Quantum Computation: Density Matrices & Mixed State

what is a "mixed state"?

Simply, probability distribution over pure quantum states.

How do we create a mixed state? And how can we use it to distinguish b/t mixed states?

Let's say we have a qubit where

p_1 = prob. of $|\psi_1\rangle$
p_2 = prob. of $|\psi_2\rangle$

we measure in basis $|u_1\rangle, |u_2\rangle$ (since qubit)

Formula: $\langle u_1 | \rho | u_2 \rangle$ $\begin{bmatrix} - & - & - \end{bmatrix} \begin{bmatrix} \rho \end{bmatrix} \begin{bmatrix} - \\ - \\ - \end{bmatrix}$

where $\underbrace{\rho = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|}$

↳ this is the DENSITY MATRIX

How do density matrices show that GLOBAL PHASE does not affect measurement?

100% prob of $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

100% prob of $-|0\rangle = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

cannot distinguish! SAME DENSITY MATRIX

QUANTUM COMPUTATION: DENSITY MATRICES & MIXED STATES

WHAT IS THE SIMPLER WAY OF GETTING THE PROB IN THE STANDARD BASIS?

LETS SAY YOUR DENSITY MATRIX:

$$\begin{bmatrix} 5/9 & 1/9 & i/9 \\ 1/9 & 2/9 & -2/9\,i \\ i/9 & 2/9\,i & 2/9 \end{bmatrix}$$

THE PROB. OF $|1\rangle = \rho_{11}$

THE PROB OF $|2\rangle = \rho_{22}$

... FOR $|3\rangle = \rho_{33}$

WHAT ARE THE PROP. OF DENSITY MATRICES?

- THEY ARE HERMITIAN $\rho^{+} = \rho$
- THEY ARE POSITIVE, WHERE $\langle u|\rho|u\rangle \geq 0$ (FOR $\forall\, |u\rangle \in \mathbb{C}^d$)
- PROBABILITES MUST ADD TO 1 OR: $\sum_{i=1}^{d} \rho_{ii} = 1$

WHAT IS THE WORD USED TO DESCRIBE THAT THE PROB. of ρ must equal 1?

the trace of a matrix ρ, also denoted $tr(\rho)$

Quantum Computation: Density Matrices & Mixed States

What is the diagonal values of a density matrix akin to?

- classical probability distribution

What is the main thing you can do to a quantum qudit state?

Apply a unitary transformation. This affects the mixed state thus the density matrix.

quantum mixed state $\xrightarrow{U}$ new mixed state

p_i prob. of $|\varphi_i\rangle$ $(i = 1..m)$ "p_i prob. of $U|\varphi_i\rangle$"

in the same way: density matrix

$$\rho = \sum_{i=1}^{m} p_i |\varphi_i\rangle\langle\varphi_i| \xrightarrow{U} U(\rho)U^{t}$$

thus $\rho \xrightarrow{U} U\rho U^{t}$

Quantum Computation: Density Matrices & Mixed State

Distinguishability of quantum states

- Recall that 2 different mixtures such as $|0\rangle, |1\rangle$ & $|+\rangle, |-\rangle$ have the same density matrix. b/c of this fact two different mixtures of quantum states can only be distinguished if they have different density matrices.

Describe "partial trace"

The formula:

$$Tr_y \rho = \sum_{\alpha \in I} (I_x \otimes \langle\alpha|)\, \rho\, (I_x \otimes |\alpha\rangle)$$

You will want to calculate the partial trace of a quantum system, if for example, you prepared two qubits X & Y and threw away qubit Y. (this concept can be extrapolated to the discarding of any # of qubits.)

HOWEVER, partial trace is also good to see parts of a quantum system in isolation.

$$Tr_{(y)} |\phi^+\rangle\langle\phi^+| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

$$\underset{2\times 1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

TOTALLY MIXED STATE