X. Parametric Equations and Polar Coordinates

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10.1 Parametrizations of Plane Curves

Parametric Equations

Definition

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point**. When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval together constitute a **parametrization** of the curve. A given curve can be represented by different sets of parametric equations.

Example: Sketch and identify the path traced by the point $P(x, y)$ if $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0.$

Sol. Make a brief table of values, plot the points, and draw a smooth curve through them.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

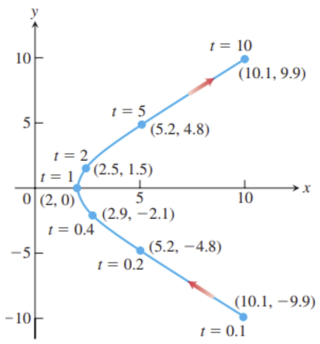

Figure 1.

Next, we eliminate the parameter t from the equations. we can find that $x - y = 2/t$ and $x + y = 2t$. We can then eliminate the parameter by multiplying these equations to get $x^2 - y^2 = 4$, which is a equation for a hyperbola. However, the parametric equations do not yield any points on the left branch of the hyperbola.

For small positive values of t, the path lies in the fourth quadrant and rises into the first quadrant as t increases, crossing the x-axis when $t = 1$. The parameter domain is $(0, \infty)$ and there is no starting point and no terminal point for the path.

Cycloids

A wheel of radius a rolls along a horizontal straight line. How can we find the parametric equations for the path traced by a point P on the wheel's circumference? The path is called a **cycloid**.

First, we take the line to be the x-axis, and roll the wheel to the right. We can use the angle t through which the wheel turns, measured in radians. The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

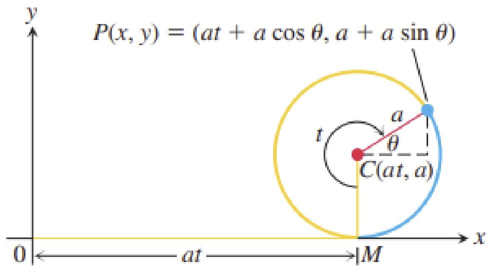

Figure 2. The position of $P(x, y)$ on the rolling wheel at angle t.

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$ in the figure. This changes the parametric equation to

$$x = a(t - \sin t), \qquad y = a(1 - \cos t).$$

Brachistochrones and Tautochrones

Consider a standart cycloid flipped about the x-axis as shown below.

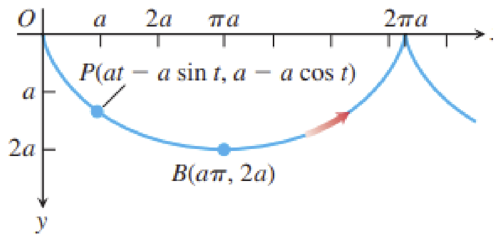

Figure 3.

Among all smooth curves joining these points, the cycloid is the curve along which a frictionless bead, subject only to the force of gravity, will slide from O to B the fastest. This makes the cycloid a **brachistochrone**(최단강하곡선). Furthermore, even if you start the bead partway down the curve toward B, it will still take the bead the same amount of time to reach B. This makes the cycloid a **tautochrone**(등시곡선).

Proving that the cycloid is a brachistochrone is done with using the *calcuslus of variations*. Proving that the cycloid is a tautochrone is done with calculating the time required to reach the point B from any point P.

10.2 Calculus with Parametric Curves

Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $\frac{dx}{dt} \neq 0$, we may divide both sides of this equation by $\frac{dx}{dt}$ to solve for $\frac{dy}{dx}$.

> **Parametric Formula for dy/dx**
>
> If all three derivatives exist and $dx/dt \neq 0$, then
>
> $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

If parametric equations define y as a twice-differentiable function of x, we can apply the equation above to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}.$$

> **Parametric Formula for d^2y/dx^2**
>
> If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,
>
> $$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Example: Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$.

Sol.

1. Express $y' = \frac{dy}{dx}$ in terms of t: $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$.
2. Differentiate y' with respect to t: $\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$
3. Divide dy'/dt by dx/dt. $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{2-6t+6t^2}{(1-2t)^3}$.

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$.

We assume the functions f and g are **continuously differentiable** (meaning they have continuous first derivatives) on the interval $[a, b]$. We also assume that the derivatives $f'(t)$ and $g'(t)$ are not simultaneously zero. This prevents the curve C from having corners and cusps, making the curva a **smooth curve**.

We subdivide the path AB into n piecec at points $A = P_0, P_1, P_2, \dots\ , P_{n-1}, P_n = B$. These points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, where $P_k = \big(f(t_k), g(t_k)\big)$. Join successive points of this subdivision by straight-line segments, and we get the segment length:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(f(t_k) - f(t_{k-1}))^2 + (g(t_k) - g(t_{k-1}))^2}.$$

If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$. By the Mean Value Theorem, there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\Delta x_k = f(t_k) - f(t_{k-1}) = f'(t_k^*)\Delta t_k,$$

$$\Delta y_k = g(t_k) - g(t_{k-1}) = g'(t_k^{**})\Delta t_k.$$

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, an approximation to the "length" of the curve AB is the sum of all the lengths L_k:

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{f'(t_k^*)^2 + g'(t_k^{**})^2}\Delta t_k.$$

Although this last sum on the right is not exactly a Riemann sum (because f' and g' are evaluated at different points), it can be shown that its limit, as the norm of the partition tends to zero and the number of segments $n \to \infty$, is the definite integral

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \sqrt{f'(t_k^*)^2 + g'(t_k^{**})^2}\Delta t_k = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Therefore, it is reasonable to define the length of the curve from A to B to be this integral.

> **Definition**
>
> If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$ and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral
>
> $$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we can write formula for arc length this way:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

A smooth curve C does not double back or reverse the direction of motion over the time interval $[a, b]$ since $(f')^2 + (g')^2 > 0$ throughout the interval. At a point where a curve does start to double back(왔던 길을 그대로 돌아가는 것) on itself, either the curve fails to be differentiable or both derivatives must simultaneously equal zero.

If there are two different parametrizations for a curve C whose length we want to find, it does not matter which one we use. However, the parametrization we choose must meet the conditions stated in the definition of the length of C.

Example: Find the length of the asteroid $x = \cos^3 t\,,\quad y = \sin^3 t\,,\quad 0 \le t \le 2\pi.$

Sol. Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t\,,\quad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t\,(-sint)]^2 = 9\cos^4 t \sin^2 t\,.$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t\,(\cos t)]^2 = 9\sin^4 4\cos^2 t\,.$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)} = 3\cos t \sin t.$$

Therefore,

$$Length\ of\ first-quadrant\ portion = \int_0^{\frac{\pi}{2}} 3\cos t \sin t\, dt = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sin 2t\, dt = \frac{3}{2}.$$

The length of the asteroid is four times this: $4(3/2) = 6$.

Example: Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Parametrically, we represent the ellipse by the equations $x = a\sin t$ and $y = b\cos t$, $a > b$ and $0 \le t \le 2\pi$. Then,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\cos^2 t + b^2\sin^2 t = a^2 - (a^2 - b^2)\sin^2 t = a^2(1 - e^2\sin^2 t)$$

Then the perimeter is given by

$$P = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2 t}\, dt.$$

The integral for P is nonelementary and is know as the *complete integral of the second kind.* We can compute its value to within any degree of accuracy using infinite series in the following way. From the binomial expansion for $\sqrt{1-x^2}$, we have $\sqrt{1 - e^2\sin^2 t} = 1 - \frac{1}{2}e^2\sin^2 t + \frac{1}{2\cdot 4}e^4\sin^4 t - \cdots$.

Then to each term in this last expression we apply the integral for $\int_0^{\pi/2} \sin^n t\, dt$ when n is even, giving the perimeter

$$\begin{aligned} P &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\sin^2 t}\, dt \\ &= 4a\left[\frac{\pi}{2} - \left(\frac{1}{2}e^2\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) - \left(\frac{1}{2\cdot 4}e^4\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right) - \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}e^6\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right) - \cdots\right] \\ &= 2\pi a\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{e^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{e^6}{5} - \cdots\right]. \end{aligned}$$

Since $e < 1$, the series on the right-hand converges by comparison with the geometric series $\sum(e^2)^n$. We do not have an explicit value for P, but we can estimate it as closely as we like by summing finitely many terms from the infinite series.

Length of a Curve $y = f(x)$

Given a continuously differentiable function $y = f(x)$, $a \le x \le b$, we can assign $x = t$ as a parameter. The graph of the function f is then the curve C defined parametrically by $x = t$ and $y = f(t)$, $a \le x \le b$, which is a special case of what we have considered in this chapter. We have $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = f'(t)$.

We know that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$, giving $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + [f'(t)]^2 = 1 + [f'(x)]^2$.

The Arc Length Differential

At Chapter 6, we define the arc length function for a parametrically defined curve $x = f(t)$ and $y = g(t)$, $a \le x \le b$, by

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}dz.$$

Then, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

The differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is often abbreviated as $ds = \sqrt{dx^2 + dy^2}$.

Example: Find the centroid of the first-quadrant arc of the asteroid in the recent example.

Sol. We take the curve's density to be $\delta = 1$ and calculate the curve's mass and moments about the coordinate axes, The distribution of mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$, A typical segment of the curve has mass $dm = 1 \cdot ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3\cos t \sin t\, dt$.

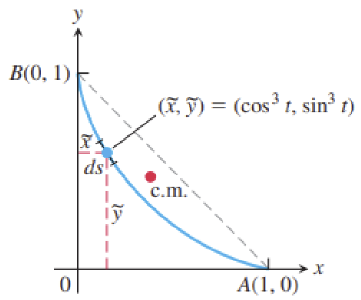

Figure 4. The centroid of the asteroid arc.

The curve's mass is

$$M = \int dm = \int_0^{\frac{\pi}{2}} 3\cos t \sin t\, dt = \frac{3}{2}.$$

The curve's moment about the x-axis is

$$M_s = \int \tilde{y} dm = \int_0^{\frac{\pi}{2}} \sin^3 t \cdot 3\cos t \sin t\, dt = 3\int_0^{\frac{\pi}{2}} \sin^4 t \cos t\, dt = 3 \cdot \frac{\sin^5 t}{5}\Big]_0^{\frac{\pi}{2}} = \frac{3}{5}.$$

It follows that $\tilde{y} = \frac{M_x}{M} = \frac{2}{5}$, and the centroid is the point $(\frac{2}{5}, \frac{2}{5})$.

Example: Find the time T_c it takes for a frictionless bead to slide along the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$ from $t = 0$ to $t = \pi$.

Sol.

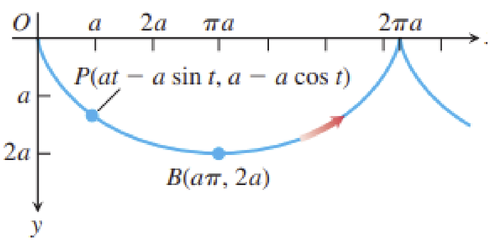

Figure 5.

We will drop a bead at point P. At the start, the kinetic energy of the bead is zero, since its velocity is zero. The work done by gravity in moving the bead is mgy, and this must equal the change in kinetic energy. That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the speed of the bead when it reaches (x, y) must be $v = \sqrt{2gy}$. That is, $\frac{ds}{dT} = \sqrt{2gy}$. then the time T_c it takes for a frictionless bead to slide along the cycloid $x = a(t - \sin t), y = a(1 - \cos t)$ from $t = 0$ to $t = \pi$ will be

$$T_c = \int_{t=0}^{t=\pi} \frac{ds}{\sqrt{2gy}}.$$

We need to express ds parametrically in terms of the parameter t. For the cycloid, $dx/dt = a(1 - \cos t)$ and $\frac{dy}{dt} = a \sin t$, so

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{a^2(2 - 2\cos t)}dt$$

Substituting for ds and y in the integrand, it follows that

$$T_c = \int_0^{\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(1 - \cos t)}}\, dt = \int_0^{\pi} \sqrt{\frac{a}{g}}\, dt = \pi\sqrt{\frac{a}{g}}.$$

This is the amount of time it takes the frictionless bead to slide down the cycloid B after it is released from an arbitrary point P.

Areas of Surfaces of Revolution

Areas of Surfaces of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

As with length, we can calculate the surface area from any convenient parametrization that meets the stated criteria.

Example: The standard parametrization of the circle of radius 1 centered at the point $(0, 1)$ in the xy-plane is

$$x = \cos t\,, y = 1 + \sin t\,, 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis.

Sol. We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2} dt$$
$$= \int_0^{2\pi} 2\pi(1 + \sin t) dt = 2\pi[t - \cos t]_0^{2\pi} = 4\pi^2.$$

10.3 Polar Coordinates

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin O** (called the **pole**) and an **initial ray** from O. Usually the positive x-axis is chosen as the initial ray. Then each point P can be located by assigning to it a **polar coordinate pair (r, θ)** in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So, we label the point P as $P(r, \theta)$.

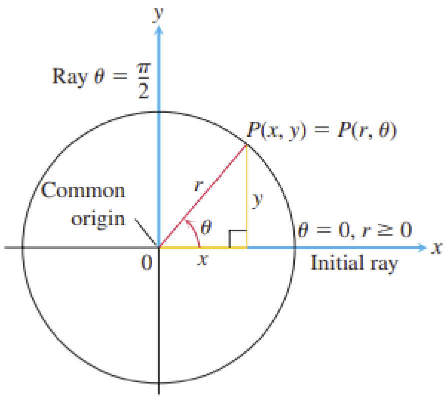

Figure 6.

Relating Polar and Cartesian Coordinates

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

Example: Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

Sol. We apply the equations relating polar and Cartesian coordinates:

$$x^2 + (y-3)^2 = 9, x^2 + y^2 - 6y = 0.$$
$$r^2 - 6r\sin\theta = 0. r = 0 \text{ or } r = 6\sin\theta$$

$$\therefore r = 6\sin\theta.$$

10.4 Graphing Polar Coordinate Equations

Symmetry

Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.
3. *Symmetry about the origin:* If the (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.

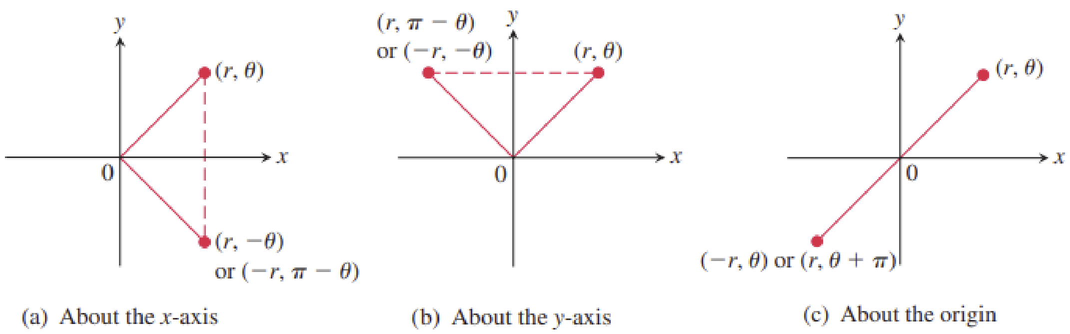

Figure 7.

Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is dy/dx. but this is not given by the formula $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(f(\theta)\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cos\theta)} = \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

Therefore, we see that dy/dx is not the same as $df/d\theta$.

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

provided $\frac{dx}{d\theta} \neq 0$ at (r, θ).

Example: Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Sol. The curve is symmetric about the x-axis because

(r, θ) on the graph $\Longrightarrow r = 1 - \cos\theta \;\Longrightarrow r = 1 - \cos(-\theta) \Longrightarrow (r, -\theta)$ on the graph.

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues from π to 2π, r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope 0 and returns to the origin with slope 0.

If we draw a smooth curve considering the symmetry, the slopes, the intercepts, we get the graph below:

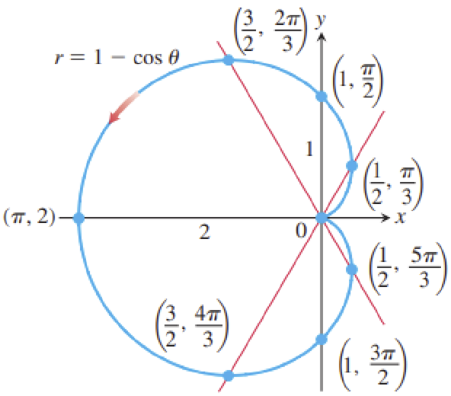

Figure 8.

The curve is called the *cardioid* because its heart shape.

10.5 Areas and Lengths in Polar Coordinates

Area in the Plane

The region OTS in the Figure below is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k / 2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^2 \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

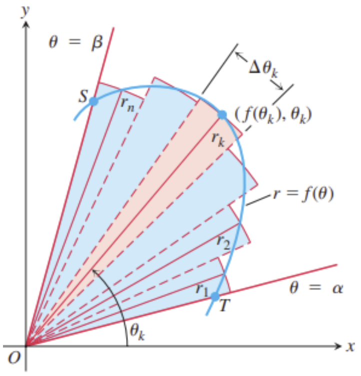

Figure 9.

The area of region OTS is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are therefore led to the following formula for the region's area:

$$A = \lim_{||P||\to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$ when $\alpha \le \theta \le \beta$, $r \ge 0$, and $\beta - \alpha \le 2\pi$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

This is the integral of the **area differential**

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Example: Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Sol. We graph the cardioid and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 d\theta = \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta$$
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi.$$

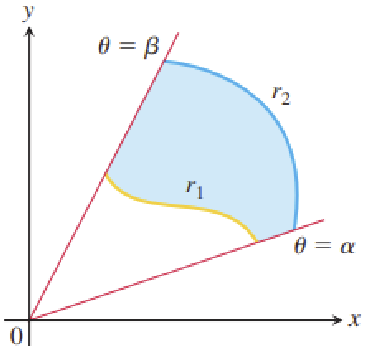

Figure 10.

To find the area of a region like the one above, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2)r_1^2 d\theta$ from the integral of $(1/2)r_2^2 d\theta$.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, $r \ge 0$, and $\beta - \alpha \le 2\pi$.

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2 d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2 d\theta = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta$$

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta.$$

The parametric length formula then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

when substituted for x and y.

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example: Find the length of the cardioid $r = 1 - \cos\theta$.

Sol. We sketch the cardioid to determine the limits of integration. The point $P(r,\theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β. With $r = 1 - \cos\theta$, $\frac{dr}{d\theta} = \sin\theta$, we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1-\cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta$$

and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = [-4\cos\theta]_0^{2\pi} = 8.$$

10.6 Conic Sections

In this section we define and review parabolas, ellipses, and hyperbolas geometrically and derive their standard Cartesian equations. These curves are called **conic sections** or **conics.**

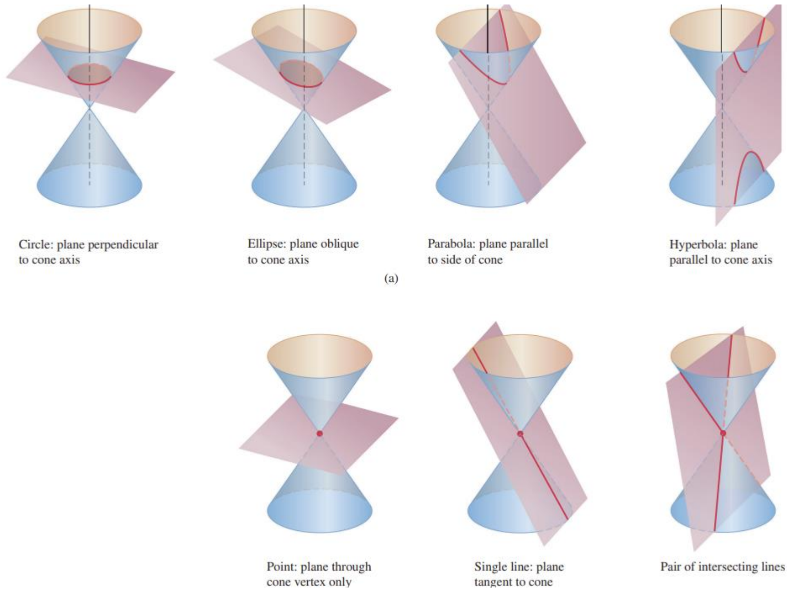

Figure 11. The standard conic sections.

Parabolas

> **Definitions**
>
> A set that consists of all the points in a plane equidistant(거리가 같다) from a given fixed point and a given fixed line in the plan is a **parabola**(포물선). The fixed point is the **focus**(초점) of the parabola. The fixed line is the **directrix**(준선).

Suppose that the focus lies at the point $F(0,p)$ on the positive y-axis and that the directrix is the line $y = -p$. In the notation of the figure, a point $P(x,y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{x^2 + (y-p)^2}, \qquad PQ = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad or \quad x^2 = 4py.$$

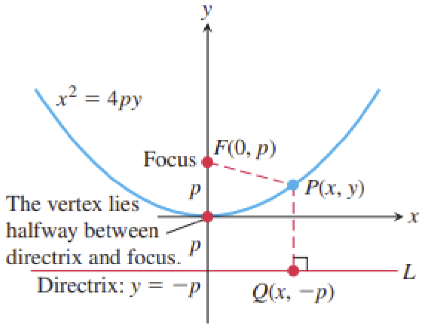

Figure 12. The standard form of the parabola $x^2 = 4py$, $p > 0$.

These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the axis of the parabola. The point where a parabola crosses its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin. The positive number p is the parabola's **focal length**.

Ellipses

Definitions

An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant term. The two fixed points are the foci of the ellipse.

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices**.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, and $PF_1 + PF_2 = 2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

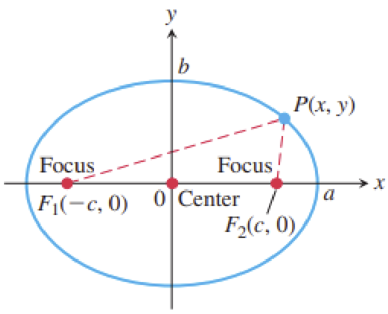

Figure 13. The standard form of a ellipse.

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Since $PF_1 + PF_2$ is greater than the length F_1F_2, the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in the equation above is positive. If we let b denote the positive square roof of $a^2 - c^2$,

$$b = \sqrt{a^2 - c^2},$$

then $a^2 - c^2 = b^2$ and the equation above takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This equation reveals that this ellipse symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y},$$

which is zero if $x = 0$ and infinite if $y = 0$.

The **major axis** of the ellipse in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number a itself is the **semimajor axis**, the number b the **semiminor axis**. The number $c = \sqrt{a^2 - b^2}$ is the **center-to-focus distance** of the ellipse. If $a = b$ then the ellipse is a circle.

Hyperbolas

Definitions

A **Hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola.

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the hyperbola's **vertices**.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, and the constant difference is $2a$, then the coordinates of a point P on the hyperbola satisfy the equation

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

This looks just like the equation for an ellipse, but now $a^2 - c^2$ is negative because $a < c$. If we let b denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}$$

then $a^2 - c^2 = -b^2$ and the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The differences between the equation for an ellipse and the equation for a hyperbola are the minus sign and the new relation $c^2 = a^2 + b^2$.

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the x-axis at the points $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

and this is infinite when $y = 0$. The hyperbola has no y-intercepts. The lines

$$y = \pm \frac{b}{a} x$$

are the two **asymptotes** of the hyperbola defined by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

10.7 Conics in Polar Coordinates

Eccentricity

Definitions

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The **eccentricity** of a parabola is $e = 1$.

Whereas a parabola has one focus and one directrix, each ellipse has two foci and two directrices. These are the lines perpendicular to the major axis at distances $\pm a/e$ from the center. From the figure below we see that a parabola has the property

$$PF = 1 \cdot PD$$

for any point P on it, where F is the focus and D is the point nearest P on the directrix. For an ellipse, it can be shown that the equations that replace the equation above are

$$PF_1 = e \cdot PD_1, \qquad PF_2 = e \cdot PD_2.$$

Here, e is the eccentricity P is any point on the ellipse, F_1 and F_2 are the foci, and D_1 and D_2 are the points on the directrices nearest P.

In both equations the directrix and focus must correspond; that is, if we use the distance from P to F_1, we must also use the distance from P to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as **directrices** for the **hyperbola** and that

$$PF_1 = e \cdot PD_1 \quad and \quad PF_2 = e \cdot PD_2.$$

Here P is any point on the hyperbola, F_1 and F_2 are the foci, and D_1 and D_2 are the points nearest P on the directrices.

In both the ellipse and the hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices. So $Eccentricity = \frac{distance\ between\ foci}{distance\ between\ vertices}$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are further apart than the vertices and the ratio is greater than 1.

The "focus-directrix" equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD$$

where e is the constant of proportionality. Then the path traced by P is

(a) a *parabola* if $e = 1$,

(b) an *ellipse* of eccentricity e if $e < 1$, and

(c) a *hyperbola* of eccentricity e if $e > 1$.

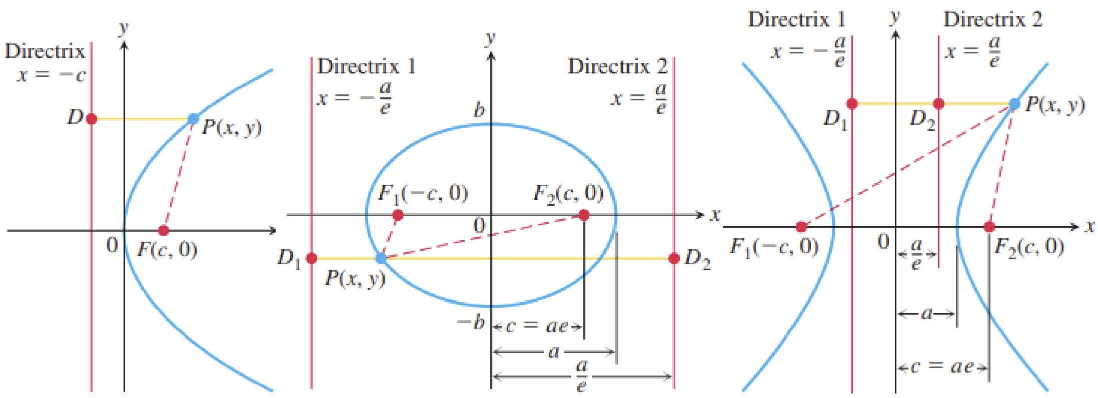

Figure 14.

Polar Equations

Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\cos\theta},$$

where $x = k > 0$ is the vertical directrix.

$$r = \frac{ke}{1 - e\cos\theta},$$

where $x = k < 0$ is the vertical directrix.

$$r = \frac{ke}{1 + e\sin\theta},$$

where $y = k > 0$ is the horizontal directrix.

$$r = \frac{ke}{1 - e\sin\theta},$$

where $y = k < 0$ is the horizontal directrix.

From the ellipse diagram below, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea.$$

From this, we find that $ke = a(1 - e^2)$. Replacing ke in the equation above by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

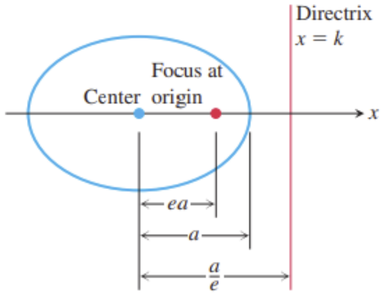

Figure 15.

> **Polar Equation for a Ellipse with Eccentricity e and Semimajor Axis a**
>
> $$r = \frac{a(1 - e^2)}{1 + e\cos\theta}.$$

Notice that when $e = 0$. the equation becomes $r = a$, which represents a circle.

Lines

> **The Standard Polar Equation for Lines**
>
> If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L *is*
>
> $$r\cos(\theta - \theta_0) = r_0.$$

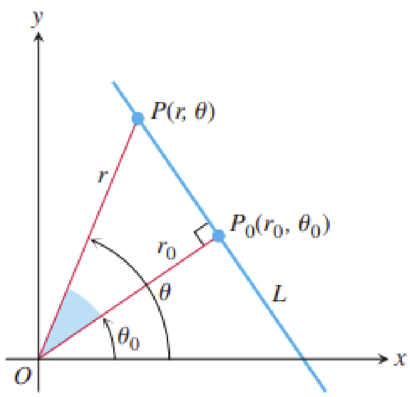

Figure 16.

Circles

The polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$ is

$$a^2 = r_o^2 + r^2 - 2r_0 r\cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$r = 2a\cos(\theta - \theta_0)$$

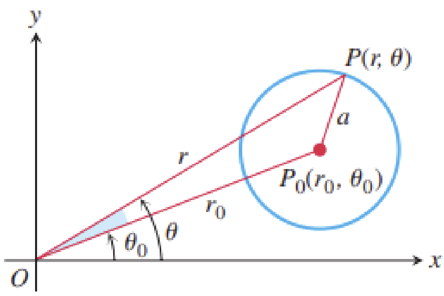

Figure 17.