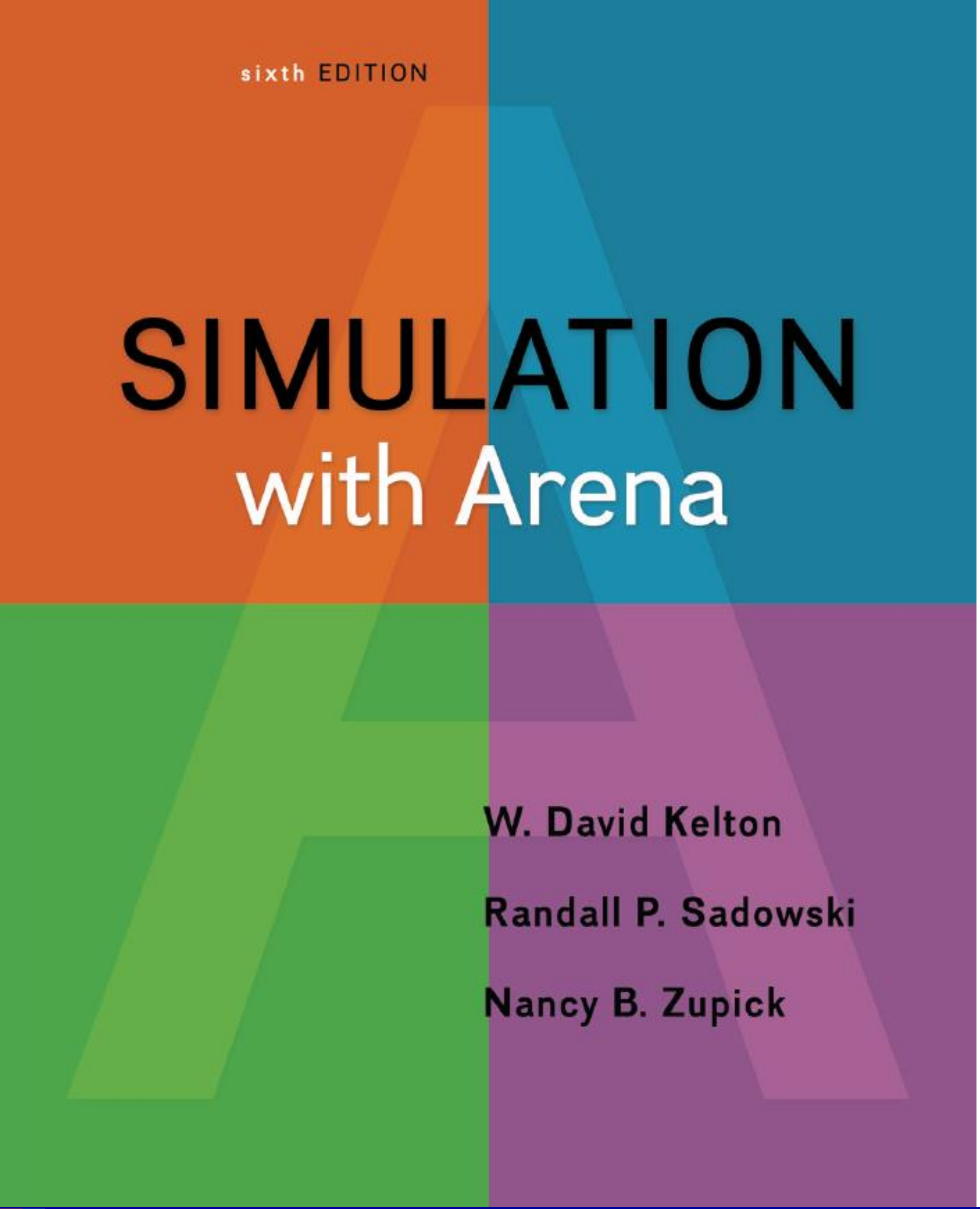


Fundamental Simulation Concepts

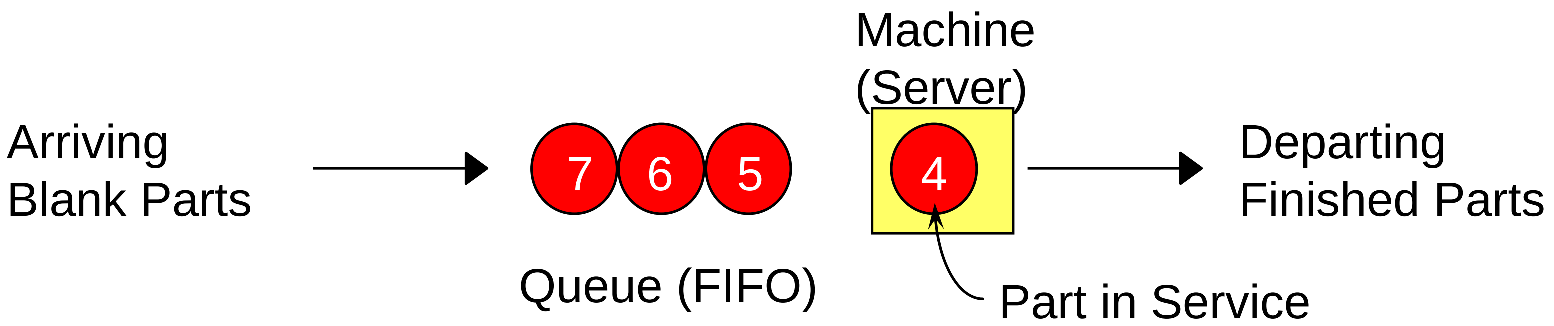
Chapter 2

Last revision March 9, 2014

What We'll Do ...

- **Underlying ideas, methods, and issues in simulation**
- **Software-independent (setting up for Arena)**
- **Example of a simple processing system**
  - Decompose problem
  - Terminology
  - Simulation by hand
  - Some basic statistical issues
- **Spreadsheet simulation**
  - Simple static, dynamic models
- **Overview of a simulation study**

The System: A Simple Processing System



- **General intent:**
  - Estimate expected production
  - Waiting time in queue, queue length, proportion of time machine is busy
- **Time units**
  - Can use different units in different places … must declare
  - Be careful to check units when specifying inputs
  - Declare *base time units* for internal calculations, outputs
  - Be reasonable (interpretation, roundoff error)

Model Specifics

- **Initially (time 0) empty and idle**
- **Base time units: minutes**
- **Input data (assume given for now …), in minutes:**

Part Number	Arrival Time	Interarrival Time	Service Time
1	0.00	1.73	2.90
2	1.73	1.35	1.76
3	3.08	0.71	3.39
4	3.79	0.62	4.52
5	4.41	14.28	4.46
6	18.69	0.70	4.36
7	19.39	15.52	2.07
8	34.91	3.15	3.36
9	38.06	1.76	2.37
10	39.82	1.00	5.38
11	40.82	.	.
.	.	.	.
.	.	.	.

- **Stop when 20 minutes of (simulated) time have passed**

Goals of Study: Output Performance Measures

- ***Total production*** **of parts over run (*P*)**
- ***Average waiting time*** **of parts in queue:**

$$\frac{\sum_{i=1}^{N} WQ_i}{N}$$

N = no. of parts completing queue wait
WQ_i = waiting time in queue of *i*th part
Know: $WQ_1 = 0$ (why?)
$N \geq 1$ (why?)

- ***Maximum waiting time*** **of parts in queue:**

$$\max_{i=1,\ldots,N} WQ_i$$

Goals of Study: Output Performance Measures (cont'd.)

- ***Time-average number of parts in queue***:

$$\frac{\int_0^{20} Q(t)\,dt}{20}$$

$Q(t)$ = number of parts in queue at time t

- ***Maximum number of parts in queue***: $\max_{0 \le t \le 20} Q(t)$
- ***Average*** and ***maximum total time in system*** of parts (a.k.a. ***cycle time***):

$$\frac{\sum_{i=1}^{P} TS_i}{P}, \qquad \max_{i=1,\ldots,P} TS_i$$

TS_i = time in system of part i

Goals of Study: Output Performance Measures (cont'd.)

- *Utilization* of machine (proportion of time busy)

$$\frac{\int_0^{20} B(t)\,dt}{20}, \quad B(t) = \begin{cases} 1 & \text{if machine is busy at time } t \\ 0 & \text{if machine is idle at time } t \end{cases}$$

- Many others possible (information overload?)

Analysis Options

- **Educated guessing**
  - Average interarrival time = 4.08 minutes
  - Average service time = 3.46 minutes
  - So (on average) parts are being processed faster than they arrive
    - System has a chance of operating in a stable way in long run, i.e., might not “explode”
    - If all interarrivals and service times were exactly at their mean, there would never be a queue
    - But data clearly exhibit variability, so a queue could form
  - If we’d had average interarrival < average service time, and this persisted, then queue would explode
  - Truth — between these extremes
  - Guessing has its limits …

Analysis Options (cont'd.)

- **Queueing theory**
  - Requires additional assumptions about model
  - Popular, simple model: *M/M/1 queue*
    - Interarrival times ~ exponential
    - Service times ~ exponential, indep. of interarrivals
    - Must have E(service) < E(interarrival)
    - Steady-state (long-run, forever)
    - Exact analytic results; e.g., average waiting time in queue is

$$\frac{\mu_S^2}{\mu_A - \mu_S}, \qquad \begin{aligned} \mu_A &= \mathrm{E(interarrival\ time)} \\ \mu_S &= \mathrm{E(service\ time)} \end{aligned}$$

  - Problems: validity, estimating means, time frame
  - Often useful as first-cut approximation

Mechanistic Simulation

- **Individual operations (arrivals, service times) will occur exactly as in reality**
- **Movements, changes occur at right “times,” in right order**
- **Different pieces interact**
- **Install “observers” to get output performance measures**
- **Concrete, “brute-force” analysis approach**
- **Nothing mysterious or subtle**
  - But a lot of details, bookkeeping
  - Simulation software keeps track of things for you

Pieces of a Simulation Model

- ***Entities***
  - "Players" that move around, change status, affect and are affected by other entities
  - *Dynamic objects* — get created, move around, leave (maybe)
  - Usually represent "real" things
    - Our model: entities are parts
  - Can have "fake" entities for modeling "tricks"
    - Breakdown demon, break angel
      - Though Arena has built-in ways to model these examples directly
  - Usually have multiple *realizations* floating around
  - Can have different types of entities concurrently
  - Usually, identifying types of entities is first thing to do in building model

Pieces of a Simulation Model (cont'd.)

- ***Attributes***
  - Characteristic of all entities: describe, differentiate
  - All entities have same attribute "slots" but different values for different entities, for example:
    - Time of arrival
    - Due date
    - Priority
    - Color
  - Attribute value tied to a specific entity
  - Like "local" (to entities) variables
  - Some automatic in Arena, some you define

Pieces of a Simulation Model (cont'd.)

- **(Global) *Variables***
  - Reflects a characteristic of whole model, not of specific entities
  - Used for many different kinds of things
    - Travel time between all station pairs
    - Number of parts in system
    - Simulation clock (built-in Arena variable)
  - Name, value of which there's only one copy for whole model
  - Not tied to entities
  - Entities can access, change variables
  - Writing on wall (rewriteable)
  - Some built-in by Arena, you can define others

Pieces of a Simulation Model (cont'd.)

- ***Resources***
  - What entities compete for
    - People
    - Equipment
    - Space
  - Entity *seizes* a resource, uses it, *releases* it
  - Think of a *resource being assigned to an entity*, rather than an entity "belonging to" a resource
  - "A" resource can have several *units* of capacity
    - Seats at a table in a restaurant
    - Identical ticketing agents at an airline counter
  - Number of units of resource can be changed during simulation

Pieces of a Simulation Model (cont'd.)

- ***Queues***
  - Place for entities to wait when they can't move on (maybe since resource they want to seize is not available)
  - Have names, often tied to a corresponding resource
  - Can have a finite capacity to model limited space — have to model what to do if an entity shows up to a queue that's already full
  - Usually watch length of a queue, waiting time in it

Pieces of a Simulation Model (cont'd.)

- ***Statistical accumulators***
  - Variables that "watch" what's happening
  - Depend on output performance measures desired
  - "Passive" in model — don't participate, just watch
  - Many are automatic in Arena, but some you may have to set up and maintain during simulation
  - At end of simulation, used to compute final output performance measures

Pieces of a Simulation Model (cont'd.)

- **Statistical accumulators for simple processing system**
  - Number of parts produced so far
  - Total of waiting times spent in queue so far
  - No. of parts that have gone through queue
  - Max time in queue we've seen so far
  - Total of times spent in system
  - Max time in system we've seen so far
  - Area so far under queue-length curve $Q(t)$
  - Max of $Q(t)$ so far
  - Area so far under server-busy curve $B(t)$

Simulation Dynamics: Event-Scheduling “World View”

- **Identify characteristic *events***
- **Decide on *logic* for each type of event to:**
  - Effect *state changes* for each event type
  - Observe statistics
  - Update times of future events (maybe of this type, other types)
- **Keep a simulation *clock*, future *event calendar***
- ***Jump* from one event to the next, process, observe statistics, update event calendar**
- **Must specify an appropriate *stopping rule***
- **Usually done with general-purpose programming language (C++, Java, Matlab, FORTRAN, etc.)**

Events for the Simple Processing System

- ***Arrival*** **of a new part to system**
  - Update time-persistent statistical accumulators (from last event to now)
    - Area under $Q(t)$
    - Max of $Q(t)$
    - Area under $B(t)$
  - "Mark" arriving part with current time (use later)
  - If machine is idle:
    - Start processing (schedule departure), Make machine busy, Tally waiting time in queue (0)
  - Else (machine is busy):
    - Put part at end of queue, increase queue-length variable
  - Schedule next arrival event

Events for the Simple Processing System (cont'd.)

- ***Departure*** **(when a service is completed)**
  - Increment number-produced stat accumulator
  - Compute & tally time in system (now – time of arrival)
  - Update time-persistent statistics (as in arrival event)
  - If queue is non-empty:
    - Take first part out of queue, compute & tally its waiting time in queue, begin service (schedule departure event)
  - Else (queue is empty):
    - Make machine idle (Note: there will be no departure event scheduled on future events calendar, which is as desired)

Events for the Simple Processing System (cont'd.)

- ***The End***
  - Update time-persistent statistics (to end of simulation)
  - Compute final output performance measures using current (= final) values of statistical accumulators
- **After each event, event calendar's top record is removed to see what time it is, what to do**
- **Also must initialize everything**

Some Additional Specifics for the Simple Processing System

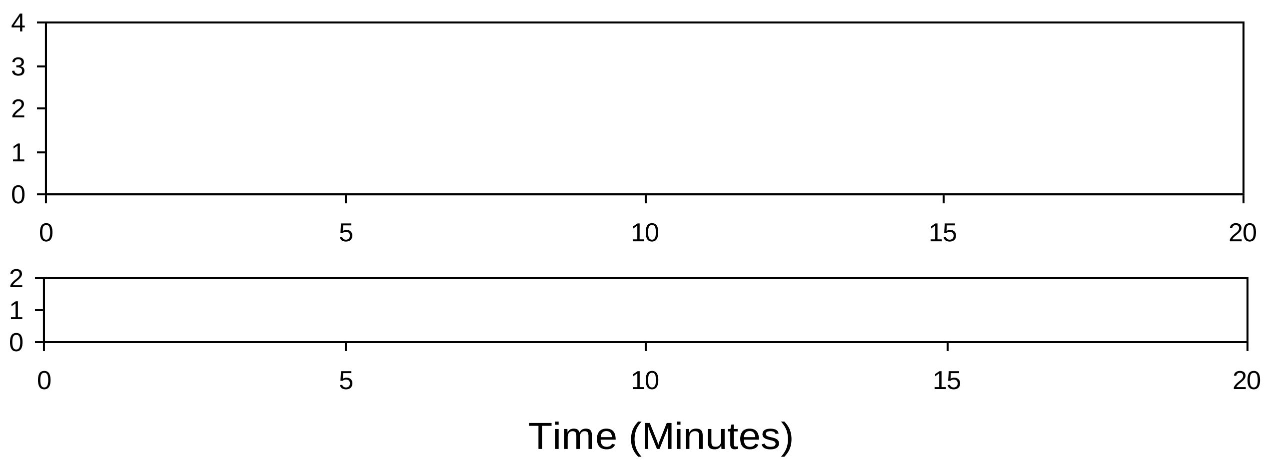
- **Simulation clock variable (internal in Arena)**
- **Event calendar: list of event *records*:**
  - [Entity No., Event Time, Event Type]
  - Keep *ranked* in increasing order on Event Time
  - Next event always in top record
  - Initially, schedule first Arrival, The End (Dep.?)
- **State variables: describe current status**
  - Server status $B(t) = 1$ for busy, 0 for idle
  - Number of customers in queue $Q(t)$
  - Times of arrival of each customer now in queue (a list of random length)

Simulation by Hand

- **Manually track state variables, statistical accumulators**
- **Use “given” interarrival, service times**
- **Keep track of event calendar**
- **“Lurch” clock from one event to next**
- **Will omit times in system, “max” computations here (see text for complete details)**

Simulation by Hand: Setup

<table>
<tr><td>System</td><td>Clock</td><td>$B(t)$</td><td colspan="2">$Q(t)$</td><td>Arrival times of custs. in queue</td><td>Event calendar</td></tr>
<tr><td>Number of completed waiting times in queue</td><td colspan="3">Total of waiting times in queue</td><td colspan="2">Area under $Q(t)$</td><td>Area under $B(t)$</td></tr>
<tr><td>$Q(t)$ graph

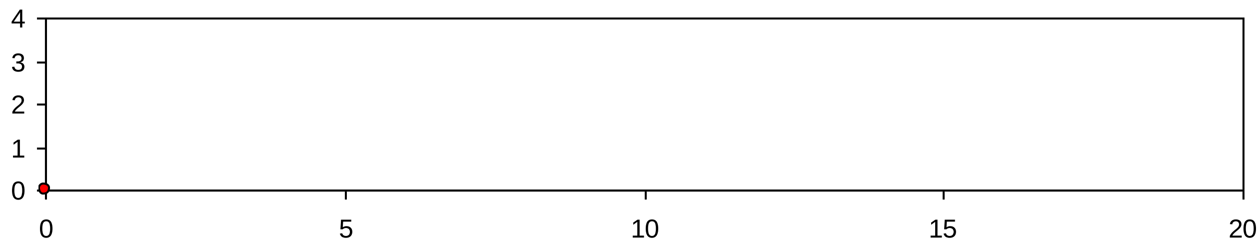
$B(t)$ graph</td><td colspan="6">
</td></tr>
<tr><td>Interarrival times</td><td colspan="6">1.73, 1.35, 0.71, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="6">2.90, 1.76, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

Simulation by Hand:
t = 0.00, Initialize

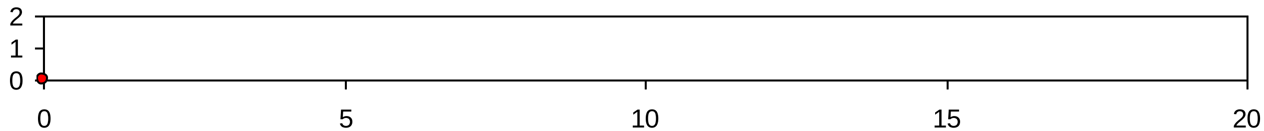
System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
	0.00	0	0	<empty>	[1, 0.00, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
0	0.00	0.00	0.00

$Q(t)$ graph



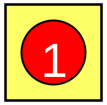
$B(t)$ graph



Time (Minutes)

Interarrival times	1.73, 1.35, 0.71, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	2.90, 1.76, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand: *t* = 0.00, Arrival of Part 1

<table>
<tr><td>System

</td><td>Clock

0.00</td><td>B(t)

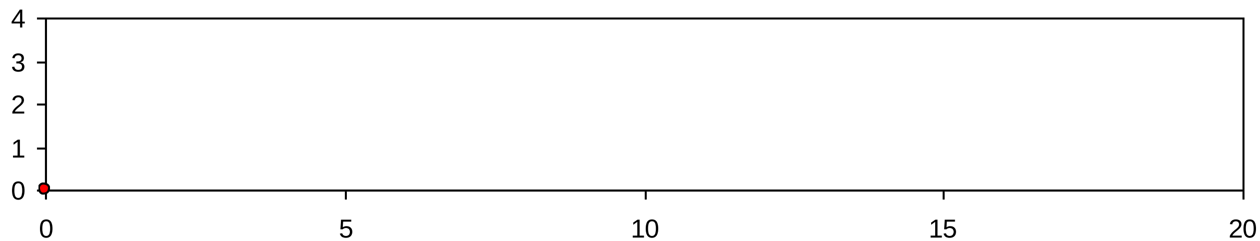
1</td><td>Q(t)

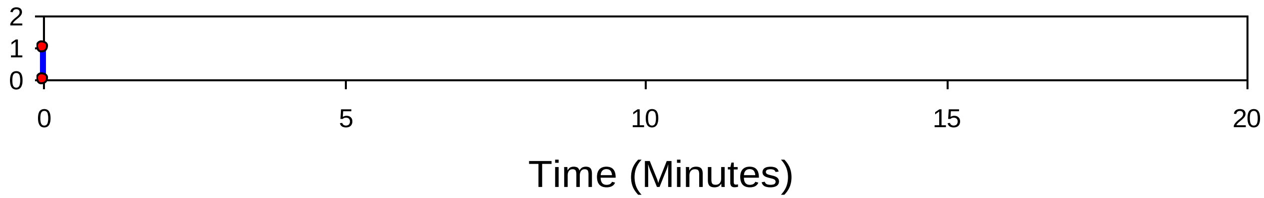
0</td><td>Arrival times of custs. in queue
<empty></td><td>Event calendar
[2, 1.73, Arr]
[1, 2.90, Dep]
[–, 20.00, End]</td></tr>
<tr><td>Number of completed waiting times in queue
1</td><td colspan="2">Total of waiting times in queue

0.00</td><td colspan="2">Area under Q(t)

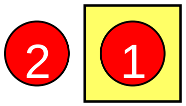
0.00</td><td>Area under B(t)

0.00</td></tr>
<tr><td>Q(t) graph

B(t) graph</td><td colspan="5">


</td></tr>
<tr><td>Interarrival times</td><td colspan="5"><s>1.73</s>, 1.35, 0.71, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="5"><s>2.90</s>, 1.76, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

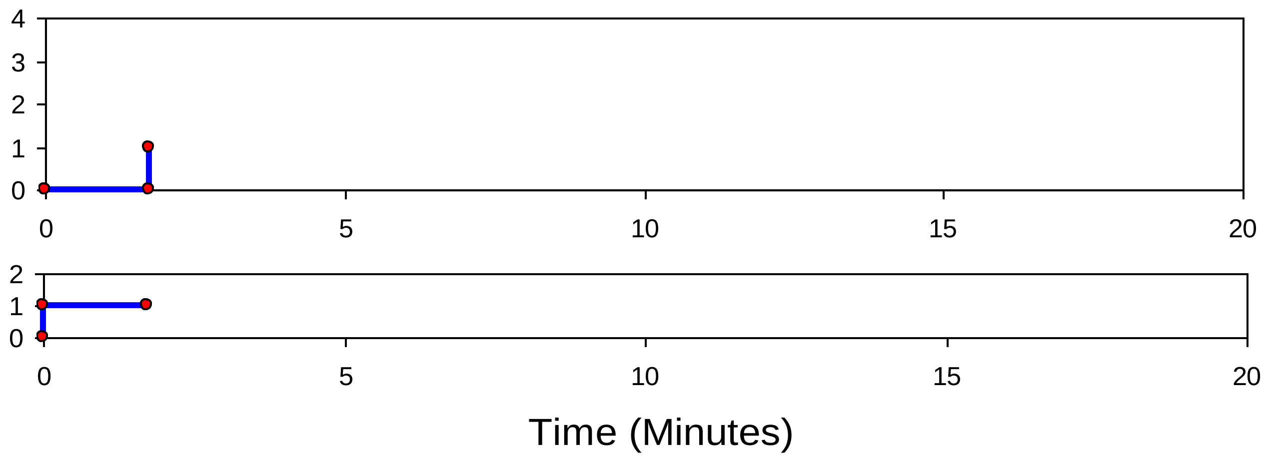
Simulation by Hand:
t = 1.73, Arrival of Part 2

System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
	1.73	1	1	(1.73)	[1, 2.90, Dep] [3, 3.08, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
1	0.00	0.00	1.73

$Q(t)$ graph

$B(t)$ graph



Interarrival times	~~1.73~~, ~~1.35~~, 0.71, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, 1.76, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand:
t = 2.90, Departure of Part 1

<table>
<tr><td>System
</td><td>Clock

2.90</td><td>B(t)

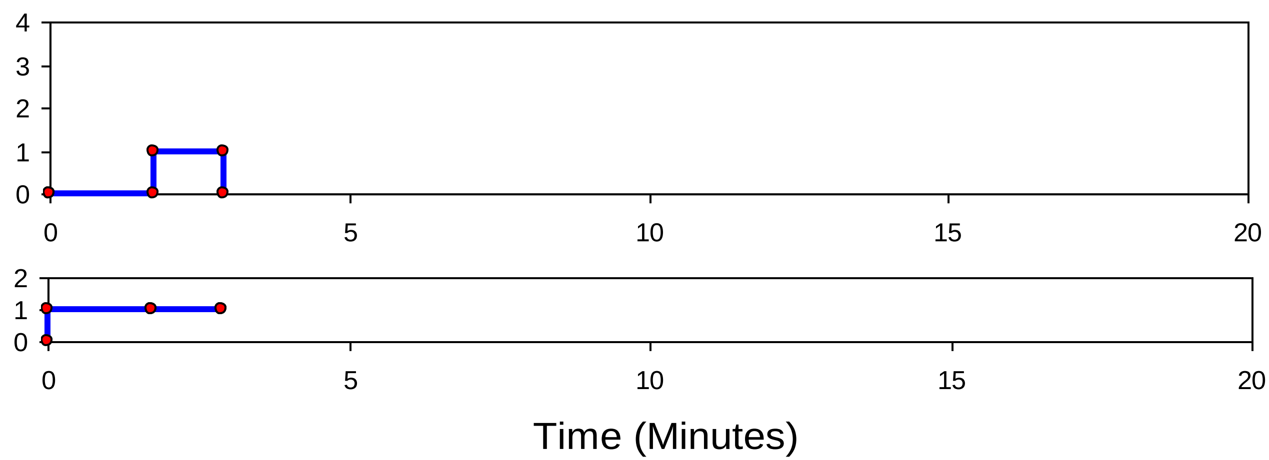
1</td><td>Q(t)

0</td><td colspan="2">Arrival times of custs. in queue
<empty></td><td>Event calendar
[3, 3.08, Arr]
[2, 4.66, Dep]
[–, 20.00, End]</td></tr>
<tr><td>Number of completed waiting times in queue
2</td><td colspan="3">Total of waiting times in queue

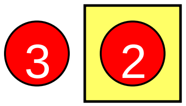
1.17</td><td colspan="2">Area under Q(t)

1.17</td><td>Area under B(t)

2.90</td></tr>
<tr><td>Q(t) graph

B(t) graph</td><td colspan="6"></td></tr>
<tr><td>Interarrival times</td><td colspan="6"><s>1.73</s>, <s>1.35</s>, 0.71, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="6"><s>2.90</s>, <s>1.76</s>, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

Simulation by Hand:
t = 3.08, Arrival of Part 3

<table>
<tr><td>System
</td><td>Clock

3.08</td><td>B(t)

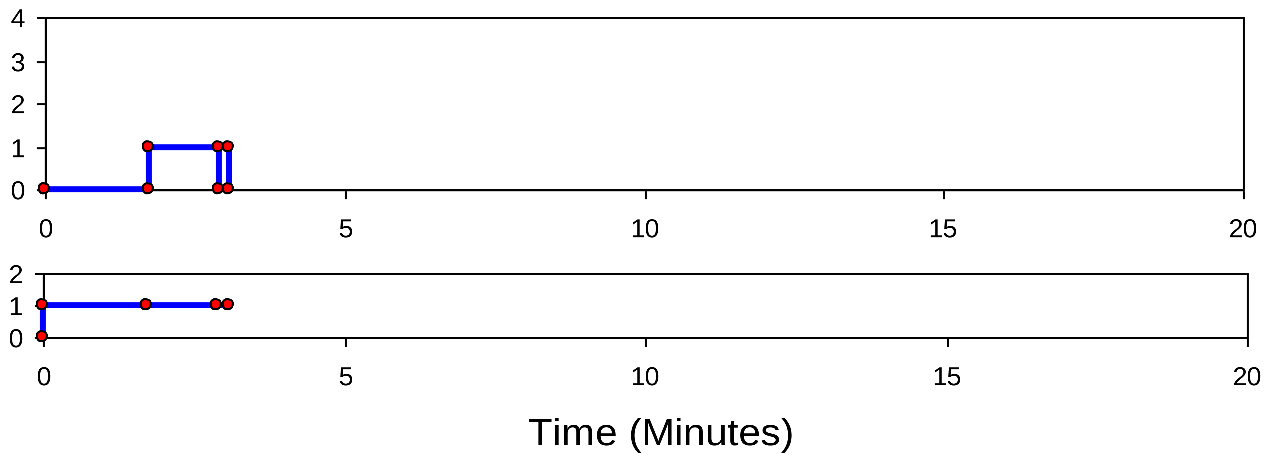
1</td><td colspan="2">Q(t)

1</td><td>Arrival times of custs. in queue
(3.08)</td><td>Event calendar
[4, 3.79, Arr]
[2, 4.66, Dep]
[–, 20.00, End]</td></tr>
<tr><td>Number of completed waiting times in queue
2</td><td colspan="3">Total of waiting times in queue

1.17</td><td colspan="2">Area under Q(t)

1.17</td><td>Area under B(t)

3.08</td></tr>
<tr><td>Q(t) graph

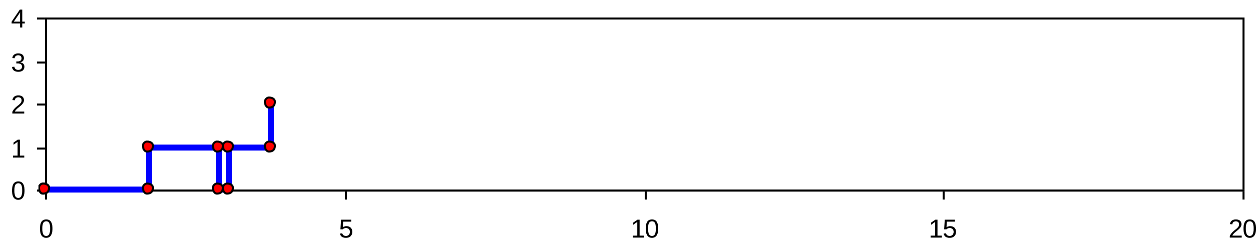
B(t) graph</td><td colspan="6"></td></tr>
<tr><td>Interarrival times</td><td colspan="6">~~1.73~~, ~~1.35~~, ~~0.71~~, 0.62, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="6">~~2.90~~, ~~1.76~~, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

Simulation by Hand:
t = 3.79, Arrival of Part 4

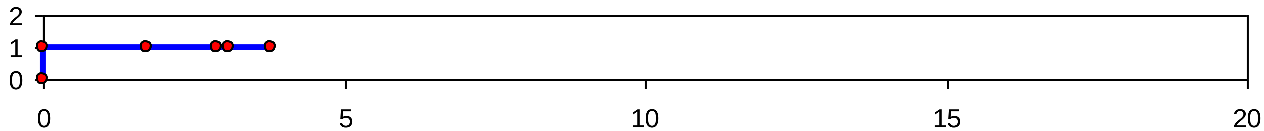
System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
4 3 $\boxed{2}$	3.79	1	2	(3.79, 3.08)	[5, 4.41, Arr] [2, 4.66, Dep] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
2	1.17	1.88	3.79

$Q(t)$ graph



$B(t)$ graph

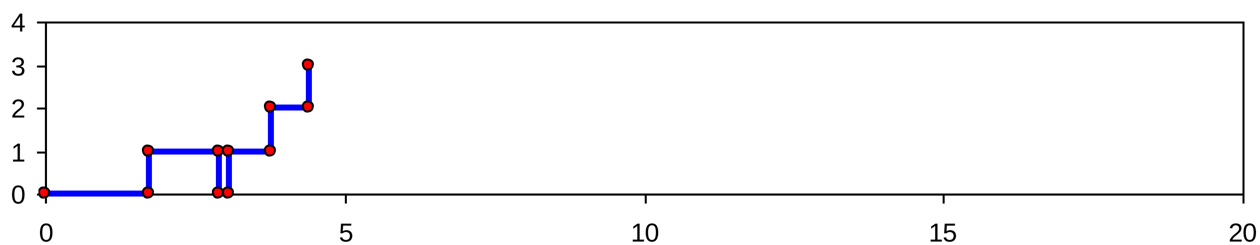
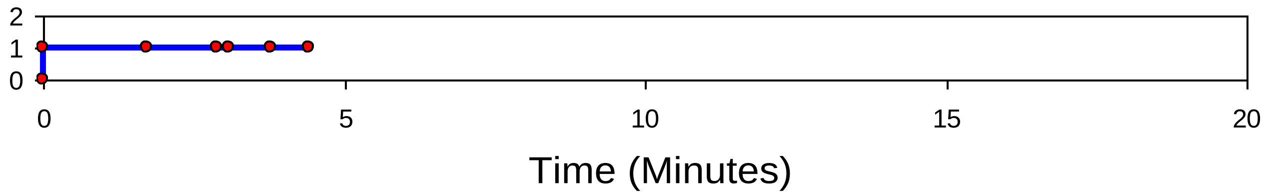


Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, 14.28, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand:
t = 4.41, Arrival of Part 5

System	Clock	*B*(*t*)	*Q*(*t*)	Arrival times of custs. in queue	Event calendar
5 4 3 2	4.41	1	3	(4.41, 3.79, 3.08)	[2, 4.66, Dep] [6, 18.69, Arr] [–, 20.00, End]

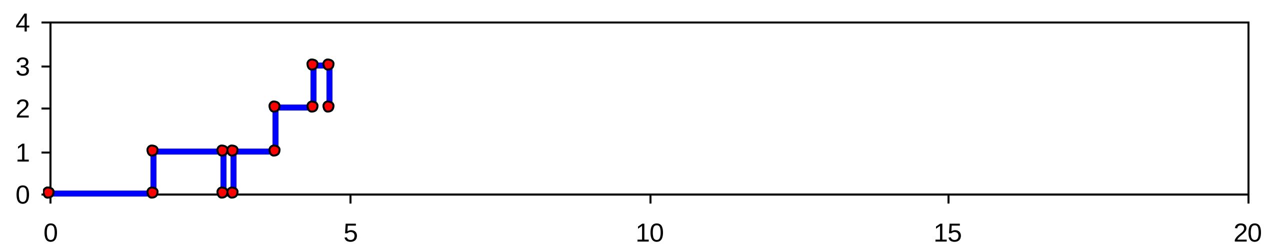
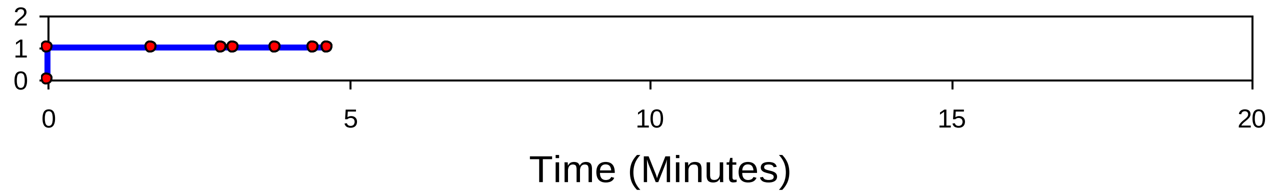
Number of completed waiting times in queue	Total of waiting times in queue	Area under *Q*(*t*)	Area under *B*(*t*)
2	1.17	3.12	4.41

Q(*t*) graph	
B(*t*) graph	
Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, 3.39, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

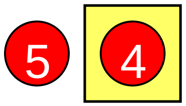
Simulation by Hand:
t = 4.66, Departure of Part 2

System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
5 4 3	4.66	1	2	(4.41, 3.79)	[3, 8.05, Dep] [6, 18.69, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
3	2.75	3.87	4.66

$Q(t)$ graph	
$B(t)$ graph	
Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, ~~3.39~~, 4.52, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

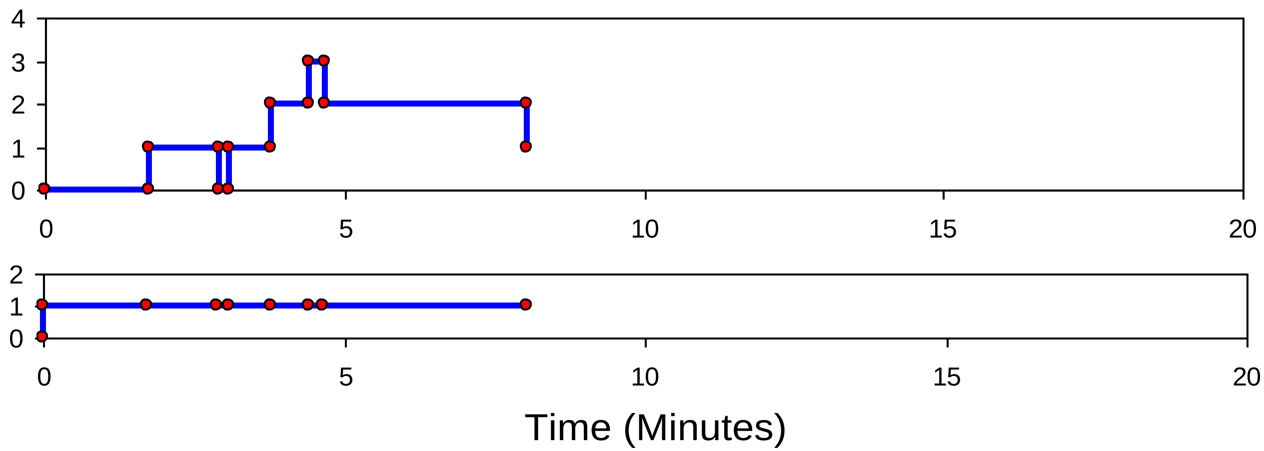
Simulation by Hand:
t = 8.05, Departure of Part 3

System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
	8.05	1	1	(4.41)	[4, 12.57, Dep] [6, 18.69, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
4	7.01	10.65	8.05

$Q(t)$ graph

$B(t)$ graph

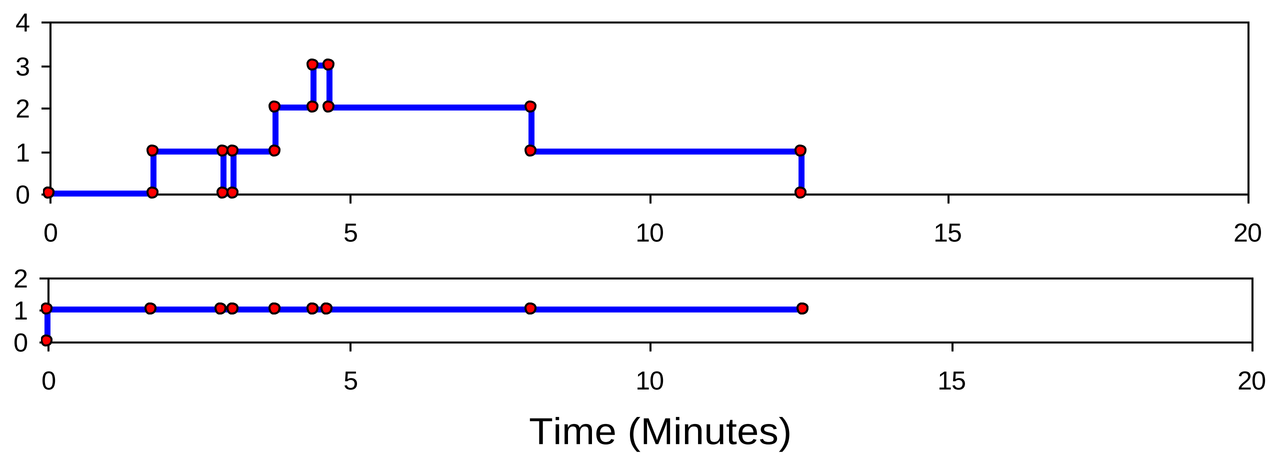


Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, 4.46, 4.36, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand:
t = 12.57, Departure of Part 4

System	Clock	*B*(*t*)	*Q*(*t*)	Arrival times of custs. in queue	Event calendar
5	12.57	1	0	()	[5, 17.03, Dep] [6, 18.69, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under *Q*(*t*)	Area under *B*(*t*)
5	15.17	15.17	12.57

Q(*t*) graph *B*(*t*) graph	
Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, ~~4.46~~, 4.36, 2.07, 3.36, 2.37, 5.38, ...

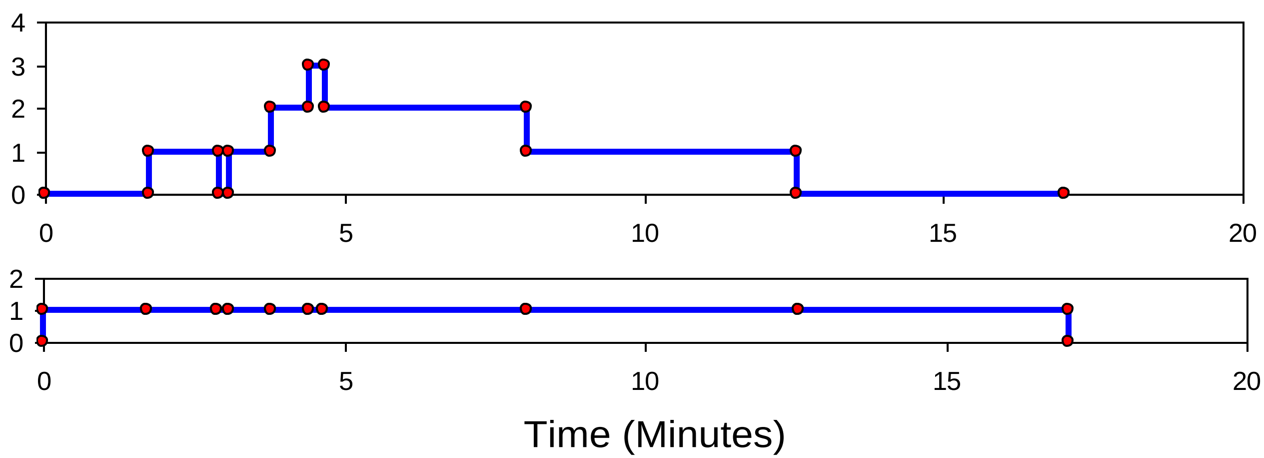
Simulation by Hand:
t = 17.03, Departure of Part 5

System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
	17.03	0	0	()	[6, 18.69, Arr] [–, 20.00, End]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
5	15.17	15.17	17.03

$Q(t)$ graph

$B(t)$ graph



Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, 0.70, 15.52, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, ~~4.46~~, 4.36, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand:
t = 18.69, Arrival of Part 6

<table>
<tr><td>System
</td><td>Clock

18.69</td><td>B(t)

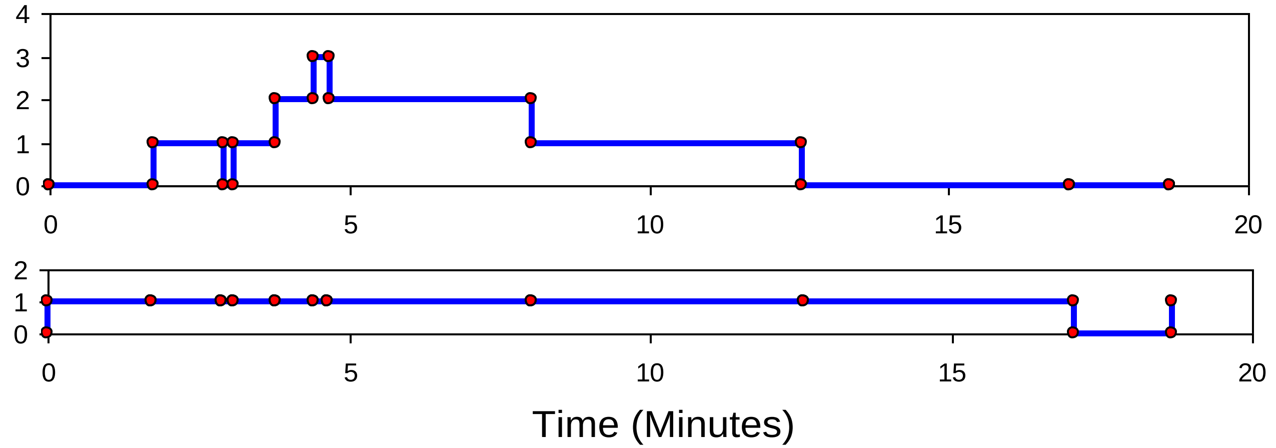
1</td><td colspan="2">Q(t)

0</td><td>Arrival times of custs. in queue
()</td><td>Event calendar
[7, 19.39, Arr]
[–, 20.00, End]
[6, 23.05, Dep]</td></tr>
<tr><td>Number of completed waiting times in queue
6</td><td colspan="3">Total of waiting times in queue

15.17</td><td colspan="2">Area under Q(t)

15.17</td><td>Area under B(t)

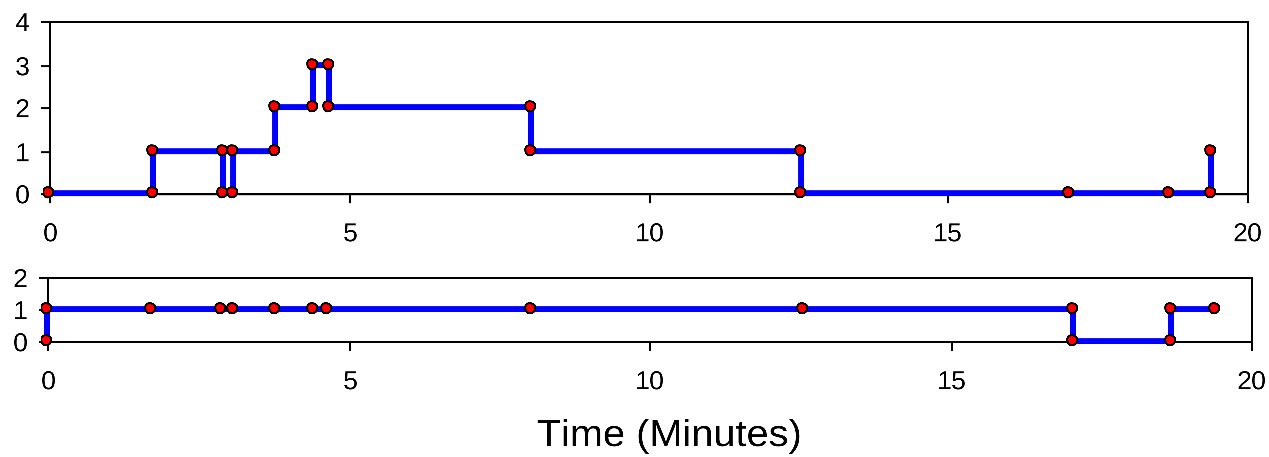
17.03</td></tr>
<tr><td>Q(t) graph

B(t) graph</td><td colspan="6"></td></tr>
<tr><td>Interarrival times</td><td colspan="6">~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, ~~0.70~~, 15.52, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="6">~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, ~~4.46~~, ~~4.36~~, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

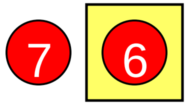
Simulation by Hand:
t = 19.39, Arrival of Part 7

System	Clock	$B(t)$	$Q(t)$	Arrival times of custs. in queue	Event calendar
7 6	19.39	1	1	(19.39)	[–, 20.00, End] [6, 23.05, Dep] [8, 34.91, Arr]

Number of completed waiting times in queue	Total of waiting times in queue	Area under $Q(t)$	Area under $B(t)$
6	15.17	15.17	17.73

$Q(t)$ graph $B(t)$ graph	
Interarrival times	~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, ~~0.70~~, ~~15.52~~, 3.15, 1.76, 1.00, ...
Service times	~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, ~~4.46~~, ~~4.36~~, 2.07, 3.36, 2.37, 5.38, ...

Simulation by Hand: *t* = 20.00, The End

<table>
<tr><td>System
</td><td>Clock

20.00</td><td>B(t)

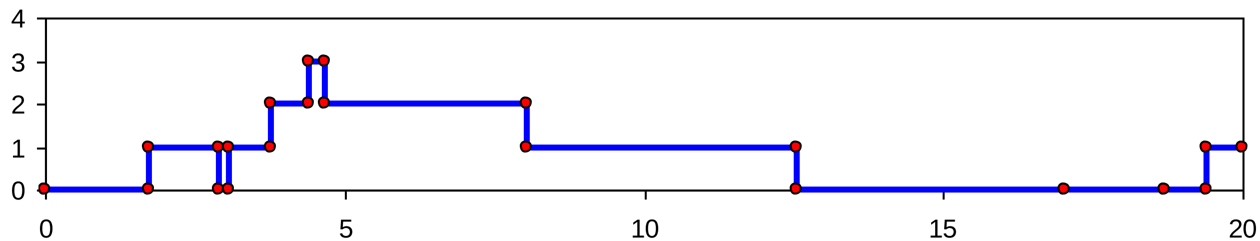
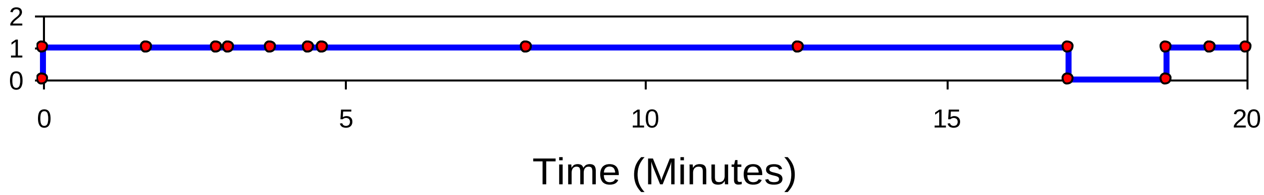
1</td><td colspan="2">Q(t)

1</td><td>Arrival times of
custs. in queue
(19.39)</td><td>Event calendar
[6, 23.05, Dep]
[8, 34.91, Arr]</td></tr>
<tr><td>Number of
completed waiting
times in queue
6</td><td colspan="3">Total of
waiting times in queue

15.17</td><td colspan="2">Area under
Q(t)

15.78</td><td>Area under
B(t)

18.34</td></tr>
<tr><td>Q(t) graph

B(t) graph</td><td colspan="6"></td></tr>
<tr><td>Interarrival times</td><td colspan="6">~~1.73~~, ~~1.35~~, ~~0.71~~, ~~0.62~~, ~~14.28~~, ~~0.70~~, ~~15.52~~, 3.15, 1.76, 1.00, ...</td></tr>
<tr><td>Service times</td><td colspan="6">~~2.90~~, ~~1.76~~, ~~3.39~~, ~~4.52~~, ~~4.46~~, ~~4.36~~, 2.07, 3.36, 2.37, 5.38, ...</td></tr>
</table>

Simulation by Hand: Finishing Up

- **Average waiting time in queue:**

$$\frac{\text{Total of times in queue}}{\text{No. of times in queue}} = \frac{15.17}{6} = 2.53 \text{ minutes per part}$$

- **Time-average number in queue:**

$$\frac{\text{Area under } Q(t) \text{ curve}}{\text{Final clock value}} = \frac{15.78}{20} = 0.79 \text{ part}$$

- **Utilization of drill press:**

$$\frac{\text{Area under } B(t) \text{ curve}}{\text{Final clock value}} = \frac{18.34}{20} = 0.92 \text{ (dimensionless)}$$

Complete Record of the Hand Simulation

Just-Finished Event			Variables		Attributes		Statistical Accumulators									Event Calendar
Entity No.	Time t	Event Type	$Q(t)$	$B(t)$	Arrival Times: (In Queue)	In Service	P	N	ΣWQ	WQ^*	ΣTS	TS^*	$\int Q$	Q^*	$\int B$	[Entity No., Time, Type]
–	0.00	Init	0	0	( )	–	0	0	0.00	0.00	0.00	0.00	0.00	0	0.00	[1, 0.00, Arr] [–, 20.00, End]
1	0.00	Arr	0	1	( )	0.00	0	1	0.00	0.00	0.00	0.00	0.00	0	0.00	[2, 1.73, Arr] [1, 2.90, Dep] [–, 20.00, End]
2	1.73	Arr	1	1	(1.73)	0.00	0	1	0.00	0.00	0.00	0.00	0.00	1	1.73	[1, 2.90, Dep] [3, 3.08, Arr] [–, 20.00 End]
1	2.90	Dep	0	1	( )	1.73	1	2	1.17	1.17	2.90	2.90	1.17	1	2.90	[3, 3.08, Arr] [2, 4.66, Dep] [–, 20.00, End]
3	3.08	Arr	1	1	(3.08)	1.73	1	2	1.17	1.17	2.90	2.90	1.17	1	3.08	[4, 3.79, Arr] [2, 4.66, Dep] [–, 20.00, End]
4	3.79	Arr	2	1	(3.79, 3.08)	1.73	1	2	1.17	1.17	2.90	2.90	1.88	2	3.79	[5, 4.41, Arr] [2, 4.66, Dep] [–, 20.00, End]
5	4.41	Arr	3	1	(4.41, 3.79, 3.08)	1.73	1	2	1.17	1.17	2.90	2.90	3.12	3	4.41	[2, 4.66, Dep] [6, 18.69, Arr] [–, 20.00, End]
2	4.66	Dep	2	1	(4.41, 3.79)	3.08	2	3	2.75	1.58	5.83	2.93	3.87	3	4.66	[3, 8.05, Dep] [6, 18.69, Arr] [–, 20.00, End]
3	8.05	Dep	1	1	(4.41)	3.79	3	4	7.01	4.26	10.80	4.97	10.65	3	8.05	[4, 12.57, Dep] [6, 18.69, Arr] [–, 20.00, End]
4	12.57	Dep	0	1	( )	4.41	4	5	15.17	8.16	19.58	8.78	15.17	3	12.57	[5, 17.03, Dep] [6, 18.69, Arr] [–, 20.00, End]
5	17.03	Dep	0	0	( )	–	5	5	15.17	8.16	32.20	12.62	15.17	3	17.03	[6, 18.69, Arr] [–, 20.00 End]
6	18.69	Arr	0	1	( )	18.69	5	6	15.17	8.16	32.20	12.62	15.17	3	17.03	[7, 19.39, Arr] [–, 20.00, End] [6, 23.05, Dep]
7	19.39	Arr	1	1	(19.39)	18.69	5	6	15.17	8.16	32.20	12.62	15.17	3	17.73	[–, 20.00, End] [6, 23.05, Dep] [8, 34.91, Arr]
–	20.00	End	1	1	(19.39)	18.69	5	6	15.17	8.16	32.20	12.62	15.78	3	18.34	[6, 23.05, Dep] [8, 34.91, Arr]

Event-Scheduling Logic via Programming

- **Clearly well suited to standard programming language (C, C++, Java, etc.)**
- **Often use "utility" libraries for:**
  - List processing
  - Random-number generation
  - Random-variate generation
  - Statistics collection
  - Event-list and clock management
  - Summary and output
- **Main program ties it together, executes events in order**

Simulation Dynamics: Process-Interaction World View

- **Identify characteristic *entities* in system**
- **Multiple copies of entities co-exist, interact, compete**
- **"Code" is non-procedural**
- **Tell a "story" about what happens to a "typical" entity**
- **May have many types of entities, "fake" entities for things like machine breakdowns**
- **Usually requires special simulation software**
  - Underneath, still executed as event-scheduling
- **View normally taken by Arena**
  - Arena translates your model description into a program in SIMAN simulation language for execution

Randomness in Simulation

- **Above was just one "replication" — a sample of size one (not worth much)**
- **Made a total of five *replications* (IID):**

Performance Measure	Replication 1	2	3	4	5	Sample Avg.	Sample Std. Dev.	95% Half Width
Total production	5	3	6	2	3	3.80	1.64	2.04
Average waiting time in queue	2.53	1.19	1.03	1.62	0.00	1.27	0.92	1.14
Maximum waiting time in queue	8.16	3.56	2.97	3.24	0.00	3.59*	2.93*	3.63*
Average total time in system	6.44	5.10	4.16	6.71	4.26	5.33	1.19	1.48
Maximum total time in system	12.62	6.63	6.27	7.71	4.96	7.64*	2.95*	3.67*
Time-average number of parts in queue	0.79	0.18	0.36	0.16	0.05	0.31	0.29	0.36
Maximum number of parts in queue	3	1	2	1	1	1.60*	0.89*	1.11*
Drill-press utilization	0.92	0.59	0.90	0.51	0.70	0.72	0.18	0.23

Substantial variability across replications

- **Confidence intervals for expected values:**
  - In general, $\bar{X} \pm t_{n-1,1-\alpha/2} s / \sqrt{n}$ (normality assumption?)
  - For expected total production, $3.80 \pm (2.776)(1.64 / \sqrt{5})$

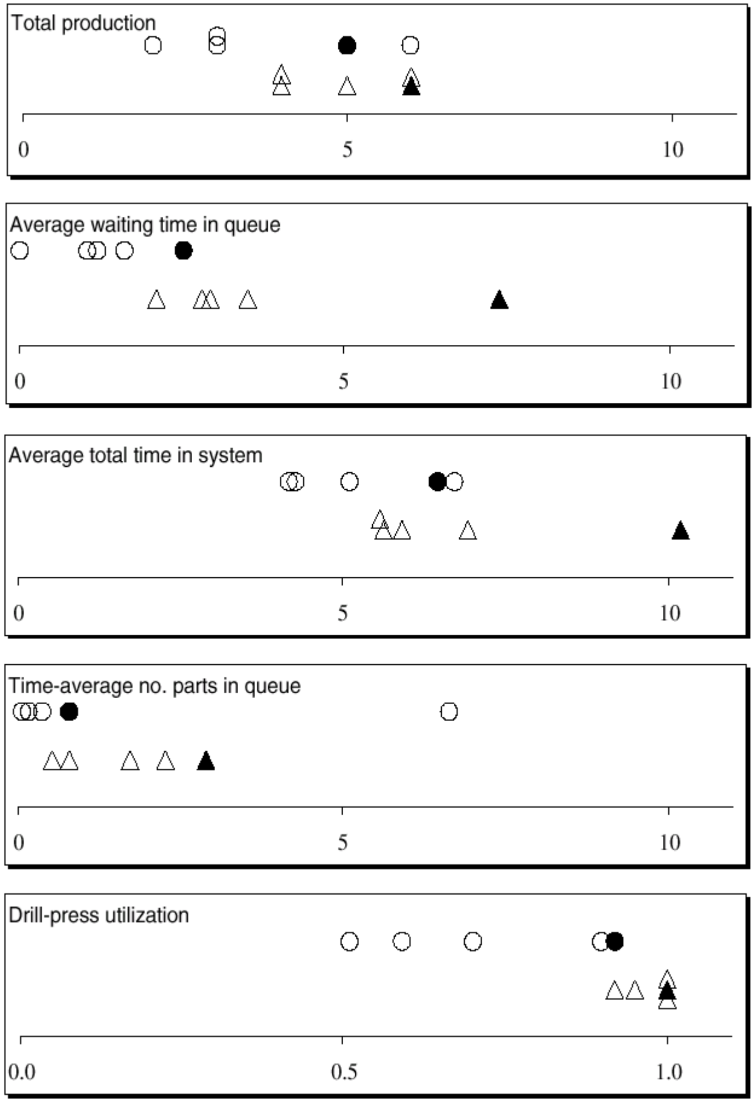
    3.80 ± 2.04

Precision?

Comparing Alternatives

- **Usually, simulation is used for more than just a single model "configuration"**
- **Often want to compare alternatives, select or search for best (via some criterion)**
- **Simple processing system: What would happen if arrival rate doubled?**
  - Cut interarrival times in half
  - Rerun model for double-time arrivals
  - Make five replications

Results: Original vs. Double-Time Arrivals



- **Original – circles**
- **Double-time – triangles**
- **Replication 1 – filled in**
- **Replications 2-5 – hollow**
- **Note variability**
- **Danger of making decisions based on one (first) replication**
- **Hard to see if there are really differences**
- **Need: Statistical analysis of simulation output data**

Simulating with Spreadsheets: Introduction

- **Popular, ubiquitous tool**
- **Can use for simple simulation models**
  - Typically, only static models
    - Risk analysis, financial/investment scenarios
  - Only (very) simplest of dynamic models
- **Two examples**
  - Newsvendor problem (static)
  - Waiting times in single-server queue (dynamic)
    - Special recursion valid only in this case

Simulating with Spreadsheets: Newsvendor Problem – Setup

- **Rupert sells daily newspapers on street**
  - Rupert buys for c = \$0.55 each, sells for r = \$1.00 each
- **Each morning, Rupert buys *q* copies**
  - q is a fixed number, same every day
- **Demand during a day: $D = \max(\lfloor X \rceil, 0)$**
  - $X \sim$ normal ($\mu = 135.7$, $\sigma = 27.1$), from historical data
  - $\lfloor X \rceil$ rounds X to nearest integer
- **If $D \leq q$, satisfy all demand, and $q - D \geq 0$ left over, sell for scrap at s = \$0.03 each**
- **If $D > q$, sells out (sells all *q* copies), no scrap**
  - But missed out on $D - q > 0$ sales
- **What should *q* be?**

Simulating with Spreadsheets: Newsvendor Problem – Formulation

- **Choose *q* to maximize expected profit per day**
  - *q* too small – sell out, miss \$0.45 profit per paper
  - *q* too big – have left over, scrap at a loss of \$0.52 per paper
- **Classic operations-research problem**
  - Many versions, variants, extensions, applications
  - Much research on exact solution in certain cases
  - But easy to simulate, even in a spreadsheet
- **Profit in a day, as a function of *q*:**

$$W(q) = \underbrace{r \min(D, q)}_{\text{Sales revenue}} + \underbrace{s \max(q - D, 0)}_{\text{Scrap revenue}} - \underbrace{cq}_{\text{Cost}}$$

  - $W(q)$ is a random variable – profit varies from day to day
- **Maximize E(*W*(*q*)) over nonnegative integers *q***

Simulating with Spreadsheets: Newsvendor Problem – Simulation

- **Set trial value of *q*, generate demand *D*, compute profit for that day**
  - Then repeat this for many days independently, average to estimate E(*W*(*q*))
    - Also get confidence interval, estimate of P(loss), histogram of *W*(*q*)
  - Try for a range of values of *q*
- **Need to generate demand $D = \max(\lfloor X \rceil, 0)$**
  - So need to generate X ~ normal ($\mu = 135.7$, $\sigma = 27.1$)
  - (Much) ahead – Sec. 12.2, generating random *variates*
  - In this case, generate $X = \Phi^{-1}_{\mu,\sigma}(U)$

    U is a random number distributed uniformly on [0, 1] (Sec. 12.1)

    $\Phi_{\mu,\sigma}$ is cumulative distribution function of normal (μ, σ) distribution

Simulating with Spreadsheets: Newsvendor Problem – Excel

- File **`Newsvendor.xls`**

All files in book: www.mhhe.com/kelton, Student Edition, **`BookExamples.zip`**

- **Input parameters in cells B4 – B8 (blue)**
- **Trial values for *q* in row 2 (pink)**
- **Day number (1, 2, ..., 30) in column D**
- **Demands in column E for each day:**

```
= MAX(ROUND(NORMINV(RAND(), $B$7, $B$8), 0), 0)
```

$$= \text{MAX}(\underbrace{\text{ROUND}}_{\text{Rounding function}}(\underbrace{\underbrace{\text{NORMINV}}_{\Phi^{-1}}(\underbrace{\text{RAND()}}_{U(0,1)\ \text{random number}}, \underbrace{\text{\$B\$7}}_{\mu}, \underbrace{\text{\$B\$8}}_{\sigma})}_{X \sim \text{normal}(\mu,\, \sigma)}, 0), 0)$$

Round to nearest integer

MAX 2nd argument

`RAND()` *is "volatile" so regenerates on any edit, or F9 key*

`$` *pins down following column or row when copying formula*

Simulating with Spreadsheets: Newsvendor Problem – Excel (cont'd.)

- **For each *q*:**
  - "Sold" column: number of papers sold that day
  - "Scrap" column: number of papers scrapped that day
  - "Profit" column: profit (+, –, 0) that day
  - Placement of "**`$`**" in formulas to facilitate copying
- **At bottom of "Profit" columns (green):**
  - Average profit over 30 days
  - Half-width of 95% confidence interval on E(W(q))
    - Value 2.045 is upper 0.975 critical point of t distribution with 29 d.f.
    - Plot confidence intervals as "I-beams" on left edge
  - Estimate of P(W(q) < 0)
    - Uses **`COUNTIF`** function
- **Histograms of *W*(*q*) at bottom**
  - Vertical red line at 0, separates profits, losses

Simulating with Spreadsheets: Newsvendor Problem – Results

- **Fine point – used same daily demands (column E) for each day, across all trial values of *q***
  - Would have been valid to generate them independently
  - Why is it better to use same demands for all *q*?
- **Results**
  - Best *q* is about 140, maybe a little less
  - Randomness in all results (tap F9 key)
    - All demands, profits, graphics change
    - Confidence-interval, histogram plots change
    - Reminder that these are random outputs, random plots
  - Higher $q \Rightarrow$ more variability in profit
    - Histograms at bottom are wider for larger *q*
    - Higher chance of both large profits, but higher chance of loss, too
    - Risk/return tradeoff can be quantified – risk taker vs. risk-averse

Simulating with Spreadsheets: Single-Server Queue – Setup

- **Like hand simulation, but:**
  - Interarrival times ~ exponential with mean $1/\lambda = 1.6$ min.
  - Service times ~ uniform on $[a, b] = [0.27, 2.29]$ min.
  - Stop when 50^{th} waiting time in queue is observed
    - i.e., when 50^{th} customer *begins* service, not exits system
- **Watch waiting times in queue $WQ_1, WQ_2, \ldots, WQ_{50}$**
  - Important – not watching anything else, unlike before
- **S_i = service time of customer i,**
  A_i = interarrival time between custs. $i - 1$ and i
- ***Lindley's recursion* (1952): Initialize $WQ_1 = 0$,**

$$WQ_i = \max(WQ_{i-1} + S_{i-1} - A_i, 0), \; i = 2, 3, \ldots$$

Simulating with Spreadsheets: Single-Server Queue – Simulation

- **Need to generate random variates: let *U* ~ U[0, 1]**
  - Exponential (mean $1/\lambda$): $A_i = -(1/\lambda) \ln(1 - U)$
  - Uniform on [*a*, *b*]: $S_i = a + (b - a)\, U$
- **File `MU1.xls`**

All files in book: www.mhhe.com/kelton, Student Edition, `BookExamples.zip`

- **Input parameters in cells B4 – B6 (blue)**
  - Some theoretical outputs in cells B8 – B10
- **Customer number (*i* = 1, 2, ..., 50) in column D**
- **Five IID replications (three columns for each)**
  - IA = interarrival times, S = service times
  - WQ = waiting times in queue (plot, thin curves)
    - First one initialized to 0, remainder use Lindley's recursion
      - Curves rise from 0, variation increases toward right
    - Creates positive autocorrelation down WQ columns
      - Curves have less abrupt jumps than if WQ_i's were independent

Simulating with Spreadsheets: Single-Server Queue – Results

- **Column averages (green)**
  - Average interarrival, service times close to expectations
  - Average WQ_i within each replication
    - Not too far from steady-state expectation
    - Considerable variation
    - Many are below it (why?)
- **Cross-replication (by customer) averages (green)**
  - Column T, thick line in plot to dampen noise
- **Why no sample variance, histograms of WQ_i's?**
  - Could have computed both, as in newsvendor; two issues:
    - Nonstationarity – what is a "typical" WQ_i here?
    - Autocorrelation – biases variance estimate, may bias histogram if run is not "long enough"

Simulating with Spreadsheets: Recap

- **Popular for static models**
  - Add-ins – @RISK, Crystal Ball
- **Inadequate tool for dynamic simulations if there's any complexity**
  - Extremely easy to simulate single-server queue in Arena – Chapter 3 main example
  - Can build very complex dynamic models with Arena – most of rest of book

Overview of a Simulation Study

- **Understand system**
- **Be clear about goals**
- **Formulate model representation**
- **Translate into modeling software**
- **Verify “program”**
- **Validate model**
- **Design experiments**
- **Make runs**
- **Analyze, get insight, document results**

More: Chapter 13