A.2

(1) Solve the equation $y''(x) = x$; $x \in (0, \pi)$ using Fourier Series s.t $y(0) = y(\pi) = 0$

Proof:- Consider the problem $-y'' = \lambda y$; $y(0) = y(\pi) = 0$. — (1)

Clearly for $\lambda \leq 0$, it is easy to see that $y(x) \equiv 0$ is the only solution.

Hence $\lambda = \mu^2 > 0 \Rightarrow y(x) = A\cos(x\mu) + B\sin(x\mu)$

$\because y(0) = y(\pi) = 0 \Rightarrow y_k(x) = \sin kx$ and $\lambda_n = k^2$; $k \in \mathbb{N}$.

Hence the eigenpair is $(k^2, \sin kx)$.

Now, since $f(x) = x$ is an odd function hence we can expand it in terms of Fourier sine series and is given by

$$f(x) = \sum_{k \geq 1} (-1)^{k+1} \frac{2}{k} \sin(kx)$$ [Equality holds due to Fourier convergence theorem for smooth map on $(0, \pi)$]

Using (1) we have,

$$k^2 y = \sum_{k \geq 1} (-1)^k \frac{2}{k} \sin(kx)$$

$$\Rightarrow y = \sum_{k \geq 1} 2 \cdot \frac{(-1)^k}{k^3} \sin(kx). \quad \blacksquare$$

(2) Show that the Chebyshev equation given by

$$(1-x^2) y'' - x y' + n^2 y = 0 ; \quad x \in [-1,1]$$

is singular SL-BVP.

Proof: Dividing with $\sqrt{1-x^2}$ we have,

$$\sqrt{1-x^2}\, y'' - \frac{x}{\sqrt{1-y^2}} y' + \frac{n^2}{\sqrt{1-x^2}} y = 0.$$

$$\Rightarrow -\left[\frac{y'}{\sqrt{1-y^2}} \right]' = \frac{n^2 y}{\sqrt{1-x^2}}.$$

Comparing with $(py')' + qy = \lambda r y$ we have,

$p(x) = 1/\sqrt{1-x^2}$; $q(x) = 0$ and $r(x) = 1/\sqrt{1-x^2}$; $\lambda = n^2$ on $[-1,1]$.

p and r blows up at end points and hence is a singular SLBVP.

(3) Let λ_1 and $\lambda_2 (> \lambda_1)$ be two eigenvalue of a regular SLBVP with eigenfunction y_1 & y_2. Show y_2 admits a zero in between two consecutive zero of y_1.

Proof :- $-(py_1')' + qy_1 = \lambda_1 r y_1$ — (I) $\to \times y_2$

$-(qy_2')' + qy_2 = \lambda_2 r y_2$ — (II) $\to \times y_1$

$$\left[p(y_1 y_2' - y_2 y_1')\right]' = (\lambda_1 - \lambda_2) y_1 y_2 r.$$

$$\Rightarrow \left[p(y_1 y_2' - y_2 y_1')\right]_{x_1}^{x_2} = (\lambda_1 - \lambda_2) \int_{x_1}^{x_2} y_1 y_2 r \quad \text{— (III)}$$

WLOG:- let $y_1(a) > 0$ and x_0 be such that $y_1(x_0) = 0$ and x_0 is first such root from a.

Then, $y_1'(x_0) < 0$.

If $x_1 = a$ and $x_2 = x_0$ in (iii) we have,

$$-p(x_0)\, y_1'(x_0)\, y_2(x_0) = (\lambda_1 - \lambda_2) \int_a^{x_0} y_1 y_2 r \, dx. \quad \text{--- (4)}$$

If one assumes, $y_2(x) > 0 \;\forall\, x \in [a, x_0]$, $\exists$ no zero of y_2 between 'a' and 'x_0' then

LHS of (4) is positive ($\because p > 0$)

RHS of (4) is negative ($\because r > 0$ & $\lambda_2 > \lambda_1$)

- a contradiction

Again if $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) < 0$ and $y_1'(x_2) > 0$, then from (4);

$$-p(x_2)\, y_1'(x_2)\, y_2(x_2) + p(x_1)\, y_1'(x_1)\, y_2(x_1) = (\lambda_1 - \lambda_2)\int_{x_1}^{x_2} y_1 y_2 r \, dx \quad \text{(*)}$$

If $y_2(x) > 0 \;\forall x \in [x_1, x_2]$; $\nexists$ any zero of y_2 between x_1 & x_2.

then RHS of (*) is positive and LHS of (*) is negative.

— a contradiction.

(4) Reduce the problem $y'' + xy' + \lambda y = 0$; $y(0) = y(1) = 0$. (1)

Integrating Factor: $\mu(x) = e^{\int x\, dx} = e^{x^2/2}$.

$\left[\mu(x) = \frac{1}{P} e^{\int \frac{Q}{P} dx}\right.$ is the integrating factor of the eqn $\left. Py'' + Qy' + \lambda R y = 0\right]$

$$\therefore e^{x^2/2} y'' + e^{x^2/2} x y' + \lambda e^{x^2/2} y = 0.$$

$$\Rightarrow -\left(e^{x^2/2} y'\right)' = \lambda e^{x^2/2} y$$

$$\therefore p(x) = e^{x^2/2} \quad \& \quad r(x) = e^{x^2/2}.$$

(5) Comment on the eigenpairs of $y'' + \lambda y = 0$; $\lambda > 0$ s.t $y'(0) = h_1 y(0)$ and $y'(\ell) = -h_2 y(\ell)$. ; $h_1, h_2 > 0$

Soln: Solution: $y(x) = A\cos\mu x + B\sin\mu x \quad (\lambda = \mu^2)$

Now, $y'(0) = h_1 y(0) \Rightarrow B\mu = h_1 y(0) = h_1 A \Rightarrow A = \dfrac{B\mu}{h_1}$.

and, $y'(\ell) = -A\mu\sin(\mu\ell) + B\mu\cos(\mu\ell) = -h_2 y(\ell) = -h_2(A\cos\mu\ell + B\sin\mu\ell)$

$$\Rightarrow B\left[-\frac{\mu^2}{h_1}\sin(\mu\ell) + \mu\cos(\mu\ell)\right] = -Bh_2\left[\frac{\mu}{h_1}\cos\mu\ell + \sin\mu\ell\right]$$

$$\Rightarrow B\left\{\left(-\frac{\mu^2}{h_1}+h_2\right)\sin\mu l+\left(\mu+\frac{h_2}{h_1}\mu\right)\cos\mu l\right\}=0$$

$$\Rightarrow \tan(\mu l)=\frac{\mu(h_1+h_2)}{\mu^2-h_1h_2}.$$

Now, since $h_1, h_2 \neq 0$ then

$$y_n=\frac{\mu_n}{h_1}\cos\mu_n x+\sin\mu_n x \quad \text{and} \quad \mu_n \sim \frac{\pi n}{l} \quad \text{as } n\to\infty.$$