

# Simultaneous modelling of movement, measurement error, and observer dependence in double observer distance sampling surveys

**Abstract:** Wildlife researchers often use detections, non-detections and recorded distances of animals encountered in transect surveys to estimate abundance. However, commonly available distance sampling estimators require that distances to target animals are made without error and that animals are stationary while sampling is being conducted. In practice these requirements are often violated. In this paper, we describe a marginal likelihood framework for estimating abundance from double observer data that can accommodate movement and measurement error. In particular, we suppose that two observers independently detect and record binned distances to observed animal groups, and that they record a binary indicator for whether animals were moving or not. Under this framework, stationary animals are subject to measurement error and moving animals are subject to both movement and measurement error. Integrating over unknown animal locations, we construct a marginal likelihood for detection, movement, and measurement error parameters. Estimates of animal abundance can then be obtained via a modified Horvitz-Thompson-like estimator. Unmodelled heterogeneity in detection probability can potentially be incorporated via observer dependence effects. We investigate the performance of our approach in a simulation study and by analyzing data from a waterfowl helicopter survey.

**Key words:** aerial survey, mark-recapture distance sampling, measurement error, movement, point independence

# 1 Introduction

Distance sampling surveys (Burnham et al., 1980; Buckland et al., 2001) are often used to estimate the abundance of wildlife populations. Historically, such surveys were implemented using a single observer who followed a transect line and recorded the perpendicular distance to each detected animal group. Assuming 100% detection on the transect line, models can be fitted to these data that estimate abundance over the surveyed area while accounting for detection probabilities that decrease with distance from the transect line.

More recently, investigators discovered that double observer surveys have some large advantages over single observer surveys. For instance, one can use the detection/non-detection records to relax the assumption of 100% detection on the transect line (Borchers et al., 1998), a crucial development for many species and sampling situations (e.g. aerial surveys). Analysis of double observer distance data is now canonically referred to as “mark-recapture distance sampling” (MRDS) because there is a detection history (i.e. binary detection/nondetection records for each observer) in addition to recorded distances. Analysis of these histories could potentially be done in a similar manner to a two sample mark-recapture experiment, although this approach ignores some additional information contained in the distribution of observed distances (Laake and Borchers, 2004).

Several different observer configurations are possible within an MRDS

estimation framework (Burt et al., 2014). In an “independent” configuration, observers search for animals independently. Under this configuration it is possible to try to account for heterogeneity in detection probabilities (e.g. visual distinctiveness of different animal groups) by modelling lack of fit between the distribution of observed distances and estimated detection probabilities as a function of distance (Laake and Borchers, 2004; Borchers et al., 2006; Buckland et al., 2010). Alternatively, in a “trial” configuration (Laake and Borchers, 2004), one observer searches ahead, while another searches closer to the survey platform. Under this configuration, detections by the first observer are used as trials for the second observer. Collecting data in this manner can be useful for reducing the biasing influence of responsive movement of animals, but at the cost of no longer being able to model heterogeneity in detection probability (Burt et al., 2014).

In this paper, we develop an integrated likelihood framework to account for movement and measurement error within an independent observer MRDS framework. Our objective is to account for the biasing effects of measurement error and responsive movement while also being able to model individual heterogeneity through an observer dependence specification. The effect of movement on distance sampling estimators has previously been examined by Glennie et al. (2015), who showed movement could cause considerable bias in distance-based abundance estimators. However, we have not found any papers that attempt to explicitly model movement. By contrast, a number of authors have proposed models that account for measurement error in specific

67 distance sampling applications (see e.g. Borchers et al., 2010, and references  
68 therein).

69 The remainder of this article is structured as follows. First, we describe  
70 a motivating data set, in which distance, detection histories, and individ-  
71 ual covariates are assembled from a double observer waterfowl aerial survey.  
72 Second, we describe a maximum marginal likelihood framework for analyzing  
73 these data. Under our framework, locations of an animal group seen by the  
74 first observer and the second observer are treated as two latent variables.  
75 Next, we illustrate application of our approach by conducting a simulation  
76 study and applying it to the waterfowl data set. We conclude with a short  
77 discussion.

## 78 **2 Waterfowl data**

79 In June of 2014, biologists conducted a pilot double observer helicopter sur-  
80 vey of Arctic bird species in the Queen Maud Gulf Migratory Bird Sanctuary  
81 (Nunavut, Canada). The birds surveyed were predominantly waterfowl, but  
82 also included cranes and ptarmigan; we refer to them collectively as water-  
83 fowl for the remainder of the paper. The purpose of this particular survey  
84 was not to estimate abundance over the whole area. Rather, researchers  
85 were interested in comparing estimates of detection probability from double  
86 observer survey methods to estimates of detection probability using single  
87 observer protocols. The survey is described in greater detail elsewhere (Al-

isauskas and Conn, 2017), but we briefly provide information relevant to the analysis provided in this paper.

In the survey, two observers on the same side of the helicopter independently detected and recorded the perpendicular distance from the transect line to each bird group they observed. Distances were binned into 6 classes: 0-40m, 40-80m, 80-120m, 120-160m, 160-200m, and 200m+. They also recorded species, the number of waterfowl in each detected group (“group”), and a binary indicator for whether the waterfowl group was flapping their wings (“moving”). These data were previously analyzed by Alisauskas and Conn (2017), who used standard MRDS methods that ignored movement and measurement error in their analysis. Their analysis suggested higher detection probabilities for moving individuals, larger group sizes, and for the front seat observer (relative to a rear seat observer). They also estimated similar species effects on detection for 7 of the 9 species analyzed; here, we pool data from these 7 species to form an illustrative dataset. This protocol led to a total of 1025 unique waterfowl group detections; 323 were detected by both observers, 353 by the front observer only, and 258 by the back observer only. Note that the the back observer’s view of the first distance bin was partially obstructed. A plot of observed distance deviations suggests responsive movement away from the aircraft for moving animal groups. There were also some minor distance discrepancies for animal groups that were not moving, which is suggestive of measurement error (Fig. 1). Our objective is to build models that formally account for movement and measurement error

111 processes.

### 112 **3 Model development**

113 Consider a double observer MRDS survey where observers independently  
114 record binned distances to detected groups of animals and a total of  $n$  an-  
115 imal groups are encountered by at least one observer (see Table 1 for a  
116 complete list of notation). We develop a two stage approach for estimat-  
117 ing abundance in the surveyed area from such data. In the first step, a  
118 marginal likelihood framework is used to simultaneously estimate parameters  
119 of detection, movement, and measurement error processes. In the second, a  
120 Horvitz-Thompson-like estimator is used to estimate abundance conditioned  
121 on parameter estimates from step 1 (a bootstrap procedure is used to quan-  
122 tify precision). For purposes of this paper we do not explicitly consider the  
123 problem of extrapolating abundance/density to a larger region (e.g. to un-  
124 surveyed locations); we touch on this issue in the Discussion.

125 In MRDS surveys with binned distances, observers record animals as  
126 occurring in one of  $n_{\mathcal{S}}$  perpendicular distance bins,  $\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{n_{\mathcal{S}}}$ . De-  
127 tection probability typically decreases with distance from the transect line,  
128 and the maximum distance bin is often set such that animals further away  
129 are poorly detected and can be ignored without greatly affecting precision of  
130 abundance estimates. Movement and measurement error introduce compli-  
131 cations: animals can potentially move into or out of  $\mathcal{S}$ , and animals outside

of  $\mathcal{S}$  can be detected in  $\mathcal{S}$ . For these reasons, the models we develop rely on augmenting  $\mathcal{S}$  with additional distance bins to allow for movement and measurement error (Fig. 2). Call this augmented set  $\mathcal{Z}$ .

Let  $y_{oi}$  be a binary indicator for whether or not the  $i$ th animal group was detected by observer  $o$ . Similarly, let  $d_{oi}$  denote the distance bin recorded by observer  $o$  for animal group  $i$  (note  $d_{oi}$  is only defined when  $y_{oi} = 1$ ). Letting bold lower case symbols denote vectors (e.g.  $\mathbf{y}_o$  gives a sequence of detections for observer  $o$ ,  $i = 1, 2, \dots, n$ ) and bold upper case symbols denote matrices, we seek to define a marginal likelihood  $[\boldsymbol{\theta} | \mathbf{Y}, \mathbf{D}, \mathbf{X}]$ , where  $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\varphi}\}$  are parameters describing detection, movement, and measurement error, and  $\mathbf{X}$  include individual covariates collected for each animal group that can be used to explain variation in detection probabilities.

### 3.1 Likelihood

To construct such a likelihood, we start with the general framework proposed by Borchers et al. (2015) for spatial mark-recapture and distance sampling surveys. Conditioning on detection, Borchers et al. suggested that the joint distribution of animal locations and detections could be written as a product of (1) a joint probability density function for the latent locations of animals, and (2) a joint probability mass function for the encounter histories conditional on location. We expand upon this framework to allow movement to affect the distribution of animal locations and to incorporate a measurement error mechanism.

154 Letting  $\mathbf{z}_o$  denote the true locations of animals when they enter the field  
 155 of view of observer  $o$ , we write the joint probability mass function of observed  
 156 data as a product of

- 157 1.  $[\mathbf{Z}|\boldsymbol{\theta}]$ , a bivariate probability mass function for the distribution of true  
 158 animal locations, given detection by at least one observer; and
- 159 2.  $[\mathbf{Y}, \mathbf{D}|\mathbf{Z}, \boldsymbol{\theta}, \mathbf{X}]$ , a model for binary detections and observed distances  
 160 given true unobserved locations and individual detection covariates;  
 161 and

162 If we knew the true locations of observed animals, we could simply base  
 163 inference on the likelihood

$$[\boldsymbol{\theta}|\mathbf{Y}, \mathbf{D}, \mathbf{X}] \propto [\mathbf{Z}|\boldsymbol{\theta}][\mathbf{Y}, \mathbf{D}|\mathbf{Z}, \boldsymbol{\theta}, \mathbf{X}].$$

164 However, we do not know the actual animal locations so instead integrate  
 165 (sum) over an augmented set of distance bins  $\mathcal{Z}$  that could plausibly have  
 166 resulted in a detection (see *Distribution of animal locations* for more discus-  
 167 sion of bin augmentation). As such, we write the joint marginal likelihood  
 168 of detection, movement, and measurement error parameters as

$$[\boldsymbol{\theta}|\mathbf{Y}, \mathbf{D}, \mathbf{X}] \propto \prod_i \left( \sum_{z_{i1} \in \mathcal{Z}} \sum_{z_{i2} \in \mathcal{Z}} [\mathbf{z}_i|\boldsymbol{\theta}][\mathbf{y}_i, \mathbf{d}_i|\mathbf{z}_i, \boldsymbol{\theta}, \mathbf{x}_i] \right). \quad (1)$$

169 We now describe each of the likelihood components in further detail.



### 170 3.1.1 Distribution of animal locations

171 The first component of the likelihood (Eq. 1) is the joint probability mass  
 172 function for the locations of group  $i$ ,  $[\mathbf{z}_i|\boldsymbol{\theta}]$  given detection by at least one  
 173 observer. We write this distribution as a function of (i) an initial state dis-  
 174 tribution, (ii) a movement kernel, and (iii) a thinning probability equivalent  
 175 to detection probability by at least one observer. Specifically, we set

$$[\mathbf{z}_i|\boldsymbol{\theta}] \propto [z_{i1}][z_{i2}|z_{i1}, \boldsymbol{\phi}]p_i^*(z_{i1}, z_{i2}|\mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\varphi}). \quad (2)$$

176 In all applications described in this paper, we make the assumption that the  
 177 first observer (typically in a front seat) detects animal groups before any re-  
 178 sponsive movement has occurred. Under this assumption, random placement  
 179 of transect lines should help ensure that perpendicular distances of animals  
 180 from the transect line are uniformly distributed in space (cf. Buckland et al.,  
 181 2001). Letting  $\pi_j$  denote the proportional diameter of distance bin  $j$  (i.e.  
 182  $\pi_j = a_j / \sum_k a_k$  where  $a_j$  is the diameter of distance bin  $j$ ), we simply have

$$[z_{i1}] = \text{Categorical}(\pi_1, \pi_2, \dots, \pi_{n_Z}),$$

183 where it is understood that “Categorical” denotes a multinomial distribution  
 184 with index 1, and  $n_Z$  is the number of latent distance bins.

185 Next, the bivariate movement pmf  $[z_{i2}|z_{i1}, \boldsymbol{\phi}]$  describes the location of  
 186 animal group  $i$  when it enters the field of view of observer 2 as a function of  
 187 the location when it was in the field of view of observer 1. We model this as

188 another categorical distribution:

$$[z_{i2}|z_{i1}, \boldsymbol{\phi}] = \text{Categorical}(\psi(z_{i1}, 1), \psi(z_{i1}, 2), \dots, \psi(z_{i1}, n_Z)).$$

189 For applications in this paper, we parameterize the movement transition ker-  
 190 nel probabilities  $\boldsymbol{\psi}$  using asymmetric kernels. Using an asymmetric kernel  
 191 can allow movement rates to be different toward and away from the transect  
 192 line (anticipating a behavioral response to the survey platform). In particu-  
 193 lar, we set

$$194 \quad \psi(z_{i1}, z_{i2}) \propto g(z_{i1}, z_{i2}|\boldsymbol{\phi}), \text{ where} \quad (3)$$

195

$$196 \quad g(z_{i1}, z_{i2}) = \begin{cases} f(\delta_{i2}|\mu = \delta_{i1}, \sigma = \phi_1) & z_{i2} < z_{i1}, m_i = 1 \\ f(\delta_{i2}|\mu = \delta_{i1}, \sigma = \phi_2) & z_{i2} \geq z_{i1}, m_i = 1 \\ 1.0 & z_{i2} = z_{i1}, m_i = 0 \\ 0.0 & z_{i2} \neq z_{i1}, m_i = 0 \end{cases} \quad (4)$$

197 Here,  $f()$  gives a probability density function; in our examples, we consider  
 198 Laplace (double exponential) and Gaussian distributions as choices for  $f()$ .  
 199 Note that  $\delta_{io}$  gives the perpendicular distance from the transect line to the  
 200 midpoint of distance bin  $z_{io}$ . Also note that we assume that stationary  
 201 animals (i.e. with  $m_i = 0$ ) do not change distance bins.

202 Finally, the thinning probability  $p_i^*(z_{i1}, z_{i2}|\mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\varphi})$  describes the prob-

203 ability of being detected by at least one observer for an animal that is in  
 204 distance bin  $z_{i1}$  at time 1 and  $z_{i2}$  at time 2. For generality, we calculate  
 205 probability as the sum of obtaining one of the three possible detection histo-  
 206 ries: 11, 10, or 01 (detected by both observers, detected by the front observer  
 207 but not the back, or detected by the back observer but not the front). In  
 208 particular,

$$\begin{aligned}
 p_i^*(z_{i1}, z_{i2} | \mathbf{x}_i, \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\varphi}) &= p_{i1}(z_{i1})\omega(z_{i1}, \mathcal{S})p_{i2}(z_{i2})\omega(z_{i2}, \mathcal{S}) + \\
 &\quad p_{i1}(z_{i1})\omega(z_{i1}, \mathcal{S}) [p_{i2}(z_{i2})(1 - \omega(z_{i2}, \mathcal{S})) + (1 - p_{i2}(z_{i2}))] + \\
 &\quad p_{i2}(z_{i2})\omega(z_{i2}, \mathcal{S}) [p_{i1}(z_{i1})(1 - \omega(z_{i1}, \mathcal{S})) + (1 - p_{i1}(z_{i1}))].
 \end{aligned}$$

209 This expression is slightly different than typically encountered in mark-  
 210 recapture calculus, as one must account for two ways of getting a 0 in a  
 211 capture history: an observer can miss the animal group, or an observer can  
 212 detect the group but determine it is out of the truncation range of the tran-  
 213 sect (i.e.  $\notin \mathcal{S}$ ). To account for the latter possibility, we make use of the  
 214 measurement error kernel  $\omega$  (Table 1), which can be parameterized simi-  
 215 larly to  $\boldsymbol{\phi}$  (see Eqs. 3-4). In applications in the paper, we consider use of  
 216 symmetric kernels (Gaussian or Laplace) with a single dispersion parameter,  
 217  $\varphi$ . Our expression for  $p_i^*$  also relies on individual- and observer-dependent  
 218 detection probabilities,  $p_{io}(z_{io})$ . In order to impart meaningful variation in  
 219 detection probability, it is useful to express these in a regression framework

220 on a logit-linear scale, such that

$$\text{logit}(\mathbf{p}) = \mathbf{X}\boldsymbol{\beta}. \quad (5)$$

221 Note that we write  $p_{io}$  as a function of  $z_{io}$  to emphasize that the design matrix  
 222  $\mathbf{X}$  will often depend on the latent position of animals.

### 223 3.1.2 Likelihood of observed detections

The next component of the the likelihood is  $[\mathbf{y}_i, \mathbf{d}_i | \mathbf{z}_i, \boldsymbol{\theta}, \mathbf{x}_i]$ , the probability of observing the particular detection history and distance bin values for animal group  $i$  conditional on animal location. Conditional on detection by at least one observer, there are again three possible types of encounter histories: 11, 10, or 01. For 11 histories, there are  $n_S^2$  combinations of possible recorded distance bins; for 10 histories, there are  $n_S$  distance bins possible for observer 1; for 01 histories, there are  $n_S$  distance bins possible for observer 2. Thus, we can view  $[\mathbf{y}_i, \mathbf{d}_i | \mathbf{z}_i, \boldsymbol{\theta}, \mathbf{x}_i]$  as a multinomial distribution with index 1 and  $n_S^2 + 2n_S$  possible outcomes. The likelihood contribution for a particular animal group  $i$  can thus be written as

$$p_i^* \times \begin{cases} p_{i1}(z_{i1})\omega(z_{i1}, d_{i1})p_{i2}(z_{i2})\omega(z_{i2}, d_{i2}) & \text{if } y_{i1} = y_{i2} = 1 \\ p_{i1}(z_{i1})\omega(z_{i1}, d_{i1}) [p_{i2}(z_{i2})(1 - \omega(z_{i2}, \mathcal{S})) + (1 - p_{i2}(z_{i2}))] & \text{if } y_{i1} = 1, y_{i2} = 0 \\ p_{i2}(z_{i2})\omega(z_{i2}, d_{i2}) [p_{i1}(z_{i1})(1 - \omega(z_{i1}, \mathcal{S})) + (1 - p_{i1}(z_{i1}))] & \text{if } y_{i1} = 0, y_{i2} = 1. \end{cases}$$

224 **3.2 Horvitz-Thompson-like abundance estimator**

225 **3.3 Extension to incorporate detection heterogeneity**

226 **4 Analysis of waterfowl data**

227 **5 Simulation studies**

228 **6 Discussion**

229 extrapolation to unsurveyed areas.

230 additive vs. multiplicative measurement error (Borchers et al., 2010)

231 influence of proportion moving (assumed 0.7 in sim study).

232 other errors (group size differences, flying/not flying, species, etc.)

233 **7 Data accessibility**

234 R scripts and data necessary to recreate analyses have been collated into  
235 an R package, which is currently available at [https://github.com/pconn/](https://github.com/pconn/MSmixture)  
236 **MSmixture**. We plan to publish the package to an online archive/repository  
237 upon acceptance.

238

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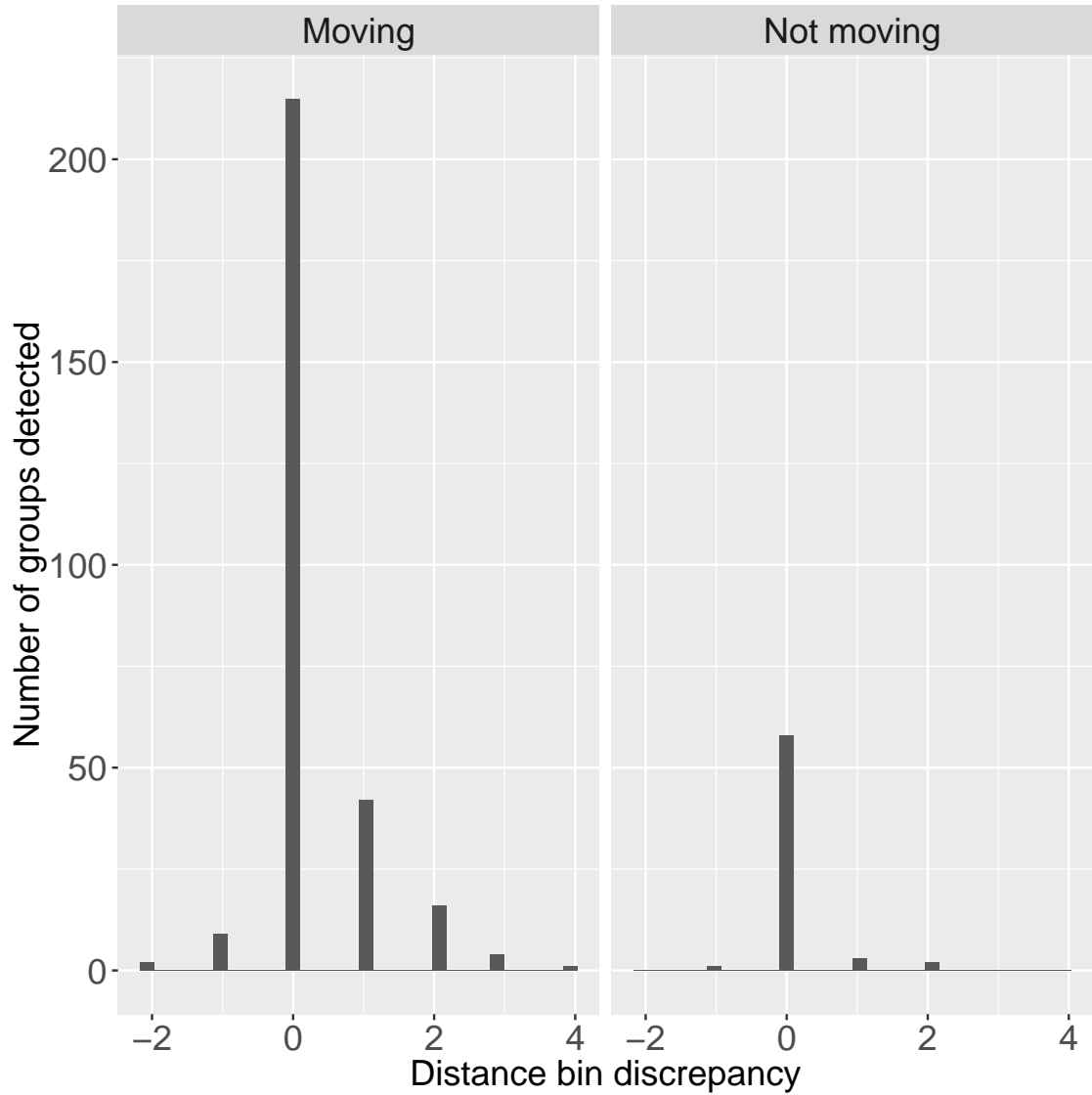


Figure 1: Distribution of distance bin discrepancies ( $d_2 - d_1$ ) for bird groups encountered by both front and back seat observers in aerial surveys. For moving birds, the distance bin observed by the back observer tended to be further away than the bin observed by the front observer. Since the second observer invariably detected birds later than the front observer, this suggests responsive movement away from the aircraft.



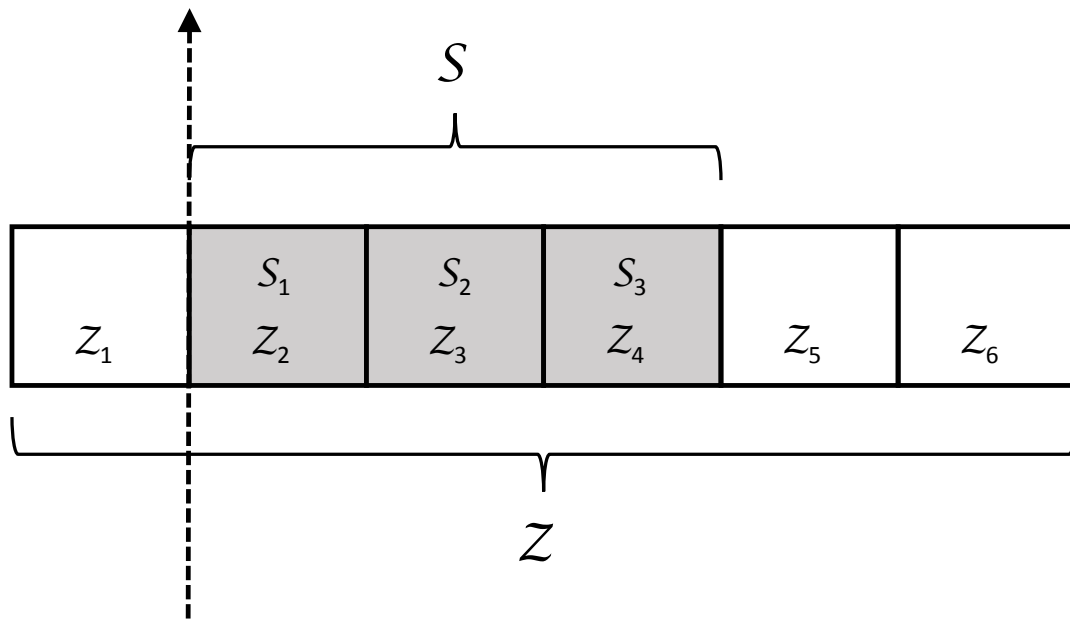


Figure 2: A depiction of observed ( $\mathcal{S}$ ) and latent ( $\mathcal{Z}$ ) distance bins that could potentially be used in analysis of a hypothetical mark-recapture distance sampling (MRDS) survey. In this example, only animals encountered in one of the three shaded distance bins to the right of the transect line (dashed line) are recorded; however, the state space is augmented with an additional three bins to account for possible animal movement and measurement error. In practice, the number of augmented distance bins that are needed will be a function of the magnitude of the movement and measurement error processes.

Table 1: Definitions of fixed and estimated quantities for the MRDS model incorporating movement and measurement error.

Quantity	Definition
<b>A. Fixed quantities</b>	
$n$	Number of animals detected by at least one observer
$y_{io}$	Binary indicator for whether animal group $i$ was detected by observer $o$
$d_{io}$	Distance bin recorded by observer $o$ for animal group $i$ (if recorded)
$m_i$	A binary indicator for whether animal group $i$ was moving when observed (a single determination is made)
$\mathbf{x}_i$	A vector of covariates used to explain variation in detection probability for group $i$
$g_i$	Number of animals in group $i$ (a single determination is made)
$\mathcal{S}$	The set of distance bins for which data are recorded, $\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{n_{\mathcal{S}}}$
$\mathcal{Z}$	The set of latent distance bins used for modeling true animal locations, $\mathcal{Z} = \mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_{n_{\mathcal{Z}}}$
$\pi_j$	Proportion of $\mathcal{Z}$ covered by latent distance bin $j$
<b>B. Parameters and functions of parameters</b>	
$z_{io}$	True (latent) distance bin of group $i$ when encountered by observer $o$
$\delta_{io}$	Perpendicular distance from the transect line to the midpoint of bin $z_{io}$
$\beta$	A vector of parameters governing logit-linear variation in detection probability
$\phi$	Parameters governing animal movement
$\varphi$	Parameters governing distance measurement error
$p_{io}(z_{io})$	Probability that observer $o$ detects group $i$ given that the group is truly in distance bin $z_{io}$
$p_i^*(z_{i1}, z_{i2})$	Probability that at least one observer detects group $i$ given the group is in distance bin $z_{i1}$ at time 1 and $z_{i2}$ at time 2
$\psi(z_{i1}, z_{i2})$	Probability that an animal that is in latent distance bin $z_{i1}$ when it passes observer 1 will be in latent distance bin $z_{i2}$ when it passes observer 2
$\omega(z, d)$	Probability that an animal group in distance bin $z$ is recorded as being in distance bin $d$
$\omega(z, \mathcal{S})$	Probability that an animal group in distance bin $z$ will have a recorded distance bin falling within $\mathcal{S}$
$\mathbf{X}$	A design matrix used to impart logit-linear structure on detection probabilities; note this will often include latent distance values, $\mathbf{z}_i$ .
$N$	True abundance of animals in the surveyed area