dx$$

$$= \int_{\mathbb{R}^3} Du_t \cdot Du + u_t u_{tt} \, dx$$

$$= -\int_{\mathbb{R}^3} u_t \Delta u - u_t u_{tt} \, dx$$

since $u_{tt} - \Delta u + a(x)u_t = 0$

$$= -\int_{\mathbb{R}^3} u_t \left(a(x) u_t\right) dx$$

$$= -\int_{\mathbb{R}^3} u_t^2 \, a(x) \, dx \leq 0.$$

Therefore $E(t)$ is a decreasing function of $t \geq 0$.

F04 #4: Let $r^2 = x_1^2 + x_2^2$. Then $r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$ and hence

$$r\dot{r} = x_1(x_2 + x_1(x_1^2+x_2^2)) + x_2(-x_1 + x_2(x_1^2+x_2^2))$$
$$= x_1^2 r^2 + x_2^2 r^2 = r^4.$$

Then $\dot{r} = r^3 \longrightarrow \frac{dr}{dt} = r^3$. Thus for some constant C,

$$-\frac{1}{2}r^{-2} = t - C.$$
$$r(t)^2 = \frac{1}{2(C-t)}.$$

Therefore ~~each solution~~ $r(t)$ blows up in finite time and hence each solution of the given autonomous system blows up in finite time.

If $x_1(0)=1$, $x_2(0)=0$, $r(0)^2 = 1$ and hence $C = \frac{1}{2}$. Therefore $r(t)^2 = \frac{1}{1-2t}$, which implies that the blow up time is $t = \frac{1}{2}$.

F04 #5: This is Dulac's Criterion.

We observe

$$\nabla\cdot(\varphi f) = \frac{\partial}{\partial x_1}\left(\frac{1}{x_1 x_2} f_1\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{x_1 x_2} f_2\right)$$
$$= \frac{\partial}{\partial x_1}\left(\frac{a - bx_2 - ex_1}{x_2}\right) + \frac{\partial}{\partial x_2}\left(\frac{-c + dx_1 - fx_2}{x_1}\right)$$
$$= -\frac{e}{x_2} - \frac{f}{x_1} < 0$$

since $x_1, x_2 > 0$.

Suppose there was a closed orbit in the 1st quadrant, let Ω be the region enclosed by this closed orbit. Then

$$0 > \int_\Omega \nabla\cdot(\varphi f)\,dx = \int_{\partial\Omega} \varphi f \cdot \nu \, d\sigma$$

Since $f = (\dot{x}_1, \dot{x}_2)$ and $\partial\Omega$ is the closed orbit represented by (x_1, x_2), $f\cdot\nu = 0$ and hence

$$\int_{\partial\Omega} \varphi(f\cdot\nu)\,d\sigma = 0.$$

This is a contradiction. Therefore there are no closed orbits in the 1st quadrant. #

F04 #6 :

Let u, v be 2 vector flds satisfying the 3 given properties. Let $w = u - v$. Then w is conservative, $\nabla \cdot w = 0$ and $|w(x)| = O(|x|^{-2})$. Since w is conservative, $\exists\, F$ s.t. $w = \nabla F$. Since $\nabla \cdot w = 0$, $\Delta F = 0$, and hence F is harmonic.

Since $|w(x)| = O(|x|^{-2})$, each derivative of F is bdd. Since each derivative of F is harmonic, by Liouville's Theorem, $\partial_{x_i} F = 0 \ \forall i$ since $w \to 0$ as $x \to \infty$. Therefore F is constant. Since $w = \nabla F = 0$, we have $u = v$.

3) Let R be s.t. $\operatorname{supp} q \subset B(0, R/2)$. For $|x| \geq 10R$,

$$|u(x)| \leq \frac{1}{4\pi} \int_{B(0,R/2)} \frac{|q(y)|}{|x-y|^2}\, dy.$$

For $|x| \geq 10R$, $|x-y| \geq |x| - |y| \geq |x| - R/2 \geq \frac{|x|}{2}$ since $|x| \geq 10R$.

Thus for these x,

$$|u(x)| \leq \frac{1}{4\pi} \frac{4}{|x|^2} \int_{B(0,R/2)} |q(y)|\, dy$$

$$\leq \frac{1}{4\pi} \frac{4}{|x|^2} \int_{\mathbb{R}^3} |q(y)|\, dy.$$

Therefore for large x,

$$|u(x)| = O(|x|^{-2}).$$

F04 #6 cont'

2) We have
$$\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{q(y)(x-y)}{|x-y|^3}\,dy = \frac{+1}{4\pi}\int_{\mathbb{R}^3}\frac{q(x-y)\,y}{|y|^3}\,dy.$$
Then
$$\nabla_x\cdot\frac{+1}{4\pi}\int_{\mathbb{R}^3} q(x-y)\frac{y}{|y|^3}\,dy = +\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|y|^3}\sum_{j=1}^{3}\partial_{x_j}\left(q(x-y)y_j\right)dy.$$
$$= +\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|y|^3}\,y\cdot\nabla_x q(x-y)\,dy.$$
$$= -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|y|^3}\,y\cdot\nabla_y\left(q(x-y)\right)dy.$$
$$= \lim_{\varepsilon\to 0}\frac{-1}{4\pi}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{1}{|y|^3}\,y\cdot\nabla_y\left(q(x-y)\right)dy.$$
$$= \lim_{\varepsilon\to 0}\frac{-1}{4\pi}\left[-\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\nabla_y\cdot\frac{y}{|y|^3}\,q(x-y)\,dy + \int_{\partial(\mathbb{R}^3\setminus B(0,\varepsilon))} q(x-y)\frac{y}{|y|^3}\cdot\nu(y)\,dy\right] \quad (*)$$
As $\frac{y}{4\pi|y|^3}$ is the gradient of the fundamental sol in $\mathbb{R}^3$,
$$-\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\nabla_y\frac{y}{|y|^3}\,q(x-y)\,dy = 0.$$
So
$$(*) = \frac{-1}{4\pi}\lim_{\varepsilon\to 0}\int_{\partial(\mathbb{R}^3\setminus B(0,\varepsilon))} q(x-y)\frac{y}{|y|^3}\cdot\nu(y)\,d\sigma(y)$$
$$= \frac{-1}{4\pi}\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} q(x-y)\frac{y}{|y|^3}\cdot\left(-\frac{y}{|y|}\right)d\sigma(y)$$

F04 #6 cont':

$$= +\frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} q(x-y)\frac{1}{\varepsilon^2}\,d\sigma(y)$$

$$= +\frac{1}{4\pi\varepsilon^2}\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} q(x-y)\,d\sigma(y) = +q(x).$$

Therefore

$$\nabla\cdot u = q.$$

1) here $F(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{q(y)}{|x-y|}dy. = -\frac{1}{4\pi}\int_{\mathbb{R}^3} q(x-y)\frac{1}{|y|}dy.$

We claim $\nabla F = u$. But this comes from

$$\nabla\frac{1}{|x|} = -\frac{x}{|x|^3}.$$

~~and here~~

$$\cancel{\nabla_x F = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\nabla_x q(x-y)\frac{1}{|y|}dy}$$

We have

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} q(x-y)\frac{y}{|y|^3}dy = -\frac{1}{4\pi}\int_{\mathbb{R}^3} q(x-y)\,\nabla_y\left(\frac{1}{|y|}\right)dy$$

$$= \lim_{\varepsilon\to 0^+} -\frac{1}{4\pi}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} q(x-y)\,\nabla_y\left(\frac{1}{|y|}\right)dy$$

$$= \lim_{\varepsilon\to 0^+} -\frac{1}{4\pi}\left[-\int_{\mathbb{R}^3\setminus B(0,\varepsilon)}\nabla_y q(x-y)\cdot\frac{1}{|y|}dy + \int_{\partial\mathbb{R}^3\setminus B(0,\varepsilon)}\frac{q(x-y)}{|y|}\nu\,d\sigma(y)\right)$$

F04 #6 cont:

Since

$$\left| \int_{\partial B(0,\varepsilon)} \frac{q(x-y)}{|y|} \frac{-y}{|y|} \, d\sigma(y) \right|$$

$$\leq \|q\|_{L^\infty} \frac{1}{\varepsilon} 4\pi\varepsilon^2 \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Thus

$$u(x) = \lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(0,\varepsilon)} \nabla_x q(x-y) \frac{1}{|y|} \, dy$$

$$= \nabla F(x).$$

#

F04 #7: Write $uu_{x_1} + u_{x_2} + u = 0$

$$u(0, x_2) = e^{-2x_2}.$$

Let $F(p,z,x) := zp_1 + p_2 + z = 0$. Then

$$D_pF = (z,1) \qquad D_zF = p_1 + 1 \qquad D_xF = 0$$

Therefore:

$$\dot p = -D_xF - D_zF\,p = (p_1+1)\,p \qquad x_1(0) = 0$$

$$\dot z = D_pF\cdot p = zp_1 + p_2 = -z. \qquad x_2(0) = x_2(0)$$

$$\dot x = D_pF = (z,1) \qquad z(0) = e^{-2x_2(0)}.$$

① $\dot z = -z \longrightarrow z(s) = z(0)e^{-s} = e^{-2x_2(0)}e^{-s}$.

② $\dot x_1 = z = z(0)e^{-s}$, $x_1(0) = 0$.

$$\hookrightarrow x_1(s) = -z(0)e^{-s} + x_1(0) + z(0)$$

$$= z(0)(1-e^{-s}) = e^{-2x_2(0)}(1-e^{-s}).$$

③ $\dot x_2 = 1$, $x_2(0) = x_2(0)$

$$\hookrightarrow x_2(s) = s + x_2(0).$$

Therefore

$$z(s) = e^{-2x_2(0)}e^{-s} = e^{-2x_2(0)}e^{x_2(0)-x_2(s)} = e^{-x_2(0)}e^{-x_2(s)}.$$

We have

$$x_1(s) = e^{-2x_2(0)}\left(1 - e^{x_2(0)-x_2(s)}\right) = \left(e^{-x_2(0)}\right)^2 - e^{-x_2(s)}\left(e^{-x_2(0)}\right)$$

Thus

$$e^{-x_2(0)} = \frac{e^{-x_2(s)} + \sqrt{e^{-2x_2(s)} + 4x_1(s)}}{2}$$

(~~since~~ we chose + since if we chose the −, we would get a contradiction as $e^{-x_2(0)} > 0$)

Therefore

$$z(s) = e^{-x_2(s)}\left[\frac{e^{-x_2(s)} + \sqrt{e^{-2x_2(s)} + 4x_1(s)}}{2}\right]$$

$$\longrightarrow u(x,t) = e^{-t}\left[\frac{e^{-t} + \sqrt{e^{-2t} + 4x}}{2}\right]$$

Note

$$u\left(\tfrac{1}{9}, \ln 2\right) = \frac{1}{2}\left[\frac{1/2 + \sqrt{1/4 + 4/9}}{2}\right] = \frac{1}{4}\left(\frac{1}{2} + \frac{5}{6}\right) = \frac{1}{3}.$$

F04 #8

We have $\int_0^\infty e^{-s\tau}\frac{dy}{d\tau}(x,\tau)\,d\tau = e^{-s\tau}y(x,\tau)\Big]_{\tau=0}^{\infty} - \int_0^\infty -se^{-s\tau}y(x,\tau)\,d\tau$

$$= -1 + s\int_0^\infty e^{-s\tau}y(x,\tau)\,d\tau.$$

Thus taking the Laplace transform, we have

$$x\frac{d}{dx}\bar{y} + \frac{d}{dx}s\bar{y} + 2\bar{y} = 0.$$

$$(x+s)\frac{d\bar{y}}{dx} + 2\bar{y} = 0. \qquad \bar{y}(0,t) = \int_0^\infty e^{-s\tau}e^{-a\tau}\,d\tau = \frac{1}{s+a}.$$

$$(\ln\bar{y})' = -\frac{2}{x+s}.$$

$$\ln\bar{y} = \ln(x+s)^{-2}$$

$$\bar{y} = (x+s)^{-2}c \longrightarrow c = \frac{s^2}{s+a}.$$

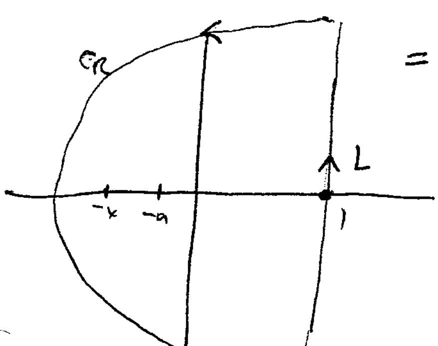
Thus

$$\bar{y}(x,s) = \frac{s^2}{(x+s)^2(s+a)}.$$

For $x, t \geq 0$,

chosen b/c poles are $s = -x, -a$ which are ≤ 0.

$$y(x,t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{1-iR}^{1+iR}\frac{e^{st}s^2}{(x+s)^2(s+a)}\,ds$$



$$= -\frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R}\frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds + \left(\operatorname*{Res}_{s=-a}\frac{e^{st}s^2}{(s+x)^2(s+a)} + \operatorname*{Res}_{s=-x}\frac{e^{st}s^2}{(s+x)^2(s+a)}\right)$$

We have

$$\left|\int_{C_R}\frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds\right| = \left|\int_{\pi/2}^{3\pi/2}\frac{e^{(1+Re^{i\theta})t}(1+Re^{i\theta})^2}{(1+Re^{i\theta}+x)^2(1+Re^{i\theta}+a)}\,iRe^{i\theta}\,d\theta\right|$$

$$\leq \frac{(R-1)^2Re^t}{(R-|x|-1)(R-|a|-1)}\int_{\pi/2}^{3\pi/2}e^{Rt\cos\theta}\,d\theta.$$

34 #8 cont:

$$\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}\,d\theta = 2\int_0^{\pi/2} e^{-Rt\sin\theta}\,d\theta$$

for $\theta \in [0, \frac{\pi}{2}]$, $\sin\theta \geq \frac{2}{\pi}\theta$. Thus $e^{-Rt\sin\theta} \leq e^{-Rt\frac{2}{\pi}\theta}$.

Then

$$2\int_0^{\pi/2} e^{-Rt\sin\theta}\,d\theta \leq 2\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}\theta}\,d\theta.$$

$$= 2\left(-\frac{\pi}{2Rt}\right)e^{-\frac{2Rt}{\pi}\theta}\Big]_{\theta=0}^{\pi/2}$$

$$= +\frac{\pi}{Rt}\left[1 - e^{-Rt}\right].$$ need $t>0$ here

Therefore

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{st}s^2}{(s+x)^2(s+a)}\,ds \to 0 \quad \text{as } t>0.$$

We also have

$$\operatorname*{Res}_{s=-a}\frac{e^{st}s^2}{(s+x)^2(s+a)} = \lim_{s\to -a}\frac{e^{st}s^2}{(s+x)^2} = \frac{e^{-at}a^2}{(x-a)^2}.$$

$$\operatorname*{Res}_{s=-x}\frac{e^{st}s^2}{(s+x)^2(s+a)} = \lim_{s\to -x}\frac{d}{ds}\frac{e^{st}s^2}{s+a} = \lim_{s\to -x}\frac{(s+a)[te^{st}s^2+e^{st}2s] - e^{st}s^2}{(s+a)^2}$$

$$= \frac{(-x+a)(te^{-xt}x^2 - e^{-xt}2x) - e^{-xt}x^2}{(a-x)^2}.$$

Therefore

$$y(x,t) = \frac{a^2e^{-at}}{(x-a)^2} + \frac{(x-a)(2x - tx^2)e^{-xt} - x^2e^{-xt}}{(x-a)^2}. \quad \#$$