CONSTRUCTIVE VERSIONS OF TARSKI'S FIXPOINT THEOREMS

and

ASYNCHRONOUS ITERATIVE METHODS FOR SOLVING A FIXPOINT SYSTEM OF EQUATIONS IN A CHAIN-COMPLETE POSET

Patrick COUSOT

University of Grenoble (France)

Let F be an isotone operator on the complete lattice L into itself. Tarski's lattice theoretical fixpoint theorem states that the set of fixpoints of F is a non-empty complete lattice for the ordering of L. We give a constructive proof of this theorem showing that the set of fixpoints of F is the image of L by a lower and an upper preclosure operator. These preclosure operators are the composition of lower and upper closure operators which are defined by means of limits of stationary transfinite iteration sequences for F. In the same way we give a constructive characterization of the set of common fixpoints of a family of commuting isotone operators.

In the second part we show that the classes of asynchronous iterative methods and asynchronous iterative methods with memory can be used to solve fixpoint systems of continuous equations in a chain-complete poset. These iterative methods correspond to parallel algorithms for solving the system of equations on a multiprocessor computer with no synchronization between the cooperating processes.

CONSTRUCTIVE VERSIONS OF TARSKI'S FIXPOINT THEOREMS and ASYNCHRONOUS METHODS FOR SOLVING A FIXPOINT SYSTEM OF ISOTONE EQUATIONS IN A CHAIN-COMPLETE POSET

Patrick COUSOT

University of Grenoble (France)

Recalling classical fixpoint theorems :

Tarski (1955) :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$F \in iso(L \to L)$ isotone operator on L into L

$fp(F) = \{x \in L : F(x) = x\}$ is a complete (hence non empty) lattice $(\sqsubseteq, lfp(F), gfp(F), \vee, \wedge)$

- $lfp(F) = \sqcap \; postfp(F) = \sqcap \{x \in L : F(x) \sqsubseteq x\}$
- $gfp(F) = \sqcup \; prefp(F) = \sqcup \{x \in L : x \sqsubseteq F(x)\}$
- $\vee \; = \lambda S. \; lfp(F \mid [\sqcup S, \top])$
- $\wedge \; = \lambda S. \; gfp(F \mid [\bot, \sqcap S])$

Kleene (1952) :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$F \in uscont(L \to L)$ upper semi continuous operator on L into L.

$$lfp(F) = \bigsqcup_{i \geq 0} F^i(\bot)$$

Devidé (1964), Hitchcock and Park (1973), Pasini (1974) :

From an abstract point of view, one can avoid the continuity hypothesis by considering transfinite iterations.

Definitions :

- upper iteration sequence for F starting with $D \in L$:

  sequence $\langle X^\delta : \delta \in \mathrm{Ord} \rangle$ defined by transfinite recursion :

  . $X^0 = D$
  . $X^\delta = F(X^{\delta-1})$ if δ is a successor ordinal
  . $X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha$ if δ is a limit ordinal

- lower iteration sequence for F starting with $D \in L$:

  . $X^0 = D$
  . $X^\delta = F(X^{\delta-1})$ if δ is a successor ordinal
  . $X^\delta = \bigsqcap_{\alpha<\delta} X^\alpha$ if δ is a limit ordinal

- Stationary iteration sequence :

  $\exists \varepsilon : \forall \beta , \quad \beta \geqslant \varepsilon \Rightarrow X^\beta = X^\varepsilon$

- limit of the (stationary) upper iteration sequence for F starting with D :

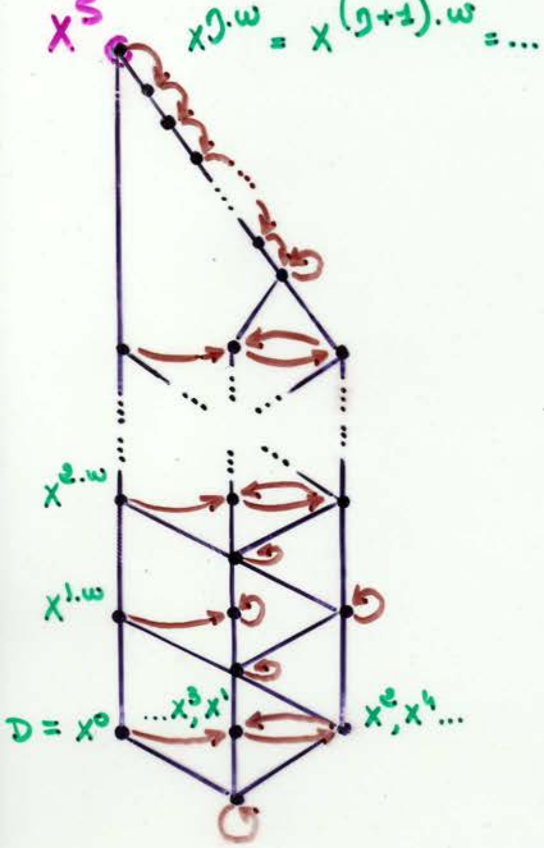
  X^ε denoted $\mathrm{luis}(F)(D)$

- limit of the (stationary) lower iteration sequence for F starting with D :

  X^ε denoted $\mathrm{llis}(F)(D)$

Note : $\mathrm{luis}(F)$ and $\mathrm{llis}(F)$ are partial functions .

Behavior of an upper iteration sequence



— In general X^{δ} and $X^{\delta'}$, $\delta \neq \delta'$ are not comparable, but:

$$X^{\beta.\omega+n} \sqsubseteq X^{\beta'.\omega+n'} \qquad \forall \beta' > \beta \ \forall n' \leq n$$

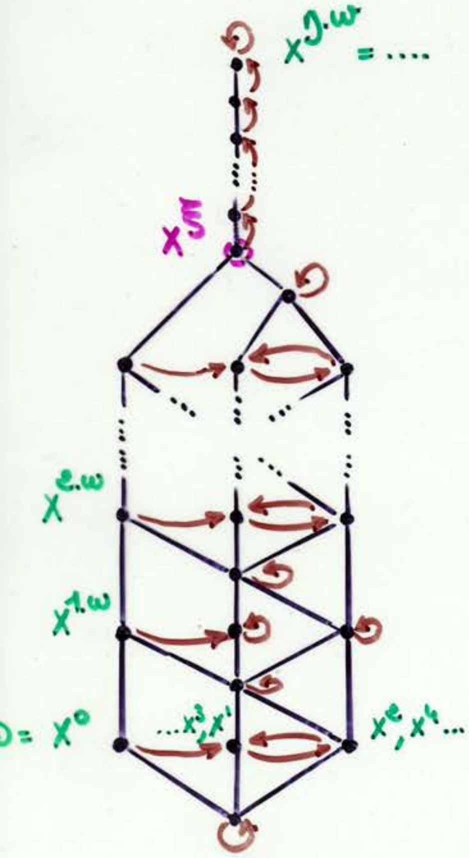
— $\langle X^{\alpha.\omega} : \alpha \in Ord \rangle$ is a stationary increasing chain, its limit $X^{\vartheta.\omega}$ is the least of the post-fixpoints of F greater than or equal to D

— $$X^{\vartheta.\omega} = \ell uis\,(\lambda X.\, X \sqcup F(X))(D) = \ell uis\,(\lambda X.\, D \sqcup F(X))(D)$$

— Let ξ be the smallest limit ordinal such that X^{ξ} and $F(X^{\xi})$ are comparable:

— If $X^{\xi} \in postfp(F)$:

- $\langle X^{\xi+\delta} : \delta \in Ord \rangle$ is a "cyclicly decreasing" sequence of post-fixpoints of F
- $\ell\ell is(F)(X^{\xi})$ is the greatest of the fixpoints of F less than or equal to X^{ξ}

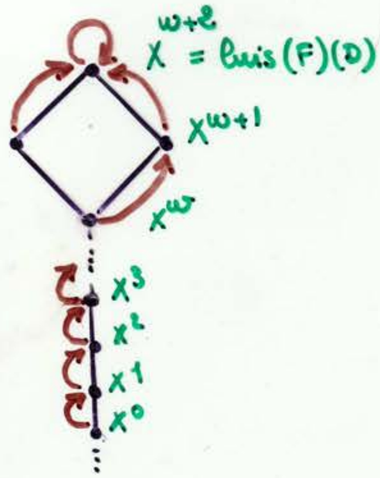


— If $X^{\xi} \in prefp(F)$:

- The increasing chain $\langle X^{\xi+\delta} : \delta \in Ord \rangle$ of elements of $prefp(F)$ is stationary.
- Its limit $\ell uis(F)(X^{\xi})$ is equal to $X^{\vartheta.\omega}$ which is the least of the fixpoints of F greater than or equal to D.

Characterization of the set of fixpoints of F

We have seen that:



- If $D \in prefp(F)$ then $luis(F)(D)$ exists, it is the least of the fixpoints of F greater than or equal to D
- The restriction of $luis(F)$ to $prefp(F)$ is an upper closure operator (isotone, idempotent, extensive).

Therefore:

1. $luis(F)(\ prefp(F)\ ) \subseteq fp(F)$
2. $\forall P \in fp(F), \exists Q \in prefp(F) : P = luis(F)(Q)$ (take $Q = P$)

   therefore $fp(F) \subseteq luis(F)(\ prefp(F)\ )$

$\Rightarrow \quad fp(F) = luis(F)(prefp(F))$

$\Rightarrow \quad fp(F) = llis(F)(postfp(F))$ (by duality)

3. $\forall D \in L, \quad D \sqsupseteq D \sqcap F(D)$

   $\Rightarrow D \in postfp(\lambda X.\ X \sqcap F(X))$

   $\Rightarrow llis(\lambda X.\ X \sqcap F(X))(D) \in fp(\lambda X.\ X \sqcap F(X))$

   but $fp(\lambda X.\ X \sqcap F(X)) = prefp(F)$ since $X \sqsubseteq F(X) \Leftrightarrow X = X \sqcap F(X)$

   $\Rightarrow llis(\lambda X.\ X \sqcap F(X))(L) \subseteq prefp(F)$

4. $\forall P \in prefp(F), \exists Q \in L : P = llis(\lambda X.\ X \sqcap F(X))(Q)$ (take $Q = P$)

   therefore $prefp(F) \subseteq llis(\ \lambda X.\ X \sqcap F(X)\ )(L)$

$\Rightarrow \quad prefp(F) = llis(\lambda X.\ X \sqcap F(X))(L)$

$\Rightarrow \quad postfp(F) = luis(\lambda X.\ X \sqcup F(X))(L)$ (by duality)

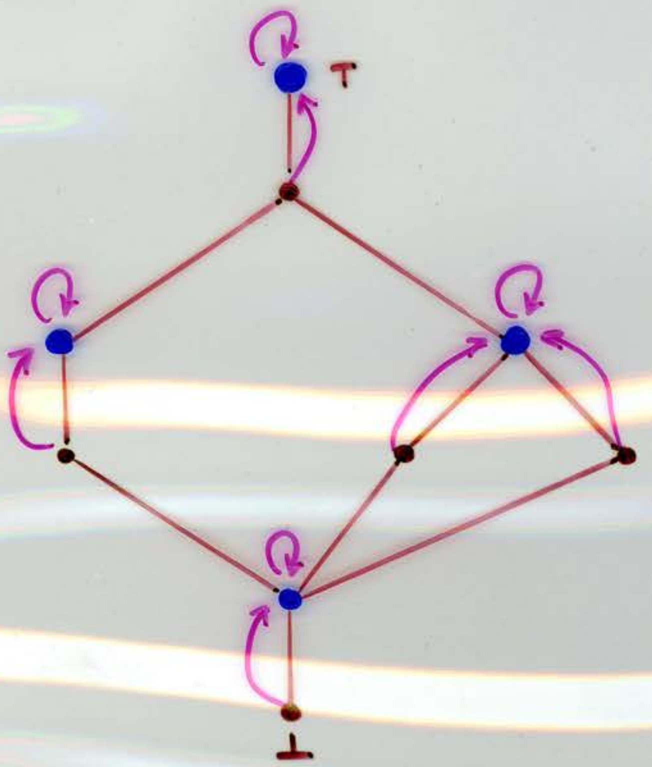
Image of a Complete Lattice by a closure operator

Ward [1948] :

$L\,(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\rho \in uclo(L \to L)$ monotone, idempotent, extensive

$\rho(L)\ (\sqsubseteq, \rho(\bot), \top, \lambda S.\,\rho(\sqcup S), \sqcap)$



by duality :

$L\,(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\rho \in lclo(L \to L)$ monotone, idempotent, reductive

$\rho(L)\ (\sqsubseteq, \bot, \rho(\top), \sqcup, \lambda S.\,\rho(\sqcap S))$

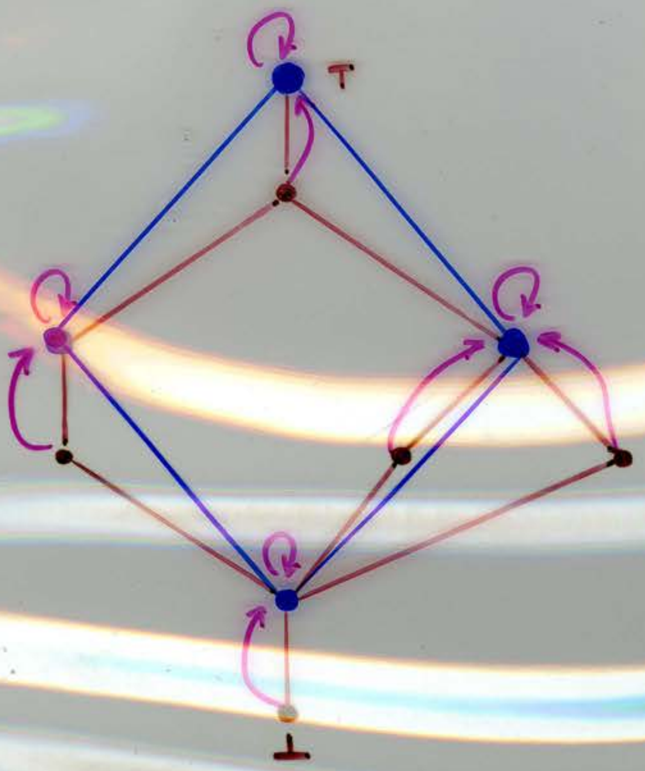
Image of a Complete Lattice by a closure operator

Ward [1948] :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\rho \in uclo(L \to L)$ monotone, idempotent, extensive

$\rho(L)\ (\sqsubseteq, \rho(\bot), \top, \lambda S.\, \rho(\sqcup S), \sqcap)$



by duality :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

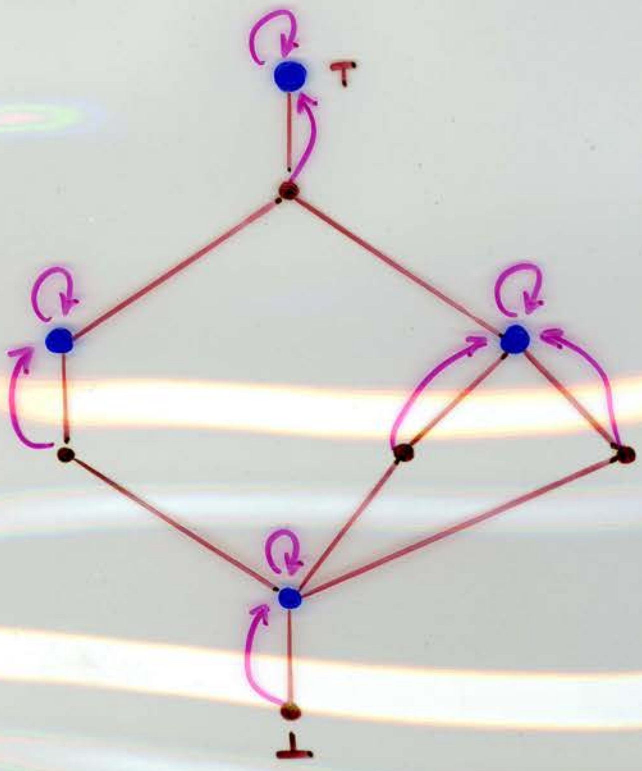
$\rho \in lclo(L \to L)$ monotone, idempotent, reductive

$\rho(L)\ (\sqsubseteq, \bot, \rho(\top), \sqcup, \lambda S.\, \rho(\sqcap S))$

Image of a Complete Lattice by a closure operator

Ward [1943]:

$$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap) \quad \text{complete lattice}$$
$$\rho \in uclo(L \to L) \quad \text{isotone, idempotent, extensive}$$
$$\overline{}$$
$$\rho(L)\ (\sqsubseteq, \rho(\bot), \top, \lambda S.\rho(\sqcup S), \sqcap)$$



by duality:

$$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap) \quad \text{complete lattice}$$
$$\rho \in lclo(L \to L) \quad \text{isotone, idempotent, reductive}$$
$$\overline{}$$
$$\rho(L)\ (\sqsubseteq, \bot, \rho(\top), \sqcup, \lambda S.\rho(\sqcap S))$$

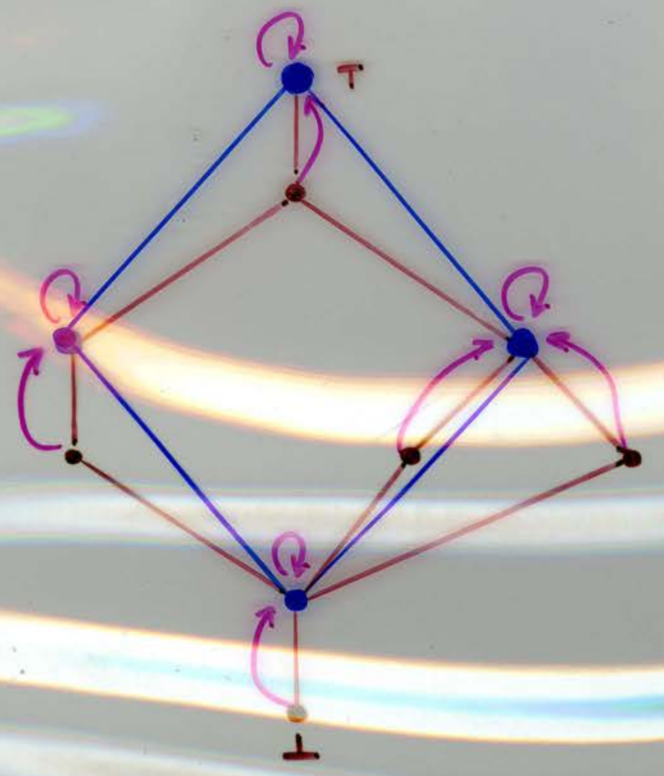
Image of a Complete Lattice by a closure operator

Ward [1948] :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\rho \in uclo(L \to L)$ isotone, idempotent, extensive

$\rho(L)\ (\sqsubseteq, \rho(\bot), \top, \lambda S . \rho(\sqcup S), \sqcap)$



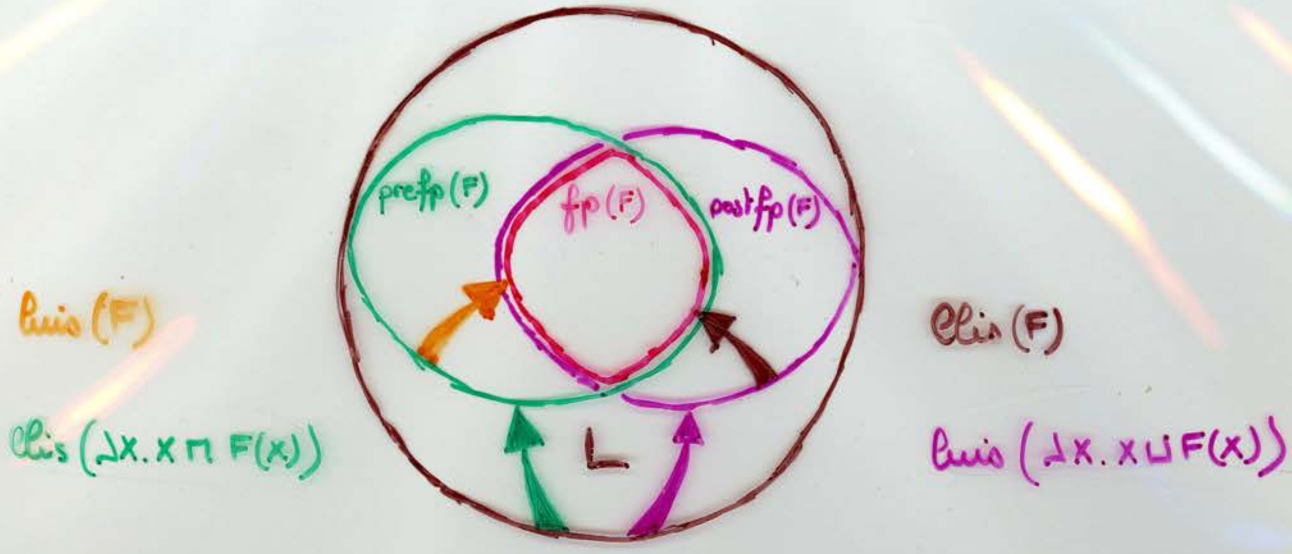
by duality :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\rho \in lclo(L \to L)$ isotone, idempotent, reductive

$\rho(L)\ (\sqsubseteq, \bot, \rho(\top), \sqcup, \lambda S . \rho(\sqcap S))$

Constructive version of Tarski's fixpoint theorem



- $\mathrm{postfp}(F) = \mathrm{lfp}^{\sqsubseteq}(\lambda X.\, X \sqcup F(X))(L)$ & $\mathrm{lfp}^{\sqsubseteq}(\lambda X.\, X \sqcup F(X)) \in \mathrm{fer}_s(L \to L)$

$\Rightarrow \mathrm{postfp}(F)\,(\sqsubseteq,\ \mathrm{lfp}^{\sqsubseteq}(\lambda X.\, X \sqcup F(X))(\bot),\ \top,\ \lambda S.\, \mathrm{lfp}^{\sqsubseteq}(\lambda X.\, X \sqcup F(X))(\sqcup S),\ \sqcap)$

- $\mathrm{prefp}(F) = \mathrm{gfp}^{\sqsubseteq}(\lambda X.\, X \sqcap F(X))(L)$ & $\mathrm{gfp}^{\sqsubseteq}(\lambda X.\, X \sqcap F(X)) \in \mathrm{fer}_i(L \to L)$

$\Rightarrow \mathrm{prefp}(F)\,(\sqsubseteq,\ \bot,\ \mathrm{gfp}^{\sqsubseteq}(\lambda X.\, X \sqcap F(X))(\top),\ \sqcup,\ \lambda S.\, \mathrm{gfp}^{\sqsubseteq}(\lambda X.\, X \sqcap F(X))(\sqcap S))$

- $\mathrm{fp}(F) = \mathrm{lfp}^{\sqsubseteq}(F)(\mathrm{prefp}(F))$ & $\mathrm{lfp}^{\sqsubseteq}(F) \in \mathrm{fer}_s(\mathrm{prefp}(F) \to \mathrm{prefp}(F))$

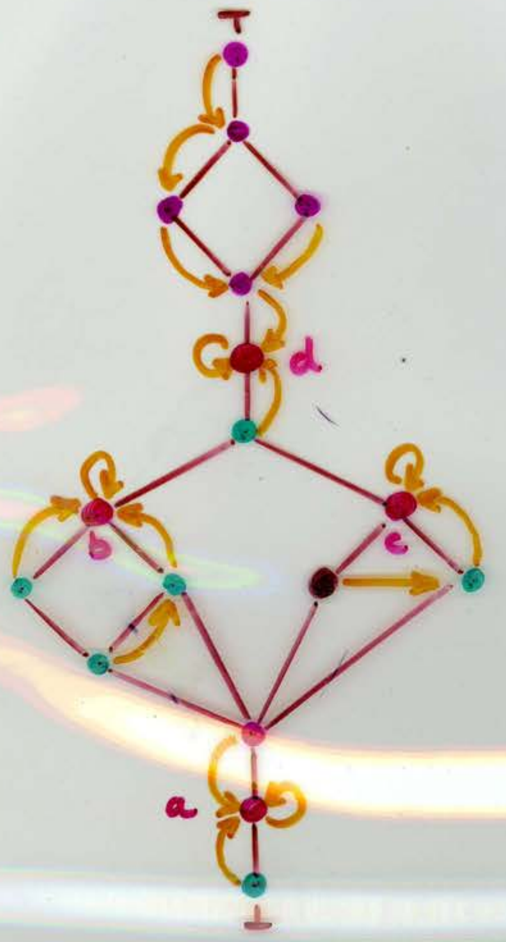
$\Rightarrow \mathrm{fp}(F)\,(\sqsubseteq,\ \mathrm{lfp}^{\sqsubseteq}(F)(\bot),\ \ldots,\ \lambda S.\, \mathrm{lfp}^{\sqsubseteq}(F)(\sqcup S),\ \ldots\ldots)$

- $\mathrm{fp}(F) = \mathrm{gfp}^{\sqsubseteq}(F)(\mathrm{postfp}(F))$ & $\mathrm{gfp}^{\sqsubseteq}(F) \in \mathrm{fer}_i(\mathrm{postfp}(F) \to \mathrm{postfp}(F))$

$\Rightarrow \mathrm{fp}(F)\,(\sqsubseteq,\ \ldots,\ \mathrm{gfp}^{\sqsubseteq}(F)(\top),\ \ldots\ldots,\ \lambda S.\, \mathrm{gfp}^{\sqsubseteq}(F)(\sqcap S))$

$\Rightarrow \mathrm{fp}(F)\,(\sqsubseteq,\ \mathrm{lfp}^{\sqsubseteq}(F)(\bot),\ \mathrm{gfp}^{\sqsubseteq}(F)(\top),\ \lambda S.\, \mathrm{lfp}^{\sqsubseteq}(F)(\sqcup S),\ \lambda S.\, \mathrm{gfp}^{\sqsubseteq}(F)(\sqcap S))$

$fp(F)\left(\sqsubseteq, luis(F)(\bot), llis(F)(\top), \lambda S.luis(F)(\sqcup S), \lambda S.llis(F)(\sqcap S)\right)$



- ↺ F
- • postfp(F)
- • prefp(F)
- • X et F(X) not comparable
- • fp(F)

$a = lfp(F) = luis(F)(\bot)$

$d = gfp(F) = llis(F)(\bot)$

$b \sqcup c \notin fp(F)$

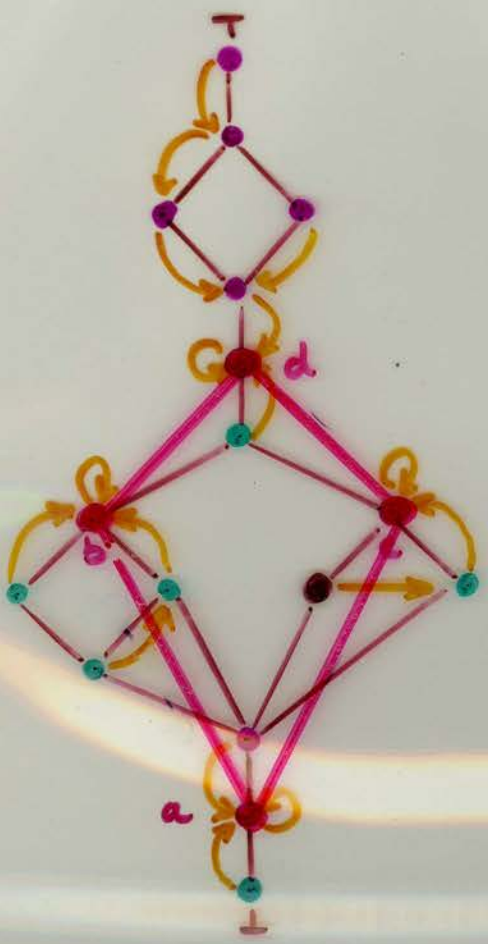
$d = luis(F)(b \sqcup c)$

$b \sqcap c \notin fp(F)$

$a = llis(F)(b \sqcap c)$

7 bis

$\mathrm{fp}(F)\Big(\sqsubseteq, \mathrm{luis}(F)(\bot), \mathrm{llis}(F)(\top), \lambda S.\mathrm{luis}(F)(\sqcup S), \lambda S.\mathrm{llis}(F)(\sqcap S)\Big)$



↑ F

• postfp(F)

• prefp(F)

• X et F(X) not comparable

• fp(F)

$a = \mathrm{lfp}(F) = \underline{\mathrm{luis}}(F)(\bot)$

$d = \mathrm{gfp}(F) = \underline{\mathrm{llis}}(F)(\bot)$

$b \sqcup c \notin \mathrm{fp}(F)$

$d = \underline{\mathrm{luis}}(F)(b \sqcup c)$

$b \sqcap c \notin \mathrm{fp}(F)$

$a = \underline{\mathrm{llis}}(F)(b \sqcap c)$

Note on Kleene fixpoint theorem and the upper semi continuity hypothesis

$$\mathrm{lfp}(F) = \mathrm{luis}(F)(\perp) = X^{\varepsilon}$$

where

$$\underbrace{X^0 = \perp,\ X^1 = F(X^0),\ \ldots\ldots,}_{\text{strictly increasing chain}}\ X^{\varepsilon},\ \underbrace{\ldots\ldots}_{\text{stationary}}$$

Let us look for a sufficient hypothesis implying $\varepsilon \leq \omega$:

$$\exists \varepsilon \leq \omega : \quad X^{\varepsilon} = X^{\varepsilon+1} = \ldots = X^{\omega} = X^{\omega+1}$$

$$\Leftrightarrow \quad X^{\omega} = F(X^{\omega})$$

$$\Leftrightarrow \quad \bigsqcup_{\alpha<\omega} X^{\alpha} = F(X^{\omega})$$

$$\Leftrightarrow \quad X^0 \sqcup \bigsqcup_{\alpha+1<\omega} X^{\alpha+1} = F(X^{\omega})$$

$$\Leftrightarrow \quad \bigsqcup_{\alpha+1<\omega} F(X^{\alpha}) = F(X^{\omega})$$

$$\Leftrightarrow \quad \bigsqcup_{\alpha<\omega} F(X^{\alpha}) = F\Big(\bigsqcup_{\alpha<\omega} X^{\alpha}\Big)$$

We can choose:

$$\{X^0 = \perp, \ldots, X^{i+1} = F(X^i), \ldots\} \Rightarrow \Big\{\bigsqcup_{i\geq 0} F(X^i) = F\Big(\bigsqcup_{i\geq 0} X^i\Big)\Big\}$$

or less generally the upper-semi-continuity hypothesis:

$$\{X^0 \sqsubseteq X^1 \sqsubseteq \ldots \sqsubseteq X^i \sqsubseteq X^{i+1} \sqsubseteq \ldots\} \Rightarrow \Big\{\bigsqcup_{i\geq 0} F(X^i) = F\Big(\bigsqcup_{i\geq 0} X^i\Big)\Big\}$$

in which cases:

$$\mathrm{lfp}(F) = \bigsqcup_{i\geq 0} F^i(\perp)$$

Constructive version of Tarski's fixpoint theorem for commuting isotone maps.

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$\{F_i : i \in I\}$ non-empty family of isotone commuting operators
$(\forall i, j \in I,\ F_i \circ F_j = F_j \circ F_i)$.

$fp(\{F_i : i \in I\}) = \{x \in L : \forall i \in I,\ F_i(x) = x\}$

$$fp(\{F_i : i \in I\}) \Big(\sqsubseteq, lfp\Big(\bigsqcup_{i \in I} F_i\Big), gfp\Big(\bigsqcap_{i \in I} F_i\Big),$$
$$\lambda S.\, luis\Big(\bigsqcup_{i \in I} F_i\Big)(\sqcup S), \lambda S.\, llis\Big(\bigsqcap_{i \in I} F_i\Big)(\sqcap S)\Big)$$

If I is well-ordered and $Card(I) \leq \omega$ that is $I = \{i_\alpha : 0 \leq \alpha \leq n\}$, then

$$fp(\{F_i : i \in I\}) = fp\Big(\mathop{\circ}_{i \in I} F_i\Big)$$

where

$$\mathop{\circ}_{i \in I} F_i = F_{i_0} \circ F_{i_1} \circ \cdots \circ F_{i_n}$$

- CHAOTIC
- ASYNCHRONOUS
- ASYNCHRONOUS WITH MEMORY

ITERATIVE METHODS for solving a fixpoint system of isotone equations on a complete lattice

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$F \in \mathrm{iso}(L^n \to L^n)$

the direct decomposition of the equation

$$X = F(X)$$

is the system of equations :

$$\begin{cases} X_1 = F_1(X_1, \ldots, X_n) \\ \ldots \\ X_n = F_n(X_1, \ldots, X_n) \end{cases}$$

The iteration sequences used in the constructive version of Tarski's fixpoint theorem correspond to Jacobi's iteration method :

$$\begin{cases} X_i^{\delta+1} = F_i(X_1^{\delta}, \ldots, X_n^{\delta}) \\ i = 1, \ldots, n \end{cases}$$

but what about other iteration strategies such as Gauss-Seidel iteration method :

$$\begin{cases} X_i^{\delta+1} = F_i(X_1^{\delta+1}, \ldots, X_{i-1}^{\delta+1}, X_i^{\delta}, \ldots, X_n^{\delta}) \\ i = 1, \ldots, n \end{cases}$$

Convergence of an iterative method

In general, the convergence of an iterative method is dependent upon the iteration strategy. For example:

$$\begin{cases} x = f(x,y) \\ y = g(x,y) \end{cases}$$

$f(x,y)$	$y = a$	$y = b$	$y = c$
$x = a$		c	
$x = b$			
$x = c$	a		c

$g(x,y)$	$y = a$	$y = b$	$y = c$
$x = a$	b	c	
$x = b$			
$x = c$		a	c

Strategy 1 :
Jacobi's successive approximations

$$\begin{bmatrix} x^0 & & = a \\ y^0 & & = b \end{bmatrix}$$

$$\begin{bmatrix} x^1 & = f(x^0,y^0) & = c \\ y^1 & = g(x^0,y^0) & = c \end{bmatrix}$$

$$\begin{bmatrix} x^2 & = f(x^1,y^1) & = c \\ y^2 & = g(x^1,y^1) & = c \end{bmatrix}$$

$$\begin{bmatrix} x^3 & = f(x^2,y^2) & = c \\ y^3 & = g(x^2,y^2) & = c \end{bmatrix}$$

$$\begin{bmatrix} x^4 & = f(x^3,y^3) & = c \\ y^4 & = g(x^3,y^3) & = c \end{bmatrix}$$

stationary

Strategy 2:
Gauss-Seidel

$$\begin{bmatrix} x^0 & & = a \\ y^0 & & = b \end{bmatrix}$$

$$\begin{bmatrix} x^1 & = f(x^0,y^0) & = c \\ y^1 & = y^0 & = b \end{bmatrix}$$

$$\begin{bmatrix} x^2 & = x^1 & = c \\ y^2 & = g(x^1,y^1) & = a \end{bmatrix}$$

$$\begin{bmatrix} x^3 & = f(x^2,y^2) & = a \\ y^3 & = y^2 & = a \end{bmatrix}$$

$$\begin{bmatrix} x^4 & = x^3 & = a \\ y^4 & = g(x^3,y^3) & = b \end{bmatrix}$$

cyclic

In the case of isotone operators on a complete lattice, can the convergence theorems for Jacobi's successive approximations be generalized to other iteration strategies?

Specification of a chaotic iteration strategy

At each step δ we give the set J^δ of components which evolve at this step:

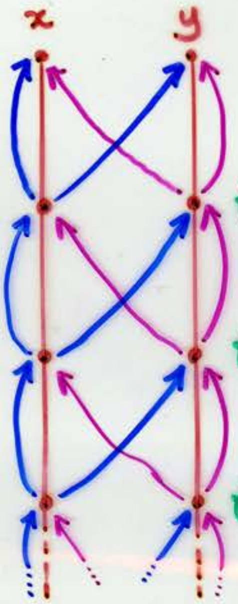
Example:

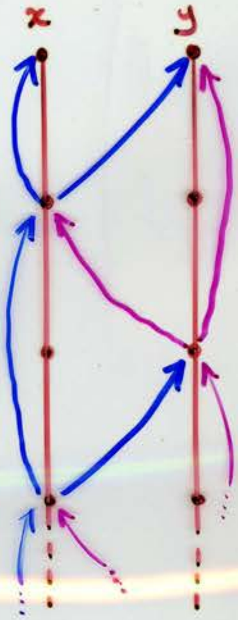
Jacobi's successive approximations:

$$\begin{cases} x^{i+1} = f(x^i, y^i) \\ y^{i+1} = g(x^i, y^i) \end{cases}$$

Gauss Seidel:

$$\begin{cases} x^{i+1} = f(x^i, y^i) \\ y^{i+1} = g(x^{i+1}, y^i) \end{cases}$$





Jacobi	J^δ	δ	J^δ	Gauss Seidel
$\begin{cases} x^0 \\ y^0 \end{cases}$		0		$\begin{cases} x^0 \\ y^0 \end{cases}$
$\begin{cases} x^1 = f(x^0, y^0) \\ y^1 = g(x^0, y^0) \end{cases}$	$\{1,2\}$	1	$\{1\}$	$\begin{cases} x^1 = f(x^0, y^0) \\ y^1 = y^0 \end{cases}$
$\begin{cases} x^2 = f(x^1, y^1) \\ y^2 = g(x^1, y^1) \end{cases}$	$\{1,2\}$	2	$\{2\}$	$\begin{cases} x^2 = x^1 \\ y^2 = g(x^1, y^1) \end{cases}$
$\begin{cases} x^3 = f(x^2, y^2) \\ y^3 = g(x^2, y^2) \end{cases}$	$\{1,2\}$	3	$\{1\}$	$\begin{cases} x^3 = f(x^2, y^2) \\ y^3 = y^2 \end{cases}$
		$\vdots$		
	$\{1,2\}$	δ	$\{1\}$ if δ odd $\{2\}$ if δ even	

Convergence of chaotic iterations :

$X = F(X)$ $\quad \begin{cases} X_i = F_i(X_1, \ldots, X_n) \\ i = 1, \ldots, n \end{cases}$

Definition :

a chaotic iteration sequence $\langle X^\delta : \delta \in \text{Ord} \rangle$ for F starting with D and defined by the strategy $\langle J^\delta : \delta \in \text{Ord} \rangle$ where

$F \in (L^n \to L^n)$

$\forall \delta \in \text{Ord}, \; J^\delta \subseteq \{1, \ldots, n\}$

is the sequence defined by transfinite recursion, as follows:

- $X^0 = D$
- $X_i^\delta = X_i^{\delta-1}$ if $i \notin J^\delta$ and δ successor ordinal
- $X_i^\delta = F_i(X^{\delta-1})$ if $i \in \underline{J^\delta}$ and δ successor ordinal
- $X^\delta = \bigsqcup_{\alpha < \delta} X^\alpha$ if δ limit ordinal.

Theorem :

$L(\sqsubseteq, \bot, \top, \sqcup, \sqcap)$ complete lattice

$F \in \text{iso}(L^n \to L^n)$ $\quad n \geq 1$

If $D \in \text{prefp}(F)$ then the chaotic iteration sequence for F starting with D and defined by any fair strategy $\langle J^\delta : \delta \in \text{Ord} \rangle$ is a stationary increasing chain, its limit is $\text{luis}(F)(D)$.

The strategy $\langle J^\delta : \delta \in \text{Ord} \rangle$ is fair if no component is abandoned for ever :

$\forall \delta \in \text{Ord}, \; \forall i \in \{1, \ldots, n\}, \; \exists \alpha \geq \delta : i \in J^\delta$

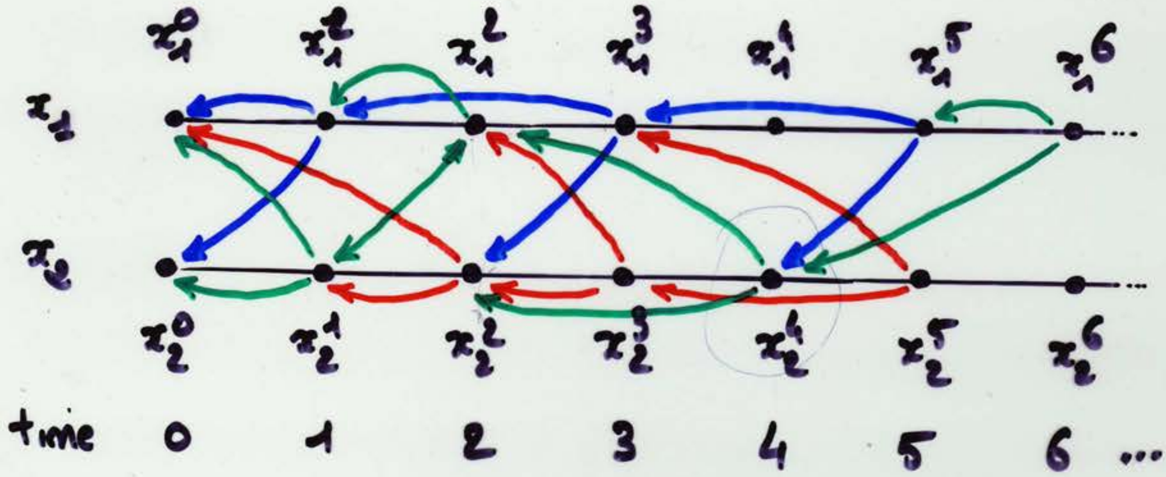
Specification of a chaotic iteration with delay / asynchronous iteration

Parallel algorithms for solving the system of equations on a multiprocessor with no synchronization between the cooperating processes.

Example :

$$\begin{cases} x_1 = F_1(x_1, x_2) \\ x_2 = F_2(x_1, x_2) \end{cases}$$

— processor 1
— processor 2
— processor 3



At time δ we define :

- The set J^δ of components which evolve at time δ
- For each component $i \in J^\delta$ modified at time δ we specify which values

$$x_1^{s_i^\delta(1)}, \dots, x_n^{s_i^\delta(n)}$$

have been used in order to compute its value

$$x_i^\delta = F_i\left(x_1^{s_i^\delta(1)}, \dots, x_n^{s_i^\delta(n)}\right) \quad \text{if } i \in J^\delta$$

Definition Chazan & Miranker (69), revised Baudet (76), revised Cousot (77)

An asynchronous iteration sequence $\langle X^\delta : \delta \in Ord \rangle$ for $F \in L^n \to L^n$ starting with $D \in L^n$ and defined by the strategy $\langle J^\delta : \delta \in Ord \rangle$, $\langle S^\delta : \delta \in Ord \rangle$ is the sequence :

- $X^0 = D$
- $X_i^\delta = X_i^{\delta - 1}$ if $i \notin J^\delta \wedge \delta$ successor ordinal
- $X_i^\delta = F_i\left(X_1^{S_i^\delta(1)}, \ldots, X_n^{S_i^\delta(n)}\right)$ if $i \in J^\delta \wedge \delta$ successor ordinal
- $X_i^\delta = \bigsqcup_{\alpha < \delta} X^\alpha$ if δ limit ordinal

Contrains :

- $\forall \delta \in Ord$, $J^\delta \subseteq \{1, \ldots, n\}$
- $\forall \delta \in Ord$, $\forall i \in J^\delta$, $S_i^\delta \in (Ord)^n$
- No component is forgotten for ever :
  $\forall \delta \in Ord$, $\forall i = 1..n$, $\exists \alpha \geq \delta$: $i \in J^\alpha$
- Fairness conditions :
  - X_i^δ is written after $X_1^{S_i^\delta(1)}, \ldots, X_n^{S_i^\delta(n)}$ have been read : $\forall \delta, \forall i \in J^\delta, \forall j = 1, \ldots, n$, $S_i^\delta(j) < \delta$
  - The evaluation of $F_i\left(X_1^{S_i^\delta(1)}, \ldots, X_n^{S_i^\delta(n)}\right)$ takes a finite amount of time (but not necessarily bounded)

Convergence theorem :

$L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ complete lattice, $F \in iso(L^n \to L^n)$

Any asynchronous iteration sequence starting with $D \in prefp(F)$ is stationary, it's limit is $luis(F)(D)$

Asynchronous iteration with memory

Let f be $\lambda X.\, g(X) \sqcup h(X)$. A possible decomposition of the computation of $luis(f)(D)$ is:

$$(1) \quad \begin{cases} X^0 = D \\ X^{\delta+1} = g(X^\delta) \sqcup Y^\delta \end{cases} \qquad \begin{cases} Y^0 = D \\ Y^{\delta+1} = h(X^\delta) \end{cases}$$

so that two synchronized processors can be used to evaluate the same component. This decomposition is not decribable by asynchronous iterations. We must use an asynchronous iteration with 2 memories as follows

$$2) \quad \begin{cases} X^0 = D \\ X^1 = D \\ X^{\delta+2} = g(X^{\delta+1}) \sqcup h(X^\delta) \\ \qquad = \boxed{F(X^{\delta+1}, X^\delta)} \end{cases}$$

where $F(X, Y) = g(X) \sqcup h(Y)$ so that $f(X) = F(X, X)$

Convergence of asynchronous iterations with memory

Definition

- $<J^\delta, \delta \in Ord>$ transfinite sequence of subsets of $\{1, \dots, n\}$ such that (a) $\{\forall \delta \in Ord, \forall i \in \{1,\dots,n\}, \exists \alpha \geq \delta : i \in J^\alpha\}$
- $<S^\delta, \delta \in Ord>$ transfinite sequence of elements of $(Ord^n)^m$ such that (b) $\{\forall i \in \{1,\dots,n\}, \forall j \in \{1,\dots,m\}, \forall \delta \in Ord, (S_j^\delta)_i < \delta\}$

  (c) $\{\forall \delta \in Ord, \forall i \in \{1,\dots,n\}, \forall j \in \{1,\dots,m\}, \exists \beta \geq \delta :$
  $\{\forall \alpha \geq \beta, \delta \leq (S_j^\alpha)_i\}\}$

  (d) $\{\forall \beta, \delta \in Ord, \{\beta \text{ limit ordinal and } \beta < \delta\} \Rightarrow$
  $\{\forall i \in \{1,\dots,n\}, \forall j \in \{1,\dots,m\}, \beta \leq (S_j^\delta)_i\}$
- Let L^n be a complete lattice, $F \in iso\left((L^n)^m \to (L^n)^m\right)$ into L^n. An asynchronous iteration with m memories for F starting with $D \in L^n$ and defined by the strategy $<J^\delta, \delta \in Ord>, <S^\delta, \delta \in Ord>$ is the sequence $<X^\delta, \delta \in Ord>$ of elements of L^n defined by transfinite recursion as follows:
  - $X^0 = D$
  - $X_i^\delta = X_i^{\delta - 1}$ if $i \in [1,n] - J^\delta$ and δ successor ordinal
  - $X_i^\delta = F_i(Z^1, \dots, Z^m)$ if $i \in J^\delta$ and δ successor ordinal
    where $\forall i \in [1,n], \forall j \in [1,m]$ $Z_i^j = X_i^{(S_j^\delta)_i}$
  - $X_i^\delta = \bigsqcup_{\alpha < \delta} X^\alpha$ if δ successor ordinal.

Convergence theorem :

Any asynchronous iteration sequence with memory starting with $D \in L^n$ such that $D \sqsubseteq F(D, \dots, D)$ is stationary its limit is equal to $luis\left(\lambda X . F(X, \dots, X)\right)(D)$.