A parametric segmentation abstract domain functor for fully automatic inference of array properties

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end-of-visit talk, joint work with

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Motivation

The problem of array content analysis

- Statically and fully automatically determine properties of array elements in finite reasonable time
- Undecidable problem $\rightarrowtail$ abstract interpretation
- Example:

```
          int n = 10;
          int i, A[n];
          i = 0;
/* 1: */
          while /* 2: */ (i < n) {
/* 3: */
             A[i] = 0;
/* 4: */
             i = i + 1;
/* 5: */
          }
/* 6: */
```

$\longleftarrow \forall\, i \in [0,n): A[i] = 0$

Contribution

- A new simple parametric array segmentation abstract domain functor
- An evaluation prototype for experimentation
- Example:

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                int n = 10;
                int i, A[n];
                i = 0;
/* 1: */
                while /* 2: */ (i < n) {
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                   A[i] = 0;
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                }
/* 6: */

p6 = <{0},[0,0],{n,10,i}>; [ i: [10,10] n: [10,10] ]
0.000713 s
```

Self-imposed constraints for solving the array content analysis problem

- A basic abstraction usable in compilers and general purpose static analyzers
- A bit like *intervals* for numerical values which
  - is simple to implement
  - has low analysis cost and so does scale up
  - answers 60 to 95% of questions e.g. in compilers
- Parametrizable (to reuse existing abstractions)
- Fully automatic (no hidden hypotheses)

The array segmentation abstraction

Which kind of invariants do we need?

```
        int n = 10;
        int i, A[n];
        i = 0;
/* 1: */
        while /* 2: */ (i < n) {
/* 3: */
           A[i] = 0;
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           i = i + 1;
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        }
/* 6: */
```

Invariant:

if `i = 0;` then
  array A not initialized
else if `i` > 0 then
  A[0] = ... = A[`i`-1] = 0
else (* `i` < 0 *)
  Impossible

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```

Disjunction (case analysis)

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Disjunction (case analysis)

Array segment

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/* 1: */
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  Impossible

Disjunction (case analysis)

Array segment

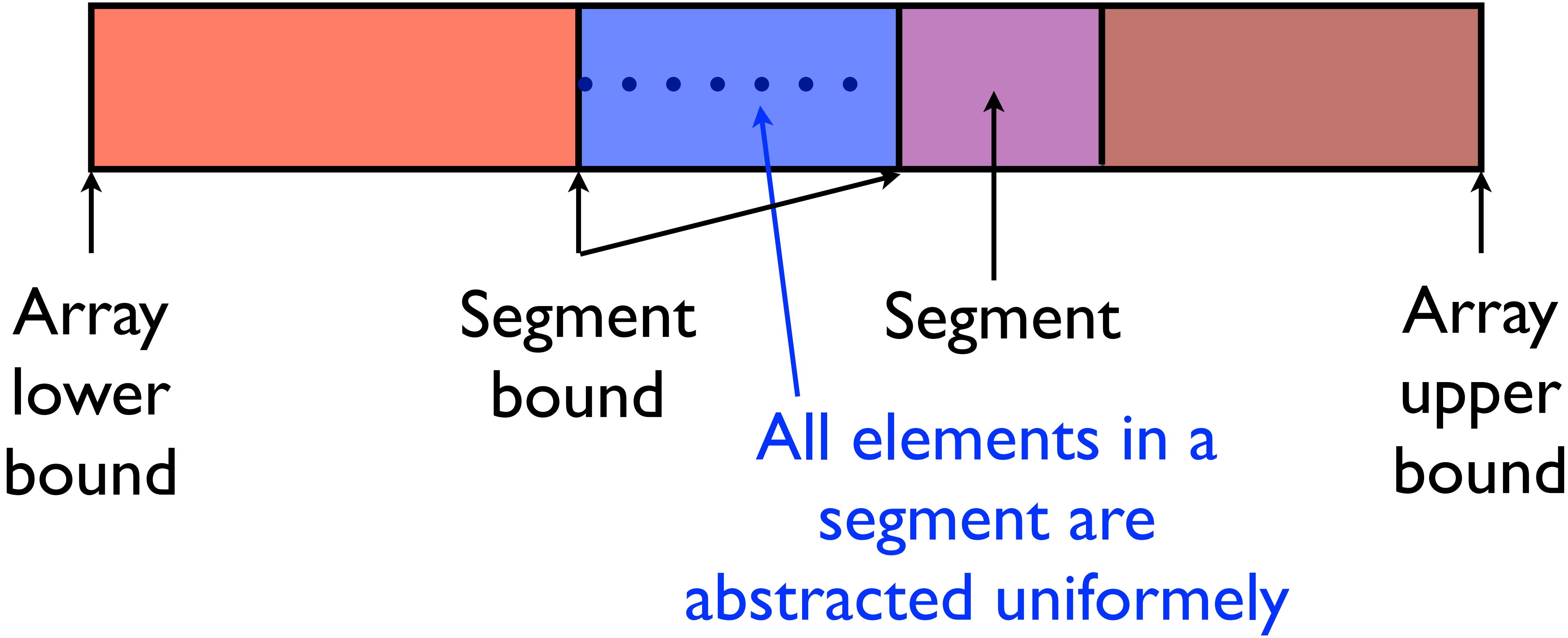
Segment bounds related to variables

The array segmentation abstract domain functor: abstract properties

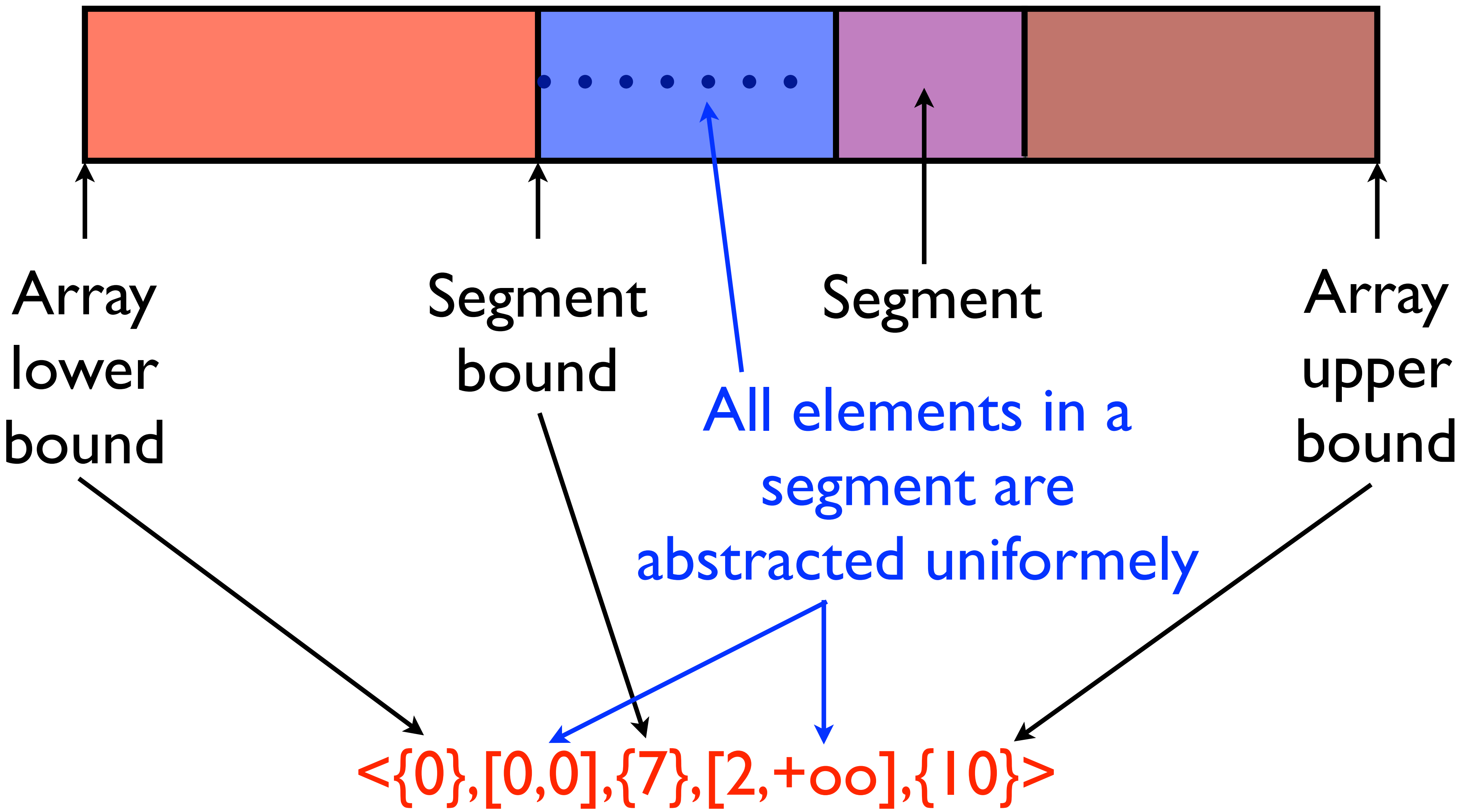
Array segmentation

- Classical array abstractions, elementwise or

- Refinement by segments



Array segmentation



-oo is min_int, +oo is max_int

Symbolic array segment bounds

- Array segments are
  - in strict increasing order of the array indices
  - delimited by sets of expressions known to have equal values

<{0},[0,1],{i-1},[2,5],{i},[6,+oo],{n,10}>

so 0 < i-1 < i < n = 10

Symbolic array segment bounds

- Refinement of the segmentation: through assignment to array elements
- Coarsening of the segmentation: through widening
- Purely symbolic (variables abstract values are not strictly necessary to handle segment limits so works for all value abstractions!)

```
          int n = 10;
          int i, A[n];
          i = n;
/* 1: */
          while /* 2: */ (0 < i) {
/* 3: */
             i = i - 1;
/* 4: */
             A[i] = 0;
/* 5: */
          }
/* 6: */
Analysis with (interval domain x top domain):
p6 = [ A: <{0},[-oo,+oo],{n,10}?> ] [ i: T n: T ]
0.000212 s
```

Top abstraction of simple variables

The explanation of this question mark ? is forthcoming

Symbolic array segment bounds (cont'd)

- symbolic, not numerical, so handles arrays of unknown size

```
          parameter int n; /* assume n>1 */
          int i, A[n];
          i = n;
/* 1: */
          while /* 2: */ (0 < i) {
/* 3: */
             i = i - 1;
/* 4: */
             A[i] = 0;
/* 5: */
          }
/* 6: */

Analysis with widening/narrowing and (arrays: interval domain x variables:
interval domain):
p6 = [ A: <{0,i},[0,0],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.001854 s
```

Array of fixed but unknown size

Todo: should work with Javascript arrays (& iterators) with -∞, +∞ bounds and segments with float limits (?).

The semantics of arrays

- The classical operational semantics (McCarthy):

  Array $\in$ Set of indices $\rightarrow$ Set of values

- Our semantics for segmentation:

  Array $\in$ Values of variables $\rightarrow$ Set of indices $\rightarrow$ Set of values

John McCarthy: Recursive Functions of Symbolic Expressions and Their Computation by Machine, Part I. C ACM 3(4): 184-195 (1960)

The semantics of arrays revisited (I)

- The classical operational semantics (John McCarthy):

  Array $\in$ Set of indices $\rightarrow$ Set of values

- Our semantics for segmentation:

  Array $\in$ Values of variables $\rightarrow$ Set of indices $\rightarrow$ Set of values

  Segments

John McCarthy: Recursive Functions of Symbolic Expressions and Their Computation by Machine, Part I. C ACM 3(4): 184-195 (1960)

Disjunctions

- Disjunctions are needed (as shown by the array initialization example)
- Disjunctive enumeration of cases leads to explosion (e.g. because of conditionals and/or loops)
- Abstract interpretation offers a *standard solution* through overapproximation (preserves soundness but not completeness)
- A simple & cheap join is needed for any efficient array content analysis abstract domain (can over-approximate the lub/disjunction)

A very simple solution for disjunction: possibly empty segments

- Disjunctions are introduced exclusively through possibly empty segments

<{0},[0,0],{i}?,[-oo,+oo],{n,10}?>

if `i = 0;` then
  block is empty (so array `A` is not initialized)
else if `i` > 0 then
  A[0] = ... = A[`i`-1] = 0
else (* `i` < 0 *)
  Impossible

The array segmentation abstract domain

$$\langle L, \ldots, \{e_1, \ldots, e_n\}\ A\ \{e'_1, \ldots, e'_m\}[?], \ldots, H\rangle$$

Segment bounds

Abstraction of array element pairs (i, v_i) within the segment

Possibility of emptiness:

- $e_1 = \ldots = e_n < e'_1 = \ldots = e'_m \longrightarrow \sqcup$
- $e_1 = \ldots = e_n \leq e'_1 = \ldots = e'_m \longrightarrow ?$

Parametrization of the array segmentation abstract domain functor

- Which symbolic expressions are used in block bounds?
- Which array abstraction is used to abstract array element values (i, v_i) within a segment?
- Which variables abstraction is used to abstract variables appearing in expressions?
- Which reductions are performed between symbolic block limits and abstractions of variables?
- Which coarseness is chosen for widenings/narrowings?

The ARRAYAL prototype

- Symbolic expressions :
  - constant
  - variable ± constant

Could be more expressive but very simple solver for

e =,<, ≤ e' !

- Array abstraction and variables abstraction, choice of
  - top
  - constant
  - parity
  - intervals
  - reduced product[(*)] (parity x intervals)
  - reduced cardinal power[(*)] of intervals by parity

Could be functors!

- 5699 lines of Ocaml (+6481 for unit tests)

Note: ARRAYAL is an abstract domain functor not a static analyzer, the abstract equations for programs of this talk have been established by hand (for lack of time for the equation generator).

(*) Patrick Cousot, Radhia Cousot: Systematic Design of Program Analysis Frameworks. POPL 1979: 269-282

The importance of parametrization

- The array segmentation abstract domain will work in any analysis context since no other information is necessary on simple variables (but for aliasing), although it can is exploited if available
- The segmentation and ordering information is inferred during the analysis (not given by the user/ or another (pre-)analysis)
- The cost/precision can be balanced by
  - appropriate abstraction of array *element and variable values*
  - degree of precision of *reductions*
- No need for any other external component

Example of reduction of array segments bounds by the variable values abstraction

```
          parameter int n; /* assume n>1 */
          int i, A[n];
          i = n;
/* 1: */
          while /* 2: */ (0 < i) {
/* 3: */
             i = i - 1;
/* 4: */
             A[i] = 0;
/* 5: */
          }
/* 6: */
```

The fact that i=0 is not taken into account

```
Analysis with widening/narrowing and (arrays: interval domain x variables:
interval domain):

Segmentation reduction ('?' elimination)? (y/n): no
p6 = [ A: <{0},[-oo,+oo],{i}?,[0,0],{n}?> ] [ i: [0,0] n: [2,+oo] ]

Segmentation reduction ('?' elimination)? (y/n): yes
p6 = [ A: <{0,i},[0,0],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.001832 s
```

Here, it is!

An analysis example

A detailed example

```
          int n = 10;
          int i, A[n];
          i = 0;
/* 1: */
          while /* 2: */ (i < n) {
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             A[i] = 0;
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             i = i + 1;
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```

```
p1 = A[n][n=10,i=0] = <{0,i},[-oo,+oo],{n,10}>; [ i: [0,0] n: [10,10] ]
p2 = ... = p5 = p6  = <>; [ i: _|_ n: _|_ ]
```

A detailed example (cont'd)

```
         int n = 10;
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A detailed example (cont'd)

```
        int n = 10;
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p3 = p2[i<n]        = <{0},[0,0],{i}?,[-oo,+oo],{n,10}>; [ i: [0,9] n: [10,10] ]
```

A detailed example (cont'd)

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p5 = p4[i=i+1]      = <{0},[0,0],{i-1}?,[0,0],{i},[-oo,+oo],{n,10}?>; [ i: [1,10] n: [10,10] ]
p2 = p2 W (p1 U p5) = <{0},[0,0],{i}?,[-oo,+oo],{n,10}?>; [ i: [0,+oo] n: [10,10] ]
```

A detailed example (cont'd)

```
        int n = 10;
        int i, A[n];
        i = 0;
/* 1: */
        while /* 2: */ (i < n) {
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p5 = p4[i=i+1]      = <{0,i-1},[0,0],{1,i},[-oo,+oo],{n,10}>; [ i: [1,1] n: [10,10] ]
p2 = p2 W (p1 U p5) = <{0},[0,0],{i}?,[-oo,+oo],{n,10}>; [ i: [0,+oo] n: [10,10] ]
p3 = p2[i<n]        = <{0},[0,0],{i}?,[-oo,+oo],{n,10}>; [ i: [0,9] n: [10,10] ]
p4 = p3[A[i]=0]     = <{0},[0,0],{i}?,[0,0],{i+1},[-oo,+oo],{n,10}?>; [ i: [0,9] n: [10,10] ]
p5 = p4[i=i+1]      = <{0},[0,0],{i-1}?,[0,0],{i},[-oo,+oo],{n,10}?>; [ i: [1,10] n: [10,10] ]
p2 = p2 W (p1 U p5) = <{0},[0,0],{i}?,[-oo,+oo],{n,10}?>; [ i: [0,+oo] n: [10,10] ]
p6 = p2[i>=n]       = <{0},[0,0],{n,10,i}>; [ i: [10,+oo] n: [10,10] ]
```

Concretization (meaning of abstract properties)

Concretization

For example $(a \in \mathbb{N} \mapsto \mathbb{Z}, i \in \mathbb{Z}, n \in \mathbb{Z})$,

$$
\begin{aligned}
& \gamma(\texttt{A:\{0\}0\{i\}?}\top\texttt{\{10,n\}?, i:[0,10], n:[10,10]}) \\
= \quad & \{\langle\langle \texttt{A},\ a\rangle,\ \langle \texttt{i},\ i\rangle,\ \langle \texttt{n},\ n\rangle\rangle \mid i \in [0,10] \wedge n = 10 \wedge \\
& \qquad (i > 0) \Rightarrow (\forall j \in [0, i-1] : a(i) = 0)\}
\end{aligned}
$$

Concretization

- **Concrete semantics of simple variables:**
  environments $\rho \in \mathbb{R}$ where $\mathbb{R} \triangleq \mathbb{X} \mapsto \mathbb{V}$ assign values $\rho(\mathtt{x})$ to variables
- **Concrete semantics of an array:**
  $T \in \mathbb{Z} \mapsto \mathbb{V}$
- **Concretization of an abstract array segmentation**

$$\gamma(\langle\, L_1, P_1, L_2[?], P_2, \ldots, L_{n-1}[?], P_{n-1}, L_n[?]\,\rangle; \overline{\rho}) \quad = \quad \bigcap_{i=1}^{n-1} \gamma(L_i, P_i, L_{i+1}[?]; \overline{\rho})$$

$$\begin{aligned}
\gamma(L, P, L'; \overline{\rho}) \quad = \quad & \{\langle T,\ \rho\rangle \mid \rho \in \gamma_v(\overline{\rho}) \wedge \forall \mathsf{e}_1, \mathsf{e}_2 \in L : \forall \mathsf{e}'_1, \mathsf{e}'_2 \in L' : \\
& \llbracket \mathsf{e}_1 \rrbracket\rho = \llbracket \mathsf{e}_2 \rrbracket\rho < \llbracket \mathsf{e}'_1 \rrbracket\rho = \llbracket \mathsf{e}'_2 \rrbracket\rho \wedge \\
& \forall j \in [\, \llbracket \mathsf{e}_1 \rrbracket\rho,\ \llbracket \mathsf{e}'_1 \rrbracket\rho\,) : T(j) \in \gamma_a(P)\}
\end{aligned}$$

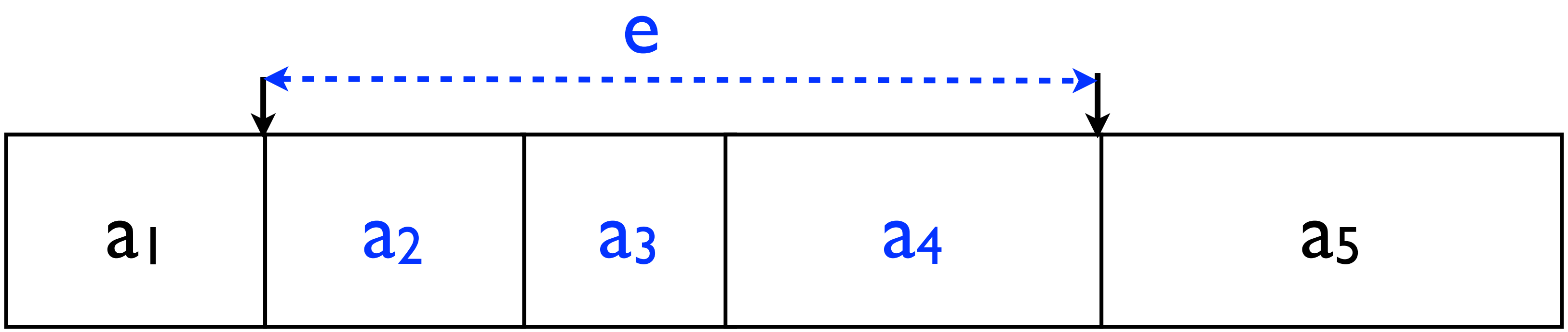
$$\begin{aligned}
\gamma(L, P, L'?; \overline{\rho}) \quad = \quad & \{\langle T,\ \rho\rangle \mid \rho \in \gamma_v(\overline{\rho}) \wedge \forall \mathsf{e}_1, \mathsf{e}_2 \in L : \forall \mathsf{e}'_1, \mathsf{e}'_2 \in L' : \\
& \llbracket \mathsf{e}_1 \rrbracket\rho = \llbracket \mathsf{e}_2 \rrbracket\rho \leq \llbracket \mathsf{e}'_1 \rrbracket\rho = \llbracket \mathsf{e}'_2 \rrbracket\rho \wedge \\
& \forall j \in [\, \llbracket \mathsf{e}_1 \rrbracket\rho,\ \llbracket \mathsf{e}'_1 \rrbracket\rho\,) : T(j) \in \gamma_a(P)\}
\end{aligned}$$

The array segmentation abstract domain functor: abstract operations

Abstract value of an array element

Value of A[e]:

1. Determine to which segment(s) of A the index e *may* belong
2. If none, signal an array overrun
3. Select the corresponding abstract value of array elements (their join if more than one)



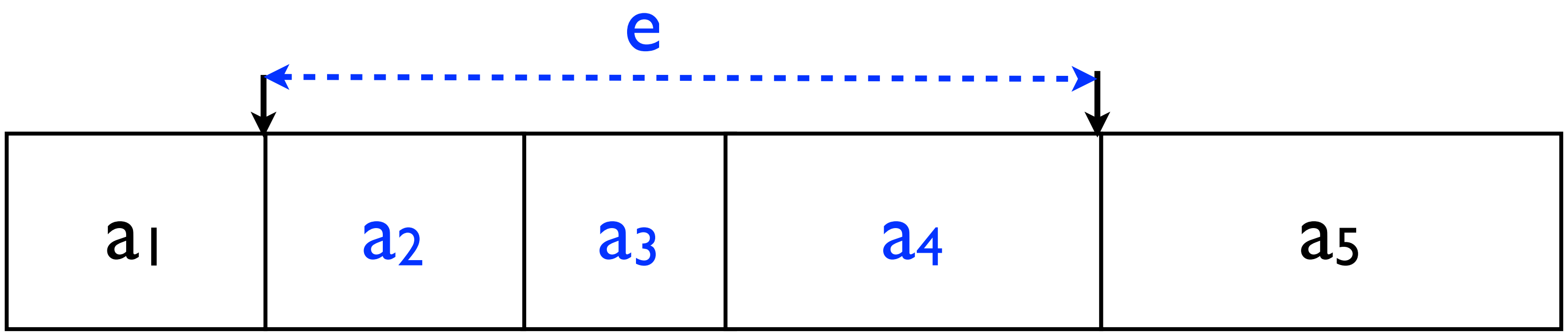
$$A[e] := a_2 \sqcup a_3 \sqcup a_4$$

Assignment to an array element

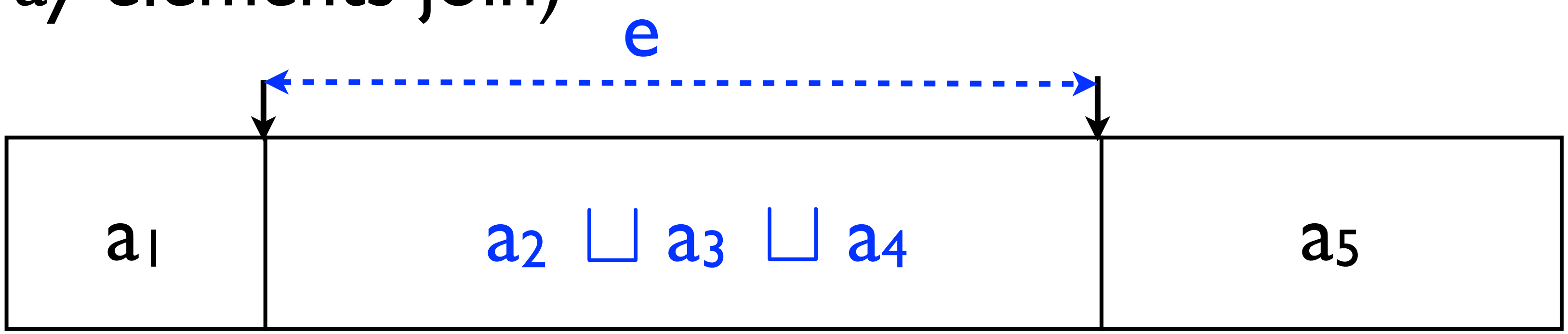
Assignment to A[e] := v

1. Determine to which segment(s) the index e may belong

2. If none, signal a array overrun



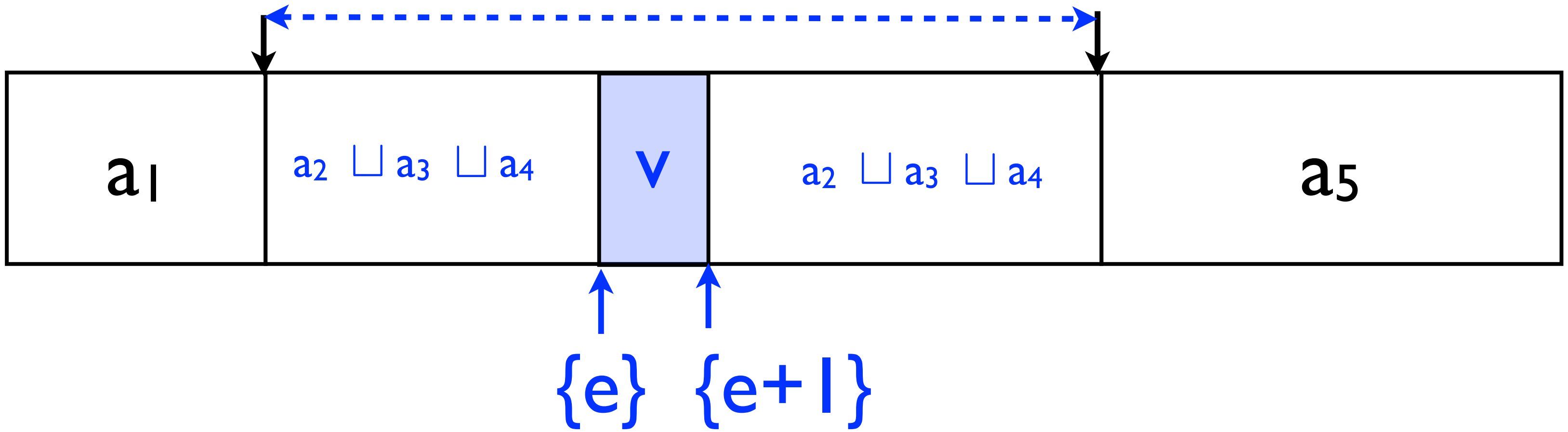
3. If more than one, join these segments (using the array elements join)



Assignment to an array element

Assignment to A[e] := v (continued)

4. Split the segment to insert abstract value v of assigned element (with special cases for assignments to segment bounds positions)



5. Adjust emptiness of resulting segments

Assignment to a simple variable

- Invertible assignment $i_{new} = e(i_{old})$ so $i_{old} = e^{-1}(i_{new})$
  - Replace i by $e^{-1}(i_{new})$ in all expressions in array segment bounds where i does appear

```
[ A: <{0},[-oo,+oo],{i},[1,+oo-1],{n}?> ] [ i: [1,+oo] n: [2,+oo] ]
i=i-1;
[ A: <{0},[-oo,+oo],{i+1},[1,+oo-1],{n}?> ] [ i: [0,+oo-1] n: [2,+oo] ]
```

- Non-invertible assignment to i = e
  - Eliminate all expressions in array segment bounds where i does appear
  - If a block limit becomes empty, join adjacent blocks
  - Add i to all block limits containing e

Conditionals on simple variables

- Test e = e’
  - Add e/e’ in segment bounds with e’/e

- Test e < e’
  - Adjust emptiness (and reduce block bounds)

Conditionals on array elements

- Access + restriction by test + assignment

Segmentwise comparison, join, meet, widening, narrowing

- For identical segmentations, binary operations are performed segmentwise
- Example: join

$$
\begin{array}{ll}
 & \langle\{0\}, [0,0], \{i\}, [0,2], \{n\}\rangle \\
\sqcup & \langle\{0\}, [1,1], \{i\}, [-1,0], \{n\}\rangle \\
= & \langle\{0\}, [0,1], \{i\}, [-1,2], \{n\}\rangle
\end{array}
$$

Segmentation unification

- For non-identical segmentations, a segment unification must be performed first:
    - By splitting segments when possible

$$\langle\{0\}, a, \{i\}, b, \{n\}\rangle \longrightarrow \langle\{0\}, a, \{i\}, b, \{j\}, b, \{n\}\rangle$$

$$\langle\{0\}, a', \{i\}, b', \{j\}, c', \{n\}\rangle \longrightarrow \langle\{0\}, a', \{i\}, b', \{j\}, c', \{n\}\rangle$$

- Otherwise, by joining adjacent segments

$$\langle\{0\}, a, \{i\}, b, \{n\}\rangle \longrightarrow \langle\{0\}, a \sqcup b, \{n\}\rangle$$

$$\langle\{0\}, a', \{j\}, b', \{n\}\rangle \longrightarrow \langle\{0\}, a' \sqcup b', \{n\}\rangle$$

(assuming i and j are incomparable with their variable abstractions and in the other array segmentations)

Example of segmentation unification in a union

$$
\begin{array}{ll}
 & \mathtt{A} : \{0, \mathtt{i}\}\top\{10, \mathtt{n}\},\ \mathtt{i} : [0, 0],\ \mathtt{n} : [10, 10] \\
\sqcup & \mathtt{A} : \{0, \mathtt{i-1}\}0\{1, \mathtt{i}\}\top\{10, \mathtt{n}\},\ \mathtt{i} : [1, 1],\ \mathtt{n} : [10, 10]
\end{array}
$$

$$
\begin{array}{ll}
\mathtt{=} & \mathtt{A} : \{0\}\bot\{\mathtt{i}\}?\top\{10, \mathtt{n}\},\ \mathtt{i} : [0, 0],\ \mathtt{n} : [10, 10] \\
\sqcup & \mathtt{A} : \{0\}0\{\mathtt{i}\}\top\{10, \mathtt{n}\},\ \mathtt{i} : [1, 1],\ \mathtt{n} : [10, 10]
\end{array}
$$

$$
\mathtt{=} \quad \mathtt{A} : \{0\}0\{\mathtt{i}\}?\top\{10, \mathtt{n}\},\ \mathtt{i} : [0, 1],\ \mathtt{n} : [10, 10]
$$

Comparison of expressions e =/≤/< e' in segment bounds

- Purely symbolically

  e.g. $x + i < y + j$ since $x=y$ & $i<j$

- Using non-relational information on variables

  e.g. $x + 1 < y$ since $x:[-\infty, 3]$ & $y: [5, +\infty]$

- Using information on (other) array segment ordering

  e.g. $x+1 < y$ since ...{x}?...{...}...{y+1}...

- Using information provided by a relational abstract domain (e.g. pentagons, DBM, octagons, sub-polyhedra, polyhedra, ...)

A few more examples

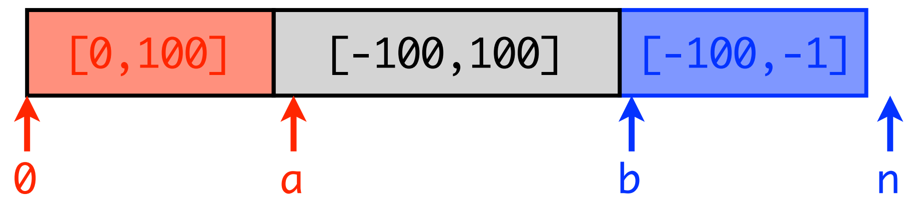
Array partitioning

```
             parameter int n  /* assume n>1 */
             var int a, b, c, A[n];
             assume A: {0}[-100,+100]{n}
             a = 0; b = 0; c = 0;
/*  1: */
             while /* 2: */ (a < n) {
/*  3: */
                if A[a] >= 0 then {
/*  4: */
                    B[b] = A[a]; b = b + 1;
/*  5: */
                } else {
/*  6: */
                    C[c] = A[a]; c = c + 1;
/*  7: */
                }
/*  8: */
                a = a + 1;
/*  9: */
             }
/* 10: */
```

```
p10 = [ A: <{0},[-100,100],{n}?> B: <{0},[0,100],{b}?,[-oo,+oo],{n}?> C: <{0},
[-100,-1],{c}?,[-oo,+oo],{n}?> ] [ a: [2,+oo] b: [0,+oo] c: [0,+oo] n:
[2,+oo] ]
0.003711 s
```

In situ array partitioning

```
           parameter int n; /* assume n>1 */
           var int a, b, x, A[n];
           assume A: {0}[-100,+100]{n}
           a = 0; b = n;
/*  1: */
           while /* 2: */ (a < b) {
/*  3: */
              if A[a] >= 0 then {
/*  4: */
                  a = a + 1;
/*  5: */
              } else {
/*  6: */
                  b = b - 1;
/*  7: */
                  x = A[a]; A[a] = A[b]; A[b] := x;
/*  8: */
              }
/*  9: */
           }
/* 10: */
```




```
Analysis with widening/narrowing and (interval domain x interval domain):
p1  = [ A: <{0,a},[-100,100],{n,b}> ] [a: [0,0] b: [2,+oo] n: [2,+oo] x: [-oo,+oo]]
p2  = [ A: <{0},[0,100],{a}?,[-100,100],{b}?,[-100,-1],{n}?> ] [a: [0,+oo] b:
[0,+oo] n: [2,+oo] x: [-oo,+oo]]
p10 = [ A: <{0},[0,100],{b,a}?,[-100,-1],{n}?> ] [a: [0,+oo] b: [0,+oo] n: [2,+oo]
x: [-oo,+oo]]
0.015378 s
```

I – Non-relational analysis on values (I)

```
        int n = 10;
        int i, A[n];
        i = 0;
/* 1: */
        while /* 2: */ (i < n) {
/* 3: */
            A[i] = 0;
/* 4: */
            i = i + 1:
/* 5: */
            A[i] = -16;
/* 6: */
            i = i + 1:
/* 7: */
        }
/* 8: */
```

Array: reduced product of parity and intervals – i.e. semantics A[i] := v_i

Variables: reduced product of parity and intervals

```
p1 = <{0,i},(T, [-oo,+oo]),{n,10}>; [ i: (e, [0,0]) n: (e, [10,10]) ]
p2 = <{0},(e, [-16,0]),{i}?,(T, [-oo,+oo]),{n,10}?>; [ i: (e, [0,+oo-1]) n: (e, [10,10]) ]
p8 = <{0},(e, [-16,0]),{n,10,i}>; [ i: (e, [10,+oo-1]) n: (e, [10,10]) ]

0.000832 s
```

II – Non-relational analysis on values (II)

```
        int n = 10;
        int i, A[n];
        i = 0;
/* 1: */
        while /* 2: */ (i < n) {
/* 3: */
            A[i] = 0;
/* 4: */
            i = i + 1:
/* 5: */
            A[i] = -16;
/* 6: */
            i = i + 1:
/* 7: */
        }
/* 8: */
```

Array: interval power parity on array elements – i.e. semantics A[i] := v_i

Variables: reduced product of parity and intervals

```
p1 = <{0,i},(o -> [-oo,+oo],e -> [-oo,+oo]),{n,10}>; [ i: (e, [0,0]) n: (e, [10,10]) ]
p2 = <{0},(o -> _|_,e -> [-16,0]),{i}?,(o -> [-oo,+oo],e -> [-oo,+oo]),{n,10}?>; [ i: (e,
[0,+oo-1]) n: (e, [10,10]) ]
p8 = <{0},(o -> _|_,e -> [-16,0]),{n,10,i}>; [ i: (e, [10,+oo-1]) n: (e, [10,10]) ]

0.00088 s
```

III – Relational analysis on (indexes x values)

```
        int n = 10;
        int i, A[n];
        i = 0;
/* 1: */
        while /* 2: */ (i < n) {
/* 3: */
            A[i] = 0;
/* 4: */
            i = i + 1:
/* 5: */
            A[i] = -16;
/* 6: */
            i = i + 1:
/* 7: */
        }
/* 8: */
```

Array: interval power parity on array elements – i.e. semantics A[i] := (i, v_i)

Variables: reduced product of parity and intervals

```
p1 = <{0,i},(o -> [-oo,+oo],e -> [-oo,+oo]),{n,10}>; [ i: (e, [0,0]) n: (e, [10,10]) ]
p2 = <{0},(o -> [-16,-16],e -> [0,0]),{i}?,(o -> [-oo,+oo],e -> [-oo,+oo]),{n,10}?>; [ i:
(e, [0,+oo-1]) n: (e, [10,10]) ]
p8 = <{0},(o -> [-16,-16],e -> [0,0]),{n,10,i}>; [ i: (e, [10,+oo-1]) n: (e, [10,10]) ]

0.001274 s
```

The semantics of arrays revisited (once again)

- The classical operational semantics (J. McCarthy):

  Array $\in$ Set of indices $\rightarrow$ Set of values

- Our semantics for relational segmentation:

  Array $\in$ Values of variables $\rightarrow$ Set of indices $\rightarrow$ Set of (index x values)

Segments

Relation between indexes and values per segment

J. McCarthy. Towards a mathematical science of computation. In C. M. Popplewell, editor, *IFIP Congress 1962*. North-Holland, 1983.

The segmentation abstract domain functor

- Our semantics for relational segmentation:

  Array $\in$ Values of variables $\rightarrow$ Set of indices $\rightarrow$ Set of (index x values)

- The abstraction functor:

Variable abstract domain → Expressions abstract domain → Array elements abstract domain → Segmentation abstract domain

Sound, automatic, terminating but incomplete...

```
          parameter int n; /* assume n>1 */
          int i, A[n];
          i = n;
/* 1: */
          while /* 2: */ (0 < i) {
/* 3: */
             i = i - 1;
/* 4: */
             A[i] = i;
/* 5: */
          }
/* 6: */
```

```
Analysis with widening/narrowing without thresholds and (interval
domain x interval domain):
[ -oo +oo ]

p6 = [ A: <{0,i},[-oo,+oo-1],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.003486 s
```

Sound, automatic, terminating but incomplete...

```
            parameter int n; /* assume n>1 */
            int i, A[n];
            i = n;
/* 1: */
            while /* 2: */ (0 < i) {
/* 3: */
                i = i - 1;
/* 4: */
                A[i] = i;
/* 5: */
            }
/* 6: */
```

i: [2,+oo] initial
i: [1,+oo-1] decrementation
i: [-oo,+oo] widening
i: [0,+oo] test & narrowing

```
Analysis with widening/narrowing without thresholds and (interval
domain x interval domain):
[ -oo +oo ]

p6 = [ A: <{0,i},[-oo,+oo-1],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.003486 s
```

Improvement ... 1st solution

- Widening/narrowing with thresholds

```
          parameter int n; /* assume n>1 */
          int i, A[n];
          i = n;
/* 1: */
          while /* 2: */ (0 < i) {
/* 3: */
             i = i - 1;
/* 4: */
             A[i] = i;
/* 5: */
          }
/* 6: */
```

```
Analysis with widening/narrowing with following thresholds and
(interval domain x interval domain):
[ -oo -1 0 1 +oo ]

p6 = [ A: <{0,i},[0,+oo-1],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.001868 s
```

Improvement … 2^{nd} solution

- Recurrent reanalysis

```
            parameter int n; /* assume n>1 */
            int i, A[n];
            i = n;
/* 1: */
            while /* 2: */ (0 < i) {
/* 3: */
               i = i - 1;
/* 4: */
               A[i] = i;
/* 5: */
            }
/* 6: */
```

```
Analysis with widening/narrowing without thresholds but with
reiteration for arrays on stabilized simple variables and (interval
domain x interval domain):
[ -oo +oo ]

p6 = [ A: <{0,i},[0,+oo-1],{n}> ] [ i: [0,0] n: [2,+oo] ]
0.002766 s
```

Principle of recurrent reanalysis

$A_0, V_0 = \mathrm{lfp}_{\perp,\perp}\ \lambda x, x'.\ x, x'\ (\nabla \times \nabla)\ F(x, x')$

$A_1, V_1 = \mathrm{gfp}_{A_0, V_0}\ \lambda x, x'.\ x, x'\ (\Delta \times \Delta)\ F(x, x')$

$A_2, V_2 = \mathrm{lfp}_{\perp, V_1}\ \lambda x, x'.\ x, x'\ (\nabla \times \sqcup)\ F(x, x')$

$A_3, V_3 = \mathrm{gfp}_{A_2, V_2}\ \lambda x, x'.\ x, x'\ (\Delta \times \sqcap)\ F(x, x')$

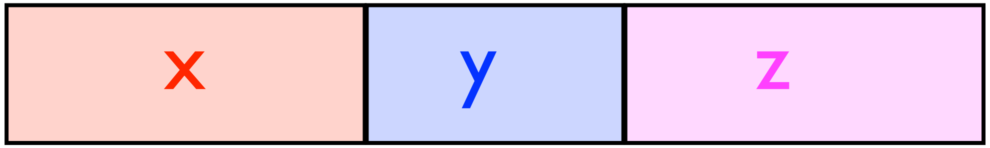
...

arrays $\times$ *variables*

Segmentation relational analyzes
(not yet implemented)

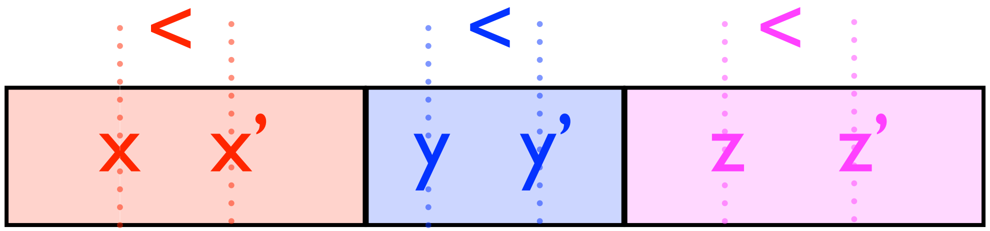
Relational analyses

- Inter-segments



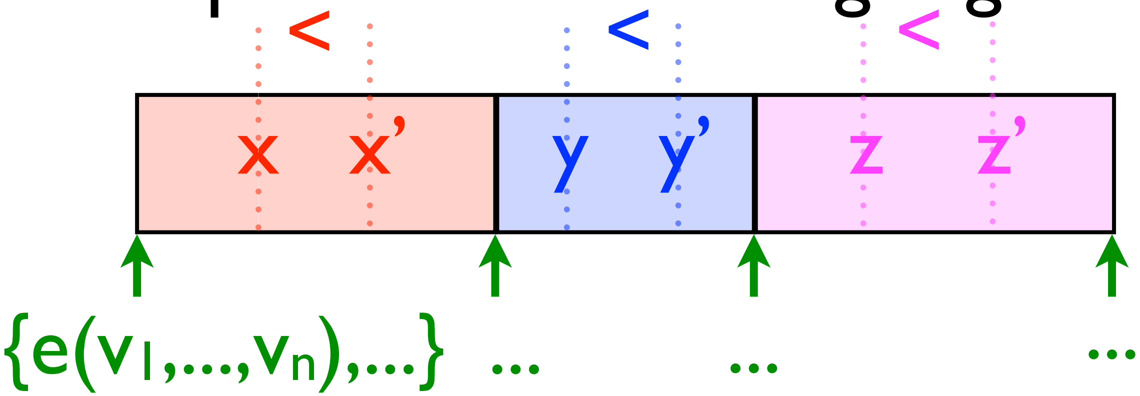
$r(x,y,z)$

- Intra/inter-segment



$r(x,x',y,y',z,z')$

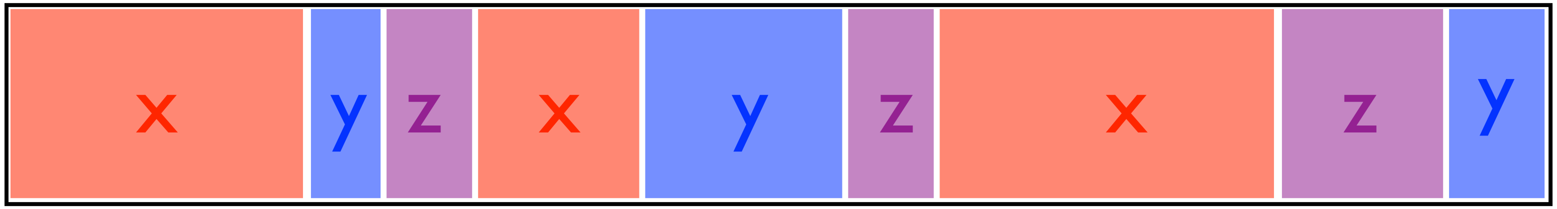
- Can also relate to variables appearing in sets of expressions delimiting segment bounds



$r(x,x',y,y',z,z',v_1,\ldots,v_n)$

Possible extensions

Partitions (or covers) instead of segments



$r(x,y,z)$

Existential instead of universal intra-segment properties

$$A{:}\langle L,\ldots,\{e_1,\ldots,e_n\}\ a\ \{e'_1,\ldots,e'_m\}[?],\ldots,H\rangle$$

- Universal:

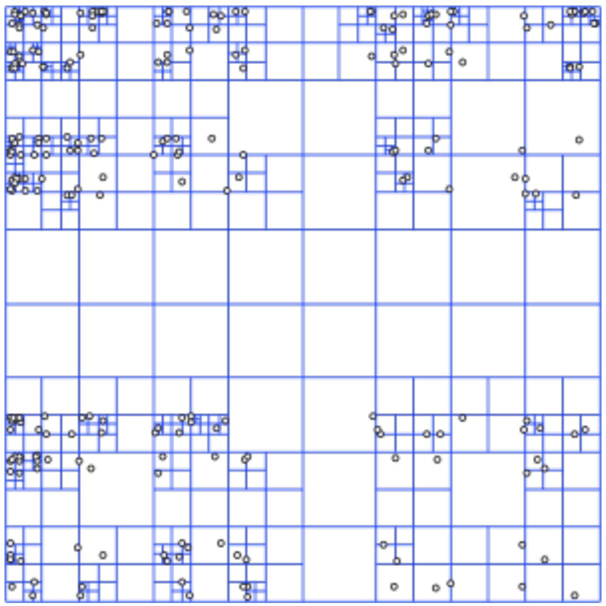
$$[\![e_1]\!] = \ldots = [\![e_n]\!] = l <[\leq] [\![e'_1]\!] = \ldots = [\![e'_m]\!] = h \wedge$$
$$\forall i: (l \leq i \leq h) \Rightarrow (A[i] \in \gamma(a))$$

- Existential:

$$[\![e_1]\!] = \ldots = [\![e_n]\!] = l <[\leq] [\![e'_1]\!] = \ldots = [\![e'_m]\!] = h \wedge$$
$$\exists i: (l \leq i \leq h) \Rightarrow (A[i] \in \gamma(a))$$

Multi-dimentional arrays

- Use vectors of expressions for each index instead of expressions in the sets delimiting segment bounds
- Order the segments by a total order on these vectors (componentwise, lexicographic,etc)
- Determining which order is more convenient requires more research
- More complex tilings (e.g. region quadtrees) are also conceivable



Related work

Related work

- Of course there are many static analyzes related to bounds of array indexes, starting from

  Patrick Cousot & Radhia Cousot. Static Determination of Dynamic Properties of Programs. IProceedings of the second international symposium on Programming, Paris, 106—130, 1976, Dunod, Paris.

- including for non-uniform alias analysis

  Stephen J. Fink, Kathleen Knobe, Vivek Sarkar: Unified Analysis of Array and Object References in Strongly Typed Languages. SAS 2000: 155-174

  Arnaud Venet: Nonuniform Alias Analysis of Recursive Data Structures and Arrays. SAS 2002: 36-51

- vectorization, parallelization, ...

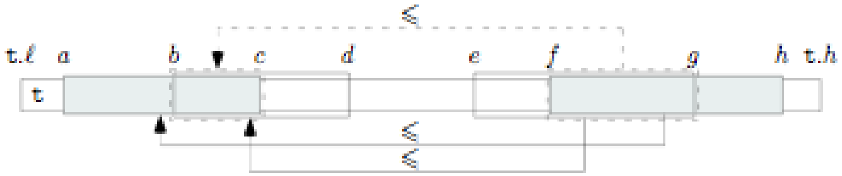
  Gerald Roth, Ken Kennedy: Dependence Analysis of Fortran90 Array Syntax. PDPTA 1996: 1225-1235

- etc, etc.

Related work (cont'd)

- Our basic inspiration: parametric predicate abstraction

P. Cousot: Verification by Abstract Interpretation. Verification: Theory and Practice. LNCS 2772, 2003: 243-26



used in many automatic abstract-interpretation-based array analyzes (often using partitions)

Denis Gopan, Thomas W. Reps, Shmuel Sagiv: A framework for numeric analysis of array operations. POPL 2005: 338-350

Nicolas Halbwachs, Mathias Péron: Discovering properties about arrays in simple programs. PLDI 2008: 339-348

Xavier Allamigeon: Non-disjunctive Numerical Domain for Array Predicate Abstraction. ESOP 2008: 163-177

Related work (cont'd)

- Predicate abstraction with refinement and/or more arbitrary forms of predicates

Cormac Flanagan, Shaz Qadeer: Predicate abstraction for software verification. POPL 2002: 191-202

Shuvendu K. Lahiri, Randal E. Bryant: Indexed Predicate Discovery for Unbounded System Verification. CAV 2004: 135-147

Shuvendu K. Lahiri, Randal E. Bryant: Constructing Quantified Invariants via Predicate Abstraction. VMCAI 2004: 267-281

Shuvendu K. Lahiri, Randal E. Bryant: Predicate abstraction with indexed predicates. ACM Trans. Comput. Log. 9(1): (2007)

Alessandro Armando, Massimo Benerecetti, Jacopo Mantovani: Abstraction Refinement of Linear Programs with Arrays. TACAS 2007: 373-388

Mohamed Nassim Seghir, Andreas Podelski, Thomas Wies: Abstraction Refinement for Quantified Array Assertions. SAS 2009: 3-18

Related work (con’d)

- Theorem prover-based with refinement and/or arbitrary forms of predicates

Ranjit Jhala, Kenneth L. McMillan: Array Abstractions from Proofs. CAV 2007: 193-206

Sumit Gulwani, Bill McCloskey, Ashish Tiwari: Lifting abstract interpreters to quantified logical domains. POPL 2008: 235-246

Laura Kovács, Andrei Voronkov: Finding Loop Invariants for Programs over Arrays Using a Theorem Prover. FASE 2009: 470-485

Evaluation criteria

Important evaluation criteria not always very clear from the array content analysis literature:

- without program restrictions ?
- fully automatic without user-given specifications and inductive invariants ??
- scales up ???
- used/usable in production-quality static analysis tools ????

Conclusion

The array segmentation abstract domain functor

- Fully automatic analysis (no hidden hypotheses)
- Simple
- Efficient (should scale up, needs further work to confirm)
- Autonomous (no required dependencies on index abstractions or other analyzes)
- Parametric (precision can be gained by precise array element/index analyzes)
- The abstract domain functor must be integrated in production-quality static analyzers(*)
- Hopefully useful!

(*) program fixpoint equations are presently encoded by hand!

Thanks to all for this very nice visit