Momentum and Collisions

Physics Club

Algebra-Based Mechanics

The following resources are useful for extra reading prior to attempting the handout or as a general introduction/review. Problems can be found from a variety of sources on the internet, and a quick search should be enough.

- Ch. 8 of *College Physics 2e* by OpenStax for a detailed explanation of momentum and collisions.
- For a video supplement, see unit 6 of Khan Academy's high school physics course.
- Unit 5 of Fiveable's AP Physics 1 study guides for a brief introduction to the concepts.
- See this document made by Flipping Physics.

Contents

1 Center of Mass

Many times, analyzing the motion of a system or object is complex as each point moves in a different way. To make the analysis easier, a special location known as the center of mass is introduced.

Definition: The center of mass is the average location of the mass in a system or object and is formally defined as,

$$\mathbf{r}_{\mathrm{cm}} = \frac{1}{M} \sum_{k=0}^{n} m_k \mathbf{r}_k \tag{1}$$

where M is the total mass and m_k and $\mathbf{r}_k$ are the mass and location of the k-th particle[a] respectively.

[a] In a system, a particle may be a set of discrete objects like the center of mass of a ball-person system. In a rigid object, a particle is roughly a very very small point with mass although calculus is required for a more formal definition.

Since $\mathbf{r}_{\mathrm{cm}}$ is a vector, we can decompose it to get the center of mass along each dimension,

$$x_{\mathrm{cm}} = \frac{1}{M} \sum_{k=1}^{n} m_k x_k, \quad y_{\mathrm{cm}} = \frac{1}{M} \sum_{k=1}^{n} m_k y_k, \quad z_{\mathrm{cm}} = \frac{1}{M} \sum_{k=1}^{n} m_k z_k$$

Example: (Adapted from HRK ed. 4 vol. 1) A particle is defined as (m, x, y) where m is the mass, x is the x-coordinate, and y is the y-coordinate. There are three weighted particles $P_1 = (3\text{ kg}, 0\text{ m}, 0\text{ m})$, $P_2 = (8\text{ kg}, 1\text{ m}, 2\text{ m})$, and $P_3 = (4\text{ kg}, 2\text{ m}, 1\text{ m})$. Where is the center of mass of these three particles?

Solution: Calculate the location of the center of mass in each dimension separately. In the x dimension,

$$x_{\mathrm{cm}} = \frac{P_{1,m}P_{1,x} + P_{2,m}P_{2,x} + P_{3,m}P_{3,x}}{P_{1,m} + P_{2,m} + P_{3,m}} = \frac{(3\text{ kg})(0\text{ m}) + (8\text{ kg})(1\text{ m}) + (4\text{ kg})(2\text{ m})}{3\text{ kg} + 8\text{ kg} + 4\text{ kg}} = 1.7\text{ m}$$

In the y dimension,

$$y_{\mathrm{cm}} = \frac{P_{1,m}P_{1,y} + P_{2,m}P_{2,y} + P_{3,m}P_{3,y}}{P_{1,m} + P_{2,m} + P_{3,m}} = \frac{(3\text{ kg})(0\text{ m}) + (8\text{ kg})(2\text{ m}) + (4\text{ kg})(1\text{ m})}{3\text{ kg} + 8\text{ kg} + 4\text{ kg}} = 1.33\text{ m}$$

Therefore, the coordinates of the center of mass are $\boxed{(1.7\text{ m}, 1.33\text{ m})}$.

Problem 1. (Source: HRK ed. 4 vol. 1) Show that the ratio of the distances x_1 and x_2 of two particles from their center of mass is the inverse ratio of their masses: that is, $x_1/x_2 = m_2/m_1$.

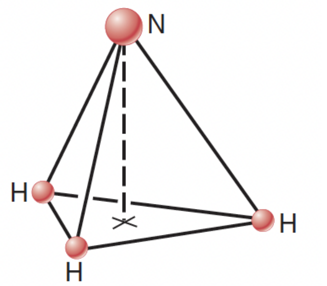
Solution. The center of mass is defined as,

$$x_{\mathrm{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Assume x_{cm} is at position 0. Then,

$$\begin{aligned}\frac{m_1x_1+m_2x_2}{m_1+m_2}&=0\\ m_1x_1+m_2x_2&=0\\ m_1x_1&=m_2x_2\\ x_1/x_2&=m_2/m_1\end{aligned}$$

Problem 2. (Source: HRK ed. 5 vol 1.) In the ammonia (NH_3) molecule, the three hydrogen (H) atoms form an equilateral triangle, the distance between centers of the atoms being 16.28×10^{-11}m, so that the center of the triangle is 9.40×10^{-11}m from each hydrogen atom. The nitrogen (N) atom is at the apex of a pyramid, the nitrogen-hydrogen distance is 10.14×10^{-11}m and the nitrogen/hydrogen atomic mass ratio is 13.9. Locate the center of mass relative to the nitrogen atom.



Solution. By symmetry, the center of mass will be at the center of the pyramid's base (equilateral triangle). Therefore, only y_{cm} must be calculated. The location of the plane of the CoM of the hydrogen atoms can be found using the Pythagorean Theorem where the hypotenuse is 10.14×10^{-11}m and one of the bases (lying normal to the dotted line) is 9.40×10^{-11}m,

$$y_h=\sqrt{(10.14\times 10^{-11}\text{ m})^2-(9.40\times 10^{-11}\text{ m})^2}=3.80\times 10^{-11}\text{ m}$$

Assuming the nitrogen atom is at position 0, the center of mass in the y dimension is,

$$y_{\text{cm}}=\frac{m_{\text{N}}y_{\text{N}}+m_{\text{H}}y_{\text{H}}}{m_{\text{N}}+m_{\text{H}}}=\frac{(13.9m_{\text{H}})(0\text{ m})+(3m_{\text{H}})(3.80\times 10^{-11}\text{ m})}{13.9m_{\text{H}}+3m_{\text{H}}}=\boxed{6.75\times 10^{-12}\text{ m}}$$

Problem 3. (Source: HRK ed. 5 vol. 1) Richard, mass 78.4 kg, and Judy, who is less massive, are enjoying Lake George at dusk in a 31.6-kg canoe. When the canoe is at rest in the placid water, they change seats, which are 2.93 m apart and symmetrically located with respect to the canoe's center. Richard notices that the canoe moved 41.2 cm relative to a submerged log and calculates Judy's mass. What is it?

Solution. The initial center of mass is,

$$\begin{aligned}x_{\text{cm}}&=\frac{1}{m_R+m_J+m_C}(m_Rx_R+m_Jx_J+m_Cx_C)\\ &=\frac{1}{m_C+110}[(78.4\text{ kg})(2.93\text{ m})+m_C(0\text{ m})+(31.6\text{ kg})(1.465\text{ m})]=\frac{276}{m_C+110}\end{aligned}$$

The canoe moved 0.412 m and the people switched places. The final center of mass is,

$$\begin{aligned} x'_{\text{cm}} &= \frac{1}{m_C+110}[(78.4\text{ kg})(0\text{ m}+0.412\text{ m})+m_C(2.93\text{ m}+0.412\text{ m})+(31.6\text{ kg})(1.465\text{ m}+0.412\text{ m})] \\ &= \frac{91+3.342m_C}{m_C+110} \end{aligned}$$

However, the center of mass does not move because there are no external forces, so $x_{\text{cm}} = x'_{\text{cm}}$. Therefore,

$$\frac{276}{m_C+110} = \frac{91+3.342m_C}{m_C+110}$$

Solving yields $m_C = \boxed{55.4\text{ kg}}$.

1.1 Velocity of the Center of Mass

Another important part of motion analysis is the velocity of the center of mass which can be used especially well when changing reference frames.

Velocity is the rate of change of position or formally,

$$\overline{\mathbf{v}} = \frac{\Delta\mathbf{r}}{\Delta t}$$

The center of mass is located at,

$$\mathbf{r}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k\mathbf{r}_k$$

The change in position of the center of mass is then,

$$\Delta\mathbf{r}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k(\mathbf{r}'_k-\mathbf{r}_k) = \frac{1}{M}\sum_{k=0}^{n} m_k\Delta\mathbf{r}_k$$

where $\mathbf{r}'_k$ is the final position of the CoM. The change in position in a time interval is,

$$\frac{\Delta\mathbf{r}_{\text{cm}}}{\Delta t} = \overline{\mathbf{v}}_{\text{cm}} = \left(\frac{1}{M}\sum_{k=0}^{n} m_k\Delta\mathbf{r}_k\right)\cdot\frac{1}{\Delta t}$$

Bringing the Δt into the summation,

$$\overline{\mathbf{v}}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k\frac{\Delta\mathbf{r}_k}{\Delta t} = \frac{1}{M}\sum_{k=0}^{n} m_k\mathbf{v}_k \tag{2}$$

which proves that the velocity of the center of mass is the weighted average of the velocities of individual particles with masses as weights, much like the position.

1.2 Acceleration of the Center of Mass

The acceleration of the CoM can be found through a similar process as the velocity.

Acceleration is the rate of change of velocity or formally,

$$\overline{\mathbf{a}} = \frac{\Delta\mathbf{v}}{\Delta t}$$

The velocity of the center of mass is,

$$\mathbf{v}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k \mathbf{v}_k$$

The change in velocity of the center of mass is then,

$$\Delta\mathbf{v}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k(\mathbf{v}_k' - \mathbf{v}_k) = \frac{1}{M}\sum_{k=0}^{n} m_k \Delta\mathbf{v}_k$$

where $\mathbf{v}_k'$ is the final velocity of the CoM. The change in position in a time interval is,

$$\frac{\Delta\mathbf{v}_{\text{cm}}}{\Delta t} = \bar{\mathbf{a}}_{\text{cm}} = \left(\frac{1}{M}\sum_{k=0}^{n} m_k \Delta\mathbf{v}_k\right)\cdot\frac{1}{\Delta t}$$

Bringing the Δt into the summation,

$$\bar{\mathbf{a}}_{\text{cm}} = \frac{1}{M}\sum_{k=0}^{n} m_k \frac{\Delta\mathbf{v}_k}{\Delta t} = \frac{1}{M}\sum_{k=0}^{n} m_k \mathbf{a}_k \tag{3}$$

which proves that the acceleration of the center of mass is the weighted average of the accelerations of individual particles with masses as weights.

> **Remark**: The derivation of the velocity and acceleration of the CoM carried out above is mathematically sound; however, due to limitations of an algebraic medium for physics, it is not completely conceptually sound. In the derivation, velocity and acceleration were defined as $\frac{\Delta\mathbf{r}}{\Delta t}$ and $\frac{\Delta\mathbf{v}}{\Delta t}$ respectively. However, these are actually the average velocity and acceleration. Hence it's more formal to label the previous expressions as averages. In reality, the **instantaneous** quantities are important, but those require calculus[a] to derive.
>
> [a]This is one of the fundamental problems of algebra based physics as the lack of mathematical rigor prevents the introduction of instantaneous quantities, a very important part of physics.

1.3 Dynamics of the Center of Mass

Now that the basic foundations of kinematics with the center of mass have been established, it's proper to turn to the dynamics.
For simplicity, consider a two particle system of m_1 and m_2. Newton's 2nd law in a general form is: $\sum \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}} = m\mathbf{a}$. The law applied to the particles is,

$$\sum \mathbf{F}_{\text{ext, 1}} + \sum \mathbf{F}_{\text{int, 1}} = m_1\mathbf{a}_1$$
$$\sum \mathbf{F}_{\text{ext, 2}} + \sum \mathbf{F}_{\text{int, 2}} = m_2\mathbf{a}_2$$

First, the internal forces are analyzed which are only the force from particle 1 on particle 2 or $\mathbf{F}_{12}$ and the force from particle 2 on particle 1 or $\mathbf{F}_{21}$. However, by Newton's 3rd law, the forces are equal and opposite thus $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Therefore, the internal forces cancel (a general result in any system). Adding the equations together (applying the forces on the system),

$$\mathbf{F}_{\text{ext, }1} + \mathbf{F}_{\text{ext, }2} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 \tag{4}$$

Eq. 3 can be re-arranged as

$$M\mathbf{a}_{\text{cm}} = \sum_{k=0}^{n} m_k\mathbf{a}_k = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_k\mathbf{a}_k$$

The left side of Eq. 4 is the net force or $\mathbf{F}_{\text{net}}$ while the right, as seen from the equation above is $M\mathbf{a}_{\text{cm}}$. Therefore, the net force is equivalent to the the total mass multiplied the acceleration of the center of mass. Formally,

$$\mathbf{F}_{\text{net}} = M\mathbf{a}_{\text{cm}} \tag{5}$$

Example: A 2 kg turtle dives horizontal off his 1 kg raft floating in his tank. If the turtle leaves the raft going 0.2 m/s relative to the ground, what speed does the raft move in the opposite direction?

Solution: There are no external forces acting on the turtle-raft system, therefore the center of mass does not move. We'll keep the CoM at position 0. The velocity of the center of mass is,

$$\mathbf{v}_{\text{cm}} = \frac{m_t\mathbf{v}_t + m_r\mathbf{v}_r}{m_t + m_r}$$

If the center of mass does not move, $\mathbf{v}_{\text{cm}} = 0$. Therefore,

$$m_t\mathbf{v}_t + m_r\mathbf{v}_r = 0$$

Solving for $\mathbf{v}_r$

$$\mathbf{v}_r = -\frac{m_t\mathbf{v}_t}{m_r} = \frac{(2\text{ kg})(0.2\text{ m/s})}{1\text{ kg}} = -0.4\text{ m/s}$$

The speed $v_r = ||\mathbf{v}_r|| = \boxed{0.4\text{ m/s}}$.

Problem 4. (Source: HRK ed. 4 vol. 1) A Chrysler with a mass of 2210 kg is moving along a straight stretch of road at 105 km/h. It is followed by a Ford with mass 2080 kg moving at 43.5 km/h. How fast is the center of mass of the two cars moving?

Solution. The velocity of the center of mass is defined as,

$$\mathbf{v}_{\text{cm}} = \frac{m_C\mathbf{v}_C + m_F\mathbf{v}_F}{m_C + m_F} = \frac{(2210\text{ kg})(105\text{ km/h}) + (2080\text{ kg})(43.5\text{ km/h})}{2210\text{ kg} + 2080\text{ kg}} = \boxed{75.18\text{ km/h}}$$

2 Linear Momentum

Just like many quantities in physics, the concept of linear momentum stems from the need to differentiate between various objects travelling at the same velocity. It's particularly important in collisions, which are discussed later.

Definition: Linear momentum for one particle is defined as,

$$\mathbf{p} = m\mathbf{v} \tag{6}$$

and for a many-particle system, as,

$$\mathbf{P} = \sum_{k=0}^{n} m_k \mathbf{v}_k = M\mathbf{v}_{\text{cm}} \tag{7}$$

Note: Linear momentum is a vector quantity with dimensions of $[M][L][T]^{-1}$ and commonly has the units of $\text{kg} \cdot \text{m/s}$ in the SI system.

Remark: Newton's 2nd law was original written using momentum as,

$$\overline{\mathbf{F}} = \frac{\Delta p}{\Delta t}$$

The following is the derivation of the more widely used form of Newton's 2nd law,

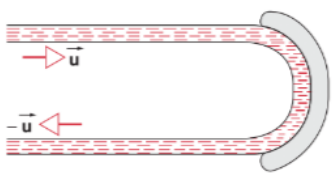
$$\overline{\mathbf{F}} = \frac{m\mathbf{v}' - m\mathbf{v}}{\Delta t} = m\frac{\Delta v}{\Delta t} = m\overline{\mathbf{a}}$$

Problem 5. A car of mass m is moving with a momentum p. How would you represent its kinetic energy in terms of these two quantities?

Solution. Re-arrange E1. 6 to get $\mathbf{v} = \mathbf{p}/m$. The kinetic energy is $\text{KE} = \frac{1}{2}m\mathbf{v}^2$. Substituting the expression for $\mathbf{v}$ in terms of momentum into the kinetic energy formula,

$$\text{KE} = \frac{1}{2}m\frac{\mathbf{p}^2}{m^2} = \boxed{\frac{\mathbf{p}^2}{2m}}$$

Problem 6. (Source: HRK ed. 5 vol. 1) A stream of water impinges on a stationary "dished" turbine blade, as shown in the figure. The speed of the water is μ, both before and after it strikes the curved surface of the blade, and the mass of water striking the blade per unit time is constant at the value μ. Find the force exerted by the water on the blade.



Solution. Force is change in momentum over time and momentum is the product of mass and velocity. Therefore,

$$\mathbf{F} = \frac{\Delta\mathbf{p}}{\Delta t} = \Delta\mathbf{v}\frac{m}{\Delta t} = \boxed{2u\mu}$$

2.1 Conservation of Linear Momentum

Conservation of linear momentum is a phenomenon that is applicable in all parts of physics from the atomic level to celestial bodies. It's quite powerful and makes analyzing systems, particularly those that involve collisions, much easier.

Definition: If the net external force on a system is zero, then the total momentum of the system is constant. Formally, if $\mathbf{F}_{\text{ext}} = 0$ then $\mathbf{P} = \text{const.}$ Since $\mathbf{P}$ is a vector, it can be decomposed into its dimensions, butomentum must be conserved **separately** in individual dimensions,

$$\mathbf{P}_{i,x} = \mathbf{P}_{f,x},\ \mathbf{P}_{i,y} = \mathbf{P}_{f,y},\ \mathbf{P}_{i,z} = \mathbf{P}_{f,z}$$

Example: A pumpkin launcher shoots out a 9 kg pumpkin at 30 m/s at 35°above horizontal. If the pumpkin launcher has a mass of 400 kg and is free to move, what is its horizontal recoil speed?

Solution: Since only the horizontal or x direction is of concern, by the property of separate dimensional conservation,

$$\cancel{m_p \mathbf{v}_{p,ix}} + \cancel{m_l \mathbf{v}_{l,ix}} = m_p \mathbf{v}_{p,fx} + m_l \mathbf{v}_{l,fx}$$

Therefore,

$$m_p \mathbf{v}_{p,fx} = -m_l \mathbf{v}_{l,fx}$$

Solving for $\mathbf{v}_{l,fx}$,

$$\mathbf{v}_{l,fx} = -\frac{m_p \mathbf{v}_{p,fx}}{m_l} = -\frac{(9\text{ kg})(30\cos 35^\circ\text{ m/s})}{400\text{ kg}} = -0.55\text{ m/s}$$

The speed is $v = ||\mathbf{v}_{l,fx}|| = \boxed{0.55\text{ m/s}}$.

Problem 7. An astronaut (86 kg) on a space walk (outside of the shuttle) throws Space Cat (4.8 kg) at a speed of 25 m/s, relative to the shuttle, at an angle of 40 degrees above horizontal away from himself. What is the speed of the astronaut after launching our feline superhero?

Solution. Since the situation is in space, the angle is extraneous information because the astronaut will just move in the opposite direction to the cat. The change in momentum of the cat is, $m_c \mathbf{v}_{c,f} = (4.8\text{ kg})(25\text{ m/s}) = 120\text{ kg}\cdot\text{ m/s}$. This change is equal to the astronauts change in momentum so,

$$m_a \mathbf{v}_{a,f} = 120 sikg\cdot\text{ m/s} \Rightarrow \mathbf{v}_{a,f} = \boxed{1.4\text{ m/s}}$$

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Problem 8. A miniature spring-loaded, radio-controlled gun is mounted on an air puck. The gun's bullet has a mass of 5.00 g, and the gun and puck have a combined mass of 120 g. With the system initially at rest, the radio controlled trigger releases the bullet causing the puck and empty gun to move with a speed of 0.500 m/s. What is the bullet's speed?

Solution. The puck-gun-bullet system has no external force acting on it. Therefore, the change in momentum of the bullet will equal the change in momentum of the puck and empty gun. The change in momentum of the puck and empty gun is 0.06 m/s. Setting this equal to the bullet's change in momentum yields $\mathbf{v}_b = \boxed{12 \text{ m/s}}$.

3 Impulse

Definition: The impulse momentum theorem states that impulse is the change in momentum. Formally,

$$J = \Delta p \tag{8}$$

or when written with force,

$$J = \mathbf{F}\Delta t \tag{9}$$

Example: (Source: HRK ed. 5 vol. 1) A 325-g ball with a speed v of 6.22 m/s strikes a wall at an angle θ of 33.0°and then rebounds with the same speed and angle. It is in contact with the wall for 10.4 ms. (a) What impulse was experienced by the ball? (b) What was the average force exerted by the ball on the wall?

Solution: For part (a), only the component of the momentum which is perpendicular to the wall changes. Therefore,

$$\mathbf{J} = \Delta\mathbf{p} = -2(0.325 \text{ kg})(6.22 \sin 33^\circ \text{ m/s})\hat{\mathbf{j}} = \boxed{-2.20 \text{ kg} \cdot \text{m/s}\hat{\mathbf{j}}}$$

Note that $\hat{\mathbf{j}}$ simply indicates the component of momentum perpendicular to the wall or the y-direction. The other component is zero. For part (b), the force is defined as $\mathbf{F} = \mathbf{J}/t$. However, the question asks for the force exerted by the ball on the wall which has the opposite impulse. Therefore,

$$\mathbf{F} = -\mathbf{J}/t = -(-2.20 \text{ kg}\hat{\mathbf{j}})/(0.0104 \text{ s}) = \boxed{211.5 \text{ N}\hat{\mathbf{j}}}$$

Problem 9. A positive impulse of 16 N-s is applied to 1.3 kg toy car. What is the speed of the car if it was initially moving at 5 m/s in the positive direction?

Solution. Impulse is the change in momentum, which means the car's final velocity can be modeled by,

$$m\mathbf{v}_f = m\mathbf{v}_i + 16$$

Solving for $\mathbf{v}_f$

$$\mathbf{v}_f = \frac{m\mathbf{v}_i + 16}{m} = \frac{(1.3 \text{ kg})(5 \text{ m/s}) + 16 \text{ N-s}}{1.3 \text{ kg}} = \boxed{17.3 \text{ m/s}}$$

4 Collisions

Collisions are one of the most common phenomenons in physics and appear in a variety of fields. They can be analyzed to a great degree of accuracy using concepts such as the CoM, momentum, and impulse. There are two types of collisions, elastic and inelastic; both are discussed below along with their various special cases.

4.1 Momentum in Collisions

The time period during which a collision occurs is so small that the impulse from any external force is negligible. Formally, $J_{\text{ext}} \ll J_{\text{collision}}$. Therefore, momentum is conserved during the time around a collision.

Example: A 20-g bullet moving at 1000 m/s is fired through a one-kg block of wood emerging at a speed of 100 m/s. If the block had been originally at rest and is free to move, what is its resulting speed?

Solution: By conservation of momentum,

$$m_b \mathbf{v}_{b,i} + \cancel{m_w \mathbf{v}_{w,i}} = m_b \mathbf{v}_{b,f} + m_w \mathbf{v}_{w,f}$$

Solving for $\mathbf{v}_{w,f}$

$$\mathbf{v}_{w,f} = \frac{m_b(\mathbf{v}_{b,i} - \mathbf{v}_{b,f})}{m_w} = \frac{0.020\text{ kg}(1000\text{ m/s} - 100\text{ m/s})}{1\text{ kg}} = \boxed{18\text{ m/s}}$$

Problem 10. Two skaters, both of mass 75 kg, are on skates on a frictionless ice pond. One skater throws a 0.3-kg ball at 5 m/s to his friend, who catches it and throws it back at 5 m/s. When the first skater has caught the returned ball, what is the velocity of each of the two skaters?

Solution. Both skaters catch the ball at 5 m/s and throw the ball in the opposite direction at 5 m/s, so the magnitude of their velocity will be the same. If we consider the system to be that of skater one and the ball then there is no external force when the first skater throws or catches the ball. Therefore, the only time the skater-ball system has an external force is when the second skater throws the ball. Thus the change in momentum of the skater-ball system is the same as the change in momentum of the ball from the second skater, which is

$$\Delta \mathbf{p}_b = m_b(\mathbf{v}_{b,f} - \mathbf{v}_{b,i}) = 0.3\text{ kg}(-5\text{ m/s} - 5\text{ m/s}) = -3\text{ kg}\cdot\text{m/s}$$

We assumed the first skater is on the left and the second skater on the right and set the right direction as positive x. Since $m_{s,1} \gg m_b$, the mass of the skater-ball system is ≈ 75 kg. Therefore, the momentum is

$$\mathbf{p}_{\text{system}} = m_{\text{system}}\mathbf{v} = -3\text{ kg}\cdot\text{m/s}$$

Solving for $\mathbf{v}$ yields an answer of $\boxed{-0.04\text{ m/s}}$ (the skaters are moving apart).

Remark: When solving physics problems, defining the system is one of the most important decisions as it can make problems very trivial. In this case, we defined the system excluding one of the skaters, perhaps unconventional, but useful nonetheless. In other cases, switching the reference frame is also handy. We could have instead solved the problem through the center of mass frame. It is vital to make these decisions as there are

> as many easy ways to solve the problem as there are hard if the right system or frame is not chosen (although for the purposes of a standard physics course, it's obvious what to choose).

4.2 Elastic Collisions

While not very applicable in everyday reality[1], elastic collisions are still important from the problem solving perspective, conceptual understanding, and are still useful on the atmoic scale.

In an elastic collision, momentum **and** energy are conserved. We'll use a two body system as an example and write its conservation equations.

The equaiton for the conservation of momentum is,

$$m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} = m_1\mathbf{v}_{1,f} + m_2\mathbf{v}_{2,f} \tag{10}$$

and the equation for the conservation of energy is,

$$\frac{1}{2}m_1\mathbf{v}_{1,i}^2 + \frac{1}{2}m_2\mathbf{v}_{2,i}^2 = \frac{1}{2}m_1\mathbf{v}_{1,f}^2 + \frac{1}{2}m_2\mathbf{v}_{2,f}^2 \tag{11}$$

Re-writting the Eq. 10 to isolate the objects on either side of the equality,

$$m_1(\mathbf{v}_{1,i} - \mathbf{v}_{1,f}) = m_2(\mathbf{v}_{2,f} - \mathbf{v}_{2,i}) \tag{12}$$

and doing the same to Eq. 11,

$$m_1(\mathbf{v}_{1,i}^2 - \mathbf{v}_{1,f}^2) = m_2(\mathbf{v}_{2,f}^2 + \mathbf{v}_{2,i}^2) \tag{13}$$

We then recognize that the expression in the parentheses in Eq. 13 is in the form $a^2 - b^2$ which is equal to $(a+b)(a-b)$. Re-writting Eq. 13,

$$m_1(\mathbf{v}_{1,i}^2 + \mathbf{v}_{1,f}^2)(\mathbf{v}_{1,i}^2 - \mathbf{v}_{1,f}^2) = m_2(\mathbf{v}_{2,f}^2 + \mathbf{v}_{2,i}^2)(\mathbf{v}_{2,f}^2 - \mathbf{v}_{2,i}^2) \tag{14}$$

Dividing Eq. 14 by Eq. 13,

$$\frac{\cancel{m_1}(\mathbf{v}_{1,i}^2 + \mathbf{v}_{1,f}^2)\cancel{(\mathbf{v}_{1,i}^2 - \mathbf{v}_{1,f}^2)} = \cancel{m_2}(\mathbf{v}_{2,f}^2 + \mathbf{v}_{2,i}^2)\cancel{(\mathbf{v}_{2,f}^2 - \mathbf{v}_{2,i}^2)}}{\cancel{m_1}\cancel{(\mathbf{v}_{1,i} - \mathbf{v}_{1,f})} = \cancel{m_2}\cancel{(\mathbf{v}_{2,f} - \mathbf{v}_{2,i})}}$$

which yields the important equation:

$$\mathbf{v}_{1,i} - \mathbf{v}_{2,i} = \mathbf{v}_{2,f} - \mathbf{v}_{1,f} \tag{15}$$

Now, we can use Eq. 12 and Eq. 15 to solve for the final velocities of the two objects[2],

$$\mathbf{v}_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\mathbf{v}_{1,i} + \left(\frac{2m_2}{m_1 + m_2}\right)\mathbf{v}_{2,i} \tag{16}$$

$$\mathbf{v}_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right)\mathbf{v}_{1,i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\mathbf{v}_{2,i} \tag{17}$$

[1] An experiment where the time interval considered is very small or the environment prevents much loss of energy can be created where an elastic model may not have a lot of error.

[2] It is left as an excercise for the reader to prove the derivation of Eq. 16 and Eq. 17.

Eq. 16 and Eq. 17 are then used to consider certain special cases,

1. **Equal masses:** When $m_1 = m_2$,

$$\mathbf{v}_{1,f} = \mathbf{v}_{2,i} \text{ and } \mathbf{v}_{2,f} = \mathbf{v}_{1,i}$$

2. **Target at rest:** When m_2 is at rest ($\mathbf{v}_{2,i} = 0$),

$$\mathbf{v}_{1,f} = \frac{m_1 - m_2}{m_1 + m_2}\mathbf{v}_{1,i} \text{ and } \mathbf{v}_{2,f} = \frac{2m_1}{m_1 + m_2}\mathbf{v}_{1,i}$$

   Cases 1 and 2 together show that the first mass is stopped and the second mass goes off with the velocity the first mass originally had.

3. **Massive target:** If $m_2 \gg m_1$,

$$\mathbf{v}_{1,f} \approx -\mathbf{v}_{1,i} + 2\mathbf{v}_{2,i} \text{ and } \mathbf{v}_{2,f} \approx \mathbf{v}_{2,i}$$

   When the massive particle is moving very slowly or at rest,

$$\mathbf{v}_{1,f} \approx -\mathbf{v}_{1,i} \text{ and } \mathbf{v}_{2,f} \approx 0$$

4. **Massive projectile:** When $m_1 \gg m_2$,

$$\mathbf{v}_{1,f} \approx \mathbf{v}_{1,i} \text{ and } \mathbf{v}_{2,f} \approx 2\mathbf{v}_{1,i} - \mathbf{v}_{2,i}$$

Example: We have two balls. The first ball has mass 0.54 kg and is traveling 7.1 m/s to the right. It collides head-on elastically with a second ball of mass 0.95kg traveling 2.8 m/s to the left. After the collision, what is the speed and direction of each ball?

Solution: Using the equation,

$$\mathbf{v}_{1,i} - \mathbf{v}_{2,i} = \mathbf{v}_{2,f} - \mathbf{v}_{1,f}$$

and solving for $\mathbf{v}_{1,f}$

$$\mathbf{v}_{1,f} = \mathbf{v}_{2,f} - \mathbf{v}_{1,i} + \mathbf{v}_{2,i}$$

Putting the expression in the conservation of momentum setup,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1(\mathbf{v}_{2,f} - \mathbf{v}_{1,i} + \mathbf{v}_{2,i}) + m_2\mathbf{v}_{2,f}$$

Solving for $\mathbf{v}_{2,f}$,

$$\mathbf{v}_{2,f} = \frac{2m_1\mathbf{v}_{1,i} + (m_2 - m_1)\mathbf{v}_{2,i}}{m_1 + m_2} = \boxed{4.4 \text{ m/s}}$$

Adding the value for $\mathbf{v}_{2,f}$ into the second equation and populating the rest of values yields $\mathbf{v}_{1,f} = \boxed{-5.5 \text{ m/s}}$

Problem 11. A 10 kg mass traveling 2 m/s meets and collides elastically with a 2 kg mass traveling 4 m/s in the opposite direction. Find the final velocities of both objects.

Solution. Using the equations,

$$\mathbf{v}_{1,f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\mathbf{v}_{1,i} + \left(\frac{2m_2}{m_1 + m_2}\right)\mathbf{v}_{2,i}$$

and

$$\mathbf{v}_{2,f} = \left(\frac{2m_1}{m_1 + m_2}\right)\mathbf{v}_{1,i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\mathbf{v}_{2,i}$$

to solve for the final velocity yields $\mathbf{v}_{1,f} = \boxed{0\text{ m/s}}$ and $\mathbf{v}_{2,f} = \boxed{6\text{ m/s}}$.

4.3 Inelastic Collisions

Inelastic collisions make up the vast majority of the collisions that occur in our daily life. In inelastic collisions, momentum is conserved, but energy is not. One special case is of interest, perfectly inelastic collisions in which the particles stick together after colliding and move with a common velocity. Using the example of a two particle system, momentum is conserved as,

$$m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} = (m_1 + m_2)\mathbf{v}$$

Therefore, the final velocity of the composite system is,

$$\mathbf{v} = \frac{m_1}{m_1 + m_2}\mathbf{v}_{1,i} + \frac{m_2}{m_1 + m_2}\mathbf{v}_{2,i}$$

When m_2 is at rest,

$$\mathbf{v} = \frac{m_1}{m_1 + m_2}\mathbf{v}_{1,i}$$

In a special case of perfectly inelastic collisions, when $\mathbf{v} = 0$, $\dfrac{\mathbf{v}_{1,i}}{\mathbf{v}_{2,i}} = \dfrac{-m_2}{m_1}$.

Example: A railroad freight car, mass 15 000 kg, is allowed to coast along a level track at a speed of 2.0 m/s. It collides and couples with a 50 000-kg loaded second car, initially at rest and with brakes released. What percentage of the initial kinetic energy of the 15 000-kg car is preserved in the two-coupled cars after collision?

Solution: The initial kinetic energy of the 15000 kg car is

$$\frac{1}{2}m_{c,1}\mathbf{v}_{c,i}^2 = \frac{1}{2}(15000\text{ kg})(2\text{ m/s})^2 = 30000\text{ J}$$

The kinetic energy of the two cars after the collision is

$$\frac{1}{2}(m_{c,1} + m_{c,2})\mathbf{v}_f^2$$

To solve for v_f^2, use conservation of momentum.

$$m_{c,1}\mathbf{v}_{c,i} = (m_{c,1} + m_{c,2})\mathbf{v}_f$$

Therefore, $\mathbf{v}_f = 0.462$ m/s. Thus the kinetic energy is

$$\frac{1}{2}(65000\text{ kg})(0.462\text{ m/s})^2 = 6936\text{ J}$$

. The percent of kinetic energy is,

$$\frac{\mathrm{KE}_f}{\mathrm{KE}_i} \times 100\% = \frac{6936\ \mathrm{J}}{30000\ \mathrm{J}} \times 100\% = \boxed{23\%}$$

Problem 12. A 2500-kg truck moving at 10.00 m/s strikes a car waiting at a traffic light, hooking bumpers. The two continue to move together at 7.00 m/s. What was the mass of the struck car?

Solution. The collision is inelastic because the two objects move together at a common velocity. By conservation of momentum,

$$\cancel{m_c \mathbf{v}_{c,i}} + m_t \mathbf{v}_{t,i} = (m_t + m_c)\mathbf{v}_f$$

. Solving for m_2,

$$m_2 = \frac{m_t(\mathbf{v}_{t,i} - \mathbf{v}_f)}{\mathbf{v}_f} = \frac{(2500\ \mathrm{kg})(10.00\ \mathrm{m/s} - 7.00\ \mathrm{m/s})}{7.00\ \mathrm{m/s}} = \boxed{1070\ \mathrm{kg}}$$

4.4 Multi-Dimensional Collisions

In multi-dimensional collisions, momentum must be conserved in each dimension separately. Thus,

$$\mathbf{P}_{x,i} = \mathbf{P}_{x,f}, \;\; \mathbf{P}_{y,i} = \mathbf{P}y, f, \;\; \mathbf{P}_{z,i} + \mathbf{P}_{z,f}$$

Example: There are two skaters. The male skater with mass 68 kg travels 15 m/s North. He approaches a 60 kg female skater who is travel 12 m/s East; they approach each other at right angles. When they meet, they hold on to each other. At what direction and speed do they move after they meet?

Solution: The two equations for the conservation of momentum are,

$$m_2\mathbf{v}_{2,x} = (m_1 + m_2)\mathbf{v}\cos\phi \tag{18}$$

$$m_1\mathbf{v}_{1,y} = (m_1 + m_2)\mathbf{v}\sin\phi \tag{19}$$

Dividing Eq. 14 by Eq. 13,

$$\frac{m_1\mathbf{v}_{1,y}}{m_1\mathbf{v}_{1,y}} = \frac{(m_1 + m_2)\mathbf{v}\sin\phi}{(m_1 + m_2)\mathbf{v}\cos\phi} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

The angle is,

$$\phi = \arctan\left(\frac{m_1\mathbf{v}_{1,y}}{m_1\mathbf{v}_{1,y}}\right) = 54.8^\circ$$

Solving for $\mathbf{v}$ in Eq. 13,

$$\mathbf{v} = \frac{m_2\mathbf{v}_{2,x}}{(m_1 + m_2)\cos\phi} = \boxed{9.8\ \mathrm{m/s}}$$

5 Problems

5.1 Multiple Choice Questions

Problem 1. A spring is compressed between two objects with unequal masses, m and M, and held together. The objects are initially at rest on a horizontal frictionless surface. When released, which of the following is true?

(A) The total final kinetic energy is zero.

(B) The two objects have equal kinetic energy.

(C) The speed of one object is equal to the speed of the other.

(D) The total final momentum of the two objects is zero.

Solution. By conservation of momentum, the total momentum before and after the masses are released must be equal. Thus, answer choice $\boxed{\text{(D)}}$ is correct.

Problem 2. Two football players with mass 75 kg and 100 kg run directly toward each other with speeds of 6 m/s and 8 m/s respectively. If they grab each other as they collide, the combined speed of the two players just after the collision would be:

(A) 2 m/s

(B) 3.4 m/s

(C) 4.6 m/s

(D) 7.1 m/s

Solution. The collision is perfectly inelastic which means,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{v}'$$

Solving for $\mathbf{v}'$,

$$\mathbf{v}' = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \frac{(75\text{ kg})(6\text{ m/s}) + (100\text{ kg})(-8\text{ m/s})}{75\text{ kg} + 100\text{ kg}} = -2\text{ m/s}$$

The speed is 2 m/s, thus answer choice $\boxed{\text{(D)}}$ is correct.

Problem 3. Two carts are held together. Cart 1 is more massive than Cart 2. As they are forced apart by a compressed spring between them, which of the following will have the same magnitude for both carts.

(A) change of velocity

(B) force

(C) speed

(D) velocity

Solution. By momentum conservation, the change in momentum of both carts must be the same. Thus, they must recieve the same impluse. Since the spring is in contact with both carts for the same amount of time, the force must also be the same. Therefore, answer choice $\boxed{\text{(B)}}$ is correct.

Problem 4. A mass m has speed v. It then collides with a stationary object of mass $2m$. If both objects stick together in a perfectly inelastic collision, what is the final speed of the newly formed object?

(A) $v/3$

(B) $v/2$

(C) $2v/3$

(D) $3v/2$

Solution. The collision is perfectly inelastic which means,

$$m\mathbf{v} + \cancel{2m\mathbf{v}_0} = (m_1 + m_2)\mathbf{v}'$$

Solving for $\mathbf{v}'$,

$$\mathbf{v}' = \frac{m\mathbf{v}}{m + 2m} = \frac{\cancel{m}\mathbf{v}}{3\cancel{m}} = \boxed{\frac{v}{3}}$$

Thus answer choice $\boxed{\text{(A)}}$ is correct.

Problem 5. A 50 kg skater at rest on a frictionless rink throws a 2 kg ball, giving the ball a velocity of 10 m/s. Which statement describes the skater's subsequent motion?

(A) 0.4 m/s in the same direction as the ball's motion.

(B) 0.4 m/s in the opposite direction of the ball's motion.

(C) 2 m/s in the same direction as the ball's motion.

(D) 2 m/s in the opposite direction of the ball's motion.

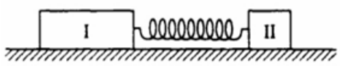
Solution. By conservation of momentum, the intial and final momenta must equal zero. Therefore,

$$m_1\mathbf{v}_1' = -m_2\mathbf{v}_2'$$

Solving for $\mathbf{v}_1'$,

$$\mathbf{v}_1' = -\frac{m_2\mathbf{v}_2'}{m_1} = -\frac{(2\text{ kg})(10\text{ m/s})}{50\text{ kg}} = -0.4\text{ m/s}$$

The velocities of the ball and skater are opposite, so answer choice $\boxed{\text{(B)}}$ is correct.



Problem 6. Two pucks are firmly attached by a stretched spring and are initially held at rest on a frictionless surface, as shown above. The pucks are then released simultaneously. If puck I has three times the mass of puck II, which of the following quantities is the same for both pucks as the spring pulls the two pucks toward each other?

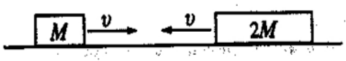
(A) Speed

(B) Magnitude of acceleration

(C) Kinetic energy

(D) Magnitude of momentum

Solution. Since the momentum before the release is zero, the momentum after must be zero too. Thus, each mass must have equal and opposite momenta. Therefore, answer choice $\boxed{\text{(D)}}$ is correct.



Problem 7. The two blocks of masses M and $2M$ shown above initially travel at the same speed v but in opposite directions. They collide and stick together. How much mechanical energy is lost to other forms of energy during the collision?

(A) $1/2Mv^2$

(B) $3/4Mv^2$

(C) $4/3Mv^2$

(D) $3/2Mv^2$

Solution. The collision is perfectly inelastic which means,

$$M\mathbf{v} + 2M - \mathbf{v} = (M + 2M)\mathbf{v}'$$

Solving for $\mathbf{v}'$,

$$\mathbf{v}' = \frac{M\mathbf{v} - 2M\mathbf{v}}{M + 2M} = \frac{-\cancel{M}\mathbf{v}}{3\cancel{M}} = -\frac{v}{3}$$

To find the energy lost, subtract the total energy after from the total energy before $(\mathrm{KE}_{1,\,\mathrm{i}} + \mathrm{KE}_{2,\,\mathrm{i}}) - (\mathrm{KE}_{1,\,\mathrm{f}} + \mathrm{KE}_{2,\,\mathrm{f}})$,

$$\left(\frac{1}{2}M\mathbf{v}^2 + \frac{1}{2}\cdot 2M\mathbf{v}^2\right) - \frac{1}{2}\cdot 3M\left(\frac{\mathbf{v}}{3}\right)^2 = \frac{3}{2}M\mathbf{v}^2 - \frac{3}{2}M\left(\frac{\mathbf{v}^2}{9}\right) = \frac{3}{2}M\mathbf{v}^2 - \frac{1}{6}M\mathbf{v}^2 = \boxed{\frac{4}{3}Mv^2}$$

Thus, answer choice $\boxed{\text{(C)}}$ is correct.

Problem 8. A boy of mass m and a girl of mass $2m$ are initially at rest at the center of a frozen pond. They push each other so that she slides to the left at speed v across the frictionless ice surface and he slides to the right. What is the total work done by the children?

(A) mv

(B) mv^2

(C) $2mv^2$

(D) $3mv^2$

Solution. By conservation of momentum $m\mathbf{v}' = -2m\mathbf{v}$. Solving for $\mathbf{v}'$,

$$\mathbf{v}' = \frac{-2\cancel{m}\mathbf{v}}{\cancel{m}} = -2v$$

The total work done is the total kinetic energy $\text{KE}_{\text{tot}} = \text{KE}_{\text{boy}} + \text{KE}_{\text{girl}}$,

$$\text{KE}_{\text{tot}} = \frac{1}{2}m(-2\mathbf{v})^2 + \frac{1}{2}\cdot 2m\mathbf{v}^2 = 2m\mathbf{v}^2 + m\mathbf{v}^2 = \boxed{3m\mathbf{v}^2}$$

Thus, answer choice $\boxed{\text{(D)}}$ is correct.

Problem 9. An object of mass M travels along a horizontal air track at a constant speed v and collides elastically with an object of identical mass that is initially at rest on the track. Which of the following statements is true for the two objects after the impact?

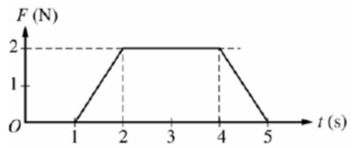
(A) The total momentum is Mv and the total kinetic energy is $\frac{1}{2}Mv^2$.

(B) The total momentum is Mv and the total kinetic energy is less thna $\frac{1}{2}Mv^2$.

(C) The total momentum is less than Mv and the total kinetic energy is $\frac{1}{2}Mv^2$.

(D) The total momentum of each object is $\frac{1}{2}Mv$.

Solution. The collision is elastic which means momentum and energy is conserved. The initial momentum of the system is $M\mathbf{v}$, so the final momentum must also be $M\mathbf{v}$. The initial kinetic energy is $1/2M\mathbf{v}^2$, which means the final kinetic energy must be $1/2M\mathbf{v}^2$. Thus, answer choice $\boxed{\text{(A)}}$ is correct.



Problem 10. A 2 kg object initially moving with a constant velocity is subjected to a force of magnitude F in the direction of motion. A graph of F as a function of time t is shown. What is the increase, if any, in the velocity of the object during the time the force is applied?

(A) 0 m/s

(B) 3.0 m/s

(C) 4.0 m/s

(D) 6.0 m/s

Solution. Increase in velocity is the change in velocity, which can be found from the change in momentum. The change in momentum is the impulse which is the area under the $\mathbf{F}$ vs. t graph. The impulse is then,

$$\frac{1}{2}(1\text{ s})(2\text{ N}) + (2\text{ s})(2\text{ N}) + \frac{1}{2}(1\text{ s})(2\text{ N}) = 6\text{ N}\cdot\text{ s}$$

From the impulse-momentum theorem, $J = \Delta p = m\Delta v$. Therefore,

$$\Delta v = \frac{J}{m} = \frac{6\text{ N}\cdot\text{ s}}{2\text{ kg}} = \boxed{3\text{ m/s}}$$

Thus, answer choice $\boxed{\text{(B)}}$ is correct.

Problem 11. (Source: HRK ed. 5 vol. 1) An object is moving in a circle at constant speed v. The magnitude of the rate of change of momentum of the object

(A) is zero

(B) is proportional to v

(C) is proportional to v^2

(D) sometimes closer to the other.

Solution. In circular motion, $\mathbf{F} = m\frac{\mathbf{v}^2}{r}$. But, $\mathbf{F} = \frac{\Delta \mathbf{p}}{\Delta t}$. Therefore, the change in momentum is proportional to $\mathbf{v}^2$, answer choice $\boxed{\text{(C)}}$.

Problem 12. (Source: HRK ed. 5 vol. 1) If the net force acting on a body is constant, what can we conclude about its momentum?

(A) The magnitude and/or the direction of **p** may change.

(B) The magnitude of **p** remains fixed, but its direction may change.

(C) The direction of **p** remains fixed, but its magnitude may change.

(D) **p** remains fixed in both magnitude and direction.

Solution. A body on which a net force acts is accelerated, which means the velocity changes. Acceleration can change either magnitude, direction, or both. Since momentum is $m\mathbf{v}$, changing the velocity results in a change in momentum. Based on the case of acceleration's effect, the magnitude and/or direction of **p** may change. Answer choice $\boxed{\text{(A)}}$ is correct.

Problem 13. A moderate force will break an egg. However, an egg dropped on the road usually breaks, while one dropped on the grass usually doesn't break. This is because for the egg dropped on the grass:

(A) the change in momentum is greater.

(B) the change in momentum is less.

(C) the time interval for stopping is greater.

(D) the time interval for stopping is less.

(E) Both choices A and C are valid.

Solution. Assume that the egg is dropped from the same height, then the impulse is equal in both cases. $\mathbf{J} = \mathbf{F}\Delta t$. The only way for the grass to prevent the egg from breaking is if it increases the time interval of the collision. Thus, $\boxed{\text{(C)}}$ is correct.

Problem 14. (Source: HRK vol. 5) Two frictionless pucks are connected by a rubber band. One of the pucks is projected across an air table, the rubber band tightens, and the second puck follows – in an apparently random way – the first puck. The center of mass of this two particle system is located

(A) at a fixed distance from one of the pucks.

(B) usually, but not always, between the two pucks.

(C) at a distance from one of the pucks that is fixed ratio to the distance between the two pucks.

(D) sometimes closer to one puck, and sometimes closer to the other.

Solution. The only forces are internal which means, the center of mass will not move. The CoM is located at a point which is the ratio of the distances between the puck. Answer choice $\boxed{\text{(C)}}$ is correct.

Problem 15. (Source: HRK ed. 5 vol 1.) A system of N particles is free from any external forces.

(a) Which of the following is true for the magnitude of the total momentum of the system?

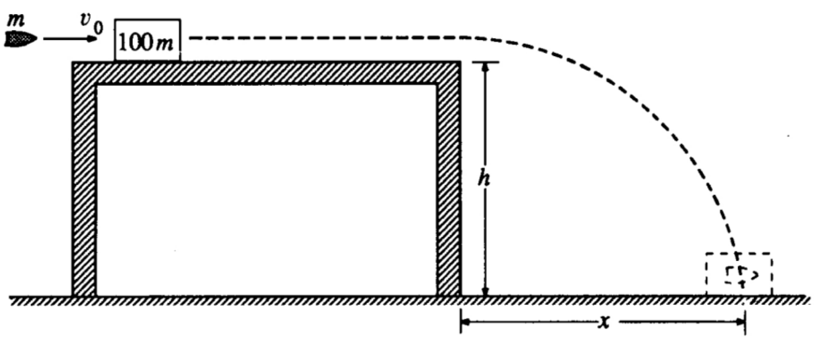
   (A) It must be zero.
   (B) It could be non-zero, but it must be constant.
   (C) It could be non-zero, and it might not be constant.
   (D) The answer depends on the nature of the internal forces in the system.

(b) Which of the following must be true for the sum of the magnitudes of the momenta of the individual particles in the system.

   (A) It must be zero.
   (B) It could be non-zero, but it must be constant.
   (C) It could be non-zero, and it might not be constant.
   (D) It could be zero, even if the magnitude of the total momentum is not zero.

Solution. For part (a), if there are no external forces, the momentum must be constant, therefore the answer is $\boxed{\text{(B)}}$. For part (b), the only constraint is that the system's momentum must be conserved, which means that the momenta of individual particles can change. Therefore, $\boxed{\text{(C)}}$ is the answer.

5.2 Free Response Questions



Problem 1. A bullet of mass m is moving horizontally with speed $\mathbf{v}_0$ when it hits a block of mass $100m$ that is at rest on a horizontal frictionless table, as shown above. The surface of the table is a height h above the floor. After the impact, the bullet and the block slide off the table and hit the floor a distance x from the edge of the table. Derive expressions for the following quantities in terms of m, h, $\mathbf{v}_0$, and appropriate constants:

a. the speed of the block as it leaves the table

b. the change in kinetic energy of the bullet-block system during impact

c. the distance x

Suppose that the bullet passes through the block instead of remaining in it.

d. State whether the time required for the block to reach the floor from the edge of the table would now be greater, less, or the same. Justify your answer.

e. State whether the distance x for the block would now be greater, less, or the same. Justify your answer.

Solution. For part (a), momentum conservation for a perfectly inelastic collision is used to find the speed as it leaves the table,

$$m\mathbf{v}_0 = m + 100m\mathbf{v}$$

Solving for $\mathbf{v}$,

$$\mathbf{v} = \frac{\cancel{m}\mathbf{v}_0}{101\cancel{m}} = \boxed{\frac{\mathbf{v}_0}{101}}$$

For part (b), $\Delta\text{KE} = \text{KE}_f - \text{KE}_i$,

$$\Delta\text{KE} = \frac{1}{2}(101m)\mathbf{v}^2 - \frac{1}{2}m\mathbf{v}_0^2 = \frac{1}{2}(101m)\left(\frac{\mathbf{v}_0}{101}\right)^2 - \frac{1}{2}m\mathbf{v}_0^2 = \boxed{-\frac{50}{101}\mathbf{v}_0^2}$$

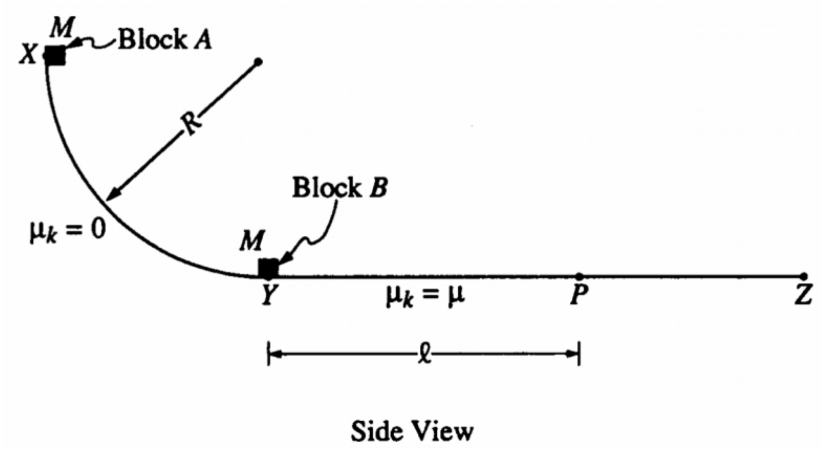
For part (c), the time t taken to fall a distance h can be used to find x using the velocity which is purely horizontal. From the kinematic equation $\Delta y = \mathbf{v}_{i,y}t - \frac{1}{2}gt^2$ (there is no initial y-velocity and Δy is negative),

$$t = \sqrt{\frac{2h}{g}}$$

Using the formula for distance $\Delta x = \mathbf{v}_{i,x}t$,

$$x = \boxed{\frac{\mathbf{v}_0}{101}\sqrt{\frac{2h}{g}}}$$

For part (d), the time required to reach the floor is independent of the horizontal velocity. The y-velocity is still zero, which means the time of flight is still the $\boxed{\text{same}}$. For part (e), when the bullet is embeded in the block, its initial momentum is completely transferred to block-bullet system. However, now the bullet passes through, less momentum is transferred to the block. Therefore, the block has less horizontal velocity which decreases its range; x is $\boxed{\text{smaller}}$.



Problem 2. A track consists of a frictionless arc XY, which is a quarter-circle of radius R, and a rough horizontal section YZ. Block A of mass M is released from rest at point X, slides down the curved section of the track, and collides instantaneously and inelastically with identical block B at point Y. The two blocks move together to the right, sliding past point P, which is a distance L from point Y. The coefficient of kinetic friction between the blocks and the horizontal part of the track is μ Express your answers in terms of M, L, μ, R, and g.

a. Determine the speed of block A just before it hits block B.

b. Determine the speed of the combined blocks immediately after the collision.

c. Assuming that no energy is transferred to the track or to the air surrounding the blocks. Determine the amount of internal energy transferred in the collision.

d. Determine the additional thermal energy that is generated as the blocks move from Y to P

Solution. For part (a), apply conservation of energy to find the final speed,

$$\cancel{\mathrm{KE}_i} + \mathrm{PE}_i = \mathrm{KE}_f + \cancel{\mathrm{PE}_f} \Rightarrow \cancel{M}gR = \frac{1}{2}\cancel{M}\mathbf{v}^2 \Rightarrow \mathbf{v} = \boxed{\sqrt{2gR}}$$

For part (b), the collision is perfectly inelastic,

$$M\mathbf{v} = 2M\mathbf{v}' \Rightarrow \mathbf{v}' = \boxed{\frac{\sqrt{2gR}}{2}}$$

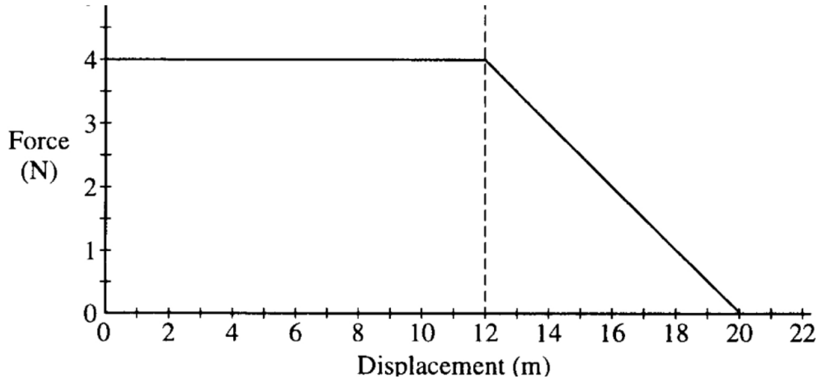
For part(c), use $\Delta\mathrm{KE} = \mathrm{KE}_f - \mathrm{KE}_i$ with the values from momentum and energy conservation,

$$\Delta\mathrm{KE} = \frac{1}{2}(2M)\left(\frac{\sqrt{2gR}}{2}\right)^2 - \frac{1}{2}(M)\sqrt{2gR}^2 = \frac{MgR}{2} - 2MgR = -\frac{MgR}{2}$$

The sign doesn't matter, so $\boxed{\frac{MgR}{2}}$ of internal energy was transferred. For part (d), the thermal energy generated is the work done by friction along the distance[3] from Y to P. Using the formula for the work done by friction, $\mathbf{F}_f \cdot d$,

$$\Delta\mathrm{E}_{\mathrm{therm}} = \mu\mathbf{F}_N \cdot L = \mu \cdot 2Mg \cdot L = \boxed{2\mu MgL}$$

[3]The diagram uses l, however, we'll use L as that was provided in the prompt.



Problem 3. A 0.20 kg object moves along a straight line. The net force acting on the object varies with the object's displacement as shown in the graph above. The object starts from rest at displacement $x = 0$ and time $t = 0$ and is displaced a distance of 20 m. Determine each of the following.

a. The acceleration of the particle when its displacement x is 6 m.

b. The time taken for the object to be displaced the first 12 m.

c. The amount of work done by the net force in displacing the object the first 12 m.

d. The speed of the object at displacement $x = 12$ m.

e. The final speed of the object at displacement $x = 20$ m.

f. The change in the momentum of the object as it is displaced from $x = 12$ m to $x = 20$ m.

Solution. For part (a), the force is constant, so $\mathbf{F} = m\mathbf{a}$ can be used. At $x = 6$ m, the force is 4 N,

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{4\text{ N}}{0.20\text{ kg}} = \boxed{20\text{ m/s}^2}$$

For part (b), the acceleration is constant, so the kinematic formula $\Delta x = \mathbf{v}_{i,x}t + \frac{1}{2}\mathbf{a}t^2$ can be used (initial velocity is zero) to find t,

$$t = \sqrt{\frac{2\Delta x}{\mathbf{a}}} = \sqrt{\frac{2(12\text{ m})}{20\text{ m/s}^2}} = \boxed{1.1\text{ s}}$$

For part (c), we can use the relationship of $W = \mathbf{F} \cdot \mathbf{r}$ to get $W = (4\text{ N}) \cdot (12\text{ m}) = \boxed{48\text{ J}}$.
For part (d), the *Work-Energy Theorem* may be applied,

$$W = \Delta\text{KE} = \text{KE}_f - \cancel{\text{KE}_i} = \frac{1}{2}m\mathbf{v}_f^2$$

Solving for $\mathbf{v}_f$,

$$\mathbf{v}_f = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2(48\text{ J})}{0.20\text{ kg}}} = \boxed{21.9\text{ m/s}}$$

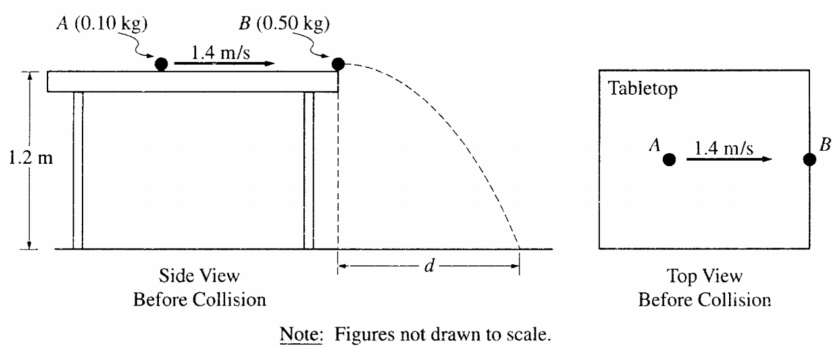
For part (e), we can again use the *Work-Energy Theorem* to find the total work done and solve for the final velocity. The work done from $x = 12$ m to $x = 20$ m is $W = \frac{1}{2}(4\text{ N})(8\text{ m}) = 16$ J. The total work is then $48\text{ J} + 16\text{ J} = 64$ J. Then the final velocity can be found,

$$\mathbf{v}'_f = \sqrt{\frac{2 \cdot W_{\text{tot}}}{m}} = \sqrt{\frac{2(64\text{ J})}{0.20\text{ kg}}} = \boxed{25.3\text{ m/s}}$$

> **Note:** If instead of energy, kinematics and Newton's 2nd Law were used, then the acceleration in the last 8 m would need to be found using the average force over that interval.

For part (f), the change in momentum is,

$$\Delta \mathbf{p} = m(\mathbf{v}'_f - \mathbf{v}_f) = (0.20\ \text{kg})(25.3\ \text{m/s} - 21.9\ \text{m/s}) = \boxed{0.68\ \text{kg}\cdot\text{m/s}}$$



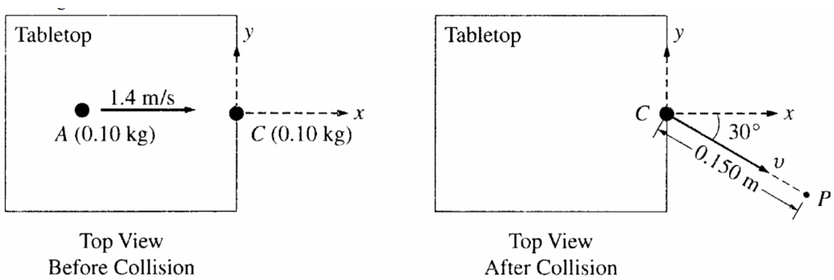
Problem 4. An incident ball A of mass 0.10 kg is sliding at 1.4 m/s on the horizontal tabletop of negligible friction as shown above. It makes a head-on collision with a target ball B of mass 0.50 kg at rest at the edge of the table. As a result of the collision, the incident ball rebounds, sliding backwards at 0.70 m/s immediately after the collision.

a. Calculate the speed of the 0.50 kg target ball immediately after the collision.

   The tabletop is 1.20 m above a level, horizontal floor. The target ball is projected horizontally and initially strikes the floor at a horizontal displacement d from the point of collision.

b. Calculate the horizontal displacement.

   In another experiment on the same table, the target ball B is replaced by target ball C of mass 0.10 kg. The incident ball A again slides at 1.4 m/s, as shown below left, but this time makes a glancing collision with the target ball C that is at rest at the edge of the table. The target ball C strikes the floor at point P, which is at a horizontal displacement of 0.15 m from the point of the collision, and at a horizontal angle of 30°from the $+x$-axis, as shown below right.



c. Calculate the speed v of the target ball C immediately after the collision.

d. Calculate the y-component of incident ball A's momentum immediately after the collision.

Solution. For part (a), we can use conservation of momentum,

$$m_A \mathbf{v}_{A,i} + \cancel{m_B \mathbf{v}_{B,i}} = m_A \mathbf{v}_{A,f} + m_B \mathbf{v}_{B,f}$$

Solving for $\mathbf{v}_{B,f}$,

$$\mathbf{v}_{B,f} = \frac{m_A(\mathbf{v}_{A,i} - \mathbf{v}_{A,f})}{m_B} = \frac{(0.10\text{ kg})[1.4\text{ m/s} - (-0.70\text{ m/s})]}{0.50\text{ kg}} = \boxed{0.42\text{ m/s}}$$

For part (b), calculate the time of flight and the distance traveled using kinematics. The time of flight t_{fl} can be found from the following equation (where h is negative),

$$h = \cancel{\mathbf{v}_{B,fz} t_{\text{fl}}} - \frac{1}{2} g t_{\text{fl}}^2 \Rightarrow t_{\text{fl}} = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1.20\text{ m})}{9.81\text{ m/s}^2}} = 0.49\text{ s}$$

Then, the displacement is,

$$d = \mathbf{v}_{B,fx} t_{\text{fl}} = (0.42\text{ m/s})(0.49\text{ s}) = \boxed{0.21\text{ m}}$$

For part (c), the time of flight is still the same as part (b), so using kinematics to solve for the x-velocity,

$$d = \mathbf{v}_{C,fx} t_{\text{fl}} \Rightarrow \mathbf{v}_{C,fx} = \frac{d}{t_{\text{fl}}} = \frac{0.15\text{ m}}{0.49s} = \boxed{0.31\text{ m/s}}$$

For part (d), the momentum is conserved in the y dimension. The initial y-momentum is zero, so the final momenta of balls A and C must be equal and opposite. Hence,

$$\mathbf{p}_{A,f} = -\mathbf{p}_{C,f} = m_C \mathbf{v}_{C,fy} = (0.10\text{ kg})(0.31\text{ m/s})\sin 30^\circ = \boxed{0.0155\text{ kg}\cdot\text{m/s}}$$