II. Limits and Continuity

극한과 연속성

2.2 Limit of a Function and Limit Laws

An Informal Description of the Limit of a Function

If $f(x)$ is *arbitrarily*(임의로) close to the number L for all x *sufficiently*(충분히) close to c, other than c itself, then we say that f approaches the limit L as x approaches c, and write

$$\lim_{x \to c} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches c is L"

But the phrases like *"arbitrarily close"* and *"sufficiently close"* are imprecise(부정확하다), making the definition above informal.

These are some examples which the function does not have a limit at $x = c$.

- The function *jumps*: For example, the unit step function has no limit as $x \to 0$ because its values jump at $x = 0$.

Example: $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (unit step function) has no limit as $x \to 0$.

- The function *grows too "large"*: The function grows arbitrarily large in absolute value as $x \to c$; therefore, the function does not stay close to *any* fixed real number. "The function is *not bounded*."

Example: $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ has no limit as $x \to 0$.

- The function *oscillates too much to have a limit*: The function oscillates and does not stay close to any single number as $x \to c$.

Example: $f(x) = \begin{cases} \sin\frac{1}{x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ has no limit as $x \to 0$.

The Limit Laws

Theorem 1 - Limit Laws

If L, M, c, and k are real numbers and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, then

1. *Sum Rule* $\quad \lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule* $\quad \lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule* $\quad \lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule* $\quad \lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule* $\quad \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

6. *Power Rule* $\lim_{x \to c}[f(x)]^n = L^n, n$ a positive integer
7. *Root Rule* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n$ a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

※ Proving of these Laws will be shown at a separate file. (Precise Definition of Limit is used)

Evaluating Limits of Polynomials(다항함수) and Rational Functions(유리함수)

Theorem 2 – Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then $\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$

Theorem 3 – Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then $\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$

Example: $\lim_{x \to -1} \frac{x^3+4x^2-3}{x^2+5} = \frac{(-1)^3+4(-1)^2-3}{(-1)^2+5} = \frac{0}{6} = 0$

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator(분모) of the rational function is not zero at the limit point c.

If the denominator is zero, canceling common factors in the numerator(분자) and denominator may reduce the fraction to one whose denominator is no longer zero at c.

Example: $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$

The Sandwich Theorem

Theorem 4 – The Sandwich Theorem (샌드위치 정리)

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval(개구간) containing c, except possibly at $x = c$ itself (포함되어도, 안되어도 상관없다). Then, the following is true.

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \Rightarrow \lim_{x \to c} f(x) = L$$

Example: Q. Given a function u that satisfies $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$, find $\lim_{x \to 0} u(x)$.

Sol. Since $\lim_{x \to 0} 1 - \frac{x^2}{4} = \lim_{x \to 0} 1 + \frac{x^2}{2} = 1$, the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$.

The theorem enables us to calculate a variety of limits. The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem. A proof will be given at a separate file. (Precise Definition of Limit is used)

2.3 The Precise Definition of a Limit

Definition of Limit

Definition 1: The Precise Definition of a Limit

Let $f(x)$ be defined on an open interval about c, except possibly at $x = c$ itself. We say that the ***limit of $f(x)$ as x approaches c is the number L***, and writes as $\lim_{x\to c} f(x) = L$, if the proposition below is true.

$$^\forall\varepsilon > 0,\ ^\exists\delta > 0\ \ such\ that\ \ 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Example: Show that $\lim_{x\to 1}(5x-3) = 2$.

Sol. Set $c = 1, f(x) = 5x - 3$, and $L = 2$ in the definition of Limit. We should choose δ that for all ε, if $0 < |x-1| < \delta$, then $|f(x) - 2| < \varepsilon$. We can find δ by working backwards:

$$|f(x) - 2| = |5x - 5| = 5|x - 1| < \varepsilon$$

$$\Rightarrow |x - 1| < \varepsilon/5.$$

Thus, we can choose $\delta = \varepsilon/5$. If $0 < |x-1| < \varepsilon/5$, then $|f(x) - 2| = 5|x-1| < 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$.

Finding Deltas Algebraically for Given Epsilons

Example: Show that $\lim_{x\to 2} x^2 = 4$.

Sol 1. (책 풀이) Solve the inequality $|x^2 - 4| < \varepsilon$:

$$|x^2 - 4| < \varepsilon$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

$$if\ \varepsilon < 4, \qquad \sqrt{4-\varepsilon} < |x| < \sqrt{4+\varepsilon}, \qquad \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

$$else, \qquad 0 < x < \sqrt{4+\varepsilon}$$

$(if\ \ \varepsilon < 4)$ To find the value of δ, take δ to be the distance from $x = 2$ to the nearer endpoint of $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$. In other words, take $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$

$(if\ \ \varepsilon > 4)$ take δ to be the distance from $x = 2$ to the nearer endpoint of $(0, \sqrt{4+\varepsilon})$. In other words, take $\delta = \min\{2,\ \sqrt{4+\varepsilon} - 2\}$

However, this solution has the process of dividing cases, making it trickier. The solution below could be better.

Sol 2. Take the $|x-2|$ out from the inequality $|x^2 - 4| < \varepsilon$.

$$|x^2 - 4| = |x+2||x-2| < \varepsilon$$

Then choose an appropriate reference for δ. We will choose 1 at this solution. Then

$$|x - 2| < 1 \Rightarrow 1 < x < 3 \Rightarrow |x + 2| < 5$$

$$|x+2||x-2| < 5|x-2|$$

Now, if we choose $\delta = \min\{1,\ \varepsilon/5\}$, the following is true.

$$|x^2 - 4| = |x+2||x-2| < 5|x-2| < 5\cdot\left(\frac{\varepsilon}{5}\right) < \varepsilon$$

2.4 One-Sided Limits

Approaching a Limit from One side

For f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches from either side. Because of this, we can say that the limit is *two-sided.*

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a *right-hand limit*(우극한) or *limit from the right*. From the left, it is a *left-hand limit*(좌극한) or *limit from the left.*

Precise Definitions of One-Sided Limits

Definition 2: Precise Definitions of One-Sided Limits

(a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has ***right-hand limit L at c***, and write $\lim_{x \to c+} f(x) = L$, if the proposition below is true.

$$^\forall\varepsilon > 0,\ ^\exists\delta > 0\ \ such\ that\ \ c < x < c + \delta \Rightarrow |f(x) - L| < \varepsilon.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that $f(x)$ has ***left-hand limit L at c***, and write $\lim_{x \to c-} f(x) = L$, if the proposition below is true.

$$^\forall\varepsilon > 0,\ ^\exists\delta > 0\ \ such\ that\ \ c - \delta < x < c \Rightarrow |f(x) - L| < \varepsilon.$$

Example: Show that $\lim_{x \to 0+} \sqrt{x} = 0$.

Sol. Set $c = 0, f(x) = \sqrt{x}$, and $L = 0$ in the definition of One-sided Limit. We should choose δ that for all ε, if $0 < x < \delta$, then $\sqrt{x} < \varepsilon$.

If we choose $\delta = \varepsilon^2$ we have $\sqrt{x} < \varepsilon$ whenever $0 < x < \varepsilon^2$, which shows that $\lim_{x \to 0+} \sqrt{x} = 0$.

Example: Show that $y = \sin(1/x)$ has no limit has no limit as x approaches zero from either side.

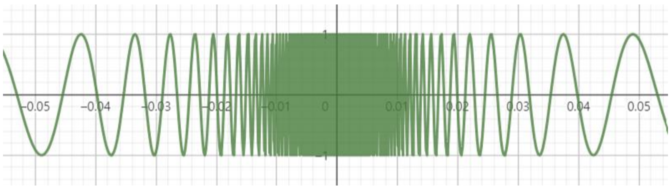

Figure 1. The graph of $y = \sin(1/x)$ near $x = 0$.

Sol. As x approaches zero, $1/x$ grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. (-1과 1 사이를 무한히 순환한다) There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true at all possible intervals of x. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Theorem 5

Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c+} f(x) = L \quad and \quad \lim_{x \to c-} f(x) = L$$

Limits Involving $(\sin\theta)/\theta$

Despite the domain of the function $f(\theta) = (\sin\theta)/\theta$ not including $x = 0$, we can find the limit of $f(\theta)$ at $x = 0$.

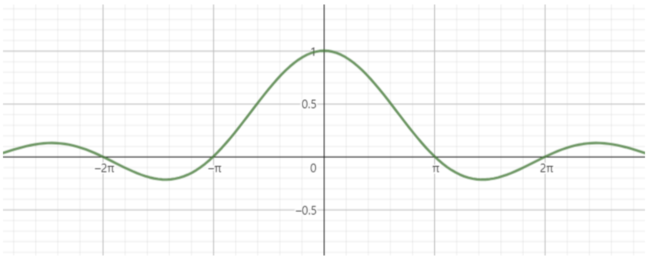

Figure 2. The graph of $f(\theta) = (\sin\theta)/\theta$ near $x = 0$.

> Theorem 6 – Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$
>
> $$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad (\theta\ in\ radians)$$

Proof Draw a circle with a radius of 1 and prove $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$ $(\theta > 0)$ by matching each one with an area from the drawing. Then since $1 > \frac{\sin\theta}{\theta} > \cos\theta$, use the Sandwich Theorem to prove that $\lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = 1$.

To consider the left-hand limit, we use that $\sin\theta$ and θ are both *odd functions*(기함수), making $\frac{\sin\theta}{\theta}$ an *even function*(우함수). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit, which means that $\lim_{\theta\to 0+}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0-}\frac{\sin\theta}{\theta} = 1$. Proving $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ $(by\ Theorem\ 5)$.

These are also true:

> Limits of some Trigonometric Functions (삼각함수)
>
> $$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1, \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0, \qquad \lim_{x\to 0}\frac{\sin Ax}{Bx} = \frac{A}{B}$$

2.5 Continuity

> Definition 3: Continuity (연속성)
>
> Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.
> The function f is ***continuous*** *at* c if
>
> $$\lim_{x\to c} f(x) = f(c).$$
>
> The function f is ***right-continuous at c (or continuous from the right)*** if
>
> $$\lim_{x\to c+} f(x) = f(c).$$
>
> The function f is ***left-continuous at c (or continuous from the left)*** if
>
> $$\lim_{x\to c-} f(x) = f(c).$$

We say that a function is ***continuous over a closed interval*** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points(내부점) of the interval. This definition also applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved.

If a function is not continuous at point c of its domain, we say that f is ***discontinuous at c,*** and that f has a discontinuity at c.

※ Note that function f can be continuous, right continuous, or left-continuous only at a point c for which $f(c)$ is defined.

Example: The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$. It is right-continuous at $x = -2$, and left-continuous at $x = 2$.

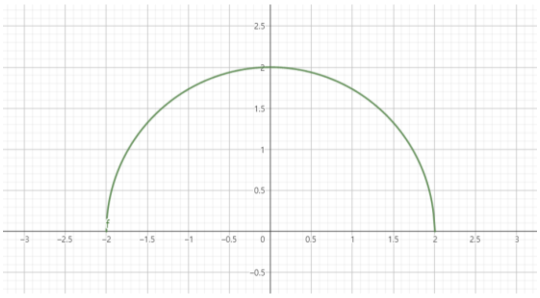

Figure 2. The graph of $f(x) = \sqrt{4 - x^2}$.

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ *if and only if* it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

Continuous Functions

We define a ***continuous function*** to be one that is continuous at every point in its domain. If a function is discontinuous at one or more points of its domain, we say it is a ***discontinuous function.***

Example: The function $f(x) = 1/x$ is a continuous function because it is continuous at every point of its natural domain. The point $x = 0$ (where the graph is discontinuous) is not in the domain of the function f.

Algebraic combinations of continuous functions are continuous wherever they are defined.

Example: Every polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$ $(by\ Theorem\ 2)$.

Example: If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined $(Q(c) \neq 0)$ $(by\ Theorem\ 3)$.

Example: The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial (which means it is continuous). Finally, at the origin, $\lim_{x\to 0}|x| = 0 = |0|$.

All six trigonometric functions are continuous wherever they are defined.

Continuity of Compositions of Functions

Theorem 7

If f is continuous at c and g is continuous at $f(c)$, then the composition $g \cdot f$ is also continuous at $x = c$.

$$\lim_{x\to c}(g \cdot f)(x) = g(f(c))$$

Example: The function $y = \sqrt{x^2 + 3x + 10}$ is continuous. The given function is the composition of the polynomial $f(x) = x^2 + 3x + 10$ with the square root function $g(x) = \sqrt{x}$, and is continuous on its natural domain. ($g(x)$ is continuous because it is a root of the continuous identity function $h(x) = x$.)

Theorem 8

If $\lim_{x\to c} f(x) = b$ and g is continuous at the point b, then

$$\lim_{x\to c} g(f(x)) = g(b)$$

Proof Let $\varepsilon > 0$ be given. Since g is continuous at b, the following is true:

$$^{\forall}\varepsilon > 0,\ ^{\exists}\delta_1 > 0 \textit{ such that } 0 < |y - b| < \delta_1 \Rightarrow |g(y) - g(b)| < \varepsilon$$

Since $\lim_{x\to c} f(x) = b$, the following is also true:

$$\textit{Choose } \varepsilon = \delta_1,\ ^{\exists}\delta > 0 \textit{ such that } 0 < |x - c| < \delta \Rightarrow |f(x) - b| < \delta_1$$

If we let $y = f(x)$, we then have that

$$\textit{Choose } \varepsilon = \delta_1,\ ^{\exists}\delta > 0 \textit{ such that } 0 < |x - c| < \delta \Rightarrow |y - b| < \delta_1 \Rightarrow |g(y) - g(b)| < \varepsilon$$

Which implies that $\lim_{x\to c} g(f(x)) = g(b)$.

Intermediate Value Theorem (IVT) for Continuous Functions

Theorem 9 – The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

$$\lim_{x\to c} g(f(x)) = g(b)$$

Proof Completeness property of the real number system (실수의 완비성 공리)

Example: Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Sol. Let $f(x) = x^3 - x - 1$. $f(1) = -1 < 0, f(2) = 5 > 0$. $f(x)$ is a polynomial, thus continuous $\Rightarrow$ IVT.

Continuous Extension to a Point

A function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & if\ x\ is\ in\ the\ domain\ of\ f \\ L, & if\ x = c. \end{cases}$$

Then the function F is continuous at $x = c$. It is called the ***continuous extension of f*** to $x = c$.

Example: Show that $f(x) = \frac{x^2+x-6}{x^2-4}, x \neq 2$ has a continuous extension to $x = 2$, and find that extension.

Sol. $f(x) = \frac{x^2+x-6}{x^2-4} = \frac{x+3}{x+2}\ for\ x \neq 2$. Thus $\lim_{x\to 2} f(x) = \frac{5}{4}$.

Let $F(x) = \frac{x+3}{x+2}$. Then $F(x) = f(x)\ for\ x \neq 2$, but is continuous at $x = 2$, having there the value of 5/4. Thus F is the continuous extension of f to $x = 2$.

2.6 Limits Involving Infinity; Asymptotes of Graphs

Finite limits as $x \to \pm\infty$

Definition 4

1. We say that $f(x)$ has the ***limit L as x approaches infinity*** and write $\lim_{x\to\infty} f(x) = L$, if

$$^{\forall}\varepsilon > 0,\ ^{\exists}M > 0\ \ such\ that\ x > M \Rightarrow |f(x) - L| < \varepsilon.$$

1. We say that $f(x)$ has the ***limit L as x approaches negative infinity*** and write $\lim_{x\to-\infty} f(x) = L$, if

$$^{\forall}\varepsilon > 0,\ ^{\exists}N > 0\ \ such\ that\ x < N \Rightarrow |f(x) - L| < \varepsilon.$$

Example: Show that $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0$.

Sol. $\lim_{x\to\infty} \frac{1}{x} = 0$: For $^{\forall}\varepsilon > 0, Choose\ M = \frac{1}{\varepsilon}\quad then\quad x > M = \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{x}\right| < \varepsilon.$

$\lim_{x\to-\infty} \frac{1}{x} = 0$: For $^{\forall}\varepsilon > 0, Choose\ N = -\frac{1}{\varepsilon}\quad then\quad x < N = -\frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{x}\right| < \varepsilon.$

Theorem 10

All the Limit Laws in Theorem 1 are true when we replace c with ∞ or $-\infty$. That is, the variable x may approach a finite number c or $\pm\infty$.

Limits at Infinity of Rational Functions

Example: $\lim_{x\to\infty} \frac{5x^2+8x+3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)+(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$, $\quad \lim_{x\to-\infty} \frac{11x+2}{2x^3-1} = \lim_{x\to-\infty} \frac{(11/x^2)+(2/x^3)}{2-(1/x^3)} = \frac{0+0}{2-0} = 0.$

Horizontal Asymptotes

> **Definition 5**
>
> A line $y = b$ is a ***horizontal asymptote*** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \qquad or \qquad \lim_{x\to-\infty} f(x) = b$$

Example: Find the horizontal asymptotes of the graph of $f(x) = \frac{x^3-2}{|x|^3+1}$.

Sol. For $x \geq 0$: $\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty}\frac{x^3-2}{x^3+1} = 1$, For $x < 0$: : $\lim_{x\to\infty}\frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty}\frac{x^3-2}{-x^3+1} = -1$.

The horizontal asymptotes are $y = -1$ and $y = 1$.

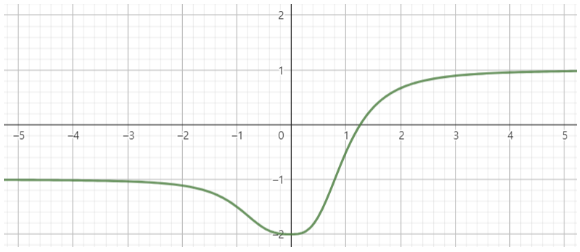

Figure 3. The graph of $f(x) = \frac{x^3-2}{|x|^3+1}$

Example: Using the Sandwich Theorem, find the horizontal asymptote of the curve $y = 2 + \frac{\sin x}{x}$.

Sol. Since $0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$ and $\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0$, we have $\lim_{x\to\pm\infty}\left|\frac{\sin x}{x}\right| = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2.$$

So the line $y = 2$ is a horizontal asymptote of the curve on both left and right.

Oblique Asymptotes

If the degree(차수) of the numerator(분자) of a rational function is 1 greater than the degree of the denominator(분모), the graph has an ***oblique*** of ***slant line asymptote***.

Example: Find the oblique asymptote of the graph of $f(x) = \frac{x^2-3}{2x-4}$.

Sol 1(책 풀이): Divide $(2x-4)$ into (x^2-3) ⇒ quotient $\left(\frac{x}{2}+1\right)$ + remainder 1

This tells us that $f(x) = \frac{x^2-3}{2x-4} = \left(\frac{x}{2}+1\right) + \left(\frac{1}{2x-4}\right)$. As $x \to \pm\infty$, the remainder goes to zero, making the slanted line $g(x) = \frac{x}{2} + 1$ an asymptote of the graph of f.

Sol 1: Let oblique asymptote $y = ax + b$. $a = \lim_{x\to\infty}\frac{x^2-3}{2x-4}\frac{1}{x} = \frac{1}{2}$, $b = \lim_{x\to\infty}\frac{x^2-3}{2x-4} - ax = \lim_{x\to\infty}\frac{2x-3}{2x-4} = 1$.

$$\therefore y = \frac{x}{2} + 1$$

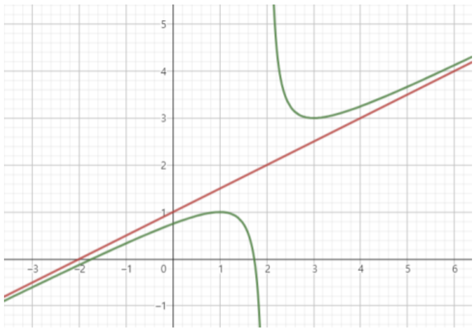

Figure 4. The graph of $f(x) = \frac{x^2-3}{2x-4}, g(x) = \frac{x}{2} + 1$

Infinite Limits

Example: $\lim_{x\to1+}\frac{1}{x-1} = \infty$ and $\lim_{x\to1-}\frac{1}{x-1} = -\infty$. $\lim_{x\to2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to2}\frac{x-2}{x+2} = 0$.

Definition 6

1. We say that $\boldsymbol{f(x)}$ ***approaches infinity as x approaches c,*** and write $\lim_{x\to c} f(x) = \infty$, if

$$^{\forall}B > 0,\ ^{\exists}\delta > 0\ \ such\ that\ 0 < |x-c| < \delta \Rightarrow f(x) > B.$$

1. We say that $\boldsymbol{f(x)}$ ***approaches negative infinity as x approaches c,*** and write $\lim_{x\to c} f(x) = -\infty$, if

$$^{\forall}(-B) > 0,\ ^{\exists}\delta > 0\ \ such\ that\ 0 < |x-c| < \delta \Rightarrow f(x) < -B.$$

Example: Prove that $\lim_{x\to0}\frac{1}{x^2} = \infty$.

Sol. Given $B > 0, \frac{1}{x^2} > B\ if\ and\ only\ if\ x^2 < \frac{1}{B}$. So, choose $\delta = \frac{1}{\sqrt{B}}$, then

$$|x| < \delta \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} \geq B$$

Vertical Asymptotes

Definition 7

A line $x = a$ is a ***vertical asymptote*** of a graph of a function $y = f(x)$ if either

$$\lim_{x\to a+} f(x) = \pm\infty \qquad or \qquad \lim_{x\to a-} f(x) = \pm\infty.$$

Example: Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$

Sol. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}, \lim_{x\to\infty} 1 + \frac{1}{x+2} = 1$. $\Rightarrow$ horizontal asymptote: $y = 1$.

Vertical asymptote is made when the denominator is zero. $\Rightarrow$ vertical asymptote: $x = -2$.

Dominant Terms

$f(x) = \frac{x^2-3}{2x-4} = \left(\frac{x}{2}+1\right) + \left(\frac{1}{2x-4}\right)$. We say that $\left(\frac{x}{2}+1\right)$ ***dominates*** when x approaches ∞ or $-\infty$, and $\left(\frac{1}{2x-4}\right)$ dominates when x approaches 2. ***Dominant terms*** like these help us predict a function's behavior.