Q1:- Find the Fourier series of $f(x) = \begin{cases} x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } -\pi \le x < 0. \end{cases}$ & $f(x+2\pi) = f(x)$

Soln:- If the Fourier series is given by

$$f(x) = a_0 + \sum_{n\geq 1} \left[a_n \cos nx + b_n \sin nx\right]$$

Then,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \pi/4$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left.\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\cos n\pi - 1\right] = \frac{(-1)^n - 1}{n^2\pi}$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n = 2k \\ -\dfrac{2}{\pi(2k-1)^2} & \text{if } n = 2k+1. \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nk}{n}\,dx\right]$$

$$= {}^{1}/_{\pi}\left[-\frac{\pi\cos(n\pi)}{n} + \frac{\sin n\pi}{n^2}\right] = \frac{(-1)^{n+1}}{n}$$

$$\therefore f(x) = {}^{\pi}/_{4} - {}^{2}/_{\pi}\sum_{k\geqslant 1}\frac{\cos(2k-1)x}{(2k-1)^2} + \sum_{k\geqslant 1}\frac{(-1)^{k+1}\sin kx}{k}$$

(2) Find the Fourier series of the function (periodically extended to $\mathbb{R}$)

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ 0 & \text{if } 1/2 \le x < 1 \\ -1 & \text{if } -1/2 \le x < 0 \\ 0 & \text{if } -1 \le x < -1/2 \end{cases}$$

Soln: $\because$ f is an odd function hence,

$$f(x) = \sum_{k \geq 1} b_k \sin\left(\frac{k\pi x}{L}\right). \quad (L = 1)$$

where,

$$b_n = 1 \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = 2 \int_{0}^{1} f(x) \sin(n\pi x)\, dx$$

$$= 2\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^{1/2} = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

So, $$b_n = \begin{cases} \dfrac{2}{(2k-1)\pi} & \text{if } n = 2k-1 \\ 0 & \text{if } n = 4k \\ \dfrac{2}{(2k-1)\pi} & \text{if } n = 2(2k-1) \end{cases}$$

Hence, $$f(x) = \frac{2}{\pi} \sum_{k \geq 1}^{\infty} \frac{\sin[(2k-1)\pi x] + \sin[2(2k-1)\pi x]}{2k-1}$$

③ An elastic string of length 4 m with fixed ends is raised by 2 m and then released from rest. Assume the displacement satisfies $25u_{xx} - u_{tt} = 0$ for $0 \le x \le 4$, $t \geq 0$. Describe the motion of the string in terms of a Fourier series solution.

Soln :- [Please assume the following :-

If, $u_{tt} - a^2 u_{xx} = 0$ then $u(x,t) = \sum_{k \geq 1} \left[a_k \cos \frac{ak\pi t}{L} + b_k \sin \frac{ak\pi t}{L} \right] \sin\left(\frac{k\pi x}{L}\right)$

in $0 \le x \le L$, $t \geq 0$

along with the boundary condition : $u(0,t) = u(L,t) = 0$]

In our case, $u(x,t) = \sum_{k \geq 1} \left[a_k \cos\left(\frac{5k\pi t}{4}\right) + b_k \sin\left(\frac{5k\pi t}{4}\right)\right] \sin\left(\frac{k\pi x}{4}\right)$

Also, $0 = u_t(x,0) = \sum_{k \geq 1} b_k \left(\frac{5k\pi}{4}\right) \sin\left(\frac{5\pi x}{4}\right)$ ($\because$ The string is released from rest)

$\therefore b_K = 0 \quad \forall\, K \in \mathbb{N}$

and, $\quad u(x,0) = \sum_{k\geq 1}^{+} a_K \sin\left(\frac{k\pi x}{4}\right) = f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 2 \\ 4-x & \text{if } 2 \leq x \leq 4 \end{cases}$

$$\therefore a_K = \frac{2}{4}\int_0^4 \sin\left(\frac{k\pi x}{4}\right) f(x)\, dx = \frac{1}{2}\int_0^2 x \sin\left(\frac{k\pi x}{4}\right) dx + \frac{1}{2}\int_2^4 (4-x) \sin\left(\frac{k\pi x}{4}\right) dx$$

$$= \frac{1}{2}\left[-\frac{4x\cos(k\pi x/4)}{k\pi} + \frac{16\sin(k\pi x/4)}{(k\pi)^2}\right]_0^2 + \frac{1}{2}\left[-\frac{16\cos\left(\frac{k\pi x}{4}\right)}{k\pi} + \frac{4x\cos\left(\frac{k\pi x}{4}\right)}{k\pi} - \frac{16\sin\left(\frac{k\pi x}{4}\right)}{(k\pi)^2}\right]_2^4$$

$$= 16/_{k^2\pi^2} \sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0, & k \text{ is even} \\ (-1)^{n-1}\dfrac{16}{(2n-1)^2\pi^2}, & k \text{ is odd} \end{cases}$$

$$\therefore u(x,t) = \frac{16}{\pi^2} \sum_{n\geq 1} \frac{(-1)^{n-1}}{(2n-1)^2} \cos\left[\frac{5(2n-1)}{4}\pi t\right] \sin\left[\frac{(2n-1)\pi x}{4}\right].$$

④ Prove that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2a_0^2 + \sum_{n\geq 1}(a_n^2 + b_n^2)$ provided f is square integrable.

$$\because f(x) = a_0 + \sum_{n\geq 1}\left[a_n \cos nx + b_n \sin nx\right]$$

$$\therefore f^2(x) = a_0^2 + a_0 \sum_{n\geq 1}\left[a_n \cos nx + b_n \sin nx\right] + \sum_{n\geq 1}\sum_{m\geq 1}\left[a_n a_m \cos nx \cos mx\right.$$

$$\left. + a_n b_m \cos nx \sin mx + a_m b_n \cos mx \sin nx + b_n b_m \sin nx \sin mx\right]$$

Integrating both sides and taking into account the orthogonality of the trigonometric system, one has.

$$\int_{-\pi}^{\pi} f^2(x)\,dx = a_0^2 \cdot 2\pi + 0 + \sum_{n\geq 1}\sum_{m\geq 1}\left[a_n a_m \pi \delta_{m,n} + b_n b_m \pi \delta_{m,n}\right]$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = 2a_0^2 + \sum_{n\geq 1}\left[a_n^2 + b_n^2\right] \qquad \boxtimes$$

⑤ Comment on the convergence of $\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ using the Fourier Series of $f(x) = x^2$ on $[-1/2, 1/2]$

Soln: $\because f$ is even, $b_n = 0 \ \forall n \in \mathbb{N}$

$\therefore a_n = 2\int_{-1/2}^{1/2} x^2 \cos\left(\frac{n\pi x}{1/2}\right) dx = \frac{(-1)^n}{n^2\pi^2}$.

Also, $a_0 = \frac{1}{2\cdot 6} = 2\cdot 4\int_0^{1/2} x^2\,dx$

$$\therefore f(x) = \frac{1}{12} + \sum_{n \geq 1} \frac{(-1)^n}{n^2 \pi^2} \cos(2n\pi x)$$

$\because f$ is continuous at '0' hence using Fourier convergence theorem one has.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$