Aaronson HW 2

[MORE FUN w/ MATRICES]

(A) 2x2 unitary matrix where diagonal are 0 but off diagonal are non-zero

NOT GATE $\rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(B) EXAMPLE of a 4x4 matrix

SWAP GATE $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(C) 3X3 matrix?

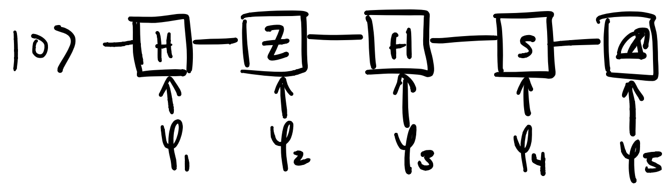
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leftarrow A$$

if you $\sqrt{A}$ then you get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$

[SINGLE QUBIT QUANTUM CIRCUITS]

* Calculate output state before & after measurement

(A) Measure in $\{|0\rangle, |1\rangle\}$



(φ_1) $H|0\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

(φ_2) $Z|\varphi_1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow$ which is $|-\rangle$

(φ_3) $H|\varphi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1+-1 & 1+-1 \\ 1+1 & -1+1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $H|-\rangle = |1\rangle$

(φ_4) $S|\varphi_3\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$

$\begin{bmatrix} 0 \\ i \end{bmatrix}$ before measurement

linear combination $0|0\rangle + i|1\rangle$

$p[\alpha] = \emptyset$

$p[\beta] = |i^2| = |\sqrt{-1}^2| = 1$

(φ_5) Since the $P[\beta]=1$ upon measurement the state will collapse to $|1\rangle$ w/ 100% certainty.

(B) Measure in $\{|+\rangle, |-\rangle\}$ basis

NOTE: Another way of understanding measurement in another basis state

$$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

$$|0\rangle = \frac{|i\rangle + |-i\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|i\rangle - |i\rangle}{\sqrt{2}}$$

$$|0\rangle - \underset{\varphi_1}{\boxed{R_{\frac{\pi}{4}}}} - \underset{\varphi_2}{\boxed{Z}} - \underset{\varphi_3}{\boxed{Y}} - \underset{\varphi_4}{\boxed{H}} - \underset{\varphi_5}{\boxed{\text{measure}}}$$

(φ_1) $R_{\pi/4}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow |+\rangle$

(φ_2) $Z|\varphi_1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(φ_3) $Y|\varphi_2\rangle = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} i = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ i \end{bmatrix}$

(φ_4) $H|\varphi_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$ state before measurement

(φ_5) measurement in $|i\rangle, |-i\rangle$ basis

$$\rightarrow \frac{\alpha+\beta}{\sqrt{2}}|i\rangle + \frac{\alpha+\beta}{\sqrt{2}}|-i\rangle$$

since state is $\begin{bmatrix} i \\ 0 \end{bmatrix}$

$\alpha = i$, $\beta = \emptyset$

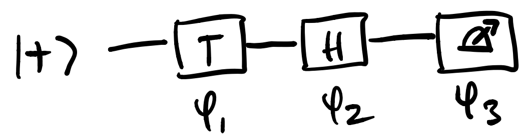
$$= \frac{i}{\sqrt{2}}|i\rangle + \frac{i}{\sqrt{2}}|-i\rangle$$

calculating probability

$$\left.\begin{aligned} p(|i\rangle) &= \left|\frac{i}{\sqrt{2}}\right|^2 \rightarrow \frac{1}{2} \\ p(|-i\rangle) &= \left|\frac{i}{\sqrt{2}}\right|^2 \rightarrow \frac{1}{2} \end{aligned}\right] - 1$$

there fore the prob. of seeing $|i\rangle$ is 50% and the prob. of seeing $|-i\rangle$ is also 50%

(c) Measure in $\{|+\rangle, |-\rangle\}$ basis



$$(\varphi_1)\ T|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\pi/4} \end{bmatrix}$$

$$(\varphi_2)\ H|\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\pi/4} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+e^{i\pi/4} \\ 1-e^{i\pi/4} \end{bmatrix}$$

before measurement

(φ_3) measurement

current state $\frac{1}{2}(1+e^{i\pi/4})|0\rangle + \frac{1}{2}(1-e^{i\pi/4})|1\rangle$

measure in $|+\rangle, |-\rangle$ basis

$$\rightarrow \frac{\frac{1}{2}(1+e^{i\pi/4}) + \frac{1}{2}(1-e^{i\pi/4})}{\sqrt{2}}|+\rangle + \frac{\frac{1}{2}(1+e^{i\pi/4}) + \frac{1}{2}(1-e^{i\pi/4})}{\sqrt{2}}|-\rangle$$

$$\frac{1}{2}\left(1+e^{i\pi/4}\right) + \frac{1}{2}\left(1+e^{i\pi/4}\right) = \frac{1}{2} + \cancel{\frac{1}{2}e^{i\pi/4}} + \frac{1}{2} - \cancel{\frac{1}{2}e^{i\pi/4}} = 1$$

$$= \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$ therefore

$$p(|+\rangle) = \frac{1}{\sqrt{2}}$$

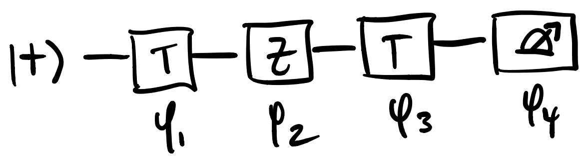
$$p(|-\rangle) = \frac{1}{\sqrt{2}}$$

correct?

$$p(|+\rangle)+p(|-\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2$$

$$= \frac{1}{2}+\frac{1}{2} = 1$$

(D) Measure in $\{|i\rangle, |-i\rangle\}$



$$(\varphi_1)\ T|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\pi/4}\end{bmatrix}$$

$$(\varphi_2)\ Z|\varphi_1\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\pi/4}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -e^{i\pi/4}\end{bmatrix}$$

$$(\varphi_3)\ T|\varphi_2\rangle = \begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\pi/4}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ e^{i\pi}\end{bmatrix}$$

$$i\frac{\pi}{4}+i\frac{\pi}{4} = 2i\,\frac{2\pi}{4} = i\,\frac{4\pi}{4} = i\pi$$

state before measurement

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\pi}}{\sqrt{2}}|1\rangle \longrightarrow$$ change basis to $|i\rangle, |-i\rangle$

$$\frac{\left(\frac{1}{\sqrt{2}}+\frac{e^{i\pi}}{\sqrt{2}}\right)}{\sqrt{2}}|i\rangle + \frac{\left(\frac{1}{\sqrt{2}}-\frac{e^{i\pi}}{\sqrt{2}}\right)}{\sqrt{2}}|-i\rangle$$

$$\frac{\left(\frac{1+e^{i\pi}}{\sqrt{2}}\right)}{\sqrt{2}}|i\rangle + \frac{\left(\frac{1-e^{i\pi}}{\sqrt{2}}\right)}{\sqrt{2}}|-i\rangle$$

$e^{i\pi} \rightarrow \underline{-1}$

$$= \frac{1+e^{i\pi}}{2}|i\rangle + \frac{1-e^{i\pi}}{2}|-i\rangle$$

$$= \frac{1+(-1)}{2}|i\rangle + \frac{1-(-1)}{2}|-i\rangle = 0|i\rangle + 1|-i\rangle$$

(ψ_4) Therefore at measure $p(|-i\rangle) = 1$ (100%)

[MISC]

(A) Normalize the state $|0\rangle + |+\rangle$

$$\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}2\\1\end{bmatrix} = \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{4}{2} + \frac{1}{2}}$$

$= \sqrt{\frac{5}{2}}$ ← normalizing factor

let's prove that this is the case. We expect that the sum of the amplitudes $\frac{\alpha + \beta}{\sqrt{\alpha^2+\beta^2}}$ will equal 1.

$\alpha = \frac{2}{\sqrt{2}}$, $\beta = \frac{1}{\sqrt{2}}$, $\sqrt{\alpha^2+\beta^2} = \sqrt{\frac{5}{2}}$

$\frac{1}{\sqrt{\frac{5}{2}}}\begin{bmatrix}2\\1\end{bmatrix}$

Normalized state

$$\frac{\frac{2}{\sqrt{2}}}{\sqrt{\frac{5}{2}}} = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5}, \quad \frac{\frac{1}{\sqrt{2}}}{\sqrt{\frac{5}{2}}} = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5}$$

$\alpha + \beta = \frac{4}{5} + \frac{1}{5} = \frac{5}{5} \longrightarrow 1$ TRUE

(B) Show that the state you normalized in part A is an eigenstate of the H gate. Eigenvalue?

$A \rightarrow H$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \times |\varphi\rangle = \lambda |\varphi\rangle$$

$$\det(A - \lambda I) = \emptyset$$

$$\det\left(\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \emptyset$$

$$\det\left(\begin{bmatrix} 1/\sqrt{2} - \lambda & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} - \lambda \end{bmatrix} \right) = \emptyset$$

$$\rightarrow \left(1/\sqrt{2} - \lambda\right)\left(-1/\sqrt{2} - \lambda\right) - \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) = \emptyset$$

$$-\frac{1}{2} - \cancel{\frac{\lambda}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}} + \lambda^2 - \frac{1}{2} = \emptyset$$

$$\lambda^2 - \frac{1}{4} = \emptyset$$

$$\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \emptyset$$

$$\lambda = -\frac{1}{2}, \frac{1}{2} \longleftarrow \text{eigen values}$$

Finding eigenspace with eigenvalues $\frac{1}{2}, -\frac{1}{2}$

$\lambda = \frac{1}{2}$ $(A - I)x$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2}-1 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2}-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \emptyset$$

$$\frac{1}{\sqrt{2}}x = \frac{1}{\sqrt{2}} - 1$$

$$x = \left(\frac{1}{\sqrt{2}} - 1\right)\sqrt{2}$$

$$= \frac{\sqrt{2}}{\sqrt{2}} - \sqrt{2}$$

$$1 - \sqrt{2}$$

$$\frac{1}{\sqrt{2}}(1-\sqrt{2})$$

$$\frac{1}{\sqrt{2}} - 1$$

$$\left(-\frac{1}{\sqrt{2}} - 1\right)(1 - \sqrt{2})$$

$$-\frac{1}{\sqrt{2}} - 1 - 1 - \sqrt{2}$$

$$2 - \sqrt{2}$$

$$\begin{bmatrix} 1/\sqrt{2}-1 & 1/\sqrt{2} \\ 1/\sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} R_1 - R_2 = \begin{bmatrix} 1/\sqrt{2}-1 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$$

$$\left(\frac{1}{\sqrt{2}} - 1\right)\left(\frac{\sqrt{2}}{1-\sqrt{2}}\right)$$

$$\left(\frac{1}{\sqrt{2}} - 1\right)x = 1$$

$$x = \frac{1}{\frac{1}{\sqrt{2}} - 1} = \frac{1}{\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}}} = \frac{1}{\frac{1-\sqrt{2}}{\sqrt{2}}} = \frac{\sqrt{2}}{1-\sqrt{2}}$$

$$= \begin{bmatrix} 1 & \frac{1}{1-\sqrt{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

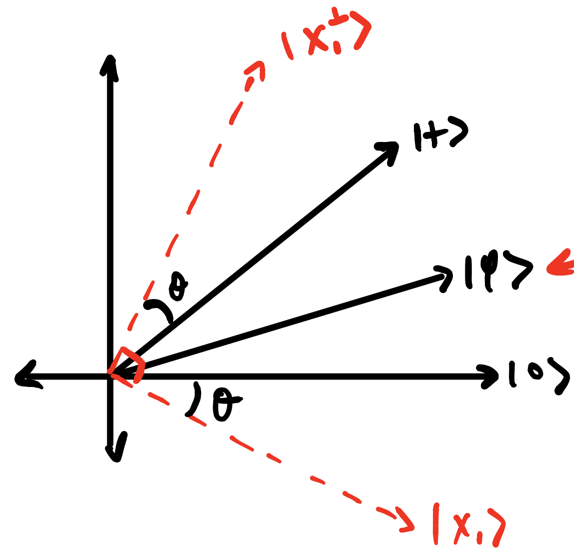
$$\cancel{\frac{1-\sqrt{2}}{\sqrt{2}}} \cancel{\left(\frac{\sqrt{2}}{1-\sqrt{2}}\right)} = 1$$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}}$$

$$x_1 + \frac{1}{1-\sqrt{2}}x_2 = \emptyset$$

[DISTINGUISHABILITY OF STATES]

(A) $|\psi\rangle$ is either $|0\rangle$ or $|+\rangle$ w/ equal probability



measured in basis $|x_1\rangle$ & $|x_1^\perp\rangle$

bisecto basis

which means that the **total** angle b/t $|\psi\rangle$ and $|x_1^\perp\rangle$ and also $|x_1\rangle$ is 45% degrees

when measured $|\psi\rangle$ will either collapse to $|x_1\rangle$ or $|x_1^\perp\rangle$

if $|\psi\rangle$ collapses to $|x_1\rangle$ then we can be certain with prob $\cos^2(\theta)$ which is the angle between the bases with $|0\rangle, |+\rangle$ respectively.