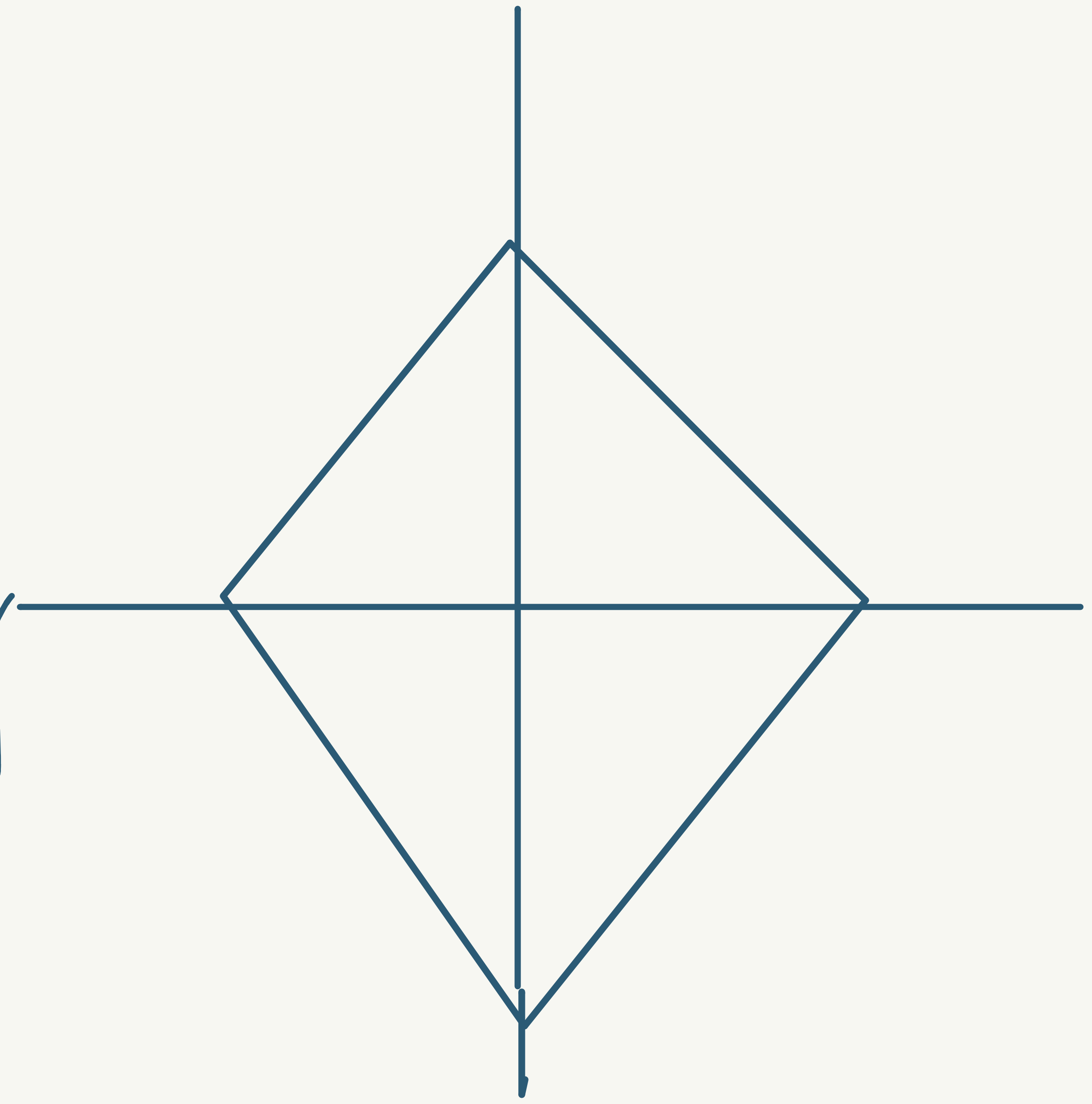
1 (a) $\Omega = \{(x,y) : |x| + |y| = 1\}$.



$\partial\Omega = \phi$

$\therefore$ Ω has an empty boundary and hence by default it is continuously differentiable.

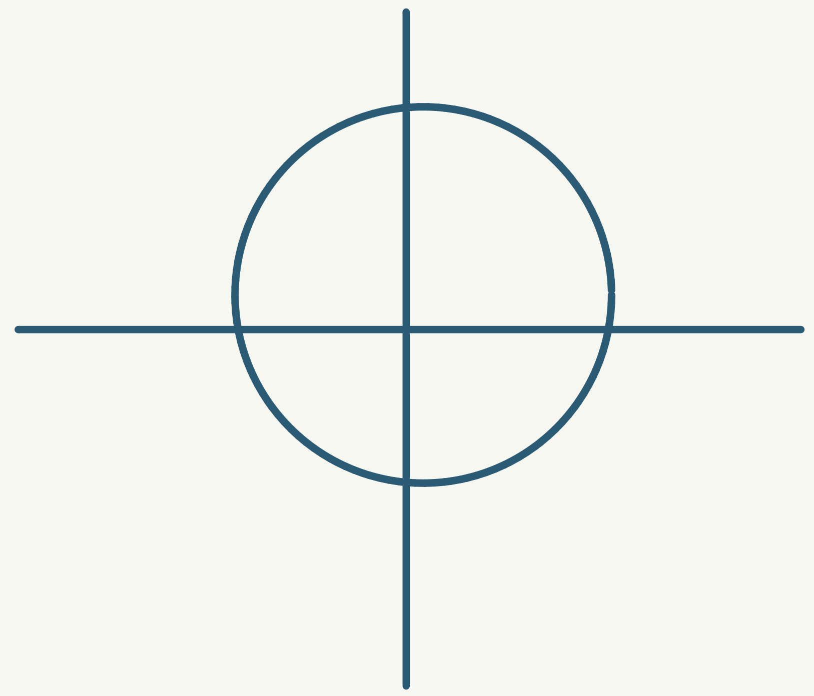
Note:- If $\Omega = \{(x,y) : |x| + |y| < 1\}$ then $\partial\Omega = \{|x| + |y| = 1\}$

Then the boundary is not C^1.

- x -

(4) Which domain has C^1-boundary.

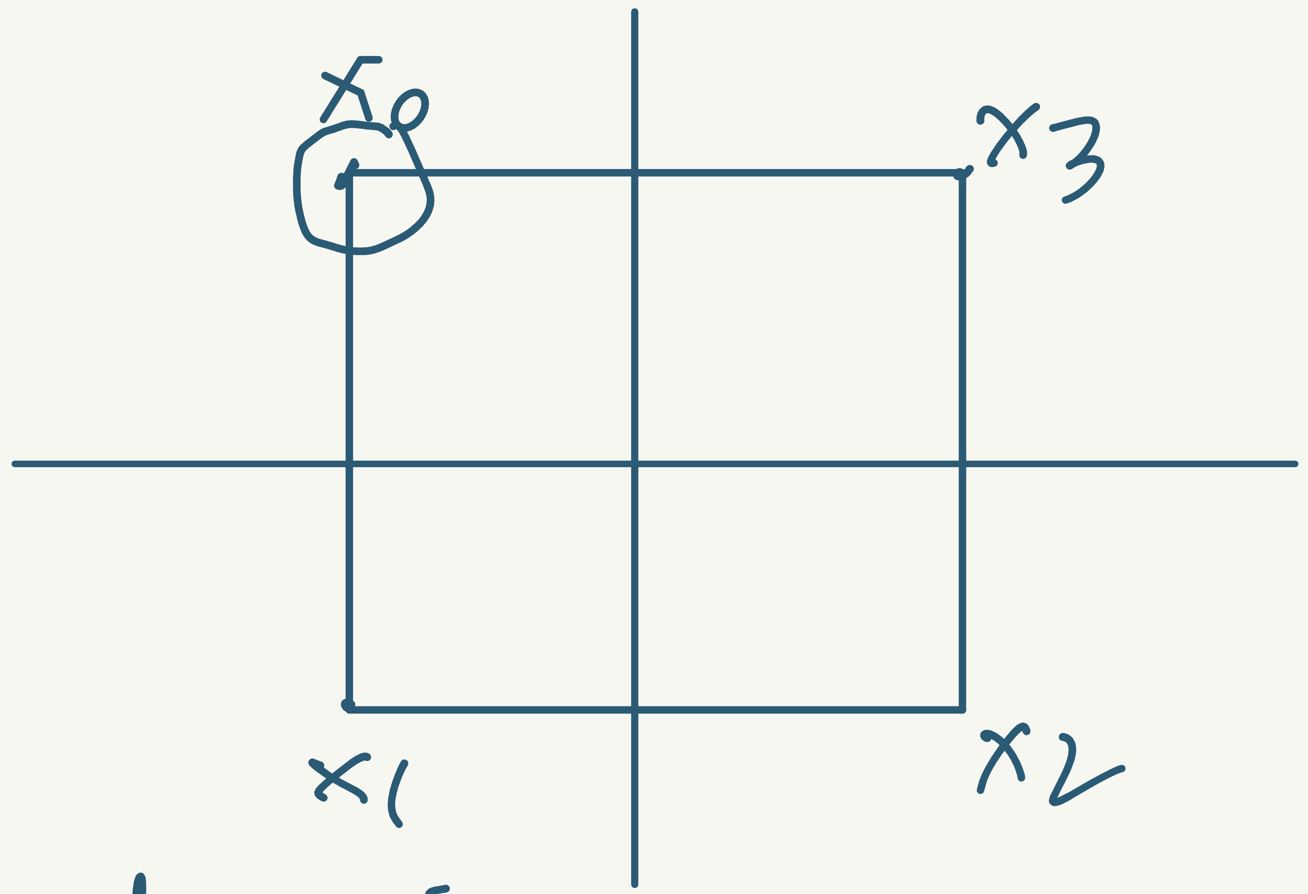
(a) $\Omega = \{x^2+y^2<1\}$



$\partial\Omega = \{x^2+y^2=1\}$.

$\partial\Omega$ locally looks like the graph of a C^1-function given by $y = \pm\sqrt{1-x^2}$. Hence, $\partial\Omega$ is C^1.

(b) $\Omega = \{\max(|x|,|y|)<1\}$



At any pt x_0, x_1, x_2, x_3 the boundary $\{\max(|x|,|y|)=1\}$ locally looks like the graph of $y=|x|$ which is not continuously differentiable. Hence, $\partial\Omega$ is not C^1.

② Classify the problem :-

(a) $u_x + u_y = 1.$

- Linear.

(b) $u_x + x u_y = u^2.$

- Semilinear.

(c) $u_x + u u_y = 0$

- Quasilinear

(d) $\operatorname{div}(|\nabla w|^{p-2} \nabla w) = 0.$

$\Rightarrow |\nabla w|^{p-4} \left\{ |\nabla w|^2 \Delta w + (p-2) \sum u_{x_i} u_{x_j} u_{x_i x_j} \right\} = 0.$

- Quasilinear.

(e) $\det(D^2 u) = 1 \Rightarrow \det \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix} = 1 \Rightarrow u_{xx} u_{yy} - u_{xy}^2 = 1$

- Fully Nonlinear.

(3) (a) $u_{xx} = u$ — (i)

$\Rightarrow \varphi''(x) = \varphi(x)$ (One can consider φ to be independent of y as (i) does not contain any derivative of y)

$\Rightarrow \varphi(x) = A e^{x} + B e^{-x}$ (Defining $u(x,y) := \varphi_y(x)$)

$\therefore u(x,y) = A(y) e^{x} + B(y) e^{-x}$. ☒

(b) $u_{xy} + u_x = 0$ — (ii)

Let, $u_x = \phi$

Then from (ii), $\phi_y(x,y) = -\phi(x,y)$ — (iii)

[$\because$ (iii) is independent of x-variable we can reduce it to an ODE]

$\Rightarrow \phi(x,y) = f(x) e^{-y} + g(y)$.

$f, g \in C^1$ are arbitrary. ☒

Substitute 'v' with 'v_{x_i}';

$$\int_\Omega u_{x_i} v_{x_i} \, dx = -\int_\Omega u v_{x_i x_i} \, dx + \int_{\partial\Omega} u v_{x_i} \nu_i \, ds \quad \text{— (ii)}$$

Summing from (i) to (n) we get,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega u \Delta v \, dx + \int_{\partial\Omega} u \frac{\partial v}{\partial \eta} \, ds. \quad \text{— (iii)} \quad (\eta = (\nu_1, .., \nu_n))$$

Interchanging u & v in (iii),

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega v \Delta u \, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \eta} \, ds \quad \text{— (iv)}$$

From (iii) & (iv);

$$\int_\Omega (u \Delta v - v \Delta u) \, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta} \right) ds.$$

$\boxtimes$

(5) Verify that $u(x,t) = x^2 + t^2$ solves wave eqn $\quad t - u_{xx} = 0$.

Clearly $u(x,t) = x^2 + t^2$ is twice continuously differentiable being the sum of two polynomial function.

Now, $u_{tt} = 2$ & $u_{xx} = 2$.

Hence, $u(x,t) = x^2 + t^2$ solves $u_{tt} - u_{xx} = 0$.

(4) $$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.$$

From Green-Gauss Theorem

$$\int_\Omega u_{x_i}\,dx = \int_{\partial\Omega} u\gamma_i\,ds \qquad (\text{for } u \in C^2 \Rightarrow uv \in C^2).$$

Replacing u by uv one has,

$$\int_\Omega u_{x_i} v = -\int_\Omega u v_{x_i}\,dx + \int_{\partial\Omega} uv\gamma_i\,ds \rightarrow (*)$$

(Integration by Parts).

⑥ Solve :- $yz_x - xz_y = 0$. — ①

Let's convert the eqn into polar coordinate.

$x = r\cos\theta$; $y = r\sin\theta$.

and define, $u(r,\theta) := z(x,y)$ (Assuming $\exists$ a solution $z(x,y)$ of ①)

Now, $z_x = u_r r_x + u_\theta \theta_x$

$$= \frac{x}{r} u_r - \frac{y}{x^2+y^2} u_\theta$$

and, $z_y = u_r r_y + u_\theta \theta_y$

$$= \frac{y}{r} u_r - \frac{x}{x^2+y^2} u_\theta.$$

$$\therefore x\left(\frac{y}{r} u_r - \frac{x}{x^2+y^2} u_\theta\right) - y\left(\frac{x}{r} u_r - \frac{y}{x^2+y} u_\theta\right) = 0$$

$\Rightarrow u_\theta = 0 \Rightarrow u(r,\theta) = f(r)$; $f \in C^1 \Rightarrow z(x,y) = f\left[(x^2+y^2)^{1/2}\right]$; $f \in C^1$

Note :- If f is identity map $\Rightarrow z^2 = x^2 + y^2$ (- a cone (surface of revolution))