VIII. Techniques of Integration

적분 방법론

8.1 Using Basic Integration Formulas

Basic Integration Formulas

1. $\int k dx = kx + C \quad (any\ number\ k)$
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x dx = e^x + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\, dx = -\cos x + C$
7. $\int \cos x\, dx = \sin x + C$
8. $\int \sec^2 x\, dx = \tan x + C$
9. $\int \csc^2 x\, dx = -\cot x + C$
10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\int \csc x \cot x\, dx = -\csc x + C$
12. $\int \tan x\, dx = \ln|\sec x| + C$
13. $\int \cot x\, dx = \ln|\sin x| + C$
14. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\, dx = \cosh x + C$
17. $\int \cosh x\, dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

8.2 Integration by Parts

Integration by Parts Formula

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

$$\int udv = uv - \int vdu$$

$$\int_a^b u(x)v'(x)dx = u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

Example: Obtain a formula that expresses the integral $\int \cos^n x\, dx$ in terms of an integral of a lower power of $\cos x$. Then derive the formula for $\int_0^{\pi/2} \cos^n x\, dx$.

Sol. We may think of $\cos^n x = \cos^{n-1} x \cdot \cos x$. Then we let $u = \cos^{(n-1)} x$ and $dv = \cos x\, dx$, so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\, dx) \quad and \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}\int \cos^n x\, dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx - (n-1)\int \cos^n x\, dx.\end{aligned}$$

This means

$$n\int \cos^n x\, dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\, dx.$$

Dividing by n gives

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x\, dx.$$

The formula above is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced.

Now calculate the definite integral:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^n x\, dx &= \frac{\cos^{n-1} x \sin x}{n}\Bigg]_0^{\frac{\pi}{2}} + \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx \\ &= \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx\end{aligned}$$

This means

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\, dx = \frac{(n-1)}{n}\cdot\frac{(n-3)}{(n-2)}\int_0^{\frac{\pi}{2}} \cos^{n-4} x\, dx = \cdots.$$

Since $\int_0^{\frac{\pi}{2}} \cos^0 x\, dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = \frac{\pi}{2}$ and $\int_0^{\frac{\pi}{2}} \cos^1 x\, dx = \sin\frac{\pi}{2} = 1$, we can conclude that

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \left(\frac{\pi}{2}\right)\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}, & n\ even \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{1\cdot 3\cdot 5\cdots n}, & n\ odd\end{cases}.$$

8.3 Trigonometric Integrals

8.4 Trigonometric Substitutions

8.5 Integration of Rational Functions by Partial Fractions

8.6 Integral Tables and CAS

8.7 Numerical Integration

In this section we study the *Trapezoidal Rule* and *Simpson's Rule.* A key goal in this analysis is to control the possible error that is introduced when computing an approximation to an integral.

Trapezoidal Approximations

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids(사다리꼴) instead of rectangles.

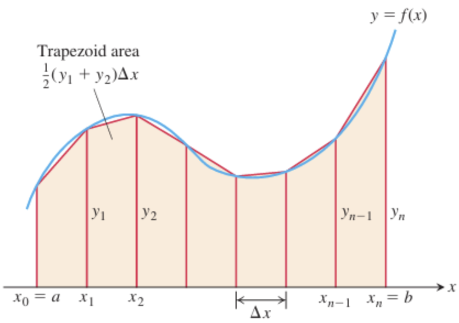

Figure 1. Trapezoid approximated on $y = f(x)$.

The subdivision points $x_0, x_1, x_2, \ldots, x_n$ is evenly spaced. We assume that the length of each subinterval is $\Delta x = \frac{b-a}{n}$. The length $\Delta x = (b-a)/n$ is called the **step size** or **mesh size.** The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{y_{i-1}+y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1}+y_i),$$

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. The area below the curve $y = f(x)$ and above the x-axis is then approximated by adding the areas of all the trapozoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0+y_1)\Delta x + \frac{1}{2}(y_1+y_2)\Delta x + \cdots + \frac{1}{2}(y_{n-2}+y_{n-1})\Delta x + \frac{1}{2}(y_{n-1}+y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + y_{n-1} + y_n) \end{aligned}$$

where $y_0 = f(a), y_1 = f(x_1), \ldots\ , y_{n-1} = f(x_{n-1}), y_n = f(b)$.

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b.

show how it leads to l`Hopital's Rule.

The Trapzoidal Rule

To approximate $\int_a^b f(x)dx$, use

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots\ , x_{n-1} = a + (n-1)\Delta x, x_n = b$$

where $\Delta x = (b-a)/n$.

Example: Use the Trapezoidal Rule with $n=4$ to estimate $\int_1^2 x^2dx$. Compare the estimate with the exact value.

Sol. Partition $[1,2]$ into four subintervals of equal length, then evaluate $y=x^2$ at each partition point. Using these y-values, $n=4$, and $\Delta x=\frac{2-1}{4}=1/4$ in the Trapezoidal Rule, we have

$$T=\frac{\Delta x}{2}(y_0+2y_1+2y_2+2y_3+y_4)=\frac{1}{8}\left(1+2\left(\frac{25}{16}\right)+2\left(\frac{36}{16}\right)+2\left(\frac{49}{26}\right)+4\right)=\frac{75}{32}=2.34375.$$

The exact value of the integral is $\int_1^2 x^2dx=\frac{7}{3}=2.333$. The T approximation overestimates the integral. The percentage error is about 0.446%.

Simpson's Rule: Approximations Using Parabolas

Another rule for approximation the definite integral of a continuous function result from using parabolas instead of the straight-line segments that produced trapezoids. As before, we partition the interval $[a,b[$ into n subintervals of equal length $h=\Delta x=(b-a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y=f(x)\geq 0$ by a parabola, as shown in the Figure below. A typical parabola passes through three consecutive points (x_{i-1},y_{i-1}), (x_i,y_i) and (x_{i+1},y_{i+1}) on the curve.

When calculating the shaded area beneath a parabola, we simplify the calculation by assuming that $x_0=-h, x_1=0$, and $x_2=h$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form $y=Ax^2+Bx+C$, so the area under it from $x=-h$ to $x=h$ is

$$A_p=\int_{-h}^{h}(Ax^2+Bx+C)dx=\left[\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right]_{-h}^{h}=\frac{2Ah^2}{3}+2Ch=\frac{h}{3}(2Ah^2+6C).$$

Since the curve passes through the three points $(-h,y_0)$, $(0,y_1)$, and (h,y_2), we also have $y_0=Ah^2-Bh+C, y_1=C, y_2=Ah^2+Bh+C$, from which we obtain $C=y_1, Ah^2-Bh=y_0-y_1, Ah^2+Bh=y_2-y_1, 2Ah^2=y_0+y_2-2y_1$.

Hence, expressing the area A_p in terms of ordinates $y_0.y_1.$ and $y_2.$ we have

$$A_p=\frac{h}{3}(2Ah^2+6C)=\frac{h}{3}((y_0+y_2-2y_1)+6y_1)=\frac{h}{3}(y_0+4y_1+y_2).$$

Now shifting the parabola horizontally to its shaded position in the Figure below does not change the area under it.

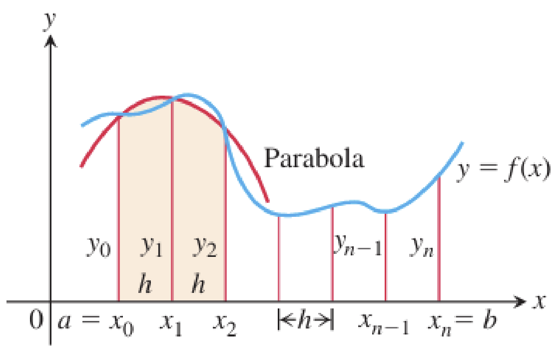

Figure 2. Parabola approximated on $y=f(x)$.

Thus, the area under the parabola through (x_0,y_0), (x_1,y_1), and (x_2,y_2) is still $\frac{h}{3}(y_0+4y_1+y_2)$. Similarly, the area under the parabola through the points (x_2,y_2), (x_3,y_3), and $(x_4.y_4)$ is $\frac{h}{3}(y_2+4y_3+y_4)$. Computing the areas under all the parabolas and adding the results gives the approximation:

$$\int_a^b f(x)dx \cong \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$
$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The result is known as Simpson's Rule. The function does not need to be positive. But the number n of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x)dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \dots\ , x_{n-1} = a + (n-1)\Delta x, x_n = b$$

where $\Delta x = (b-a)/n$. The number n is even.

Example: Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4 dx$.

Sol. Partition $[0, 2]$ into four subintervals of equal length, then evaluate $y = 5x^4$ at each partition point. Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$
$$= \frac{1}{6}\left(0 + \frac{5}{4} + 10 + \frac{405}{4} + 80\right) = 32\frac{1}{12}.$$

This estimate differs from the exact value by only $1/12$.

Error Analysis

Whenever we use an approximation technique, the issue arises as to how accurate the approximation might be. The following theorem gives formulas for estimating the errors when using the Trapezoidal Rule and Simpson's Rule. The **error** is the difference between the approximation obtained by the rule and the actual value of the definite integral $\int_a^b f(x)dx$.

Theorem 1-Error Estimates in the Trapezoidal and Simpson's Rules

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfy the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $\left|f^{(4)}\right|$ on $[a, b]$, then the error E_s in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfy the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}$$

To see why Theorem 1 is true in the case of the Trapezoidal Rule, we begin with a result from advanced calculus, which says that if f'' is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero as the square of Δx. The inequality

$$|E_T| \leq \frac{b-a}{12} \max|f''(x)|\,(\Delta x)^2$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. We usually cannot find the exact value of $\max|f''(x)|$ and must estimate an upper bound instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \leq M$ on $[a, b]$, then

$$|E_T| \leq \frac{b-a}{12} M(\Delta x)^2 = \frac{M(b-a)^3}{12n^2}.$$

To estimate the error in Simpson's Rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

approaches zero as the fourth power of Δx. This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule. The inequality

$$|E_S| = \frac{b-a}{180} \max\left|f^{(4)}(x)\right|(\Delta x)^4$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. We usually cannot find the exact value of $\max\left|f^{(4)}(x)\right|$ and must estimate an upper bound instead. If M is any upper bound for the values of $\left|f^{(4)}(x)\right|$ on $[a, b]$, then

$$|E_S| = \frac{b-a}{180} M(\Delta x)^4 = \frac{M(b-a)^5}{180n^4}.$$

8.8 Improper Integrals

Up to now, we have required definite integrals to satisfy two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. But we may encounter problems that fail to meet one or both conditions. In either case, the integrals are said to be *improper* and are calculated as limits.

Infinite Limits of Integration

Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-t/2}$ in the first quadrant. You might think this region has infinite area, but we will see that value is finite. We assign a value to the area in the following way. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$

$$A(b) = \int_0^b e^{-x/2} dx = -2e^{-\frac{x}{2}}\Big]_0^b = -2e^{-\frac{b}{2}} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$: $\lim_{b\to\infty} A(b) = \lim_{b\to\infty}\left(-2e^{-\frac{b}{2}} + 2\right) = 2.$

The value we assign to the area under the curve from 0 to ∞ is $\int_0^\infty e^{-x/2}dx = \lim_{b\to\infty}\int_0^b e^{-x/2}dx = 2.$

Definition

Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$
2. If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx.$$
3. If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.$$
where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^\infty f(x)dx$ with any convenient choice.

Example: Evaluate $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

Sol. According to the definition, we can choose $c = 0$ and write $\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$.

Next, we evaluate each improper integral on the right side of the equation above.

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Thus, $\int_{-\infty}^\infty \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$. Since $\frac{1}{1+x^2} > 0$, the improper integral can be interpreted as the finite area beneath the curve and above the x-axis.

The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if shows, the improper integral converges if $p > 1$ and diverges if $p \le 1$.

Example: For what values of p does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? What is its value if it converges?

Sol. If $p \neq 1$,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because $\lim_{b\to\infty} \frac{1}{b^{p-1}} = 0$ when $p > 1$ and ∞ when $p < 1$. Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$. If $p = 1$, the integral also diverges:

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

Integrands with Vertical Asymptotes

Another type of improper integral arises then the integrand has a vertical asymptote at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$. First we find the area of the portion from a to 1 since the curve has a vertical asymptote at $x = 0$:

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \left(2 - 2\sqrt{a}\right) = 2$$

Therefore, the area under the curve from 0 to 1 is finite and is defined to be $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2$.

Definition

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II.**

1. If $f(x)$ is continuous on $(a, b]$, and discontinuous at a, then
$$\int_a^b f(x)dx = \lim_{c\to a+} \int_c^b f(x)dx.$$
2. If $f(x)$ is continuous on $[a, b)$, and discontinuous at b, then
$$\int_a^b f(x)dx = \lim_{c\to b-} \int_a^c f(x)dx.$$
3. If $f(x)$ is discontinuous at c, where $a < c < b$, and discontinuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example: Investigate the convergence of $\int_0^1 \frac{1}{1-x} dx$.

Sol. The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$. We evaluate the integral as

$$\lim_{b\to 1-} \int_0^b \frac{1}{1-x} dx = \lim_{b\to 1-} [-\ln|1-x|]_0^b = \lim_{b\to 1-} [-\ln(1-b) + 0] = \infty.$$

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. In the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

Does the integral $\int_1^\infty e^{-x^2} dx$ converge? By definition, $\int_1^\infty e^{-x^2} dx = \lim_{b\to\infty} \int_1^b e^{-x^2} dx$. We cannot evaluate this integral directly because it is nonelementary. But we can show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2} dx$ is an increasing function of b because the area under the curve increases as b increases. Therefore, either it becomes infinite as comes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. For our function it does not become infinite: For every value of $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$ so that

$$\int_1^b e^{-x^2} dx \leq \int_1^b e^{-x} dx = -e^{-b} + e^{-1} < e^{-1}.$$

Hence, $\int_1^\infty e^{-x^2} dx = \lim_{b\to\infty} \int_1^b e^{-x^2} dx$ converges to some finite value.

The comparison of e^{-x} and e^{-x^2} was an example case of the following test.

Theorem 2- Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges.
2. If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges.

Proof If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then we have

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx, \qquad b > a.$$

From this it can be argues that

$$\int_a^\infty f(x)dx \ converges\ if \int_a^\infty g(x)dx\ converges.$$

Turning this around to its contrapositive form, this says yhay

$$\int_a^\infty g(x)dx\ diverges\ if \int_a^\infty f(x)dx\ diverges.$$

Although the theorem is stated for Type I improper integrals, a similar result is true for integrals of Type II as well.

Example: $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges because $0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2} dx$ converges.

Theorem 3-Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)dx \text{ and } \int_a^\infty g(x)dx$$

either *both converge or both diverge.*

We omit the proof of Theorem 3, which is similar to that of Theorem 2.

Example: Investigate the convergence of $\int_1^\infty \frac{1-e^{-x}}{x}dx$.

Sol. Use Limit Comparison Test with $f(x) = (1 - e^{-x})/x$ and $g(x) = 1/x$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = 1$, which is a positive finite limit. Therefore, $\int_1^\infty \frac{1-e^{-x}}{x}dx$ diverges because $\int_1^\infty \frac{1}{x}dx$ diverges.