IX. Infinite Sequences and Series

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9.1 Sequences

Representing Sequences

A sequence is a list of numbers $a_1, a_2, a_3, \dots\ , a_n, \dots$ in each order. Each of $a_1 . a_2, a_3$ and so on represents number. These are the **terms** of the sequence. The integer n is called the **index** of a_n, and indicates where a_n occurs in the list. Order is important.

Sequences can be described by writing rules that specify their terms, such as $a_n = \sqrt{n}$ or $b_n = (-1)^{n+1}\frac{1}{n}$, or by listing terms $\{a_n\} = \{\sqrt{1}, \sqrt{2}, \dots, \sqrt{n}, \dots\}$. We also sometimes write a sequence using its rule, as with $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$.

Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence such as $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\ , \frac{1}{n}, \dots\right\}$ whose terms approach 0 as n gets large. On the other hand, sequences like $\{\sqrt{1}, \sqrt{2}, \dots, \sqrt{n}, \dots\}$ have terms that get larger than any number as n increases, and sequences like $\{1, -1, 1, -1, 1, -1\ , \dots\ , (-1)^{n+1}, \dots\}$ bounce back and forth between 1 and -1, never converging on a single value.

The following definition captures the meaning of having a sequence converge to a limiting value.

> **Definitions**
>
> The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that
>
> $$|a_n - L| < \varepsilon \quad whenever \quad n > N.$$
>
> If no such number L exists, we say that $\{a_n\}$ **diverges.**
>
> If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence.

Example: Show that the sequence $\left\{\frac{1}{n}\right\}$ converges to 0: Let $\varepsilon > 0$ be given. We should show that there exists an integer N such that $\left|\frac{1}{n} - 0\right| < \varepsilon$ whenever $n > N$. The inequality $\left|\frac{1}{n} - 0\right| < \varepsilon$ will hold(만족하다) if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. If N is any integer greater than $1/\varepsilon$, the inequality will hold for all $n > N$.

The sequence $\{\sqrt{n}\}$, however, diverges, but for a different reason. As n increases, tis terms become larger than any fixed number. We describe the behavior of this sequence by writing $\lim_{n\to\infty}\sqrt{n} = \infty$. In writing infinity as the limit of a sequence, we are merely using a notation that captures the idea that a_n eventually gets and stays larger than any fixed number as n gets large. The terms of a sequence might also decrease to negative infinity.

Definitions

The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad or \quad a_n \to \infty.$$

Similarly, if for every number m there is an integer N such that for all n larger than N, $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad or \quad a_n \to -\infty.$$

A sequence may diverge without diverging to infinity or negative infinity, such as $\{1,-1,1,-1,1,-1, \dots, (-1)^{n+1}, \dots\}$.

Calculating Limits of Sequences

Theorem 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} a_n = B$.

1. Sum Rule: $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. Difference Rule $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. Constant Multiple Rule $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. Product Rule $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. Quotient Rule $\lim_{n\to\infty}(a_n / b_n) = A/B$ if $B \neq 0$

Be cautious in applying Theorem 1. It does not say, for example, that each of the sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum $\{a_n + b_n\}$ has a limit. For instance, $\{a_n\} = \{1,2,3,\dots\}$ and $\{b_n\} = \{-1,-2,-3,\dots\}$ both diverge, but their sum clearly converges to 0.

One consequence of Theorem 1 is that every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. Suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$, Then by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence $\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$ converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

The next theorem is the sequence version of the Sandwich Theorem.

Theorem 2 – The Sandwich Theorem for Sequences

Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Proof For all $\varepsilon > 0$, there exists N_1, N_2 such that $n > N_1 \to |a_n - L| < \varepsilon, n > N_2 \to |c_n - L| < \varepsilon$. Then, if $n > \max\{N_1, N_2\}$, $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \Rightarrow |b_n - L| < \varepsilon$. So, $\lim_{n\to\infty} b_n = L$.

An immediate consequence of Theorem 2 is that if $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \le b_n \le c_n$.

The application of Theorems 1 and 2 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence.

Theorem 3 – The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

Proof Let $\varepsilon > 0$. f is continuous at L. So there exists δ that $|x - L| < \delta \to |f(x) - f(L)| < \varepsilon$. Also, since $a_n \to L$ & $\delta > 0$, there exists N that $n \ge N \to |a_n - L| < \delta$. So, $n \ge N \to |a_n - L| < \delta \to |f(a_n) - f(L)| < \varepsilon$. This means $f(a_n) \to f(L)$.

Example: The sequence $\{\frac{1}{n}\}$ converges to 0. By taking $a_n = \frac{1}{n}, f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f\left(\frac{1}{n}\right) \to f(L) = 2^0 = 1.$ The sequence $\{2^{1/n}\}$ converges to 1.

Using L`Hopital's Rule

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then

$$\lim_{n\to\infty} a_n = L \qquad whenever \qquad \lim_{x\to\infty} f(x) = L$$

Proof Suppose that $\lim_{x\to\infty} f(x) = L$. Then for each positive number ε there is a number M such that

$$|f(x) - L| < \varepsilon \quad whenever \quad x > M.$$

Let N be an integer greater than M and greater than or equal to n_0. Since $a_n = f(n)$, it follows that for all $n > N$ we have

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

When we use l`Hopital's Rule to find the limit of a sequence, we often treat n as a continuous real variable and differentiate directly with respect to n. This saves us from having to rewrite the formula for a_n.

Example: Does the sequence whose nth term is $a_n = \left(\frac{n+1}{n-1}\right)^n$ converge? If so, find $\lim_{n\to\infty} a_n$.

Sol. The limit leads to the indeterminate form 1^∞. We can apply l`Hopital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n.

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n\ln\left(\frac{n+1}{n-1}\right).$$

Then, $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} = \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} = \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 3 tells us that $a_n = e^{\ln a_n} \to e^2$. The sequence converges to e^2.

Commonly Occurring Limits

Theorem 5

The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Recursive Definitions

So far, we have calculated each a_n directly from the value of n. But sequences are often defined **recursively**(재귀적으로) by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula,** for calculating any later term from terms that precede it.

Bounded Monotonic Sequences

Definitions

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \le M$ for all n. The number M is an **upper bound** for $\{a_n\}$.

If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \ge m$ for all n. The number m is a **lower bound** for $\{a_n\}$.

If m is a lower bound for $\{a_n\}$ but no number greater than m is an lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, we say that $\{a_n\}$ is an **unbounded** sequence.

If a sequence $\{a_n\}$ converges to the number L, then by definition there is a number N such that $|a_n - L| < 1$ if $n > N$. That is,

$$L - 1 < a_n < L + 1 \qquad for \;\; n > N.$$

If M is a number larger than $L + 1$ and all the finitely many numbers, $a_1, a_2, \dots\ , a_N$, then for every index n we have $a_n < M$ so that $\{a_n\}$ is bounded from above. Similarly, if m if a smaller number than $L - 1$ and all the numbers, $a_1, a_2, \dots\ , a_N$, then m is a lower bound of the sequence. Therefore, all convergent sequences are bounded.

Although it is true that every convergent sequence is bounded, there are bounded sequences that fail to converge. One example is the bounded sequence $\{(-1)^{n+1}\}$. The problem here is that some bounded sequences bounce around in the band determined by any lower bound m and any upper bound M. An important type of sequence that does not behave that way is one for which each term is at least as large, or at least as small as its predecessor.

Definitions

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \cdots$.

The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n.

The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

A nondecreasing sequence that is bounded from above always has a least upper bound. Likewise, a nonincreasing sequence bounded from below always has a greatest lower bound. These results are based on the *completeness property* of the real numbers.

Theorem 6 – The Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Proof Suppose $\{a_n\}$ is nondecreasing, L is its least upper bound, and we plot the points $(1, a_1), (2, a_2), \ldots, (n, a_n), \ldots$ in the xy-plane. If M is an upper bound of the sequence, all these points will lie on or below the line $y = M$.

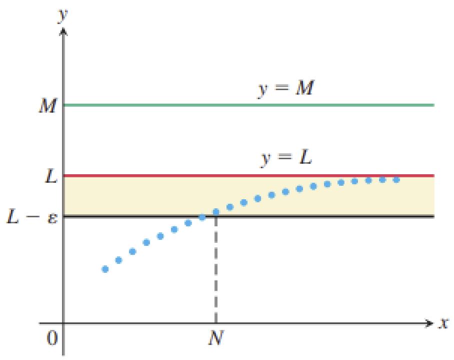

Figure 1. If the terms of a nondecreasing sequence have an upper bound M, they have a limit $L \leq M$.

The line $y = L$ is the lowest such line. None of the points (n, a_n) lines above $y = L$, but some do lie above any lower line $y = L - \varepsilon$, if ε is a positive number (because $L - \varepsilon$ is not an upper bound). The sequence converges to L because $a_n \leq L$ for all values of n, and given any $\varepsilon > 0$, there exists at least one integer N for which $a_N > L - \varepsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that $a_n \geq a_N > L - \varepsilon$ for all $n \geq N$. Thus, all the numbers a_n beyond the Nth number line within ε of L. This is precisely the condition for L to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar.

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

9.2 Infinite Series

Definitions

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

Geometric Series

Geometric Series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive or negative. If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of a. If $r = -1$, the series diverges because the nth partial sums alternate between a and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \\ s_n(1-r) &= a - ar^n = a(1-r^n) \\ s_n &= \frac{a(1-r^n)}{1-r}, \qquad (r \neq 1). \end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$, so $s_n \to a/(1-r)$ in this case. On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1-r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| > 1$, the series diverges.

The formula $a/(1-r)$ for the sum of a geometric series applies only when the summation index begins with $n =$

1 in the expression $\sum_{n=1}^{\infty} ar^{n-1}$.

Example: The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$ is a geometric series with $a = 5$ and $r = -1/4$. It converges to $\frac{a}{1-r} = \frac{5}{1+\left(\frac{1}{4}\right)} = 4$.

Example: Find the sum of the telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Sol. We look for a patter in the sequence of partial sums that might lead to s formula for s_k. The key observation is the partial fraction decomposition $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. so $\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ and $s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$. Removing parentheses and canceling adjacent terms of opposite sign collapses the sum to $s_k = 1 - \frac{1}{k+1}$. We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} - 1.$$

The nth-Term Test for a Divergent Series

We now show that $\lim_{n\to\infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let S represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the nth partial sum. When n is large, both s_n and $s_{n-1}$are close to S, so their difference, a_n, is close to zero. More formally, $a_n = s_n - s_{n-1} \to S - S = 0$. This establishes the following theorem.

Theorem 7

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 7 leads to a test for detecting the kind of divergence that occurred in Example 6.

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

Combining Series

Theorem 8

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. Difference Rule $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. Constant Multiple Rule $\sum k a_n = k \sum a_n = kA$ (any number k)

Proof The three rules for series follow from the analogous rules for sequences. To prove the Sum Rule for series, let $A_n = a_1 + a_2 + \cdots + a_n$, $B_n = b_1 + b_2 + \cdots + b_n$. Then the partial sums of $\sum(a_n + b_n)$ are

$$\begin{aligned} s_n &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = A_n + B_n. \end{aligned}$$

Since $A_n \to A$ and $B_n \to B$, we have $s \to A + B$ by the Sum Rule for sequences. The proof for the Difference Rule is similar.

To prove the Constant Multiple Rule for series, observe that the partial sums of $\sum ka_n$ form the sequence

$$s_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n,$$

which converges to kA by the Constant Multiple Rule for sequences.

As corollaries of Theorem 8, we have the following results. We omit the proofs.

1. Every nonzero constant multiple of a divergent series diverges.
2. if $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Caution: Remember that $\sum(a_n + b_n)$ can converge even if both $\sum a_n$ and $\sum b_n$ diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverges, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

The convergence or divergence of a series not affected by its first few terms. Only the "tail" of the series, the part that remains when we sum beyond some finite number of initial terms, influences whether it converges or diverges.

Reindexing

If we preserve the order of its terms, we can reindex any series without altering its convergence.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h}, \qquad \sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h}$$

We usually give preference to indexings that lead to simple expressions.

9.3 The Integral Test

Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_{n+1}$, so $s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$.

Since the partial sums form a nondecreasing sequence, the Monotonic Sequence Theorem gives the following result.

Corollary of Theorem 6

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges *if and only if* its partial sums are bounded from above.

Example: As an application of the above corollary, consider the **harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots.$$

Although the nth term $1/n$ does go to zero, the series diverges because there is no upper bound for its partial sums.

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots.$$

The sum of the third and fourth terms is greater than $\frac{1}{4}*2=\frac{1}{2}$. The sum of the next four terms is also greater than $1/2$. This continues, that in general, the sum of 2^n terms ending with $1/2^{n+1}$ is greater than $\frac{2^n}{2^{n+1}}=\frac{1}{2}$. If $n=2^k$, the partial sum s_n is greater than $k/2$, so the sequence of partial sums is not bounded from above. The harmonic series diverges.

The Integral Test

Theorem 9 – The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n=f(n)$, where f is a continuous, positive, decreasing function of x for all $x\geq N$ (N *a positive integer*). Then the series $\sum_{n=N}^{\infty}a_n$ and the integral $\int_N^{\infty}f(x)dx$ both converge, or both diverge.

Proof We establish the test for the case $N=1$. The proof for general N is similar.

Start with the assumption that f is a decreasing function with $f(n)=a_n$ for every n. This leads us to observe that the rectangles in the Figure 2-(a) below, which have areas $a_1,a_2,\ldots,a_n$, collectively enclose more area than that under the curve $y=f(x)$ from $x=1$ to $x=n+1$. That is,

$$\int_1^{n+1}f(x)dx\leq a_1+a_2+\cdots+a_n.$$

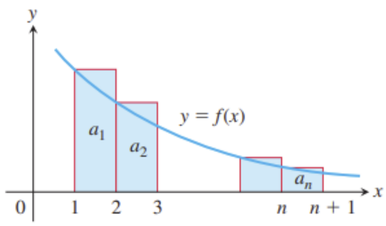

Figure 2-(a).

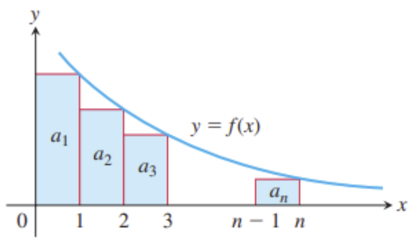

Figure 2-(b).

In Figure 2-(b) the rectangles have been faced to the left instead of the right, If we momentarily disregard the first rectangle of area a_1, we see that

$$a_2+a_3+\cdots+a_n\leq\int_1^n f(x)dx.$$

If we include a_1, we have $a_1+a_2+a_3+\cdots+a_n\leq a_1+\int_1^n f(x)dx$. Combining these results gives

$$\int_1^{n+1}f(x)dx\leq a_1+a_2+\cdots+a_n\leq a_1+\int_1^n f(x)dx.$$

These inequalities hold for each n, and continue to hold as $n \to \infty$.

If $\int_1^\infty f(x)dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^\infty f(x)dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are either both finite or both infinite.

Example: Show that the **p-series** $\sum_{n=1}^\infty \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ (p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Sol. If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1},$$

the series converges by the Integral Test. We emphasize that the sum of the p-series is not $1/(p-1)$. The series converges, but the sum is unknown.

If $p \le 0$, the series diverges by the nth-term test.

If $0 < p < 1$, then $1 - p > 0$ and $\int_1^\infty \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) = \infty$. Therefore, the series diverges by the Integral Test.

If $p = 1$, we have the divergent harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$.

In summary, we have convergence for $p > 1$ but divergence for all other values of p.

The p-series with $p = 1$ is the **harmonic series**. The p-series Test shows that the harmonic series is just barely divergent.

Error Estimation

For most convergent series, we cannot easily find the total sum. But we can estimate the sum by adding the first n terms to get s_n, but we need to know how far off s_n is from the total sum S. An approximation to a function or to a number is more useful when it is accompanied by a bound on the size of the worst possible error that could occur.

Suppose that a series $\sum a_n$ with positive terms is shown to be convergent by the Integral Test, and we want to estimate the size of the **remainder R_n** measuring the difference between the total sum S of the series and its nth partial sum s_n. That is, we wish to estimate $R_n = S - s_n = a_{n+1} + a_{n+2} + \cdots$.

To get a lower bound for the remainder, we compare the sum of the areas of the rectangles with the are under the curve $y = f(x)$ for $x \ge n$. We see that $R_n = a_{n+1} + a_{n+2} + \cdots \ge \int_{n+1}^\infty f(x)dx$. (Figure 3-(a).)

Similarly, we find an upper bound with $R_n = a_{n+1} + a_{n+2} + \cdots \le \int_n^\infty f(x)dx$. (Figure 3-(b).)

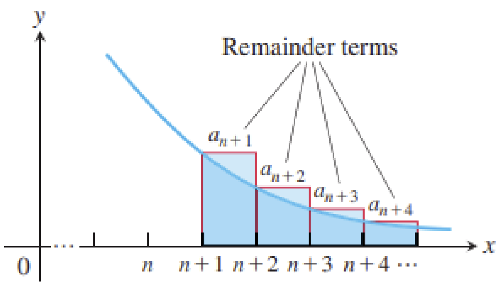

Figure 3-(a).

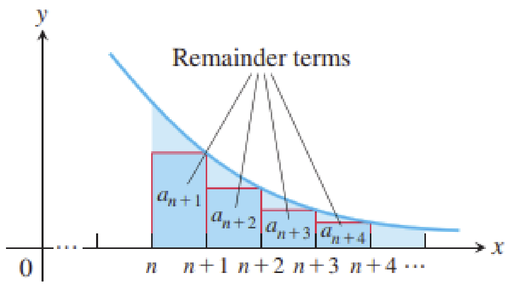

Figure 3-(b).

These comparisons prover the following result, giving bounds on the size of the remainder.

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

If we add the partial sum s_n to each side of the inequalities, we get

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq S \leq s_n + \int_{n}^{\infty} f(x)dx$$

since $s_n + R_n = S$.

Example: Estimate the sum of the series $\sum(\frac{1}{n^2})$ using the inequalities above and $n = 10$.

Sol. We have that $\int_n^{\infty} \frac{1}{x^2}dx = \frac{1}{n}$. Using this result with the inequalities above, we get $s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}$.

Taking $s_{10} \cong 1.54977$, these last inequalities give $1.64068 \leq S \leq 1.64977$. If we approximate the sum S by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cong 1.6452.$$

The error in this approximation is then less than half the length of the interval, which is less than 0.005. The true value, however, is $\pi^2/6 \cong 1.64493$.

9.4 Comparison Tests

The Direct Comparison Test

Theorem 10 – Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$ for all n. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Proof The series $\sum a_n$ and $\sum b_n$ have nonnegative terms. The Corollary of Theorem 6 tells us that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number M. The partial sums $\sum_{n=1}^{N} a_n$ are all bounded from above by $M = \sum b_n$ since $s_N = a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N \leq \sum_{n=1}^{\infty} b_n = M$.

Since the partial sums of $\sum a_n$ are bounded from above, the Corollary of Theorem 6 implies that $\sum a_n$ converges. We conclude that when $\sum b_n$ converges, then so does $\sum a_n$.

In Part (2), where we assume that $\sum a_n$ diverges, the partial sums of $\sum_{n=1}^{N} b_n$ are not bounded from above. If they were, the partial sums from $\sum a_n$ would also be bounded from above, since $a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N$, and this would mean that $\sum a_n$ converge. We conclude that if $\sum a_n$ diverges, then so does $\sum b_n$.

Example: The series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges because its nth term $\frac{5}{5n-1} > \frac{1}{n}$ is greater than the nth term of the divergent harmonic series. The series $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ converges because its terms are all positive and less than or equal to the corresponding terms of $1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$. The geometric series on the left converges and we have $1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\left(\frac{1}{2}\right)} = 3$.

The Limit Comparison Test

Theorem 11 – Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N *an integer*)

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof

- Part (1): Since $\frac{c}{2} > 0$, there exists an integer N such that $\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2}$ whenever $n > N$. Thus, for $n > N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}, \qquad \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2}, \qquad \left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n.$$

If $\sum b_n$ converges, then $\sum (3c/2) b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum (c/2) b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test.

Example: Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Sol. Because $\ln n$ grows slowly than n^c for any positive constant c, we can compare the series to a convergent p-series. To get the p-series, we see that $\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$ for n sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} = \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0. \,(l\text{'}Hopital\text{'}s\ Rule\ applied)$$

Since $\sum b_n = \sum (1/n^{5/4})$ is a p-series with $p > 1$, it converges. Therefore $\sum a_n$ converges by Part 2 of the Limit Comparison Test.

9.5 Absolute Convergence; The Ratio and Root Tests

Absolute Convergence

For a general series with both positive and negative terms, we can apply the tests for convergence studied before to the series of absolute values of its terms.

Definition

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent)** if the corresponding series of absolute values, $\sum |a_n|$, converges.

An absolutely convergent series is convergent as well, which ne now will prove.

Theorem 12 – The Absolute Convergence Test

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof For each n, $-|a_n| \le a_n \le |a_n|$ so $0 \le a_n + |a_n| \le 2|a_n|$.

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty}|a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

Example: For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$ which contains both positive and negative terms, the corresponding series of absolute values is $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots$, which converges by comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ because $|\sin n| \le 1$ for every n. The original series converges absolutely; therefore if converges.

The Ratio Test

Theorem 13 – The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ if infinite, (c) the test is *inconclusive* if $\rho = 1$.

Proof

(a) $\rho < 1$: let r be a number between ρ and 1. Then the number $\varepsilon = r - \rho$ is positive. Since

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = r, \qquad when\ n \ge N.$$

Hence

$$\begin{aligned}&|a_{N+1}| < r|a_N|,\\ &|a_{N+2}| < r|a_{N+1}| < r^2|a_N|,\\ &\vdots\\ &|a_{N+m}| < r|a_{N+m-1}| < r^m|a_M|\end{aligned}$$

Therefore,

$$\sum_{m=N}^{\infty}|a_m| = \sum_{m=0}^{\infty}|a_{N+m}| \le \sum_{m=0}^{\infty}|a_N|r^m = |a_N|\sum_{m=0}^{\infty} r^m.$$

The geometric series on the right-hand side converges because $0 < r < 1$, so the series of absolute values $\sum_{m=N}^{\infty}|a_m|$ converges by the Direct Comparison Test. Because add or deleting finitely many terms in a series does not affect its convergence or divergence property, the series $\sum_{n=1}^{\infty}|a_n|$ also converges. That is, the series $\sum a_n$ is absolutely convergent.

(b) $1 < \rho < \infty$: From some index M on,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad and \qquad |a_M| < |a_{M+1}| < |a_{M+2}| < \cdots.$$

The terms of the series do not approach zero as n becomes infinite, thus the series diverges by the nth-Term Test.

(c) $\rho = 1$: The two series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ have $\rho = 1$, yet the first series diverges, whereas the second series converges.

Example: Investigate the convergence of $\sum_{n=1}^{\infty}\frac{(2^n+5)}{3^n}$ using the Ratio Test.

Sol. $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{1}{3}\cdot\frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}} \to \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}$. The series converges absolutely (and thus converges) because $\rho = 2/3 < 1$. This does not means that $2/3$ is the sum of the series.

The Root Test

Theorem 14 – The Root Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \rho.$$

Then (a) the series *converges absolutely* if $\rho < 1$, (b) the series *diverges* if $\rho > 1$ or ρ if infinite, (c) the test is *inconclusive* if $\rho = 1$.

Proof

(a) $\rho < 1$: Choose an $\varepsilon > 0$ small enough that $\rho + \varepsilon < 1$. Since $\sqrt[n]{|a_n|} \to \rho$, the terms $\sqrt[n]{|a_n|}$ eventually get to within ε of ρ. So there exists an index M such that

$$\sqrt[n]{|a_n|} < \rho + \varepsilon \qquad when\ n \geq M.$$

Then it is also true that $|a_n| < (\rho+\varepsilon)^n$ for $n \geq M$. Now, $\sum_{n=M}^{\infty}|a_n|$ converges. Adding finitely many terms to a series does not affect its convergence and divergence, so the series $\sum_{n=1}^{\infty}|a_n|$ also converges. Therefore, $\sum a_n$ converges absolutely.

(b) $1 < \rho < \infty$: For all indices beyond some integer M, we have $\sqrt[n]{|a_n|} > 1$, so that $|a_n| > 1$ for $n > M$. The terms of the series do not converge to zero. The series diverges by the nth-Term Test.

(c) $\rho = 1$: The two series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ have $\rho = 1$, yet the first series diverges, whereas the second series converges.

Example: Consider the series with terms $a_n = \begin{cases} n/2^n, & n\ odd \\ 1/2^n, & n\ even \end{cases}$.

Sol. We apply the Root Test, finding that $\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n\ odd \\ 1/2, & n\ even \end{cases}$. Therefore, $\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}$. Since $\sqrt[n]{n} \to$

1, we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test.

9.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series.** The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots$ specifically is called the **alternating harmonic series**.

We investigate the convergence of the alternating series by applying the Alternating Series Test. This test is for *convergence* of an alternating series and cannot be used to conclude that such a series diverges.

Theorem 15 – The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$. ($\lim_{n\to\infty} u_n = 0$)

Proof We look at the case where $u_1, u_2, u_3, \ldots$ is nonincreasing, so that $N = 1$. If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L. \quad (by\ Theorem\ 6)$$

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m\to\infty} u_{2m+1} = 0, \quad \lim_{m\to\infty} s_{2m+1} = L.$$

Therefore, $\lim_{n\to\infty} s_n = L$.

Example: The alternating harmonic series clearly satisfies the three requirements of Theorem 15 with $N = 1$; it therefore converges by the Alternating Series Test. Figure below shows histograms of the partial sums of the divergent harmonic series and those of the convergent alternating harmonic series. The alternating harmonic series converges to ln 2.

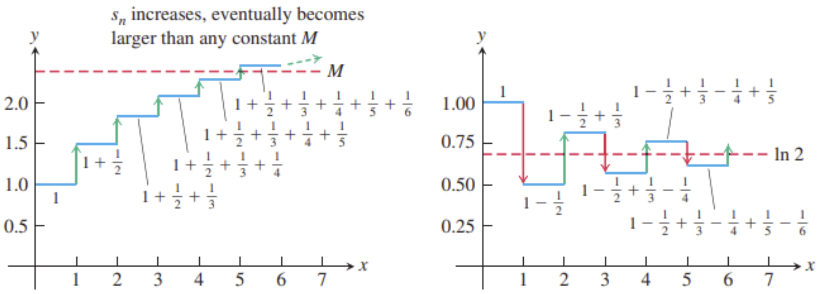

Figure 4. Convergence/Divergence of Harmonic / Alternating Harmonic Series.

Rather than directly verifying the definition $u_n \geq u_{n+1}$, a second way to show that the sequence $\{u_n\}$ is nonincreasing is to define a differentiable function $f(x)$ satisfying $f(n) = u_n$. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is nonincreasing for $x \geq N$. It follows that $f(n) \leq f(n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

Theorem 16 – The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{(n+1)}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

Proof

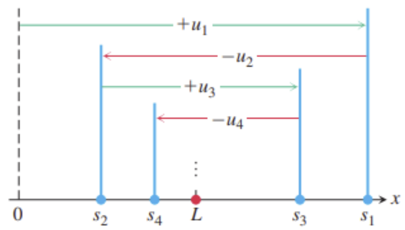

Figure 5. A graphical interpretation of the partial sums of an alternating series.

Starting from the origin of the x-axis, we lay off the positive distance $s_1 = u_1$. To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2. Since $u_2 \leq u_1$, we do not back up any farther that the origin. We continue in this seesaw pattern, backing up or going forward as the signs in the series demand. But for $n \geq N$, each forward or backward step is shorter than (or at most the same size as) the preceding step because $u_{n+1} \leq u_n$. And since the nth term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller. We oscillate back and forth across the limit L, and the amplitude of oscillation approaches zero. The limit L lies between any two successive sums s_n and s_{n+1} and hence differs from s_n by and amount less than u_{n+1}.

So, $|L - s_n| < u_{n+1}$ for $n \geq N$.

The sign of the remainder can be derived from the following equations:

The unused terms are $\sum_{j=n+1}^{\infty}(-1)^{j+1}a_j = (-1)^{n+1}(a_{n+1} - a_{n+2}) + (-1)^{n+3}(a_{n+3} - a_{n+4}) + \ldots$

$= (-1)^{n+1}[(a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots]$. Each grouped term is positive, so the remainder has the same sign as $(-1)^{n+1}$, which is the sign of the first unused term.

Conditional Convergence

Definition

A series that is convergent but not absolutely convergent is called **conditionally convergent**.

The alternating harmonic series is conditionally convergent or **converges conditionally**.

Example: p-series with $p > 1$ converges absolutely, whilst p-series with $0 < p \leq 1$ converges conditionally by the alternating series test.

Rearranging Series

Theorem 17 – The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \dots, b_n, \dots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Proof Let ε be a positive real number, let $L = \sum_{n=1}^{\infty} a_n$, and let $s_k = \sum_{n=1}^{k} a_n$. For some index N_1 and for some index $N_2 \geq N_1$, $\sum_{n=1}^{\infty} |a_n| < \frac{\varepsilon}{2}$ and $\left|s_{N_2} - L\right| < \frac{\varepsilon}{2}$. This is because:

Since $\sum_{n=1}^{\infty} |a_n|$ converges, say to M, for $\varepsilon > 0$ there is an integer N_1 such that $\left|\sum_{n=1}^{N_1-1} |a_n| - M\right| < \frac{\varepsilon}{2} \Leftrightarrow \left|\sum_{n=1}^{N_1-1} |a_n| - \left(\sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n|\right)\right| < \frac{\varepsilon}{2} \Leftrightarrow \left| -\sum_{n=N_1}^{\infty} |a_n| \right| < \frac{\varepsilon}{2} \Leftrightarrow \sum_{n=N_1}^{\infty} |a_n| < \frac{\varepsilon}{2}$.

Also, $\sum_{n=1}^{\infty} a_n$ converges to L, so for $\varepsilon > 0$ there is an integer N_2 (which we can choose greater than or equal to N_1) such that $\left|S_{N_2} - L\right| < \frac{\varepsilon}{2}$. Therefore, $\sum_{n=1}^{\infty} |a_n| < \frac{\varepsilon}{2}$ and $\left|s_{N_2} - L\right| < \frac{\varepsilon}{2}$.

Since all the terms $a_1, a_2, \dots, a_N$, appear somewhere in the sequence $\{b_n\}$, there is an index $N_3 \geq N_2$ such that if $n \geq N_3$, then $\left(\sum_{n=1}^{\infty} b_k\right) - s_{N_2}$ is at most a sum of terms a_m with $m \geq N_1$. Therefore, if $n \geq N_3$,

$$\left|\sum_{k=1}^{n} b_k - L\right| \leq \left|\sum_{k=1}^{n} b_k - s_{N_2}\right| + \left|s_{N_2} - L\right| \leq \sum_{k=N_1}^{\infty} |a_k| + \left|s_{N_2} - L\right| < \varepsilon.$$

This shows that if $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$. The series $\sum_{n=1}^{\infty} |a_n|$ converges absolutely, say to M. Thus, there exists N_1 such that $\left|\sum_{n=1}^{k} |a_n| - M\right| < \varepsilon$ whenever $k > N_1$. Now all the terms in the sequence $\{|b_n|\}$ appear in $\{|a_n|\}$. Sum together all the terms in $\{|b_n|\}$, in order, until you include all the terms $\{|a_n|\}_{n=1}^{N_1}$, and let N_2 be the largest index in the sum $\sum_{n=1}^{N_2} |b_n|$ so obtained. Then $\left|\sum_{n=1}^{N_2} |b_n| - M\right| < \varepsilon$ as well, so $\sum_{n=1}^{\infty} |b_n|$ converges to M.

Caution If we rearrange the terms of a conditionally convergent series, we can get different results.

Summary of Tests to Determine Convergence or Divergence

1. **The nth-Term Test for Divergence**: Unless $a_n \to 0$, the series diverges.
2. **Geometric series**: $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.

3. ***p*-series**: $\sum 1/n^p$ converges if $p > 1$; otherwise, it diverges.
4. **Series with nonnegative terms**: Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms**: Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember that absolute convergence implies convergence.
6. **Alternating Series**: $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

9.7 Power Series

Power Series and Convergence

Definition

A **power series about $\boldsymbol{x = 0}$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

A **power series about $\boldsymbol{x = a}$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots.$$

in which the **center $\boldsymbol{a}$** and the **coefficients $\boldsymbol{c_0, c_1, c_2, \ldots, c_n}$**, ... are constants.

We will see that a power series defines a function $f(x)$ on a certain interval where it converges. Moreover, this function will be shown to be continuous and differentiable over the interior of that interval.

The following example will show how we test a power series for convergence by using the Ratio Test to see where it converges and diverges.

Example: For what values of x do the power series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ converge?

Sol. $\left|\frac{u_{n+1}}{u_n}\right| = \frac{2n-1}{2n+1} x^2 \to x^2$. By the Ratio Test, the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$.At $x = 1$ the series becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions of convergence. Therefore, the series converges for $-1 \le x \le 1$ and diverges elsewhere.

Example: For what values of x do the power series $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\, x^2 + 3!\, x^3 + \cdots$ converge?

Sol. $\left|\frac{u_{n+1}}{u_n}\right| = (n+1)|x| \to \infty$ unless $x = 0$, so the series diverges for all values of x except $x = 0$.

Theorem 18 – The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Proof The proof uses the Direct Comparison Test, with the given series compartied to a converging geometric series.

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$ by the nth-Term Test. Hence, there is an integer N such that $|a_n c^n| < 1$ for all $n > N$, so that $|a_n| < \frac{1}{|c|^n}$ for $n > N$.

Now take any x such that $|x| < |c|$, so that $|x|/|c| < 1$. Multiplying both sides of the inequality above with $|x|^n$ gives $|a_n||x|^n < \frac{|x|^n}{|c|^n}$ for $n > N$. Since $|x|/|c| < 1$, the geometric series $\sum_{n=0}^{\infty} \frac{|x|^n}{|c|^n}$ converges. Therefore, by the DCT, the series $\sum_{n=0}^{\infty} a_n |x|^n$ converges, so the original power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < |c|$.

Now supposer that the series $\sum_{n=0}^{\infty} a_n d^n$ diverges. If x is a number with $|x| > |d|$ and the series converges at x, then the first half of the theorem shows that the series should also converges at d, contrary to our assumption. So, the series diverges for $|x| > |d|$.

The Radius of Convergence of a Power Series

Corollary to Theorem 18

The convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x. ($R = \infty$)
3. The series converges at $x = a$ and diverges elsewhere. ($R = 0$)

Proof We first consider the case where $a = 0$, so that we have a power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0. If the series converges everywhere; then we are in Case 2. If the series converges only at $x = 0$; then we are in Case 3. Otherwise, there is a nonzero number d such that $\sum_{n=0}^{\infty} c_n d^n$ diverges.

Let S be the set of values of x for which $\sum_{n=0}^{\infty} c_n x^n$ converges. The set S does not include any x with $|x| > |d|$, since Theorem 18 implies the series diverges at all such values. So, the set S is bounded. By the Completeness Property of the Real Numbers, S has a least upper bound R. Since we are not in Case 3, the series converges at some number $b \neq 0$ and, by Theorem 18, also on the open interval $(-|b|, |b|)$. Therefore, $R > 0$.

If $|x| < R$ then there is a number c in S with $|x| < c < R$, since otherwise R would not be the upper bound for S. The series converges at c since $c \in S$, so by Theorem 18 the series converges absolutely at x.

Now suppose $|x| > R$. If the series converges at x, then Theorem 18 implies it converges absolutely on the open interval $(-|x|, |x|)$, so that S contains this interval. Since R is an upper bound for S, it follows that $|x| \leq R$, which is a contradiction. So, if $|x| < R$ then the series diverges. This proves the theorem for power series centered at $a = 0$.

For a power series centered at an arbitrary point $x = a$, set $x' = x - a$ and repeat the argument above, replacing $x = x'$. Since $x' = 0$ when $x = a$, convergence of the series $\sum_{n=0}^{\infty} c_n (x')^n$ on a radius R open interval centered at $x' = 0$ corresponds to convergence of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ on a radius R open interval centered at $x = a$.

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is called the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the series.

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely, $|x-a|<R$ or $a-R<x<a+R$.
2. If R is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, of the Alternating Series Test.
3. If R is finite, the series diverges for $|x-a|>R$ because the nth term does not approach zero for those values of x.

Operations on Power Series

Theorem 19 – Series Multiplication for Power Series

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Theorem 20

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)|<R$.

Theorem 21 – Term-by-Term Differentiation

If $\sum c_n (x-a)^n$ has a radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \quad \textit{on the interval} \quad a-R<x<a+R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Proof The proof is made up of 8 parts.

a. We can show that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ has the same interval of convergence. First, we show that if $0 < t < R$, $\sum_{n=0}^{\infty} n a_n t^{n-1}$ converges. Let c that is $t < c < R$. Since $n^{1/n} \to 1$, $n^{1/n} t < c$ for large enough n. This leads to $|n a_n t^{n-1}| = |a_n| \left(n^{1/n} t\right)^n / t \leq |a_n| c^n / t$. So, by DCT, $\sum_{n=0}^{\infty} n a_n x^{n-1}$, and the radius of convergence is equal or larger than R.

   However, $\sum_{n=0}^{\infty} |n a_n x^{n-1}|$ converging means that $\sum_{n=0}^{\infty} |a_n x^{n-1}|$ also converges by DCT, so

   $\sum_{n=0}^{\infty} a_n x^n = a_0 + x \sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges. this means the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is equal or smaller than R. Adding these two arguments give you that $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} n a_n x^{n-1}$ has the same radius of convergence R, and has the same interval of convergence of $(-R, R)$.

b. We can use the MVT to show that $\frac{(x+h)^n - x^n}{h} = n c_n^{n-1}$ for some c_n between x and $x + h$ for $n = 1, 2, 3, \ldots$. Let $h(x) = x^n$. Then $h'(x) = n x^{n-1}$. Applying $a = x$ and $b = x + h$ to $h(x)$ gives the result above.

c. Now, $\left| g(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \sum_{n=1}^{\infty} n a_n x^{n-1} - \frac{\sum_{n=0}^{\infty} a_n (x+h)^n - \sum_{n=0}^{\infty} a_n x^n}{h} \right| = \left| \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h} \right| = \left| \left(\sum_{n=2}^{\infty} n a_n x^{n-1} + a_1 \right) - \left(\sum_{n=2}^{\infty} a_n \frac{(x+h)^n - x^n}{h} - 0 - a_1 \right) \right| = \sum_{n=2}^{\infty} n |a_n| \left| x^{n-1} - \frac{1}{n} \cdot \frac{(x+h)^n - x^n}{h} \right|$. By part (b), this is equal to $\sum_{n=2}^{\infty} n |a_n| |x^{n-1} - c_n^{n-1}| = |\sum_{n=2}^{\infty} n a_n (x^{n-1} - c_n^{n-1})|$. This derivation is possible because $f(x)$ and $g(x)$ has the same interval of convergence.

d. We can use the MVT to show that $\frac{x^{n-1} - c_n^{n-1}}{x - c_n} = (n-1) d_{n-1}^{n-2}$. for some d_{n-1} between x and $x + h$ for $n = 1, 2, 3, \ldots$. Let $h_1(x) = x^{n-1}$. Then $h_1'(x) = (n-1) x^{n-2}$. Applying $a = x$ and $b = c_n$ to $h_1(x)$ gives the result above.

e. It is given that $x < d_{n-1} < c_n < x + h$. So we can also see that $|x - c_n| < |h|$ and $|d_{n-1}| \leq \alpha = \max\{|x|,\ |x+h|\}$.

f. Now use the results of part (c), (d), and (e). $\left| g(x) - \frac{f(x+h) - f(x)}{h} \right| = |\sum_{n=2}^{\infty} n a_n (x^{n-1} - c_n^{n-1})| = |\sum_{n=2}^{\infty} n a_n (x - c_n)((n-1) d_{n-1}^{n-2})| \leq |\sum_{n=2}^{\infty} n a_n |h| ((n-1) \alpha^{n-2})| = |h| \sum_{n=2}^{\infty} |n(n-1) a_n \alpha^{n-2}|$

g. Repeat the procedure of part (a) to get the result.

h. Now let $h \to 0$ in part (f). Since $\sum_{n=2}^{\infty} |n(n-1) a_n \alpha^{n-2}|$ is finite, the right-hand side converges to 0. So the left-hand side should also converge to zero, which implies that $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x)$.

Example: Find $f'(x)$ and $f''(x)$ if $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, -1 < x < 1$.

Sol. $f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n x^{n-1}, -1 < x < 1$;

$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1) x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) x^{n-2}, -1 < x < 1$.

Caution Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series $\sum_{n=1}^{\infty} \frac{\sin(n! x)}{n^2}$ converges for all x. But if we differentiate term by term, we get the series $\sum_{n=1}^{\infty} \frac{n! \cos(n! x)}{n^2}$, which diverges for all x. This is not power series since it is not a sum of positive integer powers of x.

Theorem 22 – Term-by-Term Integration

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)dx = \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

Proof

At the proof of Theorem 21, We showed that the radius of convergence is same for the original series and the term-by-term differentiated series. Using this, we can derive that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ both converges for $-R < x < R$.

The results of Theorem 21 showed that the term-by-term differentiation works. Since $g(x)$ term-by-term differentiated is $f(x)$, $g'(x) = f(x)$. That is, $\int f(x)dx = g(x) + C$.

Example: The series $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$ converges on the interval $-1 < t < 1$. Therefore, $\ln(1+x) = \int_0^x \frac{1}{1+t} dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ or $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1$.

9.8 Taylor and Maclaurin Series

Series Representations

Assume that $f(x)$ is the sum of a power series about $x = a$, $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$ with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I, we obtain

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots,$$
$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots,$$

with the nth derivative being $f^{(n)}(x) = n!\, a_n + a\ sum\ of\ terms\ with\ (x-a) as\ a\ factor.$

Since these equations all hold at $x = a$, we have $f'(a) = a_1 . f''(a) = 1 \cdot 2a_2, f'''(a) = 1 \cdot 2 \cdot 3a_3$, and, in general,

$$f^{(n)}(a) = n!\, a_n.$$

These formulas reveal a pattern in the coefficients of any power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ that converges to the values of f on I. If there is such a series, then there is only one such series, and its nth coefficient is

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

If f has a series representation, then the series must be

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

Taylor and Maclaurin Series

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The Maclaurin series generated by f is often just called the Taylor series of f.

Example: Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Sol. We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$f(x) = x^{-1}, f'(x) = -x^{-2}, f''(x) = 2!\,x^{-3}, \ldots\ , f^{(n)}(x) = (-1)^n n!\,x^{-n-1},$$

so that

$$f(2) = 2^{-1} = \frac{1}{2}, f'(2) = -\frac{1}{2^2}, \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \ldots\ , \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

The Taylor series is

$$\begin{aligned} &f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots \\ &= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots. \end{aligned}$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is $\frac{1/2}{1+(x-2)/2} = \frac{1}{x}$.

The Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$.

Taylor Polynomials

Definition

Let f be a function with derivatives of order k for $k = 1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We speak of a Taylor polynomial of *order* n rather than *degree* n because $f^{(n)}(a)$ may be zero. Just as the linearization of f at $x = a$ provides the best linear approximation of f about a, the higher-order Taylor polynomials provide the "best" polynomial approximations of their respective degrees.

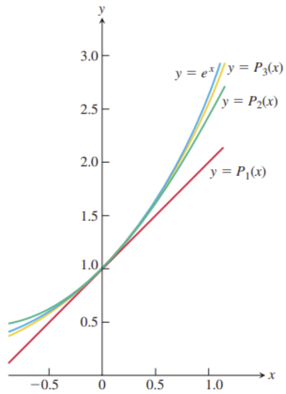

Figure 6. The graph of $f(x) = e^x$ and its Taylor polynomials

Example: Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

Sol. Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every $n = 0, 1, 2, \dots$, the Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This is also the Maclaurin series for e^x. In the next section we will see that the series converges to e^x at every x.

The Taylor polynomial of order n at $x = 0$ is $P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.

9.9 Convergence of Taylor Series

Theorem 23 – Taylor's Theorem

If f and its first n derivatives $f', f'', \dots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exist a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem. The proof is provided at the end of the section.

When we apply Taylor's Theorem, we usually want to hold a fixed and tread b as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x.

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad for\ some\ c\ between\ a\ and\ x.$$

When we state Taylor's theorem this way, it says that for each $x \in I$, $f(x) = P_n(x) + R_n(x)$. The function $R_n(x)$ is determined by the value of the $(n+1)$st derivative $f^{(n+1)}$ at a point c that depends on both a and x, and that lies somewhere between them. For any value of n we want, the equation gives both a polynomial approximation of f of that order and a formula for the error involved in using that approximation over the interval I.

Equation above is called **Taylor's formula.** The function $R_n(x)$ is called the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I.

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-a)^k.$$

Often, we can estimate R_n without knowing the value of c, as the following example shows.

Example: Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Sol. The function has derivatives of all orders throughout the interval $I = (-\infty, \infty)$. Taylor polynomial gives $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + R_n(x)$, and $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x. Since e^x is an increasing function of x, e^c lies between $e^0 = 1$ and e^x. When x is negative, so is c, and $e^c < 1$. When x is zero, $e^x = 1$ so that $R_n(x) = 0$. When x is positive, so is c, and $e^c < e^x$. Thus, for $R_n(x)$ given as above,

$$|R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \qquad when\ x \le 0,$$

and

$$|R_n(x)| \le e^x \frac{x^{n+1}}{(n+1)!} \qquad when\ x > 0.$$

Finally, because $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ for every x, $\lim_{n\to\infty} R_n(x) = 0$, and the series converges to e^x for every x. Thus,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots.$$

We can use the result with $x = 1$ to write $e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n(1)$, where for some c between 0 and 1,

$$R_n(1) = e^c \frac{1}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$

Estimating the Remainder

Theorem 24 – The Remainder Estimation Theorem

If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Example: Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

Sol. The function and its derivatives are $f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, \ldots, f^{(2k)}(x) = (-1)^k \sin x, f^{(2k+1)}(x) = (-1)^k \cos x$, so $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$. The series has only odd-powered terms and, for $n = 2k+1$, Taylor's Theorem gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x).$$

All the derivatives of $\sin x$ have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M = 1$ to obtain

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $\frac{|x|^{2k+2}}{(2k+2)!} \to 0$ as $k \to \infty$, whatever the value of x, so $R_{2k+1}(x) \to 0$ and the Maclaurin series for $\sin x$ converges to $\sin x$ for every x. Thus,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

Using Taylor Series

Example: For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ and obtain an error whose magnitude is no greater than 3×10^{-4}?

Sol. Here we can take advantage of the fact that the Taylor series for $\sin x$ is an alternating series for every nonzero value of x. According to the Alternating Series Estimation Theorem, the error in truncating

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

after $(x^3/3!)$ is no greater than

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}.$$

Therefore, the error will be less than or equal to 3×10^{-4} if $\frac{|x|^5}{120} < 3 \times 10^{-4}$ or $|x| < \sqrt[5]{360 \times 10^{-4}} \cong 0.514$.

The Alternating Series Estimation Theorem tells us something that the Remainder Estimation Theorem does not; namely, that the estimate $x - (x^3/3!)$ for $\sin x$ is an underestimate when x is positive, because then $x^5/120$ is positive.

A Proof of Taylor's Theorem

We prove Taylor's theorem assuming $a < b$. The proof for $a > b$ is nearly the same. The Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and its first n derivatives match the function f and its first n derivatives at $x = a$. We do not disturb that matching if we add another term of the form $K(x-a)^{n+1}$, where K is any constant, because such a term and its first n derivatives are all equal to zero at $x = a$. The new function

$$\phi_n(x) = P_n(x) + K(x-a)^{n+1}$$

and its first n derivatives still agree with f and its first n derivatives at $x = a$.

We now choose the value of K that makes the curve $y = \phi_n(x)$ agree with the original curve $y = f(x)$ at $x = b$. In symbols,

$$f(b) = P_n(b) + K(b-a)^{n+1}, \quad or \quad K = \frac{f(b) - P_n(b)}{(b-a)^{n+1}}.$$

With K defined, the function $F(x) = f(x) - \phi_n(x)$ measures the difference between the original function f and the approximating function ϕ_n for each x in $[a, b]$.

We now use Rolle's Theorem. First, because $F(a) = F(b) = 0$ and both F and F' are continuous on $[a, b]$, we know that $F'(c_1) = 0$ for some c_1 in (a, b).

Next, because $F'(a) = F'(c_1) = 0$ and both F' and F'' are continuous on $[a, c_1]$, we know that $F''(c_2) = 0$ for some c_2 in (a, c_1).

Rolle's Theorem, applied successively to $F'', F''', \ldots, F^{n-1}$, implies the existence of

$$\begin{array}{lll} c_3 & in\ (a, c_2) & such\ that\ F'''(c_3) = 0, \\ c_4 & in\ (a, c_3) & such\ that\ F^{(4)}(c_4) = 0, \\ \vdots & & \\ c_n & in\ (a, c_{n-1}) & such\ that\ F^{(n)}(c_n) = 0. \end{array}$$

Finally, because $F^{(n)}$ is continuous on $[a, c_n]$ and differentiable on (a, c_n), and $F^{(n)}(a) = F^{(n)}(c_n) = 0$, Rolle's Theorem implies that there is a number c_{n+1} in (a, c_n) such that $F^{(n+1)}(c_{n+1}) = 0$.

If we differentiate $F(x) = f(x) - P_n(x) - K(x-a)^{n+1}$ a total of $n+1$ times, we get $F^{(n+1)}(x) = f^{(n+1)}(x) - 0 - (n+1)!\,K$. The two recent equations give $K = \frac{f^{n+1}(c)}{(n+1)!}$ for some number $c = c_{n+1}$ in (a, b).

Combined, we get $f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$. This concludes the proof.

9.10 Applications of Taylor Series

The Binomial Series for Powers and Roots

The Taylor series generated by $f(x) = (1+x)^m$, when m is constant, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)(m-k+1)}{k!}x^k + \cdots.$$

This series, called the **binomial series**, converges absolutely for $|x| < 1$. To derive the series, we first list the function and its derivatives:

$$\begin{aligned}
&f(x) = (1+x)^m \\
&f'(x) = m(1+x)^{m-1} \\
&f''(x) = m(m-1)(1+x)^{m-2} \\
&f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \\
&\vdots \\
&f^{(k)}(x) = m(m-1)(m-2)\cdots(m-k+1)(1+x)^{m-k}
\end{aligned}$$

We then evaluate these at $x = 0$ and substitute into the Taylor series formula to obtain the series above. If m is an integer greater than or equal to zero, the series stops after $m+1$ terms because the coefficients from $k = m+1$ on are zero. If m is not a positive integer or zero, the series is infinite and converges for $|x| < 1$. To see why, let u_k be the term involving x^k. Then apply the Ratio Test for absolute convergence to see that

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{m-k}{k+1}x\right| \to |x| \qquad as\ k \to \infty.$$

Our derivation of the binomial series shows only that it is generated by $(1+x)^m$ and converges for $|x| < 1$.

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k,$$

where we define $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$, and $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$ for $k \geq 3$.

Evaluating Nonelementary Integrals

Taylor series can be used to express nonelementary integrals in terms of series.

Example: Express $\int \sin x^2\, dx$ as a power series: From the series for $\sin x$ we substitute x^2 for x to obtain

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Therefore,

$$\int \sin x^2\, dx = C + \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots.$$

Now, estimate $\int_0^1 \sin x^2\, dx$ with an error of less than 0.001. From the indefinite integral, we find that

$$\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \frac{1}{19\cdot 9!} - \cdots.$$

The series on the right-hand side alternates, and we find by numerical evaluations that $\frac{1}{11\cdot 5!} \cong 0.00076$ is the first term to be numerically less than 0.001. The sum of the preceding two terms gives $\int_0^1 \sin x^2\, dx \cong \frac{1}{3} - \frac{1}{42} \cong 0.310$. With three more terms the error shrinks to only about 1.08×10^{-9}.

Arctangents

We have

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2},$$

in which the last term comes from adding the remaining terms as a geometric series with first term $a = (-1)^{n+1} t^{2n+2}$ and ratio $r = -t^2$. Integrating both sides of the equation above from $t = 0$ to $t = x$ gives

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x),$$

where

$$R_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt.$$

The denominator of the integrand is greater than or equal to 1; hence

$$|R_n(x)| \le \int_0^{|x|} t^{2n+2} dt = \frac{|x|^{2n+3}}{2n+3}.$$

If $|x| < 1$, the right side of this inequality approaches zero as $n \to \infty$. Therefore $\lim_{n\to\infty} R_n(x) = 0$ if $|x| \le 1$ and

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \quad |x| \le 1.$$

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

Example: $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{(x-1) - \frac{1}{2}(x-1)^2 + \cdots}{x-1} = \lim_{x\to 1} \left(1 - \frac{1}{2}(x-1) + \cdots\right) = 1$. Of course, this particular limit can be evaluated using l`Hopital's Rule just as well.

Example: Evaluate $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$.

Sol. The Taylor series for $\sin x$ and $\tan x$ are $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$.

Subtracting the series term by term gives $\sin x - \tan x = -\frac{x^3}{2} - \frac{x^5}{8} - \cdots = x^3\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right)$.

Division of both sides by x^3 and taking limits then gives $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = \lim_{x\to 0}\left(-\frac{1}{2} - \frac{x^2}{8} - \cdots\right) = -\frac{1}{2}$.

Euler's Identity

A complex number is a number with the form $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$. If we substitute $x=i\theta$ in the Taylor series for e^x and use the relations $i^2=-1, i^4=1$. Then, we obtain

$$e^{i\theta}=1+\frac{i\theta}{1!}+\frac{i^2\theta^2}{2!}+\frac{i^3\theta^3}{3!}+\cdots=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right)=\cos\theta+i\sin\theta.$$

This does not prove that $e^{i\theta}=\cos\theta+i\sin\theta$ because we have not defined what it means to raise e to an imaginary power. Rather, it tells us how to define $e^{i\theta}$ so that its properties are consistent with the properties of the exponential function for real numbers.

Definition

For any real number θ, $e^{i\theta}=\cos\theta+i\sin\theta$. This equation is called **Euler's identity**. Then, $e^{i\pi}=-1$.

Summary

Frequently used Taylor Series

$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n,\quad |x|<1$

$\frac{1}{1+x}=1-x+x^2-\cdots+(-x)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n,\quad |x|<1$

$e^x=1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!},\quad |x|<\infty$

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!},\quad |x|<\infty$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!},\quad |x|<\infty$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n},\quad -1<x\le 1$

$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1},\quad |x|\le 1$