VII. Transcendental Functions

초월함수

7.1 Inverse Functions and Their Derivatives

7.2 Natural Logarithms

7.3 Exponential Functions

7.4 Exponential Change and Separable Differential Equations

7.5 Indeterminate Forms and L`Hopital's Rule

Indeterminate Form 0/0

Consider the function

$$F(x) = \frac{3x - \sin x}{x}$$

when the x approaches zero. Then both the numerator and denominator approach 0, and 0/0 is undefined. Such limits may or may not exist in general, but the limit does exist for the function $F(x)$ under discussion by applying l`Hopital's Rule.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces 0/0. We use 0/0 as a notation for an expression that does not have a numerical value, known as an **indeterminate form.** Other examples of an indeterminate form are $\infty/\infty, \infty \cdot 0, \infty - \infty$, 0^0, and 1^∞.

Theorem 5 – L`Hopital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right side of this equation exists.

Example: $\lim_{x\to0}\frac{3x-\sin x}{x}=\lim_{x\to0}\frac{3-\cos x}{1}=\frac{3-1}{1}=2.$ $\lim_{x\to0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}.$

Using L`Hopital's Rule

To find

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

by l`Hopital's Rule, we continue to differentiate $f$ and $g$, so long as we still get the form 0/0 at $x=a$. But as soon as one or the other of these derivatives is different from zero at $x=a$ we stop differentiating. L`Hopital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example: $\lim_{x\to0}\frac{1-\cos x}{x+x^2}=\lim_{x\to0}\frac{\sin x}{1+2x}\neq\lim_{x\to0}\frac{\cos x}{2}=\frac{1}{2}.$

L`Hopital's Rule applies to one=sided limits as well.

Example: $\lim_{x\to0+}\frac{\sin x}{x^2}=\lim_{x\to0+}\frac{\cos x}{2x}=\infty,$ $\lim_{x\to0-}\frac{\sin x}{x^2}=\lim_{x\to0-}\frac{\cos x}{2x}=-\infty.$

Indeterminate Forms $\frac{\infty}{\infty},\infty\cdot0,\infty-\infty$

① Form ∞/∞

It can be proven that l`Hopital's Rule can be applied to the indeterminate form ∞/∞. The proof will be given at the end of the chapter 7.5. If $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x\to a$, a may be either finite or infinite. Moreover, $x\to a$ may be replaced by the one-sided limits $x\to a^+$ or $x\to a^-$.

Example: $\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0.$

② Form $\infty\cdot0$ and $\infty-\infty$

Sometimes these forms can be handled by using algebra to convert them to a 0/0 or ∞/∞ form.

Example: $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{h\to0+}\left(\frac{1}{h}\sin h\right)=\lim_{h\to0+}\frac{\sin h}{h}=1.$ $(let\ h=1/x)$

Example: $\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}=\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to0}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=0.$

Indeterminate Powers

Limits that lead to the indeterminate forms $1^\infty,0^0$, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l`Hopital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This is also valid for one-sided limits.

If $\lim_{x\to a}\ln f(x)=L$, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}e^{\ln f(x)}=e^L.$$

Here a may be either finite or infinite.

Example: Find $\lim_{x\to\infty} x^{\frac{1}{x}}$.

Sol. The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{\frac{1}{x}}$ and find $\lim_{x\to\infty} \ln f(x)$.

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

Therefore $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^0 = 1$.

Proof of L`Hopital's Rule

The proof of l`Hopital's Rule is based on Cauchy's Mean Value Theorem. We prove Cauchy's Theorem first and then show how it leads to l`Hopital's Rule.

Theorem 6 – Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof We apply the Mean Value Theorem twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the MVT would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the MVT to the function $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}[g(x) - g(a)]$. This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0$$

so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof of L`Hopital's Rule

We first establish the limit equation for the case $x \to a^+$, The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so $\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$.

As x approaches a, c approaches a because it always lies between a and x, Therefore,

$$\lim_{x\to a+}\frac{f(x)}{g(x)}=\lim_{x\to a+}\frac{f'(c)}{g'(c)}=\lim_{x\to a+}\frac{f'(x)}{g'(x)}.$$

This establishes l`Hopital’s Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy’s MVT to the closed interval $[x,a], x<a$. These two completes the Theorem.

7.6 Inverse Trigonometric Functions

Defining the Inverse Trigonometric Functions

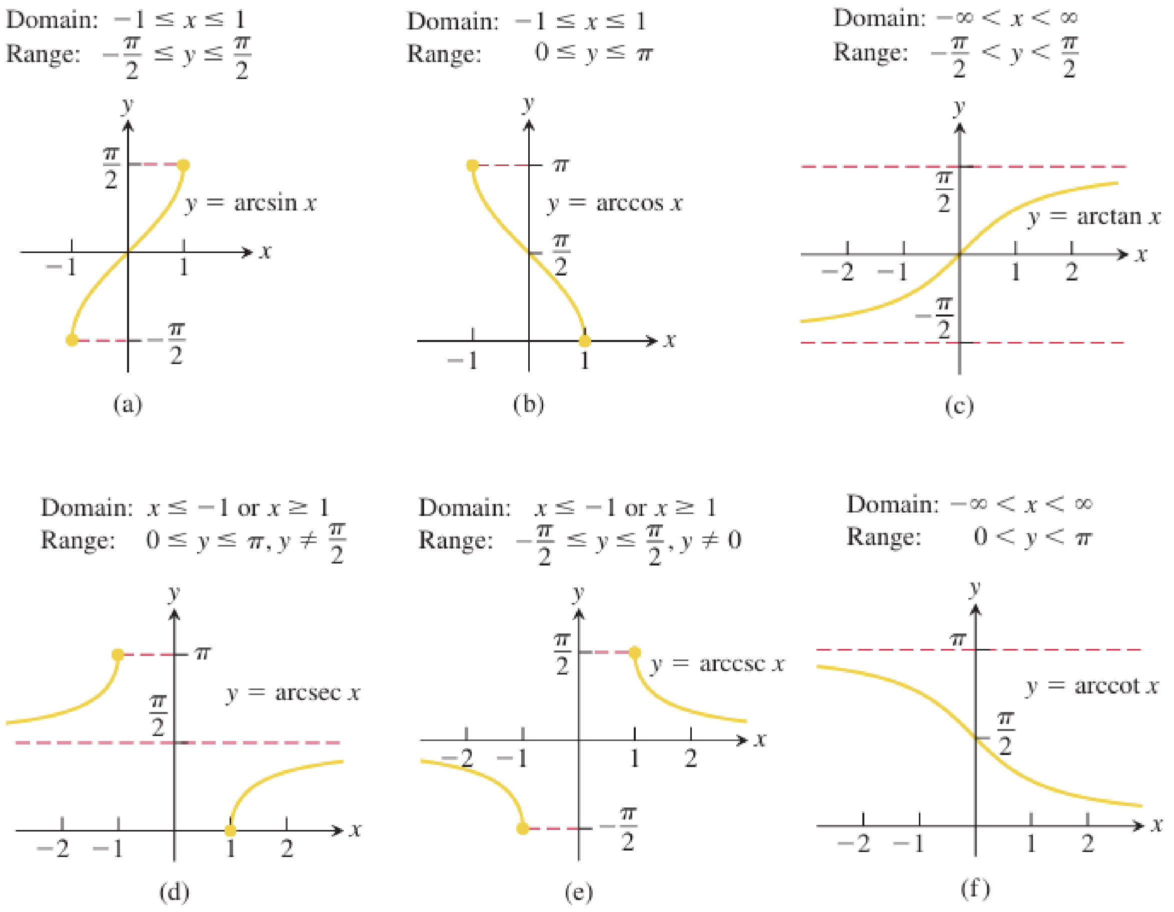


Figure 1. Graphs of the six inverse trigonometric functions and their domain & range

> **Definition**
>
> $\boldsymbol{y=\arcsin x}$ is the number in $[-\pi/2,\pi/2]$ for which $\sin y = x$.
>
> $\boldsymbol{y=\arccos x}$ is the number in $[0,\pi]$ for which $\cos y = x$.
>
> $\boldsymbol{y=\arctan x}$ is the number in $(-\pi/2,\pi/2)$ for which $\tan y = x$.
>
> $\boldsymbol{y=\operatorname{arccot} x}$ is the number in $(0,\pi)$ for which $\cot y = x$.
>
> $\boldsymbol{y=\operatorname{arcsec} x}$ is the number in $[0,\pi/2)\cup(\pi/2,\pi]$ for which $\sec y = x$.
>
> $\boldsymbol{y=\operatorname{arccsc} x}$ is the number in [-$\pi/2$,0) $\cup$ $(0,\pi/2]$for which $\csc y = x$.

The Derivative of $y = \arcsin u$

We know that the function $x = \sin y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Theorem 1 in Section 7.1 therefore assures us that the inverse function $y = \arcsin x$ is differentiable throughout the interval $-1 < x < 1$. However, we cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangents to the graph are vertical at these points.

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\cos\,(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get the general formula:

$$\boxed{\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{dx}{du}, \qquad |u| < 1.}$$

Example: $\frac{d}{dx}(\arcsin x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$

The Derivative of $y = \arctan u$

The derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$.

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\boxed{\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{dx}{du}.}$$

The Derivative of $y = \operatorname{arcsec} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, the inverse function $y = \operatorname{arcsec} x$ is differentiable.

$$\begin{aligned} y &= \operatorname{arcsec} x \\ \sec y &= x \\ \frac{d}{dx}(\sec y) &= \frac{d}{dx}x \\ \sec y \tan y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. \end{aligned}$$

To express the result in terms of x, we use the relationships $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$ to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}.$$

However, the graph shows that the slope of the graph $y = \operatorname{arcsec} x$ is always positive. Thus,

$$\frac{d}{dx}\operatorname{arcsec} x = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & if\ x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & if\ x < -1. \end{cases}$$

This can be expressed below with the absolute value symbol.

$$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

$$\frac{d}{dx}(\operatorname{arcsec} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{dx}{du}, \qquad |u| > 1.$$

Derivatives of the Other Three Inverse Trigonometric Functions

Inverse Function-Inverse Cofunction Identities

$$\arccos x = \pi/2 - \arcsin x$$
$$\operatorname{arccot} x = \pi/2 - \arctan x$$
$$\operatorname{arccsc} x = \pi/2 - \operatorname{arcsec} x$$

This means that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, $\frac{d}{dx}(\arccos x) = \frac{d}{dx}\left(\frac{\pi}{2} - \arcsin x\right) = -\frac{d}{dx}(\arcsin x) = -\frac{1}{\sqrt{1-x^2}}$

Derivatives of the Inverse Cofunctions

$$\frac{d}{dx}(\arccos u) = -\frac{1}{\sqrt{1-u^2}}\frac{dx}{du}, \qquad |u| < 1.$$
$$\frac{d}{dx}(\operatorname{arccot} u) = -\frac{1}{1+u^2}\frac{dx}{du}.$$
$$\frac{d}{dx}(\operatorname{arccsc} u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{dx}{du}, \qquad |u| > 1.$$

Integration Formulas

Integrals evaluated with Inverse Trigonometric Functions

The following formulas hold for any constant $a > 0$.

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad (Valid\ for\ u^2 < a^2)$

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad (Valid\ for\ all\ u)$

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad (Valid\ for\ |u| > a > 0)$

Example: $\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$

7.7 Hyperbolic Functions

Definitions and Identities

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad and \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

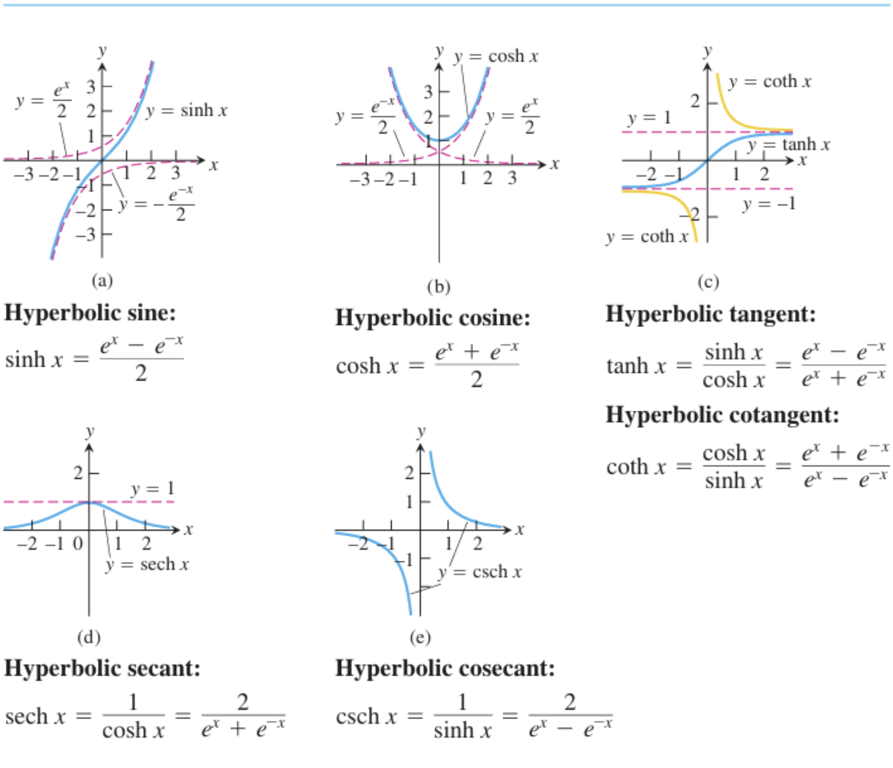


Figure 1. The six basic hyperbolic functions

Hyperbolic functions satisfy the identities listed below. Except for differences in sign, these resemble identities we know for the trigonometric functions. The identities are proved directly from the definitions.

Identities for Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \sinh 2x = 2\sinh x\cosh x \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2} \qquad \sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x \qquad \coth^2 x = 1 + \operatorname{csch}^2 x$$

For example, $2\sinh x\cosh x = 2\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^{2x}-e^{-2x}}{2} = \sinh 2x$.

Derivatives and Integrals of Hyperbolic Functions

The derivative formulas are derived from the derivative of e^u:

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) = \frac{e^u du/dx + e^{-u} du/dx}{2} = \cosh u \frac{du}{dx}$$

From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right) = -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} = -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Similarly, other derivatives of hyperbolic functions can be obtained.

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx} \qquad \frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx} \qquad \frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

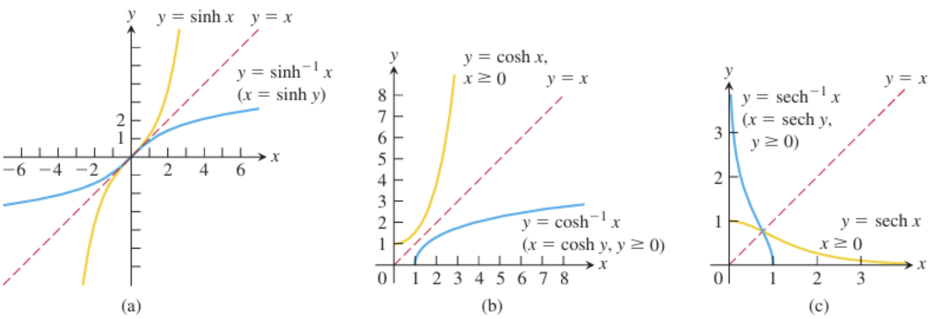
The integrals of hyperbolic functions can also be obtained.

Integrals of Hyperbolic Functions

$$\int \sinh u\, du = \cosh u + C \qquad \int \cosh u\, du = \sinh u + C \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Inverse Hyperbolic Functions



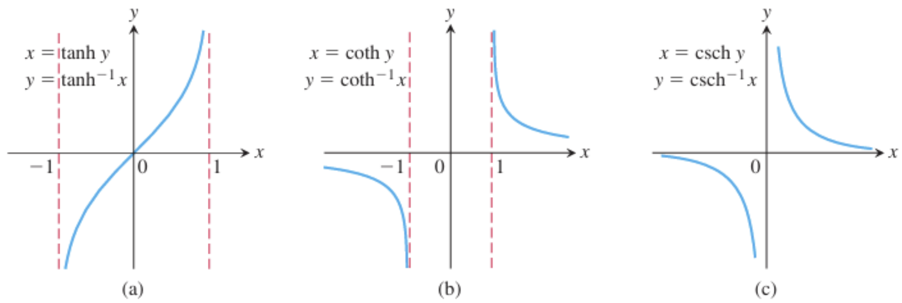


Figure 3. The graphs of the inverse hyperbolic functions.

The functions $y=\cosh^{-1}x$ and $y=\operatorname{sech}^{-1}x$ are inverses of the restricted function $y=\cosh x\,, x\geq 0$ and $y=\operatorname{sech} x\,, x\geq 0$.

Useful Identities

Identities for Inverse Hyperbolic Functions

$$\operatorname{sech}^{-1}x=\cosh^{-1}\frac{1}{x}\qquad \operatorname{csch}^{-1}x=\sinh^{-1}\frac{1}{x}\qquad \coth^{-1}x=\tanh^{-1}\frac{1}{x}$$

For example: $\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right)=\frac{1}{\cosh\left(\cosh^{-1}\frac{1}{x}\right)}=\frac{1}{\frac{1}{x}}=x$

Derivatives of Inverse Hyperbolic Functions

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}u)=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}\qquad u>1$$

$$\frac{d}{dx}(\tanh^{-1}u)=\frac{1}{1-u^2}\frac{du}{dx}\qquad |u|<1$$

$$\frac{d}{dx}(\coth^{-1}u)=\frac{1}{1-u^2}\frac{du}{dx}\qquad |u|>1$$

$$\frac{d}{dx}(\operatorname{sech}^{-1}u)=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}\qquad 0<u<1$$

$$\frac{d}{dx}(\operatorname{csch} u)=-\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}\qquad u\neq 0$$

Example: Show that $\frac{d}{dx}(\cosh^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$.

Sol. $(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}=\frac{1}{\sinh(\cosh^{-1}x)}=\frac{1}{\sqrt{\cosh^2(\cosh^{-1}x)-1}}=\frac{1}{\sqrt{x^2-1}}$, the Chain Rule gives the final result: $\frac{d}{dx}(\cosh^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$.

Integrals leading to Inverse Hyperbolic Functions

$$\int\frac{du}{\sqrt{a^2+u^2}}=\sinh^{-1}\left(\frac{u}{a}\right)+C\qquad \int\frac{du}{\sqrt{u^2-a^2}}=\cosh^{-1}\left(\frac{u}{a}\right)+C$$

$$\int\frac{du}{a^2-u^2}=\begin{cases}\frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)+C & (u^2<a^2)\\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right)+C & (u^2>a^2)\end{cases}$$

$$\int\frac{du}{u\sqrt{a^2-u^2}}=-\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right)+C\qquad \int\frac{du}{u\sqrt{a^2+u^2}}=-\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right|+C$$

Example: $\int\frac{2dx}{\sqrt{3+4x^2}}=\int\frac{du}{\sqrt{a^2+u^2}}=\sinh^{-1}\left(\frac{u}{a}\right)+C=\sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)+C$, $\int_0^1\frac{2dx}{\sqrt{3+4x^2}}=\sinh^{-1}\left(\frac{2}{\sqrt{3}}\right)=0.98665$.

7.8 Relative Rates of Growth