# Shortest path problem

**GitHub repo**: https://github.com/pchs20/shortest-path-problem

Given a directed weighted graph, this problem proposes to find a path between two vertices in the graph such that the sum of the weights of its constituent edges is minimized.

## Sets

| | | |
|---|---|---|
| $v \in V$ | Vertices of the graph. | |
| $(v1, v2) \in E$ | Edges of the graph. $(v1, v2)$ represents a directed edge from $v1$ to $v2$. | $v1, v2 \in V$ |

## Parameters

| | | |
|---|---|---|
| $w_e \in \mathbb{R}^+$ | Weight for the edge $e$. | $\forall e \in E$ |
| $v^{start} \in V$ | Initial vertex for the path. | |
| $v^{end} \in V$ | Last vertex for the path. | |

It is assumed that $v^{start}$ is not the same as $v^{end}$.

## Variables

| | | |
|---|---|---|
| $x_e \in \{0, 1\}$ | Equals 1 if the path includes the edge $e$ and equals 0 otherwise. | $\forall e \in E$ |

# Constraints

### Start of the path

Define the beginning of the path. From the selected edges of the path, the initial vertex should have exactly one more outgoing edge than ingoing.

$$\sum_{v \in V : (v^{start}, v) \in E} \left( x_{(v^{start}, v)} \right) = \sum_{v \in V : (v, v^{start}) \in E} \left( x_{(v, v^{start})} \right) + 1$$

### End of the path

Define the end of the path. From the selected edges of the path, the last vertex should have exactly one more ingoing edge than outgoing.

$$\sum_{v \in V : (v^{end}, v) \in E} \left( x_{(v^{end}, v)} \right) + 1 = \sum_{v \in V : (v, v^{end}) \in E} \left( x_{(v, v^{end})} \right)$$

### Adjacent path

The path should be formed by adjacent edges. To guarantee that, every vertex (except for the initial and last ones) should have the same ingoing and outgoing selected edges.

$$\sum_{w \in V : (v, w) \in E} \left( x_{(v, w)} \right) = \sum_{w \in V : (w, v) \in E} \left( x_{(w, v)} \right), \ \forall v \in V - \left\{ v^{start}, v^{end} \right\}$$

# Objective

Minimize the sum of the edge weights that form the path.

$$\min \sum_{e \in E} \left( x_e \cdot w_e \right)$$