

Algorithms in the Nashlib set in various programming languages

John C Nash, retired professor, University of Ottawa

Peter Olsen, retired ??

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Abstract

Algorithms from the book Nash (1979) are implemented in a variety of programming languages including Fortran, BASIC, Pascal, Python and R.

Overview of this document

A companion document **Overview of Nashlib and its Implementations** describes the process and computing environments for the implementation of Nashlib algorithms. This document gives some comments and/or details relating to the algorithms themselves or their implementations.

Algorithms 1 and 2 – one-sided SVD and least squares solution

These were two of the first algorithms to interest the first author in compact codes. At the time (1973-1978) he was working at Agriculture Canada in support of econometric modeling. More or less “regular” computers required accounts linked to official projects, but there was a time-shared Data General NOVA that offered 4K to 7K byte working spaces for data and programs in interpreted BASIC. BASIC of a very similar dialect was available also on an HP 9830 calculator. On these machines, availability of a terminal or the calculator was the only limitation to experimentation with recent innovations in algorithms. In particular, a lot of modeling was done with linear least squares regression, mostly using the traditional normal equations. The singular value decomposition and other methods such as the Householder, Givens or Gram-Schmidt approaches to the QR matrix decomposition were relatively recent innovations. However, the code for the Golub-Kahan SVD was rather long for both the hardware and the BASIC language. Instead, a one-sided Jacobi method was developed from ideas of Hestenes (1958) and Chartres (1962). Some work by Kaiser (1972) was also observed. Later workers have generally credited Hestenes with this approach, and he certainly wrote about it, but we (JN) suspect strongly that he never actually attempted an implementation. In a conversation at a conference, Chartres said that some experiments were tried, but that he believed no production usage occurred. We must remember that access to computers until the 1970s was quite difficult.

The method published in Nash (1975) and later revised in Nash and Shlien (1987) ignored some advice that Jacobi rotations should not use angles greater than $\pi/4$ (see Forsythe and Henrici (1960)). This allowed of a cyclic process that not only developed a form of the decomposition, but also sorted it to effectively present the singular values in descending order of size. This avoided extra program code of about half the length of the svd routine.

About 2 decades after Nash (1975), there was renewed interest in one-sided Jacobi methods, but rather little acknowledgment of the earlier development, and much more complicated codes. ?? How far to reference more recent developments??

Fortran

Listing

Note that this is a single precision code. Very few modern calculations are carried out at this precision. Moreover, the dialect of Fortran (Fortran 77) is now decidedly old-fashioned, though it compiles and executes just fine.

```
C&&& A1-2
C TEST ALGS. 1 & 2 J.C. NASH JULY 1978, APRIL 1989
C USES FRANK MATRIX COLUMNS
  LOGICAL ESVD,NTOL
  INTEGER N,ND,IP0S(10),NVAR,MD,I,J,K,YPOS,M
  REAL A(30,10),D(30,11),G(30),X(10),Z(10),Y(30),Q,V(10,10),EPS
  EXTERNAL FRANKM
C I/O CHANNELS
  NIN=5
  NOUT=6
  ND=10
  MD=30
  D(1,1)=5
  D(1,2)=1.0E-6
  D(1,3)=1
  Y(1)=1
  D(2,1)=6
  D(2,2)=0.999999
  D(2,3)=1
  Y(2)=2
  D(3,1)=7
  D(3,2)=2.00001
  D(3,3)=1
  Y(3)=3
  D(4,1)=8
  D(4,2)=2.9999
  D(4,3)=1
  Y(4)=4
  N=3
  M=4
  DO 30 J=1,N
  DO 25 I=1,M
    A(I,J)=D(I,J)
 25 CONTINUE
 30 CONTINUE
  ESVD=.FALSE.
  WRITE(NOUT,955)(Y(I),I=1,M)
955 FORMAT(1H ,5E16.8)
  NTOL=.FALSE.
  Q = 1e-5
  WRITE(NOUT,956)Q
956 FORMAT(' SING. VALS. .LE.',E16.8,' ARE PRESUMED ZERO')
  IF(Q.LT.0.0) STOP
C "MACHINE PRECISION" VALUE
  EPS=1E-6
  CALL A2LSVD(M,N,A,MD,EPS,V,ND,Z,NOUT,Y,G,X,Q,ESVD,NTOL)
  WRITE(NOUT,957)(J,X(J),J=1,N)
```

```

957 FORMAT(' X(',I3,')=',1PE16.8)
STOP
END
SUBROUTINE OUT(A,NA,N,NP,NOUT)
C J.C. NASH JULY 1978, APRIL 1989
INTEGER NA,N,NOUT,I,J
REAL A(NA,NP)
DO 20 I=1,N
WRITE(NOUT,951)I
951 FORMAT(' ROW',I3)
WRITE(NOUT,952)(A(I,J),J=1,NP)
952 FORMAT(1H ,1P5E16.8)
20 CONTINUE
RETURN
END
SUBROUTINE FRANKM(M,N,A,NA)
C J.C. NASH JULY 1978, APRIL 1989
INTEGER M,N,NA,I,J
C INPUTS FRANK MATRIX M BY N INTO A
REAL A(NA,N)
DO 20 I=1,M
DO 10 J=1,N
A(I,J)=AMINO(I,J)
10 CONTINUE
20 CONTINUE
RETURN
END
SUBROUTINE A3PREP(M,N1,A,NA,AIN)
C PREPARE A3 TEST
C J.C. NASH JULY 1978, APRIL 1989
C MATRIX M BY N=N1-1 IS INPUT VIA SUBROUTINE AIN
C COL. N1 IS SET TO SUM OF OTHER COLS. - UNIT SOLUTION ELEMENTS
C BUT ONLY IF M=N - OTHERWISE SIMPLY INPUT MATRIX
C NA = FIRST DIMENSION OF A
INTEGER M,N1,NA,N,J,I
REAL A(NA,N1),S
N=N1-1
CALL AIN(M,N,A,NA)
IF(M.NE.N)RETURN
DO 40 I=1,N
S=0.0
DO 30 J=1,N
S=S+A(I,J)
30 CONTINUE
A(I,N1)=S
40 CONTINUE
RETURN
END
SUBROUTINE A1SVD(M,N,A,NA,EPS,V,NV,Z,IPR)
C ALGORITHM 1 SINGULAR VALUE DECOMPOSITION BY COLUMN ORTHOGONA-
C LISATION VIA PLANE ROTATIONS
C J.C. NASH JULY 1978, FEBRUARY 1980, APRIL 1989
C M BY N MATRIX A IS DECOMPOSED TO U*Z*VT

```

```

C  A   =   ARRAY CONTAINING A (INPUT),   U (OUTPUT)
C  NA  =   FIRST DIMENSION OF A
C  EPS =   MACHINE PRECISION
C  V   =   ARRAY IN WHICH ORTHOGAONAL MATRIX V IS ACCUMULATED
C  NV  =   FIRST DIMENSION OF V
C  Z   =   VECTOR OF SINGULAR VALUES
C  IPR =   PRINT CHANNEL   IF IPR.GT.0 THEN PRINTING
C  STEP 0
      INTEGER M,N,J1,N1,COUNT
      REAL A(NA,N),V(NV,N),Z(N),EPS,TOL,P,Q,R,VV,C,S
C  UNDERFLOW AVOIDANCE STRATEGY
      REAL SMALL
      SMALL=1.0E-36
C  ABOVE IS VALUE FOR IBM
      TOL=N*N*EPS*EPS
      DO 6 I=1,N
        DO 4 J=1,N
          V(I,J)=0.0
        4   CONTINUE
          V(I,I)=1.0
        6   CONTINUE
          N1=N-1
C  STEP 1
      10  COUNT=N*(N-1)/2
C  STEP 2
      DO 140 J=1,N1
C  STEP 3
        J1=J+1
        DO 130 K=J1,N
C  STEP 4
          P=0.0
          Q=0.0
          R=0.0
C  STEP 5
          DO 55 I=1,M
            IF(ABS(A(I,J)).GT.SMALL.AND.ABS(A(I,K)).GT.SMALL)
              #      P=P+A(I,J)*A(I,K)
              IF(ABS(A(I,J)).GT.SMALL)Q=Q+A(I,J)**2
              IF(ABS(A(I,K)).GT.SMALL)R=R+A(I,K)**2
C          P=P+A(I,J)*A(I,K)
C          Q=Q+A(I,J)**2
C          R=R+A(I,K)**2
          55  CONTINUE
C  STEP 6
          IF(Q.GE.R)GOTO 70
          C=0.0
          S=1.0
          GOTO 90
C  STEP 7
      70  IF(R.LE.TOL)GOTO 120
          IF((P*P)/(Q*R).LT.TOL)GOTO 120
C  STEP 8
          Q=Q-R

```

```

      VV=SQRT(4.0*P**2+Q**2)
      C=SQRT((VV+Q)/(2.0*VV))
      S=P/(VV*C)
C   STEP 9
  90   DO 95 I=1,M
        R=A(I,J)
        A(I,J)=R*C+A(I,K)*S
        A(I,K)=-R*S+A(I,K)*C
  95   CONTINUE
C   STEP 10
      DO 105 I=1,N
        R=V(I,J)
        V(I,J)=R*C+V(I,K)*S
        V(I,K)=-R*S+V(I,K)*C
  105  CONTINUE
C   STEP 11
      GOTO 130
  120  COUNT=COUNT-1
C   STEP 13
  130  CONTINUE
C   STEP 14
  140  CONTINUE
C   STEP 15
      IF(IPR.GT.0)WRITE(IPR,964)COUNT
  964  FORMAT(1H ,I4,10H ROTATIONS)
      IF(COUNT.GT.0)GOTO 10
C   STEP 16
      DO 220 J=1,N
C   STEP 17
      Q=0.0
C   STEP 18
      DO 185 I=1,M
        Q=Q+A(I,J)**2
  185  CONTINUE
C   STEP 19
      Q=SQRT(Q)
      Z(J)=Q
      IF(IPR.GT.0)WRITE(IPR,965)J,Q
  965  FORMAT( 4H SV(,I3,2H)=,1PE16.8)
C   STEP 20
      IF(Q.LT.TOL)GOTO 220
C   STEP 21
      DO 215 I=1,M
        A(I,J)=A(I,J)/Q
  215  CONTINUE
C   STEP 22
  220  CONTINUE
      RETURN
      END
      SUBROUTINE A2LSVD(M,N,A,NA,EPS,V,NV,Z,IPR,Y,G,X,Q,ESVD,NTOL)
C   J.C. NASH   JULY 1978, FEBRUARY 1980, APRIL 1989
C   SAME COMMENTS AS SUBN A1SVD EXCEPT FOR
C   G   =   WORKING VECTOR IN N ELEMENTS

```

```

C   Y   = VECTOR CONTAINING M VALUES OF DEPENDENT VARIABLE
C   X   = SOLUTION VECTOR
C   Q   = TOLERANCE FOR SINGULAR VALUES. THOSE .LE. Q TAKEN AS ZERO.
C   ESVD = LOGICAL FLAG SET .TRUE. IF SVD ALREADY EXISTS IN A,Z,V
C   NTOL = LOGICAL FLAG SET .TRUE. IF ONLY NEW TOLERANCE Q.
      LOGICAL ESVD,NTOL
      INTEGER M,N,IPR,I,J
      REAL A(NA,N),V(NV,N),Z(N),Y(M),G(N),X(N),S,Q
C STEP 1
      IF(NTOL)GOTO 41
      IF(.NOT.ESVD)CALL A1SVD(M,N,A,NA,EPS,V,NV,Z,IPR)
      IF(IPR.GT.0)WRITE(IPR,965)(J,Z(J),J=1,N)
965  FORMAT(16H SINGULAR VALUE(,I3,2H)=,1PE16.8)
C STEP 2 VIA SUBROUTINE CALL
C ALTERNATIVE WITHOUT G
C NO STEP 3
C STEP 3 UT*Y=G
      DO 36 J=1,N
        S=0.0
        DO 34 I=1,M
          S=S+A(I,J)*Y(I)
34    CONTINUE
        G(J)=S
36    CONTINUE
C STEP 4
41  IF(Q.LT.0.0)STOP
C STEP 5
      DO 56 J=1,N
        S=0.0
        DO 54 I=1,N
          IF(Z(I).GT.Q)S=S+V(J,I)*G(I)/Z(I)
54    CONTINUE
        X(J)=S
56    CONTINUE
C STEP 6
C NEW TOLERANCE VIA NEW CALL
      RETURN
      END

```

Example output

```

gfortran ../fortran/dr0102.f
mv ./a.out ../fortran/dr0102.run
../fortran/dr0102.run < ../fortran/dr0102.txt

```

```

##      0.10000000E+01  0.20000000E+01  0.30000000E+01  0.40000000E+01
## SING. VALS. .LE.  0.99999997E-05 ARE PRESUMED ZERO
##      3 ROTATIONS
##      3 ROTATIONS
##      1 ROTATIONS
##      0 ROTATIONS
## SV( 1)=  1.37529879E+01
## SV( 2)=  1.68960798E+00
## SV( 3)=  1.18504076E-05

```

```

## SINGULAR VALUE( 1)= 1.37529879E+01
## SINGULAR VALUE( 2)= 1.68960798E+00
## SINGULAR VALUE( 3)= 1.18504076E-05
## X( 1)= 1.00434840E+00
## X( 2)= -4.34857607E-03
## X( 3)= -4.02174187E+00

```

Special implementations

Most singular value decomposition codes are much, much more complicated than Algorithm 1 of the Nashlib collection. For some work on the magnetic field of Jupiter for NASA, Sidey Timmins has used an extended (quad) precision version of the method. One of us (JN) has converted an updated algorithm (Nash and Shlien (1987)) to the Fortran 95 dialect so the multiple precision FM Fortran tools of David M. Smith (see <http://dmsmith.lmu.build/>).

?? include this code and example in the repo??

BASIC

Listing

```

5 PRINT "dr0102.bas -- Nashlib Alg 01 and 02 driver"
10 PRINT "from ENHSVA APR 7 80 -- MOD 850519, remod 210113"
20 LET E1=1.0E-7
30 PRINT "ONE SIDED TRANSFORMATION METHOD FOR REGRESSIONS VIA"
40 PRINT "THE SINGULAR VALUE DECOMPOSITION -- J.C.NASH 1973,79"
150 LET M=4
160 LET N=3
210 DIM Y(M,N+1),A(M,N),T(N,N),G(N),X(N),Z(N),U(N),B(M)
220 DIM F$(10)
230 LET F$="K"
236 PRINT "Prep matrix and RHS"
240 LET Y(1,1)=5
241 LET Y(1,2)=1.0E-6
242 LET Y(1,3)=1
243 LET B(1)=1
250 LET Y(2,1)=6
251 LET Y(2,2)=0.999999
252 LET Y(2,3)=1
253 LET B(2)=2
260 LET Y(3,1)=7
261 LET Y(3,2)=2.00001
262 LET Y(3,3)=1
263 LET B(3)=3
270 LET Y(4,1)=8
271 LET Y(4,2)=2.9999
272 LET Y(4,3)=1
273 LET B(4)=4
500 FOR I=1 TO M
510 FOR J=1 TO N-1
520 LET A(I,J)=Y(I,J)
530 NEXT J
535 quit
540 LET A(I,N)=E3
550 NEXT I

```

```

560 LET E2=N*N*E1*E1
570 PRINT
580 FOR I=1 TO N
590 FOR J=1 TO N
600 LET T(I,J)=0
610 NEXT J
620 LET T(I,I)=1
630 NEXT I
640 LET I9=0
650 IF N=1 THEN GOTO 1150
660 LET N2=N*(N-1)/2
670 LET N1=N-1
680 LET N9=N2
690 LET I9=I9+1
700 FOR J=1 TO N1
710 LET J1=J+1
720 FOR K=J1 TO N
730 LET P=0
740 LET Q=0
750 LET R=0
760 FOR I=1 TO M
770 LET P=P+A(I,J)*A(I,K)
780 LET Q=Q+A(I,J)*A(I,J)
790 LET R=R+A(I,K)*A(I,K)
800 NEXT I
810 IF Q>=R THEN GOTO 850
820 LET C=0
830 LET S=1
840 GOTO 920
850 IF (Q*R)<=0 THEN GOTO 1040
860 IF P*P/(Q*R)<E2 THEN GOTO 1040
870 LET Q=Q-R
880 LET P=2*P
890 LET V1=SQR(P*P+Q*Q)
900 LET C=SQR((V1+Q)/(2*V1))
910 LET S=P/(2*V1*C)
920 FOR I=1 TO M
930 LET V1=A(I,J)
940 LET A(I,J)=V1*C+A(I,K)*S
950 LET A(I,K)=-V1*S+A(I,K)*C
960 NEXT I
970 FOR I=1 TO N
980 LET V1=T(I,J)
990 LET T(I,J)=V1*C+T(I,K)*S
1000 LET T(I,K)=-V1*S+T(I,K)*C
1010 NEXT I
1020 LET N9=N2
1030 GOTO 1060
1040 LET N9=N9-1
1050 IF N9=0 THEN GOTO 1150
1051 REM ?? GOTO was EXIT for NS BASIC
1060 NEXT K
1070 NEXT J

```



```

1080 PRINT "SWEEP",I9,
1090 IF O1>0 THEN PRINT #01,"SWEEP ",I9," ",
1100 IF 6*INT(I9/6)<>I9 THEN GOTO 680
1110 IF O1>0 THEN PRINT #01
1120 IF I9>=30 THEN GOTO 1150
1130 PRINT
1140 GOTO 680
1150 PRINT
1160 IF O1>0 THEN PRINT #01
1170 PRINT "CONVERGENCE AT SWEEP ",I9
1180 IF O1>0 THEN PRINT #01,"CONVERGENCE AT SWEEP ",I9
1190 FOR J=1 TO N
1200 LET Q=0
1210 FOR I=1 TO M
1220 LET Q=Q+A(I,J)^2
1230 NEXT I
1240 LET Q=SQR(Q)
1250 IF Q=0 THEN GOTO 1290
1260 FOR I=1 TO M
1270 LET A(I,J)=A(I,J)/Q
1280 NEXT I
1290 LET Z(J)=Q
1300 NEXT J
1310 PRINT
1320 PRINT "SINGULAR VALUES"
1340 FOR J=1 TO N
1350 PRINT Z(J),
1370 IF 5*INT(J/5)<>J THEN GOTO 1400
1380 PRINT
1400 NEXT J
1410 PRINT
1430 PRINT "VARIABLE # OF REGRESSAND",
1440 INPUT M2
1450 IF M2<=0 THEN GOTO 350
1470 LET S1=0
1480 FOR I=1 TO M
1490 LET S1=S1+(Y(I,M2)-E3*Y(M+1,M2))^2
1500 NEXT I
1510 FOR J=1 TO N
1520 LET S=0
1530 FOR I=1 TO M
1540 LET S=S+A(I,J)*Y(I,M2)
1550 NEXT I
1560 LET G(J)=S
1570 NEXT J
1580 PRINT "ENTER TOLERANCE FOR ZERO",
1590 INPUT Q
1600 IF Q<0 THEN GOTO 1410
1610 PRINT "SINGULAR VALUES <=",Q," ARE TAKEN AS 0"
1630 LET R=0
1640 FOR I=1 TO N
1650 LET V1=0
1660 LET S=0

```

```

1670 LET P=0
1680 FOR K=1 TO N
1690 LET C=0
1700 IF Z(K)<=Q THEN GOTO 1730
1710 LET C=1/Z(K)
1720 LET V1=V1+1
1730 LET S=S+C*T(I,K)*G(K)
1740 LET P=P+(C*T(I,K))^2
1750 NEXT K
1760 LET U(I)=P
1770 LET X(I)=S
1780 LET R=R+S*S
1790 NEXT I
1800 LET X(N)=X(N)*E3
1810 PRINT
1820 PRINT "RESIDUALS"
1840 LET C=0
1850 LET S2=0
1860 FOR I=1 TO M
1870 LET S=Y(I,M2)-X(N)
1880 FOR K=1 TO N-1
1890 LET S=S-Y(I,W(K))*X(K)
1900 NEXT K
1910 PRINT S,
1930 IF 5*INT(I/5)<>I THEN GOTO 1960
1940 PRINT
1960 LET C=C+S*S
1970 IF I=1 THEN GOTO 1990
1980 LET S2=S2+(S-S3)^2
1990 LET S3=S
2000 NEXT I
2010 PRINT
2020 LET P=0
2040 IF M<=V1 THEN GOTO 2060
2050 LET P=C/(M-V1)
2060 PRINT M-V1," DEGREES OF FREEDOM"
2080 REM PRINT
2090 PRINT "SOLUTION VECTOR - CONSTANT LAST"
2110 FOR I=1 TO N
2120 LET V1=SQR(P*U(I))
2130 PRINT "X(",W(I),")=",X(I)," STD.ERR.=" ,V1,
2140 IF O1>0 THEN PRINT #01,"X(",W(I),")=",X(I)," STD.ERR.=" ,V1,
2150 IF V1<=0 THEN GOTO 2180
2160 PRINT " T=",ABS(X(I)/V1),
2170 IF O1>0 THEN PRINT #01," T=",ABS(X(I)/V1),
2180 PRINT
2190 IF O1>0 THEN PRINT #01
2200 NEXT I
2210 PRINT "SUM OF SQUARES",C," SIGMA^2",P
2220 IF O1>0 THEN PRINT #01,"SUM OF SQUARES",C," SIGMA^2",P
2230 PRINT "NORM OF SOLUTION",SQR(R)
2240 IF O1>0 THEN PRINT #01,"NORM OF SOLUTION",SQR(R)
2250 PRINT "R SQUARED=",1-C/S1," DURBIN-WATSON STAT.=" ,S2/C

```

```

2260 IF O1>0 THEN PRINT #01,"R SQUARED=",1-C/S1," DURBIN-WATSON STAT.=" ,S2/C
2270 PRINT
2280 IF O1>0 THEN PRINT #01
2290 GOTO 1580
2300 REM GET SERIES FROM FILE
2310 PRINT "FILENAME OR 'KEYBOARD' OR 'K'",
2320 INPUT G$
2330 IF LEN(G$)>0 THEN LET F$=G$
2331 REM DEFAULTS TO LAST SETTING
2340 PRINT "DATA FROM FILE :",F$
2350 IF F$="KEYBOARD" THEN 2420
2360 IF F$<>"K" THEN 2460
2370 PRINT
2380 PRINT "ENTER SERIES"
2390 FOR I=1 TO M
2400 INPUT1 Y(I,J)
2410 IF 5*INT(I/5)=I THEN PRINT
2420 NEXT I
2430 PRINT
2440 IF O1>0 THEN GOSUB 2860
2450 RETURN
2460 IF FILE(F$)=3 THEN 2490
2470 PRINT "FILE NOT FOUND OR OF WRONG TYPE"
2480 GOTO 2310
2490 OPEN #1,F$
2500 PRINT "SERIES NAME OR #",
2510 INPUT X$
2520 IF X$(1,1)="#" THEN 2770
2530 IF TYP(1)=0 THEN 2740
2540 IF TYP(1)=1 THEN 2570
2550 READ #1,C
2560 GOTO 2530
2570 READ #1,Y$
2580 IF X$<>Y$ THEN 2530
2590 I=0
2600 PRINT "SERIES:",Y$
2610 IF O1>0 THEN PRINT #01,"SERIES:",Y$
2620 IF TYP(1)<>2 THEN 2690
2630 IF I=M THEN 2690
2640 I=I+1
2650 READ#1,Y(I,J)
2660 PRINT Y(I,J),
2670 IF 5*INT(I/5)=I THEN PRINT
2680 GOTO 2620
2690 PRINT
2700 PRINT "END OF SERIES ",I," DATA POINTS"
2710 IF O1>0 THEN GOSUB 2860
2720 CLOSE #1
2730 RETURN
2740 PRINT "END OF FILE"
2750 CLOSE #1
2760 GOTO 2310
2770 X$=X$(2)

```

```

2780 P1=VAL(X$)
2790 J=0
2800 IF TYP(1)=0 THEN 2740
2810 IF TYP(1)=1 THEN 2840
2820 READ#1,C
2830 GOTO 2800
2840 J=J+1
2850 READ#1,Y$
2860 FOR I=1 TO M
2870 PRINT #01,Y(I,J),
2880 IF 5*INT(I/5)=I THEN PRINT #01
2890 NEXT I
2900 PRINT #01
2910 RETURN

```

Example output

```

bwbasic ../BASIC/dr0102.bas
echo "done"

```

```

## Bywater BASIC Interpreter/Shell, version 2.20 patch level 2
## Copyright (c) 1993, Ted A. Campbell
## Copyright (c) 1995-1997, Jon B. Volkoff
##
## dr0102.bas -- Nashlib Alg 01 and 02 driver
## from ENHSVA APR 7 80 -- MOD 850519, remod 210113
## ONE SIDED TRANSFORMATION METHOD FOR REGRESSIONS VIA
## THE SINGULAR VALUE DECOMPOSITION -- J.C.NASH 1973,79
## Prep matrix and RHS
##
## done

```

Pascal

Listing

```

Program runsvd(input,output);
{dr0102.pas == Calculation of Singular values and vectors of an arbitrary
    real matrix, solution of linear least squares approximation
    problem.

    Modifies a method due to Kaiser. See Nash and Shlien (1987): Simple
    algorithms for the partial singular value decomposition. Computer
    Journal, vol.30, pp.268-275.

    Modified for Turbo Pascal 5.0

    Copyright 1988, 1990 J.C.Nash
}

{constype.def ==
    This file contains various definitions and type statements which are
    used throughout the collection of "Compact Numerical Methods". In many

```

cases not all definitions are needed, and users with very tight memory constraints may wish to remove some of the lines of this file when compiling certain programs.

Modified for Turbo Pascal 5.0

Copyright 1988, 1990 J.C.Nash

```

}
uses Dos, Crt; {Turbo Pascal 5.0 Modules}
{ 1. Interrupt, Unit, Interface, Implementation, Uses are reserved words now.}
{ 2. System,Dos,Crt are standard unit names in Turbo 5.0.}

const
  big = 1.0E+35;    {a very large number}
  Maxconst = 25;    {Maximum number of constants in data record}
  Maxobs = 100;     {Maximum number of observations in data record}
  Maxparm = 25;     {Maximum number of parameters to adjust}
  Maxvars = 10;     {Maximum number of variables in data record}
  acctol = 0.0001;  {acceptable point tolerance for minimisation codes}
  maxm = 20;        {Maximum number or rows in a matrix}
  maxn = 20;        {Maximum number of columns in a matrix}
  maxmn = 40;       {maxn+maxm, the number of rows in a working array}
  maxsym = 210;     {maximum number of elements of a symmetric matrix
                    which need to be stored = maxm * (maxm + 1)/2 }
  reltest = 10.0;   {a relative size used to check equality of numbers.
                    Numbers x and y are considered equal if the
                    floating-point representation of reltest*x equals
                    that of reltest*y.}
  stepredn = 0.2;   {factor to reduce stepsize in line search}
  yearwrit = 1990;  {year in which file was written}

type
  str2 = string[2];
  rmatrix = array[1..maxm, 1..maxn] of real; {a real matrix}
  wmatrix = array[1..maxmn, 1..maxn] of real; {a working array, formed
                    as one real matrix stacked on another}
  smatvec = array[1..maxsym] of real; {a vector to store a symmetric matrix
                    as the row-wise expansion of its lower triangle}
  rvector = array[1..maxm] of real; {a real vector. We will use vectors
                    of m elements always. While this is NOT space efficient,
                    it simplifies program codes.}
  cgmethodtype= (Fletcher_Reeves,Polak_Ribiere,Beale_Sorenson);
  {three possible forms of the conjugate gradients updating formulae}
  probdata = record
    m      : integer; {number of observations}
    nvar   : integer; {number of variables}
    nconst: integer; {number of constants}
    vconst: array[1..Maxconst] of real;
    Ydata  : array[1..Maxobs, 1..Maxvars] of real;
    nlls   : boolean; {true if problem is nonlinear least squares}
  end;

{
  NOTE: Pascal does not let us define the work-space for the function

```

```

    within the user-defined code. This is a weakness of Pascal for this
    type of work.
}
var {global definitions}
    banner      : string[80]; {program name and description}

function calceps:real;
{calceps.pas ==
    This function returns the machine EPSILON or floating point tolerance,
    the smallest positive real number such that 1.0 + EPSILON > 1.0.
    EPSILON is needed to set various tolerances for different algorithms.
    While it could be entered as a constant, I prefer to calculate it, since
    users tend to move software between machines without paying attention to
    the computing environment. Note that more complete routines exist.
}
var
    e,e0: real;
    i: integer;
begin {calculate machine epsilon}
    e0 := 1; i:=0;
    repeat
        e0 := e0/2; e := 1+e0; i := i+1;
    until (e=1.0) or (i=50); {note safety check}
    e0 := e0*2;
{ Writeln('Machine EPSILON =',e0);}
    calceps:=e0;
end; {calceps}

function resids(nRow, nCol: integer; A : rmatrix;
    Y: rvector; Bvec : rvector; print : boolean):real;
{resids.pas
    == Computes residuals and , if print is TRUE, displays them 7
    per line for the linear least squares problem. The sum of
    squared residuals is returned.

    residual vector = A * Bvec - Y
}
var
    i, j: integer;
    t1, ss : real;
begin
    if print then
        begin
            writeln('Residuals');
        end;
    ss:=0.0;
    for i:=1 to nRow do
        begin
            t1:=-Y[i]; {note form of residual is residual = A * B - Y }
            for j:=1 to nCol do
                t1:=t1+A[i,j]*Bvec[j];
            ss:=ss+t1*t1;
        end;
    end;
end;

```

```

    if print then
    begin
        write(t1:10,' ');
        if (i = 7 * (i div 7)) and (i<nRow) then writeln;
    end;
end; {loop on i}
if print then
begin
    writeln;
    writeln('Sum of squared residuals =',ss);
end;
resids:=ss
end; {resids.pas == residual calculation for linear least squares}

```

```

Procedure matcopy(nRow ,nCol: integer; A: rmatrix; var B:wmatrix);
{matcopy.pas
  -- copies matrix A, nRow by nCol, into matrix B }
var i,j: integer;
begin
    for i:=1 to nRow do
        for j:=1 to nCol do
            B[i,j]:=A[i,j];
        end;
    end;{matcopy.pas}

```

```

Procedure PrtSVDResults( nRow, nCol:integer;
                        U, V: rmatrix; Z: rvector);
{psvdres.pas
  == routine to display svd results and print them to confile
}
var
    i, j : integer;

begin
    writeln(' Singular values and vectors:');
    for j := 1 TO nCol do
    begin
        writeln('Singular value (',j,') =', Z[j]);
        writeln('Principal coordinate (U):');
        for i := 1 to nRow do
        begin
            write(U[i,j]:10:7);
            if (7 * (i div 7) = i) and (i<nRow) then writeln;
        end;
        writeln;
        writeln('Principal component (V):');
        for i:=1 to nCol do
        begin
            write(V[i,j]:10:7);
            if (7 * (i div 7) = i) and (i<nCol) then writeln;
        end;
        writeln;
    end;

```

```

end;
end; {psvdres == print svd results via procedure PrtSVDResults }

```

```

Procedure svdtst( A, U, V: rmatrix; Z: rvector;
                  nRow, UCol, VCol: integer);

```

```

{svdtst.pas

```

```

== This routine tests the results of a singular value
decomposition calculation. The matrix A is presumed to contain
the matrix of which the purported decomposition is

```

```

    U  Z  V-transpose

```

```

This routine tests column orthogonality of U and V,
row orthogonality of V, and the reconstruction suggested
by the decomposition. It does not carry out the tests of
the Moore-Penrose inverse A+, which can be computed as

```

```

    A+ := V  Z  U-transpose.

```

```

FORTRAN codes for the conditions

```

```

    A+ A A+ = ? = A+
    A  A+ A = ? = A
    (A+ A)-transpose = ? = A+ A
    (A  A+)-transpose = ? = A  A+

```

```

are given in Nash, J.C. and Wang, R.L.C. (1986)
}

```

```

var

```

```

    i,j,k:integer;
    t1: real;
    imax, jmax: integer;
    valmax: real;

```

```

begin

```

```

    writeln('Column orthogonality of U');
    valmax:=0.0;
    imax:=0;
    jmax:=0;
    for i:=1 to UCol do
    begin
        for j:=i to UCol do
        begin
            t1:=0.0; {accumulate inner products}
            if i=j then t1:=-1;
            for k:=1 to nRow do t1:=t1+U[k,i]*U[k,j];
            if abs(t1)>abs(valmax) then
            begin
                imax:=i; jmax:=j; valmax:=t1;
            end;
        end;
    end;
end;

```



```

end;
writeln('Largest inner product is ',imax,',',jmax, '=',valmax);
writeln('Row orthogonality of U (NOT guaranteed in svd)');
valmax:=0.0;
imax:=0;
jmax:=0;
for i:=1 to nRow do
begin
  for j:=i to nRow do
  begin
    t1:=0.0; {accumulate inner products}
    if i=j then t1:=-1;
    for k:=1 to UCol do t1:=t1+U[i,k]*U[j,k];
    if abs(t1)>abs(valmax) then
    begin
      imax:=i; jmax:=j; valmax:=t1;
    end;
  end;
end;
writeln('Largest inner product is ',imax,',',jmax, '=',valmax);
writeln('Column orthogonality of V');
valmax:=0.0;
imax:=0;
jmax:=0;
for i:=1 to VCol do
begin
  for j:=i to VCol do
  begin
    t1:=0.0; {accumulate inner products}
    if i=j then t1:=-1.0;
    for k:=1 to VCol do t1:=t1+V[k,i]*V[k,j];
    if abs(t1)>abs(valmax) then
    begin
      imax:=i; jmax:=j; valmax:=t1;
    end;
  end;
end;
writeln('Largest inner product is ',imax,',',jmax, '=',valmax);
writeln('Row orthogonality of V');
valmax:=0.0;
imax:=0;
jmax:=0;
for i:=1 to VCol do
begin
  for j:=i to VCol do
  begin
    t1:=0.0; {accumulate inner products}
    if i=j then t1:=-1;
    for k:=1 to VCol do t1:=t1+V[i,k]*V[j,k];
    if abs(t1)>abs(valmax) then
    begin
      imax:=i; jmax:=j; valmax:=t1;
    end;
  end;
end;

```

```

    end;
end;
writeln('Largest inner product is ',imax,',',jmax, '=',valmax);
writeln('Reconstruction of initial matrix');
valmax:=0.0;
imax:=0;
jmax:=0;
for i:=1 to nRow do
begin
    for j:=1 to VCol do
    begin
        t1:=0;
        for k:=1 to VCol do
            t1:=t1+U[i,k]*Z[k]*V[j,k]; { U * S * V-transpose}
        {writeln('A[' ,i,',',j,']= ',A[i,j], ' Recon. =',t1, ' error=',A[i,j]-t1);}
        if abs(A[i,j]-t1)>abs(valmax) then
        begin
            imax:=i; jmax:=j; valmax:=A[i,j]-t1;
        end;
    end;
end;
writeln('Largest error is ',imax,',',jmax, '=',valmax);
end; {svdtst.pas}

```

```

{I matrixin.pas} {input or generate a matrix of reals}
{I vectorin.pas} {input or generate a vector of reals}

```

```

procedure NashSVD(nRow, nCol: integer;
    var W: wmatrix;
    var Z: rvector);

var
    i, j, k, EstColRank, RotCount, SweepCount, slimit : integer;
    eps, e2, tol, vt, p, x0, y0, q, r, c0, s0, d1, d2 : real;

procedure rotate;
var
    ii : integer;

begin
    for ii := 1 to nRow+nCol do
    begin
        D1 := W[ii,j]; D2 := W[ii,k];
        W[ii,j] := D1*c0+D2*s0; W[ii,k] := -D1*s0+D2*c0
    end;
end;

begin
    writeln('alg01.pas -- NashSVD');
    eps := Calceps;
    slimit := nCol div 4; if slimit<6 then slimit := 6;

```

```

SweepCount := 0;
e2 := 10.0*nRow*eps*eps;
tol := eps*0.1;

EstColRank := nCol; ;

for i := 1 to nCol do
begin
  for j := 1 to nCol do
    W[nRow+i,j] := 0.0;
    W[nRow+i,i] := 1.0;
  end;
end;

repeat
  RotCount := EstColRank*(EstColRank-1) div 2;
  SweepCount := SweepCount+1;

  for j := 1 to EstColRank-1 do
begin
  for k := j+1 to EstColRank do
begin
  p := 0.0; q := 0.0; r := 0.0;
  for i := 1 to nRow do
begin
  x0 := W[i,j]; y0 := W[i,k];
  p := p+x0*y0; q := q+x0*x0; r := r+y0*y0;
end;
  Z[j] := q; Z[k] := r;

  if q >= r then
begin
  if (q<=e2*Z[1]) or (abs(p)<= tol*q) then RotCount := RotCount-1

  else
begin
  p := p/q; r := 1-r/q; vt := sqrt(4*p*p + r*r);
  c0 := sqrt(0.5*(1+r/vt)); s0 := p/(vt*c0);
  rotate;
end
end
else
begin

  p := p/r; q := q/r-1; vt := sqrt(4*p*p + q*q);
  s0 := sqrt(0.5*(1-q/vt));
  if p<0 then s0 := -s0;
  c0 := p/(vt*s0);
  rotate;
end;

end;
end;
end;
writeln('End of Sweep #', SweepCount,

```

```

        '- no. of rotations performed =', RotCount);
    while (EstColRank >= 3) and (Z[EstColRank] <= Z[1]*tol + tol*tol)
        do EstColRank := EstColRank-1;
    until (RotCount=0) or (SweepCount>slimit);
    if (SweepCount > slimit) then writeln('**** SWEEP LIMIT EXCEEDED');
end;

procedure svdlss(nRow, nCol: integer;
    W : wmatrix;
    Y: rvector;
    Z : rvector;
    A : rmatrix;
    var Bvec: rvector;
    q : real);

var
    i, j, k : integer;
    s : real;

begin
    writeln('alg02.pas == svdlss');
    { write('Y:');
    for i := 1 to nRow do
        begin
            write(Y[i], ' ');
        end;
    writeln;

    for i := 1 to (nRow+nCol) do
        begin
            write('W row ', i, ':');
            for j := 1 to nCol do
                begin
                    write(W[i,j], ' ');
                end;
            writeln;
        end;
    }
    { writeln('Singular values');
    for j := 1 to nCol do
        begin
            write(Z[j]:18, ' ');
            if j=4 * (j div 4) then writeln;
        end;
    writeln;
    }
    if q>=0.0 then
        begin
            q := q*q;
            for i := 1 to nCol do
                begin
                    s := 0.0;

```

```

    for j := 1 to nCol do
    begin
        for k := 1 to nRow do
        begin
            if Z[j]>q then
                s := s + W[i+nRow,j]*W[k,j]*Y[k]/Z[j];
                { V   S+   U'   y }

            end;
        end;
        Bvec[i] := s;
    end;
    writeln('Least squares solution');
    for j := 1 to nCol do
    begin
        write(Bvec[j]:12,' ');
        if j=5 * (j div 5) then writeln;
    end;
    writeln;
    s := resids(nRow, nCol, A, Y, Bvec, true);
end;

{main program}
var
    nRow, nCol : integer;
    A, V, U : rmatrix;
    W : wmatrix; {a working matrix which will contain U Zd in the
        upper nRow rows, and V in the bottom nCol rows, where Zd
        is the diagonal matrix of singular values. That is, W
        becomes

            ( U   Zd )
            (         )
            (   V   )

        }
    Z, Zsq : rvector; {Z will contain either the squares of singular
        values or the singular values themselves}
    Y : rvector; {Y will contain the 'right hand side' of the
        least squares problem, i.e. the vector to be
        approximated }
    Bvec : rvector; {the least squares solution }
    inchar : char;
    i,j,k, imax, jmax : integer;
    t1, t2: real;

begin
    banner:='dr0102.pas -- driver for svd and least squares solution';
    {Test matrix from CNM pg 34}
    nRow:=4;
    nCol:=3;
    {Read in matrix the hard way!}

```

```

A[1,1]:=5; A[1,2]:=1.0E-6; A[1,3]:=1; Y[1]:=1;
A[2,1]:=6; A[2,2]:=0.999999; A[2,3]:=1; Y[2]:=2;
A[3,1]:=7; A[3,2]:=2.00001; A[3,3]:=1; Y[3]:=3;
A[4,1]:=8; A[4,2]:=2.9999; A[4,3]:=1; Y[4]:=4;

Matcopy(nRow,nCol, A, W); {The matrix A is copied into working array W.}
NashSVD( nRow, nCol, W, Z); {The singular value decomposition is
    computed for matrix A by columnwise orthogonalization of the
    working array W, to which a unit matrix of order nCol is added
    in order to form the matrix V in the bottom nCol rows of W.}
begin
    for j:=1 to nCol do
    begin
        Zsq[j] := Z[j];
        Z[j]:= sqrt(Z[j]);
        for i:=1 to nRow do U[i,j]:=W[i,j]/Z[j];
        for i:=1 to nCol do V[i,j]:=W[i+nRow,j];
    end;
    PrtSVDResults( nRow, nCol, U, V,Z);
    begin
        svdtst(A,U,V,Z,nRow,nCol,nCol);
        writeln('Reconstruction of initial matrix from Nash working form');
        t2:=0.0; {to store largest error in reconstruction}
        for i:=1 to nRow do
        begin
            for j:=1 to nCol do
            begin
                t1:=0.0;
                for k:=1 to nCol do
                    t1:=t1+W[i,k]*W[j+nRow,k]; { U * S * V-transpose}
                t1:=A[i,j]-t1; {to compute the residual}
                if abs(t1)>t2 then
                begin
                    t2:=abs(t1); imax:=i; jmax:=j; {to save biggest element}
                end;
            end; {loop over columns}
        end; {loop over rows}
        writeln('Largest error is ',imax,',',jmax, '=',t2);
    end; {test svd results}
end; {print results}
svdlss(nRow, nCol, W, Y, Zsq, A, Bvec, 1.0e-16);
end. {dr0102.pas == svd and least squares solution}

```

Example output

For some reason not yet understood, running the compiled Pascal program does not transfer the output to our Rmarkdown output, so we resort to saving the output and then listing it as we do program code.

```

fpc ../Pascal2021/dr0102.pas
mv ../Pascal2021/dr0102 ../Pascal2021/dr0102.run
# now execute it
../Pascal2021/dr0102.run > ../Pascal2021/dr0102.out

```

```
## Free Pascal Compiler version 3.0.4+dfsg-23 [2019/11/25] for x86_64
```

```

## Copyright (c) 1993-2017 by Florian Klaempfl and others
## Target OS: Linux for x86-64
## Compiling ../Pascal2021/dr0102.pas
## dr0102.pas(487,3) Note: Local variable "inchar" not used
## Linking ../Pascal2021/dr0102
## /usr/bin/ld.bfd: warning: link.res contains output sections; did you forget -T?
## 538 lines compiled, 0.1 sec
## 1 note(s) issued

alg01.pas -- NashSVD
End of Sweep #1- no. of rotations performed =3
End of Sweep #2- no. of rotations performed =3
End of Sweep #3- no. of rotations performed =1
End of Sweep #4- no. of rotations performed =0
Singular values and vectors:
Singular value (1) = 1.3752987437308155E+001
Principal coordinate (U):
0.3589430 0.4465265 0.5341101 0.6216916
Principal component (V):
0.9587864 0.2457477 0.1426069
Singular value (2) = 1.6896078122466185E+000
Principal coordinate (U):
-0.7557625-0.3171936 0.1213826 0.5598907
Principal component (V):
-0.2090249 0.9500361-0.2318187
Singular value (3) = 1.1885323302979959E-005
Principal coordinate (U):
-0.3286873 0.1117406 0.7626745-0.5457163
Principal component (V):
-0.1924506 0.1924563 0.9622491
Column orthogonality of U
Largest inner product is 1,3= 2.8635982474156663E-011
Row orthogonality of U (NOT guaranteed in svd)
Largest inner product is 2,2=-6.8751638785273139E-001

Column orthogonality of V

Largest inner product is 3,3=-1.1102230246251565E-016

Row orthogonality of V

Largest inner product is 3,3=-1.1102230246251565E-016

Reconstruction of initial matrix

Largest error is 4,1=-1.7763568394002505E-015

Reconstruction of initial matrix from Nash working form

Largest error is 4,1= 1.7763568394002505E-015

alg02.pas == svdlss

Least squares solution

```

```
1.0000E+000 2.4766E-006 -4.0000E+000
```

Residuals

```
-9.21E-011 -2.43E-011 7.57E-011 -1.24E-010
```

Sum of squared residuals = 3.0174571907166908E-020

For some reason, we get extra line-feed characters in the output file. They are easily removed with a text editor from the output file, but their origin is unclear. JN 2021-1-20 ??

Python

Pending ...

R

Listing

While based on Nash and Shlien (1987), the following code shows that R can be used quite easily to implement Algorithm 1. The least squares solution (Algorithm 2) is embedded in the example output.

```
Nashsvd <- function(A, MaxRank=0, cyclelimit=25, trace = 0, rotnchk=0.3) {  
  ## Nashsvd.R -- An attempt to remove tolerances from Nash & Shlien algorithm 190327  
  # Partial svd by the one-sided Jacobi method of Nash & Shlien  
  # Computer Journal 1987 30(3), 268-275  
  # Computer Journal 1975 18(1) 74-76  
  if (cyclelimit < 6) {  
    warning("Nashsvd: You cannot set cyclelimit < 6 without modifying the code")  
    cyclelimit <- 6 # safety in case user tries smaller  
  }  
  m <- dim(A)[1]  
  n <- dim(A)[2]  
  if (MaxRank <= 0) MaxRank <- n  
  EstColRank <- n # estimated column rank  
  # Note that we may simply run algorithm to completion, or fix the  
  # number of columns by EstColRank. Need ?? to fix EstColRank=0 case.??  
  V <- diag(nrow=n) # identity matrix in V  
  if (is.null(EstColRank)) {EstColRank <- n} # Safety check on number of svd  
  z <- rep(NA, n) # column norm squares -- safety setting  
  keepgoing <- TRUE  
  SweepCount <- 0  
  while (keepgoing) { # main loop of repeating cycles of Jacobi  
    RotCount <- 0  
    SweepCount <- SweepCount + 1  
    if (trace > 1) cat("Sweep:", SweepCount, "\n")  
    ## if (EstColRank == n) { EstColRank <- n - 1 } # safety  
    for (jj in 1:(EstColRank-1)) { # left column indicator  
      for (kk in (jj+1): n) { # right hand column  
        p <- q <- r <- 0.0 #  
        oldjj <- A[,jj]  
        oldkk <- A[,kk]  
        p <- as.numeric(crossprod(A[,jj], A[,kk]))  
        q <- as.numeric(crossprod(A[,jj], A[,jj]))
```



```

r <- as.numeric(crossprod(A[,kk], A[,kk]))
if (trace > 2) cat(jj, " ", kk, ": pqr", p, " ", q, " ", r, " ")
z[jj]<-q
z[kk]<-r
if (q >= r) { # in order, so can do test of "convergence" -- change to 0.2 * abs(p) for odd ca
  if ( (as.double(z[1]+q) > as.double(z[1])) && (as.double(rotnchk*abs(p)+q) > as.double(q)) )
    RotCount <- RotCount + 1
    p <- p/q
    r <- 1 - (r/q)
    vt <- sqrt(4*p*p + r*r)
    c0 <- sqrt(0.5*(1+r/vt))
    s0 <- p/(vt*c0)
    # rotate
    cj <- A[,jj]
    ck <- A[,kk]
    A[,jj] <- c0*cj + s0*ck
    A[,kk] <- -s0*cj + c0*ck
    cj <- V[,jj]
    ck <- V[,kk]
    V[,jj] <- c0*cj + s0*ck
    V[,kk] <- -s0*cj + c0*ck
  } else {
    if (trace > 2) cat(" NO rotn ")
  }
} else { # out of order, must rotate
  if (trace > 2) cat("|order|")
  RotCount <- RotCount + 1
  p <- p/r
  q <- (q/r) - 1.0
  vt <- sqrt(4*p*p + q*q)
  s0 <- sqrt(0.5*(1-q/vt))
  if (p < 0) { s0 <- -s0 }
  c0 <- p/(vt*s0)
  # rotate
  cj <- A[,jj]
  ck <- A[,kk]
  A[,jj] <- c0*cj + s0*ck
  A[,kk] <- -s0*cj + c0*ck
  cj <- V[,jj]
  ck <- V[,kk]
  V[,jj] <- c0*cj + s0*ck
  V[,kk] <- -s0*cj + c0*ck
} # end q >= r test
nup <- as.numeric(crossprod(A[,jj], A[,kk]))
# nup <- as.numeric(crossprod(A[,jj], A[,jj]))
# nur <- as.numeric(crossprod(A[,kk], A[,kk]))
if (trace > 2) cat(" new: p= ", nup, " Rel:", nup*nup/z[1], "\n")
} # end kk
} # end jj
if (trace > 0) {cat("End sweep ", SweepCount, " No. rotations =", RotCount, "\n")}
if (trace > 2) tmp <- readline("cont.?\n")
while ( (EstColRank >= 3) && (as.double(sqrt(z[EstColRank]))+sqrt(z[1]) == as.double(sqrt(z[1])) ) )
# ?? Why can we not use 2? Or do we need at least 2 cols

```

```

    EstColRank <- EstColRank - 1
    if (trace > 0) {cat("Reducing rank to ", EstColRank,"\n")} # ?? can do this more cleanly
} # end while for rank estimation
## Here may want to adjust for MaxRank. How??
if (MaxRank < EstColRank) {
  if (trace > 0) {
    cat("current estimate of sv[" ,MaxRank,"/sv[1] =",sqrt(z[MaxRank]/z[1]),"\n")
    cat("reducing rank by 1\n")
  }
  EstColRank <- EstColRank - 1
}
if ( SweepCount >= cyclelimit) {
  if (trace > 0) cat("Cycle limit reached\n")
  keepgoing <- FALSE
}
if (RotCount == 0) {
  if (trace > 1) cat("Zero rotations in cycle\n")
  keepgoing <- FALSE
}
} # End main cycle loop
z <- sqrt(z)
A <- A %*% diag(1/z)
ans <- list( d = z, u = A, v=V, cycles=SweepCount, rotations=RotCount)
ans
} # end partsvd()

```

Example output

```

# test taken from dr0102.pas
A<-matrix(0, 4,3)
A[1,]<-c(5, 1e-6, 1)
A[2,]<-c(6, 0.999999, 1)
A[3,]<-c(7, 2.00001, 1)
A[4,]<-c(8, 2.9999, 1)
print(A)

```

```

##      [,1]      [,2] [,3]
## [1,]    5 0.000001    1
## [2,]    6 0.999999    1
## [3,]    7 2.000010    1
## [4,]    8 2.999900    1

```

```

b<-c(1,2,3,4)
print(b)

```

```

## [1] 1 2 3 4

```

```

# try the R-base svd
sA <- svd(A)
sA

```

```

## $d
## [1] 1.375299e+01 1.689608e+00 1.188532e-05
##
## $u

```

```

##           [,1]      [,2]      [,3]
## [1,] -0.3589430 -0.7557625  0.3286873
## [2,] -0.4465265 -0.3171936 -0.1117406
## [3,] -0.5341101  0.1213826 -0.7626745
## [4,] -0.6216916  0.5598907  0.5457163
##
## $v
##           [,1]      [,2]      [,3]
## [1,] -0.9587864 -0.2090249  0.1924506
## [2,] -0.2457477  0.9500361 -0.1924563
## [3,] -0.1426069 -0.2318187 -0.9622491

yy <- t(sA$u) %*% as.matrix(b)
xx <- sA$v %*% diag(1/sA$d) %*% yy
xx

##           [,1]
## [1,]  1.000000e+00
## [2,] -9.005019e-12
## [3,] -4.000000e+00

# Now the Nashsvd code (this is likely NOT true to 1979 code)
source("../R/Nashsvd.R")
nsvd <- Nashsvd(A)
print(nsvd)

## $d
## [1] 1.375299e+01 1.689608e+00 1.188532e-05
##
## $u
##           [,1]      [,2]      [,3]
## [1,] 0.3589430 -0.7557625 -0.3286873
## [2,] 0.4465265 -0.3171936  0.1117406
## [3,] 0.5341101  0.1213826  0.7626745
## [4,] 0.6216916  0.5598907 -0.5457163
##
## $v
##           [,1]      [,2]      [,3]
## [1,] 0.9587864 -0.2090249 -0.1924506
## [2,] 0.2457477  0.9500361  0.1924563
## [3,] 0.1426069 -0.2318187  0.9622491
##
## $cycles
## [1] 4
##
## $rotations
## [1] 0

# Note least squares solution can be done by matrix multiplication
U <- nsvd$u
V <- nsvd$v
d <- nsvd$d
di <- 1/d
di <- diag(di) # convert to full matrix -- note entry sizes
print(di)

```

```
##           [,1]      [,2]      [,3]
## [1,] 0.07271147 0.0000000 0.00
## [2,] 0.00000000 0.5918533 0.00
## [3,] 0.00000000 0.0000000 84137.38
```

```
lsol <- t(U) %*% b
lsol <- di %*% lsol
lsol <- V %*% lsol
print(lsol)
```

```
##           [,1]
## [1,] 9.999975e-01
## [2,] 2.476918e-06
## [3,] -3.999988e+00
```

```
res <- b - A %*% lsol
print(res)
```

```
##           [,1]
## [1,] 5.027934e-11
## [2,] -1.708989e-11
## [3,] -1.166609e-10
## [4,] 8.347678e-11
```

```
cat("sumsquares = ", as.numeric(crossprod(res)))
```

```
## sumsquares = 2.339822e-20
```

```
# now set smallest singular value to 0 and in pseudo-inverse
```

```
dix <- di
```

```
dix[3,3] <- 0
```

```
lsolx <- V %*% dix %*% t(U) %*% b
```

```
# this gives a very different least squares solution
```

```
print(lsolx)
```

```
##           [,1]
## [1,] 0.2222209
## [2,] 0.7778018
## [3,] -0.1111212
```

```
# but the residuals (in this case) are nearly 0 too
```

```
resx <- b - A %*% lsolx
```

```
cat("sumsquares = ", as.numeric(crossprod(resx)))
```

```
## sumsquares = 2.307256e-09
```

Others

Pending ...

?? Could we f2c the Fortran and manually tweak to get a C code?

There is also a C version in

<https://github.com/LuaDist/gsl/blob/master/linalg/svd.c>

=====

Algorithm 3 – Givens’ decomposition

The Givens and Householder decompositions of a rectangular m by n matrix A ($m \geq n$) both give an m by m orthogonal matrix Q and an upper-triangular n by n matrix R whose product QR is a close approximation of A . At the time Nash (1979) was being prepared, the Givens approach seemed to give a more compact program code, though neither approach is large.

In practice, if one is trying to solve linear equations

$$Ax = b$$

or linear least squares problems of the form

$$Ax = b$$

then the right hand side (RHS) b can be appended to the matrix A so that the resulting working matrix

$$W = [A|b]$$

is transformed during the formation of the Q matrix into

$$W_{trans} = [R|Q'b]$$

This saves us the effort of multiplying b by the transpose of Q before we back-solve for x .

In fact, m does not have to be greater than or equal to n . However, underdetermined systems of equations do raise some issues that we will not address here.

It is therefore unnecessary to store Q , which when Nash (1979) was being prepared was a potentially large matrix. There are alternative designs of the code which could save information on the plane rotations that make up Q . Such codes can then apply the rotations to a unit matrix of the right size to reconstruct Q as needed. However, these details have largely become irrelevant in an age of cheap memory chips.

Fortran

Listing

The following listing uses the Frank matrix as a test.

```
C&&& A3
C  TEST ALGORITHM 3
C  J.C. NASH    JULY 1978, APRIL 1989
      LOGICAL SAVEQ
      CHARACTER QSAVE
      INTEGER M,N,NIN,NOUT
      REAL A(10,10),Q(10,10),EPS,S,W(10,10)
      NDIM=10
C  I/O CHANNELS
      NIN=5
      NOUT=6
      1  READ(NIN,900)M,N,QSAVE
      900  FORMAT(2I5,A1)
      WRITE(NOUT,950)M,N,QSAVE
      950  FORMAT('M=',I5,' N=',I5,' QSAVE=',A1)
      IF(M.EQ.0.OR.N.EQ.0)STOP
      SAVEQ=.FALSE.
```

```

        IF (QSAVE .EQ. "T") SAVEQ=.TRUE.
        CALL FRANKM(M,N,A,10)
        WRITE(NOUT,952)
952  FORMAT('INITIAL MATRIX')
        CALL OUT(A,NDIM,M,N,NOUT)
        DO 10 I=1,M
            DO 5 J=1,N
C          COPY MATRIX TO WORKING ARRAY
            W(I,J)=A(I,J)
        5      CONTINUE
    10      CONTINUE
C  IBM MACHINE PRECISION
        EPS=16.0**(-5)
        CALL A3GR(M,N,W,10,Q,EPS,SAVEQ)
        WRITE(NOUT,953)
953  FORMAT('FULL DECOMPOSED MATRIX')
        CALL OUT(A,NDIM,M,N,NOUT)
        IF(SAVEQ)CALL A3DT(M,N,W,NDIM,Q,NOUT,A)
        GOTO 1
        END
        SUBROUTINE A3DT(M,N,W,NDIM,Q,NOUT,A)
C  TESTS GIVENS' DECOMPOSITION
C  J.C. NASH    JULY 1978, APRIL 1989
        INTEGER M,N,NDIM,NOUT,I,J,K
        REAL A(NDIM,N),Q(NDIM,M),W(NDIM,N),S,T
        WRITE(NOUT,960)
960  FORMAT(' Q MATRIX')
        CALL OUT(Q,NDIM,M,M,NOUT)
        WRITE(NOUT,961)
961  FORMAT(' R MATRIX (STORED IN W')
        CALL OUT(W,NDIM,M,N,NOUT)
        IF(N.LT.M)GOTO 9
        S=1.0
        DO 5 I=1,M
            S=S*W(I,I)
        5      CONTINUE
        WRITE(NOUT,963)S
963  FORMAT(' DETERMINANT=',1PE16.8)
        9      CONTINUE
        T=0.0
        DO 20 I=1,M
            DO 15 J=1,N
                S=0.0
                DO 10 K=1,M
                    S=S+Q(I,K)*W(K,J)
                10      CONTINUE
                S=S-A(I,J)
                IF(ABS(S).GT.T)T=ABS(S)
            15      CONTINUE
        20      CONTINUE
        WRITE(NOUT,962)T
962  FORMAT(' MAX. DEVN. OF RECONSTRUCTION FROM ORIGINAL=',E16.8)
        RETURN

```

```

        END
        SUBROUTINE OUT(A,NDIM,N,NP,NOUT)
C   J.C. NASH   JULY 1978, APRIL 1989
        INTEGER NDIM,N,NOUT,I,J
        REAL A(NDIM,NP)
        DO 20 I=1,N
            WRITE(NOUT,951)I
951    FORMAT(' ROW',I3)
            WRITE(NOUT,952)(A(I,J),J=1,NP)
952    FORMAT(1H ,1P5E16.8)
        20 CONTINUE
        RETURN
        END
        SUBROUTINE A3GR(M,N,A,NDIM,Q,EPS,SAVEQ)
C   ALGORITHM 3  GIVENS' REDUCTION
C   J.C. NASH   JULY 1978, FEBRUARY 1980, APRIL 1989
C   M,N  = ORDER OF MATRIX TO BE DECOMPOSED
C   A    = ARRAY CONTAINING MATRIX TO BE DECOMPOSED
C   NDIM = FIRST DIMENSION OF MATRICES - NDIM.GE.M
C   Q    = ARRAY CONTAINING ORTHOGONAL MATRIX OF ACCUMULATED ROTATIONS
C   EPS  = MACHINE PRECISION = SMALLEST NO.GT.0.0 S.T. 1.0+EPS.GT.1.0
C   SAVEQ= LOGICAL FLAG SET .TRUE. IF Q TO BE FORMED
C   STEP 0
        LOGICAL SAVEQ
        INTEGER N,M,NA,MN,I,J,K,J1
        REAL A(NDIM,N),Q(NDIM,M),EPS,TOL,B,P,S,C
        MN=M
        IF(M.GT.N)MN=N
        IF(.NOT.SAVEQ)GOTO 9
        DO 5 I=1,M
            DO 4 J=1,M
                Q(I,J)=0.0
            4 CONTINUE
                Q(I,I)=1.0
        5 CONTINUE
        9 TOL=EPS*EPS
C   STEP 1
        IF(M.EQ.1)RETURN
        DO 100 J=1,MN
            J1=J+1
            IF(J1.GT.M)GOTO 100
C   STEP 2
            DO 90 K=J1,M
C   STEP 3
                C=A(J,J)
                S=A(K,J)
                B=ABS(C)
                IF(ABS(S).GT.B)B=ABS(S)
                IF(B.EQ.0.0)GOTO 90
                C=C/B
                S=S/B
                P=SQRT(C*C+S*S)
C   STEP 4

```

```

        S=S/P
C  STEP 5      IF(ABS(S).LT.TOL)GOTO 90
C  STEP 6      C=C/P
C  STEP 7
        DO 75 I=1,N
            P=A(J,I)
            A(J,I)=C*P+S*A(K,I)
            A(K,I)=-S*P+C*A(K,I)
75      CONTINUE
C  STEP 8      IF(.NOT.SAVEQ)GOTO 90
        DO 85 I=1,M
            P=Q(I,J)
            Q(I,J)=C*P+S*Q(I,K)
            Q(I,K)=-S*P+C*Q(I,K)
85      CONTINUE
C  STEP 9
90      CONTINUE
C  STEP 10
100     CONTINUE
        RETURN
        END
        SUBROUTINE FRANKM(M,N,A,NA)
C  J.C. NASH    JULY 1978, APRIL 1989
        INTEGER M,N,NA,I,J
C  INPUTS FRANK MATRIX M BY N INTO A
        REAL A(NA,N)
        DO 20 I=1,M
            DO 10 J=1,N
                A(I,J)=AMINO(I,J)
10      CONTINUE
20      CONTINUE
        RETURN
        END

```

Example output

As a precaution, we use a 1 by 1 matrix as our first test. We have seen situations where otherwise reliable programs have failed on such trivial cases.

```

gfortran ../fortran/a3.f
mv ./a.out ../fortran/a3.run
../fortran/a3.run < ../fortran/a3data.txt > ../fortran/a3out.txt

```

```

M=      1  N=      1  QSAVE=T
INITIAL MATRIX
ROW      1
      1.00000000E+00
FULL DECOMPOSED MATRIX
ROW      1
      1.00000000E+00
Q MATRIX
ROW      1

```



```

1.00000000E+00
R MATRIX (STORED IN W
ROW 1
1.00000000E+00
DETERMINANT= 1.00000000E+00
MAX. DEVN. OF RECONSTRUCTION FROM ORIGINAL= 0.00000000E+00
M= 5 N= 3 QSAVE=T
INITIAL MATRIX
ROW 1
1.00000000E+00 1.00000000E+00 1.00000000E+00
ROW 2
1.00000000E+00 2.00000000E+00 2.00000000E+00
ROW 3
1.00000000E+00 2.00000000E+00 3.00000000E+00
ROW 4
1.00000000E+00 2.00000000E+00 3.00000000E+00
ROW 5
1.00000000E+00 2.00000000E+00 3.00000000E+00
FULL DECOMPOSED MATRIX
ROW 1
1.00000000E+00 1.00000000E+00 1.00000000E+00
ROW 2
1.00000000E+00 2.00000000E+00 2.00000000E+00
ROW 3
1.00000000E+00 2.00000000E+00 3.00000000E+00
ROW 4
1.00000000E+00 2.00000000E+00 3.00000000E+00
ROW 5
1.00000000E+00 2.00000000E+00 3.00000000E+00
Q MATRIX
ROW 1
4.47213590E-01 -8.94427240E-01 9.95453036E-08 1.14146687E-07 -1.93894891E-08
ROW 2
4.47213590E-01 2.23606765E-01 -8.66025507E-01 0.00000000E+00 -1.19209290E-07
ROW 3
4.47213590E-01 2.23606795E-01 2.88675159E-01 -7.07106888E-01 -4.08248186E-01
ROW 4
4.47213590E-01 2.23606944E-01 2.88675249E-01 7.07106769E-01 -4.08248246E-01
ROW 5
4.47213590E-01 2.23606795E-01 2.88674951E-01 0.00000000E+00 8.16496611E-01
R MATRIX (STORED IN W
ROW 1
2.23606801E+00 4.02492237E+00 5.36656284E+00
ROW 2
1.92373264E-08 8.94427299E-01 1.56524777E+00
ROW 3
2.48352734E-08 1.40489522E-08 8.66025269E-01
ROW 4
4.86669869E-08 2.58095696E-08 0.00000000E+00
ROW 5
-1.40489469E-08 -4.96705121E-09 0.00000000E+00
MAX. DEVN. OF RECONSTRUCTION FROM ORIGINAL= 0.29802322E-06
M= 0 N= 0 QSAVE=

```

BASIC

Listing

The following listing also uses the Frank matrix as a test. The code has been adjusted for fixed input to allow it to be run within the `knitr` processor for Rmarkdown.

```
2 REM DIM A(10,10),Q(10,10)
10 PRINT "TEST GIVENS - GIFT - ALG 3"
12 LET M8=10
14 LET N8=10
20 DIM A(M8,N8),Q(M8,M8)
25 REM PRINT "M=",
30 REM INPUT M
32 LET M=5
40 REM PRINT "  N=",
50 REM INPUT N
52 LET N=3
70 GOSUB 1500
80 PRINT "ORIGINAL",
85 GOSUB 790
90 GOSUB 500 : REM GIVENS DECOMPOSITION
94 PRINT "FINAL ";
96 GOSUB 790
97 PRINT "FINAL ";
98 GOSUB 840
100 PRINT "RECOMBINATION "
110 FOR I=1 TO M
111   PRINT "ROW";I;":";
120   FOR J=1 TO N
130     LET S=0
140     FOR K=1 TO M
150       LET S=S+Q(I,K)*A(K,J)
160     NEXT K
170     PRINT S;" ";
210   NEXT J
220   PRINT
230 NEXT I
240 QUIT
245 REM STOP
500 REM GIVENS TRIANGULARIZATION
520 PRINT "GIVENS TRIANGULARIZATION DEC 12 77"
540 FOR I=1 TO M
545   FOR J=1 TO M
550     LET Q(I,J)=0
555   NEXT J
560   LET Q(I,I)=1
565 NEXT I
575 REM GOSUB 840: REM PRINT ORIGINAL Q MATRIX
580 LET E1=1E-7 : REM NORTH STAR 8 DIGIT -- can be changed!
585 LET T9=E1*E1
600 FOR J=1 TO N-1
605   FOR K=J+1 TO M
610     LET C=A(J,J)
615     LET S=A(K,J)
```

```

625     REM PRINT "J=",J," K=",K," A[J,J]=",C," A[K,J]=",S
630     REM PRINT "BYPASS SAFETY DIVISION ",
635     REM GOTO 660
640     LET B=ABS(C)
645     IF ABS(S)<=B THEN GOTO 655
650     LET B=ABS(S)
655     LET C=C/B
660     LET S=S/B
665     IF B=0 THEN GOTO 770
670     LET P=SQR(C*C+S*S)
680     LET S=S/P
685     IF ABS(S)<T9 THEN GOTO 770
690     LET C=C/P
695     FOR I=1 TO N
700         LET P=A(J,I)
705         LET A(J,I)=C*P+S*A(K,I)
710         LET A(K,I)=-S*P+C*A(K,I)
715     NEXT I
720     IF J=N-1 THEN GOTO 730
730     REM IF I5=0 THEN GOTO 770
735     FOR I=1 TO M
740         LET P=Q(I,J)
745         LET Q(I,J)=C*P+S*Q(I,K)
750         LET Q(I,K)=-S*P+C*Q(I,K)
755     NEXT I
770     REM Possible print point
775     NEXT K
780 NEXT J
785 RETURN
790 PRINT " A MATRIX"
795 FOR H=1 TO M
800     PRINT "ROW";H;": ";
805     FOR L=1 TO N
810         PRINT A(H,L);" ";
815     NEXT L
820     PRINT
825 NEXT H
830 PRINT
835 RETURN
840 PRINT " Q MATRIX"
845 FOR H=1 TO M
850     PRINT "ROW";H;": ";
855     FOR L=1 TO M
860         PRINT Q(H,L);" ";
865     NEXT L
870     PRINT
875 NEXT H
880 PRINT
885 RETURN
1500 REM PREPARE FRANK MATRIX IN A
1510 FOR I=1 TO M
1530 FOR J=1 TO N
1540 IF (I <= J) THEN LET A(I,J)=I ELSE LET A(I,J)=J

```

```

1550 NEXT J
1560 NEXT I
1570 RETURN
1600 END

```

Example output

As a precaution, we use a 1 by 1 matrix as our first test. We have seen situations where otherwise reliable programs have failed on such trivial cases.

```
bwbasic ../BASIC/a3.bas
```

```

## Bywater BASIC Interpreter/Shell, version 2.20 patch level 2
## Copyright (c) 1993, Ted A. Campbell
## Copyright (c) 1995-1997, Jon B. Volkoff
##
## TEST GIVENS - GIFT - ALG 3
## ORIGINAL
##   A MATRIX
## ROW 1: 1  1  1
## ROW 2: 1  2  2
## ROW 3: 1  2  3
## ROW 4: 1  2  3
## ROW 5: 1  2  3
##
## GIVENS TRIANGULARIZATION DEC 12 77
## FINAL   A MATRIX
## ROW 1: 2.2360680  4.0249224  5.3665631
## ROW 2: 0  0.8944272  1.5652476
## ROW 3: 0  0  0.7071068
## ROW 4: 0  0  0.4082483
## ROW 5: -0  -0  0.2886751
##
## FINAL   Q MATRIX
## ROW 1: 0.4472136  -0.8944272  0  0  0
## ROW 2: 0.4472136  0.2236068  -0.7071068  -0.4082483  -0.2886751
## ROW 3: 0.4472136  0.2236068  0.7071068  -0.4082483  -0.2886751
## ROW 4: 0.4472136  0.2236068  0  0.8164966  -0.2886751
## ROW 5: 0.4472136  0.2236068  0  0  0.8660254
##
## RECOMBINATION
## ROW 1: 1  1  1
## ROW 2: 1  2  2.0000000
## ROW 3: 1  2  3
## ROW 4: 1.0000000  2.0000000  3.0000000
## ROW 5: 1.0000000  2.0000000  3.0000000

```

Pascal

Listing – column-wise approach

```

program givrun(input, output);
{dr03.PAS == driver for Givens' reduction of a matrix

      Copyright 1988 J.C.Nash

```

```

}
{I constype.def}
{constype.def ==
    This file contains various definitions and type statements which are
    used throughout the collection of "Compact Numerical Methods". In many
    cases not all definitions are needed, and users with very tight memory
    constraints may wish to remove some of the lines of this file when
    compiling certain programs.

    Modified for Turbo Pascal 5.0

    Copyright 1988, 1990 J.C.Nash
}
{uses Dos, Crt;} {Turbo Pascal 5.0 Modules}
{ 1. Interrupt, Unit, Interface, Implementation, Uses are reserved words now.}
{ 2. System,Dos,Crt are standard unit names in Turbo 5.0.}

const
    big = 1.0E+35;    {a very large number}
    Maxconst = 25;    {Maximum number of constants in data record}
    Maxobs = 100;     {Maximum number of observations in data record}
    Maxparm = 25;     {Maximum number of parameters to adjust}
    Maxvars = 10;     {Maximum number of variables in data record}
    acctol = 0.0001;  {acceptable point tolerance for minimisation codes}
    maxm = 20;        {Maximum number or rows in a matrix}
    maxn = 20;        {Maximum number of columns in a matrix}
    maxmn = 40;       {maxn+maxm, the number of rows in a working array}
    maxsym = 210;     {maximum number of elements of a symmetric matrix
        which need to be stored = maxm * (maxm + 1)/2 }
    reltest = 10.0;   {a relative size used to check equality of numbers.
        Numbers x and y are considered equal if the
        floating-point representation of reltest*x equals
        that of reltest*y.}
    stepredn = 0.2;   {factor to reduce stepsize in line search}
    yearwrit = 1990;  {year in which file was written}

type
    str2 = string[2];
    rmatrix = array[1..maxm, 1..maxn] of real; {a real matrix}
    wmatrix = array[1..maxmn, 1..maxn] of real; {a working array, formed
        as one real matrix stacked on another}
    smatvec = array[1..maxsym] of real; {a vector to store a symmetric matrix
        as the row-wise expansion of its lower triangle}
    rvector = array[1..maxm] of real; {a real vector. We will use vectors
        of m elements always. While this is NOT space efficient,
        it simplifies program codes.}
    cgmethodtype= (Fletcher_Reeves,Polak_Ribiere,Beale_Sorenson);
        {three possible forms of the conjugate gradients updating formulae}
    probdata = record
        m      : integer; {number of observations}
        nvar   : integer; {number of variables}
        nconst: integer; {number of constants}
        vconst: array[1..Maxconst] of real;

```

```

        Ydata : array[1..Maxobs, 1..Maxvars] of real;
        nlls   : boolean; {true if problem is nonlinear least squares}
    end;
{
    NOTE: Pascal does not let us define the work-space for the function
    within the user-defined code. This is a weakness of Pascal for this
    type of work.
}
var {global definitions}
    banner      : string[80]; {program name and description}

function calceps:real;
{calceps.pas ==
    This function returns the machine EPSILON or floating point tolerance,
    the smallest positive real number such that 1.0 + EPSILON > 1.0.
    EPSILON is needed to set various tolerances for different algorithms.
    While it could be entered as a constant, I prefer to calculate it, since
    users tend to move software between machines without paying attention to
    the computing environment. Note that more complete routines exist.
}
var
    e,e0: real;
    i: integer;
begin {calculate machine epsilon}
    e0 := 1; i:=0;
    repeat
        e0 := e0/2; e := 1+e0; i := i+1;
    until (e=1.0) or (i=50); {note safety check}
    e0 := e0*2;
{ Writeln('Machine EPSILON =',e0);}
    calceps:=e0;
end; {calceps}

procedure givens( nRow,nCol : integer;
                  var A, Q: rmatrix);
var
    i, j, k, mn: integer;
    b, c, eps, p, s : real;
begin
    writeln('alg03.pas -- Givens',chr(39),' reduction -- column-wise');
    mn := nRow; if nRow>nCol then mn := nCol;
    for i := 1 to nRow do
        begin
            for j := 1 to nRow do Q[i,j] := 0.0;
            Q[i,i] := 1.0;
        end;
        eps := calceps;
        for j := 1 to (mn-1) do
            begin
                for k := (j+1) to nRow do

```

```

begin
  c := A[j,j]; s := A[k,j];
  b := abs(c); if abs(s)>b then b := abs(s);
  if b>0 then
    begin
      c := c/b; s := s/b;
      p := sqrt(c*c+s*s);
      s := s/p;
      if abs(s)>=eps then
        begin
          c := c/p;
          for i := 1 to nCol do
            begin
              p := A[j,i]; A[j,i] := c*p+s*A[k,i]; A[k,i] := -s*p+c*A[k,i];
            end;
          for i := 1 to nRow do
            begin
              p := Q[i,j]; Q[i,j] := c*p+s*Q[i,k]; Q[i,k] := -s*p+c*Q[i,k];
            end;
          end;
        end;
      end;
    end;
  end;
end;

Procedure Frank2(var m, n: integer; var A: rmatrix);
var
  i,j: integer;
begin
  for i:=1 to m do
    begin
      for j:=1 to n do
        begin
          write(i,' ',j,');
          if (i <= j) then
            A[i,j]:=i
          else
            A[i,j]:=j;
          writeln(A[i,j]);
        end;
      end;
    end;
end;

var
  A, Q: rmatrix;
  i, j, k, nRow, nCol : integer;
  Acopy : rmatrix;
  s : real;

begin
  banner:='dr03.pas -- driver for Givens'+chr(39)+' reduction';

```

```

nRow := 5;
nCol := 3; {Specific to this example.}
writeln('Size of problem (rows, columns)  (' ,nRow,' , ',nCol,')');
writeln('Frank matrix example');
Frank2(nRow, nCol, A);
writeln('Matrix A');
for i:=1 to nRow do
begin
  for j:=1 to nCol do
  begin
    Acopy[i,j]:=A[i,j];
    write(A[i,j]:10:5,' ');
    if (7 * (j div 7) = j) and (j<nCol) then
    begin
      writeln;
    end;
  end;
  writeln;
end;
givens(nRow,nCol,A,Q);
writeln('Decomposition');
writeln('Q');
for i:=1 to nRow do
begin
  for j:=1 to nRow do
  begin
    write(Q[i,j]:10:5,' ');
    if (7 * (j div 7) = j) and (j<nRow) then
    begin
      writeln;
    end;
  end;
  writeln;
end;
writeln('R');
for i:=1 to nRow do
begin
  for j:=1 to nCol do
  begin
    write(A[i,j]:10:5,' ');
    if (7 * (j div 7) = j) and (j<nCol) then
    begin
      writeln;
    end;
  end;
  writeln;
end;
writeln('Q*R - Acopy');
for i:=1 to nRow do
begin
  for j:=1 to nCol do
  begin
    s:=-Acopy[i,j];

```



```

    for k:=1 to nRow do s:=s+Q[i,k]*A[k,j];
    write(s:10,' ');
    if (7 * (j div 7) = j) and (j<nRow) then
    begin
        writeln;
    end;
    end;
    writeln;
end;
end. {dr03.pas == Givens' reduction driver}

```

Example output – column-wise approach

```

fpc ../Pascal2021/dr03.pas
mv ../Pascal2021/dr03 ../Pascal2021/dr03.run
../Pascal2021/dr03.run >../Pascal2021/dr03.out

```

```

Size of problem (rows, columns)  (5, 3)
Frank matrix example
1 1; 1.000000000000000E+000
1 2; 1.000000000000000E+000
1 3; 1.000000000000000E+000
2 1; 1.000000000000000E+000
2 2; 2.000000000000000E+000
2 3; 2.000000000000000E+000
3 1; 1.000000000000000E+000
3 2; 2.000000000000000E+000
3 3; 3.000000000000000E+000
4 1; 1.000000000000000E+000
4 2; 2.000000000000000E+000
4 3; 3.000000000000000E+000
5 1; 1.000000000000000E+000
5 2; 2.000000000000000E+000
5 3; 3.000000000000000E+000
Matrix A
  1.0000    1.0000    1.0000
  1.0000    2.0000    2.0000
  1.0000    2.0000    3.0000
  1.0000    2.0000    3.0000
  1.0000    2.0000    3.0000
alg03.pas -- Givens' reduction -- column-wise
Decomposition
Q
  0.44721  -0.89443   0.00000   0.00000   0.00000
  0.44721   0.22361  -0.70711  -0.40825  -0.28868
  0.44721   0.22361   0.70711  -0.40825  -0.28868
  0.44721   0.22361   0.00000   0.81650  -0.28868
  0.44721   0.22361   0.00000   0.00000   0.86603
R
  2.23607   4.02492   5.36656
  0.00000   0.89443   1.56525
  0.00000   0.00000   0.70711
  0.00000   0.00000   0.40825

```

```

-0.00000  -0.00000   0.28868
Q*R - Acopy
 1.45E-016  2.22E-016  6.95E-016
 1.45E-016 -1.03E-016 -1.11E-016
 2.81E-016  2.86E-016  2.36E-016
-1.26E-016 -4.64E-016 -8.05E-016
-2.22E-016 -2.46E-016 -5.00E-016

```

Algorithm 9 – Bauer-Reinsch matrix inversion

Wilkinson, Reinsch, and Bauer (1971), pages 45-49, is a contribution entitled **Inversion of Positive Definite Matrices by the Gauss-Jordan Method**. It hardly mentions, but appears to assume, that the matrix to be inverted is symmetric. Two Algol procedures are provided, one for a matrix stored as a square array, the other for the a matrix where only the lower triangle is stored as a single vector in row-wise order. That is, if A is of order $n=3$ and has values

```

1  2  4
2  3  5
4  5  6

```

Then the corresponding vector of $6 = n*(n+1)/2$ values is

```

1  2  3  4  5  6

```

By some exceedingly clever coding and matrix manipulation, Bauer and Reinsch developed tiny codes that invert a positive-definite matrix *in situ* using only one extra vector of length n . Thus, besides the memory to store a very small code, we need only $n*(n+3)/2$ floating point numbers and a few integers to index arrays.

Truthfully, we rarely need an explicit matrix inverse, and the most common positive-definite symmetric matrix that arises in scientific computations is the sum of squares and cross-products (SSCP) in the normal equations used for linear (or also nonlinear) least squares problems. However, the formation of this SSCP matrix is rarely the best approach to solving least squares problems. The SVD introduced in Algorithm 1 and the least squares solution in Algorithm 2 lead to better methods. (??mention A4, Choleski in A7, A8 etc.)

Despite these caveats, the Bauer-Reinsch algorithm is interesting as a historical curiosity, showing what can be done when resources are very limited.

Fortran

Listing

```

C&&& A9
C TEST ALGORITHM 9 A9GJ
C J.C. NASH JULY 1978, APRIL 1989
C USE FRANK MATRIX
  LOGICAL INDEF
  INTEGER N,N2,I,J,IJ,NOUT
  REAL A(55),X(10),S,T
  N2=55
C PRINTER CHANNEL
  NOUT=6
C MAIN LOOP
C DO 100 N=2,10,2
  N = 4
  WRITE(NOUT,950)N
950 FORMAT('OORDER=',I4,' ORIGINAL MATRIX')

```

```

C PUT IN CARDS FROM A78
C NOTE DIFFERENCES ONLY IN CALLS
  DO 20 I=1,N
    DO 10 J=1,I
      IJ=I*(I-1)/2+J
      A(IJ)=J
10    CONTINUE
20    CONTINUE
  CALL SOUT(A,N2,N,NOUT)
  CALL A9GJ(A,N2,N,INDEF,X)
  WRITE(NOUT,956)
956  FORMAT('OINVERSE')
  CALL SOUT(A,N2,N,NOUT)
  WRITE(NOUT,957)
957  FORMAT('OINVERSE OF INVERSE')
  CALL A9GJ(A,N2,N,INDEF,X)
  CALL SOUT(A,N2,N,NOUT)
C COMPUTE DEVIATION FROM ORIGINAL MATRIX
  S=0.0
  DO 50 I=1,N
    DO 40 J=1,I
      IJ=I*(I-1)/2+J
      T=ABS(J-A(IJ))
      IF(T.GT.S)S=T
40    CONTINUE
50    CONTINUE
  WRITE(NOUT,958)S
958  FORMAT('OMAX. DEVN. OF INVERSE-INVERSE FROM ORIGINAL=',1PE16.8)
C 100 CONTINUE
  STOP
  END
  SUBROUTINE SOUT(A,N2,N,NOUT)
C J.C. NASH JULY 1978, APRIL 1989
  INTEGER N2,N,NOUT,I,J,IJ,JJ
  REAL A(N2)
C PRINTS SYMMETRIC MATRIX STORED ROW-WISE AS A VECTOR
  DO 20 I=1,N
    WRITE(NOUT,951)I
951  FORMAT(' ROW',I3)
    IJ=I*(I-1)/2+1
    JJ=IJ+I-1
    WRITE(NOUT,952)(A(J),J=IJ,JJ)
952  FORMAT(1H ,1P5E16.8)
20    CONTINUE
  RETURN
  END
  SUBROUTINE A9GJ(A,N2,N,INDEF,X)
C ALGORITHM 9
C J.C. NASH JULY 1978, FEBRUARY 1980, APRIL 1989
C BAUER-REINSCH GAUSS-JORDAN INVERSION OF A SYMMETRIC, POSITIVE
C A=MATRIX - STORED AS A VECTOR -- ELEMENT I,J IN POSITION I*(I-1)/2+J
C N2=LENGTH OF VECTOR A = N*(N+1)/2
C N=ORDER OF MATRIX

```

```

C INDEF=LOGICAL FLAG SET .TRUE. IF MATRIX NOT COMPUTATIONALLY
C POSITIVE DEFINITE
C X=WORKING VECTOR OF LENGTH AT LEAST N
C DEFINITE MATRIX
C STEP 0
    LOGICAL INDEF
    INTEGER N2,N,K,KK,Q,M,Q2,JI,JQ
    REAL A(N2),S,T,X(N)
C STEP 1
    INDEF=.FALSE.
    DO 100 KK=1,N
        K=N+1-KK
C STEP 2
        S=A(1)
C STEP 3
        IF(S.LE.0.0) INDEF=.TRUE.
        IF(INDEF)RETURN
C STEP 4
        M=1
C STEP 5
        DO 60 I=2,N
C STEP 6
            Q=M
            M=M+I
            T=A(Q+1)
            X(I)=-T/S
C STEP 7
            Q2=Q+2
            IF(I.GT.K)X(I)=-X(I)
C STEP 8
            DO 40 J=Q2,M
                JI=J-I
                JQ=J-Q
                A(JI)=A(J)+T*X(JQ)
            40 CONTINUE
C STEP 9
        60 CONTINUE
C STEP 10
        Q=Q-1
        A(M)=1/S
C STEP 11
        DO 80 I=2,N
            JI=Q+I
            A(JI)=X(I)
        80 CONTINUE
C STEP 12
    100 CONTINUE
    RETURN
END

```

Example output

```
## #!/bin/bash
gfortran ../fortran/a9.f
mv ./a.out ../fortran/a9.run
../fortran/a9.run
```

```
## OORDER= 4 ORIGINAL MATRIX
## ROW 1
## 1.00000000E+00
## ROW 2
## 1.00000000E+00 2.00000000E+00
## ROW 3
## 1.00000000E+00 2.00000000E+00 3.00000000E+00
## ROW 4
## 1.00000000E+00 2.00000000E+00 3.00000000E+00 4.00000000E+00
## OINVERSE
## ROW 1
## 2.00000000E+00
## ROW 2
## -1.00000000E+00 2.00000000E+00
## ROW 3
## 0.00000000E+00 -1.00000000E+00 2.00000000E+00
## ROW 4
## 0.00000000E+00 0.00000000E+00 -1.00000000E+00 1.00000000E+00
## OINVERSE OF INVERSE
## ROW 1
## 1.00000012E+00
## ROW 2
## 1.00000024E+00 2.00000048E+00
## ROW 3
## 1.00000036E+00 2.00000072E+00 3.00000095E+00
## ROW 4
## 1.00000036E+00 2.00000072E+00 3.00000095E+00 4.00000095E+00
## OMAX. DEVN. OF INVERSE-INVERSE FROM ORIGINAL= 9.53674316E-07
```

BASIC

Listing

```
10 PRINT "ALGORITHM 9 - BAUER REINSCH INVERSION TEST"
20 N=100
40 DIM A(N*(N+1)/2),X(N)
45 LET N=4
50 GOSUB 1500
51 REM BUILD MATRIX IN A
60 GOSUB 1400
61 REM PRINT IT
70 GOSUB 1000
71 REM INVERT
80 GOSUB 1400
81 REM PRINT
90 quit
110 STOP
1000 REM ALG. 9 BAUER REINSCH INVERSION
1010 FOR K=N TO 1 STEP -1
```

```

1011     REM STEP 1
1020     S=A(1)
1021     REM STEP 2
1030     IF S<=0 THEN EXIT 1160
1031     REM STEP 3
1040     M=1
1041     REM STEP 4
1050     FOR I=2 TO N
1051         REM STEP 5
1060         Q=M
1061         M=M+I
1062         T=A(Q+1)
1063         X(I)=-T/S
1064         REM STEP 6
1070         IF I>K THEN X(I)=-X(I)
1071         REM STEP 7
1080         FOR J=Q+2 TO M
1081             REM STEP 8
1090             A(J-I)=A(J)+T*X(J-Q)
1100         NEXT J
1110     NEXT I
1111     REM STEP 9
1120     Q=Q-1
1121     A(M)=1/S
1122     REM STEP 10
1130     FOR I=2 TO N
1131         A(Q+I)=X(I)
1132     NEXT I
1133     REM STEP 11
1140 NEXT K
1141     REM STEP 12
1150 RETURN
1160 PRINT "MATRIX COMPUTATIONALLY INDEFINITE"
1170 STOP
1171     REM END ALG. 9
1400 PRINT "MATRIX A"
1410 FOR I=1 TO N
1420 FOR J=1 TO I
1430 PRINT A(I*(I-1)/2+J);
1440 NEXT J
1450 PRINT
1460 NEXT I
1470 RETURN
1500 REM FRANK MATRIX
1510 FOR I=1 TO N
1520 FOR J=1 TO I
1530 LET A(I*(I-1)/2+J)=J
1540 NEXT J
1550 NEXT I
1560 RETURN

```

Example output

```
bwbasic ../BASIC/a9.bas >../BASIC/a9.out
# echo "done"
```

Bywater BASIC Interpreter/Shell, version 2.20 patch level 2

Copyright (c) 1993, Ted A. Campbell

Copyright (c) 1995-1997, Jon B. Volkoff

ALGORITHM 9 - BAUER REINSCH INVERSION TEST

MATRIX A

1

1 2

1 2 3

1 2 3 4

MATRIX A

2

-1 2

0 -1 2

0 0 -1 1

Pascal

Listing

```
program dr09(input,output);
{dr09.pas == driver program to test procedure for the Bauer-Reinsch
  inversion of a symmetric positive definite real matrix stored
  in row-wise vector form

  Copyright 1988 J.C.Nash
}
{I constype.def}
{constype.def ==
  This file contains various definitions and type statements which are
  used throughout the collection of "Compact Numerical Methods". In many
  cases not all definitions are needed, and users with very tight memory
  constraints may wish to remove some of the lines of this file when
  compiling certain programs.

  Modified for Turbo Pascal 5.0

  Copyright 1988, 1990 J.C.Nash
}
{uses Dos, Crt;} {Turbo Pascal 5.0 Modules}
{ 1. Interrupt, Unit, Interface, Implementation, Uses are reserved words now.}
{ 2. System,Dos,Crt are standard unit names in Turbo 5.0.}

const
  big = 1.0E+35;    {a very large number}
  Maxconst = 25;    {Maximum number of constants in data record}
```

```

Maxobs = 100;      {Maximum number of observations in data record}
Maxparm = 25;      {Maximum number of parameters to adjust}
Maxvars = 10;      {Maximum number of variables in data record}
acctol = 0.0001;   {acceptable point tolerance for minimisation codes}
maxm = 20;         {Maximum number or rows in a matrix}
maxn = 20;         {Maximum number of columns in a matrix}
maxmn = 40;        {maxn+maxm, the number of rows in a working array}
maxsym = 210;      {maximum number of elements of a symmetric matrix
                    which need to be stored = maxm * (maxm + 1)/2 }
reltest = 10.0;    {a relative size used to check equality of numbers.
                    Numbers x and y are considered equal if the
                    floating-point representation of reltest*x equals
                    that of reltest*y.}
stepredn = 0.2;    {factor to reduce stepsize in line search}
yearwrit = 1990;   {year in which file was written}

type
  str2 = string[2];
  rmatrix = array[1..maxm, 1..maxn] of real; {a real matrix}
  wmatrix = array[1..maxmn, 1..maxn] of real; {a working array, formed
                    as one real matrix stacked on another}
  smatvec = array[1..maxsym] of real; {a vector to store a symmetric matrix
                    as the row-wise expansion of its lower triangle}
  rvector = array[1..maxm] of real; {a real vector. We will use vectors
                    of m elements always. While this is NOT space efficient,
                    it simplifies program codes.}
  cgmethodtype= (Fletcher_Reeves,Polak_Ribiere,Beale_Sorenson);
  {three possible forms of the conjugate gradients updating formulae}
  probdata = record
    m      : integer; {number of observations}
    nvar   : integer; {number of variables}
    nconst : integer; {number of constants}
    vconst : array[1..Maxconst] of real;
    Ydata  : array[1..Maxobs, 1..Maxvars] of real;
    nlls   : boolean; {true if problem is nonlinear least squares}
  end;

{
  NOTE: Pascal does not let us define the work-space for the function
  within the user-defined code. This is a weakness of Pascal for this
  type of work.
}

var {global definitions}
  banner      : string[80]; {program name and description}

Procedure Frank(var n: integer; var A: rmatrix; var avector: smatvec);
var
  i,j: integer;
begin
  writeln('Frank symmetric');
  for i:=1 to n do
    begin
      for j:=1 to i do
        begin

```



```

        A[i,j]:=j;
        A[j,i]:=j;
    end;
end;
end;

Procedure mat2vec(var n: integer; var A: rmatrix; var avector: smatvec);
var
    i,j,k: integer;

begin {convert to vector form}
    k:=0; {index for vector element}
    for i:=1 to n do
        begin
            for j:=1 to i do
                begin
                    k:=k+1;
                    avector[k]:=A[i,j];
                end;
            end;
        end; {matrixin}

Procedure vec2mat(var n: integer; var A: rmatrix; var avector: smatvec);
var
    i,j,k: integer;

begin {convert to matrix form}
    k:=0; {index for vector element}
    for i:=1 to n do
        begin
            for j:=1 to i do
                begin
                    k:=k+1;
                    A[i,j]:=avector[k];
                end;
            end;
        end; {matrixin}

{ I alg09.pas}
procedure brspdmi(n : integer;
                 var avector : smatvec;
                 var singmat : boolean);

var
    i,j,k,m,q : integer;
    s,t : real;
    X : rvector;

begin
    writeln('alg09.pas -- Bauer Reinsch inversion');
    singmat := false;
    for k := n downto 1 do
        begin

```

```

if (not singmat) then
begin
  s := avector[1];
  if s>0.0 then
  begin
    m := 1;
    for i := 2 to n do
    begin
      q := m; m := m+i; t := avector[q+1]; X[i] := -t/s;

      if i>k then X[i] := -X[i];
      for j := (q+2) to m do
      begin
        avector[j-i] := avector[j]+t*X[j-q];
      end;
    end;
    q := q-1; avector[m] := 1.0/s;
    for i := 2 to n do avector[q+i] := X[i];
  end
  else
    singmat := true;
  end;
end;
end;

var
  A, Ainverse : rmatrix;
  avector : smatvec;
  i, imax, j, jmax, k, n : integer;
  errmax, s : real;
  singmat: boolean;

BEGIN { main program }
  banner:='dr09.pas -- test Bauer Reinsch sym, posdef matrix inversion';
  writeln(banner);
  n:=4; {Fixed example size 20210113}
  Frank(n,A,avector);
  writeln;
  writeln('returned matrix of order ',n);
  begin
    for i:=1 to n do
    begin
      for j:=1 to n do
      begin
        write(A[i,j], ' ');
      end;
      writeln;
    end;
  end;
  mat2vec(n, A, avector);
  begin
    writeln('Symmetric matrix -- Vector form');

```

```

k := 0;
for i := 1 to n do
begin
  for j := 1 to i do
  begin
    k := k+1;
    write(avector[k]:10:5, ' ');
  end;
  writeln;
end;
end;
brspdmi(n, avector, singmat);
if singmat then halt; {safety check}
writeln('Computed inverse');
k := 0; {initialize index to smatvec elements}
for i := 1 to n do
begin
  for j := 1 to i do
  begin
    k := k+1;
    write(avector[k]:10:5, ' ');
    Ainverse[i,j] := avector[k]; {save square form of inverse}
    Ainverse[j,i] := avector[k];
    if (7 * (j div 7) = j) and (j < i) then
    begin
      writeln;
    end;
  end;
  writeln;
end;
{Compute maximum error in A * Ainverse and note where it occurs.}
errmax := 0.0; imax := 0; jmax := 0;
for i := 1 to n do
begin
  for j := 1 to n do
  begin
    s := 0.0; if i=j then s := -1.0;
    for k := 1 to n do s := s + Ainverse[i,k]*A[k,j];
    {Note: A has not been altered, since avector was used.}
    if abs(s) > abs(errmax) then
    begin
      errmax := s; imax := i; jmax := j; {save maximum error, indices}
    end;
  end; {loop on j}
end; {loop on i}
writeln('Maximum element in Ainverse * A - 1(n) = ', errmax,
        ' position ', imax, ', ', jmax);
end. {dr09.pas == Bauer Reinsch inversion}

```

Example output

```

fpc ../Pascal2021/dr09.pas
# copy to run file

```

```
mv ../Pascal2021/dr09 ../Pascal2021/dr09.run
../Pascal2021/dr09.run >../Pascal2021/dr09p.out
```

```
## Free Pascal Compiler version 3.0.4+dfsg-23 [2019/11/25] for x86_64
## Copyright (c) 1993-2017 by Florian Klaempfl and others
## Target OS: Linux for x86-64
## Compiling ../Pascal2021/dr09.pas
## Linking ../Pascal2021/dr09
## /usr/bin/ld.bfd: warning: link.res contains output sections; did you forget -T?
## 233 lines compiled, 0.1 sec
```

```
dr09.pas -- test Bauer Reinsch sym, posdef matrix inversion
Frank symmetric
```

```
returned matrix of order 4
```

```
1.0000000000000000E+000 1.0000000000000000E+000 1.0000000000000000E+000 1.0000000000000000E+000
1.0000000000000000E+000 2.0000000000000000E+000 2.0000000000000000E+000 2.0000000000000000E+000
1.0000000000000000E+000 2.0000000000000000E+000 3.0000000000000000E+000 3.0000000000000000E+000
1.0000000000000000E+000 2.0000000000000000E+000 3.0000000000000000E+000 4.0000000000000000E+000
```

```
Symmetric matrix -- Vector form
```

```
1.00000
1.00000 2.00000
1.00000 2.00000 3.00000
1.00000 2.00000 3.00000 4.00000
```

```
alg09.pas -- Bauer Reinsch inversion
```

```
Computed inverse
```

```
2.00000
-1.00000 2.00000
0.00000 -1.00000 2.00000
0.00000 0.00000 -1.00000 1.00000
```

```
Maximum element in Ainverse * A - 1(n) = 0.0000000000000000E+000 position 0,0
```

Python

WARNING: interim test only!!!!???

Listing

The Algorithm 9 code:

```
# -*- coding: utf-8 -*-
"""
CNM Algorithm 09 test

J C Nash 2021-1-12
"""

import numpy
import math
import sys
def brspdmi(Avec, n):
# =====
# Bauer Reinsch inverse of symmetric positive definite matrix stored
# as a vector that has the lower triangle of the matrix in row order
# =====
```

```

print(Avec)
X = numpy.array([ 0 ] * n) # zero vector x
for k in range(n, 0, -1):
    s = Avec[0];
    #print("s=",s)
    if (s > 0.0) :
        m = 1;
        for i in range(2,n+1):
            q = m
            m = m+i
            t = Avec[q]
            X[i-1] = -t/s
            if i>k :
                X[i-1] = -X[i-1]
        #     print("i, q, m:", i, q, m)
        for j in range((q+2), m+1):
            #         print(j)
            #         print("j-q-1=", j-q-1)
            #         print(X[j-q-1])
            Avec[j-i-1] = Avec[j-1]+t*X[j-q-1]
            q = q-1
            Avec[m-1] = 1.0/s
        for i in range(2, n+1):
            print("i ",i)
            Avec[q+i-1] = X[i-1]
    else :
        print("Matrix is singular")
        sys.exit()
    print(k,":",Avec)
return(Avec)

def FrankMat(n):
    Amat = numpy.array([ [ 0 ] * n ] * n) # numpy.empty(shape=(n,n), dtype='object')
    for i in range(1,n+1):
        #         print("i=",i)
        for j in range(1,i+1):
            #         print(j)
            Amat[i-1,j-1]=j
            Amat[j-1,i-1]=j
    return(Amat)

def smat2vec(Amat):
    n=len(Amat[0])
    n2=int(n*(n+1)/2)
    svec = [ None ] * n2
    k = 0
    for i in range(1,n+1):
        for j in range(1,i+1):
            svec[k]=Amat[i-1, j-1]
            k=k+1
    return(svec)

def svec2mat(svec):

```

```

n2=len(svec)
n=int((-1+math.sqrt(1+8*n2))/2)
print("matrix is of size ",n)
Amat = numpy.array([ [ None ] * n ] * n)
k = 0
for i in range(1,n+1):
    for j in range(1,i+1):
        Amat[i-1, j-1] = svec[k]
        Amat[j-1, i-1] = svec[k]
        k=k+1
    return(Amat)

# Main program
AA = FrankMat(4)
print(AA)
avec = smat2vec(AA)
print(avec)
n=len(AA[0])
vinv = brspdmi(avec, n)
## Computed inverse
##      2.00000
##     -1.00000      2.00000
##      0.00000     -1.00000      2.00000
##      0.00000      0.00000     -1.00000      1.00000

print(vinv)
Ainv = svec2mat(vinv)
print(Ainv)
print(AA)
print(numpy.dot(Ainv, AA))

```

Example output

```
python3 ../python/A9.py
```

```

## [[1 1 1 1]
##  [1 2 2 2]
##  [1 2 3 3]
##  [1 2 3 4]]
## [1, 1, 2, 1, 2, 3, 1, 2, 3, 4]
## [1, 1, 2, 1, 2, 3, 1, 2, 3, 4]
## i  2
## i  3
## i  4
## 4 : [1, 1, 2, 1, 2, 3, -1, -1, -1, 1.0]
## i  2
## i  3
## i  4
## 3 : [1, 1, 2, 0, 0, 2.0, -1, -1, -1, 1.0]
## i  2
## i  3

```

```

## i 4
## 2 : [1, 0, 2.0, 0, -1, 2.0, -1, 0, -1, 1.0]
## i 2
## i 3
## i 4
## 1 : [2.0, -1, 2.0, 0, -1, 2.0, 0, 0, -1, 1.0]
## [2.0, -1, 2.0, 0, -1, 2.0, 0, 0, -1, 1.0]
## matrix is of size 4
## [[2.0 -1 0 0]
##  [-1 2.0 -1 0]
##  [0 -1 2.0 -1]
##  [0 0 -1 1.0]]
## [[1 1 1 1]
##  [1 2 2 2]
##  [1 2 3 3]
##  [1 2 3 4]]
## [[1.0 0.0 0.0 0.0]
##  [0.0 1.0 0.0 0.0]
##  [0.0 0.0 1.0 0.0]
##  [0.0 0.0 0.0 1.0]]

```

R

Others

```

## remove object and run files
cd ../fortran/
rm *.o
rm *.run
rm *.out
cd ../Pascal2021/
rm *.o
rm *.run
rm *.out
cd ../BASIC
rm *.out
cd ../Documentation
## ?? others

```

```

## rm: cannot remove '*.o': No such file or directory
## rm: cannot remove '*.out': No such file or directory

```

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