# Aaronson Homework 1

[Stochastic and Unitary Matrices]

(A) - per lecture 2: stochastic matrix must fulfill the 2 properties:

1. all entries must be non-negative.
2. all columns must sum to 1.

A. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$ all positive, C2: sum = $\cancel{0}$ $\rightarrow$ not stochastic

B. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow$ all positive, all sum = 1 $\rightarrow$ stochastic

C. $\begin{pmatrix} 1 & \frac{2}{3} \\ 0 & \frac{2}{3} \end{pmatrix} \rightarrow$ all positive, $\frac{2}{3} + \frac{2}{3} = \frac{4}{3} \neq 1$, C2: sum $\neq 1$ $\rightarrow$ not stochastic

D. $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \rightarrow$ all positive, $i = \sqrt{-1} \neq 1$, C2: sum $\neq 1$ $\rightarrow$ not stochastic

E. $\begin{pmatrix} 2 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{pmatrix} \rightarrow$ not positive, all col. sum = 1 $\rightarrow$ not stochastic.

F. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow$ not positive, all columns sum $\neq 1$ $\rightarrow$ unitary

Hadamard gate

G. $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix} \longrightarrow$ not all positive
all col. sum $\neq 1$ $\rightarrow$ unitary

$\frac{3}{5}^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25}$

H. $\begin{pmatrix} \frac{3i}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3i}{5} \end{pmatrix} \rightarrow$ not all positive
all col. do not sum $\neq 1$ $\rightarrow$ not stochastic

$\left.\begin{array}{l} \left(\frac{3\sqrt{-1}}{5}\right)^2 = \frac{9 \cdot -1}{25} \\ \frac{4}{5}^2 = \frac{16}{25} \end{array}\right\}$ $-\frac{9}{25} + \frac{16}{25} = \frac{7}{25} \neq 1$
not unitary

(B) when is a stochastic matrix also a unitary matrix?

suppose square matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a+c = 1$
$b+d = 1$

pauli x matrix $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ = stochastic matrix

properties of a permutation matrix:

- square matrix
- consists of only 1's and Ø's
- each row & each column must consist of a single 1.

* this would imply that the intersection b/t stochastic

and unitary matrices are the ones that <u>only</u> contain 1's and 0's.

$A^T A = 1 \leftarrow$ unitary

(c) Example of a 2-norm matrix

Hadamard $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

preserves 4-norm of real vectors $\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow a^4 + b^4$

for example if you had input vector $|01\rangle$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|-\frac{1}{\sqrt{2}}\right|^2 =$$

$0^2 + 1^2 = \underline{1}$

$\frac{1}{2} + \frac{1}{2} = \underline{1}$

create a 4-norm preserving matrix

$$\begin{bmatrix} \frac{1}{\sqrt[4]{2}} & \frac{1}{\sqrt[4]{2}} \\ \frac{1}{\sqrt[4]{2}} & -\frac{1}{\sqrt[4]{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt[4]{2}} \\ \frac{1}{\sqrt[4]{2}} \end{bmatrix} = \left(\frac{1}{\sqrt[4]{2}}\right)^4 + \left(\frac{1}{\sqrt[4]{2}}\right)^4 =$$

$1^2 + 0^2 = 1$

$\frac{1}{2} + \frac{1}{2} = 1$

there's your matrix

[Tensor Products]

(A) $$\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cdot \frac{1}{5} \\ \frac{2}{3} \cdot \frac{4}{5} \\ \frac{1}{3} \cdot \frac{1}{5} \\ \frac{1}{3} \cdot \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{15} \\ \frac{8}{15} \\ \frac{1}{15} \\ \frac{4}{15} \end{bmatrix}$$

(B) which ones are factorable as a $2 \times 2$ tensor products

RECALL:

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} P_{00} \\ P_{01} \\ P_{10} \\ P_{11} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

$$A = \begin{array}{cc} \begin{bmatrix} \frac{2}{9} \\ \frac{1}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{bmatrix} & \begin{array}{c} ac \\ ad \\ bc \\ bd \end{array} \end{array}$$

$ac = bd$

$$\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \otimes \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$B = \begin{array}{cc} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{c} ac \\ ad \\ bc \\ bd \end{array} \end{array}$$

$ac = bc = bd$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

C. $\begin{array}{c|l} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} & \begin{array}{l} ac \\ ad \\ bc \\ bd \end{array} \end{array}$

$ac = ad = bc = bd$

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

D. $\begin{array}{c|l} \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} & \begin{array}{l} ac \\ ad \\ bc \\ bd \end{array} \end{array}$

$ac = bd \qquad ad = bc$

$$\begin{bmatrix} \frac{1}{4} \\ ? \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \text{X}$$

not factorable

E. $\begin{array}{c|l} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} & \begin{array}{l} ac \\ ad \\ bc \\ bd \end{array} \end{array}$

$ac = bc \qquad ad = bd$

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$$

(3) Prove no real 2x2 matrix such that

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & \underset{22}{\boxed{-1}} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ * & c \cdot b + \underline{d \cdot d} \end{bmatrix}$$

$d \cdot d = d^2$

$d^2$ cannot be a non-positive #

complex numbers allow us to multiply and get negative numbers

[Dirac Notation]

(A) $|\psi\rangle = \dfrac{|0\rangle + 2|1\rangle}{\sqrt{5}}$ , $|\varphi\rangle = \dfrac{2i|0\rangle + 3|1\rangle}{\sqrt{13}}$

$\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad$ $\frac{1}{\sqrt{13}}\begin{bmatrix} 2i \\ 3 \end{bmatrix}$

$\langle\psi|\varphi\rangle$ inner product

$$\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \otimes \begin{bmatrix} \frac{2i}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \cdot \frac{2i}{\sqrt{13}} \\ \frac{2}{\sqrt{5}} \cdot \frac{2i}{\sqrt{13}} \\ \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{5}} \cdot \frac{3}{\sqrt{13}} \end{bmatrix} = \begin{bmatrix} \frac{2i}{\sqrt{65}} \\ \frac{4i}{\sqrt{65}} \\ \frac{6}{\sqrt{65}} \\ \frac{6}{\sqrt{65}} \end{bmatrix}$$

(B) Quantum states are usually normalized, such that

$\langle\psi|\varphi\rangle = 1$

$|\varphi^{\dagger}\rangle$

for example if $|\varphi\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |\varphi^{\dagger}\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + 0 + 0 + 0 = 1$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

$|\varphi\rangle = 2i|0\rangle - 3i|1\rangle \longleftarrow$ not normalized

What constant, A, makes $|\varphi\rangle$ normalized?

$$|\varphi\rangle = \begin{bmatrix} 2i \\ -3i \end{bmatrix} \qquad |2i|^2 + |-3i|^2 = 4 \cdot -1 + 9 \cdot -1 = -13$$

$$\left[\frac{2}{x}i\right]^2 + \left[\frac{3}{x}i\right]^2 = 1$$

$i^2 = -1$

$$\frac{4}{x^2}i^2 + \frac{9}{x^2}i^2 = 1 \qquad \frac{-13}{x^2} = 1 \qquad x^2 = -13$$

$$-\frac{4}{x^2} - \frac{9}{x^2} = 1 \qquad x = \sqrt{-13} \rightarrow x = \sqrt{13}\,i$$

$A \overset{?}{=} \sqrt{13}\,i$

$$\hookrightarrow \left\| \frac{2i}{\sqrt{13}\,i} \right\|^2 + \left\| \frac{-3i}{\sqrt{13}\,i} \right\|^2 \overset{?}{=} 1$$

$$= \left\| \frac{2}{\sqrt{13}} \right\|^2 + \left\| \frac{-3}{\sqrt{13}} \right\| \overset{?}{=} 1$$

$$= \frac{4}{13} + \frac{9}{13} = \frac{13}{13} = 1 \checkmark$$

thus $A = \sqrt{13}\,i$, or $\left[ |\varphi\rangle = \frac{|\varphi\rangle}{\sqrt{13}\,i} \right]$

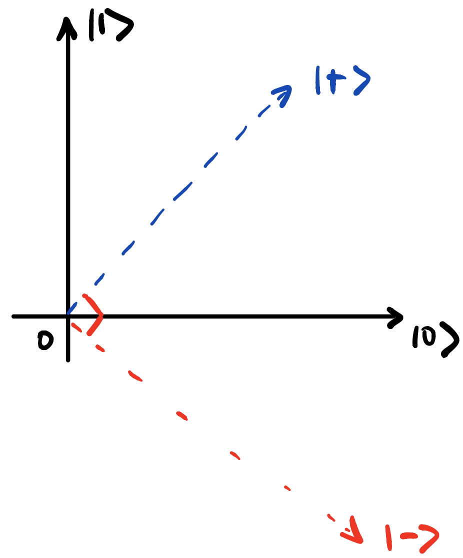
(c) $|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ , $|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$

show $|i\rangle$ and $|-i\rangle$ form an orthonormal basis for $\mathbb{C}^2$

Answer: we can lean on the standard form of a $|+\rangle$ and $|-\rangle$ vector to guide our reasoning

the $|+\rangle$ vector $= \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

the $|-\rangle$ vector $= \frac{|0\rangle - |1\rangle}{\sqrt{2}}$




the $|+\rangle$ & $|-\rangle$ basis vectors are orthonormal

the difference between the $|+\rangle/|-\rangle$ basis and the $|i\rangle/|-i\rangle$ basis is additional "i" coefficient on the $|1\rangle$ vector. Thus the $|i\rangle/|-i\rangle$ basis is also orthonormal.

(D) normalized $|\psi\rangle$ vector $= \frac{2i|0\rangle - 3i|1\rangle}{\sqrt{13}i} \Rightarrow \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$

$\longrightarrow$ transform in $\{|i\rangle, |-i\rangle\}$

use $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \times \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} i \end{bmatrix}$ $|i\rangle$ basis

$\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \times \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{2}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} i \end{bmatrix}$ $|-i\rangle$ basis

* in order to transform a vector into another basis, use an appropriate permutation matrix and multiply w/ vector.