V. Integrals

적분

5.1 Area and Estimating with Finite Sums

Area

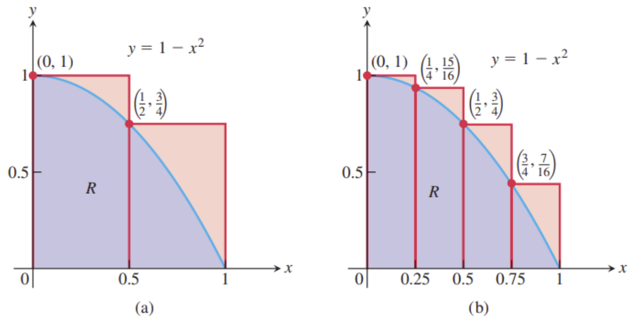

Figure 1. Upper sum estimate of the graph $y = 1 - x^2$ on $[0, 1]$.

The ***upper sum*** is obtained by taking the height of the rectangle corresponding to the maximum (uppermost) value of $f(x)$ over points x lying in the base of each rectangle. If we divide the domain $[0, 1]$ by four equal parts as shown in Figure 1-(b), these four rectangles give the approximation $A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = 0.78125$, which is still greater than A since the four rectangles contain R completely.

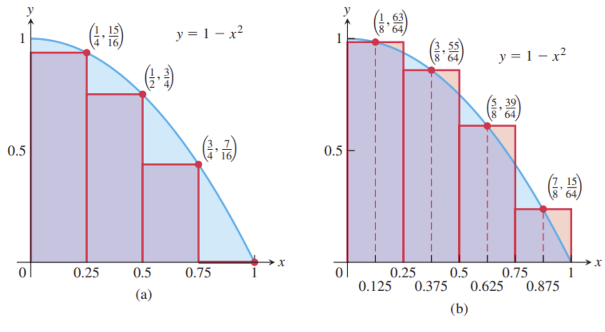

Figure 2. Lower sum / Midpoint rule estimate of the graph $y = 1 - x^2$ on $[0, 1]$.

In contrast, the ***lower sum*** is obtained by taking the height of the rectangle corresponding to the minimum (lowermost) value of $f(x)$ over points x lying in the base of each rectangle. If we divide the domain $[0, 1]$ by four equal parts as shown in Figure 2-(a), these four rectangles give the approximation $A \approx \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 0.53125$, which is smaller than the area A since the rectangles all lie inside of the region R. This implies that $0.53125 < A < 0.78125$.

You can also estimate the area by using rectangles whose heights are the values of f at the midpoint of the bases of the rectangles (As shown at Figure 2-(b)). This method of estimation is called the ***midpoint rule***.

The midpoint rule gives an estimate that is between a lower sum and an upper sum, but it is not quite so clear whether it overestimates or underestimates the true area. At Figure 2-(b), the midpoint rule estimates the area of R to be $A \approx \frac{63}{64}\cdot\frac{1}{4}+\frac{55}{64}\cdot\frac{1}{4}+\frac{39}{64}\cdot\frac{1}{4}+\frac{15}{64}\cdot\frac{1}{4}=0.671875$.

In each of the sums that we computed, the interval $[a, b]$ over which the function f is defined was subdivided into n subintervals of equal width (or length) $\Delta x = (b-a)/n$, and f was evaluated at a point in each subinterval: c_1 in the first subinterval, c_2 in the second subinterval, and so on. In each case (upper sum, lower sum, midpoint rule) the finite sums have the form $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$.

5.2 Sigma Notation and Limits of Finite Sums

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form $\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$.

Algebra Rules for Finite Sums

1. *Sum Rule:* $$\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$$
2. *Difference Rule:* $$\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}b_k$$
3. *Constant Multiple Rule* $$\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$$
4. *Constant Multiple Rule* $$\sum_{k=1}^{n}c=n\cdot c$$

Formulas for the sums of the squares and cubs of the first n integers

$$\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2$$

Limits of Finite Sums

Example: Find the limiting value of lower sum approximations to the area of the region R below the graph of $y = 1-x^2$ and above the interval $[0,1]$ on the $0,1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity.

Sol. We compute a lower sum approximation using n rectangles of equal width $\Delta x = 1/n$, and then we see what happens as $n \to \infty$. We start by subdividing $[0,1]$ into n equal width subintervals $\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{1}{2}\right],\cdots\left[\frac{n-1}{n},\frac{n}{n}\right]$. Then the sum is $f\left(\frac{1}{n}\right)\cdot\frac{1}{n}+f\left(\frac{2}{n}\right)\cdot\frac{1}{n}+\cdots+f\left(\frac{k}{n}\right)\cdot\frac{1}{n}+\cdots f\left(\frac{n}{n}\right)\cdot\frac{1}{n}.=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\cdot\frac{1}{n}=\sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\cdot\frac{1}{n}=$ $\sum_{k=1}^{n}\frac{1}{n}-\sum_{k=1}^{n}\frac{k^2}{n^3}=n\cdot\frac{1}{n}-\frac{1}{n^3}\sum_{k=1}^{n}k^2=1-\frac{2n^3+3n^2+n}{6n^3}=1-\frac{2n^2+3n+1}{6n^2}$.

So, as $n\to\infty$, $\lim_{n\to\infty}1-\frac{2n^2+3n+1}{6n^2}=1-\frac{2}{6}=\frac{2}{3}$, which means that the lower sum approximation converge to 2/3.

A similar calculation shows that the upper sum approximations also converge to 2/3. Any finite sum approximations also converges to the same value. This is because it is possible to show that any finite sum approximations is trapped between the lower and upper sum approximations. So we can define the area of the region R as this limiting value.

Riemann Sums

Begin with an arbitrary bounded function f defined on a closed interval $[a, b]$. Subdivide the interval $[a, b]$ into subintervals, not necessarily of equal widths/lengths, and form sums in the same way as for the finite approximations. To do so, we choose $n-1$ points $\{x_1, x_2, x_3, \cdots, x_{n-1}\}$ between a and b that are in increasing order, and let $x_0 = a$ & $x_n = b$. This means that $x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < b = x_n$.

The set of all of these points, $P = \{x_0, x_1, x_2, \cdots, x_n\}$, is called a ***partition*** of $[a, b]$. The partition P divides $[a, b]$ into the n closed subintervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$. $[x_{k-1}, x_k]$ is called ***kth subinterval*** (where k is an integer between 1 and n). The width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then their common width Δx is equal to $(b-a)/n$.

In each subinterval we select some point. The point chosen in the kth subinterval is called c_k. Then on each subinterval we stand a vertical rectangle that stretched from the x-axis to touch the curve at $(c_k, f(c_k))$.

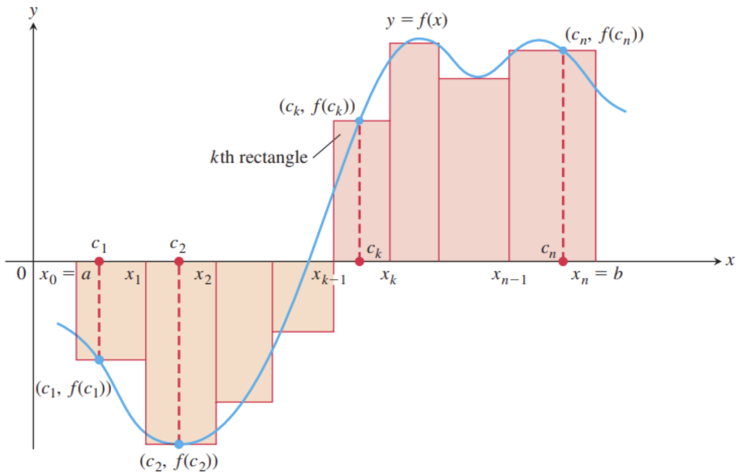

Figure 3. Procedure of calculating the Riemann Sum.

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. We sum all these products to get

$$S_p = \sum_{k=1}^{n} f(c_k) \Delta x_k.$$

The sum S_p is called a ***Riemann sum for f on the interval*** $[\boldsymbol{a}, \boldsymbol{b}]$. There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals. If we choose n subintervals all having equal width $\Delta x = (b-a)/n$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum, this leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right) \cdot \left(\frac{b-a}{n}\right).$$

We define the ***norm*** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths.

5.3 The Definite Integral

Definition of the Definite Integral

Definition 1

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the ***definite integral***(정적분) ***of*** $\boldsymbol{f}$ ***over*** $[\boldsymbol{a}, \boldsymbol{b}]$ and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:

Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - J\right| < \varepsilon.$$

If the definite integral exists, then instead writing J we write

$$\int_a^b f(x)dx.$$

This is called the ***Integral of*** $\boldsymbol{f}$ ***from*** $\boldsymbol{a}$ ***to*** $\boldsymbol{b}$. a is the Lower limit of integration, and b is the Upper limit of integration.

When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ ***converge***(수렴) to the definite integral $J = \int_a^b f(x)dx$ and that f is ***integrable***(적분 가능하다) over $[a, b]$.

In the cases where the subintervals all have equal width $\Delta x = (b-a)/n$, the Riemann sums have the form

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right).$$

where c_k is chosen in the kth subinterval. If the definite integral exists, then these Riemann sums converge to the definite integral of f over $[a, b]$, so

$$J = \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right).$$

For equal-width subintervals, $\|P\| \to 0$ is the same as $n \to \infty$. If we pick the point c_k to be the right endpoint of the kth subinterval, so that $c_k = a + k\Delta x = a + k(b-a)/n$, then the formula for the definite integral becomes

A Formula for the Riemann Sum with Equal-Width Subintervals

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right)$$

Picking c_k like this is okay because no matter how we write the integral, it is still the same number, the limit of the Riemann sums as the norm of the partition approaches zero. The variable of integration here is called a ***dummy variable.***

Integrable and Nonintegrable Functions

Theorem 1- Integrability of Continuous Functions

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

When f is continuous, we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

The following example shows a function that is not integrable over a close interval.

Example: The function

$$f(x) = \begin{cases} 1, & \mathit{if\ x\ is\ rational} \\ 0, & \mathit{if\ x\ is\ irrational} \end{cases}$$

Has no Riemann integral over $[0, 1]$. Between any two numbers there is both a rational number and an irrational number, Thus the function jumps up and down too erratically(불규칙적으로) over $[0, 1]$ to allow the region beneath its graph and above the x-axis to be approximated by rectangles, no matter how thin they are.

If we choose a partition P of $[0, 1]$, then the lengths of the intervals in the partition sum to 1; that is, $\sum_{k=1}^{n} \Delta x_k = 1$. In each subinterval $[x_{k-1}, x_k]$ there is a rational point, say c_k. Because c_k is rational, $f(c_k) = 1$. Since 1 is the maximum value that f can take anywhere, the upper sum approximation for this choice of c_k's is

$$U = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (1)\Delta x_k = 1.$$

As the norm of the partition approaches 0, these upper sum approximations converge to 1.

In such way, each subinterval $[x_{k-1}, x_k]$ there is an irrational point, say c_k. Because c_k is irrational, $f(c_k) = 0$. Since 0 is the minimum value that f can take anywhere, the lower sum approximation is

$$L = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (0)\Delta x_k = 0.$$

These lower sum approximations converge to 0 as the norm of the partition converges to 0. Thus making different choices for the points c_k results in different limits for the corresponding Riemann sums. We conclude that the definite integral of f over the interval $[0, 1]$ does not exist, and that f is not integrable over $[0, 1]$.

Properties of Definite Integrals

When f and g are integrable over the interval $[a,b]$, the definite integral satisfies the rules below.
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1. *Order of Integration*	$\int_b^a f(x)dx = -\int_a^b f(x)dx$
2. *Zero Width Interval*	$\int_a^a f(x)dx = 0$
3. *Constant Multiple*	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. *Sum and Difference*	$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. *Additivity*	$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
6. *Max-Min Inequality*	If f has maximum value max f and minimum value min f on $[a,b]$, then $(\min f)(b-a) \leq \int_a^b f(x)dx \leq (\max f)(b-a)$.
7. *Domination*	If $f(x) \geq g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$. If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx \geq 0$.

Area Under the Graph of a Nonnegative Function

Definition 2

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the ***area under the curve*** $\boldsymbol{y = f(x)}$ ***over*** $[\boldsymbol{a}, \boldsymbol{b}]$ is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

Example: Compute $\int_a^b x\,dx$ and find the area A under $y = x$ over the interval $[0,b]$, $b > 0$.

Sol. Choose partition $P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \cdots, \frac{nb}{n}\right\}$ and $c_k = \frac{kb}{n}$. So $\sum_{k=1}^n f(c_k)\Delta x_k = \sum_{k=1}^n \frac{kb}{n}\frac{b}{n} = \frac{b^2}{n^2}\sum_{k=1}^n k = \frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2}{2}\left(1+\frac{1}{n}\right)$. As $n \to \infty$ and $\|P\| \to 0$, the last expression converges to $\frac{b^2}{2}$. Therefore, $\int_a^b x\,dx = \frac{b^2}{2}$.

Average Value of a Continuous Function

Definition 3

If f is integrable on $[a,b]$, then its ***average value on*** $[\boldsymbol{a}, \boldsymbol{b}]$**,** which is also called its ***mean,*** is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx.$$

5.4 The Fundamental Theorem of Calculus

Mean Value Theorem for Definite Integrals

Theorem 2-The Mean Value Theorem for Definite Integrals

If f is continuous at $[a,b]$, then at some point c in $[a,b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx.$$

Proof If we divide both sides of the Max-Min Inequality by $(b-a)$, we obtain $\min f \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \max f$. Since f is continuous, the IVT for Continuous functions sats that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)dx$ at some point c in $[a,b]$. The continuity is important here.

Fundamental Theorem, Part 1

Theorem 3-The Fundamental Theorem of Calculus, Part 1

If f is continuous at $[a,b]$, then $F(x)=\int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b), and its derivative is $f(x)$.

$$F'(x)=\frac{d}{dx}\int_a^x f(t)dt$$

Proof $F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)dt-\int_a^x f(t)dt\right]=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt$. According to the Mean Value Theorem for Definite Integrals, there is some number c in $[x,x+h]$ such that $\frac{1}{h}\int_x^{x+h} f(t)dt=f(c)$. As $h\to 0$, $x+h$ approaches x, which forces c to approach x also. Since f is continuous at x, $f(c)$ therefore approaches $f(x)$. Hence $F'(x)=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt=\lim_{h\to 0} f(c)=f(x)$, making F differentiable at x. Since differentiability implies continuity, this also shows that F is continuous on the open interval (a,b). Then we only need to show that F is also continuous at $x=a$ and $x=b$, which could be shown by considering the one-sided limits at each point.

Fundamental Theorem, Part 2 (The Evaluation Theorem)

Theorem 4-The Fundamental Theorem of Calculus, Part 2

If f is continuous at $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx=F(b)-F(a).$$

Proof By Part 1, an antiderivative of f exists: $G(x)=\int_a^x f(t)dt$. Thus, if F is any derivative of f, then $F(x)=G(x)+C$ for some constant C for $a<x<b$. Since both F and G are continuous on $[a,b]$, this equality also applies when $x=a$ and $x=b$ (this can be shown by taking one-sided limits).

$$F(b)-F(a)=(G(b)+C)-(G(a)+C)=G(b)-G(a)=\int_a^b f(x)dx-\int_a^a f(x)dx=\int_a^b f(x)dx.$$

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a, b]$, we need to do only two things: 1. Find and antiderivative F for f, and 2. Calculate $F(b) - F(a)$ to evaluate $\int_a^b f(x)dx$. This process is much easier than using a Riemann sum computation.

The Integral of a Rate

Theorem 5-The Net Change Theorem

The net change in a differentiable function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

Total Area

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

5.5 Indefinite Integrals and the Substitution Method

Substitution: Running the Chain Rule Backwards

Theorem 6-The Substitution(치환) Rule

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on $I, then$

$$\int f\big(g(x)\big) \cdot g'(x)dx = \int f(u)du.$$

Proof By the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ whenever F is an antiderivative of f, because

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big) \cdot g'(x) = f\big(g(x)\big) \cdot g'(x).$$

If we make the substitution $u = g(x)$, then

$$\int f\big(g(x)\big) \cdot g'(x)dx = \int \frac{d}{dx}F\big(g(x)\big)\, dx = F\big(g(x)\big) + C = F(u) + C = \int F'(u)du = \int f(u)du.$$

The Substitution Method to evaluate $\int f\big(g(x)\big) \cdot g'(x)dx.$

1. Substitute $u = g(x)$ and $du = \left(\frac{du}{dx}\right) dx = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by $g(x)$.

Example: Find $\int 5\sec^2(5x+1)\,dx$.

Sol. Substitute $u = 5x+1$ and $du = 5dx$. Then $\int 5\sec^2(5x+1)\,dx = \int \sec^2 u\,du = \tan u + C$

$$\therefore \int 5\sec^2(5x+1)\,dx = \tan(5x+1) + C.$$

5.6 Definite Integrals Substitutions and the Area Between Curves

The Substitution Formula

Theorem 7-Substitution in Definite Integrals

If g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof Let F denote any antiderivative of f. Then, $\int_a^b f(g(x))\cdot g'(x)dx = F(g(x))]_{x=a}^{x=b} = F(g(b)) - F(g(a)) = F(u)]_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)du.$

Definite Integrals of Symmetric Functions

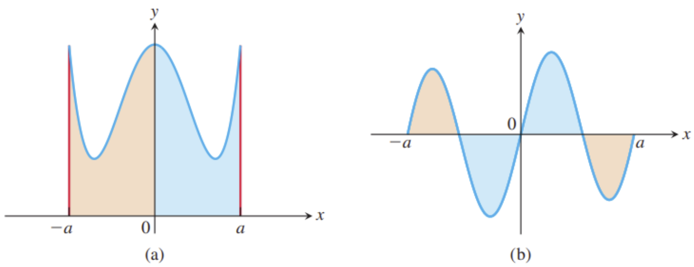

Figure 4. Definite Integrals of Symmetric Functions

Theorem 8

Let f be continuous on the symmetric interval $[-a,a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
(b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Proof of Part (a) $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = -\int_0^{-a} f(x)dx + \int_0^a f(x)dx = -\int_0^a f(-u)(-du) + \int_0^a f(x)dx = \int_0^a f(u)du + \int_0^a f(x)dx = 2\int_0^a f(x)dx$. Proof of Part (b) is also similar.

Areas Between Curves

Definition 4

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the ***area of the region between the curves*** $\boldsymbol{y = f(x)}$ ***and*** $\boldsymbol{y = g(x)}$ ***from*** $\boldsymbol{a}$ ***to*** $\boldsymbol{b}$ is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx.$$

Example: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Sol. The two curves meet at $x = -1$ and $x = 2$. So the area between the curves is $A = \int_a^b [f(x) - g(x)]dx = \int_{-1}^2 [(2 - x^2) - (-x)]dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}$.

Integration with Respect to y

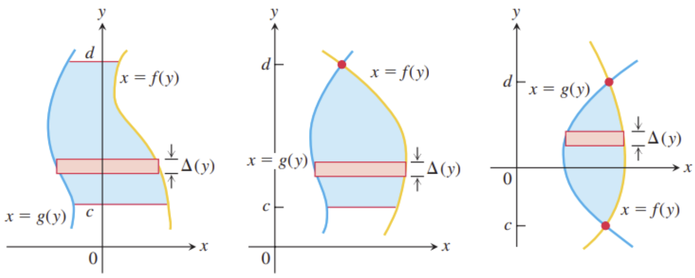

Figure 5. Bounding curves that are described by functions of y.

For regions like these, use the formula $A = \int_c^d [f(y) - g(y)]dy$ instead.