Final Project for Financial Securities and Markets

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1 Problem Statement

We would like to buy a contract paying at maturity T the amount in USD:

$$max\{0, (\frac{S(t)}{S(0)} - k) * (k' - \frac{Libor(T-\Delta, T-\Delta, T)}{Libor(0, T-\Delta, T)})\} \tag{1}$$

with:
• $S(t)$ the Nikkei-225 spot price quantoed from JPY into USD
• $L(t, T-\Delta, T)$ is the 3-month USD LIBOR rate between $T-\Delta$ and T
• T the expiration date (e.g. 3 year)
• k, k' given relative strike prices (e.g. both could be 1 or ...)

Provide a pricing routine (e.g. Python script) calculating the price of this contract, taking as inputs the deal term (T, k, k') and relevant market data.

Explain your assumptions and methodology/ choices clearly in an accompany write-up

2 Problem Analysis

The given contract payoff is calculated by the following parameters:
• $S(t)$ - NIKKEI-225 index price at time t quoted in USD
• $S(0)$ - NIKKEI-225 index price at the contract time, $t = 0$
• $Libor(0, T-\Delta, T)$ - the three-month USD libor at the contract time, $t = 0$
• $Libor(T-\Delta, T-\Delta, T)$ - the three-month USD LIBOR at the time $T-\Delta$ where T is the maturity time.
• k and k' are constants

Since $S(0)$ and $Libor(0, T-\Delta, T)$ are known at the time of the contract, and k and k' are constants, our pricing routines will calculate or simulate $S(t)$ and three-month USD LIBOR at time $T-\Delta$. In this report, we will use Black-Scholes to model equity S(t) and two different short rate models to simulate three-month USD LIBOR and compare the results.

3 Methodology

3.1 Equity Market models

We assume the equity price and the exchange rate can never be negative, thus the dynamics of the NIKKEI-225 index and the exchange rate value processes will both be driven by geometric brownian motion. We let X_t denote the exchange rate from JPY to USD at time t, and define:

$$X_t = \frac{\text{the units of USD}}{\text{the units of JPY}}$$

Given the contract is quoted in USD, we let USD be the domestic currency and JPY be the foreign currency. These are denoted by superscript d and f respectively.

The dynamics of the exchange rate under the Q^d martingle measure is:

$$dX_t = X_t(r_t^d - r_t^f)dt + X_t\sigma_t^1 dW_t^1 \tag{2}$$

where

- X_t is the exchange rate from JPY to USD at time t
- r_t^d is the risk-free rate in the domestic currency (USD)
- r_t^f is the risk-free rate in the foreign currency (JPY)
- σ_t^1 is the volatility of the exchange rate change (JPY to USD) at time t

The dynamics of NIKKEI-225 index value process quoted in JPY are:

$$dS_t^f = S_t^f(r_t^f - q)dt + S_t^f\sigma_t^2 dW_t^2 \tag{3}$$

where

- S_t^f is the NIKKEI-225 index price at time t quoted in JPY
- r_t^f is the risk-free rate in the foreign currency, JPY
- q is the dividend ratio of the NIKKEI-225 index
- σ_t^2 is the volatility of the NIKKEI-225 index price change in JPY at time t

Please note that W_t^1 and W_t^2 are correlated brownian motions and we denote their correlation by ρ_{12}, thus $dW_t^1 dW_t^2 = -\rho_{12}dt$

We let S_t^d denote the value process for the NIKKEI-225 index price quoted in USD and we have $S_t^d = S_t^f * \frac{1}{X_t}$. Then the complete model for equity is as below:

$$\begin{aligned} dX_t &= X_t(r_t^d - r_t^f)dt + X_t\sigma_t^1 dW_t^1 \\ dS_t^f &= S_t^f(r_t^f - q)dt + S_t^f\sigma_t^2 dW_t^2 \\ S_t^d &= S_t^f * \frac{1}{X_t} \end{aligned} \tag{4}$$

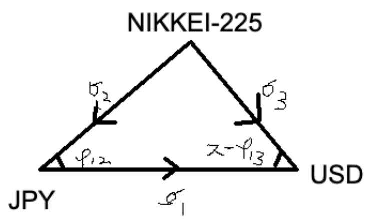
First, we apply Ito formula to calculate $d\frac{1}{X_t}$, then we have

$$
\begin{aligned}
d\frac{1}{X_t} &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \frac{1}{X_t}}d\frac{1}{X_t} + \frac{1}{2}\frac{\partial^2 f}{\partial \frac{1}{X_t}^2}(d\frac{1}{X_t})^2 \\
&= (r_t^d - q + \sigma_1^2)\frac{1}{X_t}dt - \sigma_1\frac{1}{X_t}dW_t^1
\end{aligned} \tag{5}
$$

Then we apply Ito formula to calculate dynamics to $S_t^f = \frac{S_t^d}{X_t}$ and we receive,

$$
\begin{aligned}
d\frac{S_t^d}{X_t} &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \frac{1}{X_t}}d\frac{1}{X_t} + \frac{\partial f}{\partial S_t^d}dS_t^d + \frac{1}{2}\frac{\partial^2 f}{\partial \frac{1}{X_t}^2}(d\frac{1}{X_t})^2 + \frac{1}{2}\frac{\partial^2 f}{\partial \frac{1}{X_t}\partial S_t^d}(d\frac{1}{X_t})dS_t^d + \frac{1}{2}\frac{\partial^2 f}{\partial {S_t^d}^2}(dS_t^d)^2 \\
&= 0 + S_t^d d\frac{1}{X_t} + \frac{1}{X_t}dS_t^d + 0 + 0 + dS_t^d d\frac{1}{X_t} \\
&= (r_t^f - q + \sigma_1^2 + \rho_{12}\sigma_1\sigma_2)S_t^f dt + S_t^f(\sigma_2 dW_t^2 + \sigma_1 dW_t^1)
\end{aligned} \tag{6}
$$

According to [1], we can use a triangle below to represent the correlation among W^1, W^2 and W^3.



The three vertices in the triangle above represent NIKKEI-225, USD and JPY. The edge between two vertices have an arrow pointing to the base currency. For example, if NIKKEI-225 quoted in USD, then the edge connecting two vertices of NIKKEI-225 and USD will have an arrow pointing from Nikkei-225 to USD, and the arrow pointing from vertex JPY to USD represents the exchange rate from JPY to USD. Additionally, the angle between two edges is $\rho_{v^1v^2}$ or $\pi - \rho_{v^1v^2}$ if the vertex of the angle has two arrows pointing towards it. From this correlation triangle, we have

$$
\begin{aligned}
dW_t^1 dW_t^2 &= -\rho_{12}dt \\
\sigma_3^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 \\
\sigma_2^2 &= \sigma_1^2 + \sigma_3^2 + 2\rho_{13}\sigma_1\sigma_3
\end{aligned} \tag{7}
$$

Which yields

$$
\sigma_1^2 + \rho_{12}\sigma_1\sigma_2 = \rho_{13}\sigma_1\sigma_3 \tag{8}
$$

We substitute equation (8) into equation (6), we get the following:

$$
\begin{aligned}
dS_t^d &= d\frac{S_t^f}{X_t} \\
&= (r_t^f - q + \sigma_1^2 + \rho_{12}\sigma_1\sigma_2)S_t^f dt + S_t^f(\sigma_2 dW_t^2 + \sigma_1 dW_t^1) \\
&= (r_t^f - q + \rho_{13}\sigma_1\sigma_3)S_t^f dt + S_t^f(\sigma_2 dW_t^2 + \sigma_1 dW_t^1)
\end{aligned} \tag{9}
$$

Where
- r_t^f is the risk-free rate in JPY
- q is the dividend yield of NIKKEI-225 index
- ρ_{13} is the correlation between the exchange rate and the NIKKEI-225 price change in USD
- σ_1 is the volatility of the exchange rate change, X_t
- σ_3 is the volatility of the NIKKEI-225 Index price converted to USD, S_t^d

If we let σ_3 and dW_t^3 be the volatility and the driving force of S_t^d, then the equation (9) becomes:

$$
\begin{aligned}
dS_t^d &= (r_t^f - q + \rho_{13}\sigma_1\sigma_3)S_t^f dt + S_t^f \sigma_3 dW_t^3 \\
\sigma_3 dW_t^3 &= \sigma_2 dW_t^2 + \sigma_1 dW_t^1
\end{aligned} \tag{10}
$$

The solution to equation (10) given as below:

$$
S(t)^d = S(0)^d exp\{(r_t^f - q - \rho_{13}\sigma_1\sigma_3 - \frac{1}{2}\sigma_3^2)t + \sigma_3\sqrt{T}\epsilon\} \tag{11}
$$

In our simulation, we can calculate S(T) using the following equations:

$$
\begin{aligned}
One-step: S(T)^d &= S(0)^d exp\{(r^f - q - \rho_{13}\sigma_1\sigma_3 - \frac{1}{2}\sigma_3^2)T + \sigma_3\sqrt{T}\epsilon\} \\
Multi-step: S_{t+1} - S_t &= (r_f - q - \rho_{13}\sigma_1\sigma_3)S_t dt + \sigma_3 S_t\sqrt{dt}\epsilon
\end{aligned} \tag{12}
$$

Where
- r^f is the risk-free rate for NIKKEI-225 index during time range [0, T] in JPY
- ϵ is a random sequence with a standard normal distribution.

4 Short Rate Models

Given the contract in equation (1), the calculation of contract payoff is linked to equity performance and LIBOR performance. Thus, we need to calculate the three-month USD LIBOR at time $t = T - \Delta$, $Libor(0, T - \Delta, T)$, as well as the equity price. $Libor(0, T - \Delta, T)$ is the three-month USD LIBOR which is known at time $t = 0$. The three-month USD LIBOR at time $t = T - \Delta$ can be calculated by equations below given by Bjork [2]:

$$
\begin{aligned}
L(t;S,T) &= -\frac{p(t,T)-p(t,S)}{(T-S)p(S,T)} \\
L(T-\Delta,T-\Delta,T) &= -\frac{p(T-\Delta,T)-p(T-\Delta,T-\Delta)}{(T-S)p(T-\Delta,T)} \\
&= -\frac{p(T-\Delta,T)-1}{\Delta * p(T-\Delta,T)}
\end{aligned} \tag{13}
$$

And the bond price $p(T-\Delta,T)$ can be computed from Risk-Neutral formula:

$$
\begin{aligned}
p(t,T) &= E^Q_{t,r}[exp\{-\int_t^T r_s ds\}] \\
p(T-\Delta,T) &= E^Q_{t,r}[exp\{-\int_{(T-\Delta)}^T r_s ds\}]
\end{aligned} \tag{14}
$$

Where r_s is the short rate at time $T-\Delta+ds$ in time period $[T-\Delta,T]$. Next, we will provide two models that can be used to calculate short rate, Hull-White and Log-normal LIBOR. In the simulation section, we provide results from Monte-Carlo simulations based on the Hull-White and Log-normal LIBOR models and offer the implementation source code at the end.

4.1 Hull-White

Let assume the short rate dynamics is modelled by Hull-White model under martingle measure Q^d, given by [2]

$$dr(t) = (\theta(t) - ar(t))dt + \sigma_r dV^{Q^d}(t)$$

where
- a is a constant
- σ_r is the volatility of short rate
- V^{Q^d} is a brownian motion under martingale measure, Q^d, and
- $\theta(t)$ is defined as below:

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\sigma_r^2}{2a}(1-e^{-2at})$$

The simulation includes five steps:

Step 1. Calculate instantaneous forward rate $f(t,T)$
In this step, we use Nelson-Siegel model [4] to help calculate instantaneous forward rate. The Nelson-Siegel model is extensively used by central banks and monetary policy makers [3]. This model transformed the nonlinear estimation problem into a simple linear problem by fixing the shape parameter that causes the nonlinearity.

In [4], Nelson and Siegel specify the forward rate curve f(r) as follows

$$f(\tau) = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ e^{-\frac{\tau}{\lambda}} \\ (\frac{\tau}{\lambda})e^{-\frac{\tau}{\lambda}} \end{bmatrix} \tag{15}$$

where τ is time to maturity, β_0, β_1, β_2 and λ are coefficients, with $\lambda > 0$. and our $f(0, T)$ can be obtained from the equation below :

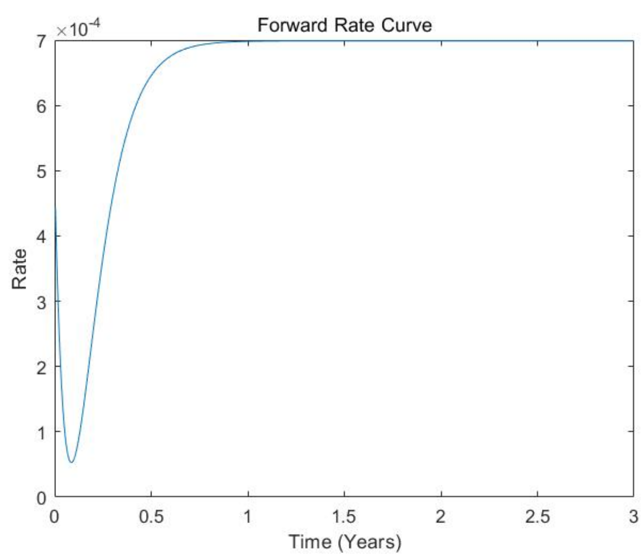
$$f(t, T) = \beta_0 + \beta_1 e^{\frac{-T}{\lambda}} + \beta_2 \frac{T}{\lambda} e^{\frac{-T}{\lambda}}$$

In our simulation, the zero-coupon bond rate, c_T, is defined by

$$c_T = \beta_0 + \beta_1 \frac{\lambda}{T}(1 - e^{\frac{-T}{\lambda}}) + \beta_2(\frac{\lambda}{T}(1 - e^{\frac{-T}{\lambda}}) - e^{\frac{-T}{\lambda}}) \tag{16}$$

If we set λ as a fixed value, then we can run a linear regression to evaluate the three other parameters, β_0, β_1, β_2, with different maturities, T, in [0,10]. We proceed a grid search on λ. For each λ, we use a linear regression to find different sets of β_0, β_1, β_2. Then choose the one with highest R^2 that is a statistical measure of how well the regression predictions approximate the real data points.

We fed linear regressions with zero-coupon bond rates with maturities from one month to 1 year avalable on May 10, 2021[6]. The resulting parameters, β_0, β_1 and β_2, generate the forward rate curve as below:



$$maturity = [1/12, 2/12, 3/12, 6/12, 1];$$
$$rate = [0.02, 0.01, 0.02, 0.04, 0.05]$$

Step 2. Calculate $\theta(t)$
After calculating the forward rate curve, we are now able to calculate $\theta(t)$ in the Hull-White Model

$$\theta(t) = \frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\sigma_r^2}{2a}(1 - e^{-2at})$$

where

$$f(0,t) = \beta_0 + \beta_1 e^{\frac{-t}{\lambda}} + \beta_2 \frac{t}{\lambda} e^{\frac{-t}{\lambda}}$$

and

$$\frac{\partial f(0,t)}{\partial T} = \frac{1}{\lambda} e^{\frac{-t}{\lambda}} (-\beta_1 + \beta_2(1 - \frac{t}{\lambda}))$$

Step 3. Given $\theta(t)$, calculate sequence $r(t)$

$$r_{t+1} - r_t = (\theta(t) - ar_t)dt + \sigma_r \sqrt{dt}\tilde{\epsilon}$$

where $\tilde{\epsilon}$ is a standard normal variable correlated with another standard normal variable, ϵ, that is used in the equity simulation. We calculate $\tilde{\epsilon}$ by first constructing two independent standard normal sequences x_1 and x_2. Then we calculate ϵ and $\tilde{\epsilon}$ by Cholesky decomposition:

$$\epsilon = x_1$$

$$\tilde{\epsilon} = \rho_{sr} x_1 + \sqrt{1 - \rho_{sr}^2} x_2$$

Step 4. Calculate $p(T - \Delta, T)$
From Bjork[2] it is given:

$$p(T-\Delta, T) = \frac{p(0,T)}{p(0, T-\Delta)} exp\{B(T-\Delta,T)f(0,T-\Delta) - \frac{\sigma_r^2}{4a} B^2(T-\Delta,T)(1-e^{-2at}) - B(T-\Delta,T) r_{T-\Delta}\} \tag{17}$$

where

$$B(T - \Delta, T) = \frac{1}{a}(1 - e^{-a\Delta}) \tag{18}$$

and

$$\begin{aligned} p(0,T) &= \frac{1}{(1 + c_T)^T} \\ c_T &= \beta_0 + \beta_1 \frac{\lambda}{T}(1 - e^{\frac{-T}{\lambda}}) + \beta_2(\frac{\lambda}{T}(1 - e^{\frac{-T}{\lambda}}) - e^{\frac{-T}{\lambda}}) \end{aligned} \tag{19}$$

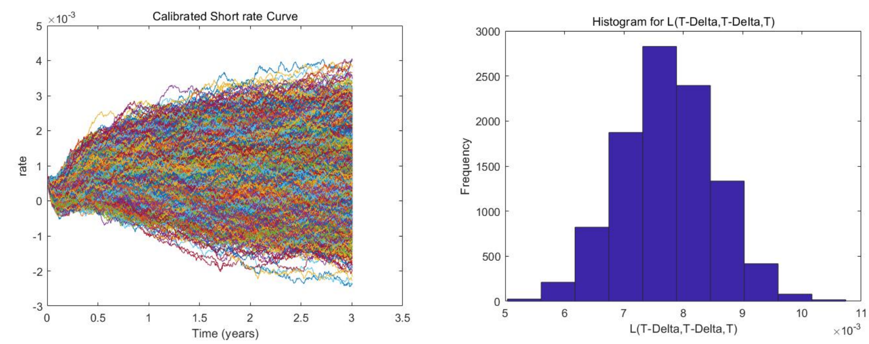
and $p(0, T - \Delta)$ is calculated by interpolation of the yield curve.

Step 5. Calculate LIBOR rate

Given $p(T-\Delta)$, we are using the following equation to calculate $L(T-\Delta, T-\Delta, T)$

$$L(T-\Delta, T-\Delta, T) = \frac{1 - p(T-\Delta, T)}{\Delta p(T-\Delta, T)} \tag{20}$$

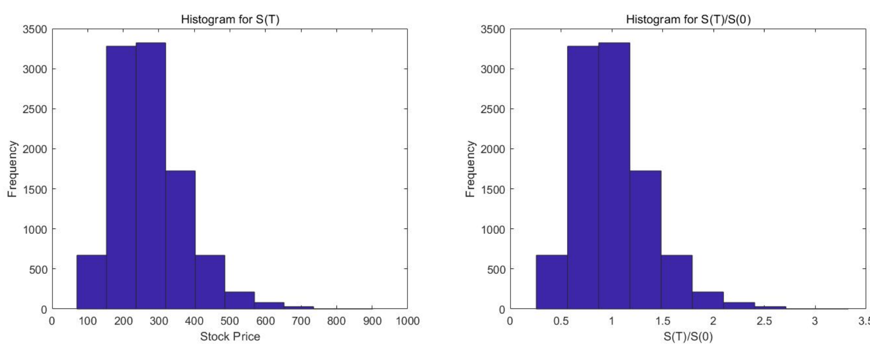
4.1.1 Simulation Results



(a) Calibrated Short Rate Curve

(b) LIBOR Outcomes

Figure 1: Monte Carlo Simulations from Hull-White



(a) S(T)

(b) S(T)/S(0)

Figure 2: Monte Carlo Simulation for Equity Correlated with Hull-White

4.2 Log Normal LIBOR

Given that LIBOR rate, $L(t, T-\Delta, T)$, is a martingale under Q^T with

$$\frac{dL}{L} = \sigma_L dV^{Q^T} \tag{21}$$

In order to simulate the contract payoff, we also need to simulate the equity price, S and $\frac{dS}{S}$, at the same martingale measure, Q^T. The equity dynamics under Q^T given as below:

$$\frac{dS}{S} = (r_f - q - \rho_{sx}\sigma_s\sigma_x + \sigma_s\sigma_p\rho_{sp})dt + \sigma_s dW^{Q^T} \tag{22}$$

Where
• r_f is the risk-free rate in JPY
• q is the dividend yield of the equity, NIKKEI-225 index
• ρ_{sx} is the correlation between the equity and the exchange rate
• σ_s is the volatility of the equity under martingle measure Q^T
• σ_x is the volatility of the exchange rate
• ρ_{sp} is the correlation between the equity and the zero-coupon bond
• σ_p is the volatility of the zero-coupon bond

The quanto correction is $r_f - q - \rho_{sx}\sigma_s\sigma_x$ and $\sigma_s\sigma_p\rho_{sp}$ is the drift term derived from Girsanov's likelihood kernal where $L = \frac{dQ^T}{dQ}$ and $\frac{dL}{L} = \varphi dW^Q$ where $\varphi = \sigma_p - 0$, and is the difference between the two numeraires, Q and Q^T. Please note that under Q, the numeraire is the money account, B, whose volatility is zero, while under Q^T, the numeraire is p(t, T). As a result, σ_p is the volatility of the zero coupon bond with maturity T, and ρ_{sp} is the correlation between the equity and the zero coupon bond with maturity T. For example, if we set the contract's maturity to be three years, then the correlation, ρ_{sp}, is between the equity and the three year zero-coupon bond.

During our simulation, we let ρ_{sL} be the corrleation between the equity and LIBOR, and assume $\rho_{sL} = -\rho_{sp}$. Other coeffecients, r_f, σ_s, σ_x and ρ_{sx} are derived from market data, where
• r_f is the risk-free foreign interest rate, we use Japan 3 Year government bond yield, -0.00138.
• σ_s is the volatility of the NIKKEI-225 equity, 0.1994, calculated from NI225 one year price curve converted in USD
• σ_x is the volatility of the exchange rate USD/JPY, -0.3049, calculated from one year USD/JPY exchange rate data
• ρ_{sx} is the correlation between NIKKEI equity in USD and the exchange rate USD/JPY, calculated from NI225 one year price curve and exchange rate data

We then simulate the equity price at maturity, S_T, with a one-step Monte-Carlo by equation below:

$$S_T = S_0 exp\{(r_f - q - \rho_{sx}\sigma_s\sigma_x + \sigma_s\sigma_p\rho_{sp} - \frac{1}{2}\sigma_s^2)T + \sigma_s\sqrt{T}\epsilon\} \tag{23}$$

We simulate LIBOR rate at time $t = T - \Delta$ with equation below:

$$\begin{aligned} L(T-\Delta, T-\Delta, T) &= L(0, T-\Delta, T) exp\{\sigma_L \sqrt{T-\Delta}\tilde{\epsilon} - \frac{1}{2}\sigma_L^2 (T-\Delta)\} \\ L(0, T-\Delta, T) &= \frac{-p(0,T) - p(0, T-\Delta)}{\Delta * p(0,T)} \end{aligned} \tag{24}$$

ϵ and $\tilde{\epsilon}$ from equations, (23) and (24), are correlated via ρ_{sL} and simulated using a modified version of Cholesky decomposition as below:

$$\begin{aligned} \epsilon &= X_1, and \\ \tilde{\epsilon} &= \rho X_1 + X_2\sqrt{1-\rho^2} \end{aligned} \tag{25}$$

where X_1 and X_2 are independent random samples from two normal distributions.

Based on the LIBOR portion above, equation (24) and the ratio of $\frac{S(T)}{S(0)}$ from our one-step simulation of S(t), equation (23) we can then calculate the payoff (π) as below

$$max[0, (\frac{S(T)}{S(0)} - k) * (k' - \frac{L(T-\Delta, T-\Delta, T)}{L(0, T-\Delta, T)})] \tag{26}$$

where we assume $k = 1$ and $k' = 1$. We then discount back using a constant discount rate of 0.32% which is the 3-year US Treasury bond.

4.2.1 Simulation Results

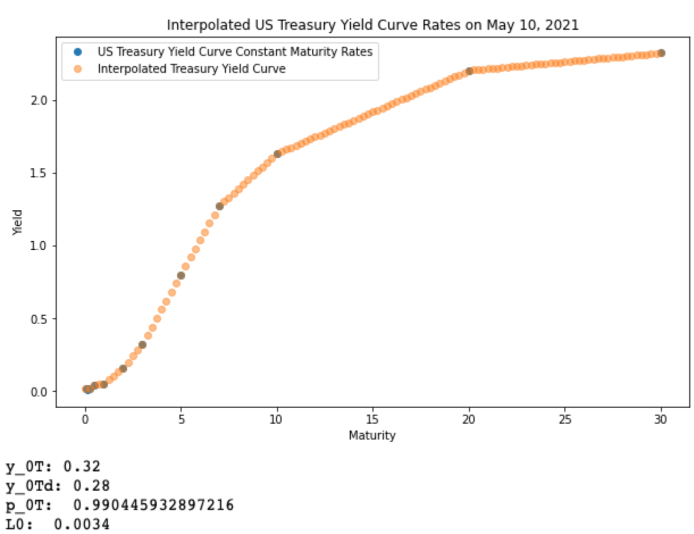


Figure 3: Interpolated US Treasury Yield Curve Rates

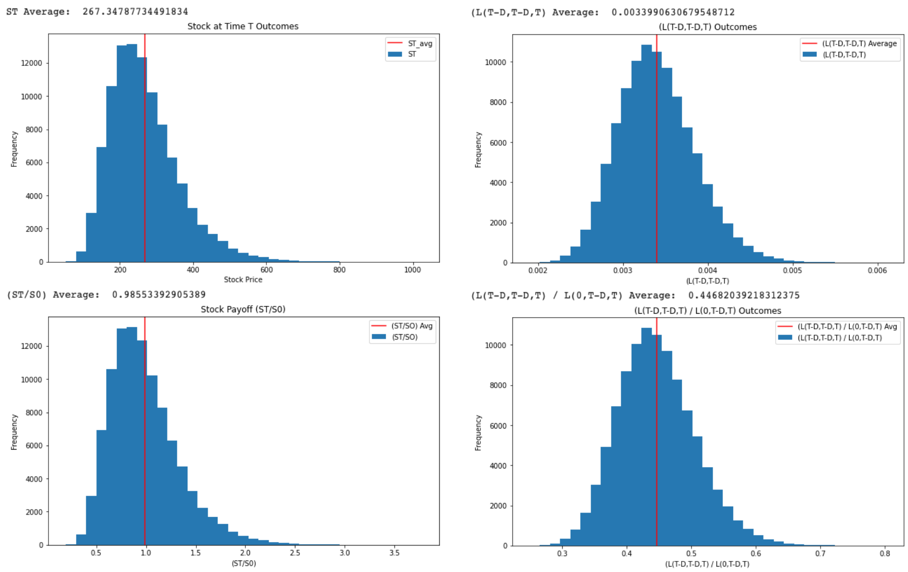


Figure 4: Simulations on Equity v.s. LIBOR

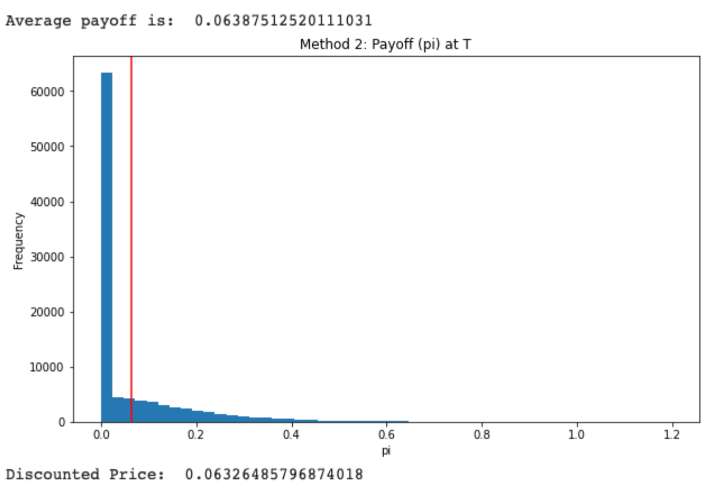


Figure 5: Payoff Outcomes

5 Data Analysis and Visualization

The following table presents the final results from two methods: 1) Hull-White, 2) Log-Normal LIBOR.

Table 1:
Simulation Results:
Hull-White v.s. Log-Normal LIBOR

Model	k	k'	$S(T)/S(0)$ avg	S(T)/S(0) - k	$L(0,T-\Delta,T)$	$L(0,T-\Delta,T)$	Payoff Avg	Contract Price Avg
Hull-White Approach	1	1	1.0112	0.0112	-0.0077	-0.0076	0.2906	0.2896
Log-Normal LIBOR Approach	1	1	0.9855	-0.0144	0.0076	0.0034	0.0643	0.0636

The results from method 1 (Hull-White), \$0.29, and method 2 (Log-Normal LIBOR), \$0.06, show a moderate difference of \$0.25. Since the inputs outlined in section 4.2 are the same across both models, we believe the difference in pricing is largely due to the numerical techniques used to simulate the terminal values for the equity and for the LIBOR portions of the contract. First, the two models, differ in the number of steps used in the simulation, with method 1 (Hull-White) using a multi-step Euler scheme to arrive at the terminal values, while method 2 (Lognormal) leverages a one-step Monte-Carlo.

Despite these differences, the results for the equity portion of the contract, $\frac{ST}{S0}$, are quite similar: method 1: 1.0112 v.s. method 2: 0.9855. This is likely due to the minimal differences between the models for ST, which both assume log normal dynamics. Besides the number of steps in the simulation for the equity portion of the contract, the only difference from a model perspective is the Girsanov change of measure term which modifies the drift for method 2, and is not present in method 1. However, this difference is small since the Girsanov change of measure term is approximately 0.0065. As a result, we do not see a large difference in the result for $\frac{ST}{S0}$ across the two methodologies (method 1: 1.0112, method 2: 0.9855).

In the LIBOR portion of the contract, we do see larger differences in the results for $\frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)}$ (hereafter "LIBOR ratio") with method 1 showing - 1.013 and method 2 showing 0.4468. This is likely due to the significant differences in the model assumptions (method 1: Hull-White vs. method 2: Black Lognormal), as well as the mechanics of the simulation (multi-step vs. one step). In method 2 specifically, the simulation for the numerator of the LIBOR ratio has only two sources of variability, the parameter (σ_L) and $\tilde{epsilon}$, which is a correlated normal random variable with a correlation coefficient ($\rho_{sL} = 0.65$).

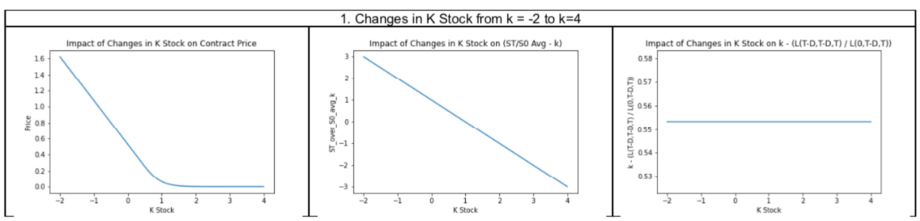
As a result, in method 2, there is a limited distribution of terminal values for the LIBOR Ratio, since we assume that σ_L is relatively small (0.05) and $L(0, T-\Delta, T)$ is known at time 0. The sensitivity analysis for method 2 below confirms this result, where we show in scenario 4 the results of changes in ρ_{sl} from -1.00 to 1.00, while holding all other variables in the contract constant. Per the graph below, changing ρ_{sl} from -1.00 to 1.00 only results in small changes to the LIBOR ratio from 0.4471 to 0.44685. Additionally, in scenario 5, we run a similar analysis for σ_l, where we toggle σ_l from 0% to 100%. Although this is an unreasonably large range, given our assumed σ_l is 5%, the range 0% to 100% is helpful for demonstration purposes. Once again, the results show that σ_l has a limited impact on the change in the LIBOR Ratio, with the LIBOR Ratio only shifting from 0.4441 to 0.4469 as σ_l goes from 0% to 100%.

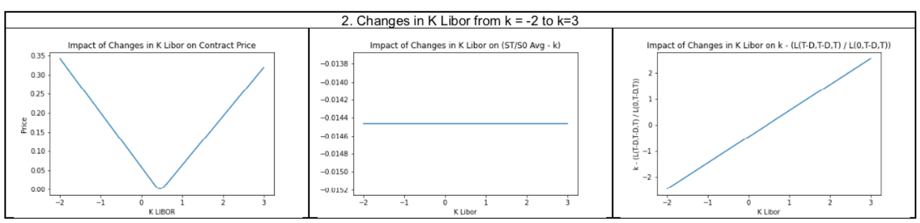
Keeping the above variables constant, we plot the impact of the following changes on

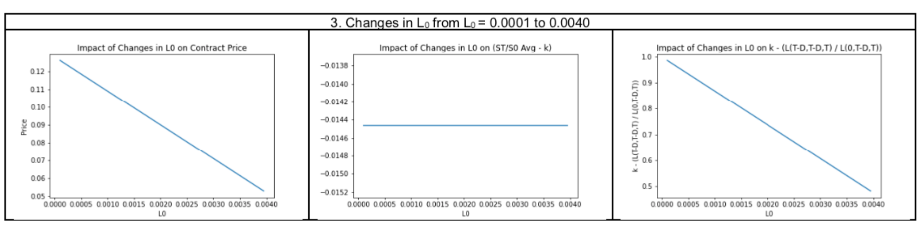
- contract price,
- $\frac{S(T)}{S(0)} - k$,
- $k - \frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)}$

Changes in the scenarios that follow are:

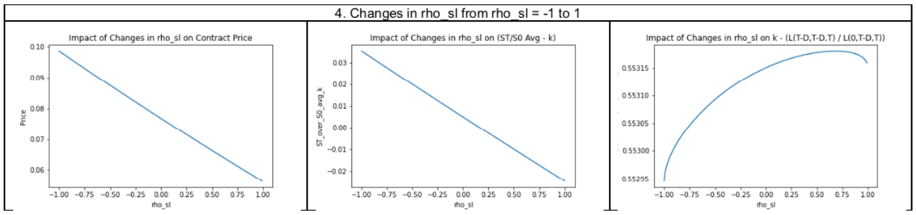
- changes in K Stock from $k = -2$ to $k = 4$,
- changes in K Libor from $k = -2$ to $k = 3$,
- changes in L_0 from $L_0 = 0.0001$ to $L_0 = 0.0040$,
- changes in ρ_{sl} from $\rho_{sl} = -1$ to $\rho_{sl} = 1$
- changes in σ_l from $\sigma_l = 0$ to $\sigma_l = 100\%$
- changes in σ_p from $\sigma_p = 0$ to $\sigma_p = 200\%$
- changes in T from $T = 0$ to $T = 3.5$ years
- changes in q from $q = 0$ to $q = 50\%$



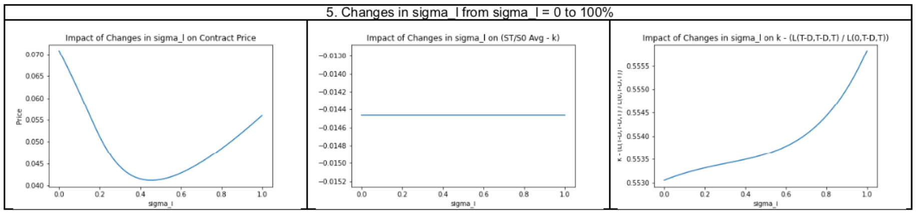




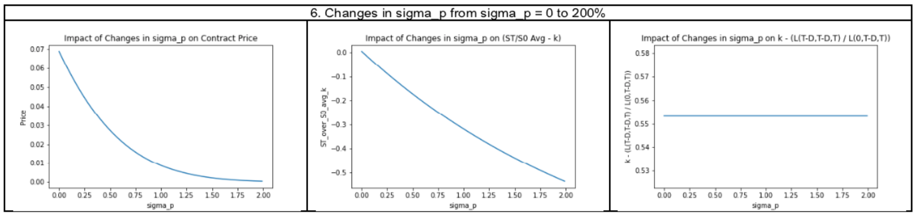
4. Changes in rho_sl from rho_sl = -1 to 1



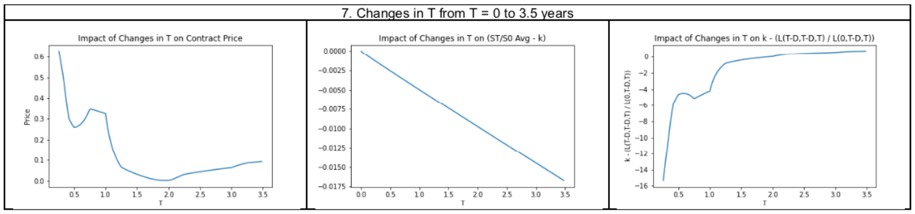
5. Changes in sigma_l from sigma_l = 0 to 100%



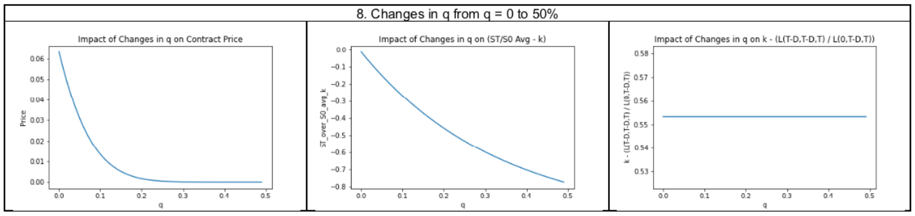
6. Changes in sigma_p from sigma_p = 0 to 200%



7. Changes in T from T = 0 to 3.5 years

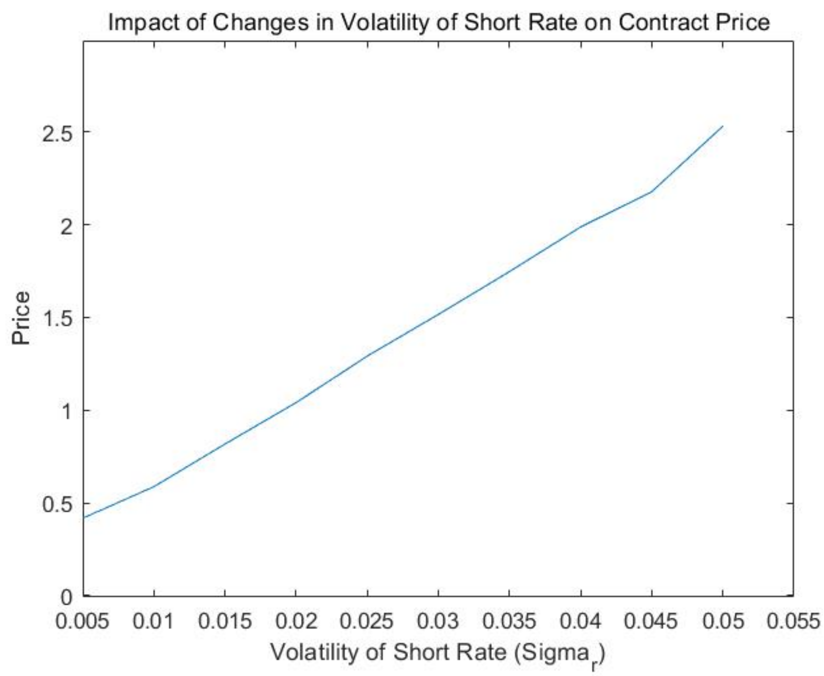


8. Changes in q from q = 0 to 50%

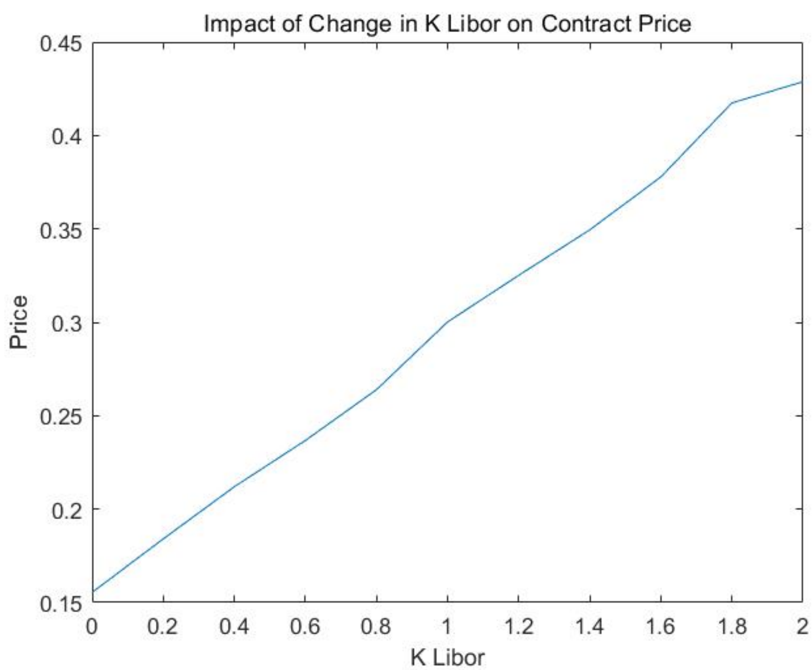


We conducted sensitivity analysis for our Hull-White simulation with regards to parameters, σ_r, k_{libor} and k_{stock}. Our sensitivity analysis tests each parameter with different values while keep other parameters fixed.

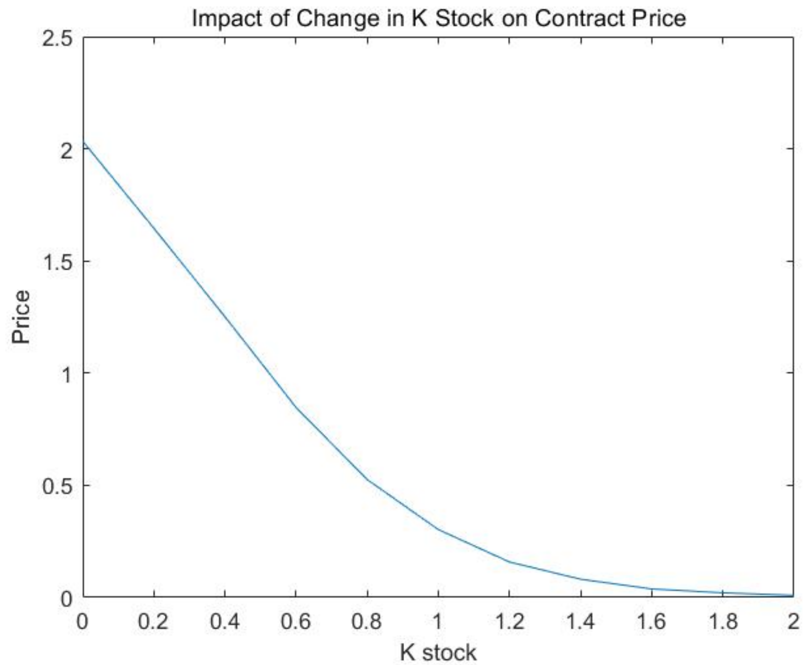
We observed the contract price is increasing with the short rate's volatility simutaneously. The simutaneous increasing property, we think, is contributed by our simulation time step where $dt = 1$ day, where the volatility of interest rate is recorded every day. The contract price is linked to the 3-month USD LIBOR relative performance, $\frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T)}$. The volatility of short rate can be accumulated during the index tenor that is 3 months. The following graph shows how the contract price is changing with short rate volatility.



We test different scenarios by setting k_{libor}'s value in range of [0,2] and make the other parameters fixed. Interestingly, we find the contract price is simutaneously increasing with k_{libor}. The reason is straightforward due to the payoff contract is defined as $max\{0, (\frac{S(T)}{S(0)} - k_{stock})(k_{libor} - \frac{L(T-\Delta,T-\Delta,T)}{L(0,T-\Delta,T))}\}$ which is a linear function. The k_{libor} directly contributing to the payoff.


Impact of Change in K Libor on Contract Price
0.45
0.4
0.35
0.3
0.25
0.2
0.15
Price
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
K Libor

Similarly, according to the payoff contract, the parameter, k_{stock}, contributes to the contract payoff inversely. When k_{stock} is increasing, the contrct payoff is decreasing.



6 Source Codes

Source codes of simulations are accessible from the following links:

1) Hull-White simulation - equity and LIBOR implementations

2) Log-Normal LIBOR simulation- equity and LIBOR implementation and sensitivity analysis

References

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