IV. Applications of Derivatives

미분의 활용

4.1 Extreme Values of Functions on Closed Intervals

Definition 1

Let f be a functions with domain D. Then f has an ***absolute*** maximum(최댓값) value on D at a point c if

$$f(x) \le f(c) \qquad for\ all\ x\ in\ D$$

and an ***absolute minimum***(최솟값) value on D at c if

$$f(x) \ge f(c) \qquad for\ all\ x\ in\ D.$$

Maximum and minimum values are called ***extreme values***(극값) of the function f. Absolute maxima or minima are also referred to as ***global*** maxima or minima. Function defined by the same equation or formula can have different extrema (maximum or minimum values), depending on the domain. A function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

Example:

Function rule	Domain D	Absolute extrema on D
$y = x^2$	$(-\infty, \infty)$	No absolute maximum Absolute minimum of 0 at $x = 0$
	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
	$(0, 2)$	No absolute extrema

Theorem 1-The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum m at $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m, f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (실수의 완비성 공리) and we will not give it here.

Local (Relative) Extreme Values

> ### Definition 2
>
> A function f has a ***local maximum***(극댓값) value at a point c within its domain D if $f(x) \le f(c)$ for all $x \in D$ lying in some open interval containing c.
> A function f has a ***local minimum***(극솟값) value at a point c within its domain D if $f(x) \ge f(c)$ for all $x \in D$ lying in some open interval containing c.

If the domain of f is the closed interval $[a, b]$, then f has a local maximum at the endpoint $x = a$ if $f(x) \le f(a)$ for all x in some half-open interval $[a, a + \delta), \delta > 0$. Likewise, f has a local maximum at an interior point $x = c$ if $f(x) \le f(c)$ for all x in some open interval $(c - \delta, c + \delta), \delta > 0$, and a local maximum at the endpoint $x = b$ if $f(x) \le f(b)$ for all x in some half-open interval $(b - \delta, b], \delta > 0$. The inequalities are reversed for local minimum values.

Local extrema are also called relative extrema. Some functions such as $f(x) = \sin\frac{1}{x}$ have infinitely many local extrema, even over a finite interval. An absolute maximum is also a local maximum. This is same for minimum values.

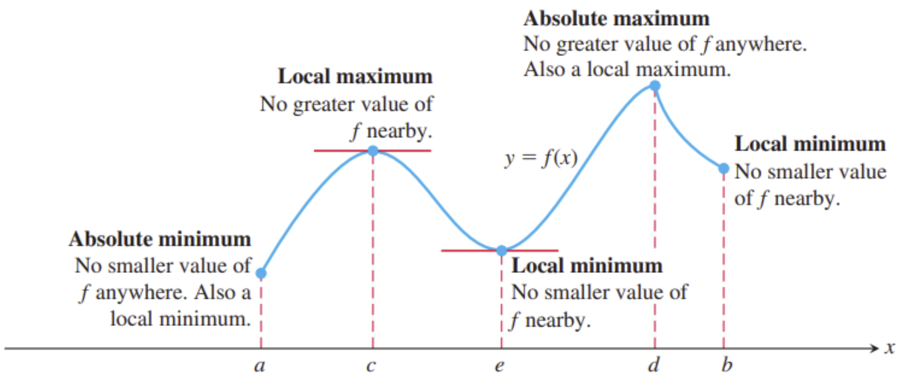

Figure 1. Extreme values of function $y = f(x)$ at domain $D = [a, b]$.

Finding Extrema

> ### Theorem 2-The First Derivative Theorem for Local Extreme Values
>
> If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then
>
> $$f'(c) = 0.$$

Proof Suppose that f has a local maximum value at $x = c$ so that $f(x) - f(x) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$. This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$.

When we examine these limits separately, we find that $f'(c) = \lim_{x \to c+} \frac{f(x)-f(c)}{x-c} \le 0$ $(\because (x - c) > 0, f(x) \le f(c))$

And $f'(c) = \lim_{x \to c+} \frac{f(x)-f(c)}{x-c} \ge 0$ $(\because (x - c) < 0, f(x) \le f(c))$. Together, this means that $f'(c) = 0$.

To prove it for local minimum values, use $f(x) \ge f(c)$, which reverses the inequalities above.

Definition 3

An interior point of the domain of a function f where f' is zero or undefined is a ***critical point*** of f.

(주의: critical point를 적으라 하면 (x, y)와 같이 점의 좌표를 적지 말고 $x = c$ 같이 domain상의 x좌표를 적어야 함.)

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical point of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

Example: Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Sol. Evaluate the function at the critical points and endpoint and take the largest and smallest of the resulting values.

The first derivative $f'(x) = \frac{2}{3}x^{-1/3}$ has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are: $f(0) = 0, f(-2) = (-2)^{2/3}, f(3) = 3^{2/3}$.

So, the function's absolute maximum value is $3^{2/3} \cong 2.08\ at\ x = 3$, and the absolute minimum value is 0 at $x = 0$.

4.2 The Mean Value Theorem

Rolle's Theorem

Theorem 3-Rolle's Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof $y = f(x)$ is continuous over the closed interval $[a, b]$. By EVT, f attains both an absolute maximum value M and an absolute minimum m at $[a, b]$. If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$. If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$, it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b).

The Mean Value Theorem

Theorem 4-The Mean Value Theorem

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$**.** Then there is at least one number c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof Let $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Then $g(x)(The\ secant\ line)$ goes through both $(a, f(a))$ and $(b, f(b))$. Then let $h(x) = f(x) - g(x)$. The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$, being continuous on $[a, b]$ and differentiable on (a, b) $(\because f, g\ also\ satisfies\ both)$. Also $h(a) = h(b) = 0$. Therefore $h'(c) = 0$ at some point $c \in (a, b)$ by Rolle's Theorem.

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \quad \Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}.$$

> **Corollary(따름정리) 1**
>
> If $f'(x) = 0$ at each and every point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Proof We want to show that f has a constant value on the interval (a, b). Suppose that there's x_1 and x_2 which are any two points in (a, b) with $x_1 < x_2$. j satisfies the hypotheses(가설, 전제조건), so the following is true:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

which $c \in (x_1, x_2)$. Since $f'(x) = 0$ for every x in (a, b), $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0$, $\therefore f(x_2) = f(x_1)$. This implies that f is a constant function $f(x) = C$.

> **Corollary 2**
>
> If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

Proof At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is $h'(x) = f'(x) - g'(x) = 0$. Thus, $h(x) = C$ on (a, b) by Corollary 1. That is, $f(x) - g(x) = C$ on (a, b), so $f(x) = g(x) + C$.

4.3 Monotonic Functions and the First Derivative Test

Increasing Functions and Decreasing Functions

A function that is increasing or decreasing on an interval is said to be ***monotonic*** on the interval.

> **Corollary 3**
>
> Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
>
> If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
> If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. For some c in (x_1, x_2). The sign(부호) of the right-hand side of this equation is the same as this sign of $f'(c)$, because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b).

First Derivative Test for Local Extrema

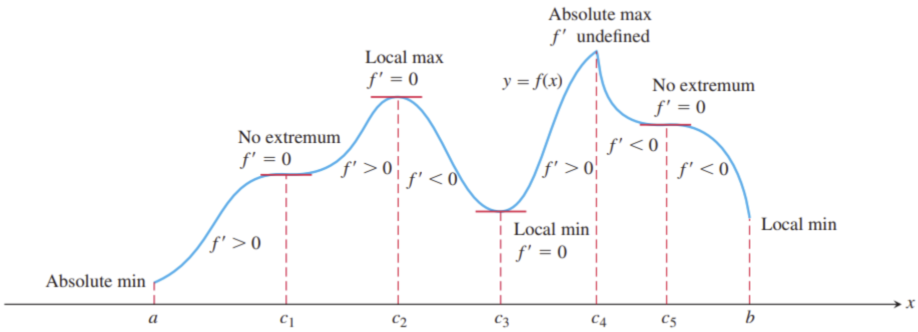

Figure 2. The critical points and the first derivative' sign changes

> **First Derivative Test for Local Extrema**
>
> Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right.
>
> 1. If f' changes from negative to positive at c, then f has a local minimum at c.
> 2. If f' changes from positive to negative at c, then f has a local maximum at c.
> 3. If f' does not change sign at c(that is, f' is positive on both sided of c or negative on both sides), then f has no local extremum at c.

Proof Part (1): Since the sing of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c. Part (2) and (3) are proved similarly.

4.4 Concavity and Curve Sketching

> **Definition 4**
>
> The graph of a differentiable function $y = f(x)$ is
>
> (a) ***concave***(오목하다, ↔convex) ***up*** on an open interval I if f' is increasing on I.
>
> (b) ***concave down*** on an open interval I if f' is decreasing on I.

> **The Second Derivative Test for Concavity**
>
> Let $y = f(x)$ be twice-differentiable on an interval I.
>
> 1. If $f'' > 0$ on I, the graph of f over I is concave up.
> 2. If $f'' < 0$ on I, the graph of f over I is concave down.

Points of Inflection

Definition 5

A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a ***point of inflection***(변곡점). At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

However, there can be a case which inflection point does not occur even though both derivatives exist and $f'' = 0$. The following example is the case:

Example: The curve $y = x^4$ has no inflection point at $x = 0$. Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. The curve is concave up everywhere.

A point of inflection can occur even though neither the first nor the second derivative exists.

Example: The graph $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$. However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there.

Therefore, second derivative not existing or being zero does not guarantee an inflection point occurring there. Sketching the curve can help identifying inflection points.

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, looking for the sing of f'' can help determine the presence of local extrema.

Theorem 5- Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) - 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither,

Proof Part (1): If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive from negative at c so f has a local maximum at c by the First Derivative Test. The proof of Part (2) is similar.

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key point, such as the intercepts(절편) and the points found in Steps 3-5.
8. Sketch the curve together with ant asymptotes that exist.

4.6 Newton's Method

Procedure for Newton's Method

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+!} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad if\ f'(x_n) \neq 0.$$

Example: Approximate the positive root of the equation $f(x) = x^2 - 2 = 0$.

Sol: With $f(x)$ and $f'(x) = 2x$, Apply the Newton's Method. Then $x_{n+1} = x_n - \frac{x_n^2-2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$ Then starting from the starting value $x_0 = 1$, repeat the process to calculate the next approximation.

4.7 Antiderivatives

Finding Antiderivatives

Definition 6

A function F is an ***antiderivative***(역도함수) of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example: The function $F(x) = x^2$ is an antiderivative of $f = 2x$ because $F'(x) = (x^2)' = 2x = f(x)$.

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The functions $x^2 + C$, where C is an ***arbitrary constant***(임의의 상수), form all the antiderivatives of $f(x) = 2x$.

Theorem 8

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Antiderivative formulas, k a nonzero constant

$$x^n: \ \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$

$$\sin kx: \ -\frac{1}{k}\cos kx + C \qquad \cos kx: \ \frac{1}{k}\sin kx + C$$

$$\sec^2 kx: \ \frac{1}{k}\tan kx + C \qquad \csc^2 kx: \ -\frac{1}{k}\cot kx + C$$

$$\sec kx \tan kx: \ \frac{1}{k}\sec kx + C \qquad \csc kx \cot kx: \ -\frac{1}{k}\csc kx + C$$

	Function	General antiderivative
Constant Multiple Rule	$kf(x)$	$kF(x)+C,\quad k\ a\ constant$
Sum or Difference Rule	$f(x)\pm g(x)$	$F(x)\pm G(x)+C$

Example: Find the general antiderivative of $f(x)=\frac{3}{\sqrt{x}}+\sin 2x$.

Sol: We have that $f(x)=3g(x)+h(x)$ if we let $g(x)=\frac{1}{\sqrt{x}}$ and $h(x)=\sin 2x$. Since $G(x)=2\sqrt{x}$ is an antiderivative of $g(x)$, and $H(x)=-\frac{1}{2}\cos 2x$ is an antiderivative of $h(x)$, from the Constant Multiple Rule and the Sum Rule, we the get that $F(x)=3G(x)+H(x)+C=6\sqrt{x}-\frac{1}{2}\cos 2x+C$ is the general antiderivative formula for $f(x)$, where C is an arbitrary constant.

Initial Value Problems and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation $\frac{dy}{dx}=f(x)$. This is called ***differential equation***(미분방정식), since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition $y(x_0)=y_0$. This condition means the function $y(x)$ has the value y_0 when $x=x_0$. The combination of a differential equation and an initial condition is called an ***initial value problem***(초깃값 문제).

Tho most general antiderivative $F(x)+C$ of the function $f(x)$ gives the ***general solution***(일반해) $y=F(x)+C$ of the differential equation $\frac{dy}{dx}=f(x)$. The general solution gives *all* the solutions of the equation. We solve the differential equation by finding its general solution. We then solve the initial value problem by finding the ***particular solution***(특수해) that satisfies the initial condition $y(x_0)=y_0$.

Indefinite Integrals

Definition 7

The collection of all antiderivatives of f is called the ***indefinite integral***(부정적분) of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an ***integral sign.*** The function f is the ***integrand***(피적분함수) of the integral, and x is the ***variable of integration.***