VI. Applications of Definite Integrals

정적분의 활용

6.1 Volumes Using Cross-Sections

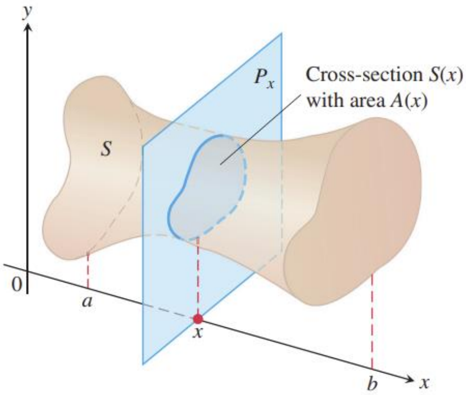


Figure 1. A cross-section $S(x)$ of the solid S.

A ***cross-section*** of a solid S is the planar region formed by intersecting S with a plane. Suppose that we want to find the volume of a solid S like the one pictured above. At each point x in the interval $[a, b]$ we form a cross-section $S(x)$ by intersecting S with a plane perpendicular(수직) to the x-axis through the point x, which gives a planar region whose area is $A(x)$. IF A is a continuous function of x, then the volume of the solid S is the definite integral of $A(x)$. This method of computing volumes is known as the ***method of slicing***.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width Δx_k and slice the solid. The partition points are $a = x_0 < x_! < \cdots < x_n = b$. Then the volume of the kth slab could be approximated as $A(x_k)\Delta x_k$. The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. The approximation given by this Riemann sum converges to the definite integral of $A(x)$ as $n \to \infty$, as shown at the definition below.

Definition 1

The ***volume*** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)dx.$$

Solids of Revolution: The Disk Method

The solid generated by rotating a planar region about an axis in its plane is called a ***solid of revolution.*** The area is then $A(x) = \pi[R(x)]^2$, which $R(x)$ is the radius of the disk. Therefore, the definition of volume gives us the following formula.

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx.$$

This method for calculating the volume of a solid of revolution is called the ***disk method***.

Example: The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Sol. $V = \int_a^b \pi[R(x)]^2 dx = \int_0^4 \pi x dx = \left[\frac{\pi x^2}{2}\right]_0^4 = 8\pi$.

Volume by Disks for Rotation About the y-Axis

$$V = \int_c^d A(y)dy = \int_c^d \pi[R(y)]^2 dy.$$

Example: Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Sol. The cross-sections are perpendicular to the line $x = 3$ and have y-coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. $V = \int_c^d \pi[R(y)]^2 dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2 dy = \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\pi\sqrt{2}}{15}$.

Solids of Revolution: The Washer Method

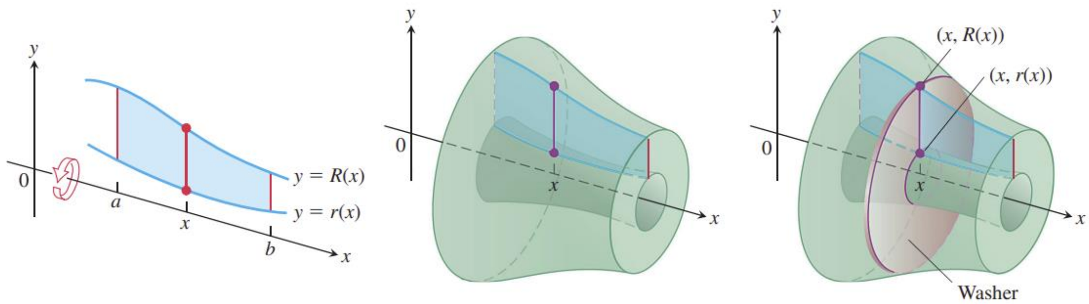


Figure 2. Solid of revolution that have cross-sections with holes.

The washer's area is the area of a circle of radius $R(x)$ minus the area of a circle of radius $r(x)$:

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives us the following formula.

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)dx.$$

Volume by Washers for Rotation About the y-Axis

$$V = \int_c^d A(y)dy = \int_c^d \pi([R(y)]^2 - [r(y)]^2)dy.$$

This method for calculating the volume of a solid of revolution is called the ***washer method*** (the solid resembles a circular washer).

Example: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid, Find the volume of the solid,

Sol. The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$.

$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)dy = \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$

6.2 Volumes Using Cylindrical Shells

Slicing with Cylinders

Slice the volume to n cylindrical shells. Each shell sits over a subinterval $[x_{k-1}, x_k]$ in the x-axis. The thickness of the shell is $\Delta x_k = x_k - x_{k-1}$. Then you can calculate the volume of the shell by multiplying circumference of the shell, the height, and the thickness Δx_k. Summing together the volumes of the individual cylindrical shells over the interval will then give the Riemann sum. Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ will lead to The Shell Method.

The Shell Method

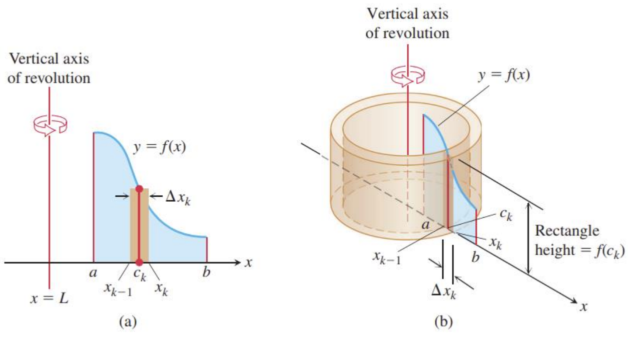


Figure 3. Applying the Shell Method to calculate the volume of the solid of revolution.

Suppose that the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$. We assume $a \geq L$, so the vertical line may touch the region but cannot pass through it. We generate a solid S by rotating this region about the line L.

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$. Choose a point c_k in each subinterval $[x_{k-1}, x_k]$. Then the volume of the cylindrical shell generated by the kth subinterval part of the graph is $\Delta V_k = 2\pi(c_k - L)f(c_k)\Delta x_k$.

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n shells: $V = \sum_{k=1}^{n} \Delta x_k$. The limit of this Riemann sum, as $\Delta x_k \to 0$ and $n \to \infty$ will lead to the Shell Method.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \binom{shell}{radius}\binom{shell}{height} dx.$$

Example: The region bounded by the curve $y = \sqrt{x}$, the $x - axis$, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Sol. $V = \int_a^b 2\pi \binom{shell}{radius}\binom{shell}{height} dx = \int_0^4 2\pi\,(x)(\sqrt{x})dx = 2\pi\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{128\pi}{5}.$

6.3 Arc length

Length of a Curve $y = f(x)$

Suppose the curve whose length we want to find is the graph of the function $y = f(x)$ from $x = a$ to $x = b$. To derive an integral formula for the length of the curve, assume that f has a continuous derivative at every point of $[a, b]$. Such a function is called smooth, and its graph is a ***smooth curve*** because it does not have any breaks, corners, or cusps.

We partition the interval $[a, b]$ into n subintervals with $a = x_0 < x_! < \cdots < x_n = b$. If $y_k = f(x_k)$, then the corresponding point $P_k(x_k, y_k)$ lies on the curve. Connect successive points P_{k-1} *and* P_k with straight-line segments. If we set $\Delta x_k = x_k - x_{k-1}$ and $\Delta y_k = y_k - y_{k-1}$, then the line segment has a length of $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. so the length of the curve is approximated by the sum $\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

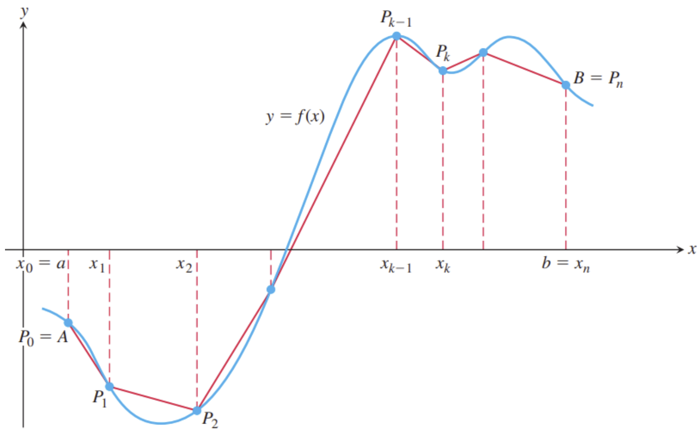


Figure 3. Calculating the length of a curve by partitioning.

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. The Mean Value Theorem tells us that there is a point c_k, with $x_{k-1} < c_k < x_k$, such that $\Delta y_k = f'(c_k)\Delta x_k$.

Substituting this for Δy_k , the sums will take the form $\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2} = \sum_{k=1}^n \sqrt{1 + \left(f'(c_k)\right)^2}\,\Delta x_k$. This is a Riemann sum whose limit we can evaluate. Because $\sqrt{1 + \left(f'(x)\right)^2}$ is continuous at $[a, b]$, the limit of the Riemann sum exists and has the value

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \left(f'(c_k)\right)^2}\,\Delta x_k = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx.$$

Definition 2

If f' is continuous on $[a,b]$, then the ***length (arc length)*** of the curve $y=f(x)$ from the point $A=(a,f(a))$ to the point $(b,f(b))$ is the value of the integral

$$L=.\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Example: Find the length of the graph of $f(x)=\frac{x^3}{12}+\frac{1}{x}, 1\le x\le 4$.

Sol. $f'(x)=\frac{x^2}{4}-\frac{1}{x^2}, 1+\left(f(x)\right)^2=\frac{x^4}{16}+\frac{1}{2}+\frac{1}{x^4}=\left(\frac{x^2}{4}+\frac{1}{x^2}\right)^2$. The length of the graph over $[1,4]$ is $L=\int_1^4\left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx=\left[\frac{x^3}{12}-\frac{1}{x}\right]_1^4=6$.

Dealing with Discontinuities in dy/dx

Even if the derivative dy/dx does not exist at some point on a curve, it is possible that dx/dy could exist. This can happen, for example, when a curve has a vertical tangent. In this case, we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation.

Formula for the Length of $x=g(y), c\le y\le d$

If g$'$ is continuous on $[c,d]$, then the ***length (arc length)*** of the curve $y=g(y)$ from the point $A=(g(c),c)$ to the point $(g(d),d)$ is the value of the integral

$$L=.\int_c^d \sqrt{1+(g'(y))^2}\,dy=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

The Differential Formula for Arc Length

If $y=f(x)$ and if f' is continuous on $[a,b]$, then by the FTC we can define a new function

$$s(x)=\int_a^x \sqrt{1+\left(f'(t)\right)^2}\,dt.$$

The function $s(x)$ is continuous and measures the length along the curve $y=f(x)$ from the initial point $P_0(a,f(a))$ to the point $Q(x,f(x))$ for each $x\in[a,b]$. The function s is called the ***arc length function*** for $y=f(x)$. From the FTC, the function s is differentiable on (a,b) and

$$\frac{ds}{dx}=\sqrt{1+\left(f'(x)\right)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Then the differential of arc length is

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{dx^2+dy^2}.$$

Example: Find the arc length function for the curve in the previous example, with $A=\left(1,\frac{13}{12}\right)$ as the starting point.

Sol. $s(x)=\int_a^x\sqrt{1+\left(f'(t)\right)^2}\,dt=\int_a^x\left(\frac{t^2}{4}+\frac{1}{t^2}\right)dt=\left[\frac{t^3}{12}-\frac{1}{t}\right]_1^x=\frac{x^3}{12}-\frac{1}{x}+\frac{11}{12}$.

6.4 Areas of Surfaces of Revolution

Defining the Surface Area

Suppose we want to find the area of the surface area swept out by revolving the graph of a nonnegative continuous function $y = f(x), a \leq x \leq b$, about the x-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs.

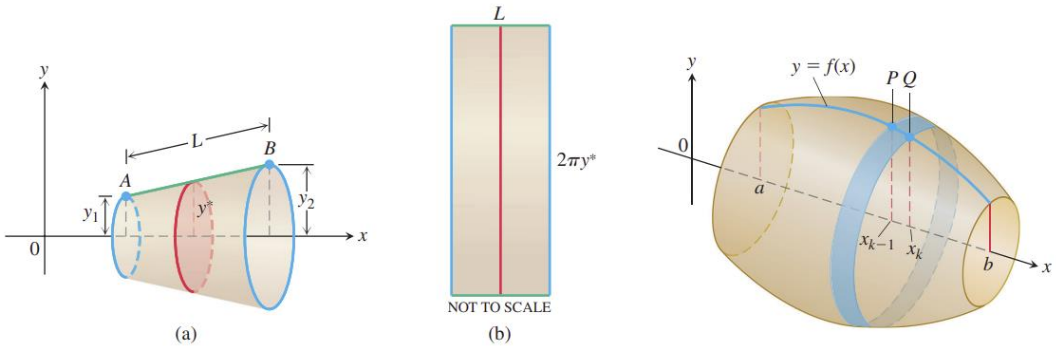


Figure 4. Defining the Surface Area of a solid of revolution.

As the arc PQ revolves about the x-axis like shown above, the line segment joining P and Q sweeps out a frustum(절두체) of a cone whose axis lies along the x-axis. The surface area of this frustum approximates the surface area of the band swept out by the arc PQ. The surface area of the frustum of the cone is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q, and L is its length. Since $f \geq 0$, the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$Frustrum\ surface\ area = 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum $\sum_{k=1}^{n} \pi \cdot (f(x_{k-1}) + f(x_k)) \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. We expect the approximation to improve as the partition of $[a, b]$ becomes finer. If the function f is differentiable, then by the MVT, there is a point $(c_k, f(c_k))$ on the curve between P and Q where $f'(c_k) = \frac{\Delta y_k}{\Delta x_k}, \Delta x_k = f'(c_k)\Delta x_k$. So

$$\sum_{k=1}^{n} \pi \cdot \big(f(x_{k-1}) + f(x_k)\big) \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \pi \cdot \big(f(x_{k-1}) + f(x_k)\big) \cdot \sqrt{1 + \big(f'(c_k)\big)^2}\Delta x_k.$$

These sums are not the Riemann sums of any function. However, the points x_{k-1}, x_k, and c_k are very close to each other, and so we expect that as the norm of the partition of $[a, b]$ goes to zero, the sums converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

> **Definition 3**
>
> If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the ***area of the surface*** generated by revolving the graph of $y = f(x)$ about the x-axis is
>
> $$S = \int_a^b 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\, dx = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}, 1 \le x \le 2$, about the x-axis.

Sol. $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2} = \frac{\sqrt{x+1}}{\sqrt{x}}$, $S = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}dx = 4\pi \cdot \left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3}-2\sqrt{2}\right)$.

Revolution About the y-Axis

Surface Area for Revolution About the y-Axis

If $x = g(x) \ge 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\,dy = \int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

6.6 Moments and Centers of Mass

Masses Along a Line

Imagine masses $m_1, m_2, \cdots, m_n$ on a rigid x-axis supported by a fulcrum(지렛대) at the origin. The ***torque***, which is a turning effect, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. A positive torque induces a counterclockwise turn. The sum of the torques measures the tendency(경향) of a system to rotate about the origin. This sum is called the ***system torque***.

$$System\ torque = \sum_{k=1}^{n} m_k g x_k = g\sum_{k=1}^{n} m_k x_k$$

The torque is the product of the gravitational acceleration g, which is a feature of the environment in which the system happens to reside, and the number $\sum_{k=1}^{n} m_k x_k$, which is a feature of the system itself. This number $\sum_{k=1}^{n} m_k x_k$ is called the ***moment of the system about the origin***. It is the sum of the ***moments*** $m_1x_1, m_2x_2 \cdots$ of the individual masses.

$$M_0 = Moment\ of\ system\ about\ origin = \sum_{k=1}^{n} m_k x_k$$

We usually want to know where to place the fulcrum to make the system balance, which means the torques add to zero. The torque of each mass when the fulcrum is at $x = \bar{x}$ is:

$$Torque\ of\ m_k\ about\ \bar{x} = \begin{pmatrix} signed\ distance \\ of\ m_k\ from\ \bar{x} \end{pmatrix}\begin{pmatrix} downward \\ force \end{pmatrix} = (x_k - \bar{x})m_k g.$$

Then the sum could be denoted as:

$$\sum (x_k - \bar{x})m_k g = g\left(\sum x_k m_k - \bar{x}\sum m_k\right) = 0,$$

When solved for $\bar{x}$,

$$\bar{x} = \frac{\sum x_k m_k}{\sum m_k} = \frac{system\ moment\ about\ origin}{system\ mass}$$

The point $\bar{x}$ is called the system's ***center of mass***

Thin Wires

Suppose that we have a straight wire located on interval $[a, b]$ on the x-axis. This wire is not homogeneous(균일/균질), but rathe the density varies continuously from point to point. If a short segment of a rod containing the point x with length Δx has mass Δm, then the density at x is given by $\delta(x) = \lim_{\Delta x \to 0} \Delta m/\Delta x$, or $\delta = dm/dx$.

Partition the interval $[a, b]$ into finitely many subintervals $[x_{k-1}, x_k]$. If we take n subintervals and replace the portion of a wire along a subinterval of length Δx_k containing x_k by a point mass located at x_k with mass $\Delta m_k = \delta(x_k)\Delta x_k$, then we obtain a collection of point masses that have approximately the same total mass and moment as the original wire. The mass M of the wire and the moment M_0 are approximated by the Riemann sums:

$$M \approx \sum_{k=1}^{n} \Delta m_k = \sum_{k=1}^{n} \delta(x_k)\Delta x_k, \qquad M_0 \approx \sum_{k=1}^{n} x_k \Delta m_k = \sum_{k=1}^{n} x_k \delta(x_k)\Delta x_k$$

By taking a limit of these Riemann sums as the length of the intervals in the partition approaches zero, we get integral formulas for the mass and the moment of the wire about the origin. The mass M, moment about the origin M_0, and center of mass $\bar{x}$ are

$$M = \int_a^b \delta(x)dx, \qquad M_0 = \int_a^b x\delta(x)dx, \qquad \bar{x} = \frac{M_0}{M} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}.$$

Masses Distributed over a Plane Region

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k). The mass of the system is $M = \sum m_k$. Each mass m_k has a moment about each axis. The moments of the entire system about the two axes are

$$Moment\ about\ x - axis: M_x = \sum m_k y_k,$$

$$Moment\ about\ y - axis: M_y = \sum m_k x_k.$$

Then the x and y-coordinate of the system's center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}.$$

As far as the balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's ***center of mass***.

Thin, Flat Plates

Imagine that the plate occupying a region in the xy-plane is cut into thin strips parallel to one of the axes. The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moments of the strip about the x and y-axis are then $\tilde{y}\Delta m, \tilde{x}\Delta m$ respectively. Then

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\Delta m}{\sum \Delta m}\ and\ \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\Delta m}{\sum \Delta m}.$$

These sums are Riemann sums for integrals, and they approach these integrals in the limit as the strips become narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\Delta m}{\int \Delta m}\ and\ \bar{y} = \frac{\int \tilde{y}\Delta m}{\int \Delta m}$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy-Plane

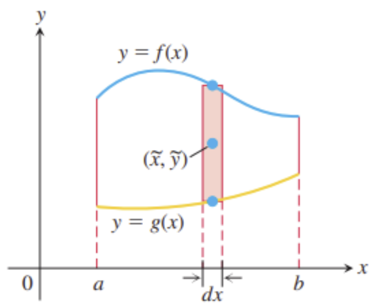
$$\textit{Moment about the } x-\textit{axis}: \quad M_x = \int \tilde{y}\Delta m$$

$$\textit{Moment about the } y-\textit{axis}: \quad M_y = \int \tilde{x}\Delta m$$

$$\textit{Mass}: \quad M = \int dm$$

$$\textit{Center of mass}: \quad \bar{x} = \frac{M_y}{M}, \bar{y} = \frac{M_x}{M}$$

Plates Bounded by Two Curves



center of mass: $(\tilde{x}, \tilde{y}) = \left(x, \frac{1}{2}[f(x) + g(x)]\right)$

length: $f(x) - g(x)$, width: dx, area: $dA = [f(x) = g(x)]dx$

mass: $dm = \delta dA = \delta[f(x) = g(x)]dx$

$$M_y = \int \tilde{x}\Delta m = \int_a^b \delta x[f(x) - g(x)]dx,$$

$$M_x = \int \tilde{y}\Delta m = \int_a^b \frac{\delta}{2}[f^2(x) - g^2(x)]dx.$$

$$\bar{x} = \frac{1}{M}\int_a^b \delta x[f(x) - g(x)]dx$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}[f^2(x) - g^2(x)]dx$$

Centroids

When the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, the center of mass is the ***centroid*** of the shape.

Fluid Forces and Centroids

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

$$F = w\bar{h}A.$$

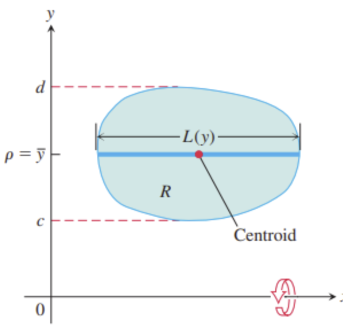
The Theorems of Pappus

Theorem 1- Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A.$$

Proof Draw the axis of revolution as the x-axis with the region R in the first quadrant. Let $L(y)$ denote the length of the cross-section of R perpendicular to the y-axis at y. Assume $L(y)$ to be continuous.



By the method of cylindrical shells,

$$V = \int_c^d 2\pi(shell\ radius)(shell\ height)dy = 2\pi\int_c^d yL(y)dy\,.$$

The y-coordinate of $R's$ centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}dA}{A} = \frac{\int_c^d yL(y)dy}{A}.$$

So that $\int_c^d yL(y)dy = A\bar{y}$.

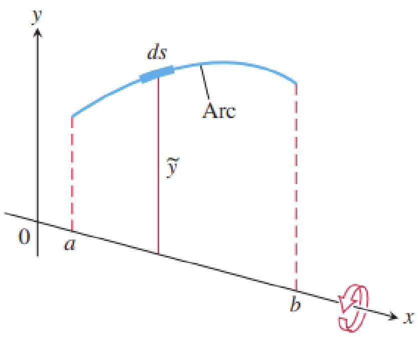
Substituting $A\bar{y}$ for the last integral in the first equation gives $V = 2\pi\bar{y}A$. With $\rho = \bar{y}$, we have $V = 2\pi\rho A$.

Theorem 1- Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length L of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L.$$

Proof Draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant. The area of the surface generated by the arc is



$$S = \int_{x=a}^{x=b} 2\pi y ds = 2\pi\int_{x=a}^{x=b} y ds.$$

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y}ds.}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} yds}{L}.$$

Hence $\int_{x=a}^{x=b} yds = \bar{y}L$.

Substituting $\bar{y}L$ for the last integral in the first equation gives $S = 2\pi\bar{y}L$. Since $\rho = \bar{y}$, we have $S = 2\pi\rho L$.