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Abstract Interpretation of Graphs

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Introduction

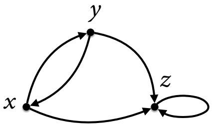
Objective

- Some abstract domains use path-based graph algorithms (zones, octagons, *etc*.)
- These algorithms are abstract interpretations of path finding algorithms (and so share a common algebraic structure)
- Was shown for the Bellman–Ford–Moore algorithm [Sergey, Midtgaard, and Clarke, 2012]
- We illustrate for the Floyd-Roy-Warshall shortest distance algorithm in a weighted graph
- more complicated since a naïve abstraction of a path by its length yields a n^4 instead of the n^3 Floyd-Roy-Warshall algorithm (n is the number of vertices of the finite graph)

Paths of a graph

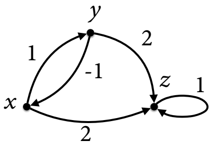
[Weighted] graphs

- (directed) graph $G = \langle V,\ E\rangle$



$$G = \left[\begin{array}{rcl} V & = & \{x, y, z\} \\ E & = & \{\langle x,\ y\rangle, \langle x,\ z\rangle, \langle y,\ x\rangle, \\ & & \langle y,\ z\rangle, \langle z,\ z\rangle\} \end{array}\right] \qquad \mathbf{G} = \begin{array}{c|ccc} & x & y & z \\ \hline x & 0 & 1 & 1 \\ y & 1 & 0 & 1 \\ z & 0 & 0 & 1 \end{array}$$

- Weighted graph $G = \langle V,\ E,\ \boldsymbol{\omega}\rangle$ with weights $\boldsymbol{\omega} \in E \to \mathbb{G}$ in a group $\langle \mathbb{G},\ 0,\ +\rangle$ (extended with ∞)



$$\left[\begin{array}{ll} \boldsymbol{\omega}(\langle x,\ y\rangle) = 1 & \boldsymbol{\omega}(\langle x,\ z\rangle) = 2 \\ \boldsymbol{\omega}(\langle x,\ z\rangle) = 2 & \boldsymbol{\omega}(\langle y,\ x\rangle) = -1 \\ \boldsymbol{\omega}(\langle y,\ z\rangle) = 2 & \boldsymbol{\omega}(\langle z,\ z\rangle) = 1 \end{array}\right. \qquad \mathbf{G} = \begin{array}{c|ccc} & x & y & z \\ \hline x & \infty & 1 & 2 \\ y & -1 & \infty & 2 \\ z & \infty & \infty & 1 \end{array}$$

(Finite non-empty) paths of a graph $G = \langle V, E\rangle$

$$\Pi(G) \quad \triangleq \quad \{x_1 \ldots x_n \in V^n \mid n > 0 \land \forall i \in [1, n[\,.\, \langle x_i,\ x_{i+1}\rangle \in E\}$$

- Many possible recursive definitions:
  - path $=$ arc $\mid$ path $\odot$ arc
  - path $=$ arc $\mid$ arc $\odot$ path
  - path $=$ arc $\mid$ path $\mid$ path $\odot$ path
  - path $=$ path $\mid$ path $\odot$ path $\quad \& \quad$ arc $\subseteq$ path
  - ...

Fixpoint characterization of the paths of a graph

Theorem 1 The paths of a graph $G = \langle V,\ E\rangle$ are

$$
\begin{aligned}
\Pi(G) &= \mathsf{lfp}^{\subseteq}\, \overrightarrow{\mathscr{F}}_{\Pi}, & \overrightarrow{\mathscr{F}}_{\Pi}(X) &\triangleq E \cup X \circledcirc E & \text{(1.a)}\\
&= \mathsf{lfp}^{\subseteq}\, \overleftarrow{\mathscr{F}}_{\Pi}, & \overleftarrow{\mathscr{F}}_{\Pi}(X) &\triangleq E \cup E \circledcirc X & \text{(1.b)}\\
&= \mathsf{lfp}^{\subseteq}\, \overleftrightarrow{\mathscr{F}}_{\Pi}, & \overleftrightarrow{\mathscr{F}}_{\Pi}(X) &\triangleq E \cup X \circledcirc X & \text{(1.c)}\\
&= \mathsf{lfp}^{\subseteq}_{E}\, \widehat{\mathscr{F}}_{\Pi}, & \widehat{\mathscr{F}}_{\Pi}(X) &\triangleq X \cup X \circledcirc X & \text{(1.d)}
\end{aligned}
\qquad \square
$$

$\circledcirc$ is the concatenation of sets of finite paths

$$P \circledcirc Q \ \triangleq\ \{x_1 \ldots x_n y_2 \ldots y_m \mid x_1 \ldots x_n \in P \wedge y_1 y_2 \ldots y_m \in Q \wedge x_n = y_1\}. \qquad (2)$$

Path problems

Path problem

- Classical definition: *path problems* are solved by graph algorithms that have the same algebraic structure
- Abstract interpretation: A *path problem* in a graph $G = \langle V,\ E\rangle$ consists in specifying/computing an abstraction $\alpha(\Pi(G))$ of its paths $\Pi(G)$ defined by a Galois connection

$$\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xrightleftharpoons[\alpha]{\gamma} \langle A,\ \sqsubseteq,\ \sqcup\rangle.$$

Fixpoint characterization of a path problem

- A path problem can be solved by a fixpoint definition/computation.

Theorem 2 Let $G = \langle V,\ E\rangle$ be a graph with paths $\Pi(G)$ and $\langle\wp(V^{>1}),\ \subseteq,\ \cup\rangle \xleftarrow[\alpha]{\gamma} \langle A,\ \sqsubseteq,\ \sqcup\rangle$.

$$
\begin{array}{rclcrcll}
\alpha(\Pi(G)) & = & \mathsf{lfp}^{\sqsubseteq}\ \overrightarrow{\mathcal{F}}^{\sharp}_{\Pi}, & & \overrightarrow{\mathcal{F}}^{\sharp}_{\Pi}(X) & \triangleq & \alpha(E) \sqcup X \mathbin{\overline{\circledcirc}} \alpha(E) & \text{(Th.2.a)}\\
& = & \ldots & & & & & \text{(Th.2.b)}\\
& = & \ldots & & & & & \text{(Th.2.c)}\\
& = & \mathsf{lfp}^{\sqsubseteq}_{\alpha(E)}\ \widehat{\mathcal{F}}^{\sharp}_{\Pi}, & & \widehat{\mathcal{F}}^{\sharp}_{\Pi}(X) & \triangleq & X \sqcup X \mathbin{\overline{\circledcirc}} X & \text{(Th.2.d)}
\end{array}
$$

where $\alpha(X) \mathbin{\overline{\circledcirc}} \alpha(Y) = \alpha(X \circledcirc Y)$. □

- The proof is by calculational design using the classical exact fixpoint abstraction with commutation

Path problem 1: paths between any two vertices

- Projection abstraction

$$\alpha^{\circ\!-\!\circ}(X) \quad \triangleq \quad \boldsymbol{\lambda}\,(y, z) \centerdot \{x_1 \ldots x_n \in X \mid y = x_1 \wedge x_n = z\}$$

such that

$$\langle \wp(V^{>1}),\ \subseteq,\ \cup\rangle \xleftrightarrows[\alpha^{\circ\!-\!\circ}]{\gamma^{\circ\!-\!\circ}} \langle V \times V \rightarrow \wp(V^{>1}),\ \dot{\subseteq},\ \dot{\cup}\rangle \tag{3}$$

- Paths between any two vertices

$$\mathsf{p} \quad \triangleq \quad \alpha^{\circ\!-\!\circ}(\Pi(G))$$

Fixpoint characterization of the paths of a graph between any two vertices

Theorem 3 Let $G = \langle V,\ E\rangle$ be a graph. The paths between any two vertices of G are $\mathsf{p} = \alpha^{\circ\!\!-\!\!\circ}(\Pi(G))$ such that

$$
\begin{aligned}
\mathsf{p} &= \mathsf{lfp}^{\dot\subseteq}\, \overrightarrow{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \dot{E} \mathbin{\dot\cup} \mathsf{p} \mathbin{\ddot{\circledcirc}} \dot{E} && \text{(Th.3.a)}\\
&= \ldots && && \text{(Th.3.b)}\\
&= \ldots && && \text{(Th.3.c)}\\
&= \mathsf{lfp}^{\dot\subseteq}_{\dot{E}}\, \widehat{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}, & \widehat{\mathscr{F}}^{\circ\!-\!\circ}_{\Pi}(\mathsf{p}) &\triangleq \mathsf{p} \mathbin{\dot\cup} \mathsf{p} \mathbin{\ddot{\circledcirc}} \mathsf{p} && \text{(Th.3.d)}
\end{aligned}
$$

where $\dot{E} \triangleq \boldsymbol{\lambda}\, x, y \cdot (E \cap \{\langle x,\ y\rangle\})$ and $\mathsf{p}_1 \mathbin{\ddot{\circledcirc}} \mathsf{p}_2 \triangleq \boldsymbol{\lambda}\, x, y \cdot \bigcup_{z\in V} \mathsf{p}_1(x, z) \circledcirc \mathsf{p}_2(z, y)$. $\square$

- The proof is by calculational design using the classical exact fixpoint abstraction with commutation

Path problem 2: Elementary paths and cycles

- A cycle is *elementary* if and only if it contains no internal subcycle (*i.e.* subpath which is a cycle).
- A path is *elementary* if and only if it contains no subpath which is an internal cycle (so an elementary cycle is an elementary path).
- The only vertices that can occur twice in an elementary path are its extremities in which case it is an elementary cycle.
- Notation: $\mathsf{elem?}(x_1 \ldots x_n)$
- Abstraction

$$\alpha^{\ni}(P) \quad \triangleq \quad \{\pi \in P \mid \mathsf{elem?}(\pi)\}.$$

$$\langle \wp(V^{>1}),\ \subseteq\rangle \xleftarrow[\alpha^{\ni}]{\gamma^{\ni}} \langle \wp(V^{>1}),\ \subseteq\rangle \qquad \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xleftarrow[\dot{\alpha}^{\ni}]{\dot{\gamma}^{\ni}} \langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle$$

Fixpoint characterization of the elementary paths of a graph

Theorem 4 Let $G = \langle V, E\rangle$ be a graph. The elementary paths between any two vertices of G are $\mathsf{p}^{\ni} \triangleq \alpha^{\circ\!\!-\!\circ} \circ \alpha^{\ni}(\Pi(G))$ such that

$$
\begin{array}{rcllcll}
\mathsf{p}^{\ni} & = & \mathsf{lfp}^{\dot\subseteq}\, \overrightarrow{\mathscr{F}}^{\ni}_{\Pi}, & \overrightarrow{\mathscr{F}}^{\ni}_{\Pi}(\mathsf{p}) & \triangleq & \dot{E} \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \dot{E} & \text{(Th.4.a)}\\
 & = & \ldots & & & & \text{(Th.4.b)}\\
 & = & \ldots & & & & \text{(Th.4.c)}\\
 & = & \mathsf{lfp}^{\dot\subseteq}_{\dot{E}}\, \widehat{\mathscr{F}}^{\ni}_{\Pi}, & \widehat{\mathscr{F}}^{\ni}_{\Pi}(\mathsf{p}) & \triangleq & \mathsf{p} \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p} & \text{(Th.4.d)}
\end{array}
$$

where $\dot{E} \triangleq \boldsymbol{\lambda}\, x, y \centerdot (E \cap \{\langle x, y\rangle\})$ and $\mathsf{p}_1 \mathbin{\dot{\circledcirc}^{\ni}} \mathsf{p}_2 \triangleq \boldsymbol{\lambda}\, x, y \centerdot \bigcup_{z \in V} \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \text{elem-conc?}(\pi_1, \pi_2)\}$. $\square$

- Proof by calculational design using the classical exact fixpoint abstraction
- (Th.4.d) is almost Floyd-Roy-Warshall but in n^4! (n number of vertices)

Iteration multiple abstraction

Exact abstraction of iterates (intuition)

$$
\begin{tikzcd}[column sep=large]
\langle \mathcal{A}, \preccurlyeq, 0, \curlyvee \rangle & \overline{x}^0 = 0 \arrow[r, "\overline{f}_0"] & \overline{x}^1 \arrow[r, "\overline{f}_1"] & \overline{x}^2 \arrow[r] & \overline{x}^i \arrow[r, "\overline{f}_i"] & \overline{x}^{i+1} \arrow[r, "\overline{f}_{i+1}"] & \overline{x}^{i+2} \arrow[r, dashed] & \bullet \arrow[r, dash, dashed] & \overline{x}^\omega = \curlyvee_{i\in\mathbb{N}} \overline{x}^i \\
\langle \mathcal{C}, \sqsubseteq, \bot, \bigsqcup \rangle & x^0 = \bot \arrow[u, bend left, "\alpha_0"] \arrow[r, "f_0", "\sqsubseteq"'] & x^1 \arrow[u, bend left, "\alpha_1"] \arrow[r, "f_1", "\sqsubseteq"'] & x^2 \arrow[u, bend left, "\alpha_2"] \arrow[r, dashed, "\sqsubseteq"'] & x^i \arrow[u, bend left, "\alpha_i"] \arrow[r, "f_i", "\sqsubseteq"'] & x^{i+1} \arrow[u, bend left, "\alpha_{i+1}"] \arrow[r, "f_{i+1}", "\sqsubseteq"'] & x^{i+2} \arrow[u, bend left, "\alpha_{i+2}"] \arrow[r, dashed, "\sqsubseteq"'] & \bullet \arrow[u, bend left] \arrow[r, dash, dashed, "\sqsubseteq"'] & x^\omega = \bigsqcup_{i\in\mathbb{N}} x^i \arrow[u, bend left, "\alpha_\omega"]
\end{tikzcd}
$$

If

- $\alpha_0(\bot) = 0$
- $\alpha_{i+1} \circ f_i = \overline{f}_i \circ \alpha_i$
- $\alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x_i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x_i)$ for all increasing chains $\langle x_i \in \mathcal{C}, i \in \mathbb{N}\rangle$.

then $\alpha_\omega(x^\omega) = \overline{x}^\omega$.

Exact abstraction of iterates (formally)

Theorem 5 Let $\langle C, \sqsubseteq, \perp, \bigsqcup\rangle$ be a cpo, $\forall i \in \mathbb{N} \,.\, f_i \in C \to C$ be such that $\forall x, y \in C \,.\, x \sqsubseteq y \Rightarrow f_i(x) \sqsubseteq f_{i+1}(y)$ with iterates $\langle x^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ defined by $x^0 = \perp$, $x^{i+1} = f_i(x^i)$, $x^\omega = \bigsqcup_{i\in\mathbb{N}} x^i$. Then these concrete iterates and $f \triangleq \dot{\bigsqcup}_{i\in\mathbb{N}} f_i$ are well-defined.

Let $\langle \mathcal{A}, \preccurlyeq, 0, \curlyvee\rangle$ be a cpo, $\forall i \in \mathbb{N} \,.\, \overline{f}_i \in \mathcal{A} \to \mathcal{A}$ be such that $\forall \overline{x}, \overline{y} \in \mathcal{A} \,.\, \overline{x} \preccurlyeq \overline{y} \Rightarrow \overline{f}_i(\overline{x}) \preccurlyeq \overline{f}_{i+1}(\overline{y})$ with iterates $\langle \overline{x}^i, i \in \mathbb{N} \cup \{\omega\}\rangle$ defined by $\overline{x}^0 = 0$, $\overline{x}^{i+1} = \overline{f}_i(\overline{x}^i)$, $\overline{x}^\omega = \curlyvee_{i\in\mathbb{N}} \overline{x}^i$. Then these abstract iterates and $\overline{f} \triangleq \dot{\curlyvee}_{i\in\mathbb{N}} \overline{f}_i$ are well-defined.

For all $i \in \mathbb{N} \cup \{\omega\}$, let $\alpha_i \in C \to \mathcal{A}$ be such that $\alpha_0(\perp) = 0$, $\alpha_{i+1} \circ f_i = \overline{f}_i \circ \alpha_i$, and $\alpha_\omega(\bigsqcup_{i\in\mathbb{N}} x_i) = \curlyvee_{i\in\mathbb{N}} \alpha_i(x_i)$ for all increasing chains $\langle x_i \in C, i \in \mathbb{N}\rangle$. It follows that $\alpha_\omega(x^\omega) = \overline{x}^\omega$.

If, moreover, $\forall i \in \mathbb{N} \,.\, f_i \in C \xrightarrow{uc} C$ is upper-continuous then $x^\omega = \mathsf{lfp}^{\sqsubseteq} f$.

Similarly $\overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$ when the $\overline{f}_i$ are upper-continuous.

If both the f_i and $\overline{f}_i$ are upper-continuous then $\alpha_\omega(\mathsf{lfp}^{\sqsubseteq} f) = \alpha_\omega(x^\omega) = \overline{x}^\omega = \mathsf{lfp}^{\preccurlyeq} \overline{f}$. $\square$

Back to the elementary path problems

Elementary paths of finite graphs $G = \langle V,\ E\rangle \qquad (|V| = n > 0)$

- Elementary paths in are of length at most $n+1$ so the fixpoint iterates in Theorem 4 converge in at most $n+2$ iterates.
- If $V = \{z_1 \ldots z_n\}$ is finite, then the elementary paths of the $k+2^{\text{nd}}$ iterate can be restricted to $\{z_1, \ldots, z_k\}$.
- Applying Theorem 5 with

$$\begin{aligned}
\alpha^{\ni}_0(\mathsf{p}) &\triangleq \mathsf{p} \\
\alpha^{\ni}_k(\mathsf{p}) &\triangleq \boldsymbol{\lambda}\, x, y \centerdot \{\pi \in \mathsf{p}(x, y) \mid \mathsf{V}(\pi) \subseteq \{z_1, \ldots, z_k\} \cup \{x, y\}\}, \quad k \in [1, n] \\
\alpha^{\ni}_k(\mathsf{p}) &\triangleq \mathsf{p}, \quad k > n
\end{aligned} \tag{9}$$

$$\langle V \times V \to \wp(V^{>1}),\ \dot{\subseteq}\rangle \xleftarrow[\alpha^{\ni}_k]{\gamma^{\ni}_k} \langle V \times V \to \bigcup_{k=2}^{n+1} V^k,\ \dot{\subseteq}\rangle. \tag{10}$$

we get an iterative algorithm.

Iterative characterization of the elementary paths of a finite graph

Theorem 6 Let $G = \langle V,\ E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\mathsf{p}^{\partial} = \mathsf{lfp}^{\dot\subseteq}\ \vec{\mathscr{F}}^{\partial}_{\Pi} = \vec{\mathscr{F}}^{\partial n+2}_{\pi} \quad \text{where} \quad \vec{\mathscr{F}}^{\partial}_{\Pi}(\mathsf{p}) \triangleq \dot E \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot\circledcirc^{\partial}} \dot E \text{ in (Th.4.a)} \qquad \text{(Th.6.a)}$$

$$\vec{\mathscr{F}}^{\partial 0}_{\pi} \triangleq \dot\varnothing, \quad \vec{\mathscr{F}}^{\partial 1}_{\pi} \triangleq \dot E,$$

$$\vec{\mathscr{F}}^{\partial k+2}_{\pi} \triangleq \dot E \mathbin{\dot\cup} \vec{\mathscr{F}}^{\partial k+1}_{\pi} \mathbin{\dot\circledcirc_{z_{k+1}}} \dot E, \quad k \in [0, n], \quad \vec{\mathscr{F}}^{\partial k+1}_{\pi} = \vec{\mathscr{F}}^{\partial k}_{\pi}, \quad k \geqslant n+2$$

$$= \ldots \qquad \text{(Th.6.b)}$$

$$= \ldots \qquad \text{(Th.6.c)}$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\ \widehat{\mathscr{F}}^{\partial}_{\Pi} = \widehat{\mathscr{F}}^{\partial n+1}_{\pi} \quad \text{where} \quad \widehat{\mathscr{F}}^{\partial}_{\Pi}(\mathsf{p}) \triangleq \mathsf{p} \mathbin{\dot\cup} \mathsf{p} \mathbin{\dot\circledcirc^{\partial}} \mathsf{p} \text{ in (Th.4.d)} \qquad \text{(Th.6.d)}$$

$$\widehat{\mathscr{F}}^{\partial 0}_{\pi} \triangleq \dot E, \quad \widehat{\mathscr{F}}^{\partial k+1}_{\pi} \triangleq \widehat{\mathscr{F}}^{\partial k}_{\pi} \mathbin{\dot\cup} \widehat{\mathscr{F}}^{\partial k}_{\pi} \mathbin{\dot\circledcirc^{\partial}_{z_{k+1}}} \widehat{\mathscr{F}}^{\partial k}_{\pi}, \quad k \in [0, n],$$

$$\widehat{\mathscr{F}}^{\partial k+1}_{\pi} = \widehat{\mathscr{F}}^{\partial k}_{\pi}, \quad k \geqslant n+2$$

$$\mathsf{p}_1 \mathbin{\dot\circledcirc_z} \mathsf{p}_2 \triangleq \boldsymbol{\lambda}\, x, y \bullet \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge z \notin \{x, y\}\}$$

$$\mathsf{p}_1 \mathbin{\dot\circledcirc^{\partial}_z} \mathsf{p}_2 \triangleq \boldsymbol{\lambda}\, x, y \bullet \{\pi_1 \odot \pi_2 \mid \pi_1 \in \mathsf{p}_1(x, z) \wedge \pi_2 \in \mathsf{p}_2(z, y) \wedge \text{elem-conc?}(\pi_1, \pi_2)\}. \qquad \square$$

Iterative characterization of an *over-approximation* of the elementary paths of a finite graph

Corollary 7 Let $G = \langle V, E\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$. Then

$$\mathsf{p}^{\partial} = \ldots \qquad \text{(Cor.7.c)}$$

$$= \mathsf{lfp}^{\dot\subseteq}_{\dot E}\, \widehat{\mathcal{F}}^{\partial}_{\Pi} \ \dot\subseteq\ \widehat{\mathcal{F}}^{n+1}_{\pi} \qquad \text{(Cor.7.d)}$$

$$\text{where} \quad \widehat{\mathcal{F}}^{0}_{\pi} \triangleq \dot E, \quad \widehat{\mathcal{F}}^{k+1}_{\pi} \triangleq \widehat{\mathcal{F}}^{k}_{\pi} \mathbin{\dot\cup} \widehat{\mathcal{F}}^{k}_{\pi} \mathbin{\dot{\circledcirc}_{z_k}} \widehat{\mathcal{F}}^{k}_{\pi}$$

$\square$

replacing $\dot{\circledcirc}^{\partial}_{z}$ by $\dot{\circledcirc}_{z}$ (with no check that concatenated paths are elementary).

Path problem 3: shortest distances between any two vertices of a weighted graph $G = \langle V, E, \boldsymbol{\omega}\rangle$ on a group $\langle \mathbb{G}, 0, +\rangle$

- The weight of a path is
$$\boldsymbol{\omega}(x_1 \ldots x_n) \triangleq \sum_{i=1}^{n-1} \boldsymbol{\omega}(\langle x_i, x_{i+1}\rangle) \tag{6}$$
- The minimal weight of a set of paths is
$$\boldsymbol{\omega}(P) \triangleq \min\{\boldsymbol{\omega}(\pi) \mid \pi \in P\}. \tag{7}$$
- Galois connection $\langle \wp(\bigcup_{n\in\mathbb{N}^+} V^n), \subseteq\rangle \xleftarrow[\boldsymbol{\omega}]{} \!\!\!\!\rightarrow \langle \mathbb{G} \cup \{-\infty, \infty\}, \geqslant\rangle$ extended pointwise to
$$\langle V \times V \rightarrow \wp(\bigcup_{n\in\mathbb{N}^+} V^n), \dot{\subseteq}\rangle \xleftarrow[\dot{\boldsymbol{\omega}}]{} \!\!\!\!\rightarrow \langle V \times V \rightarrow \mathbb{G} \cup \{-\infty, \infty\}, \dot{\geqslant}\rangle. \tag{8}$$
- The distance $\mathsf{d}(x, y)$ between an origin $x \in V$ and an extremity $y \in V$ is the length $\boldsymbol{\omega}(\mathsf{p}(x, y))$ of the shortest path between these vertices

$\mathsf{d} \triangleq \dot{\boldsymbol{\omega}}(\mathsf{p}) = \dot{\boldsymbol{\omega}}(P)$ provided $\mathsf{p} \dot{\subseteq} P$ and no cycle has a strictly negative weight

Iterative characterization of the shortest path length of a graph

Theorem 8 Let $G = \langle V, E, \boldsymbol{\omega}\rangle$ be a finite graph with $V = \{z_1, \ldots, z_n\}$, $n > 0$ weighted on the totally ordered group $\langle \mathbb{G}, \leqslant, 0, +\rangle$ with no strictly negative weight.

Then the distances between any two vertices are

$$\mathrm{d} = \dot{\boldsymbol{\omega}}(\mathrm{p}) = \widehat{\mathscr{F}}_{\delta}^{\,n+1} \qquad \text{where} \tag{Th.8}$$

$$\widehat{\mathscr{F}}_{\delta}^{\,0}(x, y) \triangleq \llbracket \langle x, y\rangle \in E \mathrel{?} \boldsymbol{\omega}(x, y) \mathrel{:} \infty \rrbracket,$$

$$\widehat{\mathscr{F}}_{\delta}^{\,k+1}(x, y) \triangleq \llbracket z_k \in \{x, y\} \mathrel{?} \widehat{\mathscr{F}}_{\delta}^{\,k}(x, y) \mathrel{:} \min(\widehat{\mathscr{F}}_{\delta}^{\,k}(x, y), \widehat{\mathscr{F}}_{\delta}^{\,k}(x, z_k) + \widehat{\mathscr{F}}_{\delta}^{\,k}(z_k, y)) \rrbracket \quad \square$$

Proof by calculational design based on Theorem 5.

Roy-Floyd-Warshall shortest distances of a graph

Algorithm 9 The Roy-Floyd-Warshall algorithm computes the shortest distances $\dot{\boldsymbol{\omega}}(\mathsf{p}) \in V \times V \to \mathbb{G} \cup \{-\infty, \infty\}$ in a finite graph with no cycle with strictly negative weight:

$$
\begin{array}{l}
\text{for } x, y \in V \text{ do} \\
\qquad \mathrm{d}(x, y) := \text{if } \langle x,\ y\rangle \in E \text{ then } \boldsymbol{\omega}(x, y) \text{ else } \infty \\
\quad \text{done;} \\
\text{for } z \in V \text{ do} \\
\quad \text{for } x, y \in V \text{ do} \\
\qquad \mathrm{d}(x, y) := \min(\mathrm{d}(x, y), \mathrm{d}(x, z) + \mathrm{d}(z, y)) \\
\quad \text{done} \\
\text{done.}
\end{array}
$$

Conclusion

Conclusion

- The Roy-Floyd-Warshall algorithm is an abstract interpretation of a concrete path finding algorithm
- The abstraction is different at each fixpoint iteration (Theorem 5), which is unusual.
- Path problems have been observed to have a common algebraic structure
- This is because the primitive structure $\langle \wp(V^{>1}), E, \cup, \circledcirc \rangle$ is preserved by the abstractions

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The End, Thank you

Happy birthday Manual