III. Derivatives

미분

3.1 Tangent Lines and the Derivative at a Point

Finding a Tangent Line to the Graph of a Function

> Definition 1
>
> The ***slope of the curve*** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number (provided(가정) the limit exists)
>
> $$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
>
> The ***tangent line***(접선) to the curve at P is the line through P with this slope.

Rates of Change: Derivative at a Point

The expression $\frac{f(x_0+h)-f(x_0)}{h}$ is called the ***difference quotient of f at x_0 with increment h*** (f의 x_0에서 h 증가에 따른 증가율)

> Definition 2
>
> The ***derivative of a function f at a point x_0***(x_0에서 f의 미분계수), denoted $f'(x_0)$, is
>
> $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
>
> provided this limit exists.

The following are all interpretations for the limit of the difference quotient $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

- The slope of the graph of $y = f(x)$ at $x = x_0$
- The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$
- The rate of change of $f(x)$ with respect to(x에 대한 $f(x)$의 변화율) at $x = x_0$
- The derivative $f'(x_0)$ at $x = x_0$

3.2 The Derivative as a Function

Definition 3

The ***derivative***(미분계수/도함수) of a function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

provided this limit exists.

If f' exists at a particular x, we say that ***f is differentiable (has a derivative) at x***(x에서 미분 가능하다). If f' exists at every point in the domain of f, we call f differentiable (미분 가능하다).

The derivative $f'(x)$ can be also denoted as below:

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called ***differentiation.*** To emphasize(강조) the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation like below as an another way to denote the derivative $f'(x)$.

$$\frac{d}{dx} f(x)$$

Example: Differentiate $f(x) = \frac{x}{x-1}$.

Sol. $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{-h}{(x+h-1)(x-1)} = \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$

Notation

There are many ways to denote the derivative of a function $y = f(x)$, like shown as below.

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) \ldots etc$$

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}.$$

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is ***differentiable on an open interval*** (finite or infinite) if it has a derivative at each point of the interval. It if ***differentiable on a closed interval*** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} \quad (Right - hand\ derivative\ at\ a)$$

$$\lim_{h \to 0-} \frac{f(b+h) - f(b)}{h} \quad (Left - hand\ derivative\ at\ b)$$

exists at the endpoints.

Example: Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

Sol. To the right of the origin, when $x > 0$, $\frac{d}{dx}(|x|) = \frac{dx}{dx} = 1$.

To the left of the origin, when $x < 0$, $\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = -1$.

The two branches of the graph come together at an angle at the origin, forming a non-smooth corner. There is no derivative at the origin because the one-sided derivatives differ there:

Right-hand derivative of $|x|$ at zero $= \lim_{h \to 0+} \frac{|0+h|-|0|}{h} = \lim_{h \to 0+} \frac{|h|}{h} = 1$

Left-hand derivative of $|x|$ at zero$= \lim_{h \to 0-} \frac{|0+h|-|0|}{h} = \lim_{h \to 0-} \frac{|h|}{h} = -1$.

When Does a Function Not Have a Derivative at a Point?

A function can fail to have a derivative at a point for many reasons, as the examples below, where the graph has:

- A corner, where the one-sided derivatives differ
- A cusp, where the slope of the graph approaches ∞ from one side and $-\infty$ from the other
- A vertical tangent line, where the slope of the graph approaches $\infty\ or - \infty$ from both sides
- A discontinuity
- Wild(심한) oscillation (like the graph $y = x \sin\frac{1}{x}\ at\ x \to 0$

Differentiable Functions Are Continuous

Theorem 1-Differentiability Implies Continuity

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$, or equivalently, that $\lim_{h \to 0} f(c+h) = f(c)$.

If $h \neq 0$, then $f(c+h) = f(c) + \big(f(c+h) - f(c)\big) = f(c) + \frac{f(c+h)-f(c)}{h} \cdot h$

Now take limits as $h \to 0$. Then $\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h \to 0} h = f(c) + f'(c) \cdot 0 = f(c)$.

Similar arguments with one-sided limits show that if f has a derivative from one side(right or left) at $x = c$, then f is continuous from that side at $x = c$.

※ Note that The converse(역) of Theorem 1 is false.

3.3 Differentiation Rules

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Power Rule

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Derivative Rules

1. Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

2. Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

3. Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

4. Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. The function f'' is called the ***second derivative*** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

If y'' is differentiable, its derivative, $y''' = d^3y/dx^3$, is the ***third derivative*** of y with respect to x.

The names continue, with $y^{(n)} = d^ny/dx^n$ denoting the ***nth derivative*** y with respect to x for $n \in \mathbb{N}$.

3.5 Derivatives of Trigonometric Functions

The derivatives of Basic Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

3.6 The Chain Rule

Theorem 2-The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'\big(g(x)\big).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

Where dy/du is evaluated at $u = g(x)$.

Power Chain Rule

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the Power Chain Rule:

$$\frac{d}{dx}(u)^n = nu^{n-1}\frac{du}{dx}.$$

3.9 Linearization and Differentials

Linearization

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable, passes through $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

As long as this line remains close to the graph of f as we move off the point tangency, $L(x)$ gives a good approximation to $f(x)$.

Definition 4

If f is differentiable at $x = a$, then the approximating function $L(x) = f(a) + f'(a)(x - a)$ is the ***linearization*** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L if the ***standard linear approximation*** of f at a. The point $x = a$ is the ***center*** of the approximation.

Example: Find the linearization of $f(x) = \sqrt{x+1}$ at $x = 0$.

Sol: Since $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

So $f(x)$ at $x = 0.01$ can be approximated as $L(0.01) = 1.005$.

Differentials

Definition 5

Let $y = f(x)$ be a differentiable function. The ***differential*** dx is an independent variable. The ***differential*** dy is

$$dy = f'(x)dx$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx.

Example: Find dy is $y = x^5 + 37x$, and find the value of dy when $x = 1$ and $dx = 0.2$.

Sol: $dy = (5x^4 + 37)dx$, substitute $x = 1 \textit{ and } dx = 0.2$ in the expression for dy. Then $dy = 8.4$.

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at appoint a and want to estimate how much this value will change if we move to a nearby point $a + dx$. If $dx = \Delta x$ is small, Δy is approximately equal to the differential dy. Since $f(a + dx) = f(a) + \Delta y$, the differential approximation gives $f(a + dx) = f(a) + dy$ when $dx = \Delta x$. Thus, the approximation $\Delta y = dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a)dx$.

Example: Use differentials to estimate $7.97^{1/3}$.

Sol: $let\ y = x^{1/3}$. $dy = \frac{1}{3x^{2/3}}dx$. We set $a = 8$, then $dx = -0.03$. Approximating with the differential gives $f(7.97) = f(8) + dy = 8^{1/3} + \frac{1}{3(8)^{\frac{2}{3}}}(-0.03) = 1.9975$. The true value $7.97^{1/3}$ is 1.997497 $(very\ close)$.

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. The true change will be $\Delta f = f(a + \Delta x) - f(a)$, and the differential estimate will be $df = f'(a)\Delta x$. The ***Approximation Error*** is the subtraction of these two:

$$Approx.Error = \Delta f - df = \Delta f - f'(a)dx = \left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right) \cdot \Delta x$$

$let\ \varepsilon = \frac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)$, then the Error can be denoted as $\varepsilon \cdot \Delta x$.

As $\Delta x \to 0$, the difference quotient $\frac{f(a+\Delta x)-f(a)}{\Delta x}$ approaches $f'(a)$, so $\varepsilon \to 0$. When Δx is small, the approximation error $\varepsilon \cdot \Delta x$ is also small.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta x + \varepsilon\,\Delta x$$

In which $\varepsilon \to 0$ as $\Delta x \to 0$.

Proof of the Chain Rule

Let g is differentiable at x_0 and f is differentiable at $g(x_0)$. Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y. Then

$$\Delta u = g'(x_0)\Delta x + \varepsilon_1 \Delta x = (g'(x_0) + \varepsilon_1)\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

$$\Delta y = f'(u_0)\Delta u + \varepsilon_2 \Delta u = (f'(u_0) + \varepsilon_2)\Delta u$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations gives

$$\Delta y = (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\Delta x.$$

So,

$$\frac{\Delta y}{\Delta x} = (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1) = f'(u_0)g'(x_0) + \varepsilon_1 f'(u_0) + \varepsilon_2 g'(x_0) + \varepsilon_1\varepsilon_2.$$

Since $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$, the last three terms on the right vanish in the limit, leaving

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0).$$

Sensitivity to Change

The equation $df = f'(x)dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx. As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways: absolute, relative, and percentage.

	True	Estimated
Absolute change	$\Delta f = f(a+dx) - f(a)$	$df = f'(a)dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$