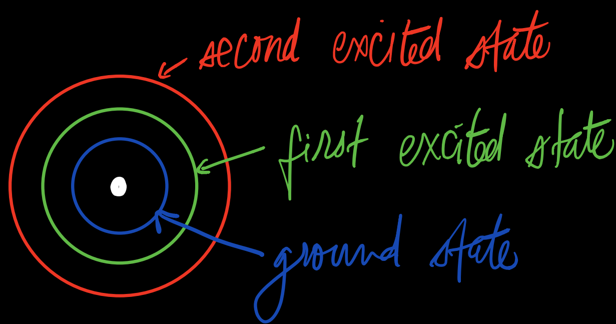
# Umesh - Quantum Computation Course

K - level system:

energy of an atom




[superposition principle:]

$$|\varphi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \dots$$
$$\alpha_n|k-1\rangle$$

generalized k-level state

Note: $\alpha_j \in \mathbb{C}$ ← complex numbers

and further $\sum_{j=0}^{k-1} |\alpha_j|^2 = 1$

meaning the sum of the amplitudes must equal 1 ← think probabilities

Measurement Axiom:

$P[j] = |\alpha_j|^2$

probability that the state of the electron is in state j.

New state $= |\varphi'\rangle = |j\rangle$

For example:

$$|\varphi\rangle = \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle - \frac{1}{2}|1\rangle + \frac{i}{2}|2\rangle$$

Now measure:

$P[0] = \left|\frac{1}{2} + \frac{1}{2}i\right|^2 \longrightarrow \frac{1}{2}$ thus $|\varphi'\rangle = |0\rangle$

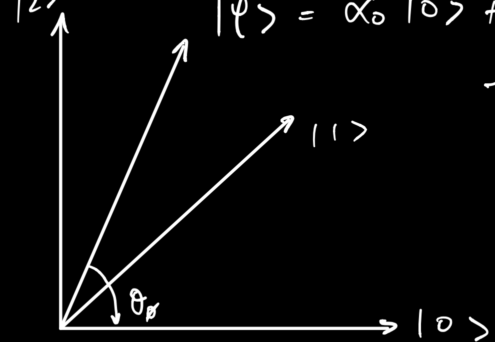
$P[1] = \left|-\frac{1}{2}\right|^2 \longrightarrow \frac{1}{4}$ thus $|\varphi'\rangle = |1\rangle$

$P[2] = \left|\frac{1}{2}i\right|^2 \longrightarrow \frac{1}{4}$ thus $|\varphi'\rangle = |2\rangle$

# K-level system: Geometric Interpretation

Superposition Principle: $|\psi\rangle \in \mathbb{C}^k$

* State is a unit vector in a hilbert space $\mathbb{C}^k$.




$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle \rightarrow \begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix} \in \mathbb{C}^3$

three dimensional vector space

$\theta$ = the angle made by $|\psi\rangle$ and the state that you are measuring against.

$P[0] = \cos^2\theta_\phi \longrightarrow |\psi\rangle = \begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix} \quad |\varphi\rangle = \begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix} \quad |\psi\rangle \,\&\, |\varphi\rangle \in \mathbb{C}^3$

$\cos\theta = |[\text{inner product}]|$

$\cos^2\theta = |[\text{inner product}]|^2$

complex conjugate

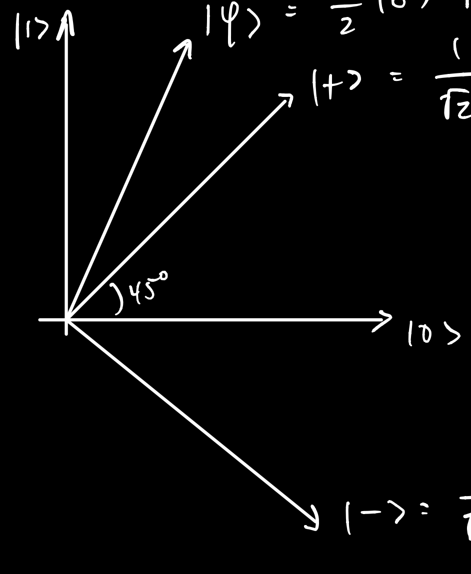
inner product: $\bar{\alpha}_0\beta_0 + \bar{\alpha}_1\beta_1 + \bar{\alpha}_2\beta_2$

OR $[\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2] \cdot \begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\end{bmatrix}$

## Example: Measurement in sign basis

$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$




$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

What is the probability that $|\psi\rangle$ will be $|+\rangle$?

$P[+]$ = inner product of $|\psi\rangle \,\&\, |+\rangle \rightarrow$

$$\left[ \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \right]^2$$

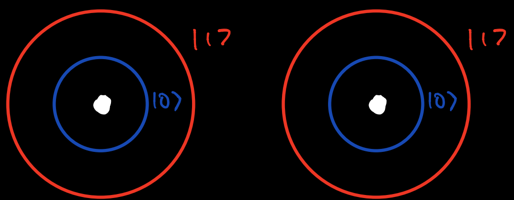
Easy to get $P[-]$ - it is just the conjugate of $P[+]$

so $P[-] = \frac{2-\sqrt{3}}{4}$

$$= \left[\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right]^2 = \frac{1+3+2\sqrt{3}}{8}$$

$$= \frac{2+\sqrt{3}}{4} \leftarrow P[+]$$

System of two qubits




Now there are four total possible states

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \quad \alpha_x \in \mathbb{C}$$

$$\sum |\alpha_x|^2 = 1$$

Measurement:

$P[00] = |\alpha_{00}|^2$

New state $\rightarrow |\psi'\rangle = |00\rangle$

... so on for the other states

Q: What is the result of measuring just the first qubit?

initial state $|\psi\rangle = \left(\frac{1}{2} + \frac{1}{2}i\right)|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}i|11\rangle$

recall: $|\psi\rangle = \boxed{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle} + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$P[0] = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$$\left[\; |\psi'\rangle = \frac{\left(\frac{1}{2} + \frac{i}{2}\right)|00\rangle + \frac{1}{2}|01\rangle}{\sqrt{\frac{3}{4}}} \;\right]$$

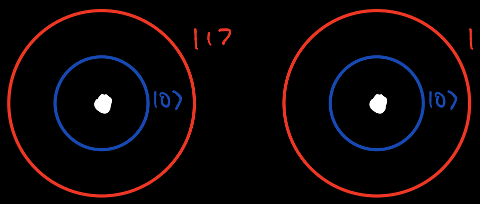
$P[0]$ = the sum of probabilities where the 1st qubit is "0"

$$= |\alpha_{00}|^2 + |\alpha_{01}|^2$$

New state? $|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$

normalized state for $P[0]$

# Entanglement




← composite system

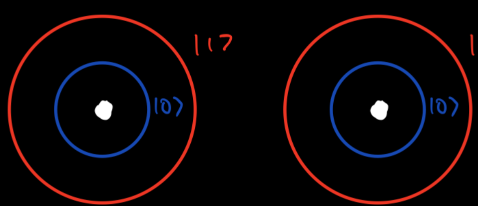
$$(\alpha_0|0\rangle+\beta_0|1\rangle)\times(\beta_0|0\rangle+\beta_1|1\rangle)$$

$$\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_2|11\rangle$$

for example: $\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle\right)$

$$=\frac{1}{2\sqrt{2}}|00\rangle+\frac{\sqrt{3}}{2\sqrt{2}}|01\rangle+\frac{1}{2\sqrt{2}}|10\rangle+\frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$$

# Bell State




$$|\varphi\rangle=\frac{1}{\sqrt{2}}|00\rangle+\frac{1}{\sqrt{2}}|11\rangle$$

$$=(\alpha_0|0\rangle+\alpha_1|1\rangle)(\beta_0|0\rangle+\beta_1|1\rangle$$

$$=\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle$$

recall: $\alpha_0\beta_0=\frac{1}{\sqrt{2}}=\alpha_1\beta_1$ , $\alpha_0=0$ OR $\beta_1=0$

this cannot be! Contradiction

Conclusion: we cannot factor this into two distinct state for each qubit

## Measuring the Bell State

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \leftarrow$$ only two possible states. the qubits are entangled.

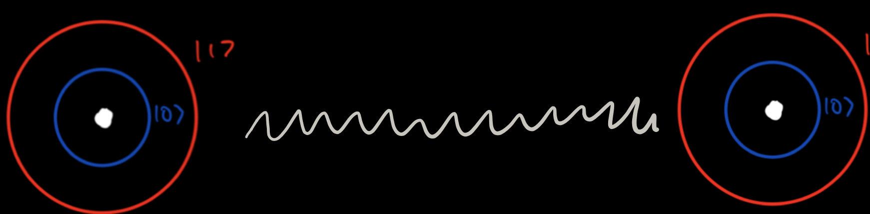
$$P[0] = \frac{1}{2} \rightarrow |\psi'\rangle = |00\rangle$$

$$P[1] = \frac{1}{2} \rightarrow |\psi'\rangle = |11\rangle$$

- What does entanglement mean? In brief, if the measurement of the first qubit is $|0\rangle$ the second qubit will also be $|0\rangle$ and the same for $|1\rangle$

## EPR Paradox

Rewriting Bell State in sign basis




$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$$

mathematic demonstration of the Bell state

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left[\,|00\rangle + |11\rangle\right] \Rightarrow \text{Bell state}$$

## Einstein, Podolsky, Rosen Paradox (1935)

the three proposed a paradox against traditional views of Quantum mechanics — specifically the uncertainty principle. The nature of two entangled qubits demonstrates in the following scenario:

- two entangled qubits are transported light years away from each other. (A & B). If we measure A to be in position $|0\rangle$ then B will 100% also be in $|0\rangle$, and vice versa with $|1\rangle$. Since information cannot be communicated between A & B, we must conclude that A & B pre-determined its states before they were separated. This contradicts the uncertainty principle.

Conclusion from the three is that the theory of quantum mechanics is incomplete.

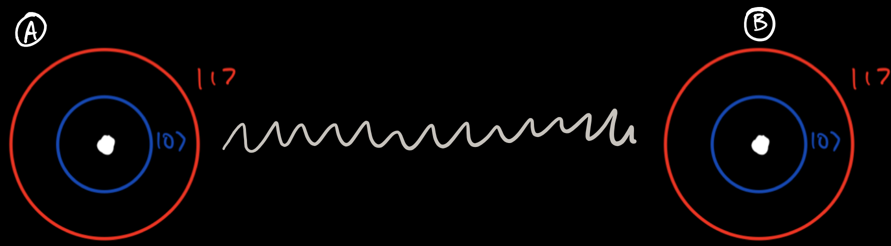
## Bell and EPR → review again

Bell 1965 → there is an experiment that distinguishes b/t the predictions of quantum mechanics and any local realism theory.

local realism $E \leq \frac{3}{4}$

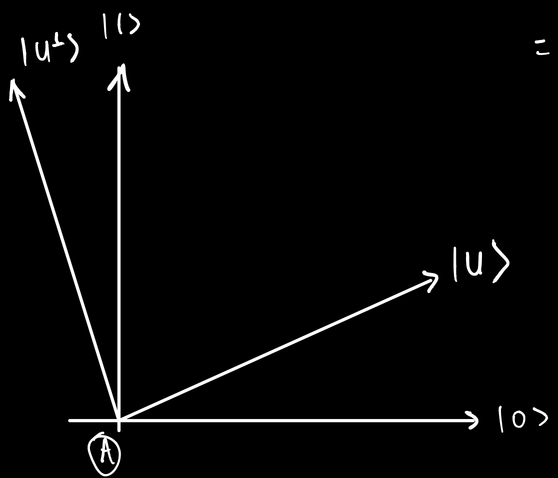
quantum mechanics $E = \cos^2\left[\frac{\pi}{8}\right] \approx .85$

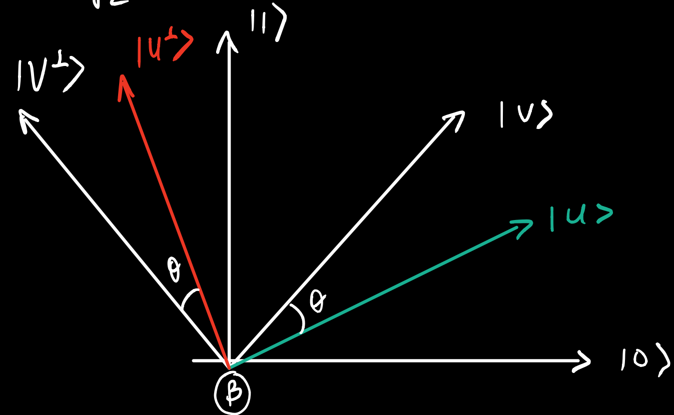
# Rotational Invariance of Bell State




$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$= \frac{1}{\sqrt{2}}|uu\rangle + \frac{1}{\sqrt{2}}|u^\perp u^\perp\rangle$$ generalized form




* $|u\rangle$ & $|v\rangle$ are in different basis vectors

$P[u] = \frac{1}{2}$ $\quad |\psi'\rangle = |uu\rangle$

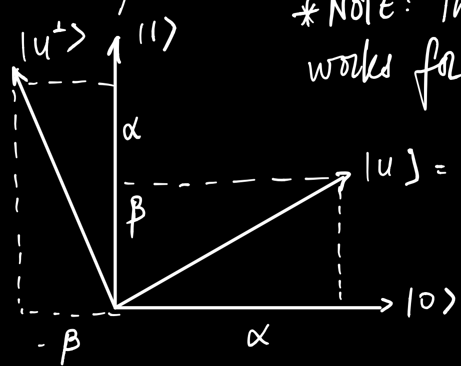
$P[u^\perp] = \frac{1}{2}$ $\quad |\psi'\rangle = |u^\perp u^\perp\rangle$

$P[v] = \cos^2\theta$

$P[v^\perp] = \cos^2\theta$

let's prove this:

*NOTE: This only works for real numbers




$|u\rangle = \alpha|0\rangle + \beta|1\rangle$

$|u^\perp\rangle = -\beta|0\rangle + \alpha|1\rangle$

recall: $\alpha^2+\beta^2=1$ b/c normalization

$$= \frac{1}{\sqrt{2}}\left[|uu\rangle + |u^\perp u^\perp\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[(\alpha|0\rangle+\beta|1\rangle)(\alpha|0\rangle+\beta|1\rangle) + (-\beta|0\rangle+\alpha|1\rangle) + (-\beta|0\rangle+\alpha|1\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}\left[(\alpha^2+\beta^2)|00\rangle + (\alpha^2+\beta^2)|11\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$ which is our original Bell State.

## CHSH Inequality

CHSH → Clauser, Horn, Shimony, Holt

| | Alice | Bob |
|---|---|---|
| Input | $x \in \{0,1\}$ | $y \in \{0,1\}$ |
| Output | $a$ | $b$ |

classical: successful with probability $\frac{3}{4}$

Logic: if $x=y=1$ output $a \neq b$
all other cases output $a=b$

Entanglement as a resource:
- cannot use entanglement to communicate information, but can use it to create non-local correlations!

| | Alice | Bob |
|---|---|---|
| Input | $x$ | $y$ |
| Output | $a$ | $b$ |

Alice & Bob share entangled qubits

Condition: $xy = a+b \mod 2$
$= a \oplus b$

Also, in order to beat the classical probability of $3/4$, Alice and Bob must be correct in every outcome.

# Quantum Systems

Axioms of Quantum Mechanics:

Axiom 1: Superposition principle
Axiom 2: Measurement
Axiom 3: Unitary evolution

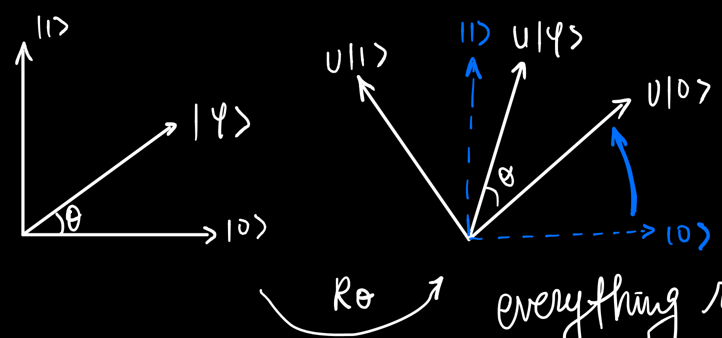
[Superposition]: allowable states of k-level system: unit vector in a k-dimensional complex space.

[Measurement]:

- specified by choosing an orthonormal basis.
- prob. of each outcome is the square of the length of the projection on to the corresponding basis vector.
- the state collapses to the observed basis vector.

[Evolution]:

- how does the state rotate over time?
  by rotation of the Hilbert space.




everything rotates by angle θ

$|0\rangle \xrightarrow{R_\theta} \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$

$|1\rangle \xrightarrow{R_\theta} -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$

we see $\longrightarrow$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^T$$

Also: $R_\theta \cdot R_{-\theta} = R_{-\theta} \cdot R_\theta = I$

Important relationship → unitary transformations

# Unitary Transformations

dagger, not "t"

$U$ is unitary iff $U \cdot U^\dagger = U^\dagger \cdot U = I$ ← identity matrix

$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \longrightarrow U^\dagger = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$

$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = I$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Q: What does a unitary matrix do?

$U|0\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = a|0\rangle + b|1\rangle$

$U|1\rangle = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} = c|0\rangle + d|1\rangle$

* the inner product of $U|0\rangle$ & $U|1\rangle$ is $\emptyset$. why? because $U|0\rangle$ & $U|1\rangle$ are orthogonal to each other.

Q: how do you get the length of a vector?

Multiply the vector by its conjugate transpose.

* $U$ preserves inner products: $\underbrace{|\varphi\rangle \text{ \& } |\psi\rangle}_{\text{inner product}}$

$\langle\varphi|\varphi\rangle$

$[\bar{a} \ \bar{b}] \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 1$

$\underbrace{U|\varphi\rangle \text{ \& } U|\psi\rangle}$

↓

get inner product

$U|\varphi\rangle$ needs to be a "bra", to do that $\langle\varphi|U^\dagger$

now form the inner product $\langle\varphi|U^\dagger U|\psi\rangle$. recall that $U^\dagger U = I$

thus $\langle\varphi|\psi\rangle$, which equals the original inner product of $\langle\varphi|\psi\rangle$

# Single Qubit Gates

## [ "Bit Flip Gate" ]

$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{X} \longrightarrow \alpha_1|0\rangle + \alpha_0|1\rangle$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

## [ "Phase Flip Gate" ]

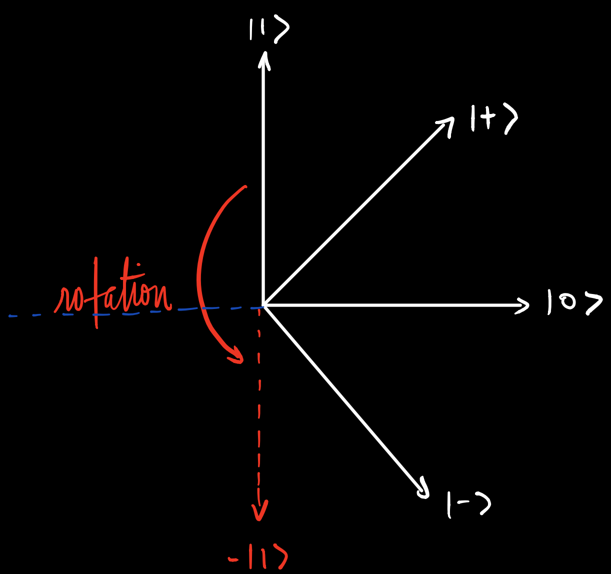
$$\alpha_0|0\rangle + \alpha_1|1\rangle \longrightarrow \boxed{Z} \longrightarrow \alpha_0|0\rangle - \alpha_1|1\rangle$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so... what does this do? observe the effect on the $|+\rangle$ state

$$Z|+\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



" Hadamard transform"

$\longrightarrow \boxed{H} \longrightarrow$

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|0\rangle \to H|0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \to |+\rangle$$

$$|1\rangle \to H|1\rangle \longrightarrow |-\rangle$$

* $HH|0\rangle = |0\rangle$ no loss of information

Cool chart that shows relationship between different gates

$$\begin{tikzcd} |0\rangle \arrow[r, "X"] \arrow[d, dash, "H"'] & |1\rangle \arrow[d, dash, "H"] \\ |+\rangle \arrow[r, dash, "Z"'] & |-\rangle \end{tikzcd}$$

$X = HZH$

$Z = HXH$

Tensor Products $\rightsquigarrow$ unentangled state

tensor product

particle A $(|u\rangle)$ | particle B $(|v\rangle)$

$\alpha_0|0\rangle + \alpha_1|1\rangle$ | $\beta_0|0\rangle + \beta_1|1\rangle$

$\in \mathbb{C}^2$ $\quad \otimes \quad$ $\in \mathbb{C}^2$

composite $A+B = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \in \mathbb{C}^4$

other properties: $(|u_1\rangle + |u_2\rangle)(|v\rangle) = |u_1\rangle \otimes |v\rangle + |u_2\rangle \otimes |v\rangle$

Inner products of tensors: for example

$|u_1\rangle \otimes |v_1\rangle \qquad |u_2\rangle \otimes |v_2\rangle$

inner product

$$*\left[\langle u_1 | u_2\rangle * \langle v_1 | v_2\rangle\right]*$$

$$u_1 = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$u_2 = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$$

$$u = u_1 \otimes u_2$$

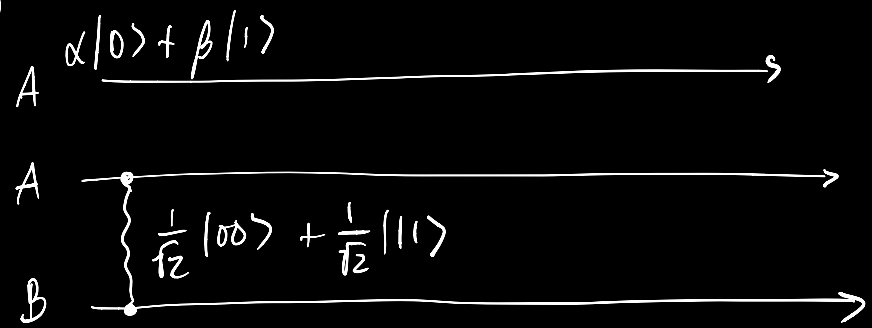
$$\begin{bmatrix} a \cdot u_2 & c \cdot u_2 \\ b \cdot u_2 & d \cdot u_2 \end{bmatrix}$$

easy way to understand.

How can Alice teleport her qubit to Bob?

Approach:

Alice can have her own qubit, then apply CNOT on her side of an entangled qubit with Bob. like this:




$$(\alpha|0\rangle + \beta|1\rangle) \otimes \left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \longleftarrow$$ composite state of Alice's qubit and entangled qubit.

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

now apply CNOT

changed b/c CNOT

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}\boxed{|101\rangle} + \frac{\beta}{\sqrt{2}}\boxed{|110\rangle}$$

Review again

A B

A B

if Alice's entangled qubit is $\emptyset \to \alpha|00\rangle + \beta|11\rangle$

$1 \to \alpha|01\rangle + \beta|10\rangle$

Notes about Exponential Growth & Qubits

one qubit $\to \in \mathbb{C}^2$ two possible outcomes

two qubits $\to \in \mathbb{C}^4$ four possible outcomes

3 qubits $\to \in \mathbb{C}^8$ 8 possible outcomes

Exponential

Exponential growth comes from tensor products

hmm... brain juices are flowing...


composite final state

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle \otimes |0\rangle$$

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|100\rangle$$

$\downarrow$ 1st $\emptyset$ $\qquad$ $\downarrow$ 1st 1

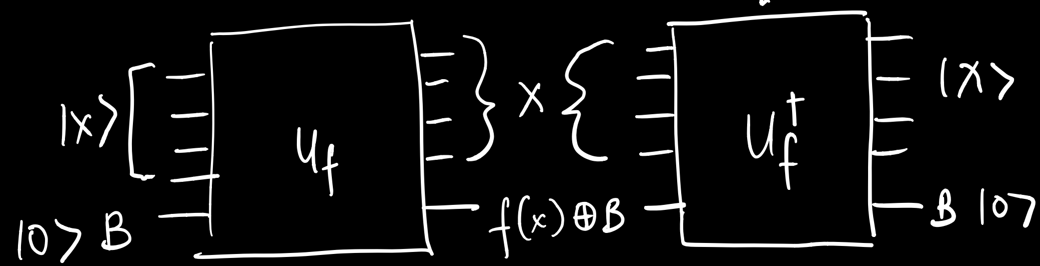
$|\varphi'\rangle = |000\rangle$ $\qquad$ $|\varphi'\rangle = |100\rangle$

# Universal quantum gate set

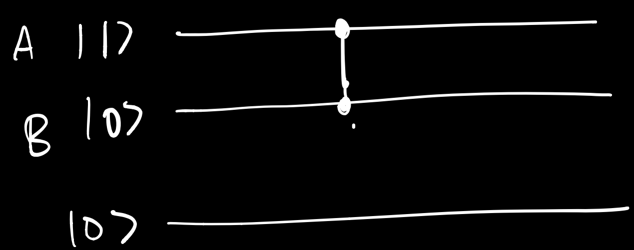
- CNOT, Z, H, X, $\frac{\pi}{8}$ rotation can make ANY quantum circuits

# Reversible Computation - why is this necessary? Junk prevents interference.

inverse of $U_f$




CSWAP = AND gate → TRUE!




0 & 0 = 0
1 & 0 = 0
1 & 1 = 1
0 & 1 = 0

A = 0 → 0
A = 1 → B

# on Quantum Algorithms

$f(x) = f(x \oplus s)$  $s = 101$

100 = 011
+ 101
00

## General Computational Process

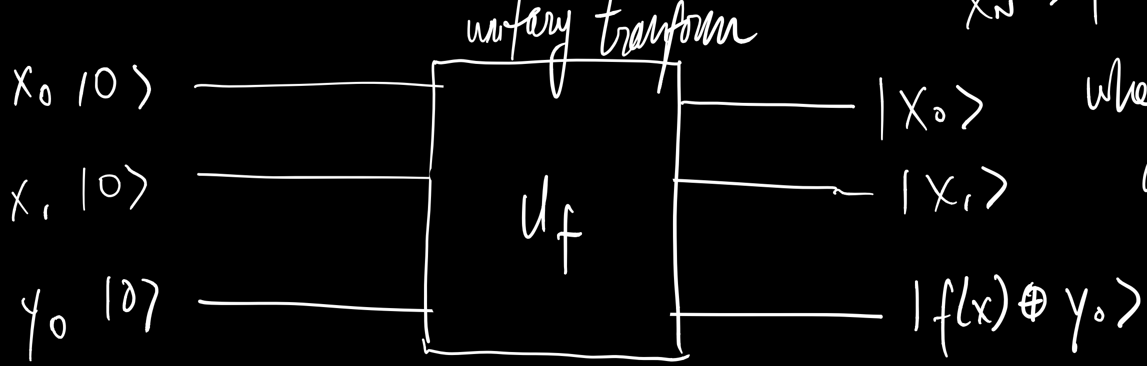
The effect of $U_f$ (unitary operation) on $x_N$ input qubits and $y_M$ output qubits.

$$U_f\left(|x\rangle_N |y\rangle_M\right) = |x\rangle_N \, |\underbrace{y \oplus f(x)}\rangle_M$$

for example

$x_N \rightarrow |00\rangle$, $y_M \rightarrow |0\rangle$

where $f(x)$ is the answer

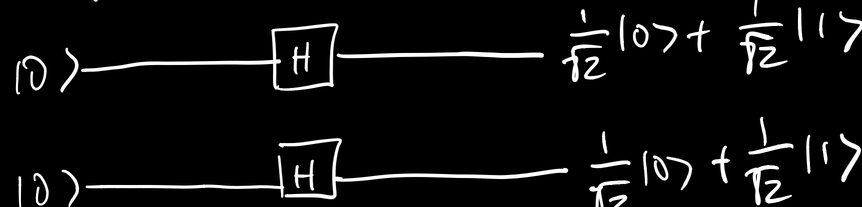



Composite form of this quantum state:

$$\underbrace{|x_0\rangle \otimes |x_1\rangle} \otimes |f(x) \oplus y_0\rangle$$

$$|x\rangle_N \otimes |f(x) \oplus y\rangle_M$$

## more of Hadamard – personal musings & practice




composite form of qubit state

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \rightarrow \text{measure}$$

$$\left\| \frac{1}{2}^2 + \frac{1}{2}^2 + \frac{1}{2}^2 + \frac{1}{2}^2 \right\|$$

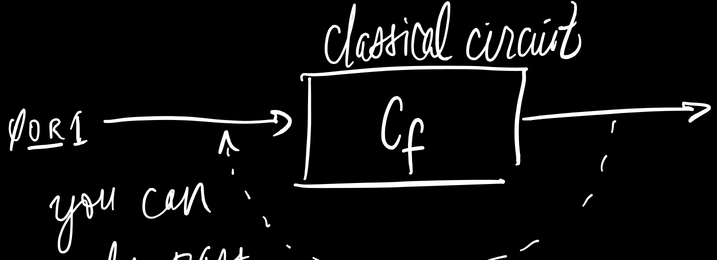
$$\left\| \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right\| = 1$$

$$U(|\psi\rangle|0\rangle)$$

$$= U|\psi\rangle \cdot U|0\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

## Thinking about Deutsch Algorithm

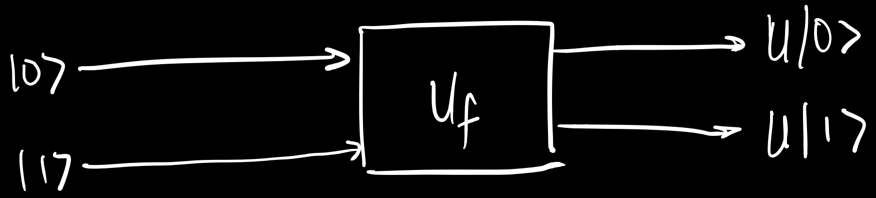
In a classical circuit




you can only pass one bit at a time

need to run twice to figure out the circuit design of $C_f$

In quantum circuits




get both states so you can determine

The input qubits

$$(H\otimes H)(X\otimes X)(|0\rangle|0\rangle)$$

$$(H\otimes H)(X|0\rangle\cdot X|0\rangle) = (H\otimes H)(|1\rangle|1\rangle)$$

$$= (H|1\rangle\otimes H|1\rangle)$$

$$= \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\otimes\left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle + \tfrac{1}{\sqrt{2}}|11\rangle$$

$$= \tfrac{1}{\sqrt{2}}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \leftarrow \text{input state}$$

recall: $U_f(|x\rangle|y\rangle) = |x\rangle|y\oplus f(x)\rangle$

$$= \tfrac{1}{\sqrt{2}}(U_f|00\rangle - U_f|01\rangle - U_f|10\rangle + U_f|11\rangle)$$

$$= \tfrac{1}{\sqrt{2}}(|0\rangle|f(0)\rangle - |0\rangle|1\oplus f(0)\rangle - |1\rangle|0\oplus f(1)\rangle + |1\rangle|1\oplus f(1)\rangle)$$

Now if $f(0)=f(1)$

$$= \tfrac{1}{2}(|0\rangle(|f(0)\rangle - |1\oplus f(0)\rangle) - |1\rangle(|0\oplus f(1)\rangle + |1\oplus f(1)\rangle))$$

$$= \tfrac{1}{2}((|0\rangle - |1\rangle)\cdot(\overbrace{|f(0)\rangle - |\underbrace{1\oplus f(0)}_{\hat{f}(0)}\rangle + |f(1)\rangle}^{\text{equal if } f(0)=f(1)} + |\overbrace{1\oplus f(1)}^{\tilde{f}(1)}\rangle))$$

# QUANTUM FOURIER TRANSFORM

$\omega$ ← omega

$$\omega = \cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8} \longrightarrow e^{\frac{2\pi}{8}i}$$

**Recall:** Simon's Algorithm

$f(x) = f(x \oplus s)$ ← solution = find "s"

**Period Finding:** $f(x) = f(x + r)$ ← this is the period

similar to Simon's algorithm, except... instead of using hadamard transform you use fourier transform

**Reference:**

"Algorithms": Dasgupta, Papa..., Vizirani

www.cs.berkeley.edu/~vazirani/algorithms.html

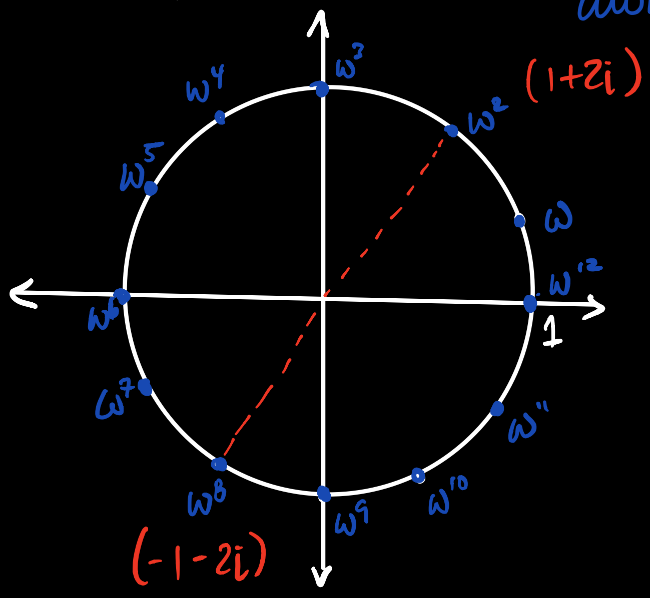
Ch.1: Modular arithmetic - pg. 25
Ch.2: (2nd half) Fast Fourier transform - pg. 62
Ch.10: Quantum factoring. pg. 299

## NTH ROOT OF UNITY

↳ what does this mean

find solutions to $x^n = 1$, answer: n total for example, let's say $n = 12$

divided into n equal pieces




$1 + w + w^2 \cdots w^{n-1} = \emptyset$

w's cancel each other

$\bar{w} = \cos\frac{2\pi}{n} - i\sin\frac{2\pi}{n} \rightarrow w^{-1}$

<u>understanding</u> $e^{ix} \rightarrow -1$

recall: $\boxed{e^{ix} = \cos(x) + i\sin(x)}$ ← this relationship

if $e^{i\pi} = \cos(\pi) + i\sin(\pi) \Rightarrow -1 + 0 \Rightarrow -1$

$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) \Rightarrow 0 + i(1) \Rightarrow i$

$$QFT_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^{N-1} \\ 1 & w^2 & w^4 & w^6 & \cdots \\ 1 & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}$$

where $N = 2^n$

$\alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$

$QFT_N$

↓ the output is based on the

$\beta_0|0\rangle + \beta_1|1\rangle + \beta_2|2\rangle + \beta_3|3\rangle$ amplitude of $|\alpha_i|^2$

the quantum fourier transform

position in the fourier transform matrix!

$$|x\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \underbrace{e^{\frac{2\pi i jk}{N}}} |p\rangle$$

basis vector

this entire expression evaluates to some $e^{ix}$