Abstract Interpolation by Dual Narrowing

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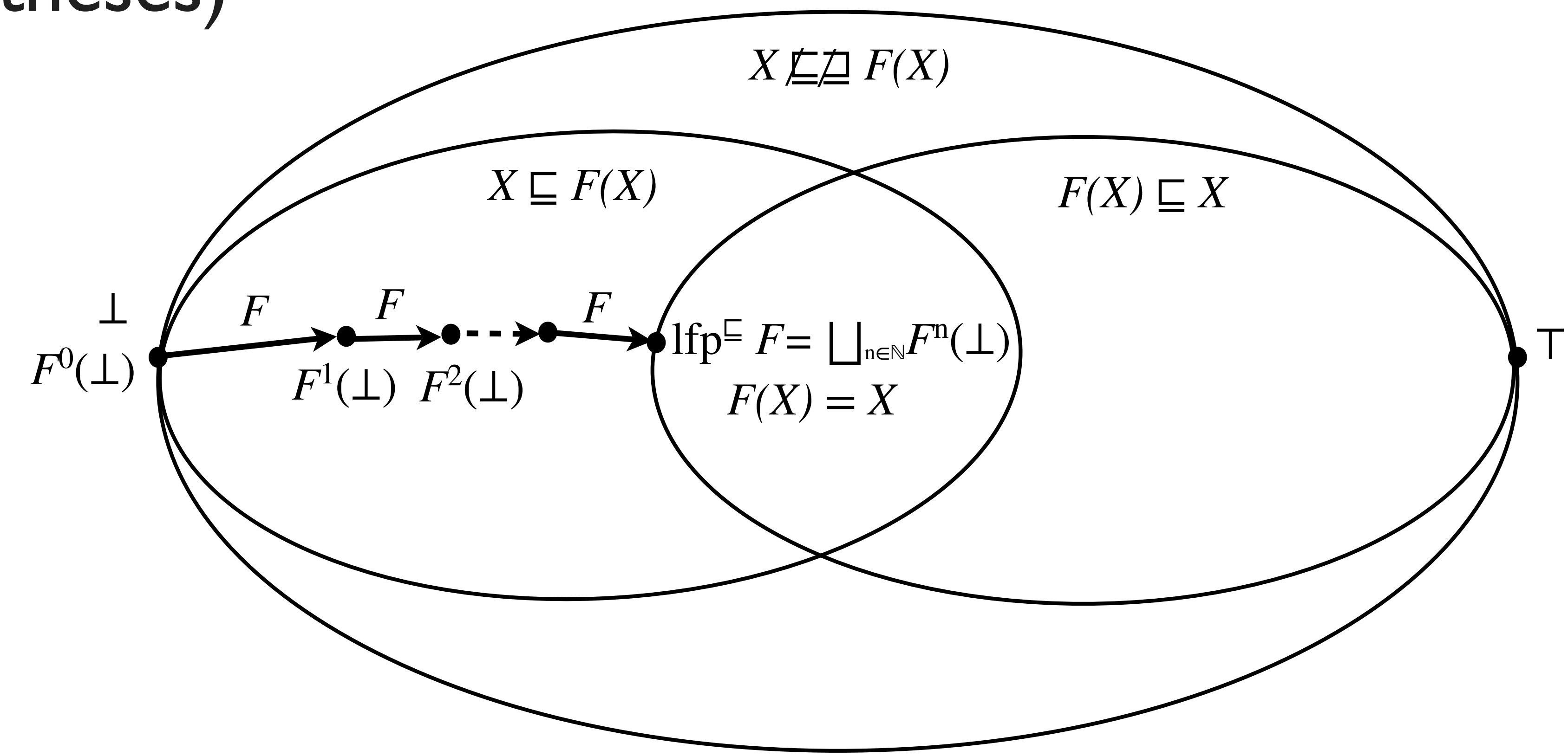
Abstract Interpreters

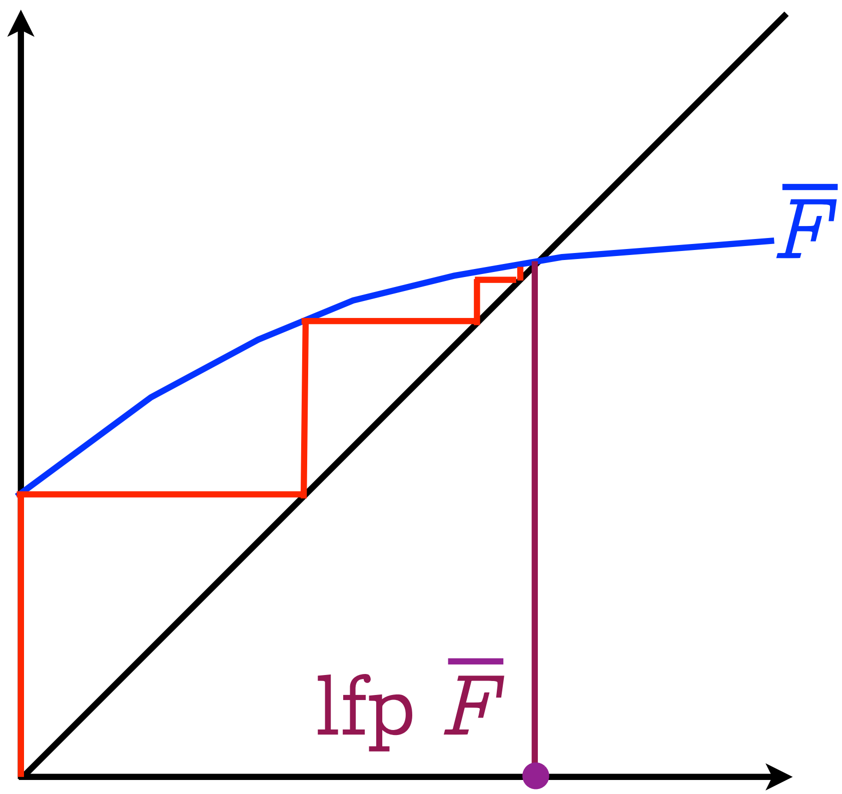
- Transitional abstract interpreters: proceed by induction on program steps
- Structural abstract interpreters: proceed by induction on the program syntax
- Main problem: over/under-approximate fixpoints in non-Noetherian abstract domains

Fixpoints

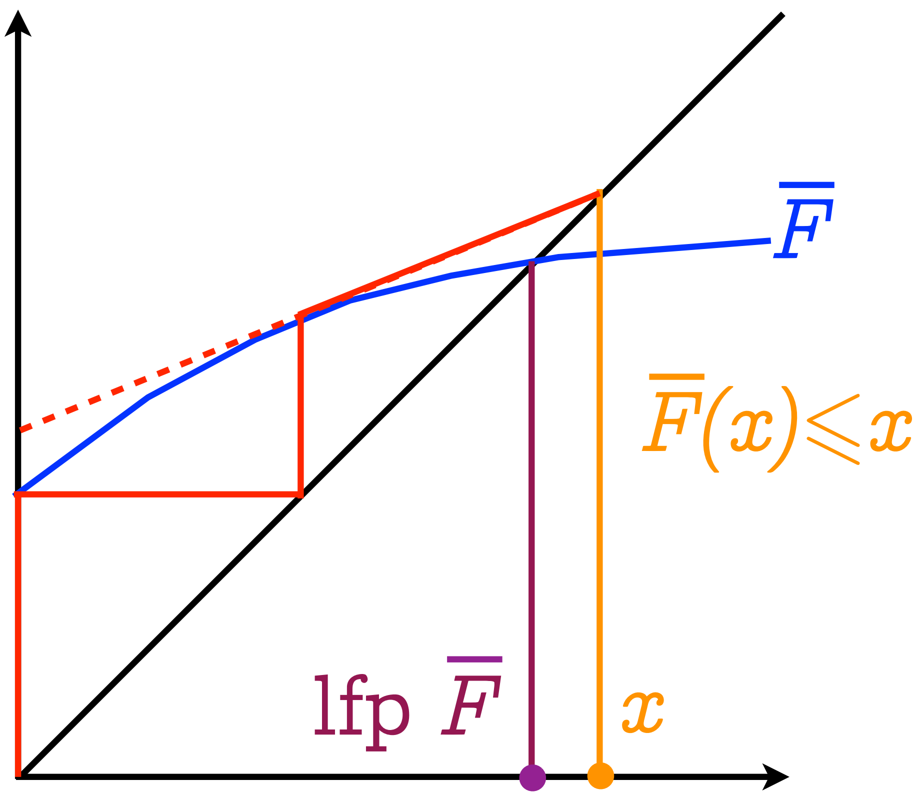
- Poset $\langle D, \sqsubseteq, \perp, \sqcup \rangle$
- Transformer: $F \in D \longmapsto D$
- Least fixpoint: $\mathrm{lfp}^{\sqsubseteq} F = \bigsqcup_{n \in \mathbb{N}} F^n(\perp)$ (under appropriate hypotheses)



Convergence acceleration with widening



Infinite iteration



Accelerated iteration with widening

(e.g. with a widening based on the derivative as in Newton-Raphson method(*))

(*) Javier Esparza, Stefan Kiefer, Michael Luttenberger: Newtonian program analysis. J. ACM 57(6): 33 (2010)

Extrapolation by Widening

- $X^0 = \bot$ (increasing iterates with widening)

  $X^{n+1} = X^n \mathbin{\nabla} F(X^n)$ when $F(X^n) \not\sqsubseteq X^n$

  $X^{n+1} = X^n$ when $F(X^n) \sqsubseteq X^n$

- Widening ∇:
  - $Y \sqsubseteq X \mathbin{\nabla} Y$ (extrapolation)
  - Enforces convergence of increasing iterates with widening, limit X^ℓ

Example of widenings

- Primitive widening [1,2]

```
(x ∇̄ y) = cas x ∈ Va, y ∈ Va dans
    + □, ? => y ;
    + ?, □ => x ;
    + [n1,m1],[n2,m2] =>
         [si n2 < n1 alors -∞ sinon n1 fsi ;
              si m2 > m1 alors +∞ sinon m1 fsi] ;
fincas ;
```

$$[a_1, b_1]\ \bar{\nabla}\ [a_2, b_2] = [\underline{\text{if}}\ a_2 < a_1\ \underline{\text{then}}\ -\infty\ \underline{\text{else}}\ a_1\ \underline{\text{fi}},\ \underline{\text{if}}\ b_2 > b_1\ \underline{\text{then}}\ +\infty\ \underline{\text{else}}\ b_1\ \underline{\text{fi}}]$$

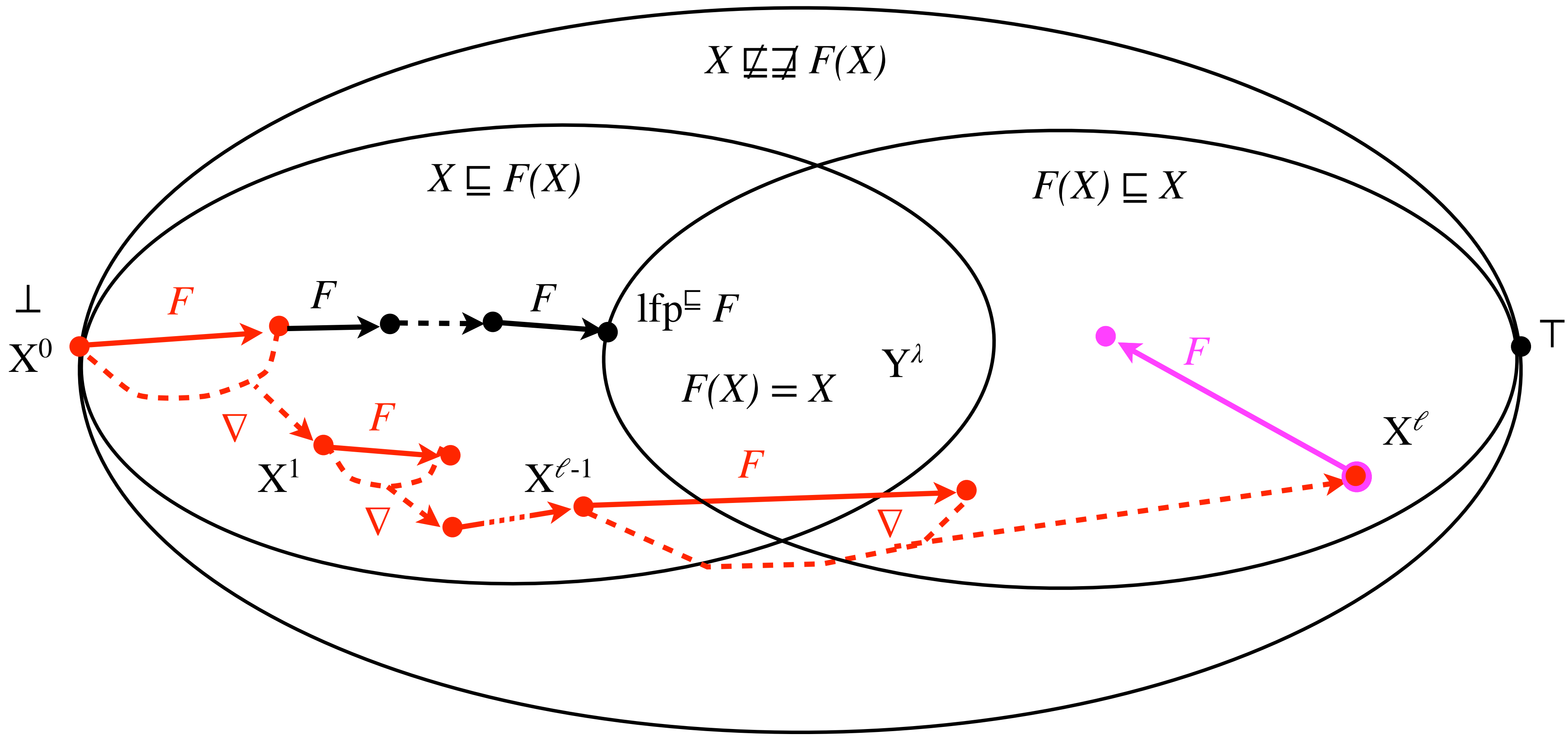
- Widening with thresholds [3]

$$\forall x \in \bar{L}_2, \perp \nabla_2(j)\, x = x\, \nabla_2(j) \perp = x$$

$$[l_1, u_1]\, \nabla_2(j)\, [l_2, u_2]$$
$$= [\mathit{if}\ 0 \leq l_2 < l_1\ \mathit{then}\ 0\ \mathit{elsif}\ l_2 < l_1\ \mathit{then}\ -b-1\ \mathit{else}\ l_1\ \mathit{fi},$$
$$\mathit{if}\ u_1 < u_2 \leq 0\ \mathit{then}\ 0\ \mathit{elsif}\ u_1 < u_2\ \mathit{then}\ b\ \mathit{else}\ u_1\ \mathit{fi}]$$

[1] Patrick Cousot, Radhia Cousot: Vérification statique de la cohérence dynamique des programmes, Rapport du contrat IRIA-SESORI No 75-032, 23 septembre 1975.
[2] Patrick Cousot, Radhia Cousot: Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints. POPL 1977: 238-252
[3] Patrick Cousot, Semantic foundations of program analysis, Ch. 10 of Program flow analysis: theory and practice, N. Jones & S. Muchnich (eds), Prentice Hall, 1981.

Extrapolation with widening



Interpolation with narrowing

- $Y^0 = X^\ell$ (decreasing iterates with narrowing)

  $Y^{n+1} = Y^n \mathbin{\Delta} F(Y^n)$ when $F(Y^n) \sqsubset Y^n$

  $Y^{n+1} = Y^n$ when $F(Y^n) = Y^n$

- Narrowing Δ:
  - $Y \sqsubseteq X \implies Y \sqsubseteq X \mathbin{\Delta} Y \sqsubseteq X$ (interpolation)
  - Enforces convergence of decreasing iterates with narrowing, Y^λ

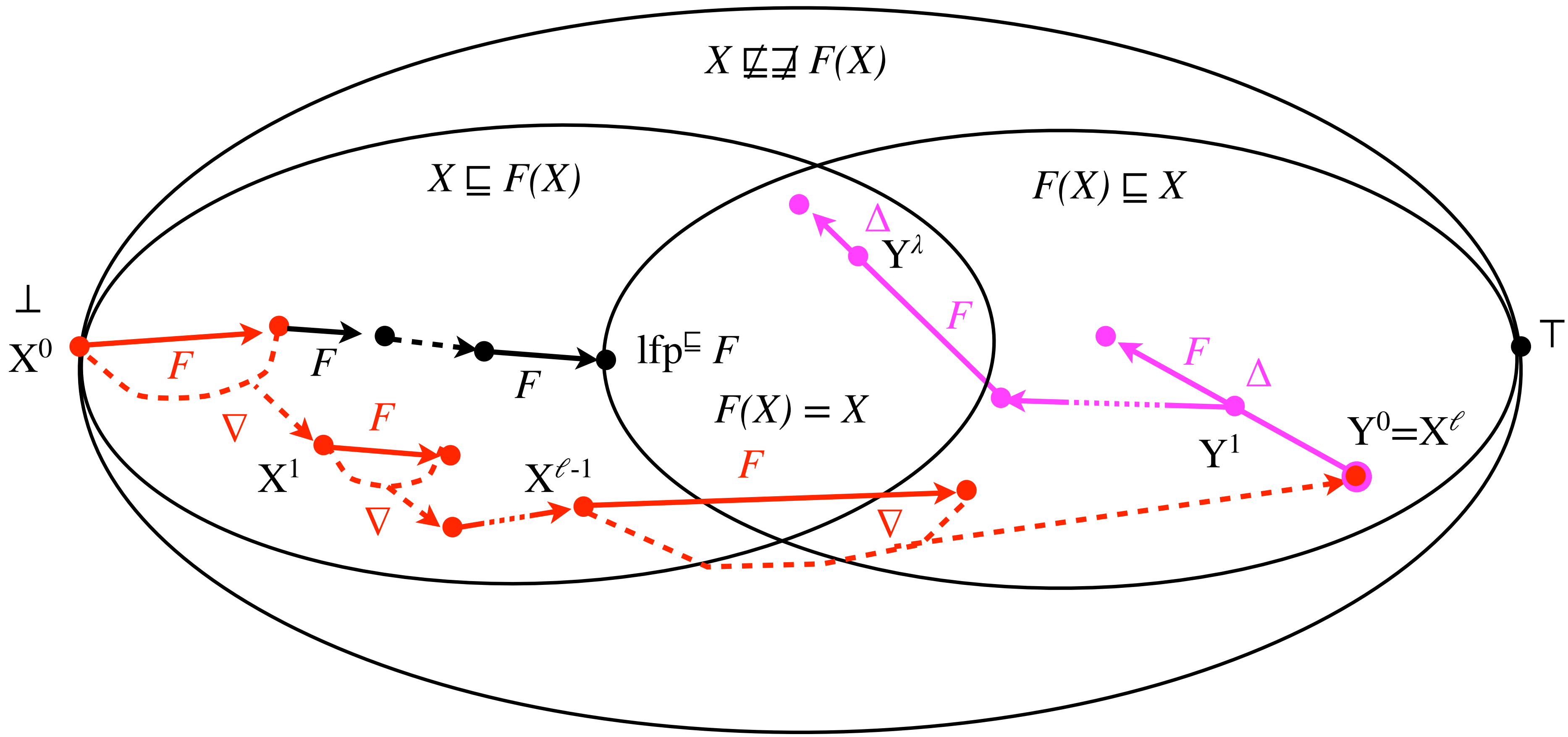
Example of narrowing

- [2]

$$
\begin{aligned}
&[a_1, b_1]\ \bar{\Delta}\ [a_2, b_2] = \\
&\quad [\underline{\text{if}}\ a_1 = -\infty\ \underline{\text{then}}\ a_2\ \underline{\text{else}}\ \text{MIN}\ (a_1, a_2), \\
&\qquad \underline{\text{if}}\ b_1 = +\infty\ \underline{\text{then}}\ b_2\ \underline{\text{else}}\ \text{MAX}\ (b_1, b_2)]
\end{aligned}
$$

[2] Patrick Cousot, Radhia Cousot: Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints. POPL 1977: 238-252

Interpolation with narrowing



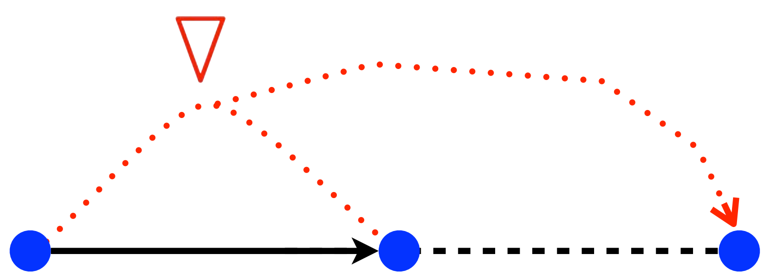
Duality

-

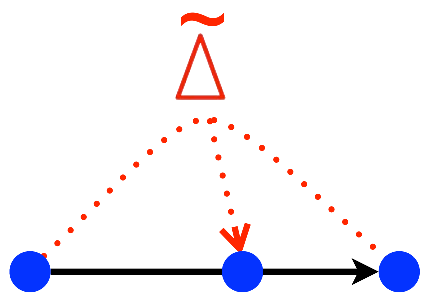
	Convergence above the limit	Convergence below the limit
Increasing iteration	Widening ∇	Dual-narrowing $\widetilde{\Delta}$
Decreasing iteration	Narrowing Δ	Dual widening $\widetilde{\nabla}$

Extrapolators (∇, $\widetilde{\nabla}$) and interpolators (Δ, $\widetilde{\Delta}$)

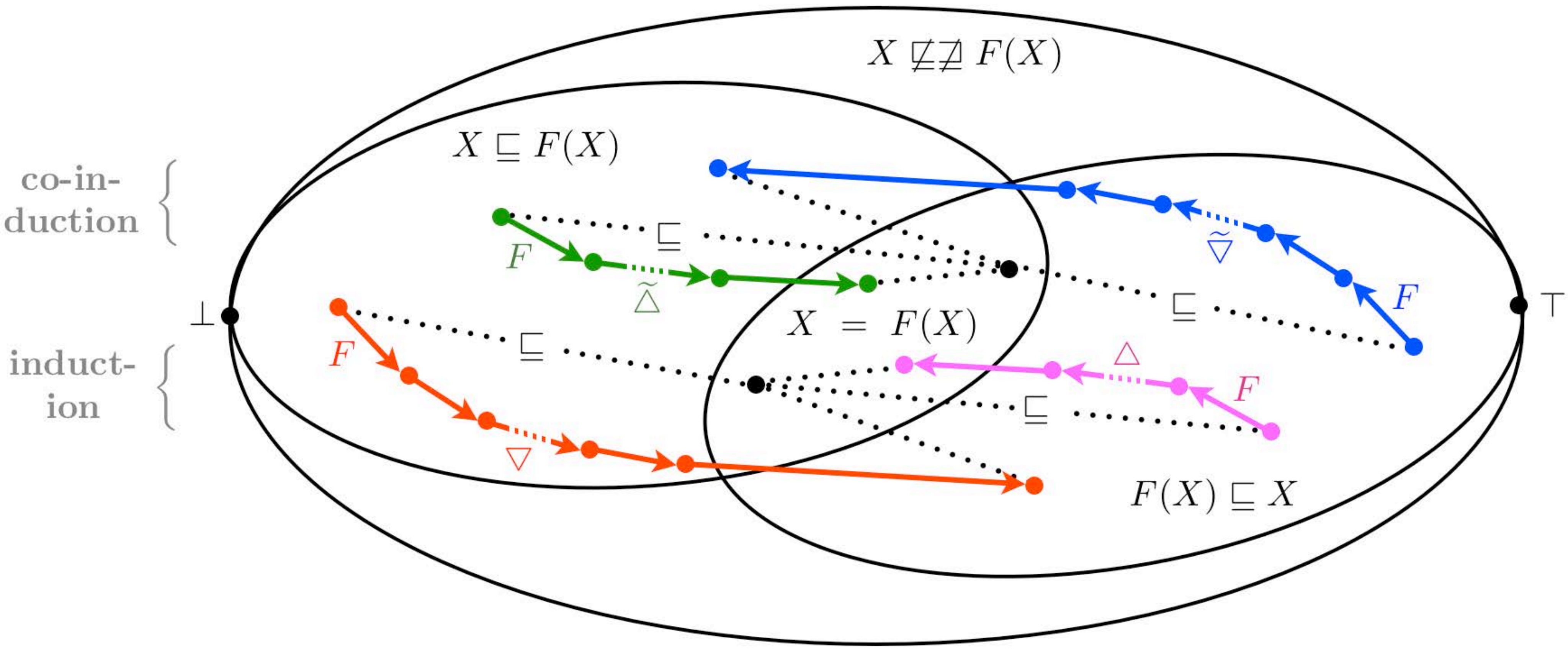
- **Extrapolators**:



- **Interpolators**:



Extrapolators, Interpolators, and Duals



Interpolation with dual narrowing

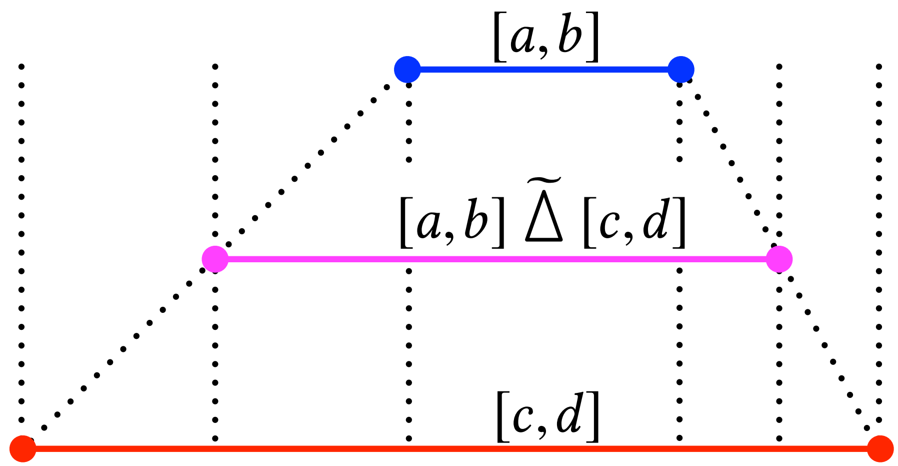
- $Z^0 = \perp$ (increasing iterates with dual-narrowing)

  $Z^{n+1} = F(Z^n) \mathbin{\widetilde{\Delta}} Y^\lambda$ when $F(Z^n) \not\sqsubseteq Z^n$

  $Z^{n+1} = Z^n$ when $F(Z^n) \sqsubseteq Z^n$

- Dual-narrowing $\widetilde{\Delta}$:
  - $X \sqsubseteq Y \implies X \sqsubseteq X \mathbin{\widetilde{\Delta}} Y \sqsubseteq Y$ (interpolation)
  - Enforces convergence of increasing iterates with dual-narrowing

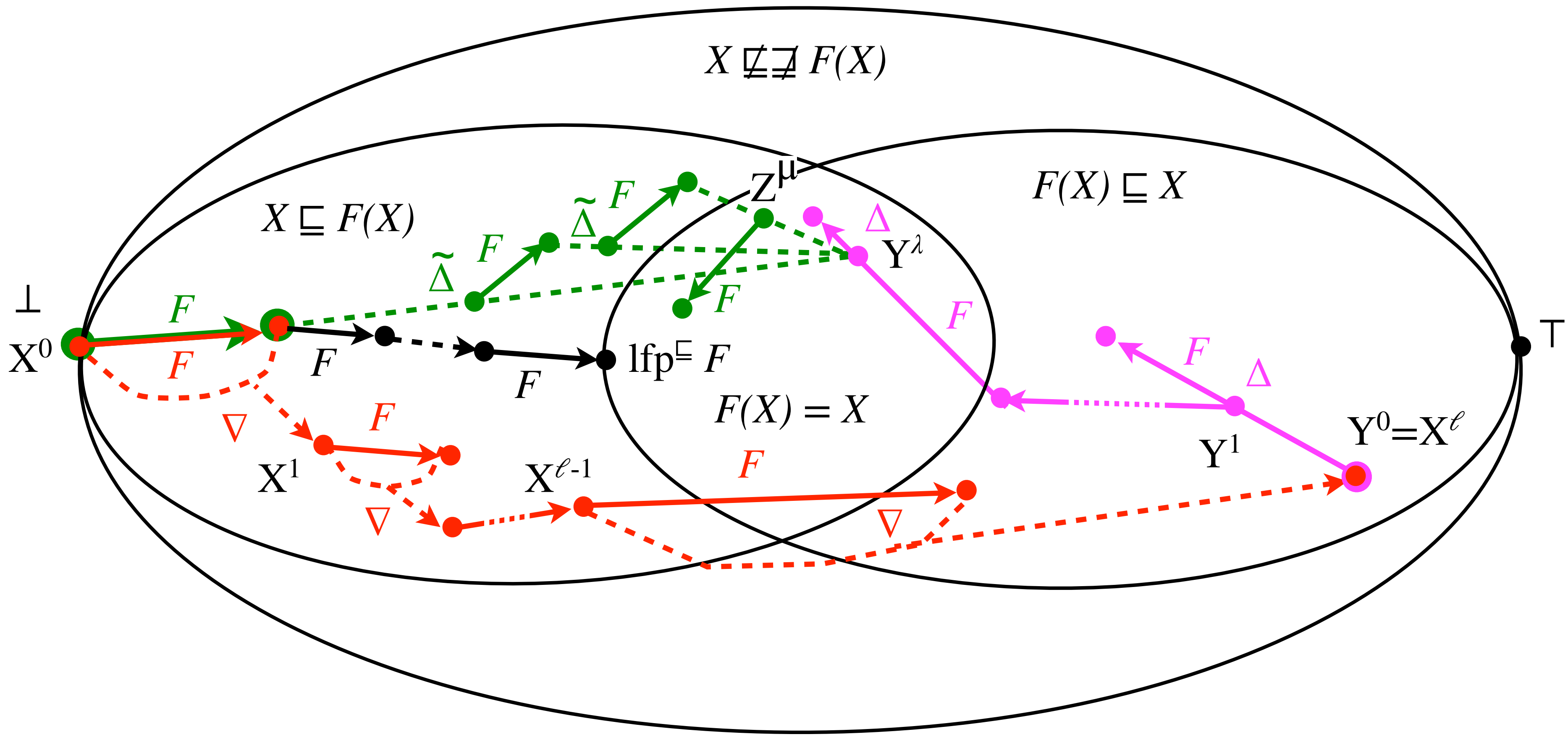
Example of dual-narrowing

- 

- $[a,b]\ \widetilde{\Delta}\ [c,d] \triangleq [\llparenthesis\, c = -\infty \;?\; a \;:\; \lfloor (a+c)/2 \rfloor \,\rrparenthesis, \llparenthesis\, d = \infty \;?\; b \;:\; \lceil (b+d)/2 \rceil \,\rrparenthesis]$

- The first method we tried in the end 70's with Radhia
  - Slow
  - Does not easily generalize (e.g. to polyhedra)

Interpolation with dual-narrowing

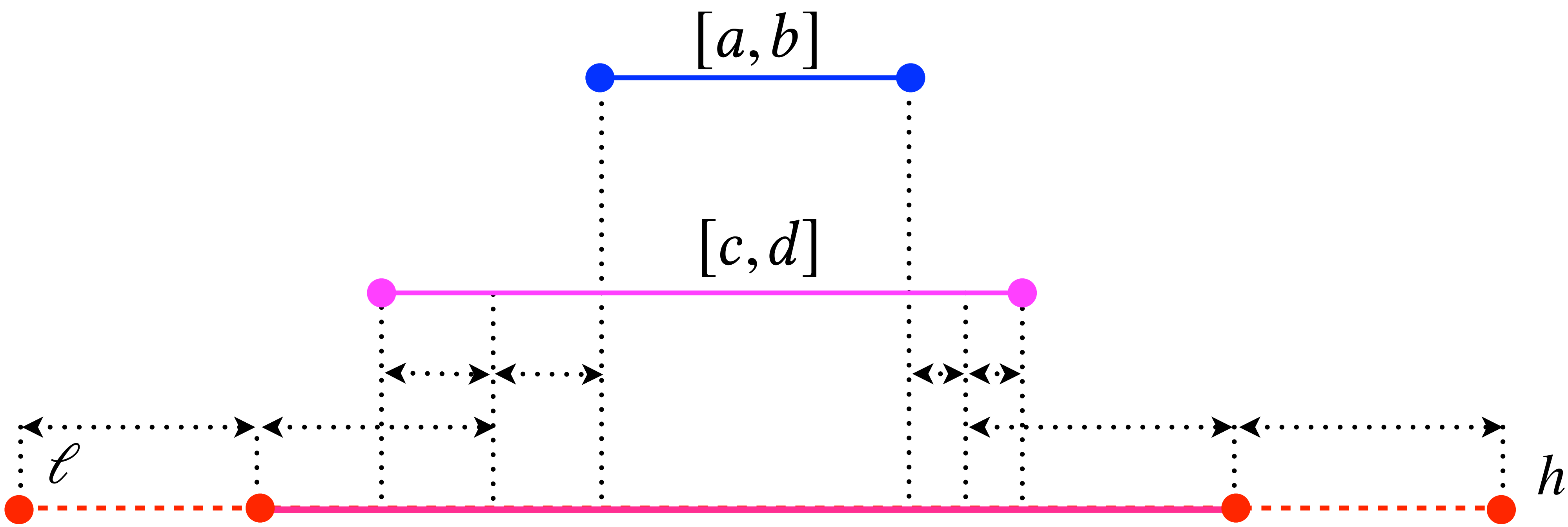


Relationship between narrowing and dual-narrowing

- $\widetilde{\Delta} = \Delta^{-1}$
- $Y \sqsubseteq X \implies Y \sqsubseteq X \mathbin{\Delta} Y \sqsubseteq X$ (narrowing)
- $Y \sqsubseteq X \implies Y \sqsubseteq Y \mathbin{\widetilde{\Delta}} X \sqsubseteq X$ (dual-narrowing)
- Example: Craig interpolation
- Why not use a bounded widening (bounded by B)?
  - $F(X) \sqsubseteq B \implies F(X) \sqsubseteq F(X) \mathbin{\widetilde{\Delta}} B \sqsubseteq B$ (dual-narrowing)
  - $X \sqsubseteq F(X) \sqsubseteq B \implies F(X) \sqsubseteq X \mathbin{\nabla_B} F(X) \sqsubseteq B$ (bounded widening)

Example of widenings (cont'd)

- Bounded widening (in $[\ell, h]$):



$$[a,b] \mathbin{\nabla_{[\ell,h]}} [c,d] \triangleq \left[\frac{c+a-2\ell}{2}, \frac{b+d+2h}{2}\right]$$

Conclusion

- Abstract interpretation in infinite domains is traditionally by iteration with widening/narrowing.
- We shown how to use iteration with dual-narrowing.
- These ideas of the 70's generalize Craig interpolation from logic to arbitrary abstract domains.

The End, Thank You